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CONTINUATION OF THEORETICAL AND EXPERIMENTAL RESEARCH ON DIGITAL ADAPTIVE CONTROL SYSTEM

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FOREWORD

This Digital Adaptive Control System research continuation was sponsored by the National Aeronautics and Space Administration, Langley Research Center, under Contract No. NAS1-6669.

These studies were conducted by the Electronics and Space Division of the Emerson Electric Company at St. Louis, Missouri in the period September 16, 1966 through February 16, 1967. The principal investigator for this research activity was Dr. John Zaborszky, Emerson Consultant. Mr. Richard Marsh was the Project Engineer.

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ABSTRACT

A new digital adaptive control system first presented in reference 1 is further developed for the effective control of a priori unknown plants. Only the desired and actual plant output states are assumed to be measureable. A flyable digital computer of conventional capabilities is the central control agent. The primary control criterion is the minimization of a weighted norm of the output state vector predicted one control interval into the future.

Two alternate methods for the representation of unknown linear nonstationary plants based upon linear interpolation are investigated. More than 600 control efficacy simulations of a representative plant spectrum through fifth order are analyzed. An updating criterion, wherein the interpolation representation of the plant is recalculated only as required to maintain effective plant control, is developed and experimentally tested. A new recursive procedure for the inversion of a type of matrix encountered in the calculation of the interpolation representation of unknown plants is developed.

A first order Volterra series representation of unknown linear stationary and linear nonstationary plants is developed. The representation is reduced to working equational form.

Non-linear plant representations by linear interpolation and by interpolation over quadratic forms are developed. Control efficacy simulations utilizing the linear and non-linear interpolation are made and the two representations are compared. Control using either representation is demonstrated.

A data truncation study is made in which the number of significant figures available in the plant output state data is assumed to be limited. Control simulations are made in which the truncated data is used to periodically update the interpolation representation of unknown linear stationary plants. The effect of truncating the data at six, five, four, and three significant figures is experimentally explored.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

SECTION 1

INTRODUCTION AND SUMMARY

The research here described is an analytical and experimental investigation of a particular adaptive control concept. The method is conveniently designated DACS (Digital Adaptive Control System). The areas of study are cognate to those presented in reference 1 and to a large degree are experimental extensions of related areas. Summary treatment is given to those analytical areas first presented in reference 1 which are considered to be expositive to the material presented here. All analytical developments originating during the most recent study are covered in full detail.

The DACS approach is characterized by:

> The assumption that the particular plant under control is a priori unknown, except by its membership in one of several broad plant classifications.
>
> That the control actions be derived from computation by an on-line digital control computer of conventional capabilities.

DEFINITION

"An adaptive control system is here defined as a control system which is capable of monitoring its own performance with respect to a given index of performance and modifying its behavior by closed-loop action in such a manner as to optimize the index of performance or approach the optimum condition," (reference 2).

APPLICABILITY

The impetus towards the evolution and use of adaptive systems comes from the existance of a class of control problems which are a priori undescribable by reason of:

Unpredictability - e.g. the unforeseen failure of a component in a space mission.

Excessive complexity of description - e.g. certain chemical processes.

Analytical intractability - e.g. many problems in fluid dynamics.

Extreme varience - e.g. the control of high speed aircraft.

The inadequacy of conventional control systems to these problems is predictable to the extent that conventional design is customized to a postulated a priori description.

1.1 DACS CONCEPT

The following principles are innate to the DACS concept:

The system is to be adaptive in the following sense. It is to permit effective control of a variety of physical plants without a priori knowledge of the usual plant descriptors (pole-zero configurations, describing functions, etc.). It is assumed that the only knowledge of the plant under control is what can be inferred from measurements made during the sequence of control actions.*

The primary control agent is an on-line digital computer of conventional capabilities. The research consists primarily in the determination of analytical methods resulting in reasonably simple algorithms for such computer centered control.

Digital computer control implies a sample-and-hold process. The sampling period is one of two primary DACS parameters. It is designated the "Decision Interval" and symbolized by T.

Using state space notation, the state vector components are restricted to the plant output variables and their real time derivatives. This choice reflects the data accessability of an unknown plant.

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* While this research has been conducted with the stated objective of unknown plant control, many of the methods are applicable to the more usual practical case of partial and/or inexact plant descriptions. They do not preclude and indeed profit by the use of any available plant descriptions.

The primary control criterion in the DACS concept is the minimization of a weighted norm of the output error state predicted one decision (sampling) interval into the future. The second primary DACS parameter controls the relative weighting of error components in the norm. It is designated the "Weighting Coefficient" and symbolized by h.

The following assumptions have been made in the present studies, but are not necessarily inherent in the concept:

The single input-single output plant has been exclusively investigated. This is primarily a matter of analytical convenience, and the methods can be extended to multivariate control.

A multistate controller has been postulated. No necessity for the continuum of control forces has been established, and selection from a quantized set is not excluded.

DACS FUNCTIONAL FLOW DIAGRAM

Figure 1-1 is a flow diagram illustrating the DACS functional operations. Note that with the exception of control force application and possible data conversion, all of the indicated functions are performed by an on-line digital control computer.

1.2 RESULTS OF PREVIOUS INVESTIGATIONS

Prior to the current research program, the DACS concept had been investigated and developed in considerable detail under sponsorship of National Aeronautics and Space Administration, Langley Research Center, Contract No. NAS1-5127, May 26, 1965 - May 25, 1966 (reference 1). Previous to this work, the DACS concept was initially developed and investigated in some detail under sponsorship of National Aeronautics and Space Administration Contract No. NASW-599, February 1, 1963 - January 31, 1964 (references 3, 4, 5, 6, and 7). The following summary of major conclusions establishes the background for the current research:

An equational basis was established for the digital computer control of an unknown plant. The early methods (references 3, 4, 5, 6, and 7) were partially empirical and in a strict sense limited to linear stationary plants whose transfer functions contained no zeroes. During the more recent research (reference 1) a method utilizing interpolation

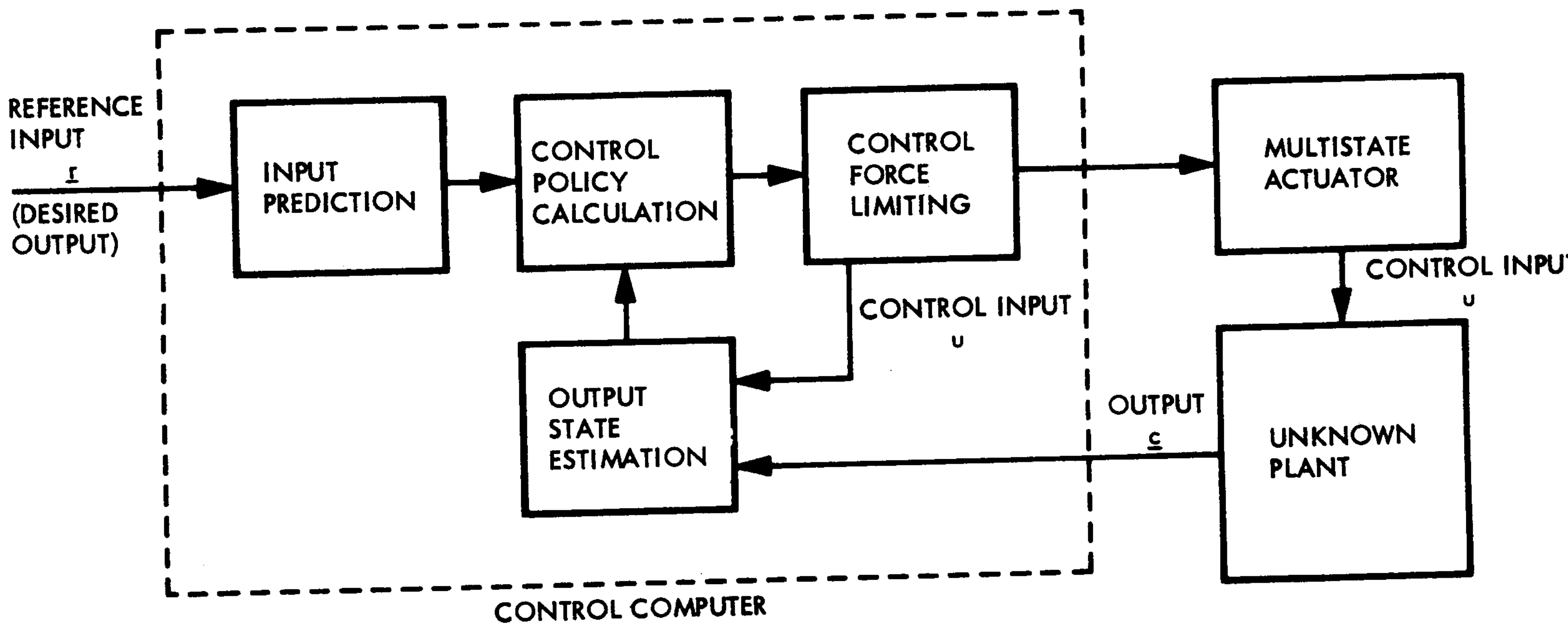

FIGURE 1-1 DACS FUNCTIONAL FLOW DIAGRAM

over values of a measured basis vector (references 8 and 9) was investigated and found to possess extremely general applicability. Extension to linear stationary plants whose transfer functions contain zeroes was demonstrated on a precise analytical basis. Further, the interpolative procedure showed promise for the representation of nonstationary (time-varying) as well as non-linear plants. Linear and non-linear forms of the interpolation method were reduced to "working" equational form for linear stationary, linear nonstationary, and non-linear plant control.

The early control methods were tested by hybrid simulation on a set of linear stationary plants of low order (fourth order or less with poles only). The linear interpolation representation was investigated on a set of linear stationary pole and pole-zero plants through ninth order. It was shown to be generally superior to the earlier plant descriptors. The general feasibility of the extension of linear interpolative representations to linear nonstationary and non-linear plants was demonstrated by simulation experiments.

The general stability of the DACS control policy over a representative set of linear stationary plants through ninth order was studied using a Liapunov function derived from the control policy. The region of stability of the control policy in the T-h plane * using several plant descriptors was established.

The Volterra series representation of unknown linear and non-linear plants (reference 10) was developed and reduced to working equational form for the second order truncation case.

A technique for start-up of the interpolation method based on matrix pseudoinversion was studied. This technique is applicable to cases where no initial plant information or data measurements is available.

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* The T-h plane is that defined by all possible sets of values of the two DACS parameters. Because both parameters are restricted to positive values (the sampling interval, T, is innately positive and only positive weighting coefficients, h, are considered), the T-h plane is confined to the first quadrant. The region of stability is defined as that set of T-h points for which stable operation of the control policy is possible.

Two methods of "learning" in the form of DACS parameter optimization were postulated and given preliminary investigation. The optimized parameters are T and h. The more usual method of plant parameter adjustment should not be inferred.

1.3 OBJECTIVES

At the initiation of the NAS1-6669 research, the following four tasks were made the primary objectives of the research effort.

Task I - Extend the study of linear time-varying plants of order through fifth using the interpolative procedure . . .

Task II - Extend the study of non-linear plants using the non-linear interpolative procedure . . .

Task III - Investigate the measurement accuracy required by the linear interpolative procedure . . .

Task IV - Develop the Volterra series (R = 1) control equations applicable to linear stationary and nonstationary plants.

Tasks I and II define the objects of study with the scope of Task I bearing the most weight. Tasks III and IV fall in the general area of Task I as they are related to the study of linear plants.

1.4 METHODS OF INVESTIGATION

The methods of investigation were combinations of:

Problem identification and definition.

Preliminary theoretical studies.

Reduction of theoretical methods to working forms.

Analysis of simulation results and correlation with theoretical method.

Validation or modification of theoretical methods on the basis of simulation results.

All simulation results of this research were obtained by digital computation on an IBM 7094 computer. Existing computer programs were used whenever possible. Most of the experimental extensions were implemented through relatively minor modifications and additions to the existing programs.

1.5 SUMMARY OF THEORETICAL EXTENSIONS

The primary theoretical extensions made under this contract were:

A criterion for updating the interpolation representation of plant response matrices is developed. The criterion establishes a basis whereby updating occurs only when the interpolative estimates become inaccurate as could be the case when the plant is time-varying. A summary of the criterion appears in paragraph 2.1.

Several methods for including direct approximation of plant time variation in the interpolation method are devised and discussed. The study is summerized in Appendix C.

A new recursive procedure for inversion of certain types of matrices is developed in which only simple arithmetic operations are involved. The procedure is particularly applicable to the type of matrices encountered using the interpolation method. A summary appears in Appendix D with an illustrative example.

The Volterra series representation of unknown linear stationary and nonstationary plants of reference 10 has been reduced to working equational form for the first order ($R = 1$) truncation case. A summary appears in Appendix A.

1.6 SUMMARY OF EXPERIMENTAL RESULTS

The existence of over 80 graphs in this report, many synoptic of extensive data sets, indicates the extent of the experimental investigations. The conclusions formulated must be confined to the defined scope of the experiments but in many cases they include a fairly general set of situations.

LINEAR TIME-VARYING PLANTS

The greatest bulk of data was obtained on approximately 50 linear time-varying plants of orders three through five. Appendix B identifies the most extensively studied plants.

Prediction Methods. - A primary objective of the experimental investigation was evaluation of the relative efficacy of two alternate interpolative representations of linear time-varying plants. The interpolative representations are individually characterized by the following features:

Stationary Basis Vector - This type of interpolative plant representation is analytically exact for linear stationary plants only. The data point sets consist of measured values of the output and as many of the higher state vector elements as possible. These measurements along with the associated control forces form the basis over which the interpolation is performed. No explicit allowance for the time variation of the plant is included. This interpolative representation depends upon frequent updating of the data point sets and re-interpolation to maintain an adequate plant description.

Time-Varying Basis Vector - This type of interpolative plant representation includes an explicit linear term to approximate the time variation of the plant. A running time base is used wherein an arbitrary time reference is used and the data point sets are identified with the time at which they occur. When updating the interpolative estimate of the plant, the time base is re-established to correspond to the time at which the measurements contained in the interpolation matrices were made. This method of explicit approximation of the plant time variations introduces little additional complication over the stationary basis vector as the dimension of the interpolation matrices is increased by only one.

Type of Plant Time Variations Considered. - The type of plant time variation studied in the most depth was sinusoidal variation of the plant differential equation coefficients. This choice was made primarily because a periodic type of variation allowed the study of relatively fast rates of variation while simultaneously allowing the limiting of the range of the coefficient values within practical values. In order to systematize the experimental approach, only one derivative coefficient was allowed to be time-varying for each plant. To a lesser extent, linear time variation of the differential equation coefficients of the plant was also studied.

Stability Investigations. - The first experiments were stability determinations for a set of time-varying plants in which approximately 400 regulator control simulations were made to ascertain the stability of particular T-h points. A limited region of the T-h plane was studied using both types of interpolative representations. The results indicated stable operation of the control policy was possible at all the T-h points investigated for all of the plants studied using the stationary basis vector. Use of the time-varying basis vector resulted in only a few cases of instability. No great preference of one of the basis vector descriptions over the other was observed in the experimental results.

Control Simulations. - With the range of stability established in a regulator sense, approximately 200 control simulations were made using a sinusoid as the desired output. Both the stationary and time-varying basis vectors were used in the control simulations. An updating monitor or criterion was used in which updates were made only at times when the interpolative plant description became inaccurate. No preference of one of the interpolative representations over the other was evident from these studies as both yielded about the same "tracking" results and both suffered from inaccurate updates with about equal occurrence.

NON-LINEAR PLANTS

The plant spectrum of the non-linear experiments was limited to those describable by the Van der Pol equation. Three values of the coefficient of the damping term in the differential equation were considered corresponding to a relatively linear plant, a moderately non-linear plant, and a highly non-linear plant.

Prediction Methods. - A primary objective of the experimental investigations was to evaluate the control efficacy of three interpolative plant representations. The three interpolative representations are individually characterized by the following features:

> Linear Basis Vector - This interpolative representation is characterized by interpolation over a set of linear basis functions. This same basis vector was used in the study of linear stationary (reference 1) and linear time-varying plants. Its usefulness in the representation of non-linear plants relies on a piecewise linear approximation of the non-linear plant response characteristics, the small size of the interpolation matrices corresponding to small data point sets, and highly frequent updating of the interpolative estimate of the plant.
>
> Non-linear Basis Vector - The non-linear interpolative representations studied utilize interpolation over a set of base functions of first and second degree. Linear, square and cross terms of the state vector elements and the control forces comprise the data point sets. Two types of non-linear basis vectors were studied. One of the two basis vectors included the cross products of the state vector elements whereas the other did not.

Regulator Control Simulations. - Control simulations in which the desired output state is zero were made for the three non-linear Van der Pol plants studied at five T-h points defining the same region of the T-h plane as was investigated in the linear time-varying plant studies. A substantial number of the T-h points yielded satisfactory regulator performance for the two least non-linear plants using all three interpolation basis vector descriptions. The regulator responses of the most non-linear plant demonstrated that a limit exists beyond which the linear basis vector does not yield a satisfactory plant description. Only two T-h points using the non-linear basis vector which included state vector element cross terms yielded what could be judged satisfactory regulator results.

Trajectory Control Simulations. - A limited set of trajectory experiments were performed using the three types of interpolation basis vector descriptions. Satisfactory tracking was demonstrated for the most linear of the three plants with the linear basis vector yielding the best results. The tracking capabilities demonstrated by the three basis vector descriptions became more degraded as the two more non-linear Van der Pol plants were included in the trajectory control simulation experiments.

Overall, the feasibility of the control of a non-linear plant using the three interpolative representations has been demonstrated although the results could not be interpreted as being extremely positive. Any conclusions must be qualified by the limited nature of the experimentation.

DATA TRUNCATION STUDY

A short study was conducted in which limited accuracy of the measured data was considered. The primacy of a study of this nature is self-evident as any practical control situation precludes unlimited measurement accuracy. The format of the experimentation was to truncate the number of significant figures that was assumed to be available in the measurements of the elements of the state vector. Full accuracy knowledge of the control forces was assumed as this is a computer derived quantity.

The plant spectrum consisted of a set of eight third and fourth order linear stationary plants of various pole configurations. The objective of the experimentation was to determine the effect of data truncation on the control of the set of linear stationary plants using the interpolation method with updating.

Two types of data truncation were studied. The first type involved truncation of all elements of the state vector and the second involved truncation of only the derivative elements of the state vector. It was observed that in both types of truncation experiments six significant figures in the data were necessary to provide control results equivalent to untruncated control results. Both types of data truncation yielded somewhat degraded results as the number of significant figures in the data was lowered successively down to three. With three significant figures of data, both types of truncation yielded divergent control results. The results using five significant figures were somewhat better when only the derivative elements of the state vector were truncated. Control in this situation is feasible although some inaccurate interpolation estimates were obtained. Truncation at four significant figures yielded approximately equivalent results for both types of truncation. While control is possible with four significant figures, the susceptibility to inaccurate interpolation estimation is fairly great.

1.7 ORGANIZATION OF THE REPORT

The following sections of this report have been organized so as to make possible eclectric sampling on the basis of reader interest. Table 1.1 is a guide to such reading.

TABLE 1.1 READER'S GUIDE

AREA OF PRIMARY INTEREST	PERTINENT PARTS OF REPORT
Linear Stationary Plants	Section 4 and Appendices A and D.
Linear Nonstationary Plants	Section 2 and Appendices A, B, C, and D.
Non-Linear Plants	Section 3 and Appendix D.
Control Theory	Paragraphs 2.1, 3.1 and 4.1, Appendices A, C, and D.
Nonanalytic Survey	Section 1, paragraphs 2.3, 3.3, and 4.3, and Section 5.
Data Truncation Study	Section 4.

SECTION 2

CONTROL OF LINEAR TIME-VARYING PLANTS

2.1 PLANT AND SYSTEM EQUATIONS

Because much of the research effort discussed here is a direct extension of work previously documented (reference 1), no attempt will be made to present a comprehensive development of the linear time-varying plant and system equations. Much of text and equations will be of a summary nature. The reader may find more complete discussions and equational developments in reference 1.

For the sake of analytical convenience, the study has been limited to the single input - single output plant. The concepts presented here are in no way inherently limited to the single variable control problem and extension to multivariate control is direct. Also, because of a need to limit the scope of the study and the primacy of certain factors over others the class of plants studied has been restricted to those which are not sensitive to derivatives of the input. An analogous restriction for a class of linear stationary plants would be to limit the study to plants whose transfer functions contain poles but no zeroes. This restriction is not particularly severe as it is demonstrated in reference 1 that inclusion of plants sensitive to derivatives of the input leads to only slight additional complication.

The type of time-varying plant studied may be qualitively described as possessing a single primary input with perhaps several secondary inputs. Those quantities which are insensitive to the primary input are termed plant parameters and those which are affected by the primary input are termed plant variables. To preserve the single input concept, it is assumed that the effect of the secondary inputs may be considered independently of the primary signal. Only those quantities which are parameters of the plant as far as the primary input is concerned are assumed to be affected by the secondary inputs. The parameter variations due to the secondary inputs may be described as functions of time and the plant is termed time-varying in the sense of posessing time-varying parameters.

METHOD OF ANALYSIS

The physical plant is assumed to be describable by a linear differential equation with coefficients which are continuous functions of time:

$$L(p,t)\ c(t) = B_0(t)\ m(t) \tag{2-1}$$

The plant is assumed to possess a single input, m(t), and a single output, c(t). The quantity, L(p,t), is a linear differential operator of order n and the derivative coefficients are functions of time. Although equation 2-1 suffices to describe the class of plants which are included in this study, it is advantageous to write the mathematical description in state space notation*:

$$\dot{\underline{x}}(t) = \underline{H}(t)\ \underline{x}(t) + \underline{g}(t)\ u(t) \tag{2-2}$$

The state variable, $\underline{x}(t)$, is identified on a one to one basis with the output of the plant, c(t), and its first n-1 derivatives. This defines $\underline{x}(t)$ as the n vector:

$$\underline{x}'(t) = \lfloor c(t)\ \dot{c}(t)\ \ldots\ \overset{n-1}{c}(t) \rfloor \tag{2-3}$$

The particular choice of the state variable is dictated by the a priori unknown plant assumption of this study.

* Vectors will be denoted by small Roman or Greek letters, matrices by Roman or Greek capitals, and transposed vectors and matrices are denoted by primes.

Because the plant differential equation possesses no derivatives of the input, the remaining terms in equation 2-2 are specific cases of the more general case where derivatives of the control input are present (reference 1).

$$\underline{g}'(t) = \lfloor 0 \; 0 \; . \; . \; . \; 0 \; B_0(t) \rfloor \qquad (2\text{-}4)$$

$$u(t) = m(t) \qquad (2\text{-}5)$$

$$\underline{H}(t) = \begin{bmatrix} \ddots & \ddots & & 0 \\ & \ddots & 1 & \ddots \\ & & \ddots & \ddots \\ -A_0(t) & -A_1(t) & \cdots\cdots & -A_{n-1}(t) \end{bmatrix} \qquad (2\text{-}6)$$

Note that $\underline{g}(t)$ is a vector and $u(t)$ is a scalar, whereas, in the more general case $\underline{g}(t)$ is a rectangular matrix and $u(t)$ is a vector. The $A_i(t)$ elements of the matrix in equation 2-6 are the time-varying coefficients of derivatives on the left hand side of equation 2-1. The matrix, $\underline{H}(t)$, is square of order n.

Equation 2-2 is termed the plant dynamic equation and is a first order vector differential equation. Equations 2-1 and 2-2 are equivalent according to the definitions of equations 2-3 through 2-6.

THE STATE EQUATION

The general continuous solution of the plant dynamical equation 2-2 is given by:

$$\underline{x}(t) = \underline{F}(t,t_o)\, \underline{x}(t_o) + \int_{t_o}^{t} \underline{F}(t,\tau)\, \underline{g}(\tau)\, u(\tau)\, d\tau \qquad (2\text{-}7)$$

where $\underline{F}(t,t_o)$ is the matrix solution of the free (homogeneous) differential equation and is termed the state transition matrix. The quantity, $\underline{x}(t_o)$, is the value of the state variable at $t = t_o$. Equation 2-7 is valid for any $t \geq t_o$. In the case of time-varying systems the transition matrix cannot be expressed as a simple exponential form as can be done for linear stationary systems. The usefulness of the transition matrix in time-varying analysis is therefore limited except for special cases where simple solutions exist.

Because the control action is effected by an on-line digital computer, the control functions, $u(t)$ (plant inputs), are piecewise constant over decision or sampling intervals of equal lengths (T seconds). Due to the piecewise constant nature of the control input, it is desirable to place the solution of the plant dynamical equation in a discrete or sampled form compatible with the sampling interval of the control action. Considering the control input (plant input) to be constant over the time intervals $kT \leq t < (k+1)T \quad k = 0,1,2, \ldots,$ a convenient form for the discrete solution is given in equation 2-8:

$$\underline{x}((k+1)T) = \underline{F}((k+1)T,kT)\ \underline{x}(kT) + \underline{\lambda}((k+1)T,kT)\ u_k \tag{2-8}$$

Equation 2-8 relates the state at the end of the kth interval, $\underline{x}((k+1)T)$, to that at the beginning, $\underline{x}(kT)$. The transition matrix, $\underline{F}(t,t_o)$, and the integral of equation 2-7 are evaluated for a decision interval length of T seconds. However, the transition matrix and the forced response vector of the plant are not constant matrices for constant T as is the case for linear stationary plants. The kth control input, u_k, is constant over the interval $kT \leq t < (k+1)T$.

In order to avoid possible confusion with the notation used here versus that used in reference 1 it is pointed out that because no derivatives of the input are assumed to be present in the plant differential equation in this study, the state variable, $\underline{x}(t)$, is continuous when discontinuities occur in the input to the plant. For this reason, no distinction is made between the instants in time just before and just after sampling instants. The kT^-, kT°, kT^+ notation of reference 1 is therefore dropped and the following equalities exist:

$$\underline{\lambda}((k+1)T,kT) = \underline{b}_1((k+1)T,kT)$$

$$\underline{b}_2((k+1)T,kT) = \underline{0} \tag{2-9}$$

Because thre is no need to deal with two forced response vectors, $\underline{\lambda}((k+1)T,kT)$ will be used exclusively.

THE CONTROL POLICY

The general objective of the control policy is to align the output state of the system with some desired output state. It will be assumed that the function describing the desired output is analytic on some open interval (t_a,t_b) * except for, at most, a finite number of discontinuities and that it is accessible for measurement or is known in advance. The desired output state will be defined as:

$$\underline{r}'(t) = \left\lfloor r(t) \quad \dot{r}(t) \; . \; . \; . \; . \overset{n-1}{r}(t) \right\rfloor \tag{2-10}$$

where the elements of the desired output state vector and the actual output state vector elements possess the same derivative relationship with respect to one another.

Figure 2-1 shows a representative time history segment of the type of control system under study. The first elements of the desired and actual state vectors ($\underline{r}(t)$ and $\underline{c}(t)$ respectively) are shown in the upper part of the figure and a typical control input sequence is shown in the lower part of the figure. Graphs of other elements of the desired and actual state vectors would be similar as the control input sequence seeks to align the two states.

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* The open interval (t_a,t_b) will be considered to contain that interval of time during which control of the plant is desired.

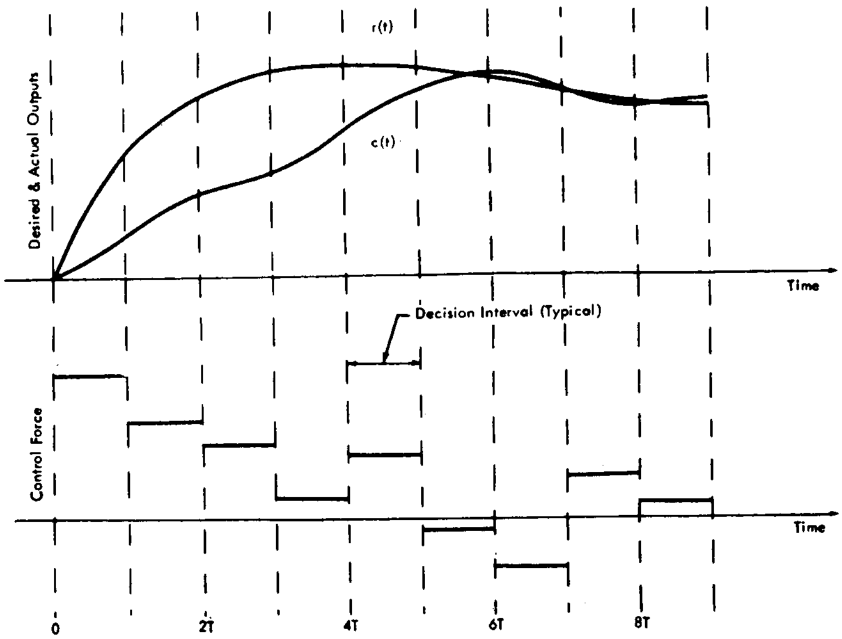


FIGURE 2-1 REPRESENTATIVE TIME HISTORY OF CONTROL SYSTEM

The General Control Policy Equation. - The control policy could be identified by a variety of performance criteria, however, the following relatively simple criterion will be used (references 1, 3, 4, 5, 6, and 7).

$$\underset{u_k}{\text{Min}} \left[Q_k \right] = \underset{u_k}{\text{Min}} \left[\underline{e}'((k+1)T) \; \underline{K} \; \underline{e}((k+1)T) \right] \tag{2-11}$$

where $\underline{K}$ is a positive definite, symmetric constant matrix and the error state vector is defined as:

$$\underline{e}(t) = \underline{r}(t) - \underline{x}(t) \tag{2-12}$$

The control algorithm is to select a constant control input, u_k, to be applied during the interval $kT \leq t < (k+1)T$ such that the positive definite quadratic form, Q_k, is minimized. Such a control input selection is necessary for each decision interval.

The practical control law is obtained by substituting the state equation 2-8 into equation 2-11, equating the first derivative with respect to u_k to zero, and solving for an explicit expression for u_k. A proof that this yields a minimum for Q_k is given in reference 1. The expression for u_k is given by:

$$u_k = \frac{\underline{\lambda}'((k+1)T,kT)\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}((k+1)T,kT)\ \underline{x}(kT)\right]}{\underline{\lambda}'((k+1)T,kT)\ \underline{K}\ \underline{\lambda}((k+1)T,kT)} \qquad (2\text{-}13)$$

Equation 2-13 is the general form of the control policy equation.

The Weighting Matrix. - The constant matrix $\underline{K}$ introduced in the quadratic form, Q_k, of equation 2-11 performs the function of a weighting function on the state variable components. The particular form used in this study is a diagonal matrix defined by equation 2-14:

$$\underline{K} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & (hT)^{2i} & & \\ & & & \ddots & \\ & & & & (hT)^{2n} \end{bmatrix} \qquad (2\text{-}14)$$

where n is the order of the system. The matrix $\underline{K}$ is obviously symmetric and positive definite; therefore, it fulfills two necessary conditions required by the control law.

The matrix has as its basis previous Emerson studies (references 1, 3, 4, 5, 6, and 7). The origin of the form is extensive studies with linear stationary plants. The extension to time-varying plants is direct.

ESTIMATION AND PREDICTION

The response of the plant during a decision interval may be broken down into two parts. The first part may be termed the free response which would occur in the absence of a control input. The second part may be termed the forced response, or that part of the response which is due to the control input. The state equation:

$$\underline{x}((k+1)T) = \underline{F}((k+1)T,kT)\ \underline{x}(kT) + \underline{\lambda}((k+1)T,kT)\ u_k \tag{2-15}$$

may be interpreted in this light by considering the first term on the right-hand side of the equation to be the free response and the second to be the forced response. An appropriate control input, u_k, is calculated by the control policy, equation 2-13, so as to better align the actual and desired output states then would be the case if no control input was applied and the response during the interval was exclusively the free response. In order to calculate the control input, u_k, to apply during the decision interval $kT \leq t < (k+1)T$, predictions of what the free response during the interval will be as well as the sensitivity of the plant to a control input must be obtained. As can be ascertained by examining the control policy, equation 2-13, the prediction problem takes the form of estimating $\underline{F}((k+1)T,kT)$ and $\underline{\lambda}((k+1)T,kT)$.

If exact knowledge of the plant is available, the prediction problem is relatively trivial as the plant equations can be used directly. If the plant equations are unknown (as is assumed in this study) then prediction must be based upon some sort of estimating or fitting technique. Two such estimating techniques have been studied. The first is termed the first order Volterra series method and is discussed in Appendix A. A second estimating technique based on an interpolative procedure has been studied in the most depth. The basic interpolation equational development is reviewed in Appendix C. For more complete treatments the reader is referred to references 1 and 8.

Interpolation Prediction Without t. - In terms of the interpolation method, the estimate of the state at t = (k+1)T is given by the equation:

$$\underline{\tilde{x}}((k+1)T) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k,\underline{\eta}_k) \tag{2-16}$$

where the tilde indicates an estimated value.

The basis vector in this case is given by:

$$\underline{\phi}'(u_i\underline{\eta}_i) = \left[\underline{x}'(iT)\;\; u_i\right] \tag{2-17}$$

The matrix of basis vectors, $\underline{\Phi}$, consists of an appropriate set of $\underline{\phi}_i$'s which need not be consecutive and ${}_D\underline{X}$ consists of a corresponding set of state variable measurements.

Define a partitioned matrix $\underline{B}$ which is subdivided into two submatrices:

$$\underline{B} = {}_D\underline{X}\;\;\underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \;\middle|\; \underline{\varphi}_1\right] \tag{2-18}$$

where if the assumed order of the matrix is p, $\underline{\Theta}_1$ is a pth order square matrix and $\underline{\varphi}_1$ is a p vector. In terms of the submatrices, the estimate of $\underline{x}((k+1)T)$ is:

$$\underline{\tilde{x}}((k+1)T) = \underline{\Theta}_1\,\underline{x}(kT) + \underline{\varphi}_1\,u_k \tag{2-19}$$

where $\underline{\Theta}_1$ and $\underline{\varphi}_1$ are constant matrices.

This interpolation representation is most directly applicable to stationary systems as time variations in the plant are not accounted for directly. The usefulness of the technique resides in the fact that it is basically a fitting technique wherein a set of data points are fitted by a multinomial. The fact that the plant may be time-varying does not alter the basic fitting process. If measurement of the full state vector is possible, then estimates of all of the state variables may be obtained and p = n.

Time variations of the plant may be accounted for by periodically updating the interpolation matrices and obtaining new values for $\underline{\Theta}_1$ and $\underline{\varphi}_1$. The updating procedure consists of shifting new, more current data into $\underline{\Phi}$, the matrix of basis vectors, corresponding data into ${}_D\underline{X}$, and obtaining a new $\underline{B}$ matrix. If the plant is relatively slowly time-varying updating may be required relatively infrequently.

Interpolation Prediction With t. - A second interpolation representation studied includes a linear term in t in the basis vector in order to account for the time variation of the plant more directly. The estimate of $\underline{x}((k+1)T)$ is still given by:

$$\tilde{\underline{x}}((k+1)T) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k, \underline{\eta}_k) \tag{2-20}$$

However, the basis vector is now given by:

$$\underline{\phi}'(u_i, \underline{\eta}_i) = \left\lfloor \underline{x}(iT) \quad u_i \quad (i+1)T \right\rfloor \tag{2-21}$$

This method of accounting for time variation of the plant is one of several discussed in more detail in Appendix C.

Because the basis vector, $\underline{\phi}_i$, contains one more term than the first case presented, one more interval of data is required and the interpolation matrices $\underline{\Phi}$ and ${}_D\underline{X}$ will be increased in order by one. The $\underline{B}$ matrix is now partitioned into three submatrices:

$$\underline{B} = {}_D\underline{X}\,\underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \;\middle|\; \underline{\varphi}_1 \;\middle|\; \underline{\varphi}_2\right] \tag{2-22}$$

where, if the assumed order of the matrix is p, $\underline{\Theta}_1$ is a pth order square matrix and $\underline{\varphi}_1$ and $\underline{\varphi}_2$ are p vectors. The interpolation estimate of $\underline{x}((k+1)T)$ in terms of the submatrices is given by:

$$\tilde{\underline{x}}((k+1)T) = \underline{\Theta}_1\,\underline{x}(kT) + \underline{\varphi}_1\,u_k + \underline{\varphi}_2\,(k+1)T \tag{2-23}$$

where $\underline{\Theta}_1$, $\underline{\varphi}_1$ and $\underline{\varphi}_2$ are constant matrices.

Time variation of the plant is now approximated by the third term on the right-hand side of equation 2-23. The vector, $\underline{\varphi}_2$, is multiplied by the value of running time referenced to some arbitrary time base. There is no inherent time base in the interpolation method, however, it is convenient to assign the zero time reference to the oldest data in the matrix of basis vectors $\underline{\Phi}$. Thus, whenever an update is made the time base is shifted and the running time is measured from this new zero time reference.

Interpolation Control Policy Equations. - The practical implementation of the control policy equation 2-13 depends upon which interpolation representation of the discrete state equation is used. If Interpolation Prediction without t in the basis vector is used the control policy equation is:

$$u_k = \frac{\underline{\varphi}_1' \, \underline{K} \left[\underline{r}((k+1)T) - \underline{\Theta}_1 \, \underline{x}(kT)\right]}{\underline{\varphi}_1' \, \underline{K} \, \underline{\varphi}_1} \qquad (2\text{-}24)$$

If the Interpolation Prediction with t in the basis vector is used the control policy equation is:

$$u_k = \frac{\underline{\varphi}_1' \, \underline{K} \left[\underline{r}((k+1)T) - \underline{\Theta}_1 \, \underline{x}(kT) - \underline{\varphi}_2 \, (k+1)T\right]}{\underline{\varphi}_1' \, \underline{K} \, \underline{\varphi}_1} \qquad (2\text{-}25)$$

Updating: - As has been mentioned in previous paragraphs, updating is one of the ways in which the Interpolation Prediction accuracy is maintained if the time variation of the plant is such as to render the interpolation matrices generated from past data prohibitively inaccurate. One way of maintaining the accuracy is to arbitrarily update every interval or every mth interval where m is a specified constant. This fixed updating procedure is rather inefficient as no flexibility exists which allows for changing rates of time variation. If the interpolation estimates are sufficiently accurate so as to provide adequate control of the system it would seem reasonable to continue to use them until such time as the control performance becomes degraded. A measure of the control performance is provided by the the Euclidean output state error norm:

$$E = \left\| \underline{e}(kT) \right\| \qquad (2\text{-}26)$$

The allowable magnitude of the output error norm would depend upon the required accuracy of the system but, in general, good control performance should result in small values for the output error norm. The difficulty in accessing the control performance and the need for updating in terms of the value of the error norm is its high degree of dependence upon the regularity of the desired output.

An alternate method of determing when updating is necessary is to compare the predicted states based upon the interpolation estimates of the plant matrices with the actual states as measured at the end of the decision intervals. One possible updating criterion is to require that the predicted state lie within a closed hypersurface centered on the actual state in the state space. A prediction error vector may be defined:

$$\underline{\delta}' = \lfloor \delta_1 \delta_2 \cdot \cdot \cdot \delta_n \rfloor \tag{2-27}$$

where δ_i is the allowable error threshold in the ith component of the state vector. An appropriate norm of $\underline{\delta}$ may then be used as the criterion for updating:

$$\Delta = \left\| \underline{\delta} \right\|^2_{\underline{H}} \geq \left\| \underline{x}((k+1)T) - \underline{\tilde{x}}((k+1)T \right\|^2_{\underline{H}} \tag{2-28}$$

where $\underline{H}$ defines the norm. If $\underline{H} = \underline{I}$ Δ corresponds to the Euclidean norm.

This criterion for updating is studied for the case where all of the δ_i were set equal to the same value defining a hypersphere and the norm matrix $\underline{H}$ was made the unity matrix. The experimental results are presented in paragraph 2.2.

SOME COMMENTS ON STABILITY

As was pointed out in reference 1, the subject of stability of time-varying systems is one about which the definitive word has not been said yet. Although many of the basic theorems which apply to the second method of Liapunov are valid for time-varying systems, the difficulties involved with forming usable Liapunov functions are formidable. The formulation of a general Liapunov function for the spectrum of time-varying plants considered in this study is impractical. For this reason, stability must be determined on an individual basis for each of the plants.

There is a strong tendency to discuss time-varying systems in terms of poles and zeroes which move about in the complex plane. This concept is not without its pitfalls because unless the speed of the time variation is extremely slow as compared with the time constants of the system no such connection exists. Certainly, describing stability of a system in terms of the movements of the 'poles' back and forth between the left and right half planes can be misleading.

Because of the difficulty of forming Liapunov functions for the systems considered, stability will be judged on the basis of control simulations. The "frozen plant" concept with ephemeral pole locations will be consistantly avoided.

2.2 EXPERIMENTAL STUDIES

The purpose of this paragraph is to present the experimental response characteristics of a selected set of linear time-varying plants controlled by the DACS Control Policy. The Interpolation Prediction method was used throughout this experimentation because it displayed the most positive results in previous DACS research (reference 1).

The objectives of this experimental program are:

To determine the control performance of the control system using the Interpolation Prediction method both with and without time considered in the basis vector for a selected set of linear time-varying plants of order through five.

To investigate the effectiveness of monitor controlled updating for the Interpolation Prediction method both with and without time considered in the basis vector.

To investigate the effect of the weighting factor, h, on the control system response for the Interpolation Prediction method both with and without time considered in the basis vector.

To compare the control performance of the Interpolation Prediction method both with and without time considered in the basis vector with respect to stability and tracking ability.

The objectives are considered along with the appropriate experimental procedures in the following paragraphs.

LINEAR TIME-VARYING PLANTS

In order to accomplish these objectives, it was necessary to select a limited but representative spectrum of linear time-varying plants of order through five. Such a set of plants was assembled by using previous DACS research to select a number of low order plants as an experimental starting point, and, by judicious selection, extend the set of plants to higher order. The resultant set consisted of approximately fifty time-varying plants (see Appendix B). This experimentation considered both linear and sine time variations. Several ranges and rates of time variation of single derivative coefficients were studied for each plant in various degrees of depth.

STABILITY BOUNDARY AND REGULATOR RESULTS

Previous research (references 1, 3, 4, 5, 6, and 7) has noted in some detail the two parameters of our control system, namely, the decision interval, T, and the weighting factor, h. In the linear stationary plant studies which are reported in the above references, the T-h stability boundaries were analytically established by Liapunov's second method. The Liapunov function formulated was applicable to any linear stationary plant, and so provided a very convenient method of determining the stability boundaries for the system controlling any such plant. Many of the basic theorems which apply to the second method of Liapunov are valid for time-varying systems. However, because of the great degree of difficulty associated with forming a usable Liapunov function, and because of the large number of time-varying plants considered by this study the system stability was established by experimental methods. Since the establishment of system stability for many T-h combinations or points in the T-h plane would require a considerable number of control simulations and, therefore, would be quite costly, the stability of only a limited area in the T-h plane was investigated by this experimentation.

This experimental study consisted of approximately 400 control simulations on selected time-varying plants. All of the control simulations were conducted with either +10 or +1 units as the initial condition of each element of the state vector. The desired output state was zero. Also, throughout this investigation system parameters associated with updating the interpolation estimates of the plant response matrices were constant; i.e. new data was shifted into the matrix of basis vectors every interval, and the interpolation estimates were recalculated every interval. Thus, the system stability at any T-h point was experimentally established by noting the control performance for regulator runs with all possible interacting system parameters judiciously fixed at constant values.

The following experiments were conducted using Interpolation Prediction which is not a self-starting method. Therefore, a start-up procedure was utilized to allow Interpolation Prediction control to start at any desired initial system state (reference 1). This procedure is of no significant importance, since it provided only an artificial method for starting the control simulations at a selectable initial state.

This start-up simulation phase is not included in the graphs of the runs. At the start of each of the runs shown in this report, the interpolation estimates of the system response matrices have been predetermined, and so the control is effected by the interpolation method during the entire run. Also, these figures are reproductions of the graphs plotted by the computer program. The axes have been relabeled and the data points have been connected using straight line approximation.

Stationary Basis Vector Results. - The experimental investigation of the limited portion of the T-h stability plane was first conducted for Interpolation Prediction using the linear basis vector without time. In other words, the basis vector did not contain any explicit time dependence term to account for the plant time variation, and is referred to as a stationary basis vector.

The portion of the T-h plane experimentally investigated consisted of the five points shown in the plane of Figure 2-2. These particular T-h combinations were selected for the area of investigation because they

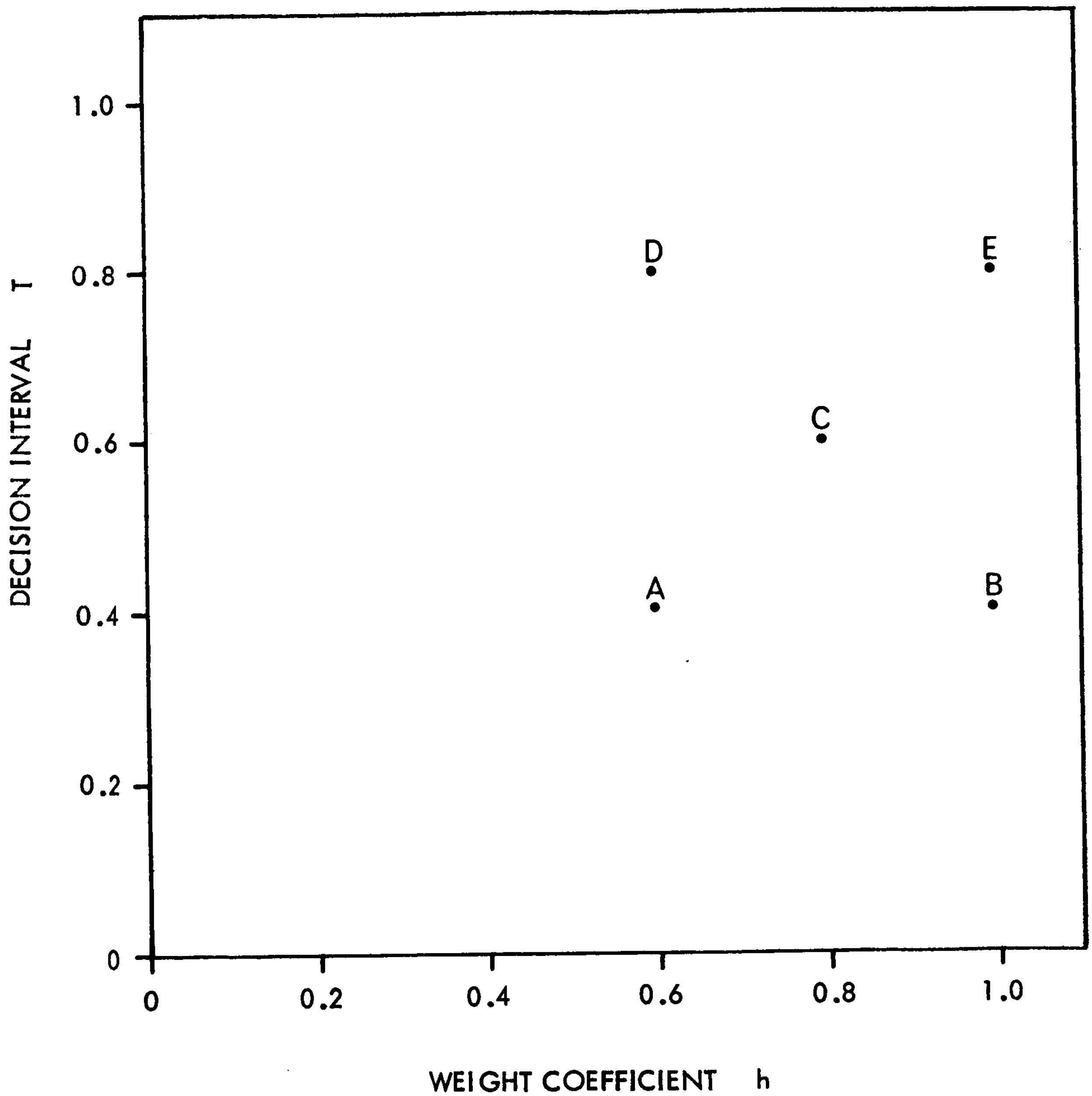


FIGURE 2-2 THE T-h STABILITY PLANE

provided the most satisfactory control performance for linear stationary plants of order through five (reference 1). Regulator response runs were obtained for each of these points (A through D) for the plants presented in Appendix B. Typical third order system results are presented by Figures 2-3, 2-4, 2-5 and 2-6, and illustrate the type of control performance obtained at T-h points A, B, D and E. Figures 2-7, 2-8, and 2-9 show typical control results for fourth and fifth order time-varying systems. For the sake of completeness, it must be pointed out that a number of the regulator response runs suffered from occurrences of poor interpolation estimates of the system matrices due either to ill conditioning of the matrix of basis vectors or simply inadequate description of the plant. However, in all such cases, the interpolation estimates subsequently improved to the degree that the controlled response could be judged stable or unstable.

The experimental results, like those presented and discussed in the last paragraph, indicated that all five T-h combinations provided stable control performance for all the plants in Appendix B.

Time-Varying Basis Vector Results. - An identical experimental study was conducted for Interpolation Prediction using the time-varying basis vector. In this case, running time was included explicitly in the basis vector to account in a linear sense for the time variation of the plant.

The same limited area of the T-h plane (see Figure 2-2) was again investigated by regulator response runs for the same time-varying plants. As before, typical third order system results are shown for the T-h points A, B, D and E in Figures 2-10, 2-11, 2-12, and 2-13. Figures 2-14, 2-15, and 2-16 illustrate typical control results for fourth and fifth order time-varying systems. It should be pointed out that occurrences of poor interpolation estimates of the system matrices similar to that noted for the stationary basis vector were encountered in the time-varying basis vector experimentation. However, such occurrences were again self-correcting and did not greatly interfere with the stability conclusions.

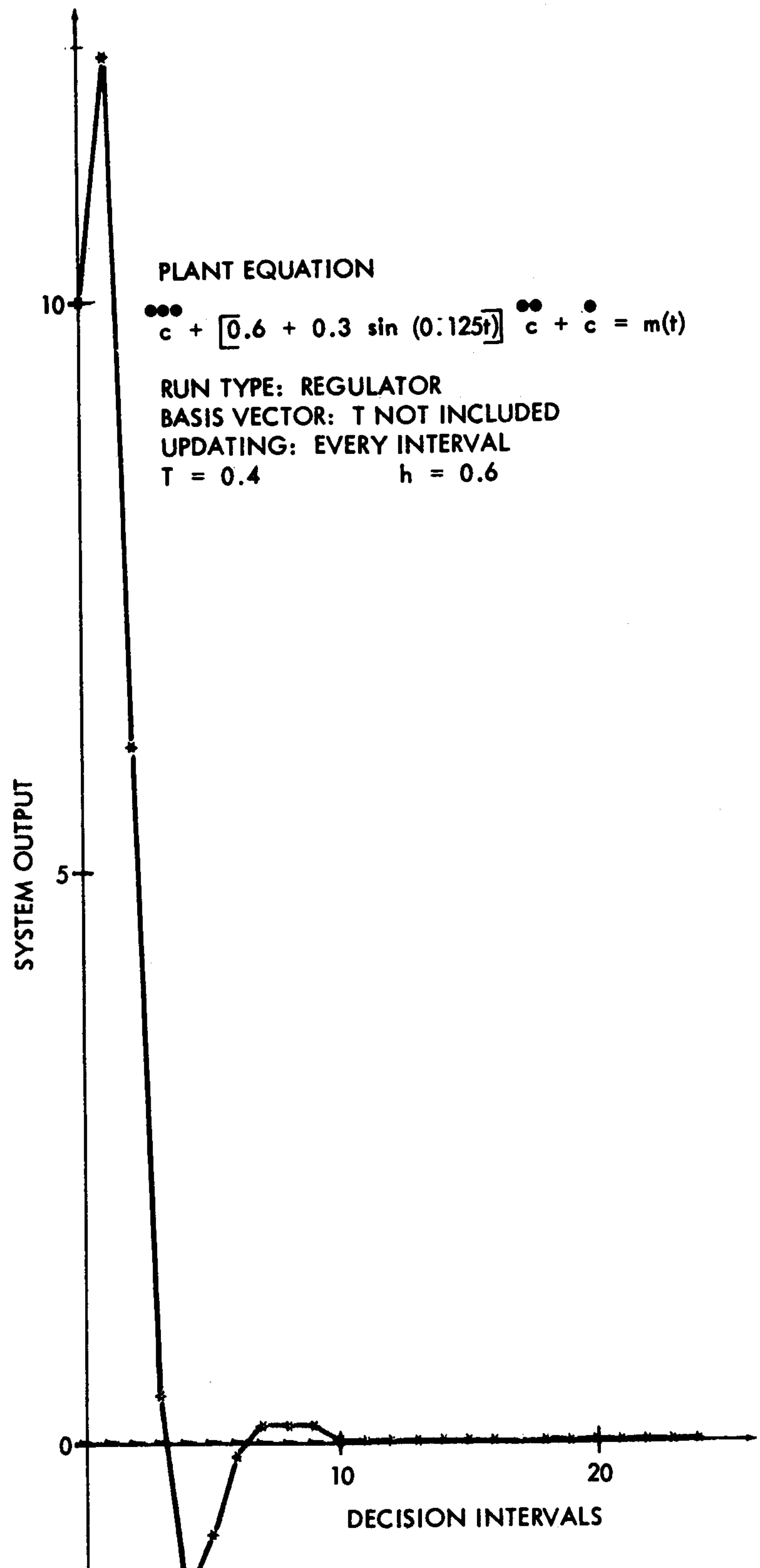


FIGURE 2-3 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

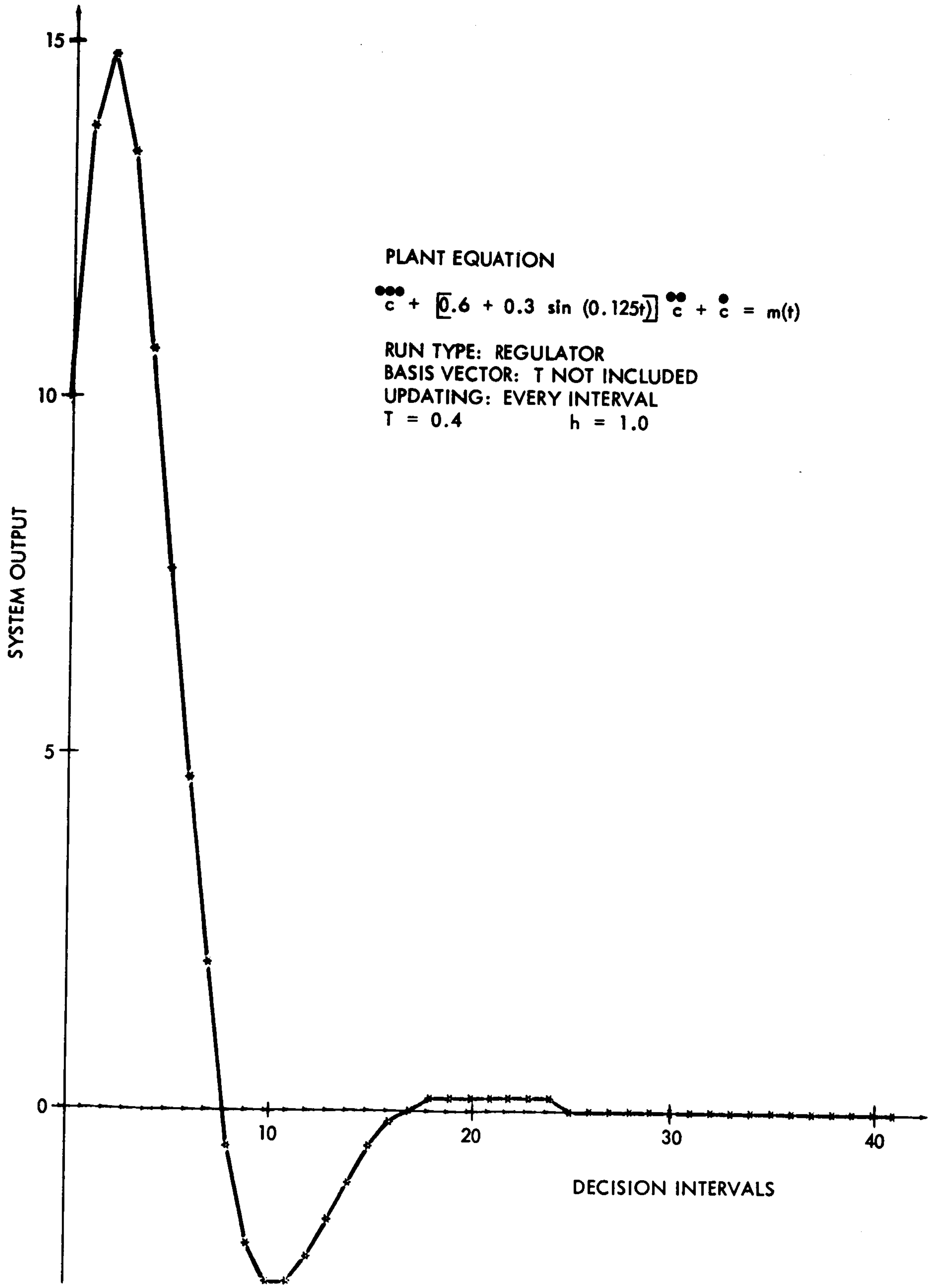


FIGURE 2-4 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

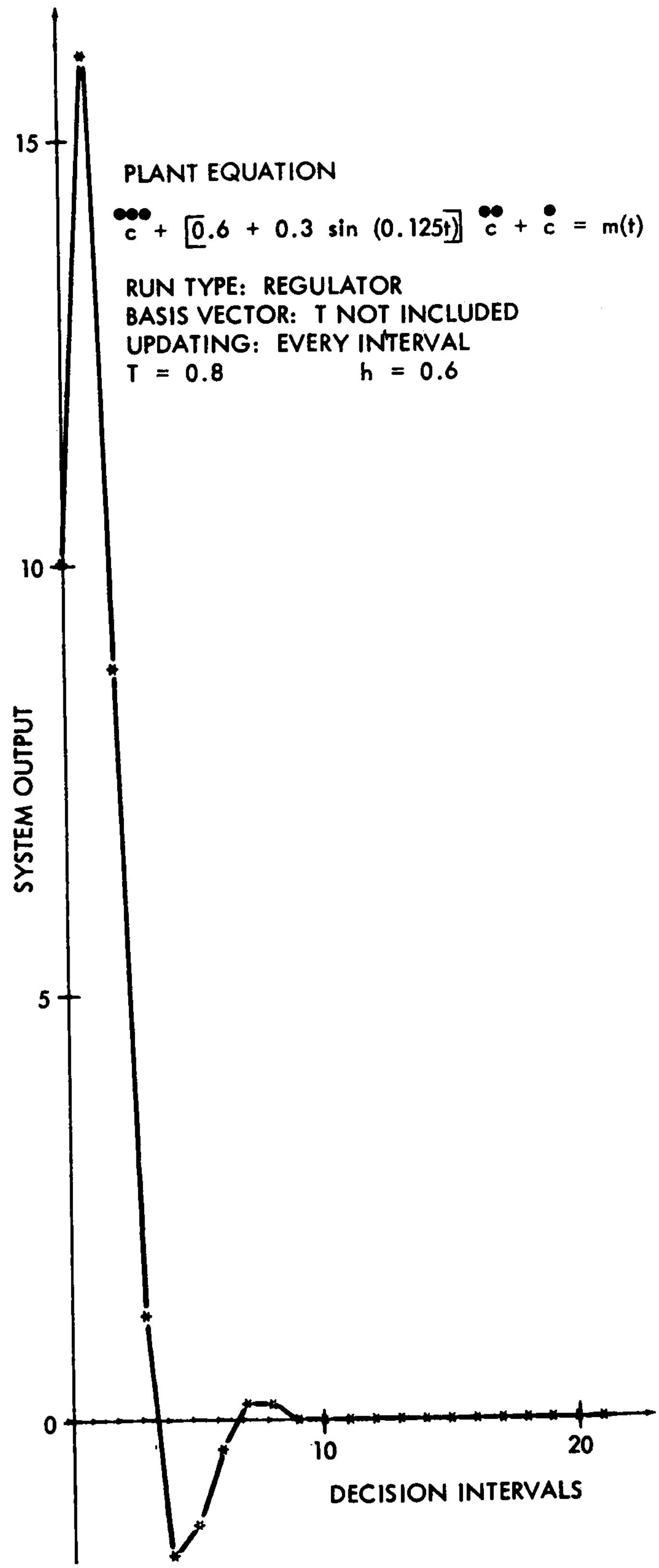


FIGURE 2-5 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

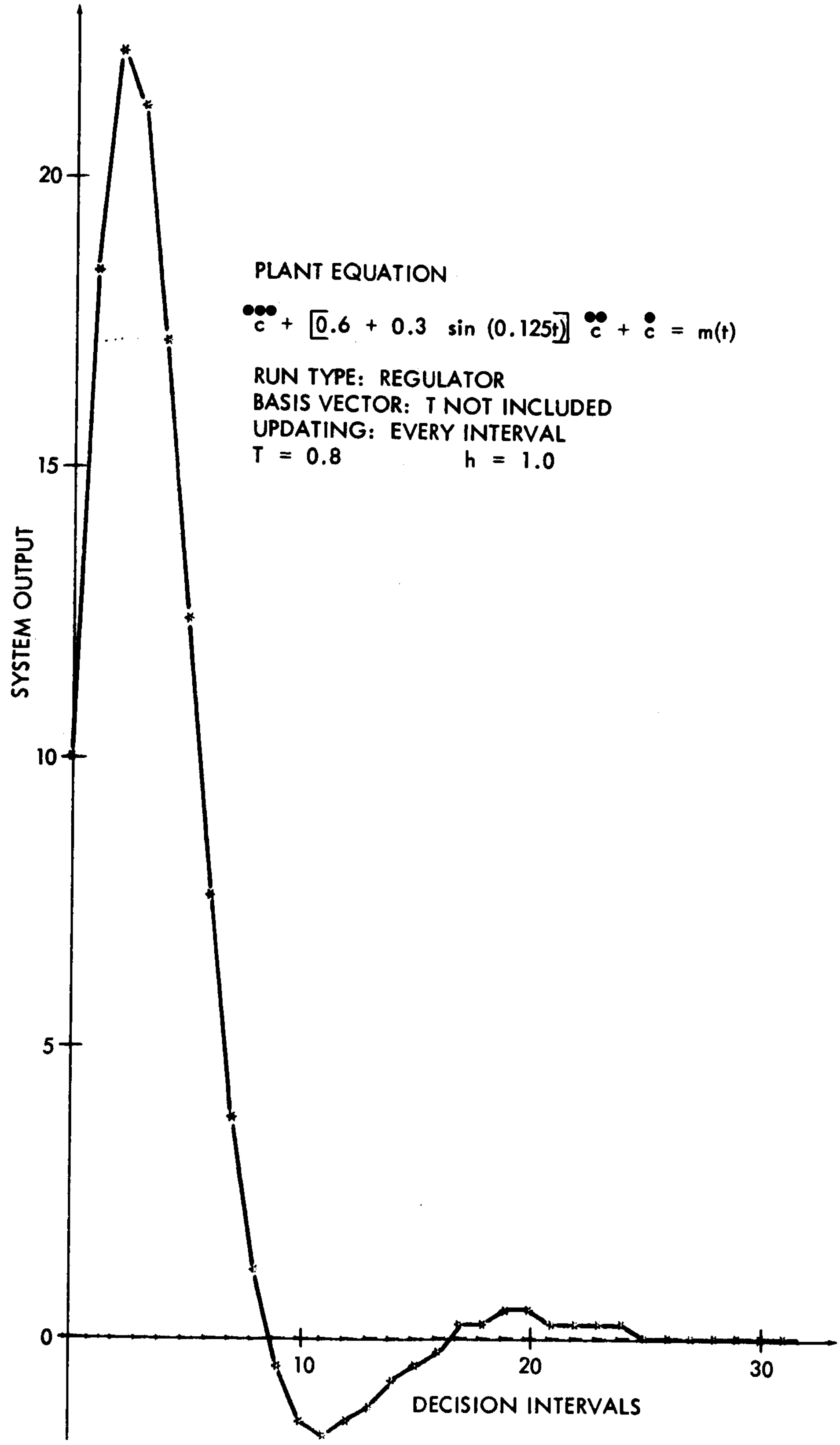


FIGURE 2-6 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

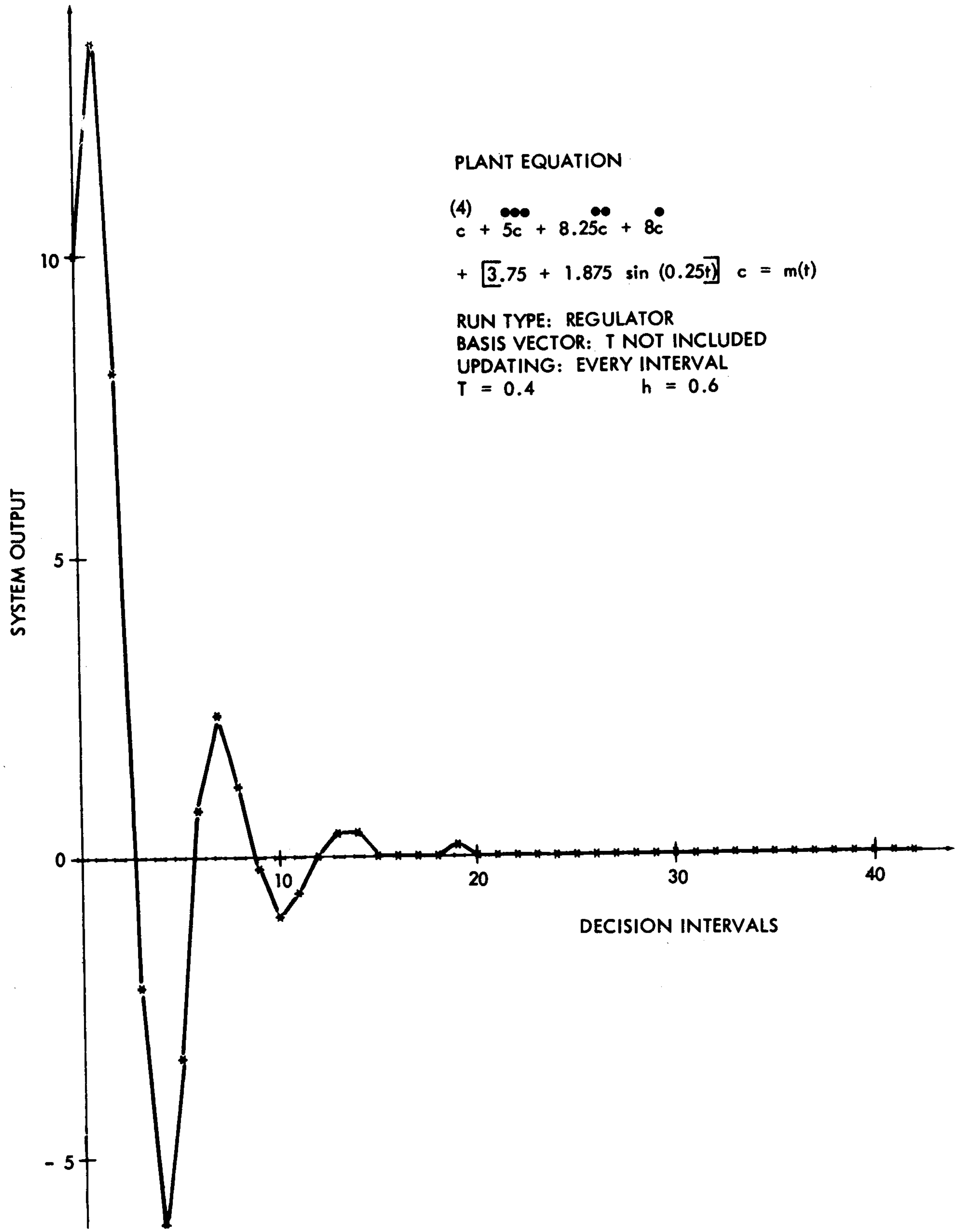


FIGURE 2-7 REGULATOR RESPONSE OF A 4TH ORDER SYSTEM

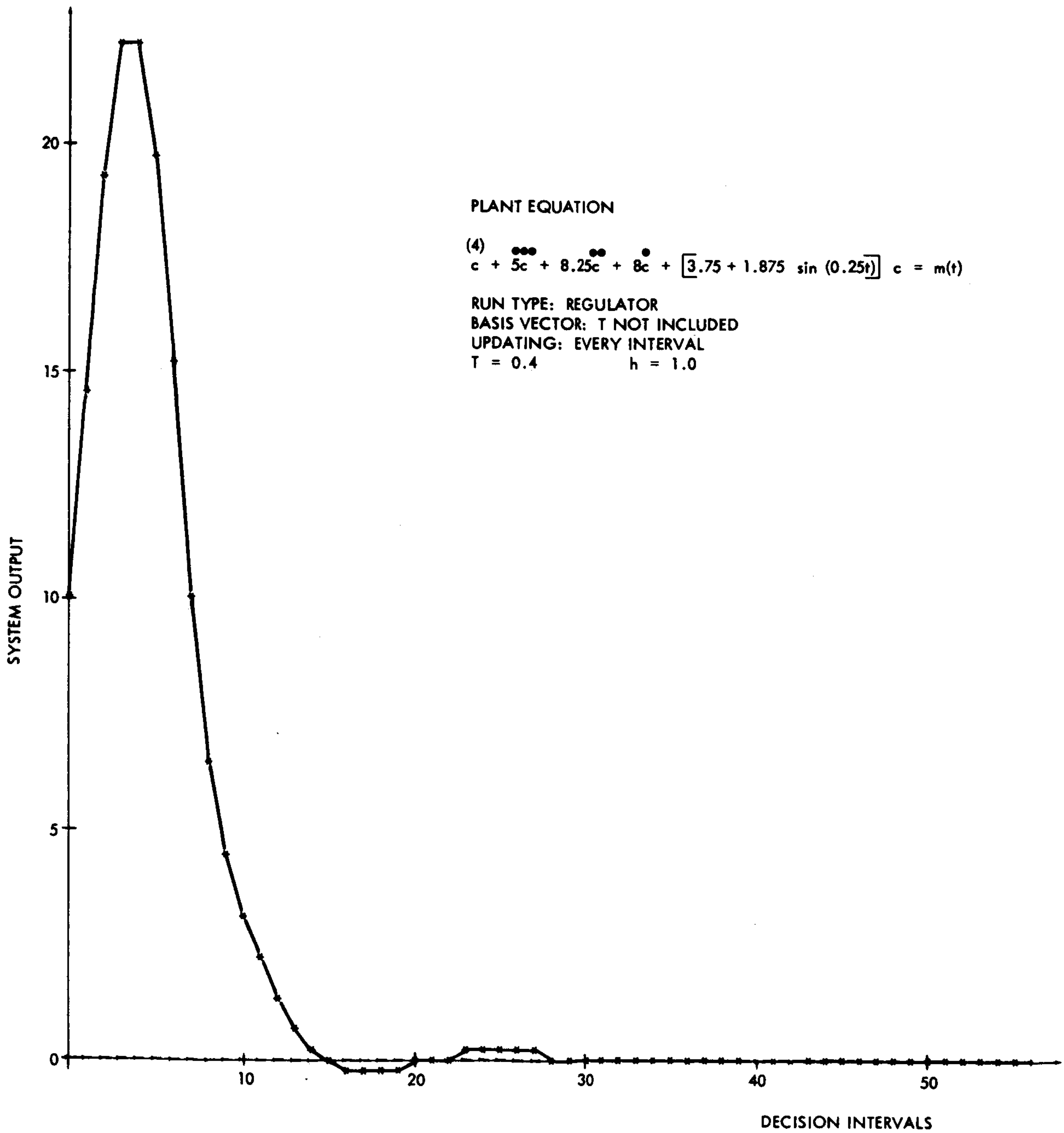


FIGURE 2-8 REGULATOR RESPONSE OF A 4TH ORDER SYSTEM

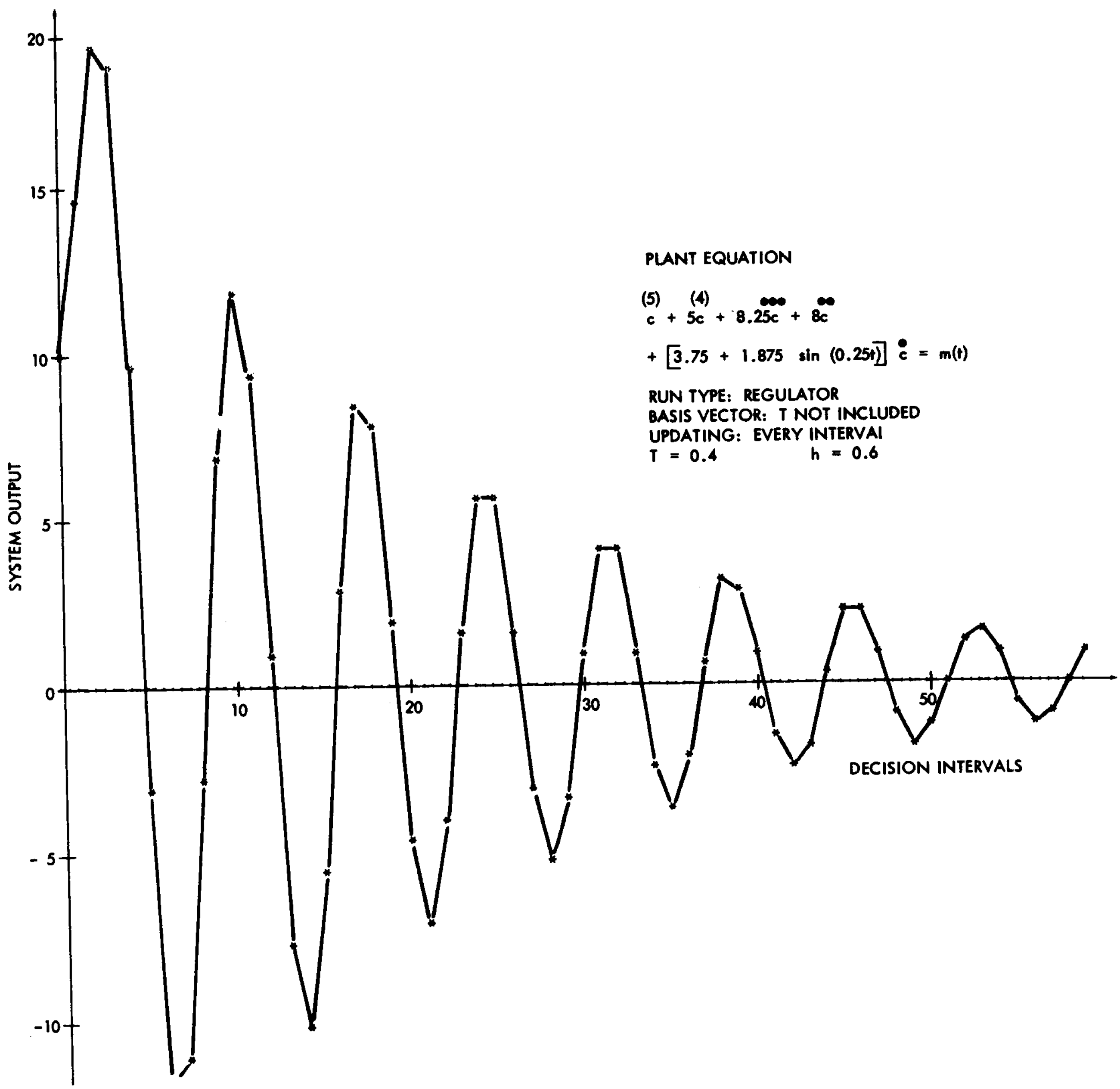


FIGURE 2-9 REGULATOR RESPONSE OF A 5TH ORDER SYSTEM

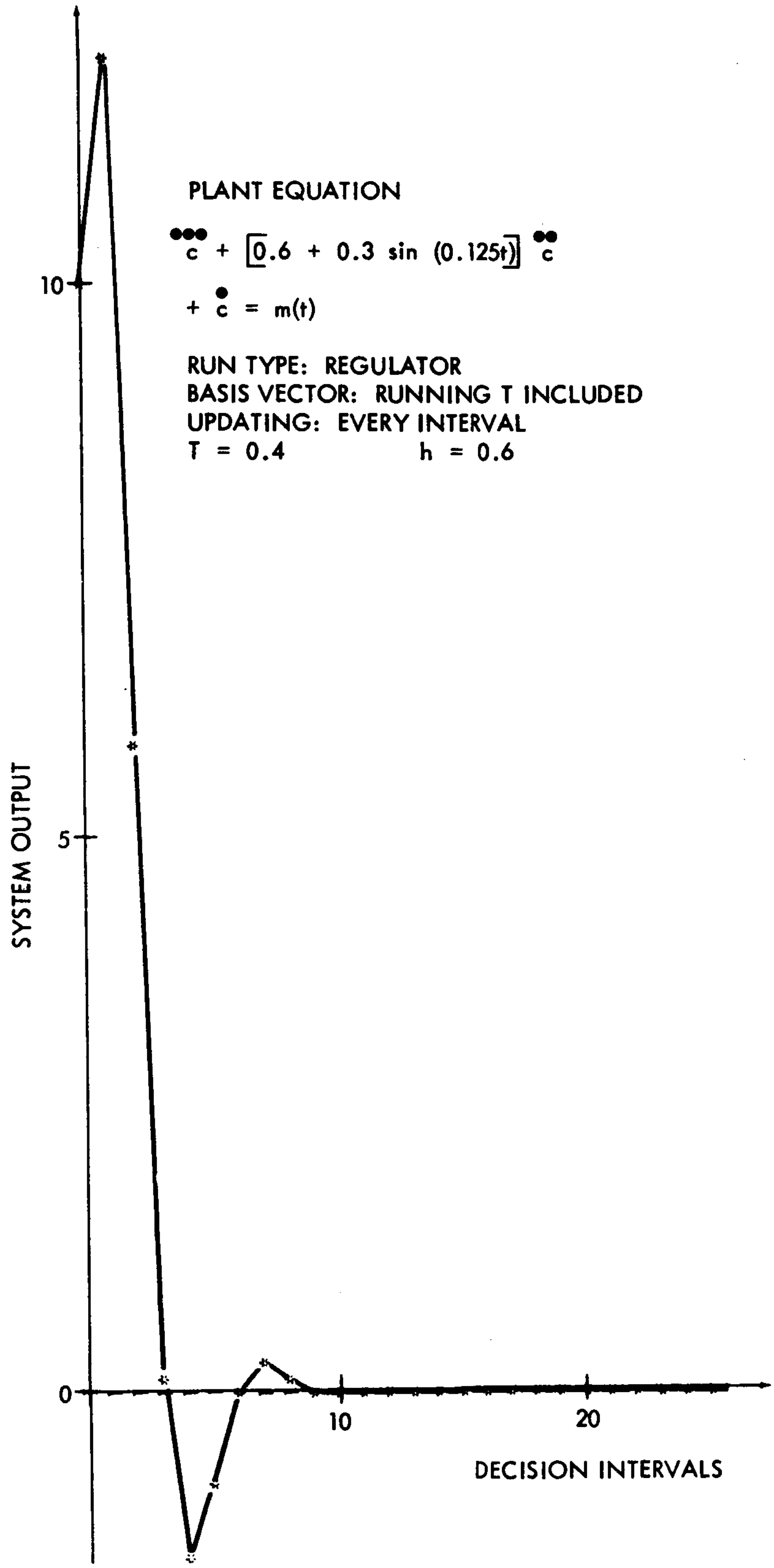


FIGURE 2-10 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

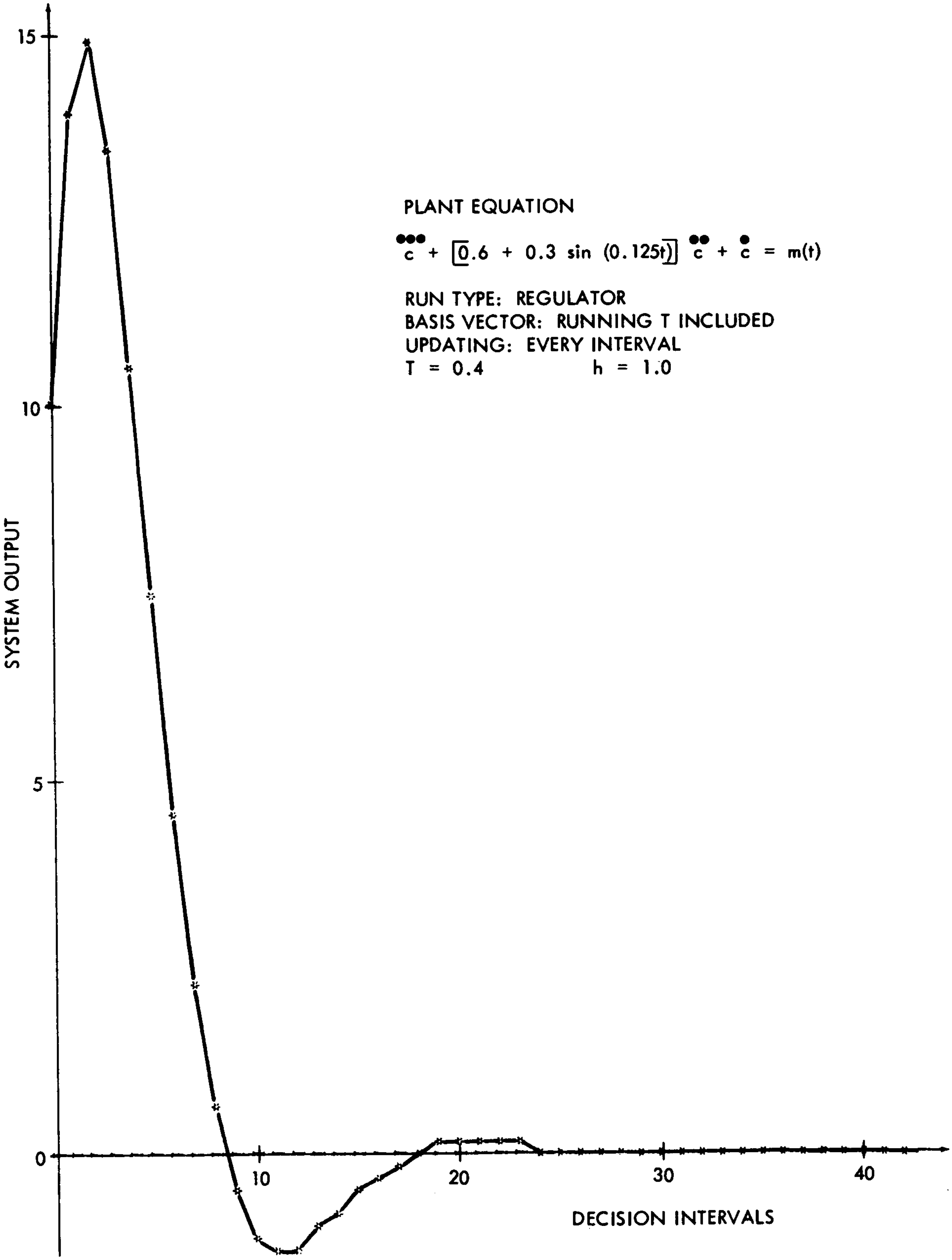


FIGURE 2-11 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

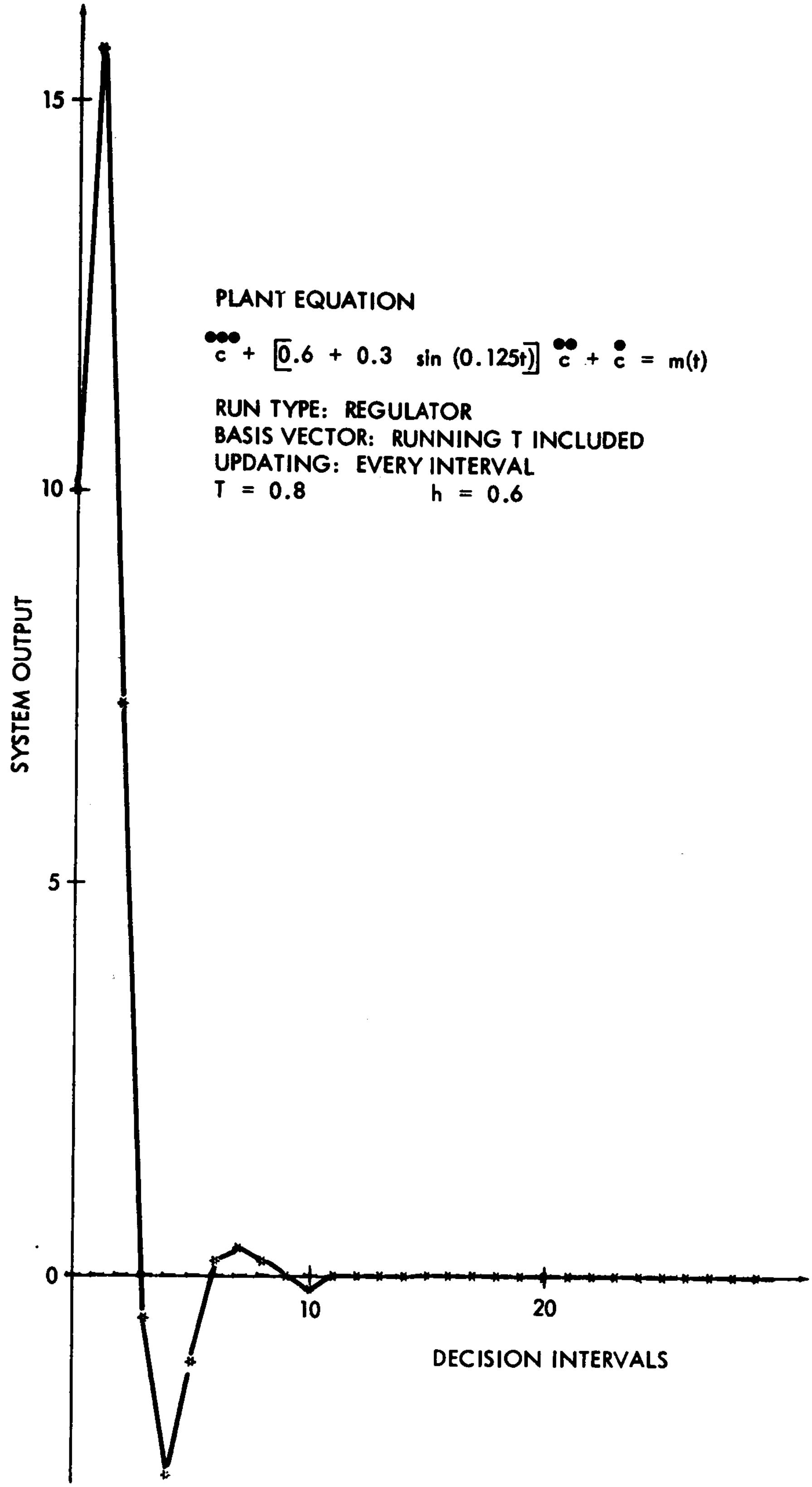


FIGURE 2-12 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

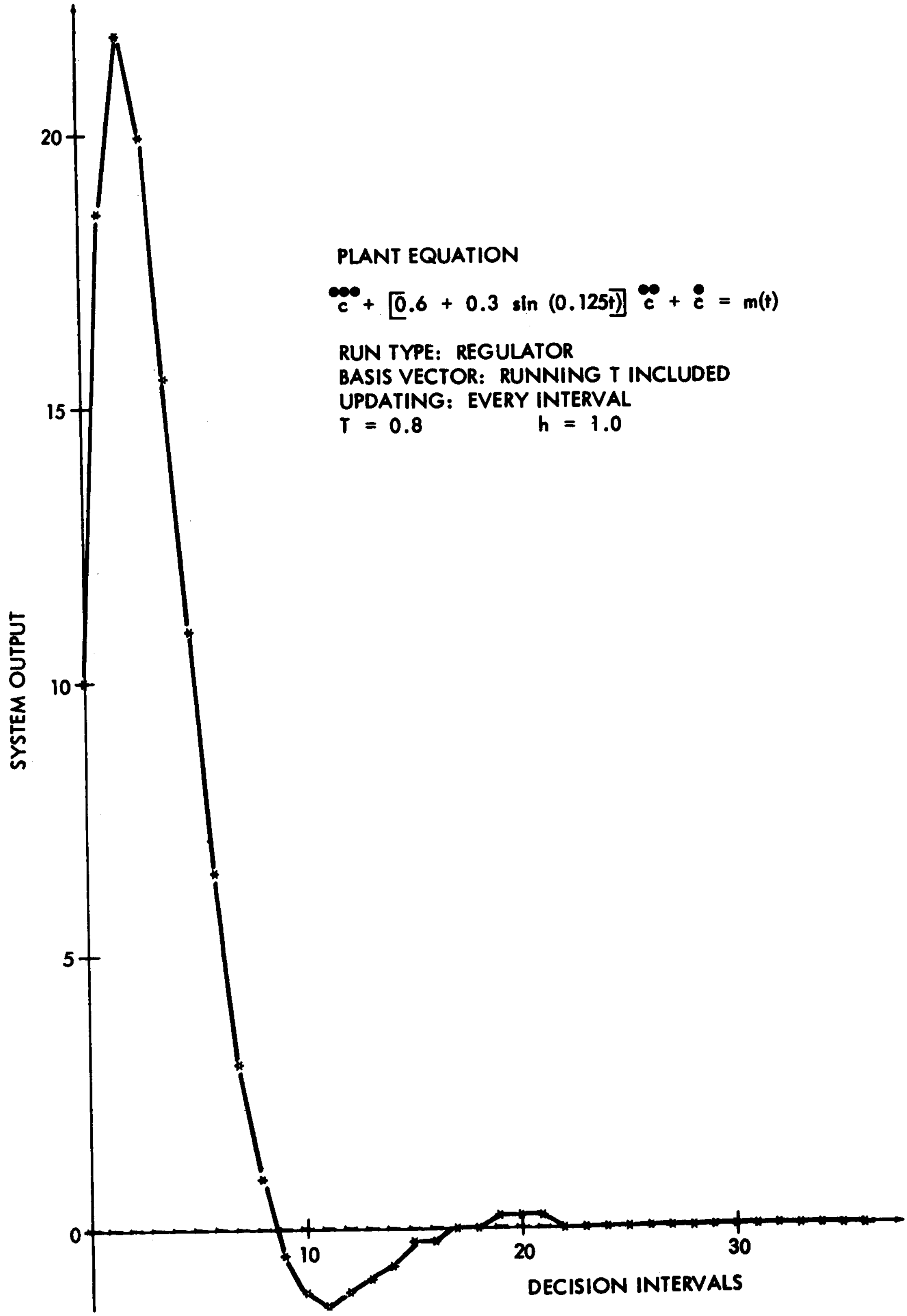


FIGURE 2-13 REGULATOR RESPONSE OF A 3RD ORDER SYSTEM

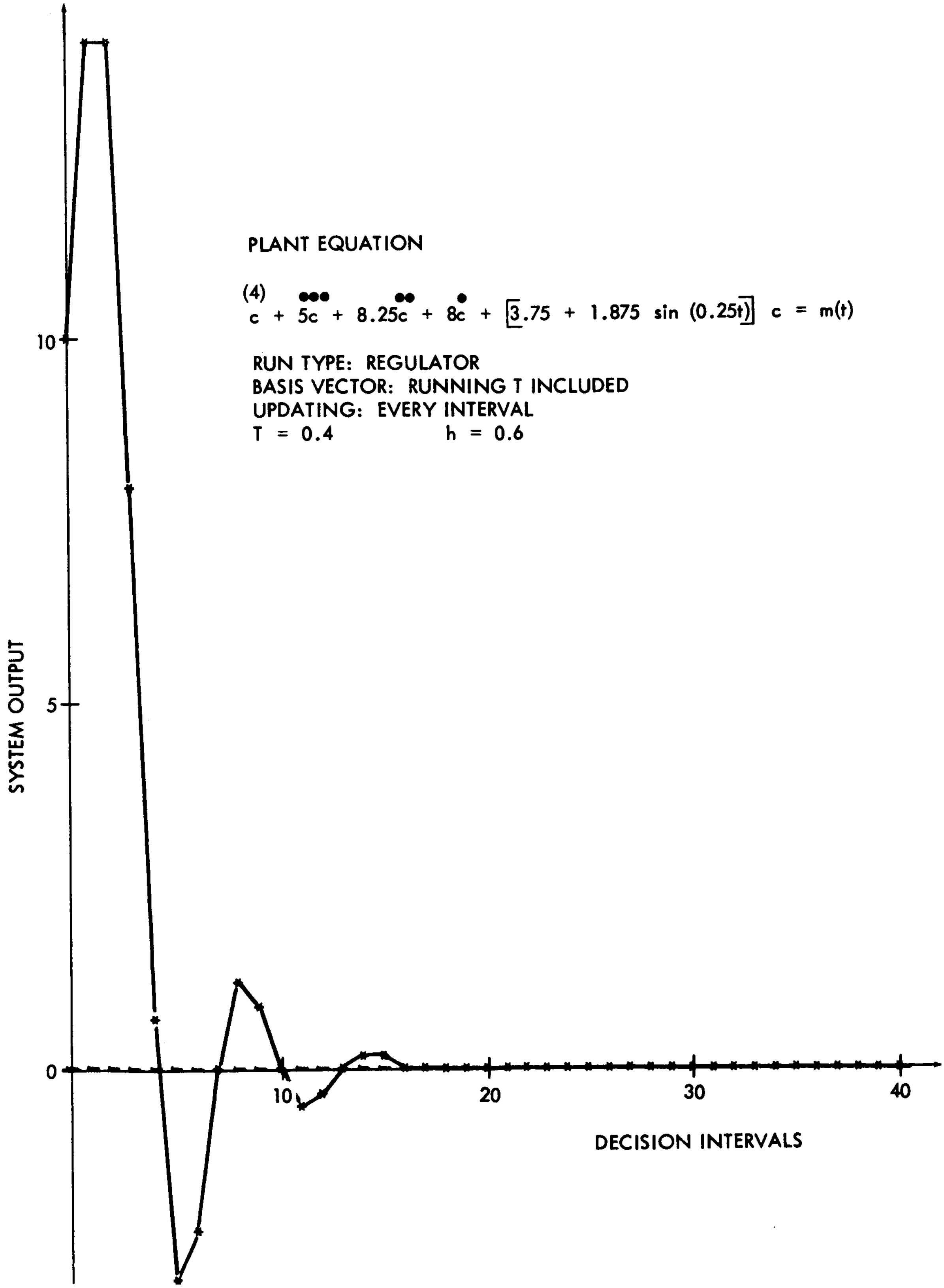


FIGURE 2-14 REGULATOR RESPONSE OF A 4TH ORDER SYSTEM

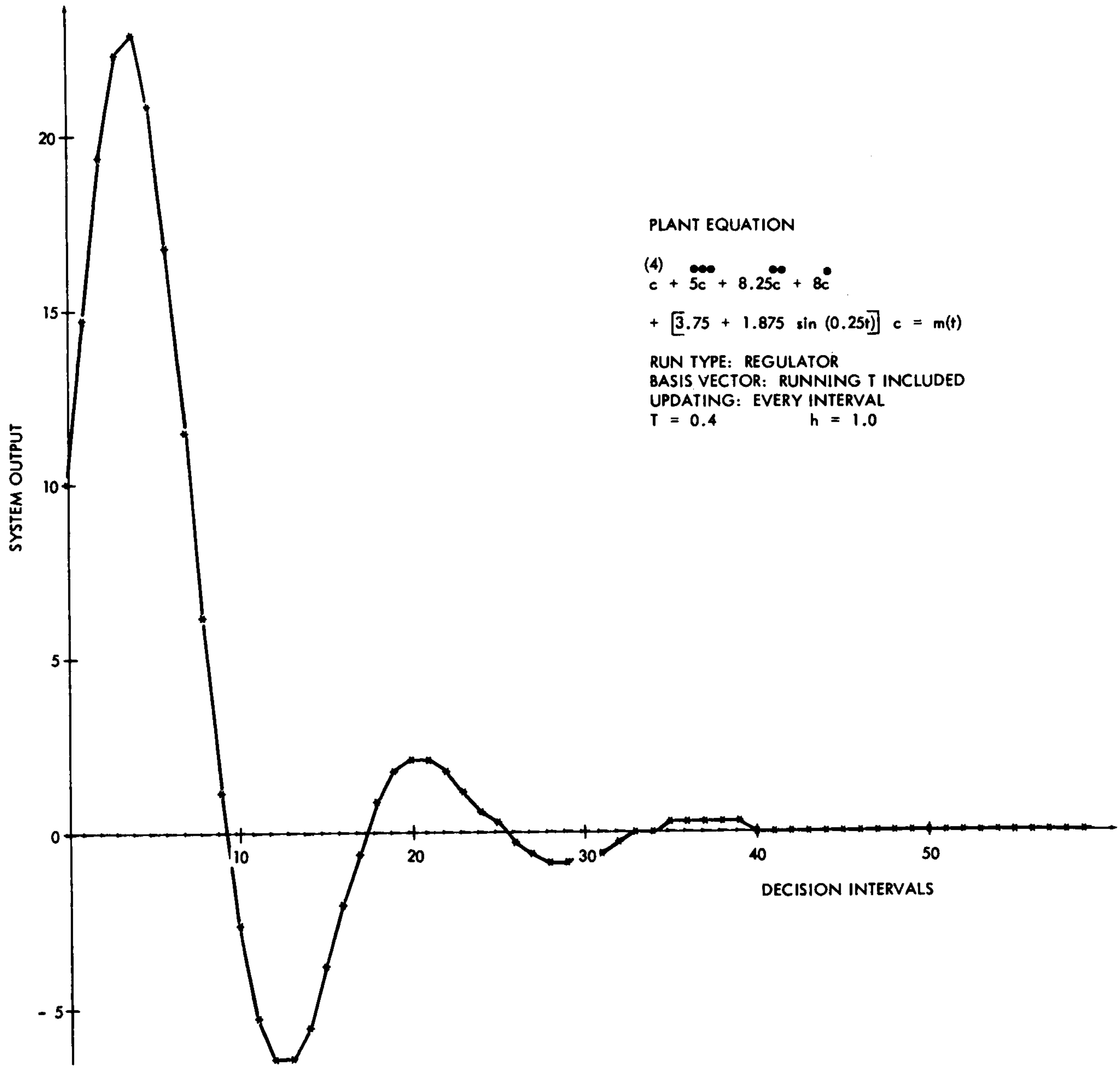


FIGURE 2-15 REGULATOR RESPONSE OF A 4TH ORDER SYSTEM

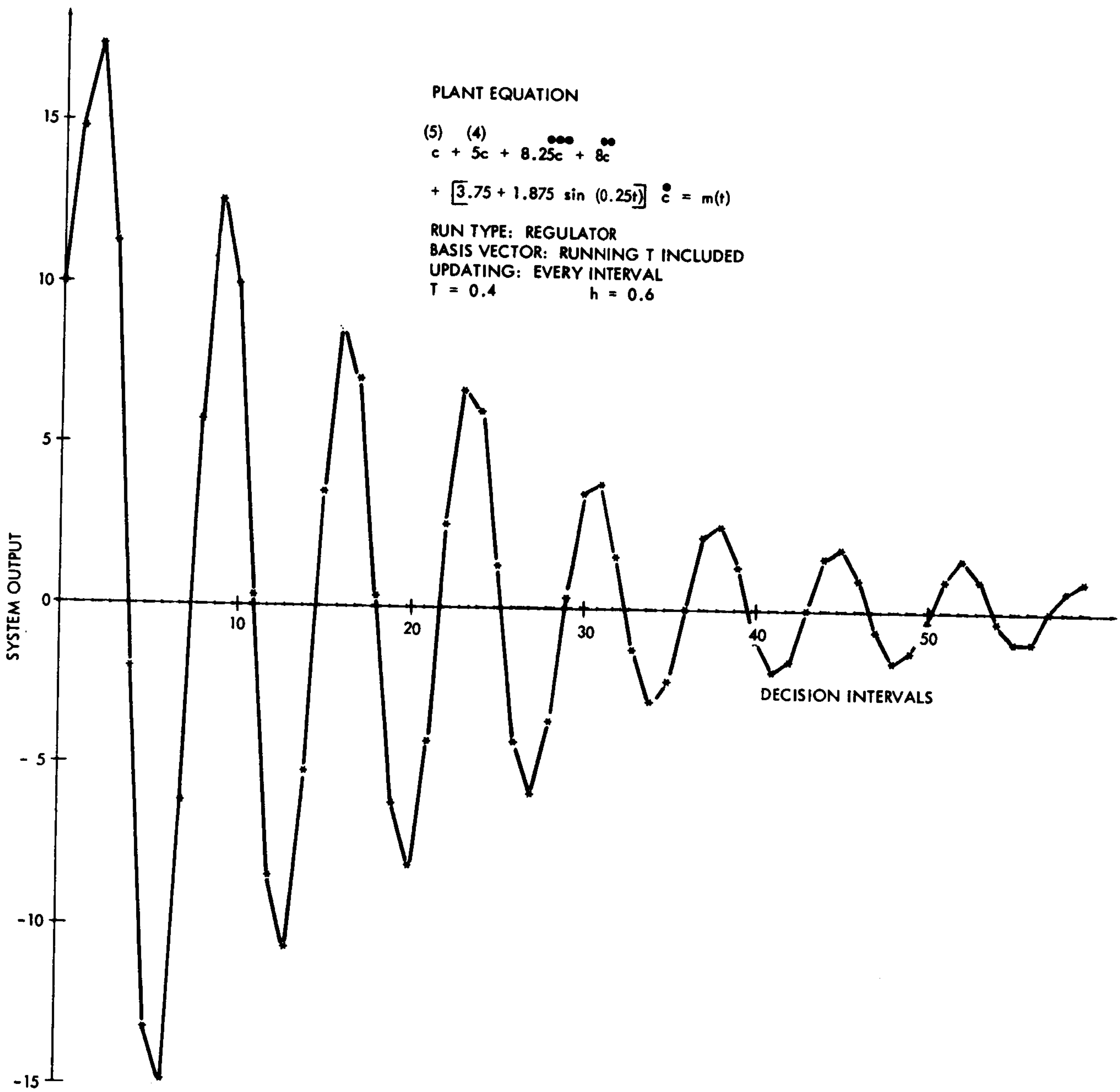


FIGURE 2-16 REGULATOR RESPONSE OF A 5TH ORDER SYSTEM

The overall experimental results indicated that all five T-h combinations provided stable control performance for all but three of the plants. These three exceptions were plants 3-11, 4-17, and 5-6 of Appendix B. The unstable points for these plants are listed below:

Plant Number	Unstable Points
3-11	D
4-17	D and E
5-6	D and E

MONITOR CONTROLLED UPDATING RESULTS

In the previously DACS research results (reference 1) as well as in the experimental results just presented, the interpolation estimates of the plant matrices were updated (recalculated) every n intervals, where $n \geq 1$ was a fixed system parameter. This research resulted in the conclusion that if the actual and desired output states are identical to some degree of satisfaction no new interpolation estimates should be obtained for system control. This conclusion along with the observation that time-varying systems require frequent updating until the desired output state is realized and then less frequent updating in order to avoid occurrences of matrix ill conditioning and/or poor plant description was noted in reference 1. These experimental conclusions and observations implied the desirability and need for some monitoring procedure to decide when updating is desirable for continuing satisfactory system control. Such an updating monitor was developed and presented in paragraph 2.1.

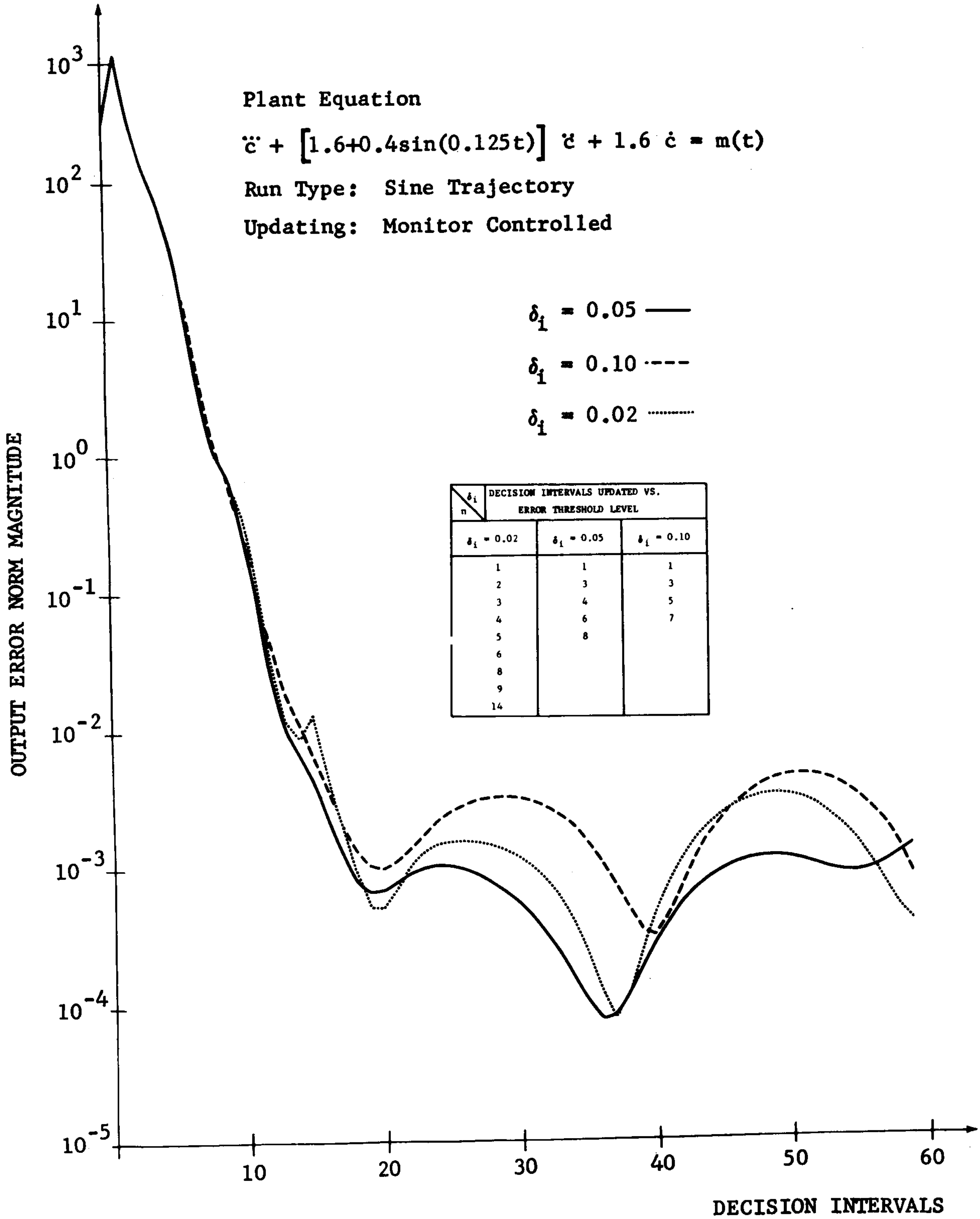
This experimental investigation utilized approximately one-half of the plants presented in Appendix B. The majority of the plants selected were of the faster time variation plant set. In every case three allowable error threshold or δ_i levels were experimentally investigated for Interpolation Prediction using the stationary basis vector description. These allowable error threshold levels were: δ_i = 0.02. 0.05, and 0.10. It should be noted that this experimentation was conducted only for the stationary basis vector case, since previous regulator results did not indicate a preference between the two descriptions. Therefore, intensive investigation was conducted for only the stationary description rather than slight investigation for both descriptions.

Throughout this particular experimentation the desired output was a sine trajectory and the stable T-h combination was T = 0.6, h = 0.8. New data was shifted into the matrix of basis vectors every interval. Also, the same experimental start-up procedure utilized in the previous stability boundry investigation and described for the linear stationary case with fixed updating (reference 1) was used in this study.

Figures 2-17 and 2-18 present typical experimental results. The graphs in the figures present the unweighted error norm versus the number of decision intervals for all three allowable error threshold levels. Also, each figure contains a list of the decision intervals where the monitor requested updating of the interpolation estimates of the plant matrices. Several very obvious observations are: as plant order increases more frequent updating is required for any δ_i value; as the value of δ_i is lowered (less allowable error) more frequent updating is required for a plant of any order. Other observations concerning the δ_i value and actually reached error norm are not so clear cut or by any means general. These results are most certainly biased by the fact that this study was completely devoted to sine time variations using sine trajectories for the desired output state. However, one general conclusion is that the value of δ_i bears no relationship to the frequency or point of occurrence of poor interpolation estimates of the plant matrices. About equal occurrence of such poor estimation was noted for all three δ_i values throughout this study.

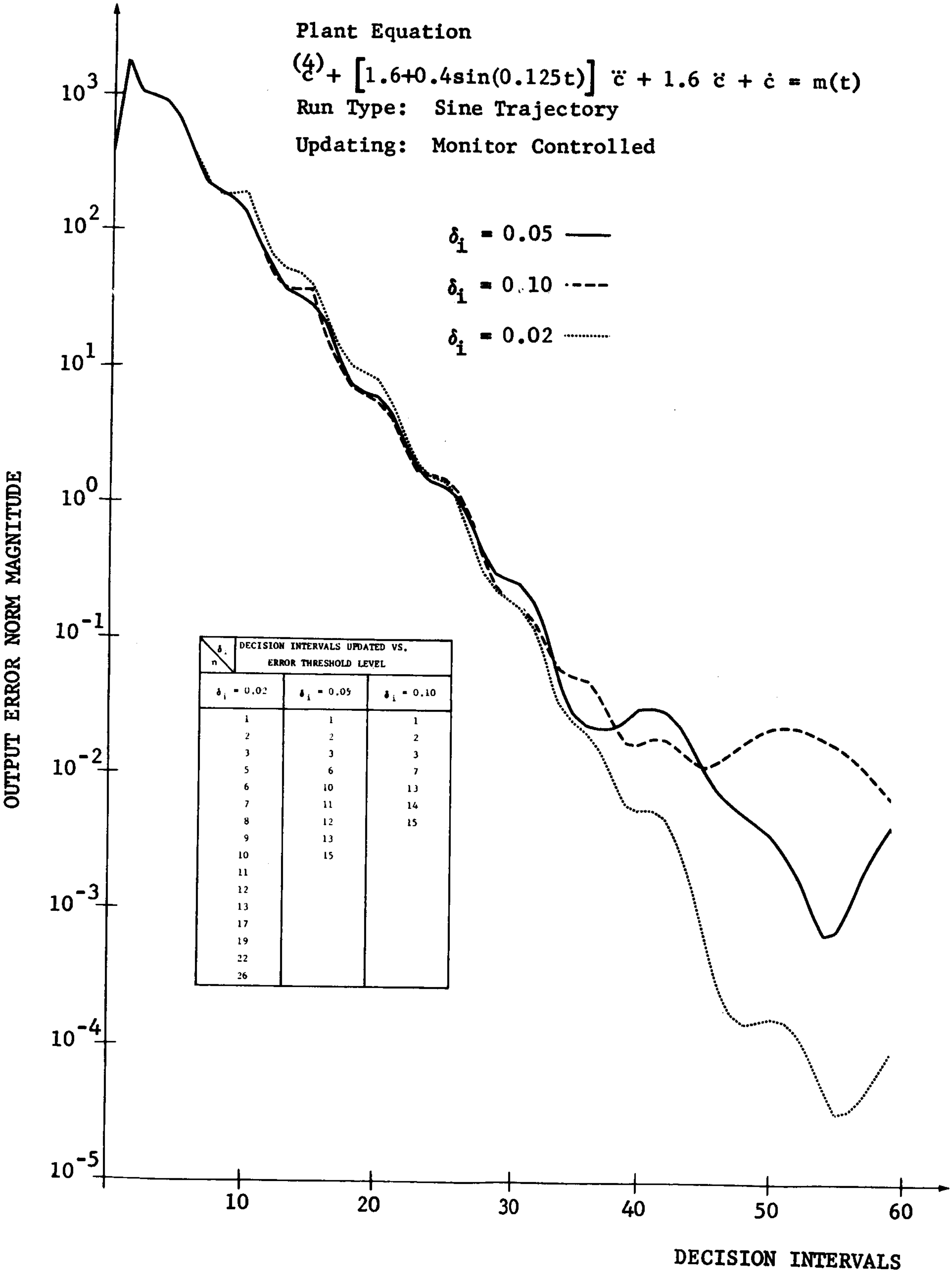
TRAJECTORY RESULTS

Interpolation Prediction control performance both with the stationary and time-varying basis vector description was examined using sine and displaced sine desired outputs for time-varying plants through fifth order. These plants are presented in Appendix B and it may be noted that only plants with sine time variation of a single coefficient were used for this study.



δ_i / n: DECISION INTERVALS UPDATED VS. ERROR THRESHOLD LEVEL		
$\delta_i = 0.02$	$\delta_i = 0.05$	$\delta_i = 0.10$
1	1	1
2	3	3
3	4	5
4	6	7
5	8	
6		
8		
9		
14		

FIGURE 2-17 OUTPUT ERROR NORM MAGNITUDE OF A 3RD ORDER SYSTEM



δ_i \ n	DECISION INTERVALS UPDATED VS. ERROR THRESHOLD LEVEL	
$\delta_i = 0.02$	$\delta_i = 0.05$	$\delta_i = 0.10$
1	1	1
2	2	2
3	3	3
5	6	7
6	10	13
7	11	14
8	12	15
9	13	
10	15	
11		
12		
13		
17		
19		
22		
26		

FIGURE 2-18 OUTPUT ERROR NORM MAGNITUDE OF A 4TH ORDER SYSTEM

This experimental study consisted of approximately 200 control simulations with the selected time-varying plants. The updating monitor with a δ_i value of 0.05 was used throughout the study for both the stationary and time-varying basis vector descriptions. Also, new data was shifted into the matrix of basis vectors every interval for both descriptions. As in the studies described in previous paragraphs of this section, the same start-up procedure utilized in other DACS research (reference 1) was used for this experimentation. All the following experimentation was conducted with either +10 or +1 units as an initial condition for each element of the state vector.

Stationary Basis Vector Results. - Typical results of the third order time-varying system control experiments are given in Figures 2-19 through 2-24 for a sine trajectory desired output. The first four, Figures 2-19, 2-20, 2-21, and 2-22, illustrate the type of control performance obtained at the points A, B, D and E of the T-h plane (See Figure 2-2). The last two, Figures 2-23 and 2-24, show the control results for a plant configuration with two speeds of time variation. These latter two figures show in different degrees tracking problems due to poor interpolation estimates of the plant matrices being used for one or more intervals.

Figures 2-25, 2-26, and 2-27 illustrate the typical control performance for three of the five T-h points investigated for fourth order time-varying systems. Figure 2-27 demonstrates the problem of using poor interpolation estimates of the plant matrices for one or more intervals. This problem area is even more evident in Figure 2-28 which shows the control performance of a fourth order system with a time-varying gain.

The last two figures, 2-29 and 2-30, are typical control results for the fifth order time-varying systems. The control performance is not unexpected, since the control performance for linear stationary systems of fifth order was approximately equivalent (reference 1). The very slow and oscillatory controlled response is typical of fifth or higher order systems, and is not related to poor interpolation estimation of the plant matrices.

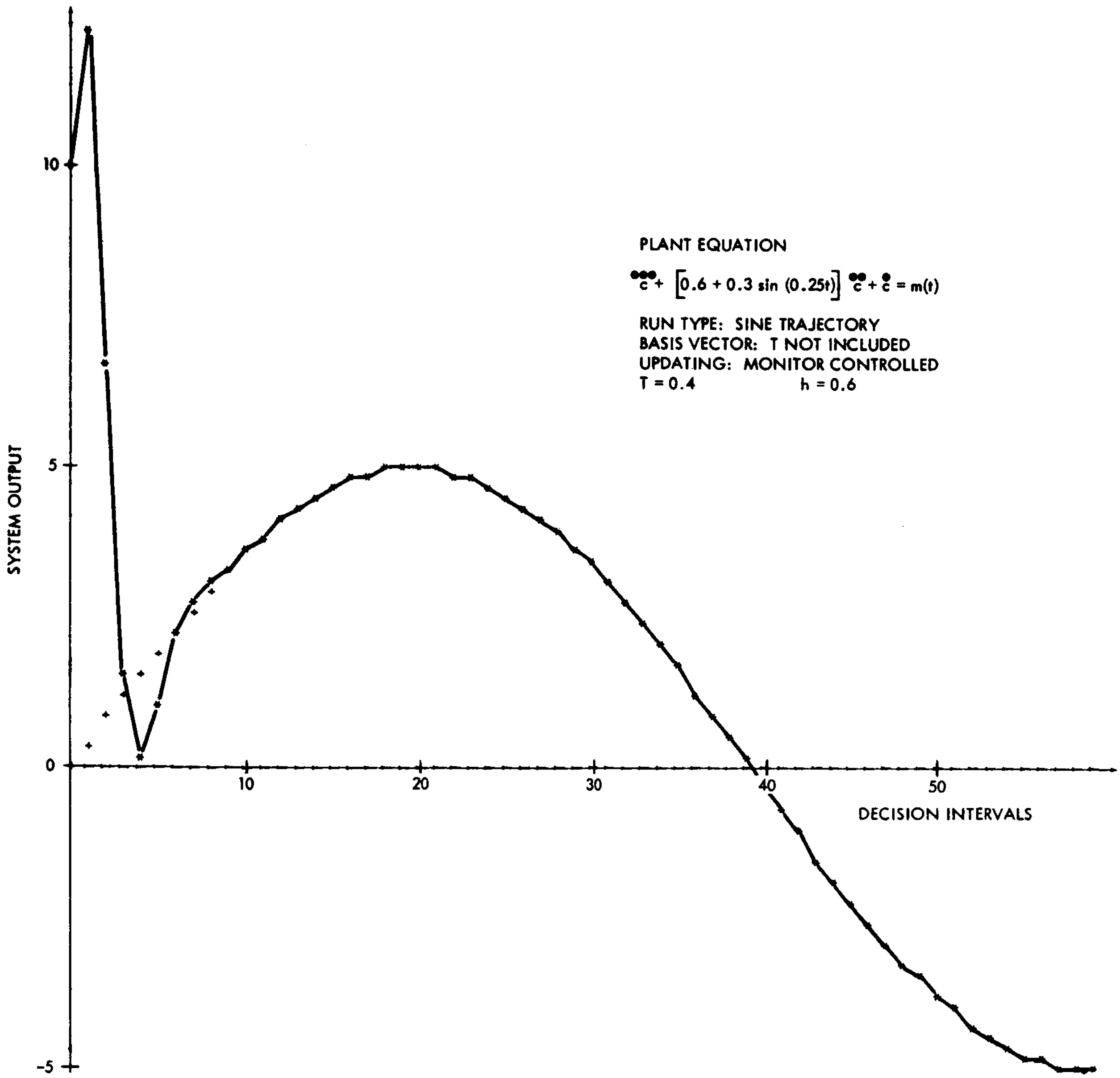


FIGURE 2-19 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

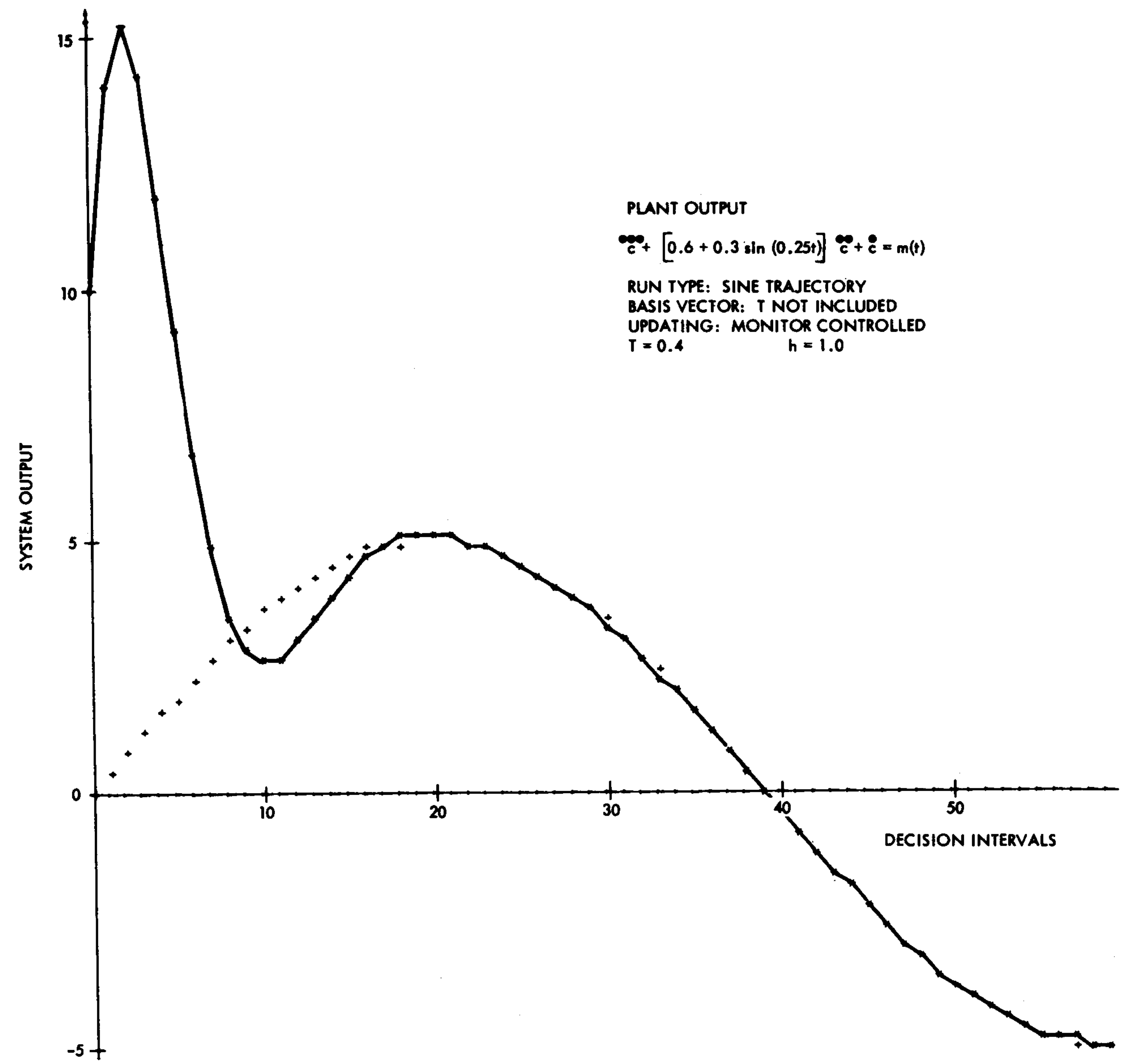


FIGURE 2-20 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

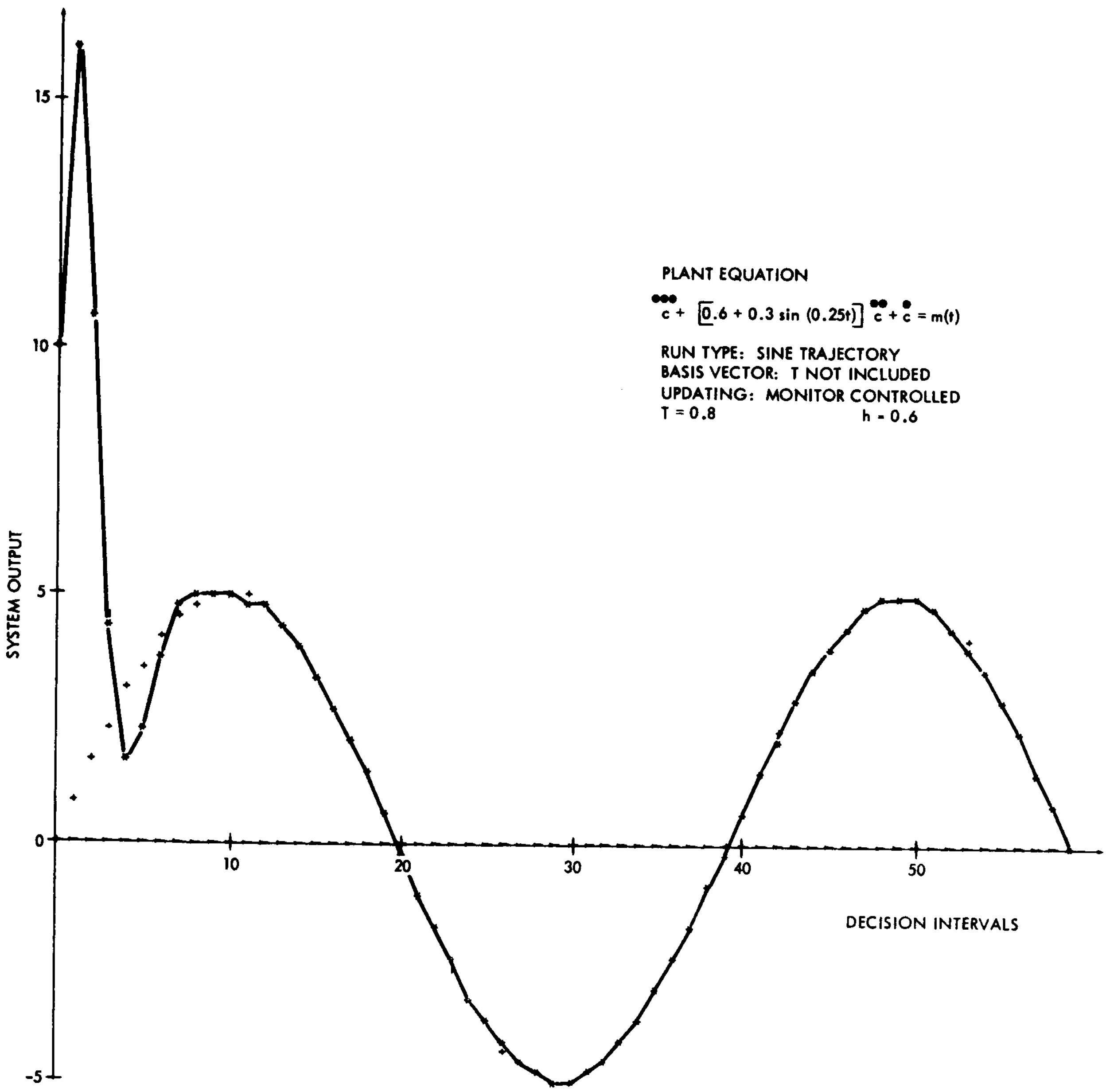


FIGURE 2-21 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

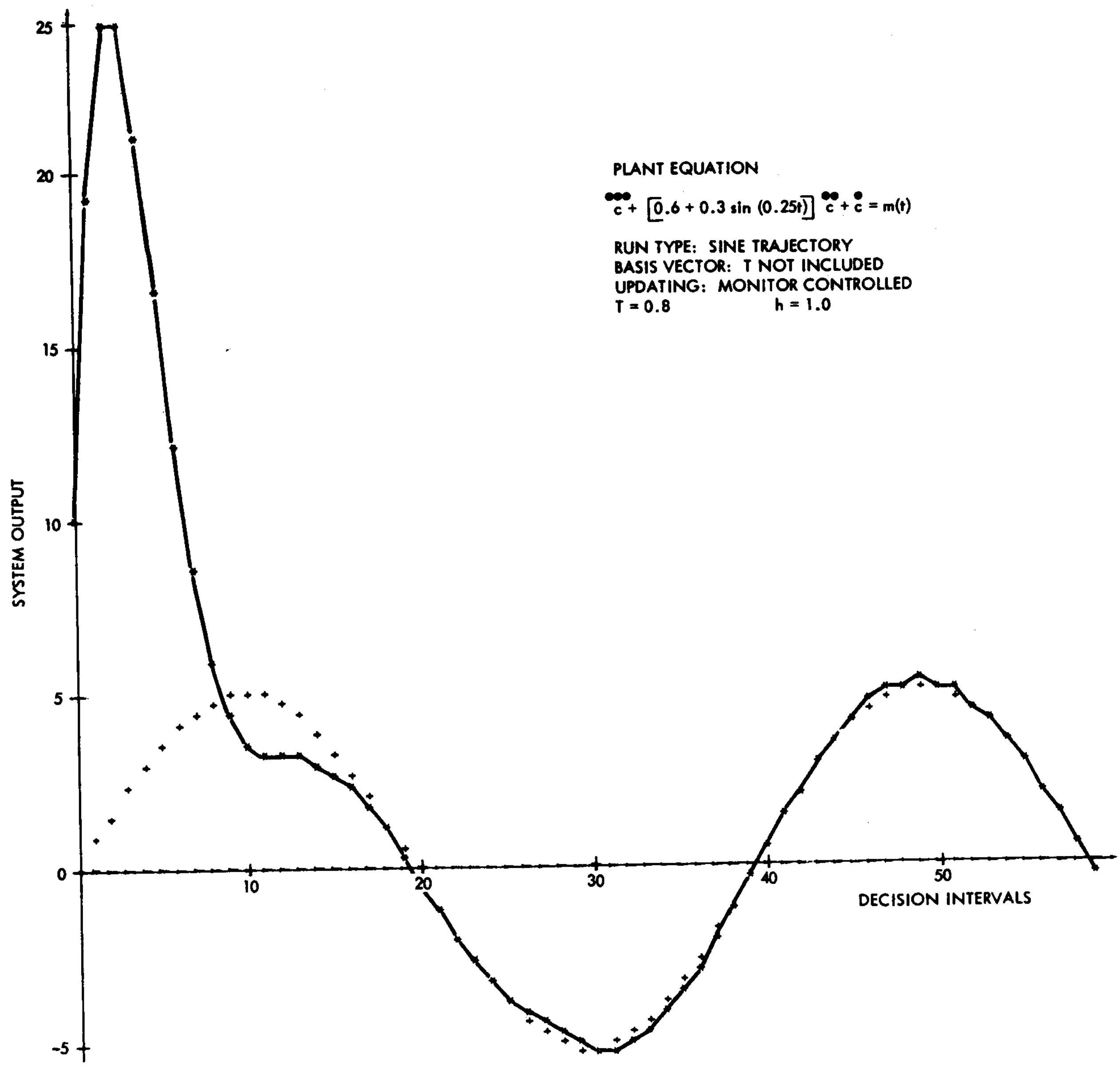


FIGURE 2-22 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

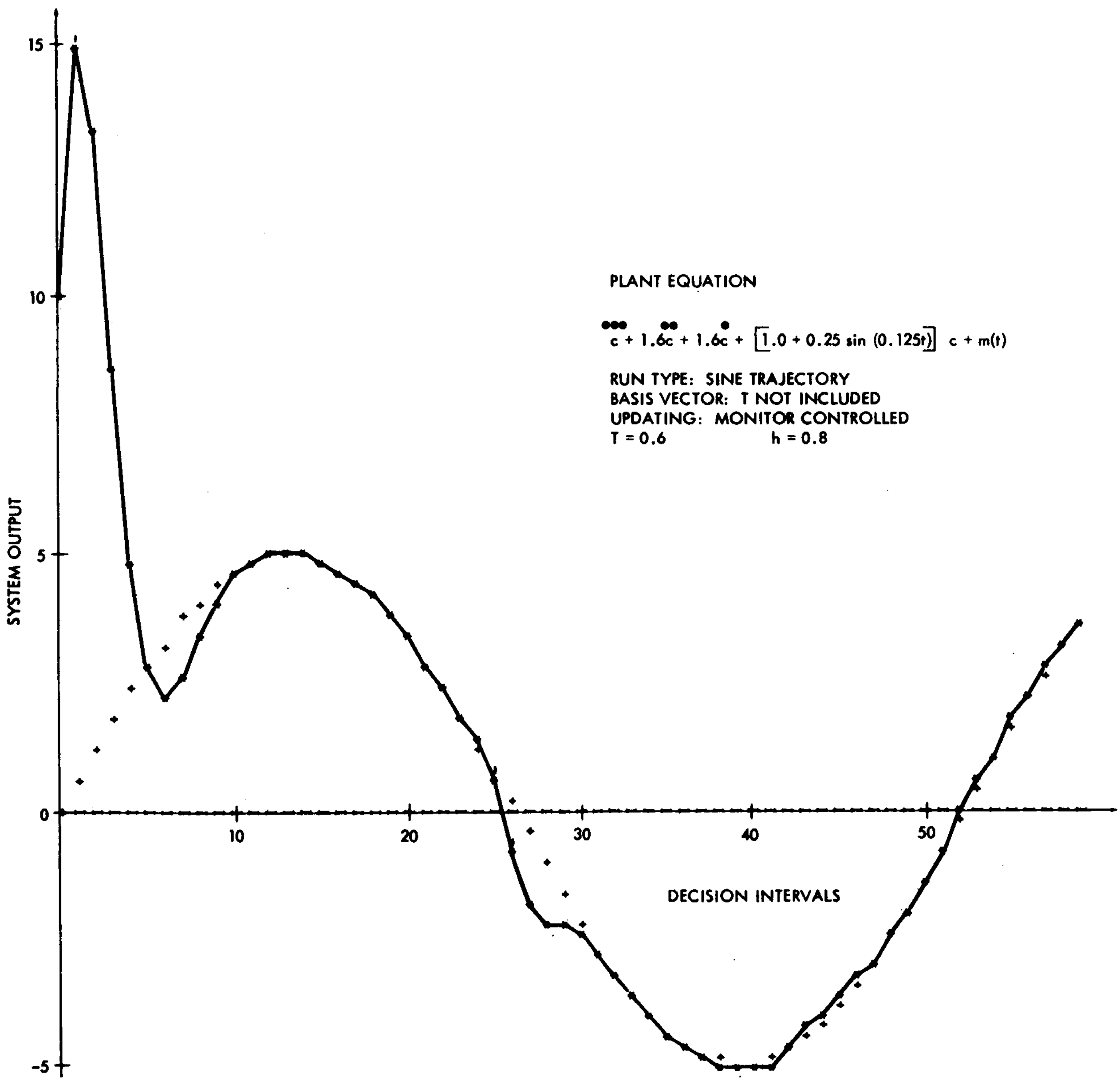


FIGURE 2-23 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

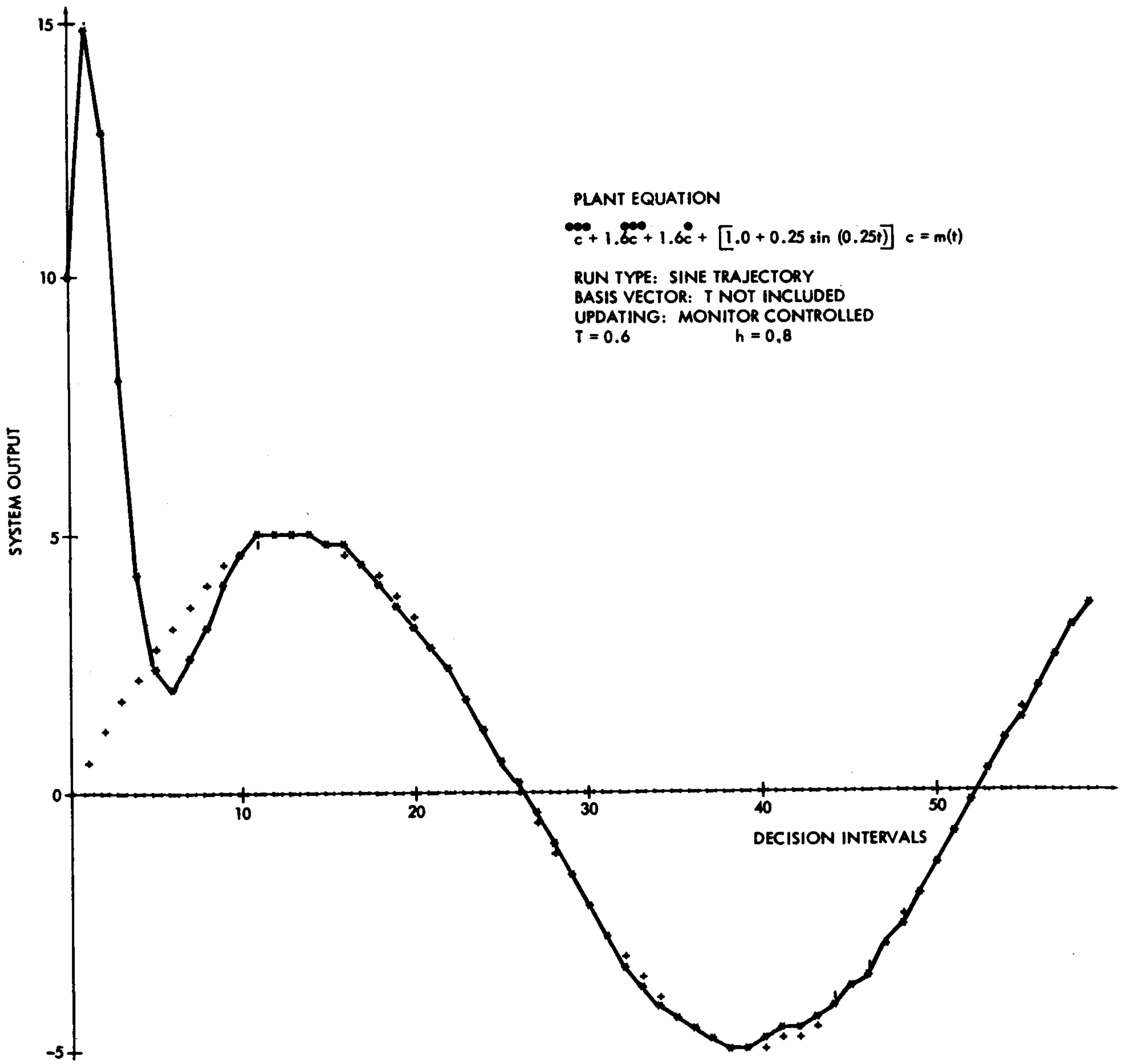


FIGURE 2-24 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

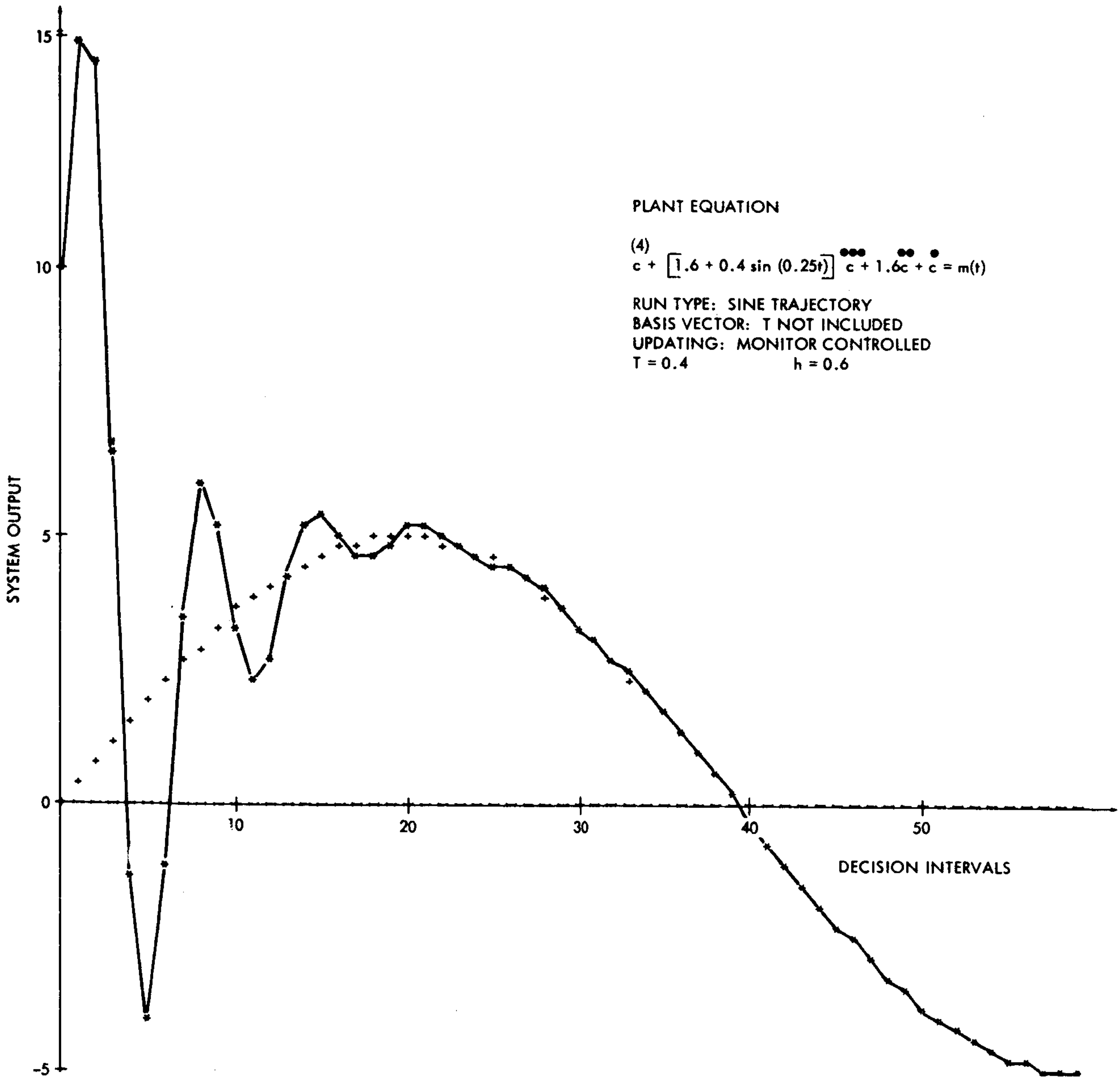


FIGURE 2-25 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

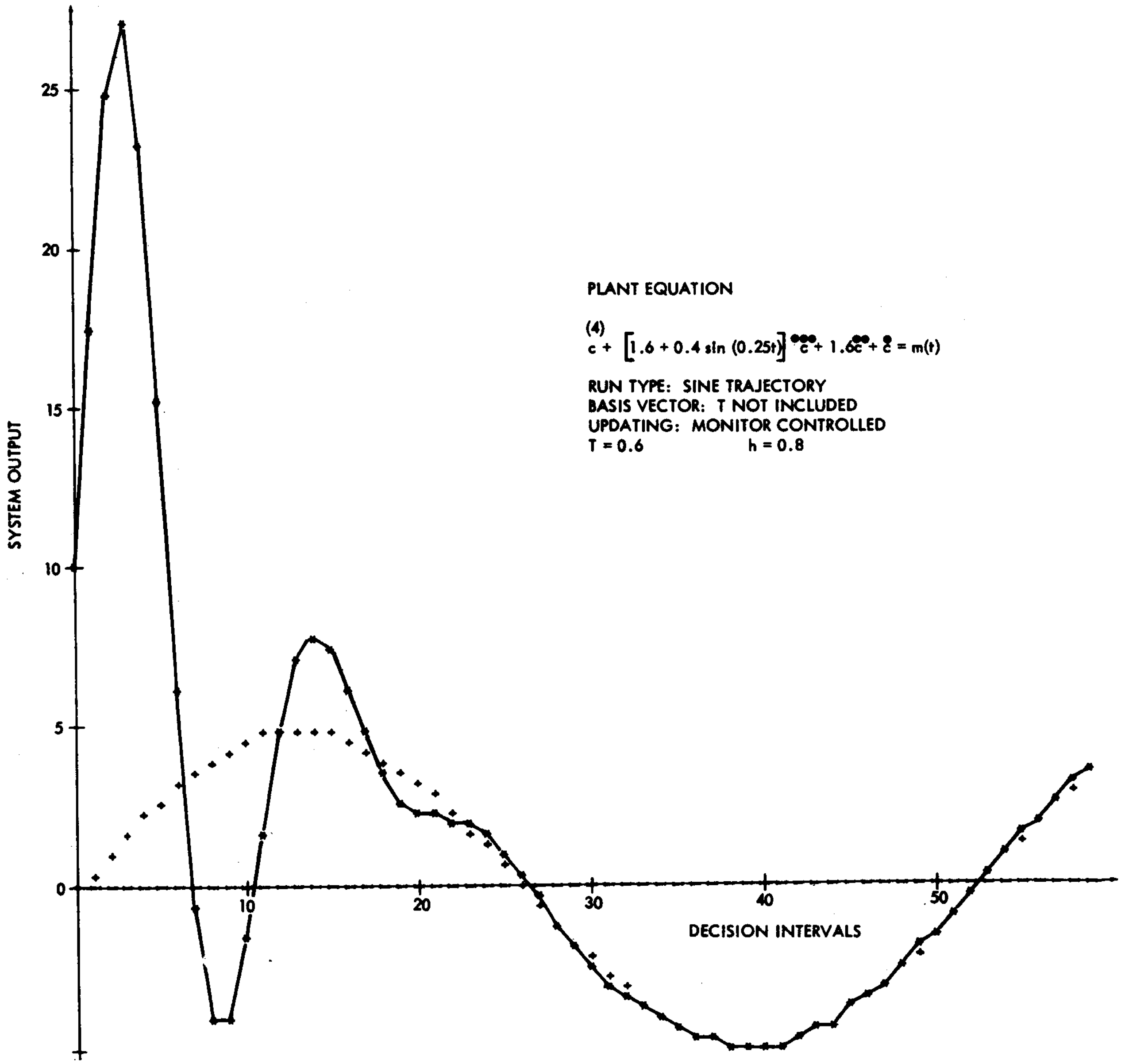


FIGURE 2-26 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

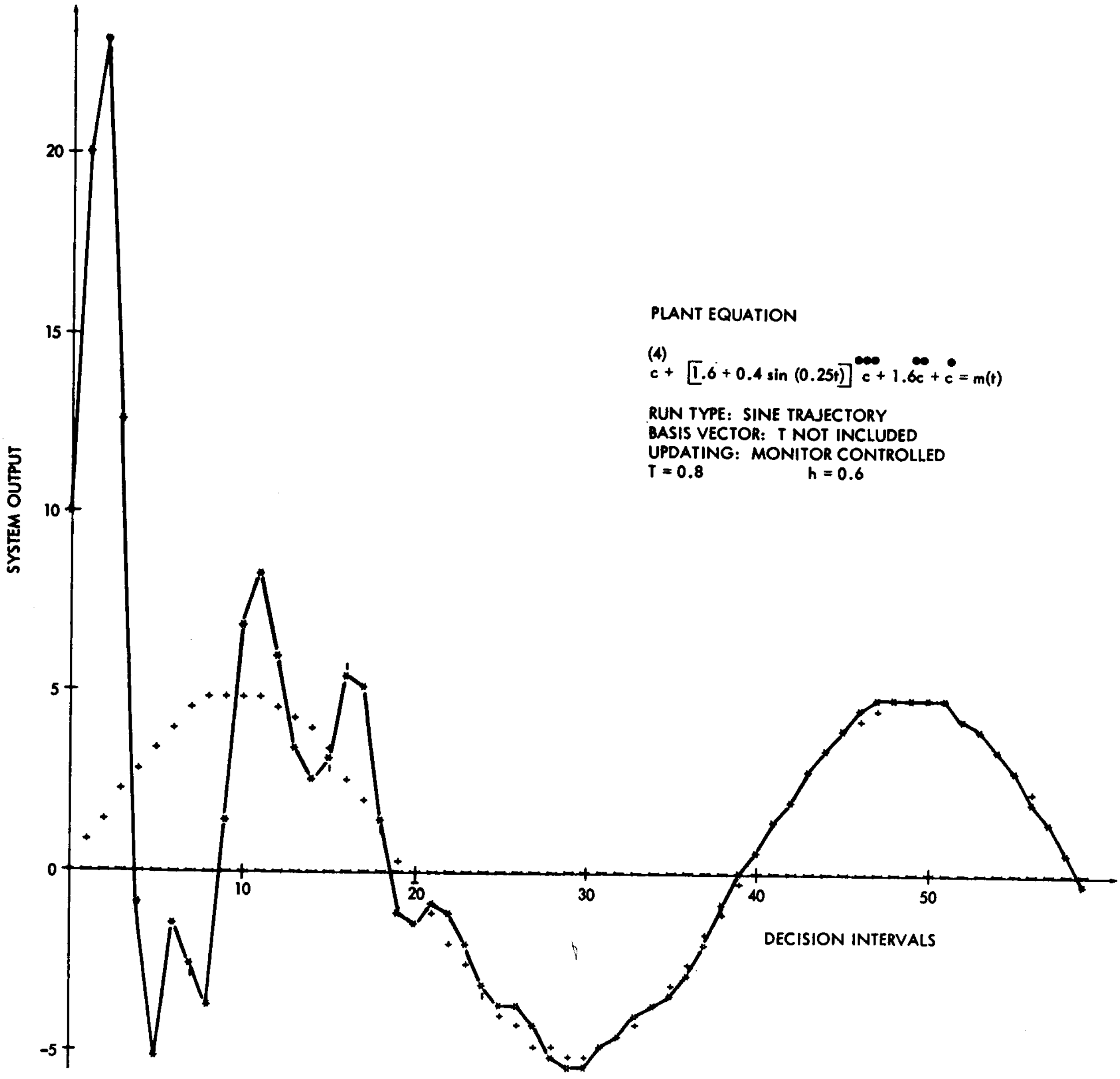


FIGURE 2-27 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

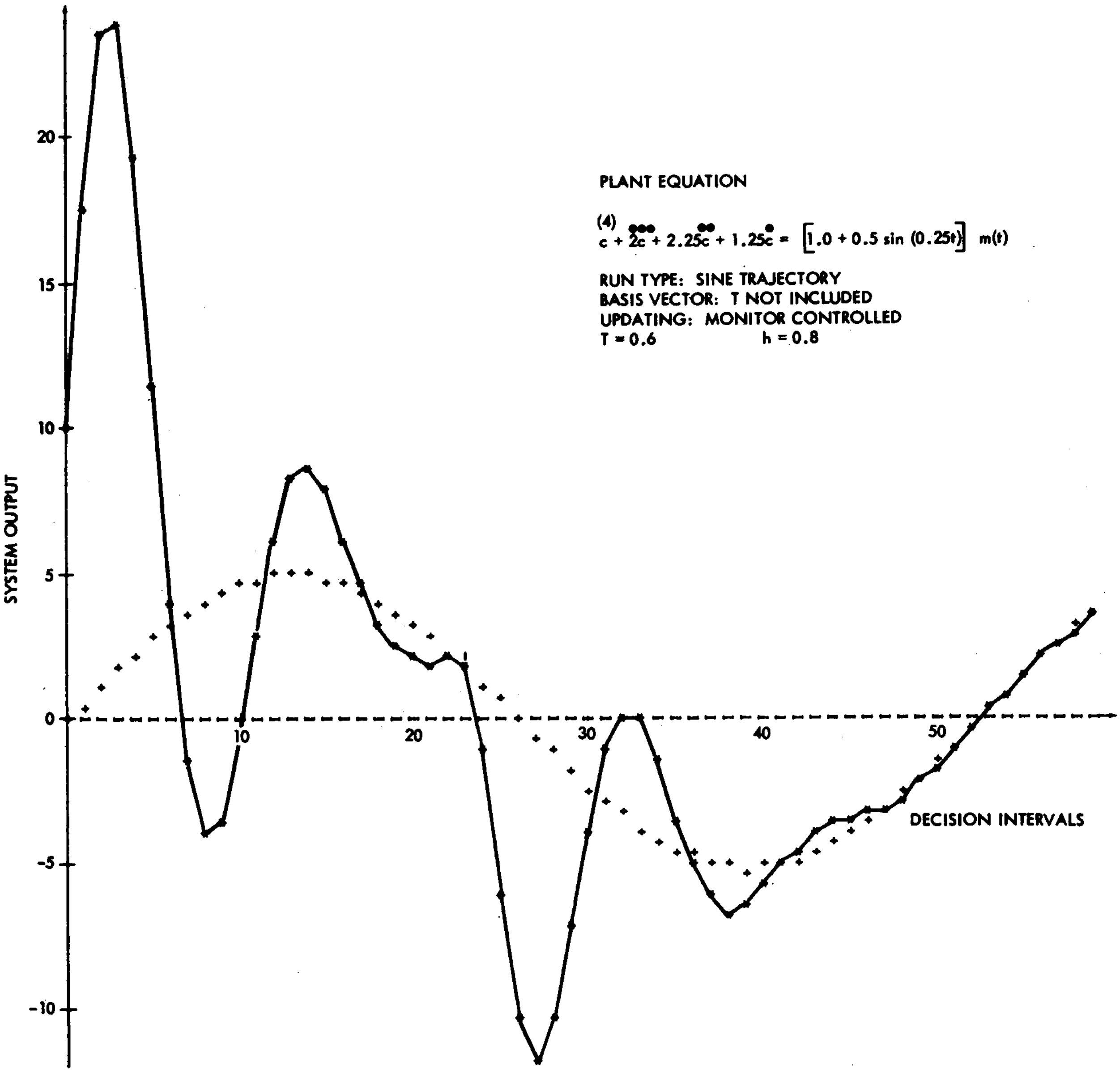


FIGURE 2-28 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

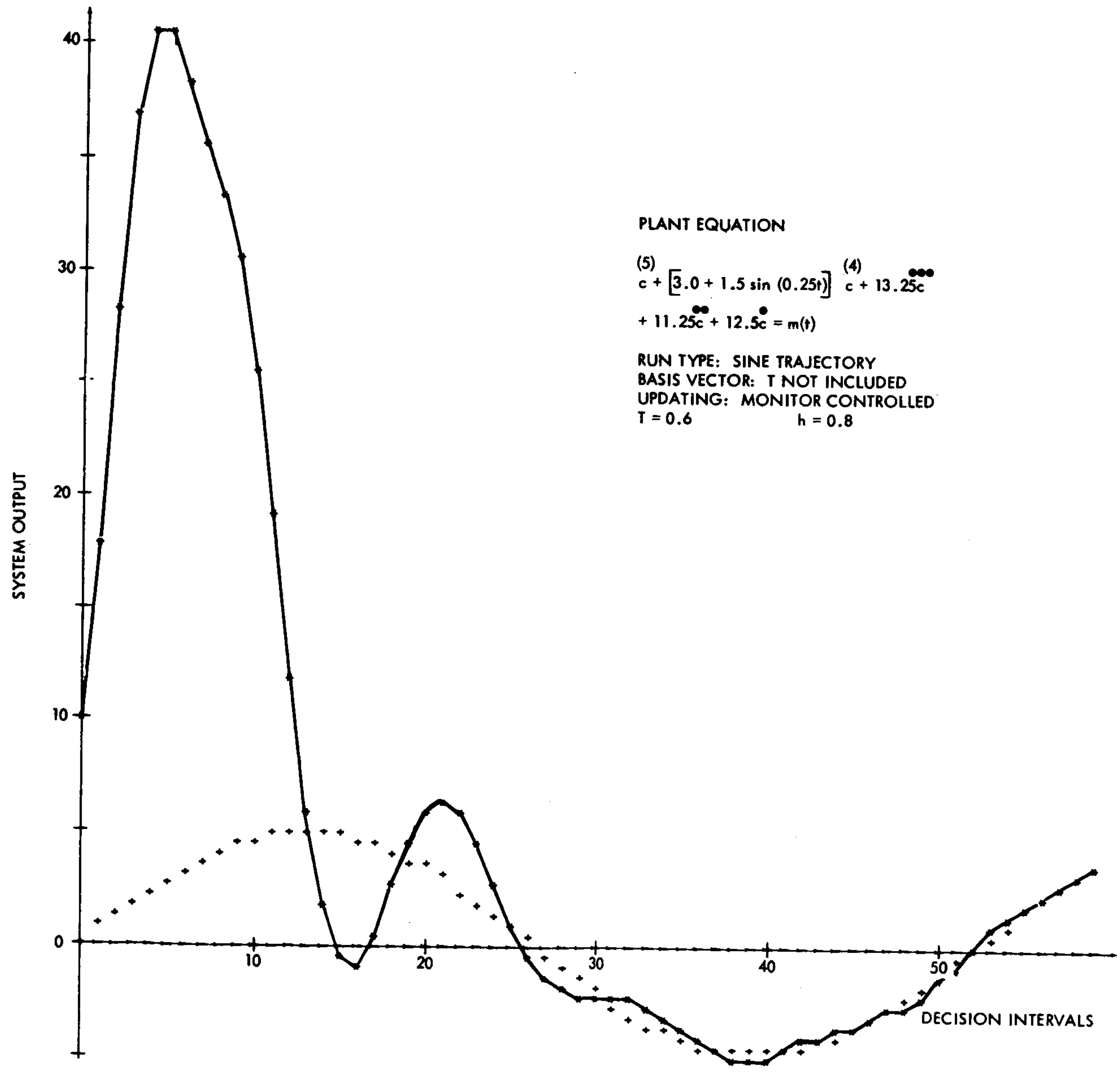


FIGURE 2-29 TRAJECTORY RESPONSE OF A 5TH ORDER SYSTEM

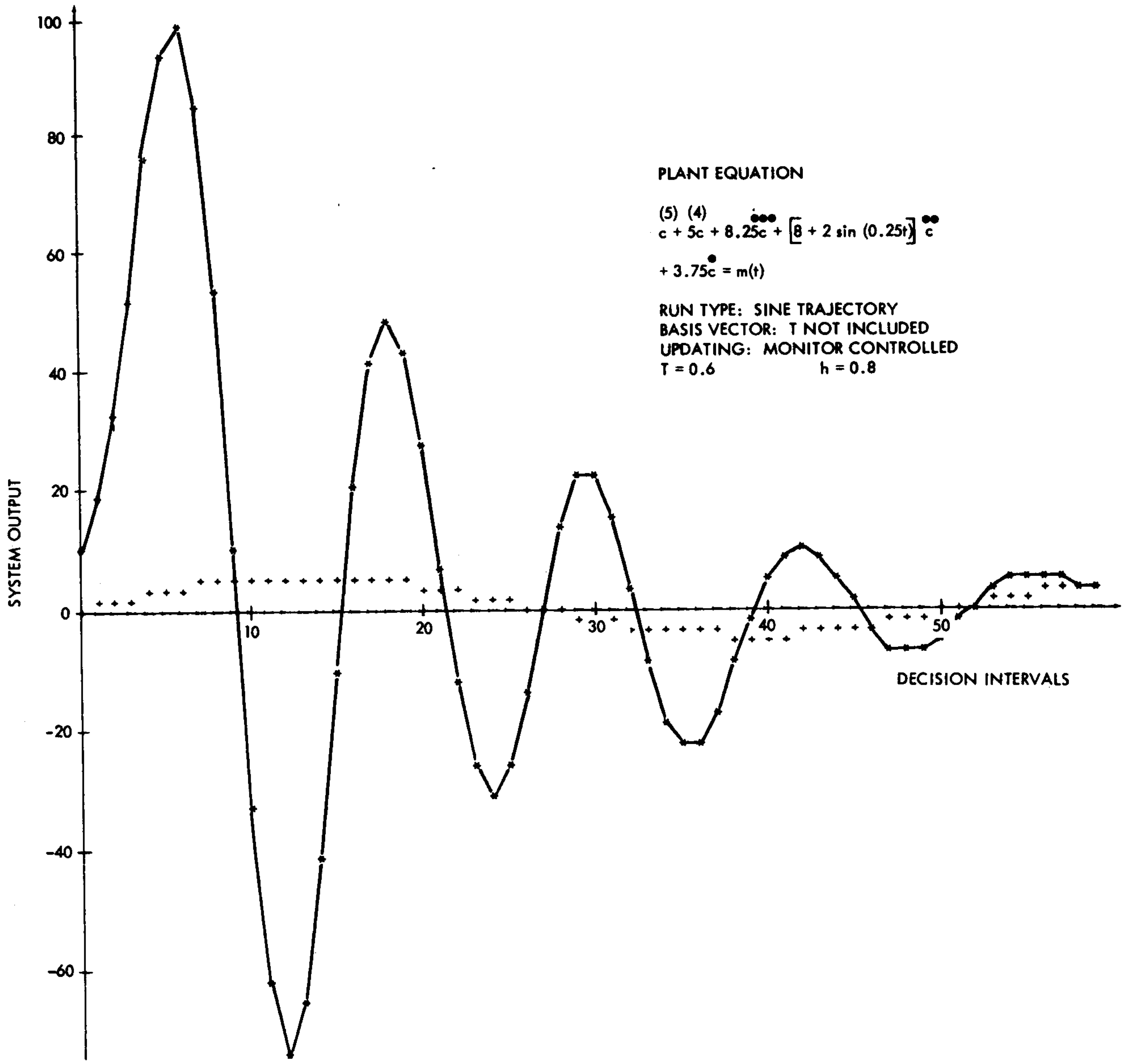


FIGURE 2-30 TRAJECTORY RESPONSE OF A 5TH ORDER SYSTEM

In summary, the experimental results just presented provided no really unexpected observations. The problem area of poor interpolation estimates of the plant matrices did somewhat isolate itself from the ill conditioning problem noted in the regulator runs. Occurrences of degraded tracking were evident, and could usually be traced back to inadequate estimates of the plant matrices. This fact is easily noted by examination of Figure 2-28. The update monitor requested recalculation of the interpolation estimates of the plant matrices during the most severe time variation associated with the sine time-varying coefficient. This occured between 22 and 26 Decision Intervals on the plot of Figure 2-28. The interpolation estimates of the plant matrices were inadequate and so poor control performance resulted until better estimates were obtained at a later time. However, in general, the trajectory results were as good as those results obtained in the stability investigation.

<u>Time-Varying Basis Vector Results</u>. - The four figures, 2-31, 2-32, 2-33, and 2-34, present control results obtained with Interpolation Prediction using the time-varying basis vector description. This set of figures are typical of those obtained for the third order time-varying systems at the upper and lower T-h point combinations considered during the stability investigation. The next two figures, 2-35 and 2-36, present the same T-h combinations for the same plant configuration except that the speed of the time variation is twice as fast in the second figure. Also, it may be noted that Figures 2-35 and 2-36 illustrate different degrees of the tracking problem associated with using poor interpolation estimates of the plant matrices for one or more intervals.

Typical fourth order time-varying system control results are presented in Figures 2-37, 2-38, and 2-39 for three of the five T-h combinations considered in this experimentation. The problem of poor interpolation estimation is quite evident in Figure 2-37 of this set as well as in Figure 2-40. Figure 2-40 presents the control performance of a fourth order system with a time-varying gain.

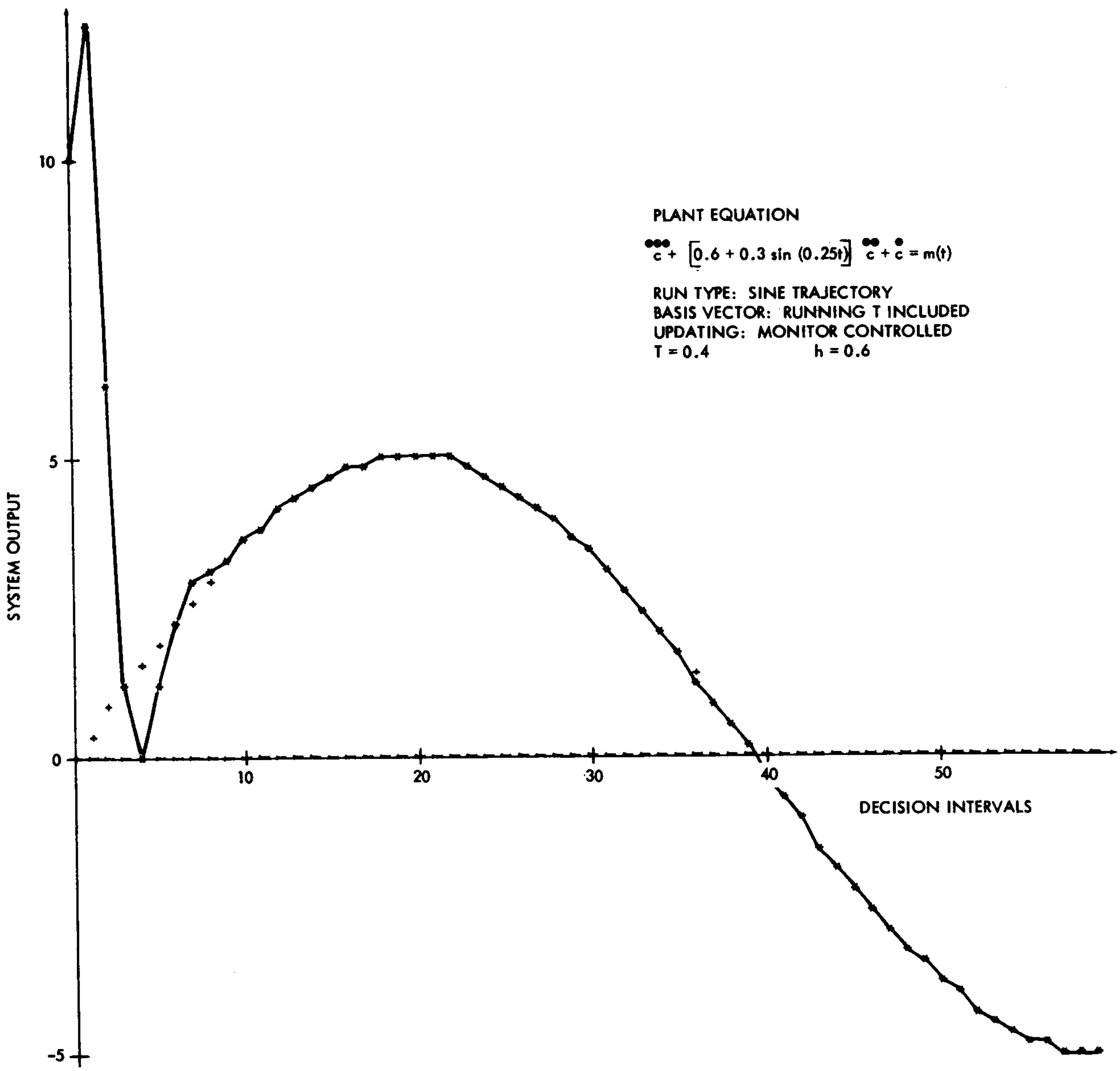


FIGURE 2-31 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

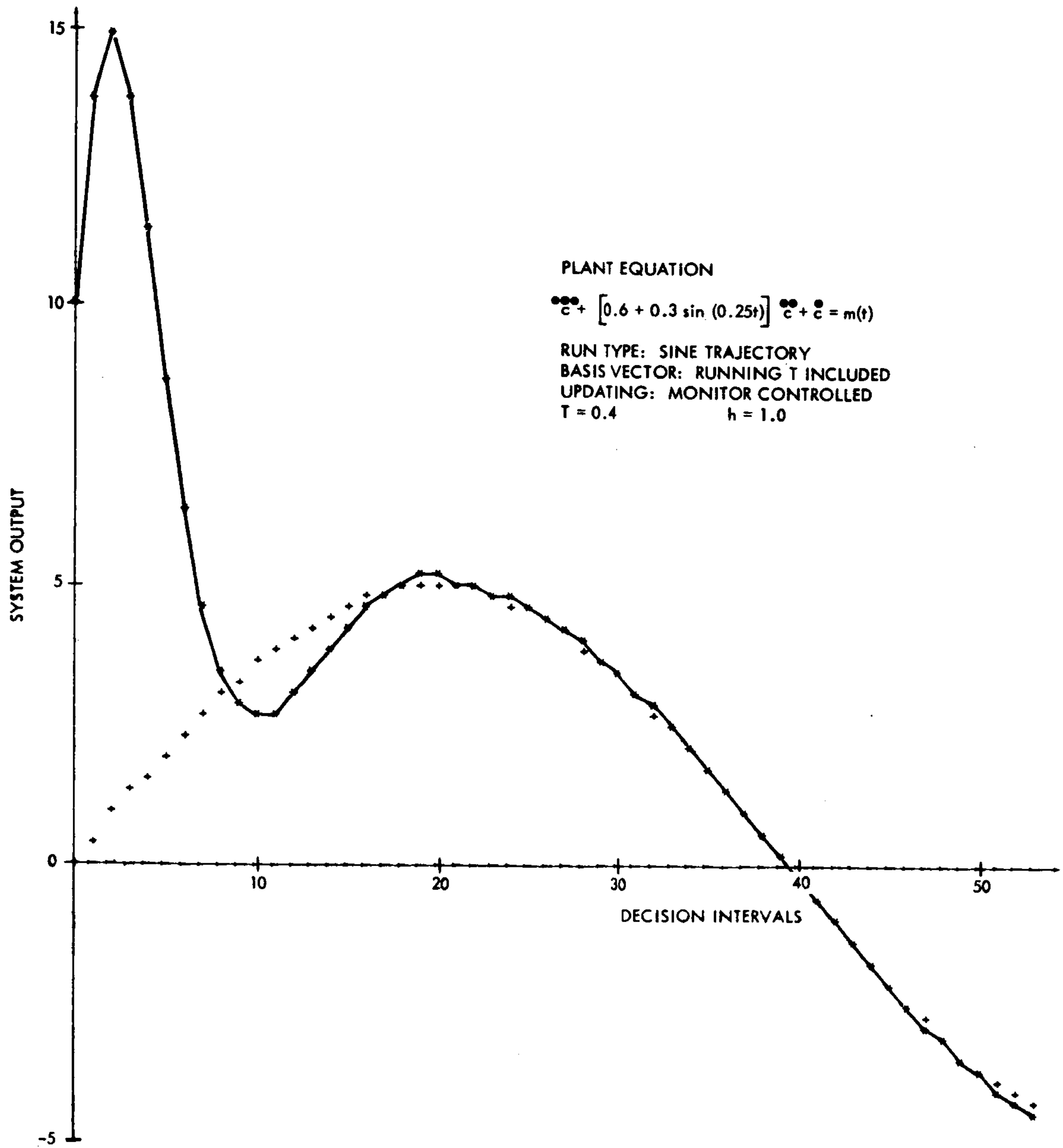


FIGURE 2-32 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

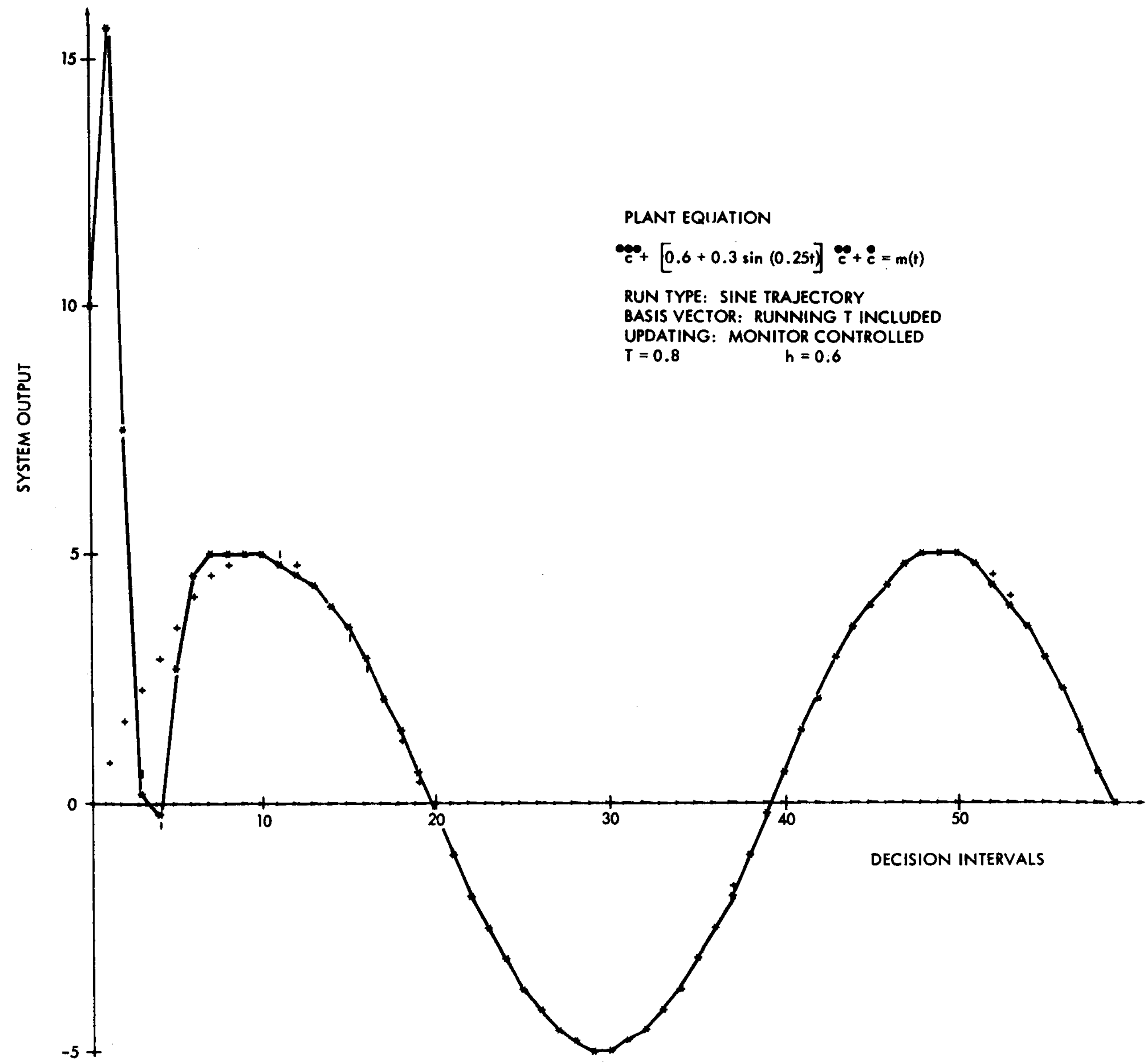


FIGURE 2-33 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

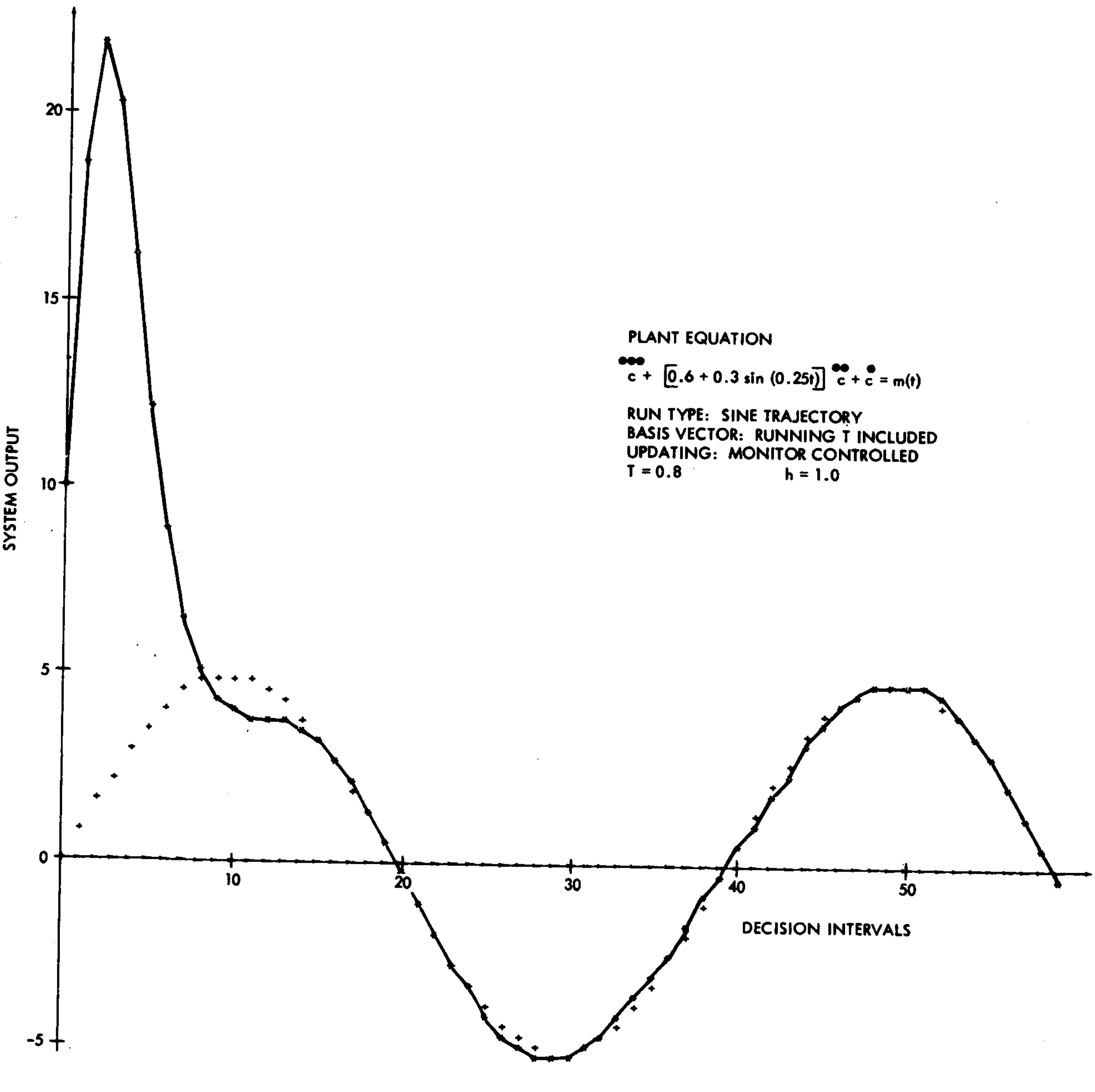


FIGURE 2-34 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

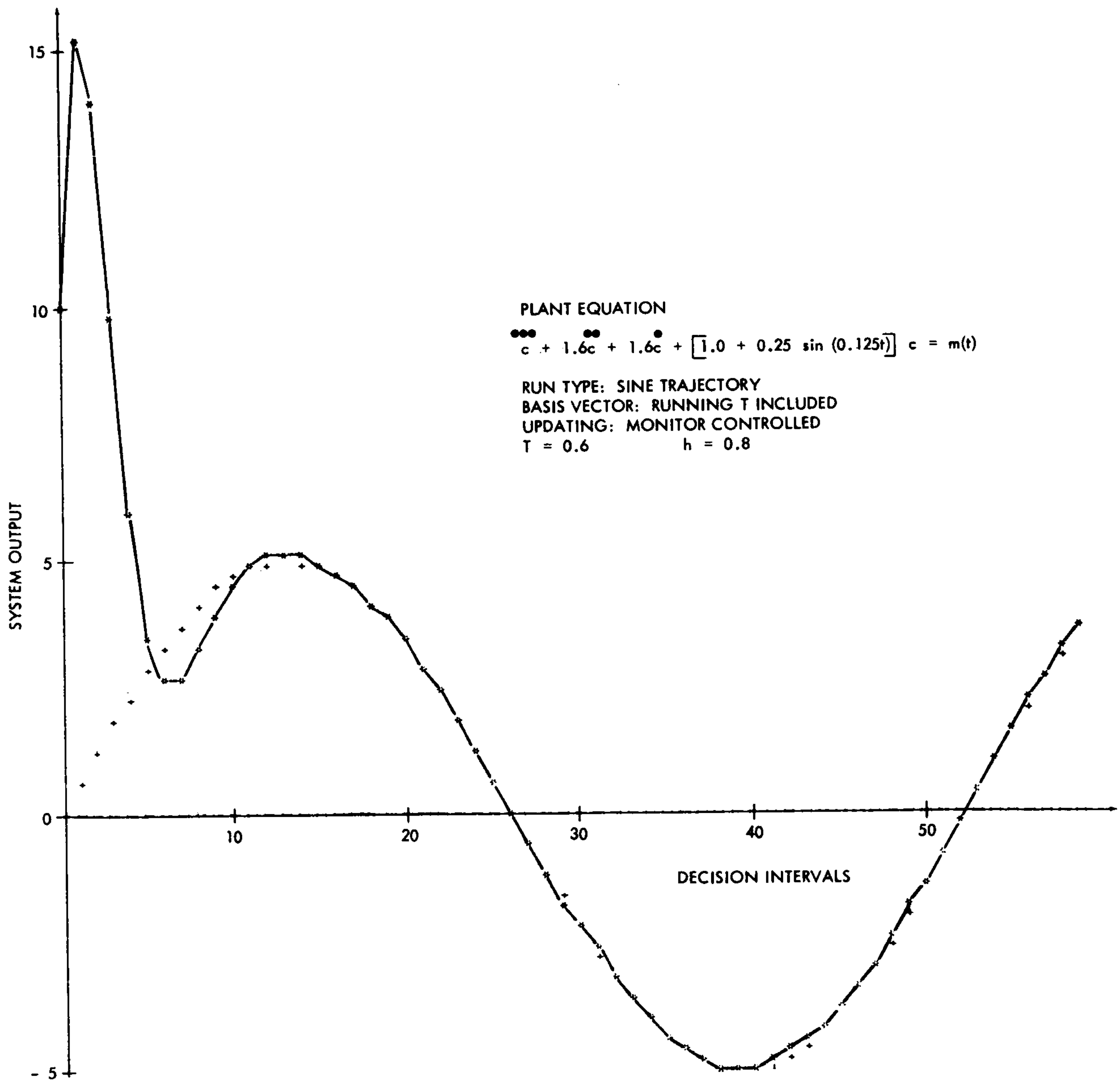


FIGURE 2-35 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

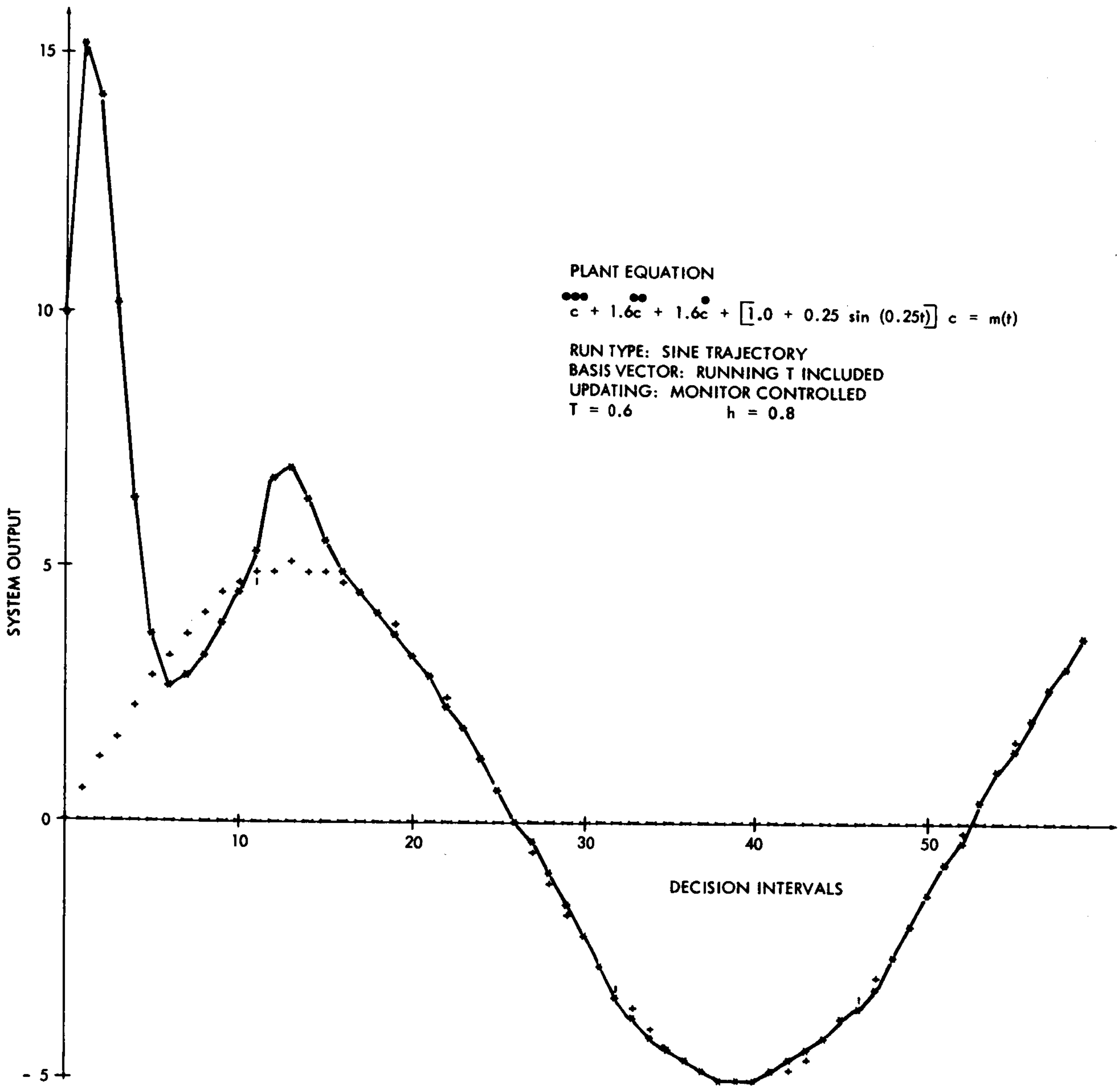


FIGURE 2-36 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

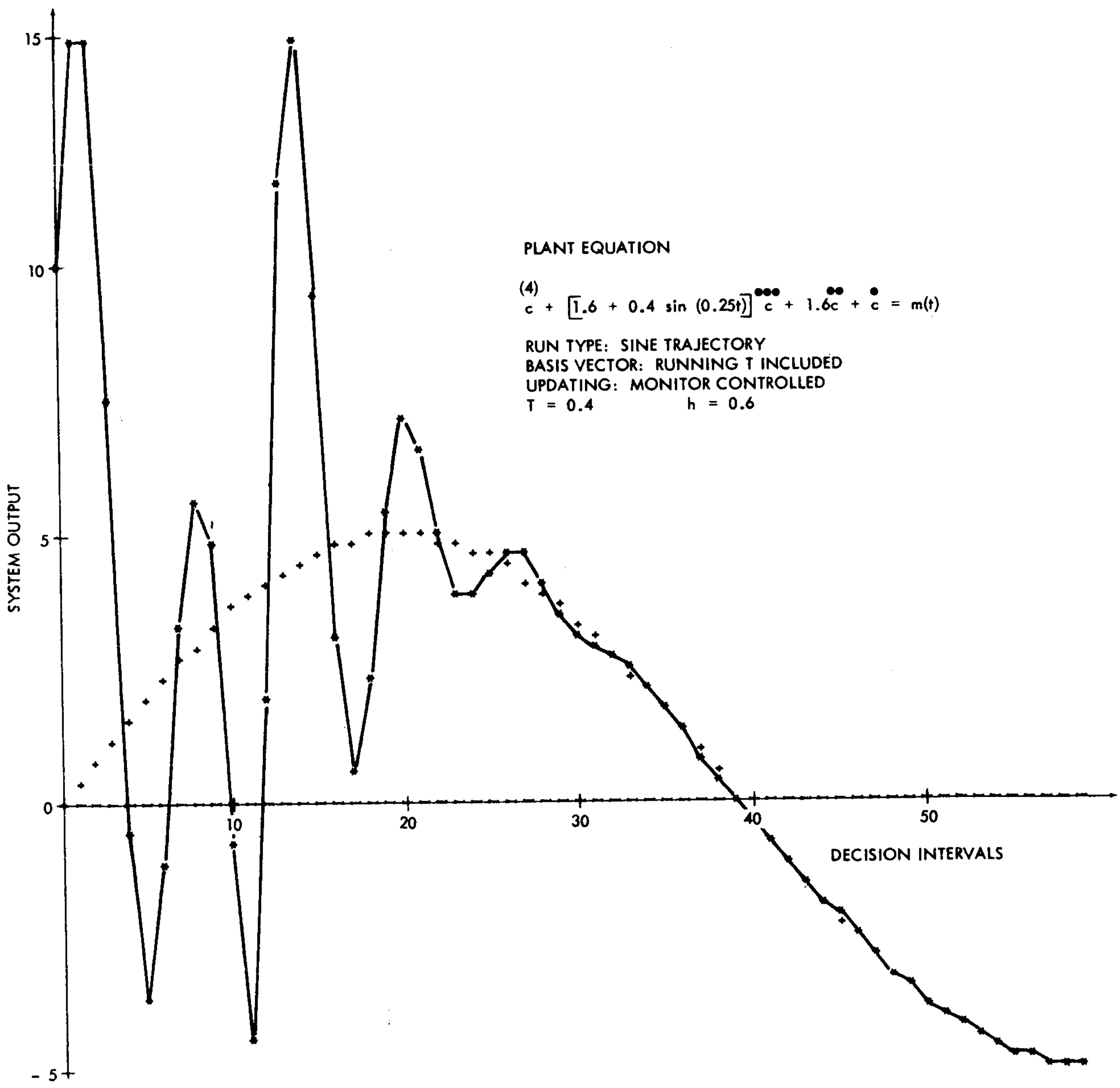


FIGURE 2-37 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

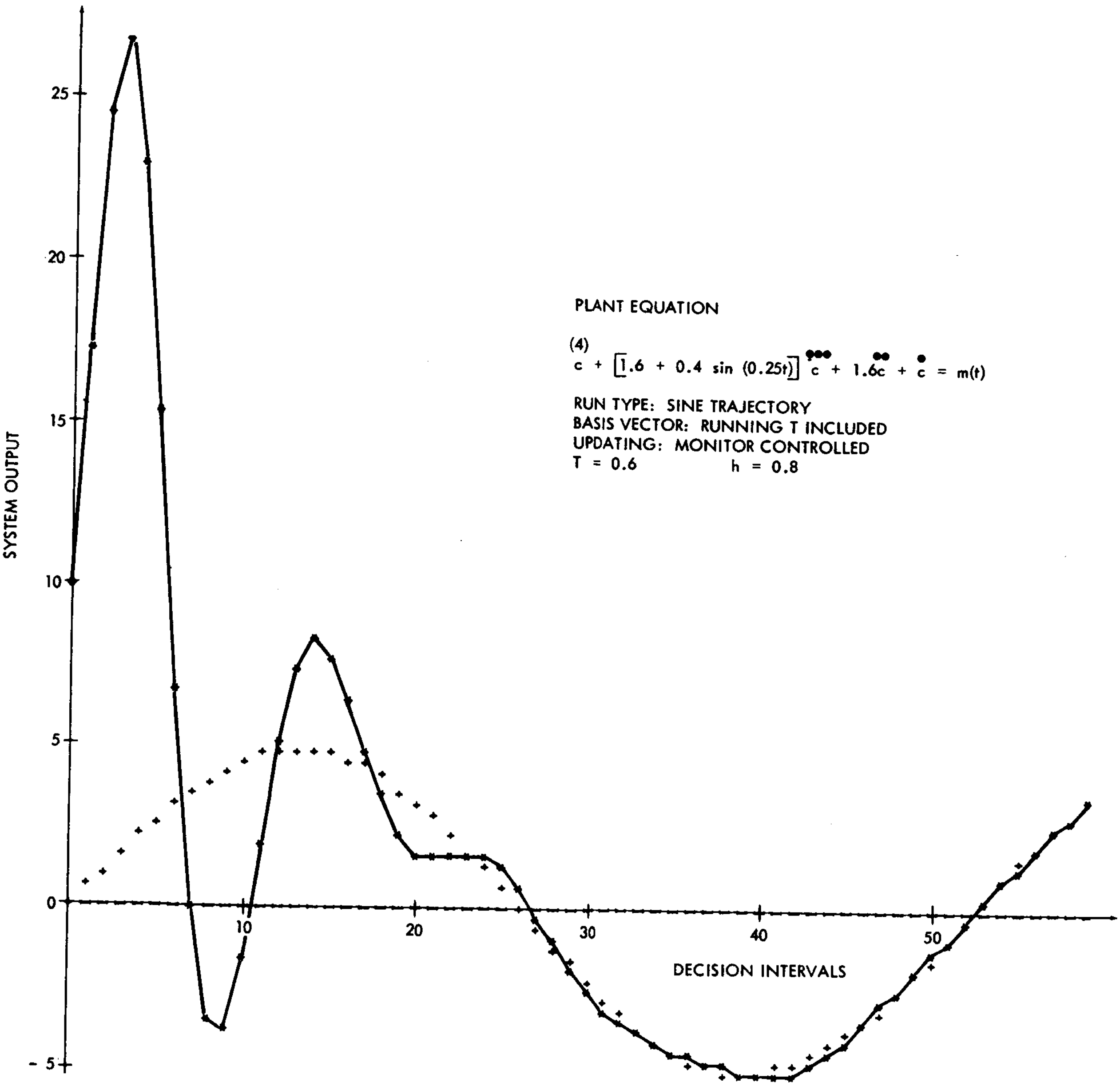


FIGURE 2-38 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

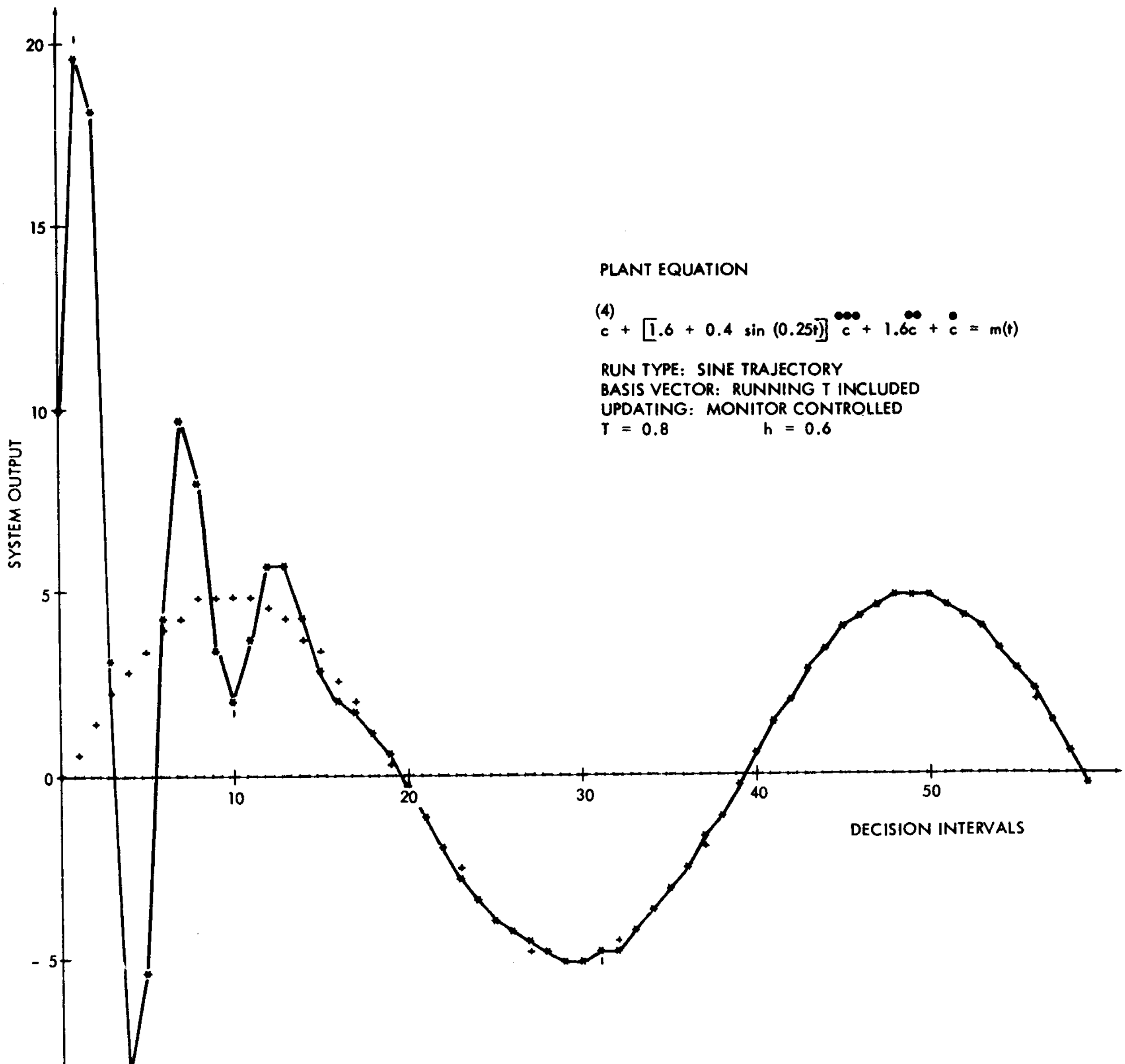


FIGURE 2-39 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

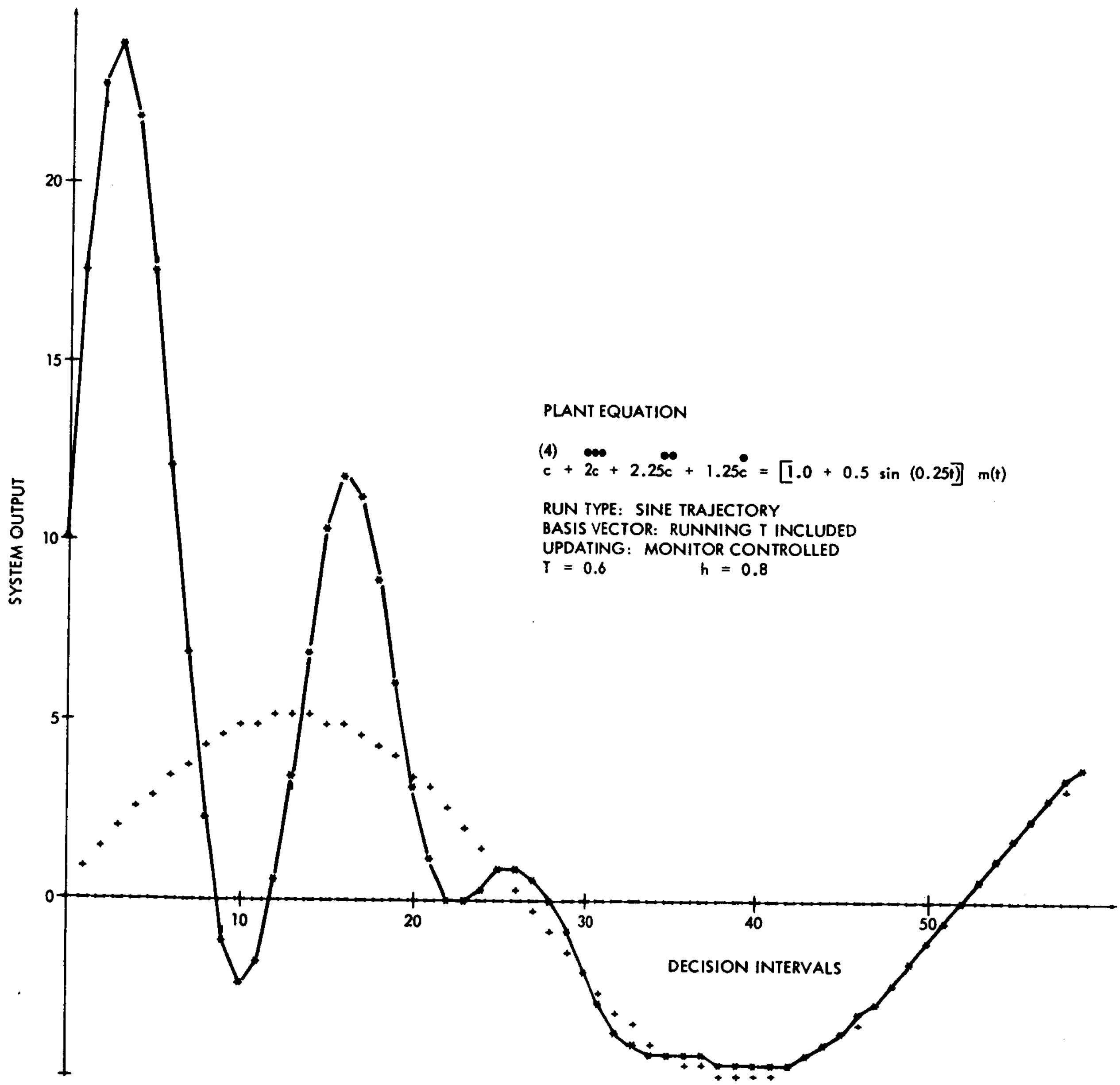


FIGURE 2-40 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM

Figures 2-41 and 2-42 demonstrate the typical control results obtained for the fifth order time-varying systems. As noted for the stationary basis vector results, these control responses are not really unexpected events. Such control results were very similar to those observed for the linear stationary systems of fifth and higher order (reference 1).

In general, this experimentation uncovered no new or unexpected results. The same problem of poor interpolation estimates of the plant matrices was again noted as it was for the regulator runs. In summary, the trajectory results were as good as the results obtained in the stability investigation. The time-varying and stationary basis vector descriptions provided very similar control results in every aspect of comparison.

WEIGHTING FACTOR PARAMETER STUDY RESULTS

The two control system parameters of most interest are the length of the decision interval, T, and the weighting factor, h. These parameters were discussed in previous paragraphs, and in past DACS research (references 1, 3, 4, 5, 6, and 7). This particular study was concerned with experimentally establishing the effect of the weighting factor on the control system performance for time-varying plants.

The results of this study are very similar to those obtained for linear stationary plants (reference 1). Typical control results are illustrated by Figures 2-3, 2-4, 2-5, 2-6, 2-19, 2-20, 2-21, and 2-22 for the stationary basis vector description. Figures 2-10, 2-11, 2-12, 2-13, 2-31, 2-32, 2-33, and 2-34 show typical control results for the time-varying basis vector description. These results demonstrate that for a given value of the decision interval, T, the smaller values of the weighting factor, h, tend to make the controlled output response faster.

STATIONARY VS TIME-VARYING BASIS VECTOR RESULTS

The experimental results presented in the preceding paragraphs have covered in some detail the particular investigations conducted on the

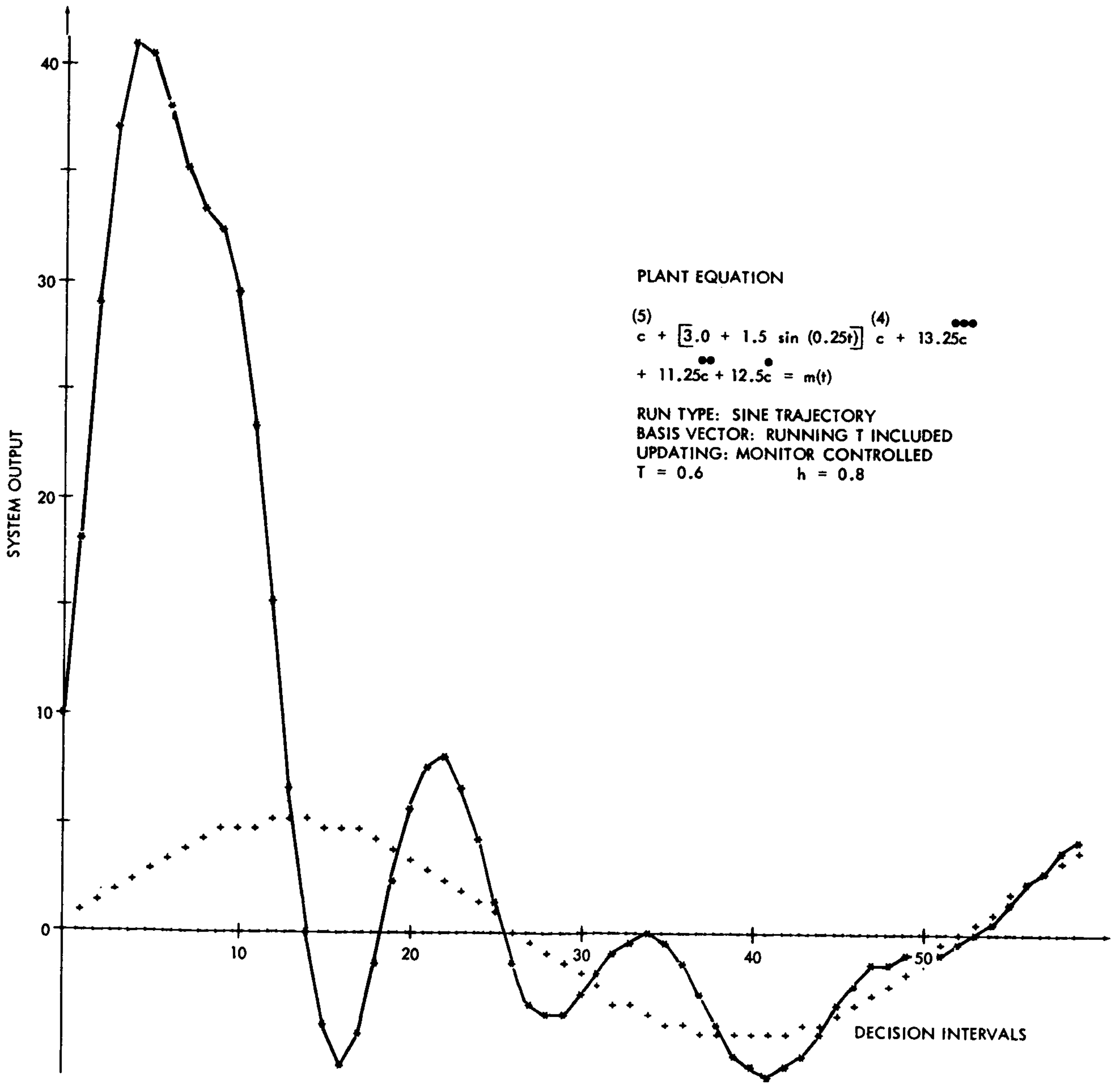


FIGURE 2-41 TRAJECTORY RESPONSE OF A 5TH ORDER SYSTEM

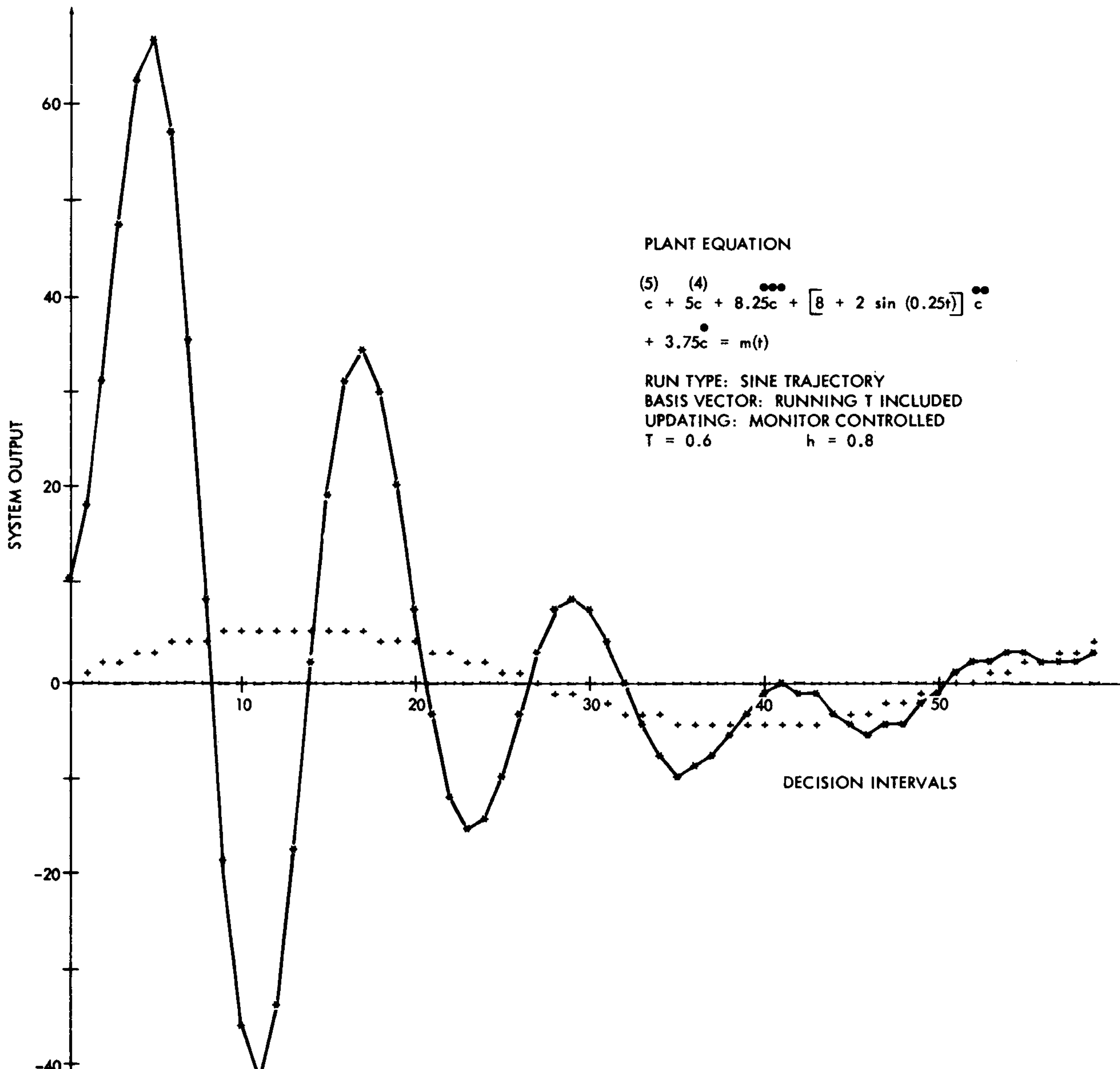


FIGURE 2-42 TRAJECTORY RESPONSE OF A 5TH ORDER SYSTEM

selected time-varying plants. These investigations were in most cases equally divided between the Interpolation Prediction with the stationary and with the time-varying basis vector descriptions. In review the following experimental studies were made for these descriptions:

> Demonstration of regulator control with fixed updating for all selected time-varying plants for five T-h combinations.
>
> Establishment of the T-h stability for a limited area of the T-h plane for all selected time-varying plants. The stability was established by the experimental regulator control results.
>
> Demonstration of trajectory (sine and displaced sine desired outputs) tracking ability with the updating monitor incorporated for selected time-varying plants.
>
> Establishment of the effect of the weighting factor, h, on control system performance for the selected time-varying plants.

Upon close review of these experimental investigation results, a preference for either the stationary or the time-varying basis vector description is not evident. This may be illustrated by noting that the stationary basis vector description provided completely stable performance for all T-h combinations for all the selected time-varying plants. The time-varying description yielded a total of five unstable T-h combinations for three of the plants. However, the time-varying basis vector description provided slightly better trajectory tracking as was noted from the unweighted error norm. These observations indicate no clearcut advantage of either description. Moreover, both descriptions were equally plagued by occurrences of poor interpolation estimates of the plant matrices.

In order to establish if the time-varying basis vector description is better with respect to controlling a broader spectrum of time-varying plants, a short experimental study was conducted on several third order plants. The particular set of plants considered are listed in Appendix B under the B.5 heading. It should be noted that these plants possess linear time variation, whereas those considered previously contained sine time variation of a single coefficient.

This investigation consisted of Interpolation Prediction control runs for both basis vector descriptions. The desired output was a sine trajectory, the updating monitor was utilized with a δ_i value of 0.05, and new data was shifted into the matrix of basis vectors every interval. Also, as in all previous studies, the same start-up procedure was utilized as is presented in reference 1. For the sake of completeness, it should be noted that an initial condition of +10 units was used for each element of the state vector.

Stationary Basis Vector Results. - Figure 2-43 through 2-48 present a summary of the experimental results obtained using this description. Figure 2-43 illustrates very satisfactory control performance. However, as the speed of the time variation is increased (Figures 2-44, and 2-45), the tracking is somewhat degraded by a few occurrences of the use of poor interpolation estimates of the plant matrices for one or more intervals. Figure 2-46 demonstrates that for a higher speed of time variation the stationary basis vector description produces poor control results. Control performance for even higher speeds of time variation is demonstrated by Figures 2-47 and 2-48.

Time-Varying Basis Vector Results. - The control performance of this description is summarized by Figures 2-49 through 2-54. Satisfactory control is observed in Figures 2-49 and 2-50. However, as the speed of the time variation is increased the tracking performance is somewhat degraded as may be noted by Figures 2-51 and 2-52. Again such occurrences are due to poor interpolation estimates of the plant matrices being used for one or more intervals. Figure 2-53 illustrates a speed of time variation where poor control performance is observed to a greater degree. Figure 2-54 presents the control results obtained for the fastest time variation considered in this study.

Comparison. - The results of this limited study indicate that the time-varying basis vector description does provide adequate control for time-varying plants for which the stationary description control was somewhat degraded. However, these results by no means establish in general the point or speed of variation at which the time-varying description becomes superior.

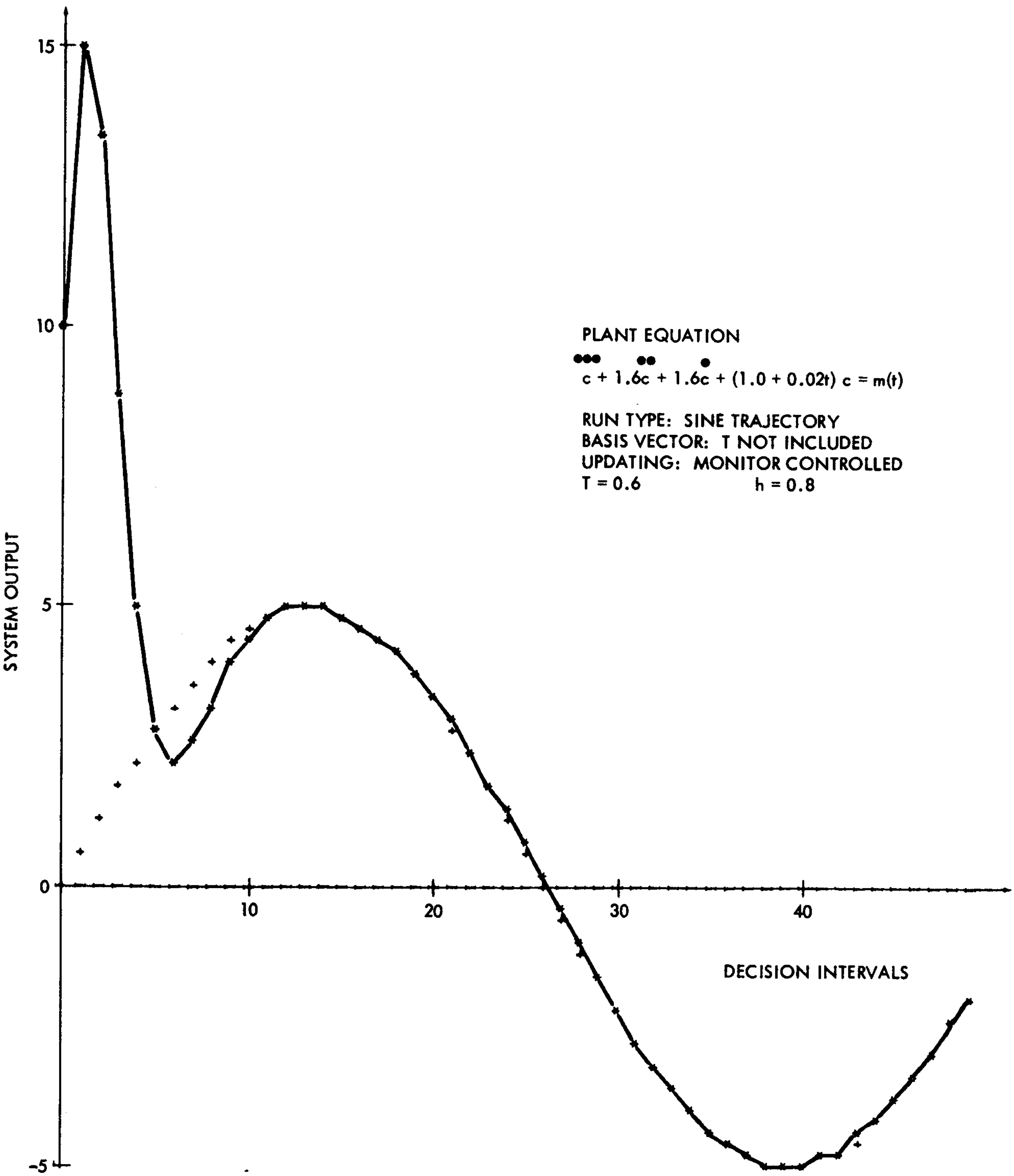


FIGURE 2-43 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

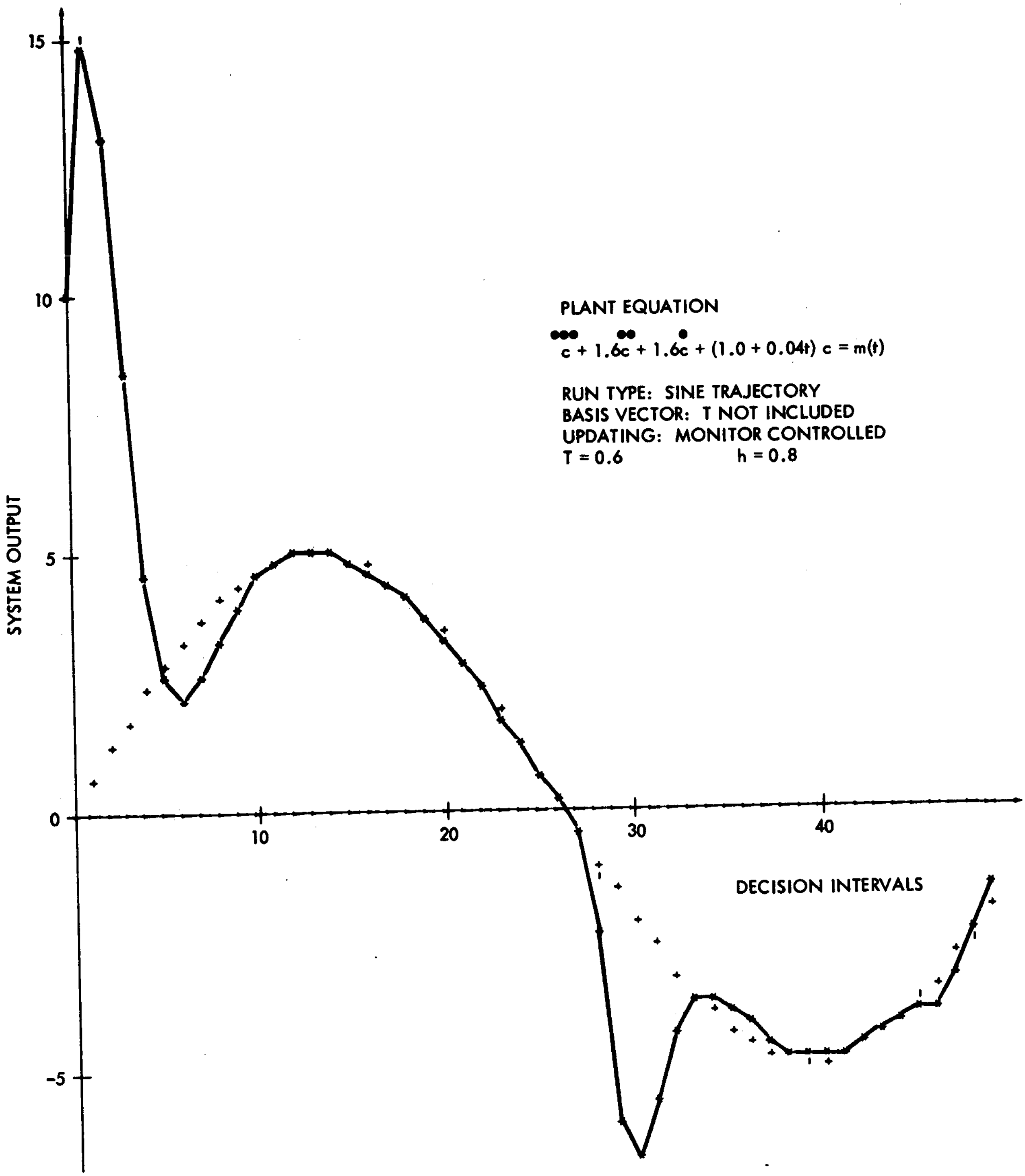


FIGURE 2-44 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

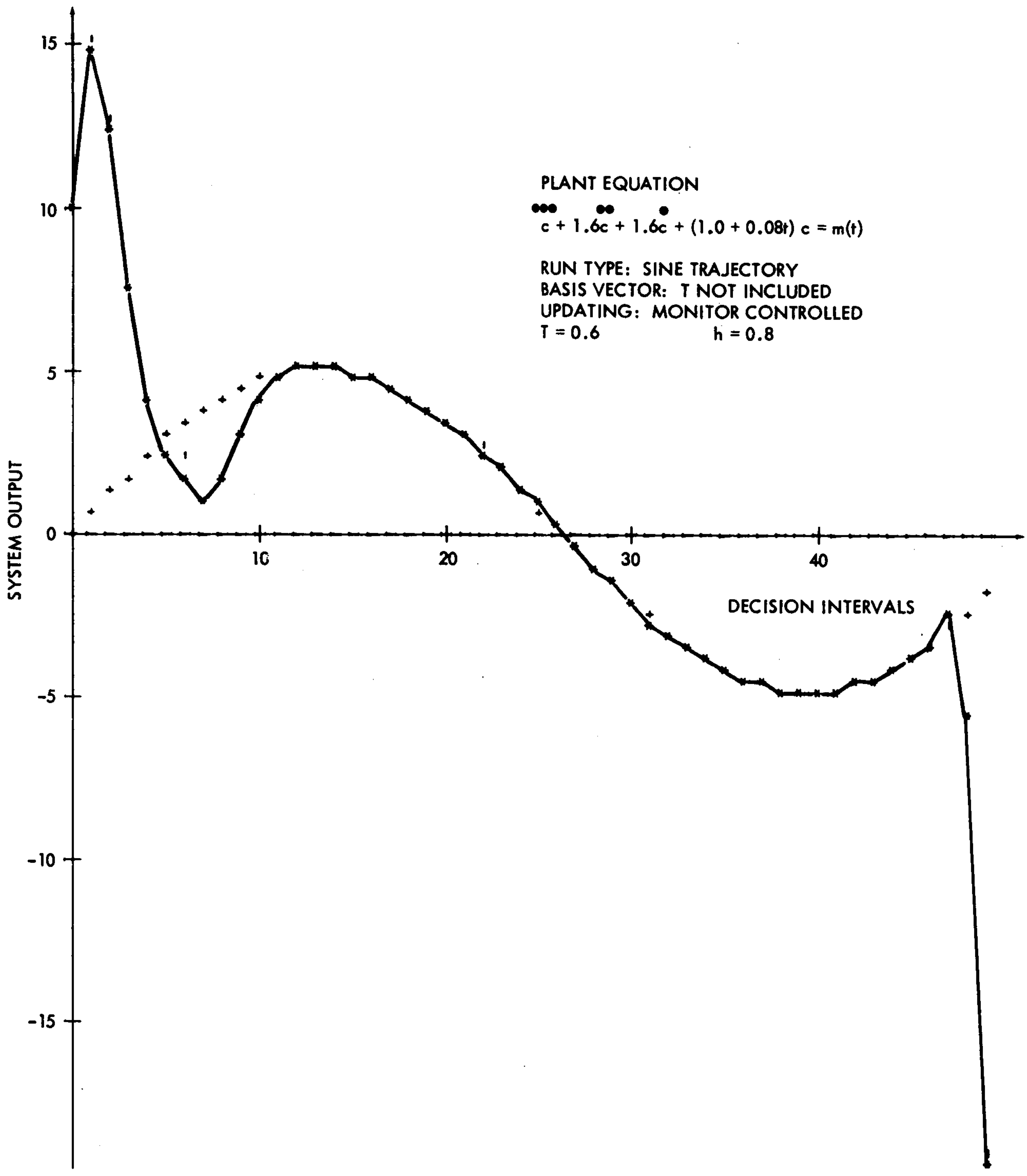


FIGURE 2-45 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

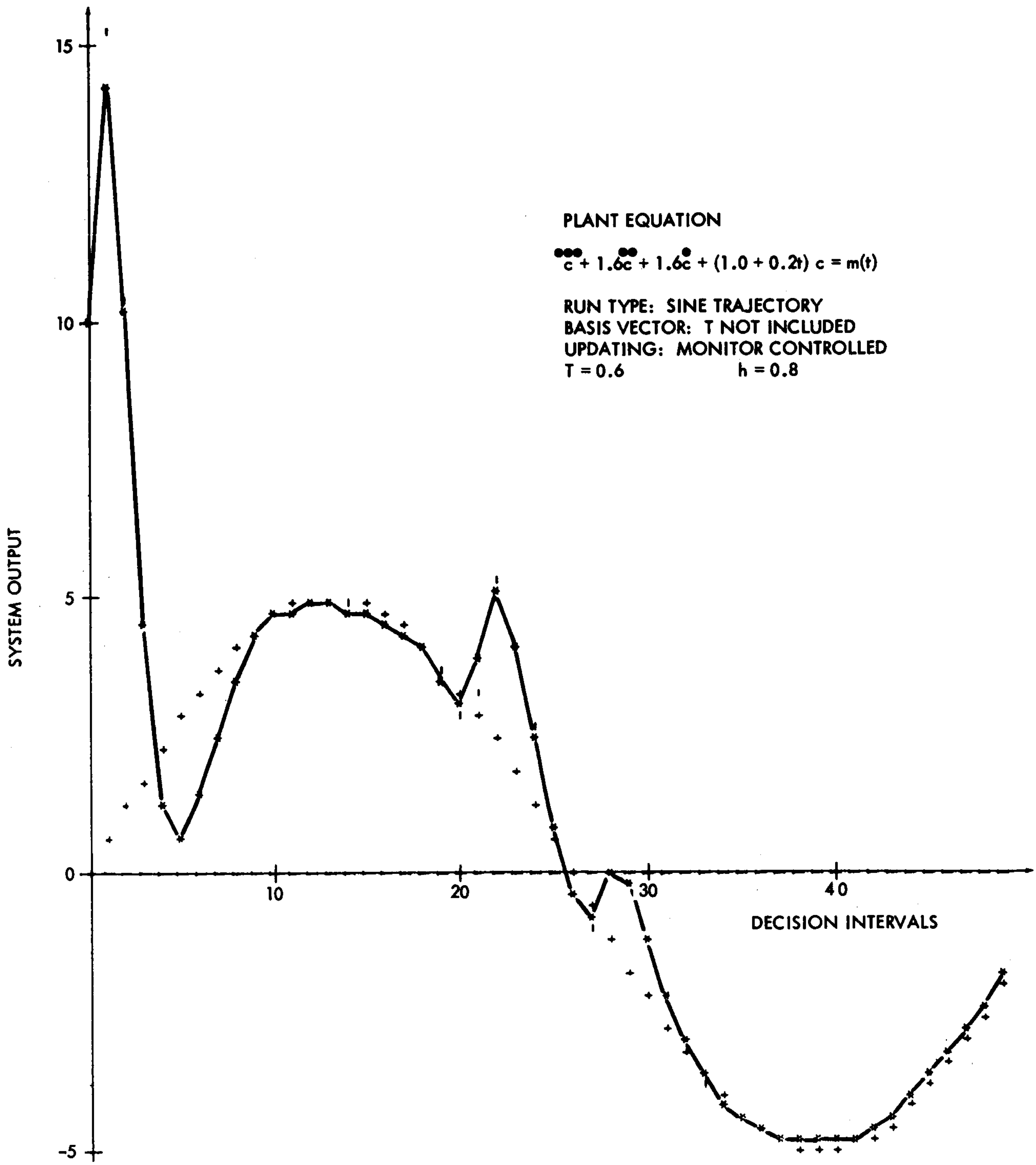


FIGURE 2-46 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

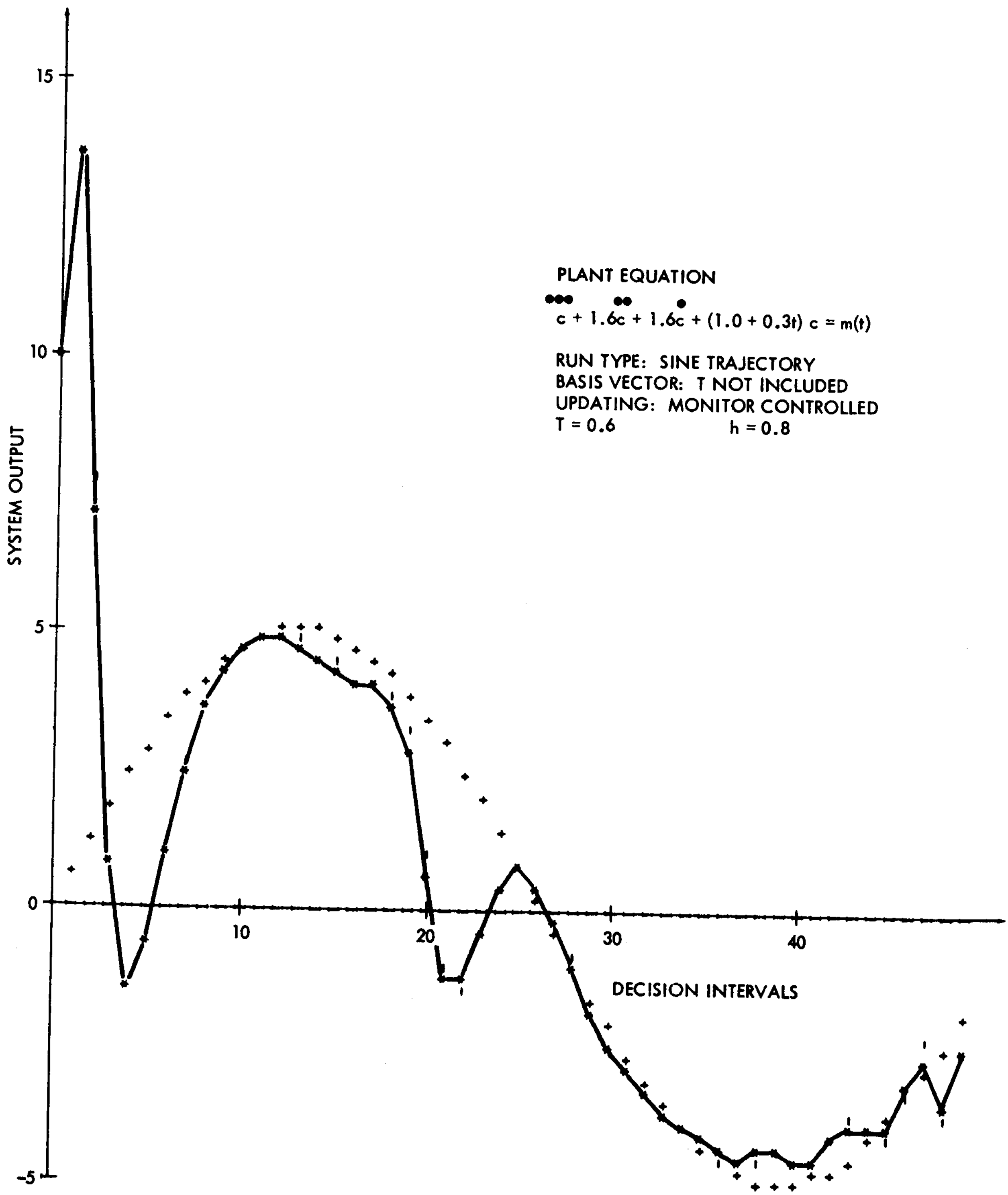


FIGURE 2-47 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

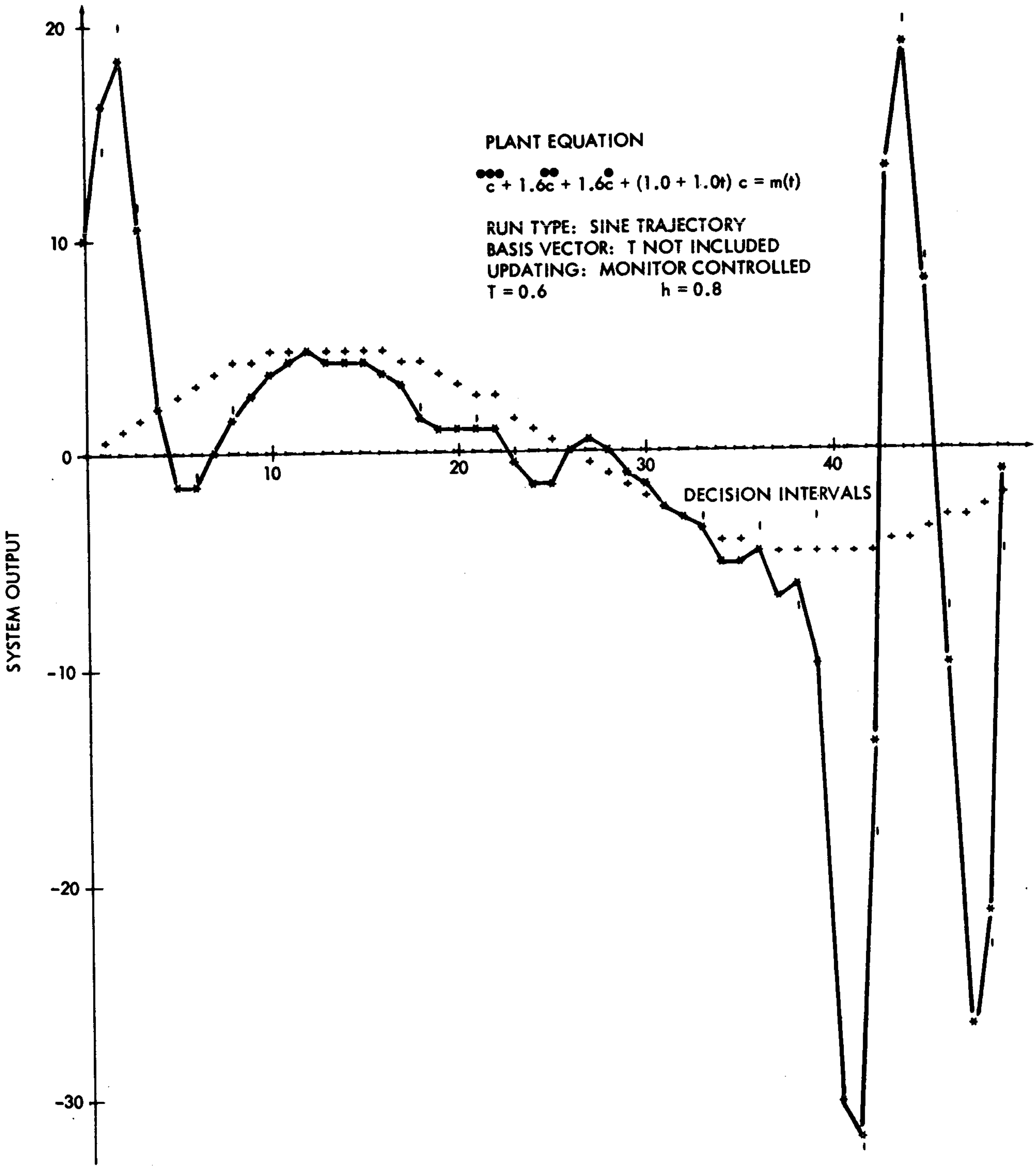


FIGURE 2-48 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

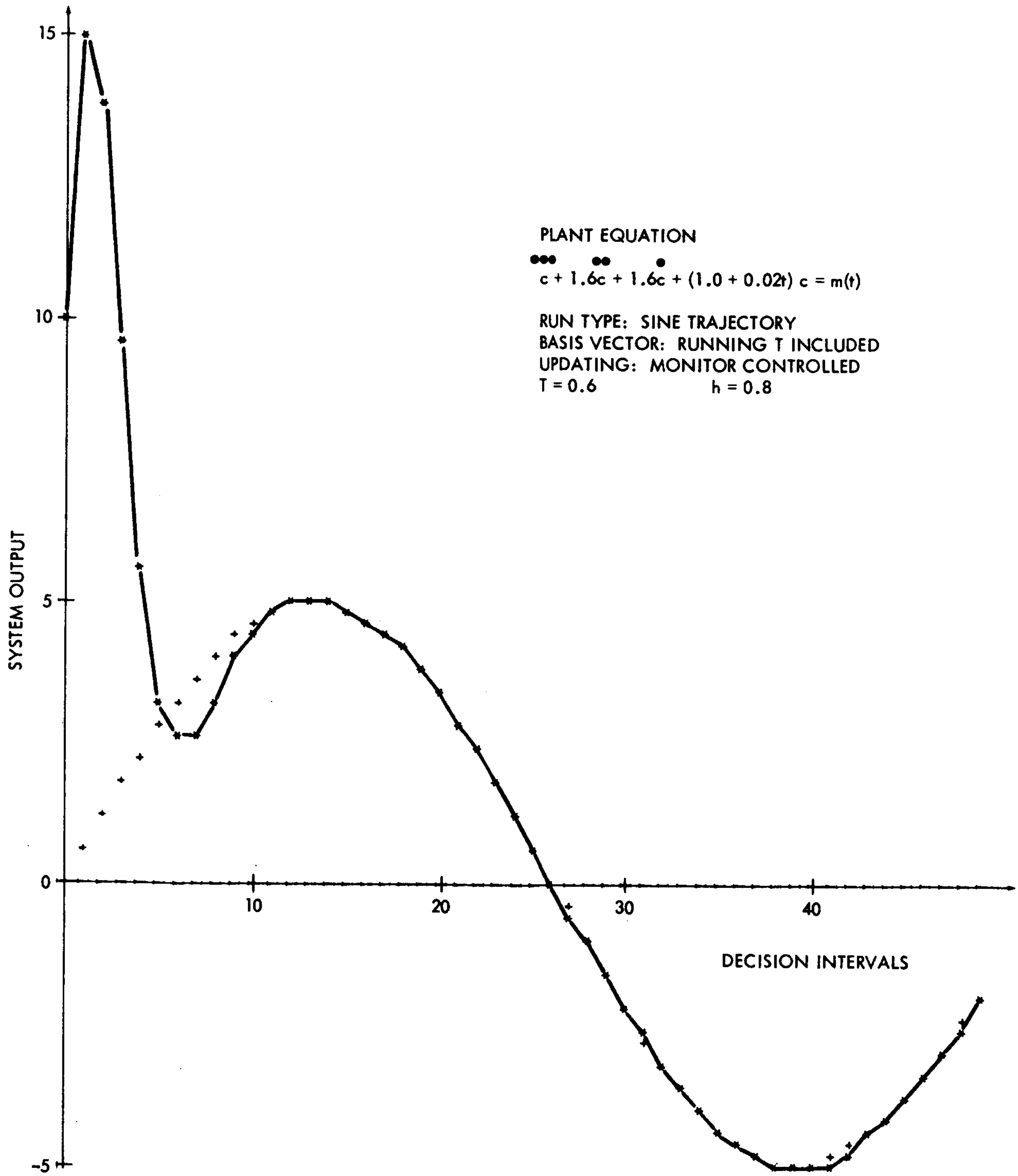


FIGURE 2-49 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

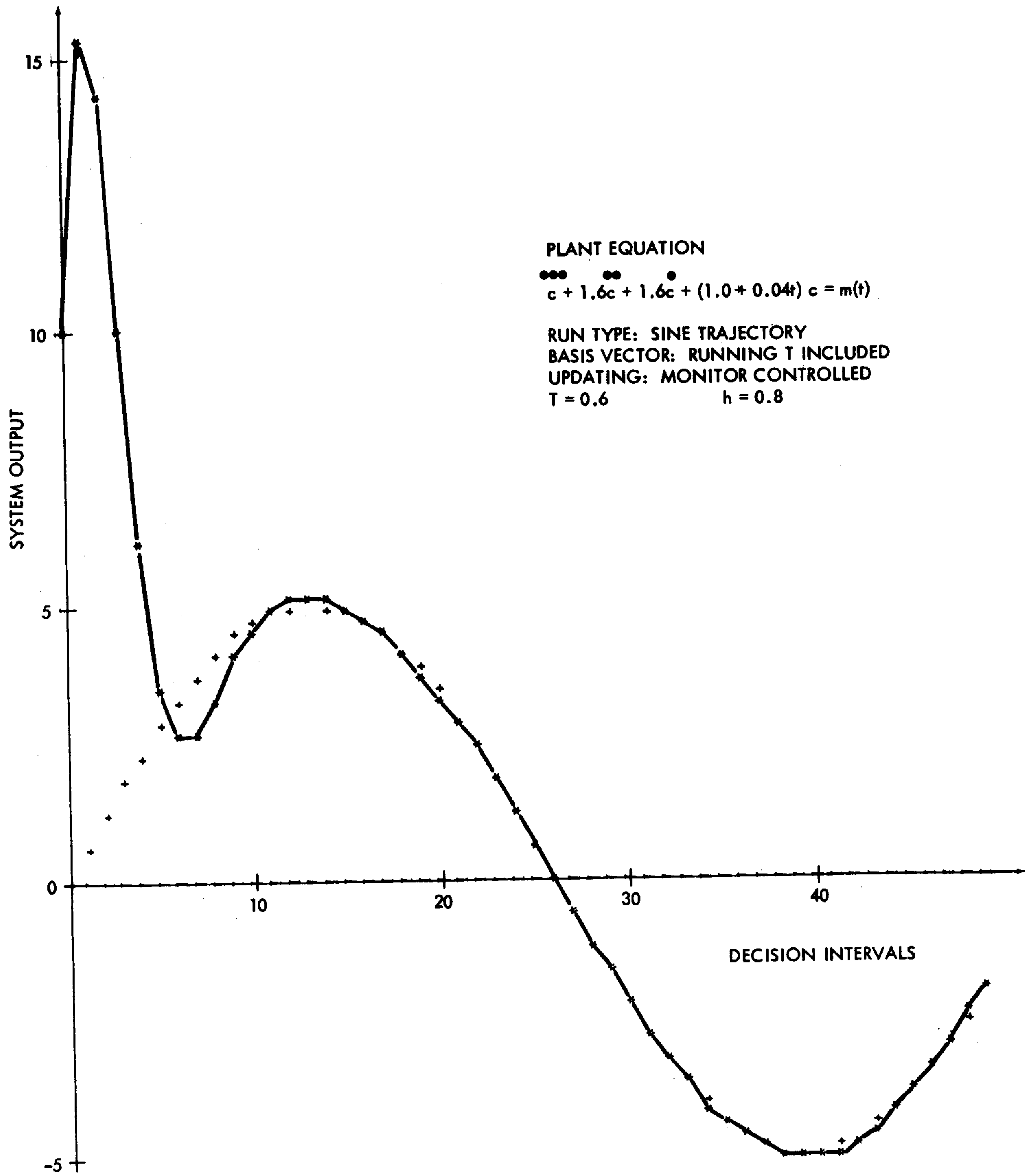


FIGURE 2-50 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

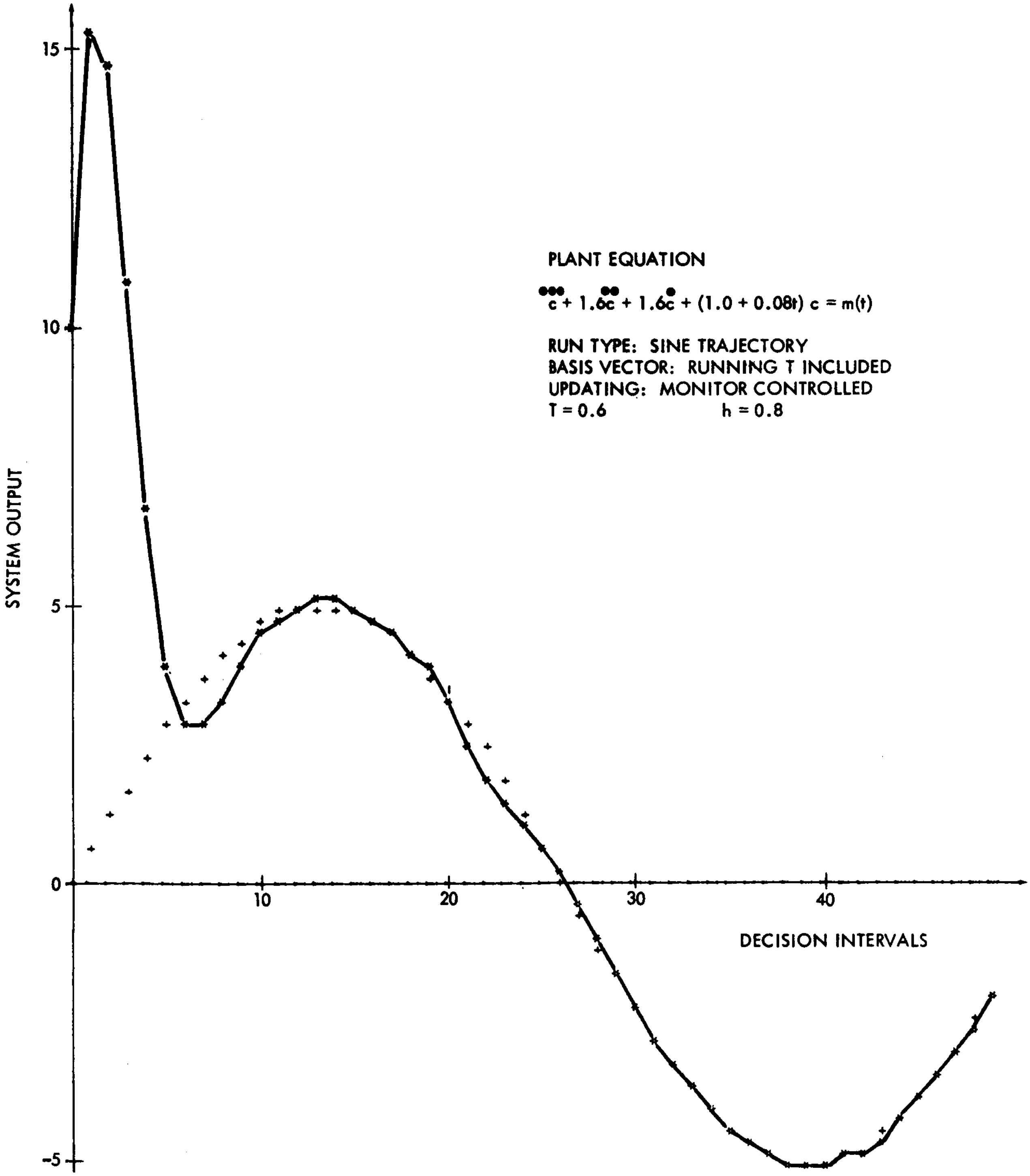


FIGURE 2-51 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

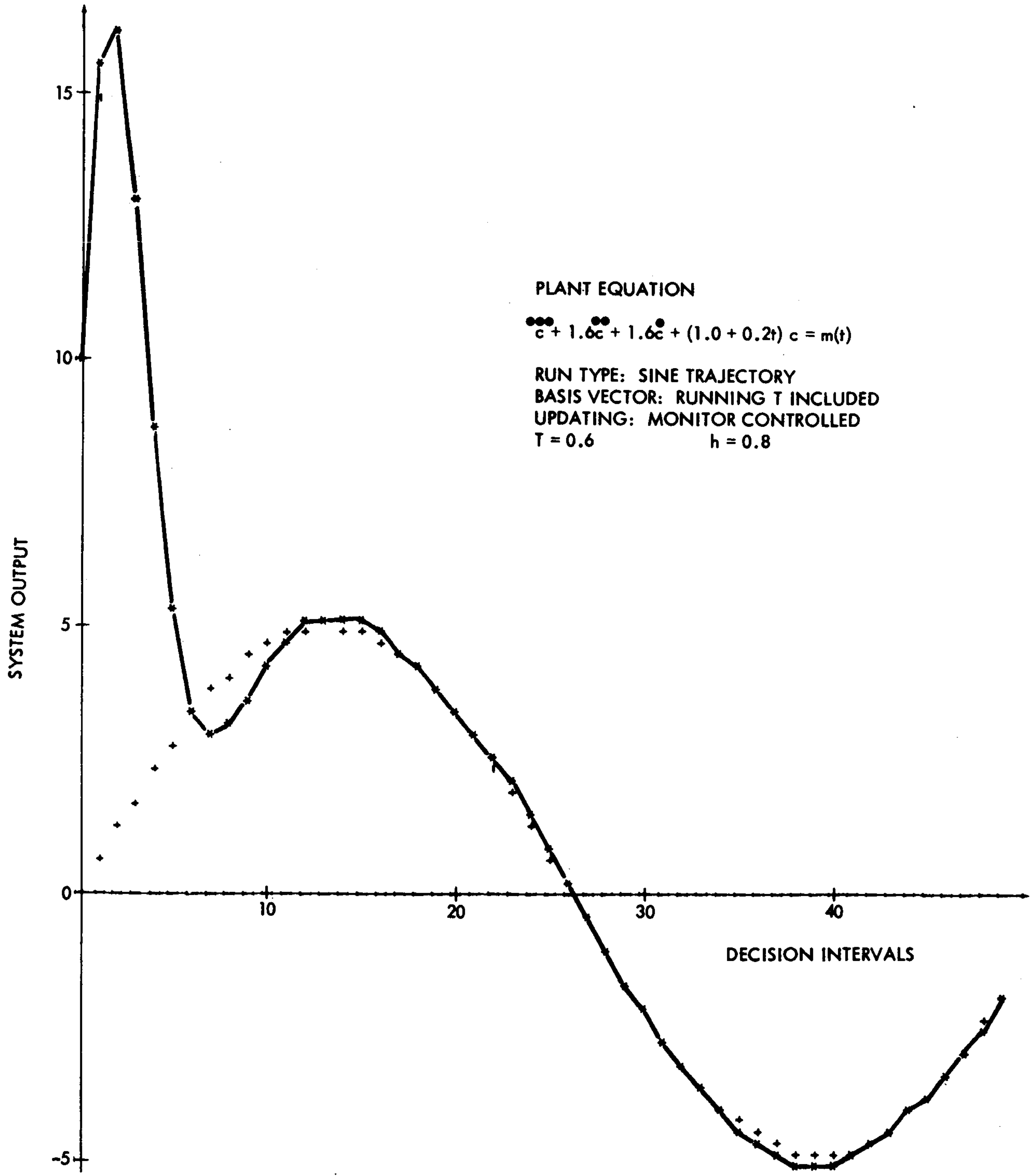


FIGURE 2-52 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

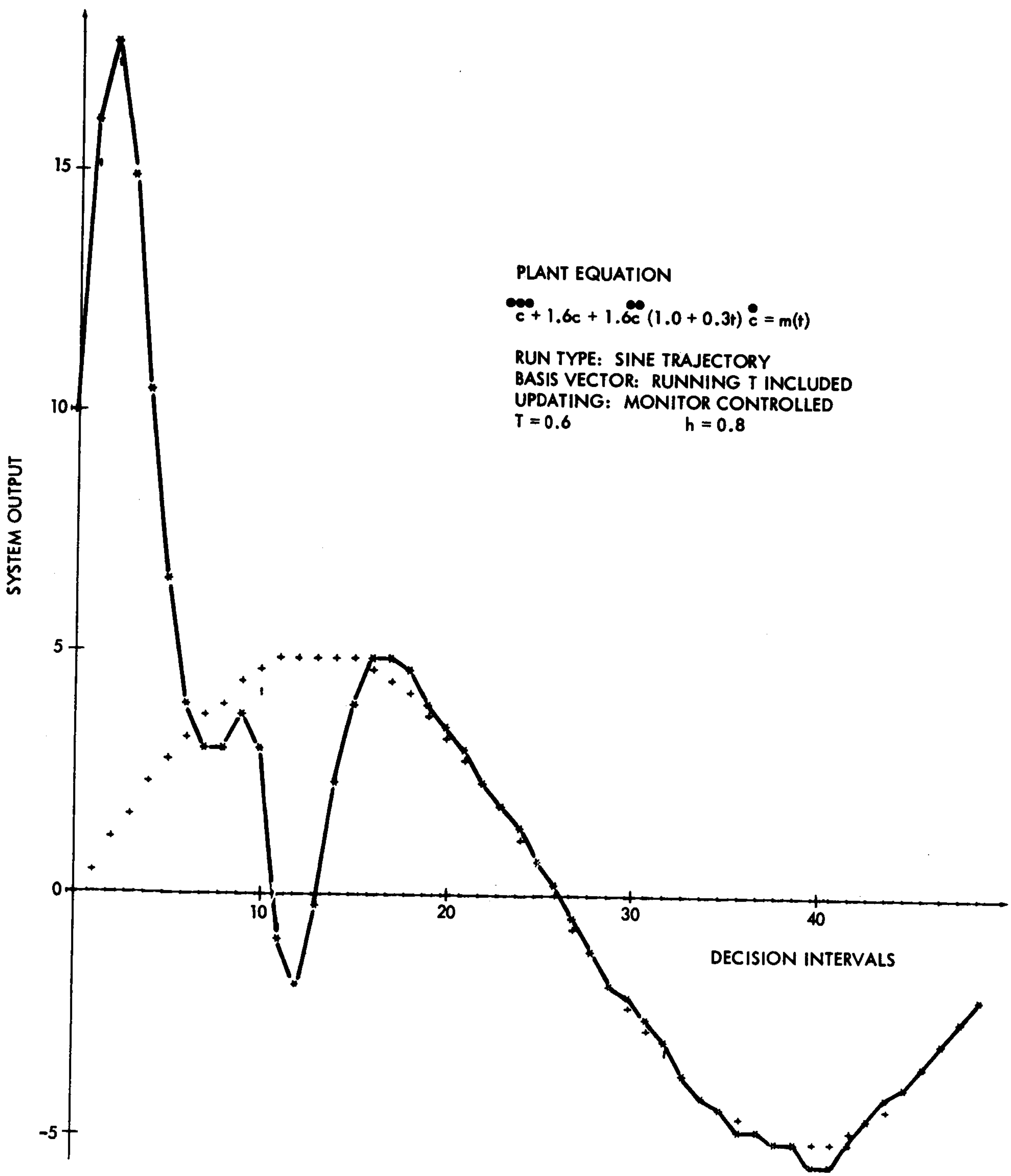


FIGURE 2-53 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

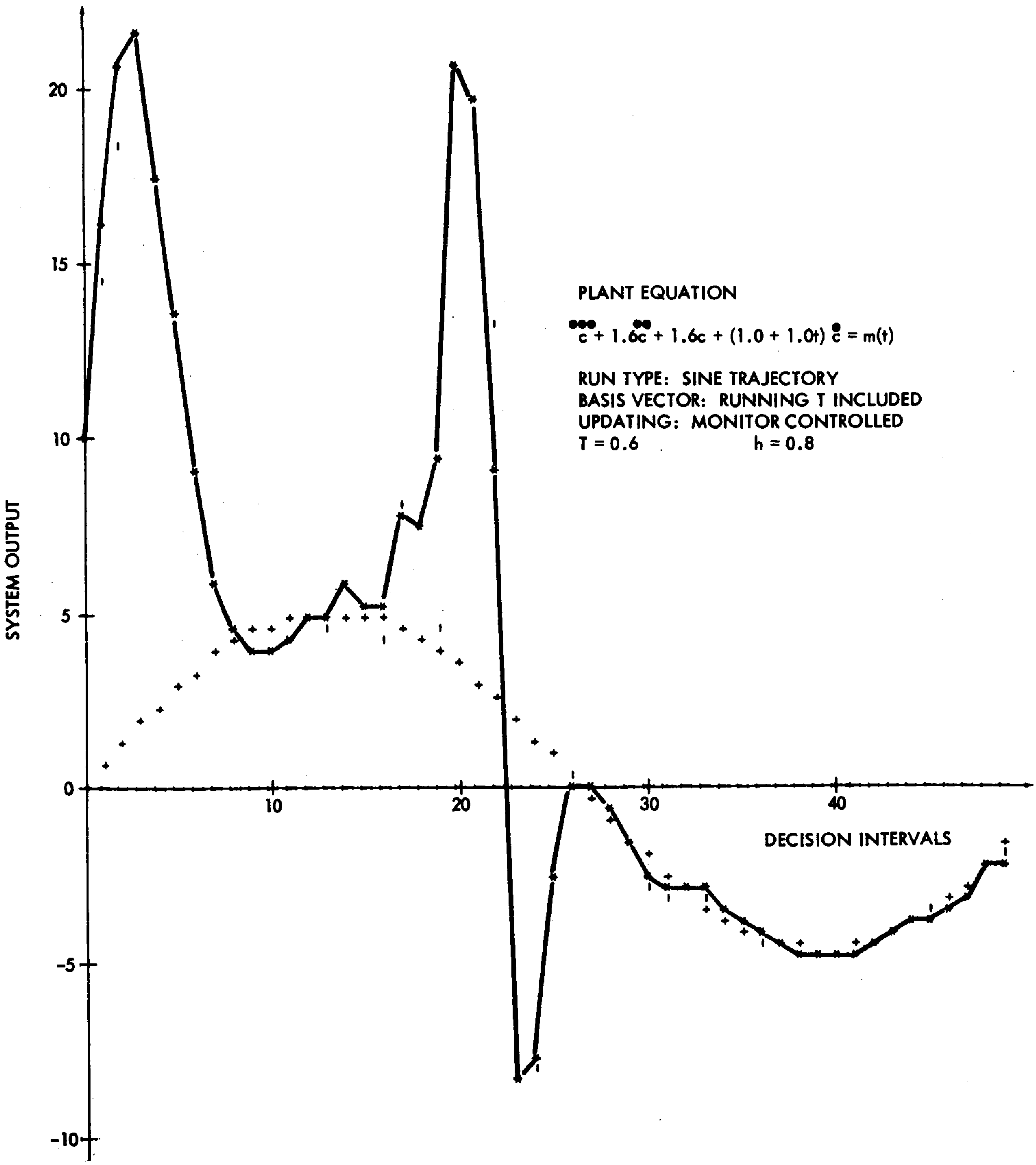


FIGURE 2-54 TRAJECTORY RESPONSE OF A 3RD ORDER SYSTEM

2.3 SUMMARY OF ANALYTICAL AND EXPERIMENTAL RESULTS

ANALYTICAL STUDIES

The mathematical tools for the study of the control policy as applicable to time-varying plants were developed in previous DACS research (reference 1) and in paragraph 2.1. These included a mathematical description of the particular plants considered, a state equation relating the values of the state variable of the plant at the sampling times, and the control policy equation by which the control input to the plant is calculated. Involved in implementing the control policy is the prediction of future states of the plant output without assuming any a priori knowledge of the plant. Throughout the time variation investigation Interpolation Prediction has been utilized with the basis vector being one of three forms. The stationary basis vector approach uses the linear basis vector without any explicit time dependence term included; the time-varying basis vector includes an explicit time dependence term (running time); the time augmented basis vector includes an explicit time dependence term (constant time). The first and last forms of the basis vector were introduced and utilized in earlier research (reference 1) whereas the first and second were presented and studied in greater depth in the research presented in this report.

An updating monitor criterion is developed in paragraph 2.1. This criterion determines when updating (recalculation of the interpolation estimates of the plant matrices) is necessary, and is based on a comparison of the predicted and measured states at the end of the decision intervals. Also, of interest is the recursive method of matrix inversion presented in Appendix D. This method is indeed applicable to the matrix inversion associated with the interpolative procedure.

EXPERIMENTAL STUDIES

Since this research extension was mainly of an experimental nature, a very large amount of data was compiled during the study. A representative set of the experimental data is presented in paragraph 2.2. Approximately 400 regulator control responses were made for about fifty linear time-varying plants of orders three through five. These were used to establish the T-h stability of five T-h combinations for each plant with both the stationary and time-varying basis vector descriptions. Also, over 200 trajectory control simulations were made to test the control performance characteristics of these two basis vector descriptions.

Numerous regulator and trajectory control simulations were made on selected plants to study such areas of interest as:

- The updating monitor criterion
- The weighting factor h
- The possible preference of the time-varying basis vector over the stationary description.

The results of these studies are considered in the following paragraphs.

Stability Boundaries. - The stability results of both the stationary and time-varying basis vector descriptions were very similar. For all five T-h combinations examined, the stationary basis vector description provided stable regulator control, and so experimental stability was established for the selected set of plants. The time-varying description yielded a few unstable combinations, and only for three of the plants considered. Also, occurrences of matrix ill conditioning and/or poor estimation were noted about equally for both descriptions. Therefore, both descriptions are about equally likely to provide satisfactory control from the stability standpoint. This conclusion applies only to the set of plants considered in this study, and any further extension or generality is not be assumed at this stage.

Tracking Capability. - Trajectory control simulations were conducted for both the stationary and time-varying basis vector descriptions. The desired output was either a sine or displaced sine trajectory. The results of this study indicate that both of the plant descriptions provide about the same degree of tracking capability. Also, both basis vector descriptions were plagued by about equal occurrences of poor interpolation estimates of the plant matrices. Therefore, as was noted in the stability boundary investigation, neither description shows any clear cut advantage over the other. The most worthwhile conclusion from this investigation is that both descriptions provide adequate control for the selected plants.

Updating Monitor. - The particular updating monitor criterion examined by this study accomplished the required task. The most obvious and perhaps the most expected conclusion is that the frequency of updating for any plant is definitely a function of the allowable error threshold level. The lower this level is the more frequent the requested updating of the interpolation estimates of the plant matrices. Also, the higher order plants require more frequent updating. The most important conclusion is that monitor controlled updating is feasible.

Weighting Factor. - The weighting matrix (see reference 1 and/or paragraph 2.1) has the effect of weighting the state vector of the plant so as to control the importance of the higher order state vector elements. The experimental results of paragraph 2.2 show similar types of responses to those of the linear stationary studies (reference 1). That is, for both basis vector descriptions the larger values of the weighting factor, h, make the response more sluggish as the control policy seeks to control more precisely the higher order state variable elements.

Stationary vs Time-Varying Basis Vector. - The summary and conclusions did not provide a strong preference for either the stationary or time-varying basis vector description. In general, both descriptions are about equal in all aspects considered in these studies. In order to possibly demonstrate the preference of one description over the other, an investigation was conducted on a different type of time-varying plant, i.e., plants with a linear time variation.

The results of this study indicate that the time-varying basis vector description does perhaps provide satisfactory control over a wider spectrum of plants. However, this is definitely not a general conclusion because of the very limited depth of this particular study.

SECTION 3

CONTROL OF NON-LINEAR PLANTS

3.1 PLANT AND SYSTEM EQUATIONS

As was the case in Section 2, much of the research effort discussed here is a direct extension of work previously documented (reference 1). Much of the text and equations will be of a summary nature where the more complete discussions and equational developments appear in reference 1.

METHOD OF ANALYSIS

The dynamics of the plant are assumed to be describable by the following very general type of non-linear differential equation:

$$L(p,t)\ c(t) + F(t,c(t),\ \dot{c}(t),\ \ldots \overset{n-1}{c}(t)) = M(p,t)\ m(t) \qquad (3\text{-}1)$$

where the plant is assumed to have a single input, m(t), and a single output, c(t). L(p,t) and M(p,t) are linear differential operators with time-variable coefficients. The function, F, is a non-linear function of its arguments.

An input-output relationship for the plant is conveniently expressed in terms of a functional relationship:

$$c(t) = T\left[m(t)\right] \qquad (3\text{-}2)$$

Volterra (reference 11) presents a proof that if the functional, $T\left[m(t)\right]$, is continuous, it may be approximated to any desired degree of accuracy over finite time intervals by a finite series of the form:

$$c(t) = y(t) + \sum_{j=1}^{J} \int_0^t \cdots \int_0^t h_j\ (t,\tau_1,\ \ldots\ \tau_j)\ m(\tau_1)\ \ldots\ m(\tau_j)\ d\tau_1\ \ldots\ d\tau_j \qquad (3\text{-}3)$$

The essential restrictions are that the system produce continuous and bounded outputs for continuous and bounded inputs. If $T\left[m(t)\right]$ can be represented exactly by a converging infinite series ($J = \infty$) of the form of equation 3-3, it is called analytic (reference 12). Volterra and George (reference 13) show that equation 3-3 may be interpreted as a functional generalization of the Taylor series expansion for the analytic functional.

The particular control method under study assumes the existence of a functional relationship of the form of equation 3-3 but makes no attempt to identify it. The control element instead senses the current response of the system along with the current sensitivity to control inputs and extrapolates this into the near future.

By considering the functional $T\left[m(t)\right]$ to be analytic in the interval over which it is being approximated, it is convenient to expand the input-output relationship to a vector relationship in which the successive state variables possess a derivative relationship.

Such a convenient way of representing the state equation is:

$$\underline{x}((k+1)T) = \underline{x}_k((k+1)T) + \underline{A}_k((k+1)T)\ \underline{u}_k \qquad (3\text{-}4)$$

where:

$$\underline{x}_k((k+1)T) = \underline{y}((k+1)T) + \sum_{i=0}^{k-1} \underline{A}_i((k+1)T)\ \underline{u}_i \qquad (3\text{-}5)$$

and

$$\underline{x}'(t) = \left\lfloor c(t)\ \dot{c}(t)\ .\ .\ .\ \overset{n-1}{c}(t)\right\rfloor \qquad (3\text{-}6)$$

Equation 3-5 defines the first term of the right-hand side of equation 3-4 as the current response of the plant due to that response, $\underline{y}((k+1)T)$, which would occur in the absense of any control inputs, plus that due to all past control forces previous to u_k. Superposition is not implied as the $\underline{A}_i$'s will depend upon $\underline{y}$ and the previously applied u_i's. The second term on the right-hand side of equation 3-4 is the current sensitivity of the plant to the control input $\underline{u}_k$. Again, superposition is not implied as $\underline{A}_k$ is not a unique constant of the plant, but is a function of past states, $\underline{y}(t)$, and the past control inputs.

PLANT FUNCTIONAL REPRESENTATION USING INTERPOLATION

Interpolation is a particularly simple way of selecting a continuous functional that coincides with the system functional at measured data points when no information is available regarding the dynamic relations of the plant. Two methods were studied for the approximation of the functional of non-linear plants.

Linear Interpolation. - For non-linear systems which may be considered to be relatively linear on a piecewise basis, interpolation over linear terms may provide an adequate piecewise description of the plant. It would be expected that the linear interpolation matrices would require frequent updating as the non-linear response of the plant is approximated by a linear combination of basis vector functions over short intervals.

If the non-linear plant can be considered to be stationary, then the first form of the linear basis vector discussed in paragraph 2.1 would be appropriate. The interpolation estimate of the state at t = (k+1)T would be given by (reference 1):

$$\underline{\tilde{x}}((k+1)T) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k,\underline{\eta}_k) \tag{3-7}$$

and the basis vector would be given by:

$$\underline{\phi}'(u_i,\underline{\eta}_i) = \lfloor \underline{x}'(iT)\;\; u_i \rfloor \tag{3-8}$$

Non-Linear Interpolation. - A second form of interpolation studied is one in which second order terms are included in the basis vector. The interpolation estimate of the state at t = (k+1)T is still given by equation 3-7, however, two possibilities exist for the basis vector as given in equations 3-9 and 3-10:

$$\underline{\phi}'(u_i,\underline{\eta}_i) = \lfloor \underline{x}'(iT)\;\; \underline{x}'^2(iT)\;\; u_i\underline{x}(iT)\;\; u_i\;\; u_i^2 \rfloor \tag{3-9}$$

$$\underline{\phi}'(u_i,\underline{\eta}_i) = \lfloor \underline{x}'(iT)\;\; \underline{x}'^2(iT)\;\; \underline{x_j x_k}'(iT)\;\; u_i\underline{x}(iT)\;\; u_i\;\; u_i^2 \rfloor \tag{3-10}$$

The basis vector of equation 3-10 includes the cross product of the state variable elements whereas the basis vector of equation 3-9 does not.

THE INTERPOLATION MATRICES

The interpolative procedure would depend upon which of the three basis vectors is used.

Linear Interpolation Matrices. - The linear interpolation matrices would be identical in form to those presented in paragraph 2.1. They are repeated here for the sake of completeness. The $\underline{B}$ matrix would be factored into two submatrices:

$$\underline{B} = {}_{D}\underline{X} \cdot \underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \;\vdots\; \underline{\Phi}_1 \right] \tag{3-11}$$

The linear interpolation estimate of the state equation would be:

$$\underline{\tilde{x}}((k+1)T) = \underline{\Theta}_1 \; \underline{x}(kT) + \underline{\Phi}_1 \; u_k \tag{3-12}$$

where $\underline{\Theta}_1 \underline{x}(kT)$ is the estimate of $\underline{x}_k((k+1)T)$ of equation 3-4 and $\underline{\Phi}_1$ is the estimate of $\underline{A}_k((k+1)T)$.

Non-Linear Interpolation Matrices. - The form of the non-linear interpolation matrices would depend upon which of the two non-linear basis vectors is used. The equations presented will be for the basis vector of equation 3-10 which includes the cross products of the state variable elements. The differences in the equations for the basis vector of equation 3-9 will be noted as they occur.

The $\underline{B}$ matrix is factored into six submatrices:

$$\underline{B} = {}_{D}\underline{X} \; \underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \;\vdots\; \underline{\Theta}_2 \;\vdots\; \underline{\Theta}_3 \;\vdots\; \underline{\Theta}_4 \;\vdots\; \underline{\Phi}_1 \;\vdots\; \underline{\Phi}_2 \right] \tag{3-13}$$

where if the assumed order of $\underline{x}(t)$ is p, the order of $\underline{\Theta}_1$, $\underline{\Theta}_2$, and $\underline{\Theta}_4$ is p x p, the order of $\underline{\Theta}_3$ is p x $\frac{p(p-1)}{p}$, and $\underline{\Phi}_1$, and $\underline{\Phi}_2$ are p vectors.

The interpolation estimate of the first term on the right-hand side of equation 3-4 is given by:

$$\underline{x}_k((k+1)T) = \underline{\Theta}_1 \; \underline{x}(kT) + \underline{\Theta}_2 \; \underline{x}^2(kT) + \underline{\Theta}_3 \; \underline{x_i x_j}(kT) \tag{3-14}$$

If the basis vector without state variable cross terms is used, the term involving $\underline{\Theta}_3$ will be missing in equation 3-14 and $\underline{B}$ is factored into five submatrices rather than six with the $\underline{\Theta}_3$ submatrix being excluded.

The estimate of the second term of the right-hand side of equation 3-4 is given by:

$$\tilde{\underline{A}}_k((k+1)T) = \left[(\underline{\Theta}_4\underline{x}(kT) + \underline{\varphi}_1) \,\vdots\, (\underline{\varphi}_2)\right] \tag{3-15}$$

and:

$$\underline{u}'_k = \lfloor u_k \; u_k^2 \rfloor \tag{3-16}$$

The quantity $\tilde{\underline{x}}_k((k+1)T)$ is a p vector, the order of $\tilde{\underline{A}}_k((k+1)T)$ is p x 2, and $\underline{u}_k$ is a 2 x 1 vector.

The non-linear interpolation estimate of the state equation is given by:

$$\tilde{\underline{x}}((k+1)T) = \tilde{\underline{x}}_k((k+1)T) + \tilde{\underline{A}}_k((k+1)T)\, \underline{u}_k \tag{3-17}$$

where $\tilde{\underline{x}}_k$, $\tilde{\underline{A}}_k$ and $\underline{u}_k$ are defined by equation 3-14, 3-15, and 3-16 respectively.

THE CONTROL POLICY

The control policy for the control of non-linear plants is the same as that for the linear plants:

$$\min_{u_k}\left[Q_k\right] = \min_{u_k}\left[\underline{e}'((k+1)T)\underline{K}\; \underline{e}((k+1)T)\right] \tag{3-18}$$

where the error state vector $\underline{e}(t)$ is defined as it was for the linear studies:

$$\underline{e}(t) = \underline{r}(t) - \underline{x}(t) \tag{3-19}$$

The final form of the control policy equation will depend upon whether the linear or one of the non-linear interpolation basis vectors is utilized.

Linear Control Policy Equation. - The control policy equation using the linear basis vector is identical in form with that presented in paragraph 2.1. It is obtained by substituting the linear estimate of the state equation, equation 3-12, into the quadratic form, equation 3-18, setting the first derivative of the resultant equation with respect to u_k equal to zero and solving for u_k. The final equation for u_k is given by:

$$u_k = \frac{\underline{\phi}_1{}' \; \underline{K} \left[\underline{r}((k+1)T) - \underline{\Theta}_1 \; \underline{x}(kT) \right]}{\underline{\phi}_1{}' \; \underline{K} \; \underline{\phi}_1} \tag{3-20}$$

Non-Linear Control Policy Equation. - The manner in which the non-linear control policy equation is derived is identical to that used in deriving the linear control equation. The interpolation estimate of the state equation, equation 3-17, is substituted into equation 3-18, the derivative of the resultant equation with respect to u_k is set equal to zero, and the derivative equation is solved for u_k. In the non-linear case the resultant control equation is a cubic in terms of u_k:

$$\begin{aligned}(2\,{}_k\underline{a}'_{oo} \; \underline{K} \; {}_k\underline{a}_{oo})\, u_k^{\,3} + (3\,{}_k\underline{a}'_{oo} \; \underline{K} \; {}_k\underline{a}_{o})\, u_k^{\,2} \\ + (2\,{}_k\underline{a}'_{oo} \; \underline{K} \; {}_k\underline{x}_{\blacktriangle} + {}_k\underline{a}'_{o} \; \underline{K} \; {}_k\underline{a}_{o})\, u_k + {}_k\underline{a}'_{o} \; \underline{K} \; {}_k\underline{x}_{\blacktriangle} = 0\end{aligned} \tag{3-21}$$

where the following notational definitions apply:

$$\begin{aligned}\underline{\widetilde{A}}_k((k+1)T) &= \left[{}_k\underline{a}_{o} \;\; {}_k\underline{a}_{oo} \right] \\ {}_k\underline{x}_{\blacktriangle} &\equiv \underline{\widetilde{x}}_k((k+1)T) - \underline{r}((k+1)T)\end{aligned} \tag{3-22}$$

The fact that equation 3-21 is a cubic guarantees that at least one real root exists which will minimize the quadratic form Q_k.

COMPARISON OF THE INTERPOLATION METHODS

Reviewing the equations involved with the three types of interpolation basis vectors shows that the linear form is by far the simplest. An explicit solution for the control force, u_k, is possible in the linear case whereas both non-linear forms involve the solution of a cubic equation. The size of the interpolation matrices are smallest for the linear case and are substantially greater in the two non-linear cases. The critical matrix as far as size is concerned is $\underline{\Phi}$ the matrix of basis vectors. This matrix must be inverted whereas the other matrices are involved in matric multiplications and additions which are simpler operations. The formulas for the order, d, of the $\underline{\Phi}$ matrices are given below. The assumed order of the system is p.

Linear Basis Vector:

$$d = p + 1 \tag{3-23}$$

Non-Linear Basis Vector Without Cross Terms:

$$d = 3p + 2 \tag{3-24}$$

Non-Linear Basis Vector With Cross Terms:

$$d = 3p + \frac{p(p-1)}{2} + 2 \tag{3-25}$$

Table 3.1 lists the dimension of $\underline{\Phi}$ for several system orders:

TABLE 3.1 DIMENSIONS OF $\underline{\Phi}$

SYSTEM ORDER	LINEAR $\underline{\Phi}$	NON-LINEAR $\underline{\Phi}$ WITHOUT CROSS TERMS	NON-LINEAR $\underline{\Phi}$ WITH CROSS TERMS
2	3	8	9
3	4	11	14
4	5	14	20
5	6	17	27

3.2 EXPERIMENTAL STUDIES

This section presents the experimental results of the non-linear control studies. The interpolation method of plant functional representation discussed in paragraph 3.1 have been used throughout the simulation experiments.

OBJECTIVES

The objectives of the experimental program were:

> To investigate the feasibility of the control of non-linear plants using a non-linear interpolation basis vector.
>
> To establish if a region exists in the T-h plane within which satisfactory control of a non-linear plant is obtained. A corollary to this objective is to observe if any systematic trends exist with respect to values of the T-h control system parameters.
>
> To investigate the control of a non-linear plant with the linear interpolation control policy and to compare these results with those obtained using the non-linear interpolation basis vectors.

The accomplishment of these objectives along with the experimental procedures used is discussed in the following paragraphs.

NON-LINEAR PLANTS

It was first necessary to select a set of non-linear plants to use as vehicles for the experimentation. This is not a menial task as the area of non-linear control systems is vast and rather highly segmented. There are many methods of analysis and synthesis which apply to limited classes of non-linear problems but no general method which applies to all. If the problem of selecting a representative spectrum of linear stationary and linear time-varying plants is considered difficult, a corresponding selection of non-linear plants is virtually impossible. The set of non-linear plants was therefore limited to those characterized by the Van der Pol equation:

$$\ddot{c} - \epsilon(1 - c^2)\,\dot{c} + c = 0 \tag{3-26}$$

While this may seem restrictive, the available time and funds allowed only a very limited look at the feasibility of the control of non-linear plants. It was decided that rather than define a very limited set of experiments for each of several non-linear plants, it would be more instructive to confine the experiments to a more exhaustive set with one type of plant. The Van der Pol "plant" was chosen because it is easy to control the degree of non-linearity by appropriate selection of the coefficient, ϵ , the non-linear damping term. The ability to control the non-linearity of the plant is certainly not a unique property of the Van der Pol equation (i.e. an otherwise linear plant with a non-linear spring constant) so that the choice is arbitrary in this respect.

The free response of the Van der Pol equation for three values of ϵ is shown in Figure 3-1. In all three cases the initial conditions were $x = 1.0$ and $\dot{x} = 1.0$ which placed the initial state inside the characteristic limit cycle which the Van der Pol equation always assumes regardless of the initial conditions. The output response in all three cases grows in magnitude until it oscillates between ±2 units which is characteristic of the Van der Pol limit cycle.

The following paragraphs consider the control of the Van der Pol "plant" as defined by equation 3-27:

$$\ddot{c} - \epsilon(1 - c^2)\,\dot{c} + c = m(t) \qquad (3\text{-}27)$$

where m(t) is the input or forcing function determined by the control policy discussed in paragraph 3.1.

T-h PLANE AND REGULATOR RESULTS

Previous research (references 1, 3, 4, 5, 6, and 7) has established the primacy of the two control or design parameters of the DACS control concept in determining controlled response characteristics of many systems for various values of the decision interval length, T, and the weighing factor, h. For linear stationary systems, a region of stable operation of the control policy in the T-h plane was established using a Liapunov function. The concept of stable operation of the control policy in the T-h plane was extended to linear time-varying systems in reference 1 and paragraph 2.2 where control simulations were performed at various T-h points to ascertain the regions of stability. Further extension to the controlled response characteristics of non-linear systems in the T-h plane is now made with the following restriction. Conclusions as to stability

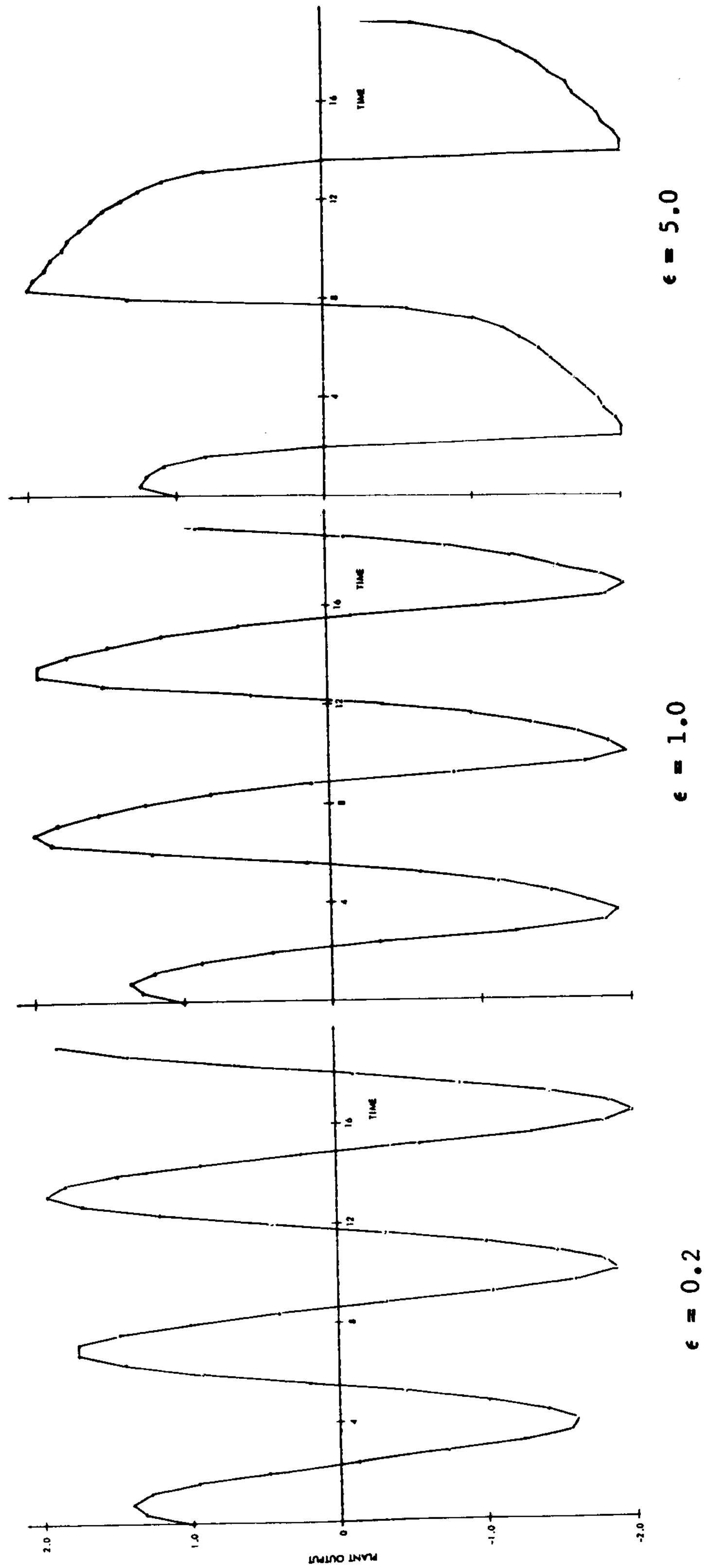


FIGURE 3-1 FREE RESPONSE OF THREE VAN DER POL PLANTS

are not made and performance is assessed only as being satisfactory or unsatisfactory. The inexactness of any estimation technique (including the interpolation method) in describing the non-linear system response characteristics and resultant high dependence on the system initial conditions and subsequent trajectory precludes general stability conclusions. As was the case in paragraph 2.2, only a limited region of the T-h plane was investigated because of economic considerations.

The first set of experiments conducted were concerned with the regulator response characteristics of the Van der Pol plant ($\underline{r}(t) = \underline{0}$ for all t) for three different values of ϵ (the coefficient of the damping term) using the three types of interpolation basis vectors discussed in paragraph 3.1. As in all of the previous control simulations presented using the interpolation procedure, a start-up phase is necessary as the procedure is not self starting. The start up method used in these studies is identical to that discussed in paragraph 2.2 of reference 1 concerning "Interpolation Prediction Control-With Updating". Simply stated, the start-up procedure computes a set of initial interpolation matrices while allowing the arbitrary selection of a set of system initial conditions for the control simulation. The controlled response graphs presented (i.e. Figure 3-8) are reproductions of the graphs plotted by the computer simulation program. The axes have been relabeled to make them more readable and the data points have been connected with straight lines. The start-up phase is not shown on the graphs.

Regulator Results For ϵ Of 0.2. - This value of ϵ yields a system which is the most nearly linear of those considered. The T-h planes for the three types of basis vectors are shown in Figures 3-2, 3-3, and 3-4. For the sake of brevity, the basis vectors are identified as follows:

- Type One - linear basis vector
- Type Two - non-linear basis vector without state variable cross terms
- Type Three - non-linear basis vector with state variable cross terms

The definitions of the three basis vectors have been presented in equations 3-8, 3-9, and 3-10 respectively of paragraph 3.1. They will be

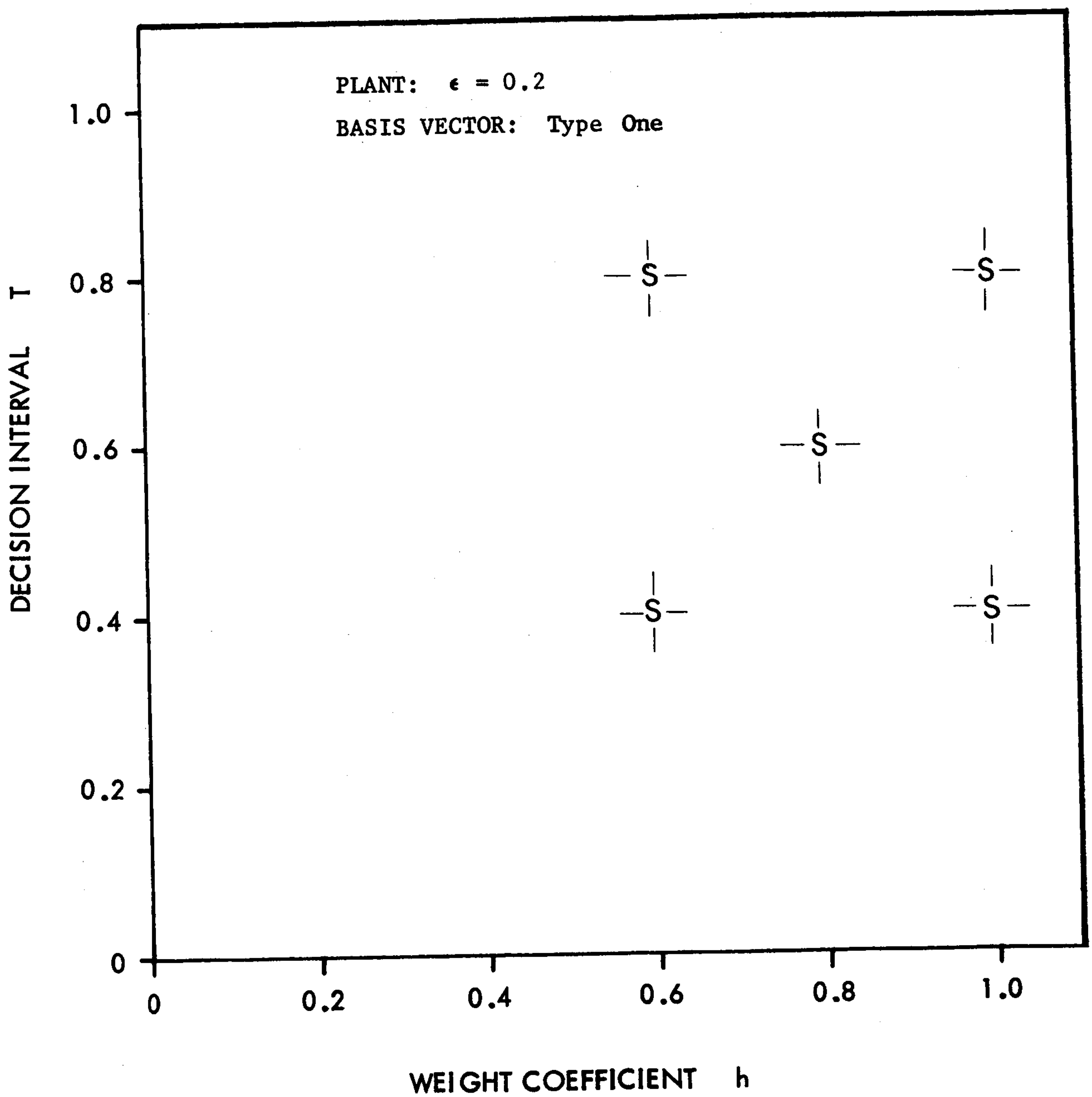


FIGURE 3-2 T-h PLANE OF THE TYPE ONE BASIS VECTOR

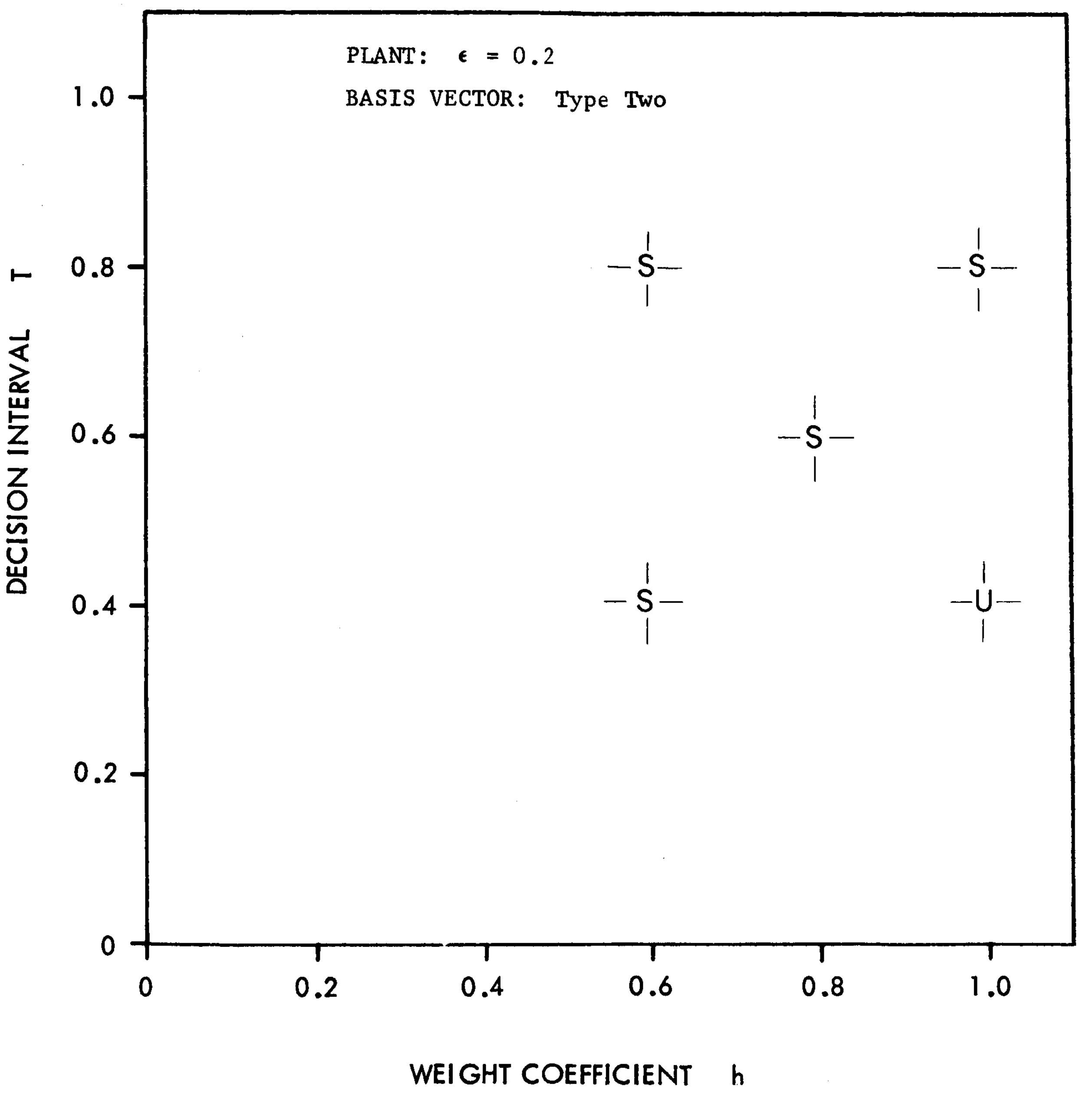


FIGURE 3-3 T-h PLANE OF THE TYPE TWO BASIS VECTOR

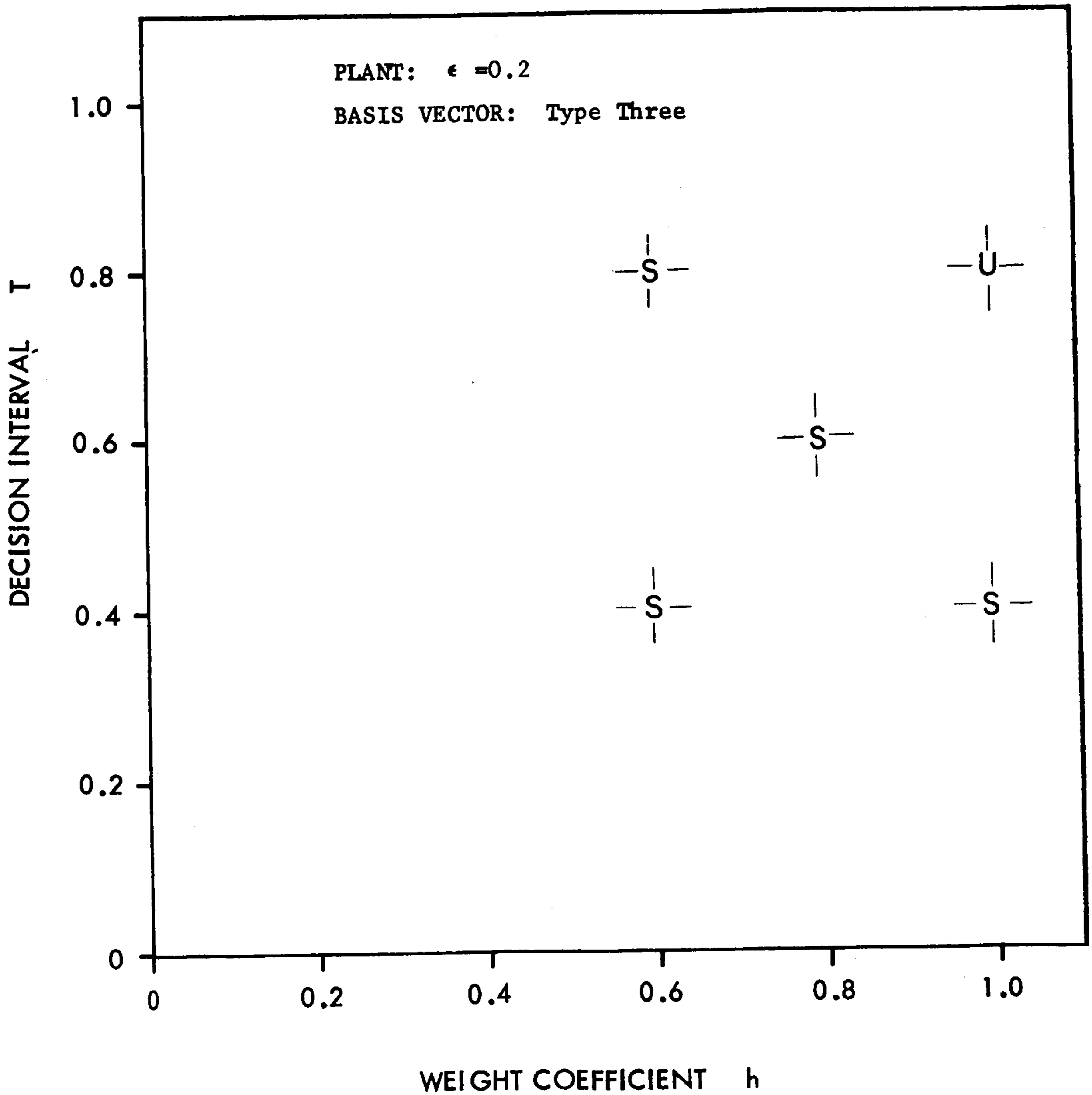


FIGURE 3-4 T-h PLANE OF THE TYPE THREE BASIS VECTOR

referred to by type only henceforth with the inferred definition of type being that given above.

One of two letters, "S" or "U", occurs on the T-h planes at points where regulator control simulators were made. The letter "S" indicates satisfactory performance at a particular point and "U" indicates unsatisfactory performance. Performance is judged upon whether the system output was driven to $\underline{0}$ during the control simulation. In all cases the initial state of the system was:

$$\underline{x}'(0) = \lfloor 1.0 \quad 1.0 \rfloor \tag{3-28}$$

In all cases, new data was shifted into the interpolation matrices every interval and the interpolation estimates were updated every interval.

Figure 3-2 shows that all five points investigated using the Type One (linear) basis vector yielded satisfactory control performance. In Figures 3-3 and 3-4 only one point gave unsatisfactory performance for each of the two non-linear basis vector descriptions. Although different points yielded the unsatisfactory performance in these cases, they both occurred at the largest value of h investigated.

Regulator Results For ϵ Of 1.0. - This value of ϵ yields a system which is of an intermediate nature as far as the degree of non-linearity of the three plants studied. The parameters of the experimentation (types of basis vectors, initial conditions, etc.) are the same for this value of ϵ as those described for $\epsilon = 0.2$.

The T-h planes corresponding to the three types of basis vector descriptions are shown in Figures 3-5, 3-6 and 3-7. Figure 3-5 shows that all five points using the Type One (linear) basis vector yielded satisfactory performance. Figure 3-6 shows that the Type Two basis vector yielded one unsatisfactory point and Figure 3-7 shows that two unsatisfactory points occurred using the Type Three basis vector. Unsatisfactory performance was obtained at the middle point of the plane with both of the non-linear basis vectors.

The actual regulator control simulations for one T-h point (T = 0.4, h = 1.0) using the three different basis vectors are presented in Figures 3-8, 3-9 and 3-10. The graphs are typical of the type of performance obtained

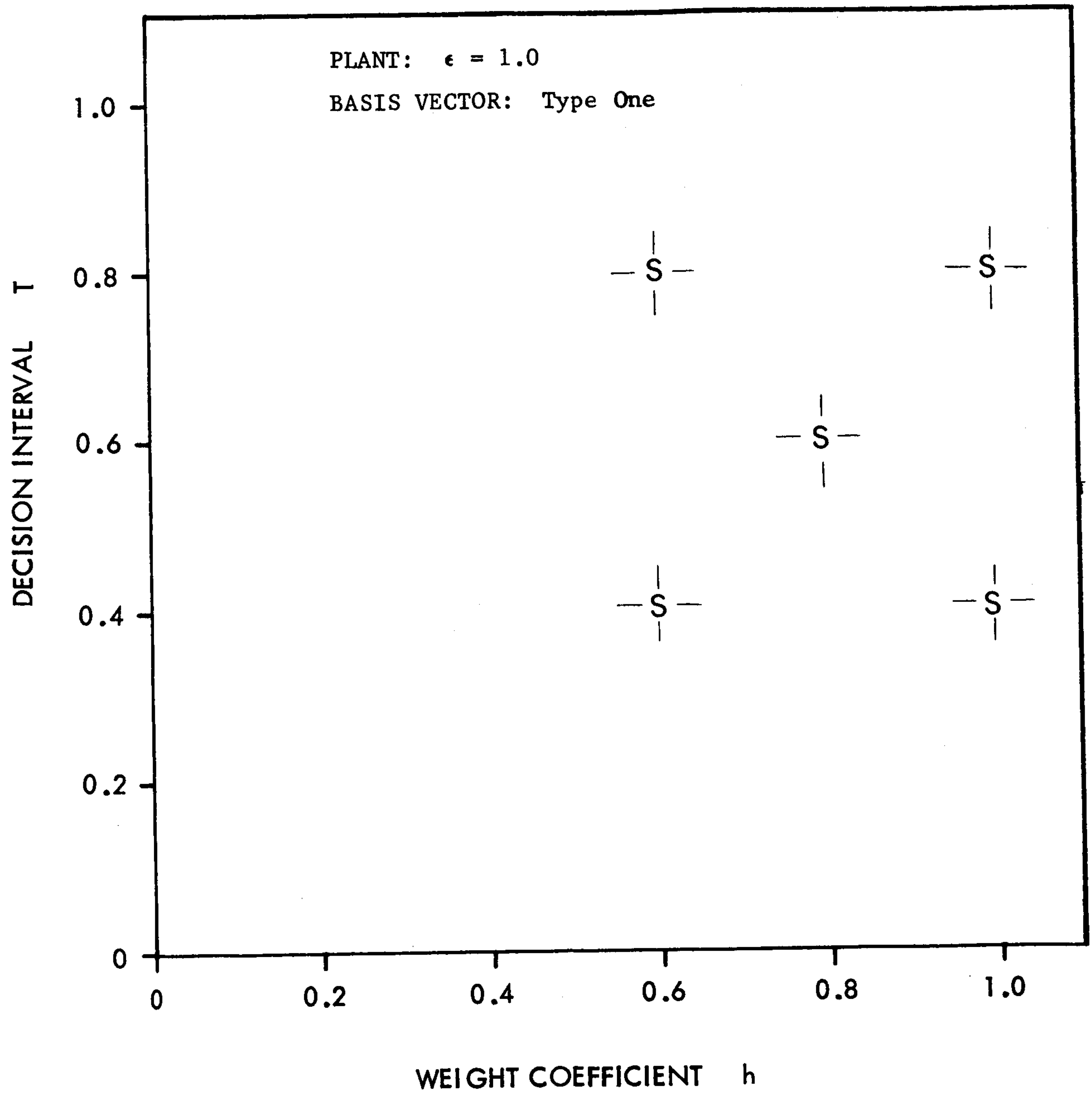


FIGURE 3-5 T-h PLANE OF THE TYPE ONE BASIS VECTOR

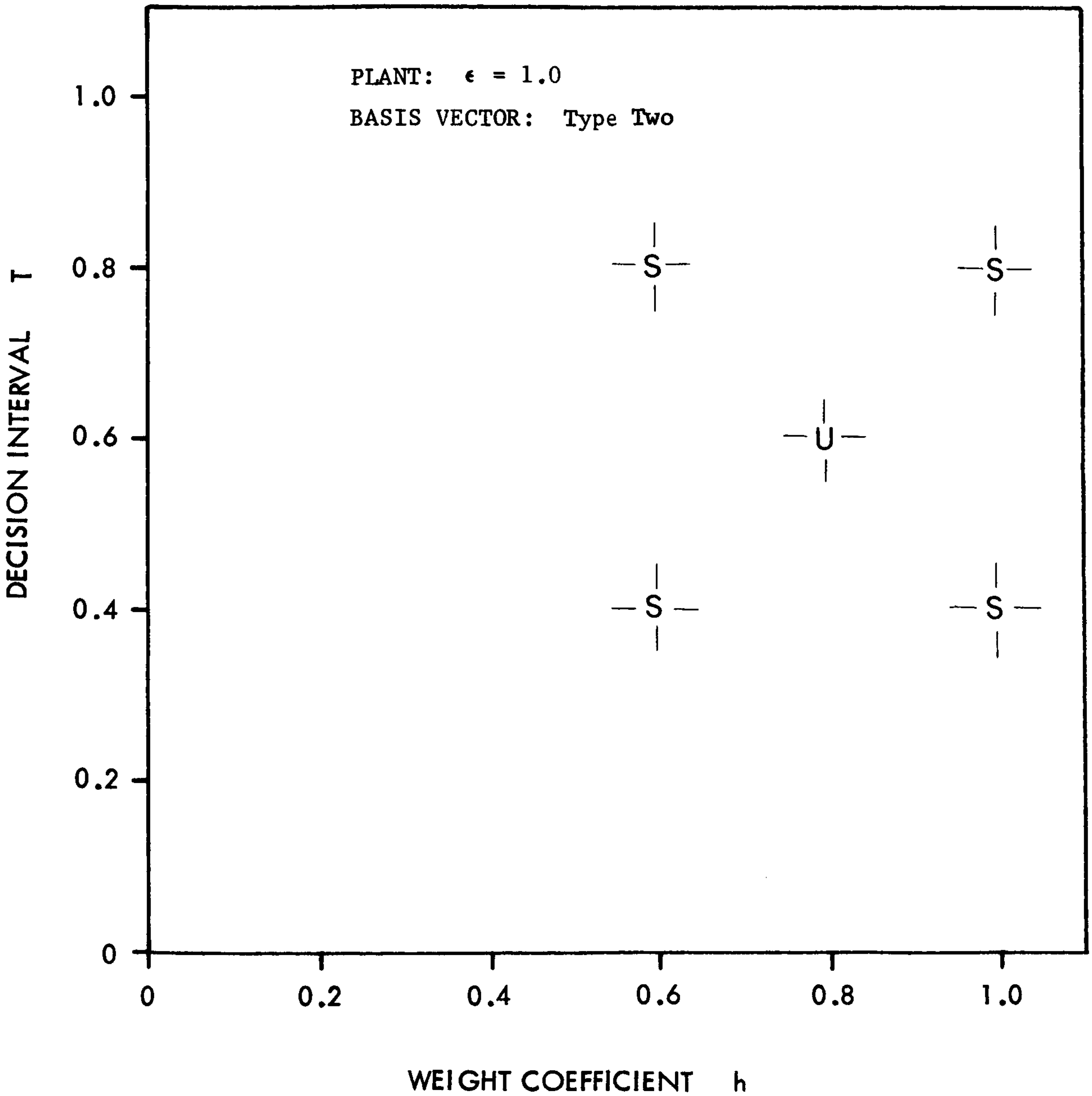


FIGURE 3-6 T-h PLANE OF THE TYPE TWO BASIS VECTOR

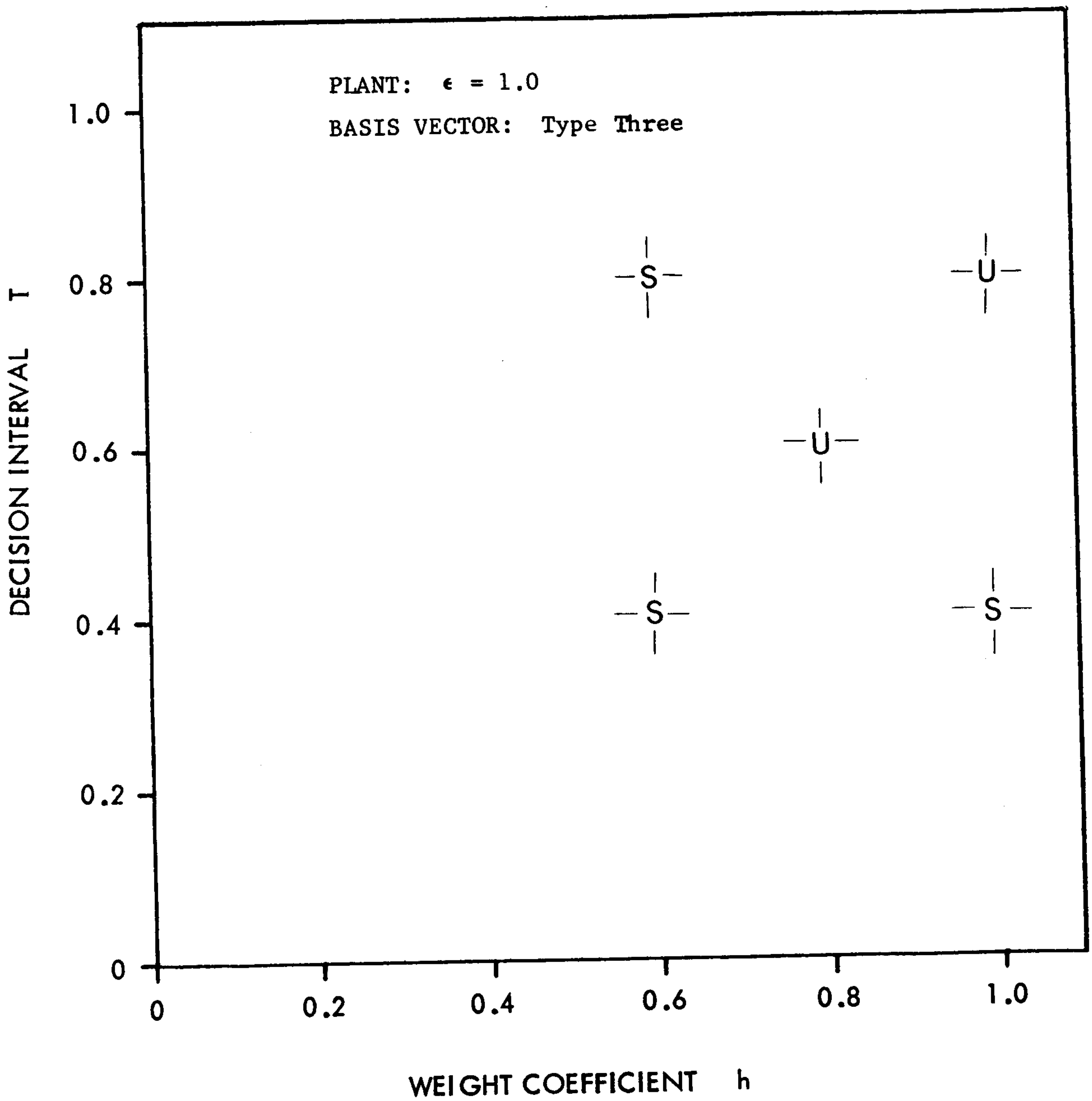


FIGURE 3-7 T-h PLANE OF THE TYPE THREE BASIS VECTOR

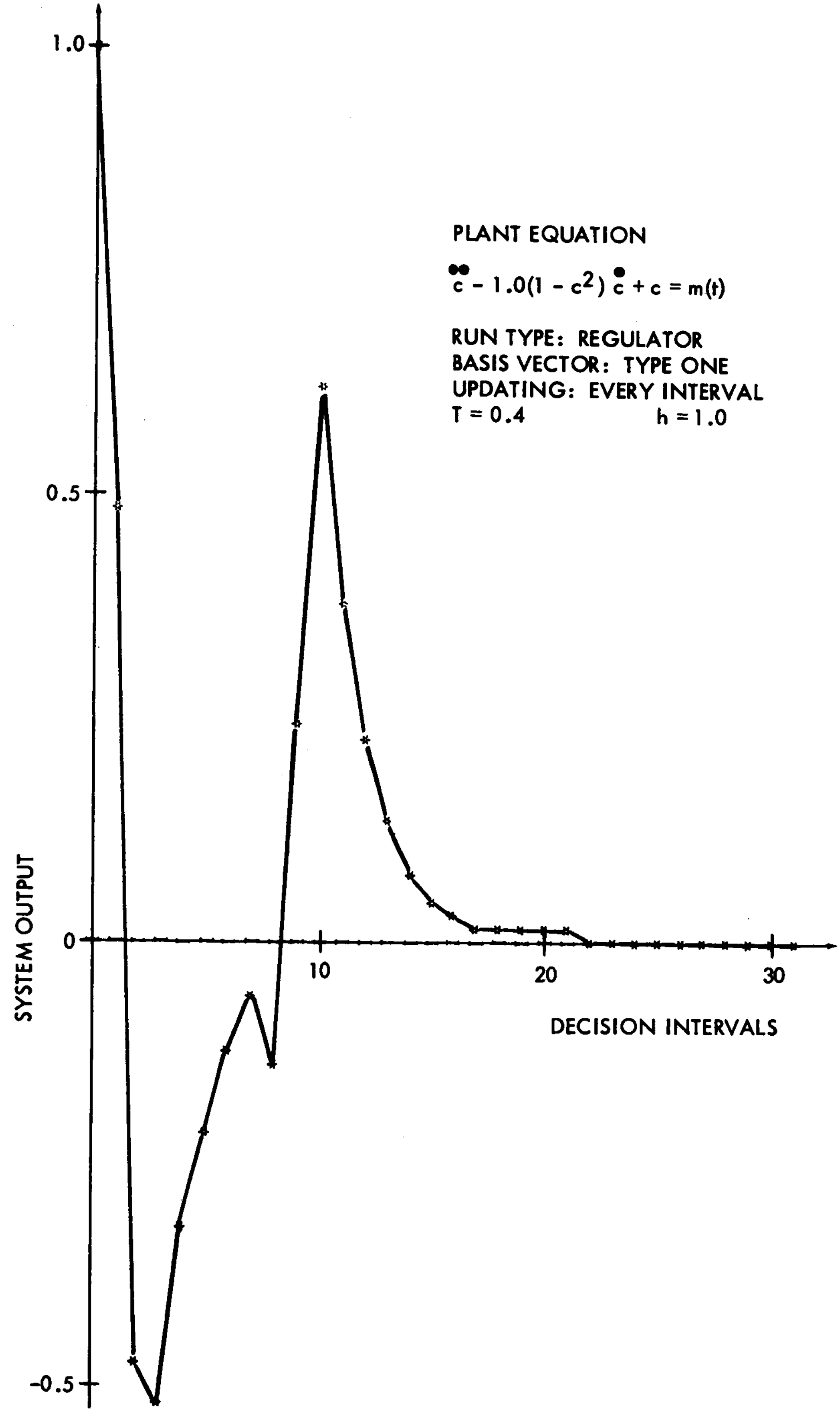


FIGURE 3-8 REGULATOR RESPONSE OF A VAN DER POL PLANT - TYPE ONE BASIS VECTOR

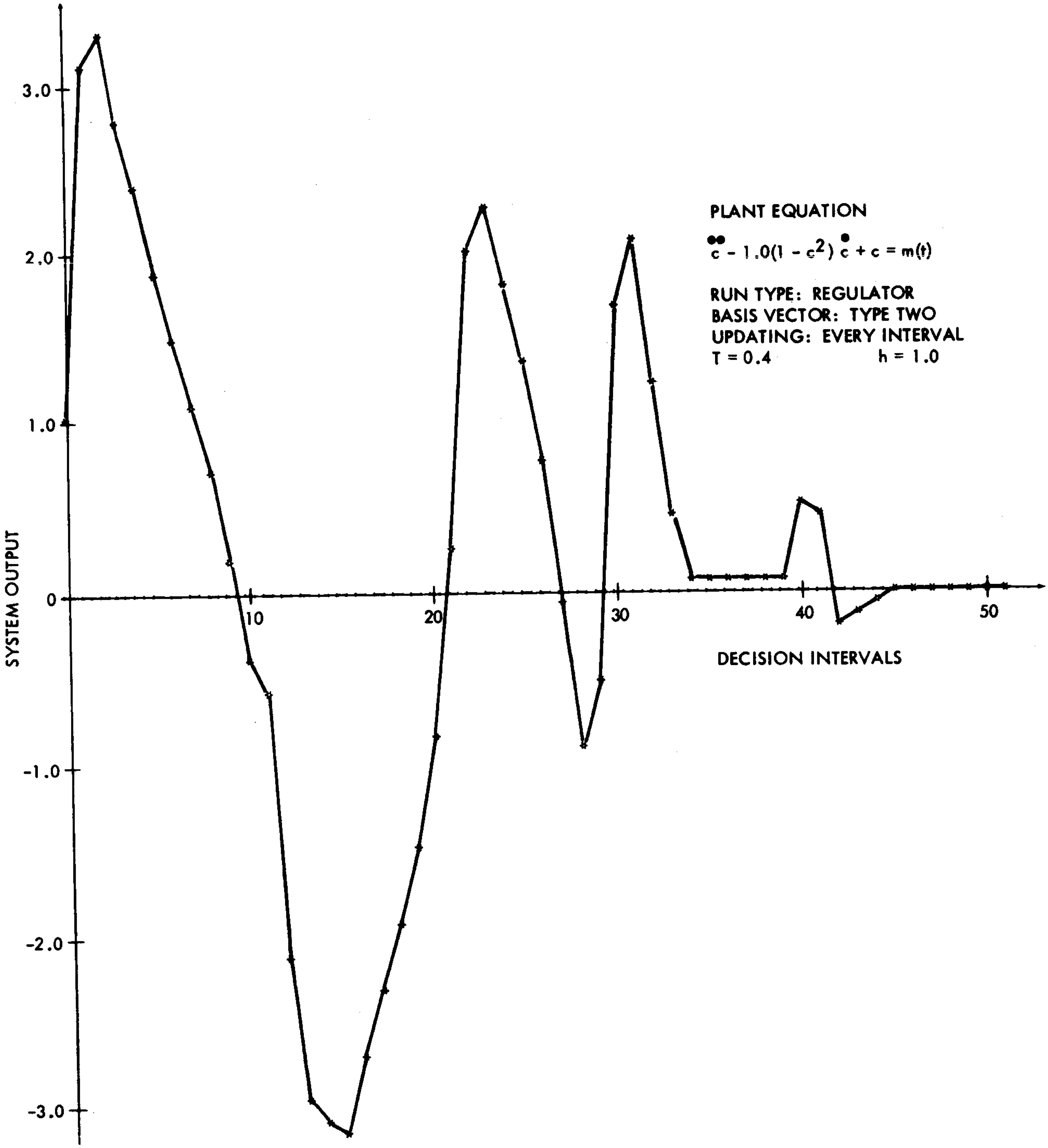


FIGURE 3-9 REGULATOR RESPONSE OF A VAN DER POL PLANT - TYPE TWO BASIS VECTOR

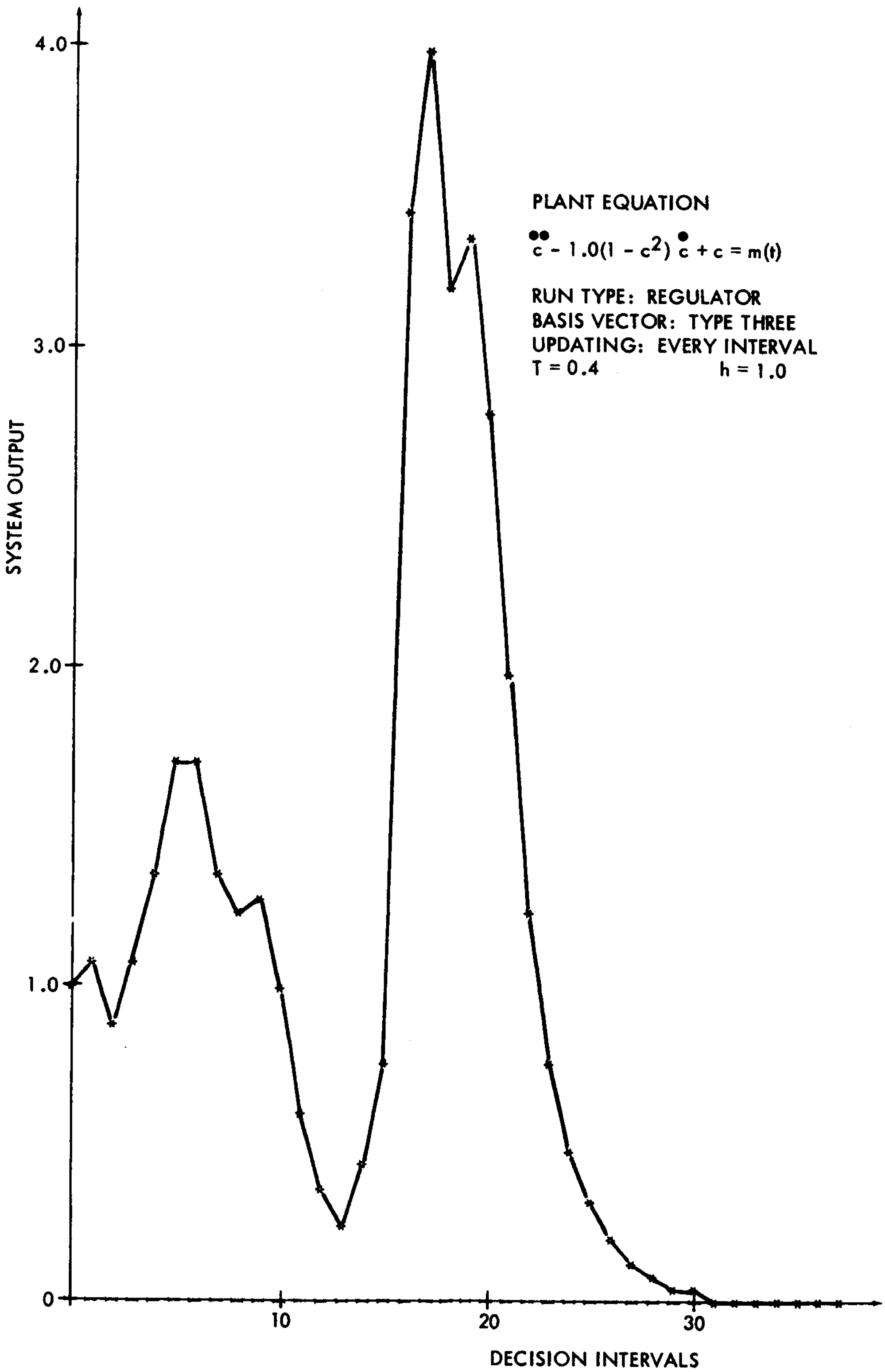


FIGURE 3-10 REGULATOR RESPONSE OF A VAN DER POL PLANT - TYPE THREE BASIS VECTOR

at a T-h point classified as satisfactory in all three ϵ cases investigated. The actual regulator responses are shown for this value of ϵ only so as to unify the presentation of the data as much as possible. Typical unsatisfactory performance yields a controlled response graph in which the system output diverges.

In all three figures (3-8, 3-9 and 3-10) the response is somewhat irregular before converging to zero. The response of Figure 3-9 for a Type Two basis vector is the most extreme in this regard.

The regulator control performance simulations for a second point in the T-h plane (T = 0.8, h = 0.6) of the ϵ = 1.0 Van der Pol plant are shown in Figures 3-11a, 3-11b, and 3-11c for Type One, Type Two, and Type Three basis vectors respectively. The response in all three cases is more regular than that obtained for the first T-h point.

Regulator Results For ϵ Of 5.0. - This is the largest ϵ value studied. Again, the parameters of the experimentation are the same as those of the ϵ = 0.2 case.

The T-h planes corresponding to the three types of basis vector descriptions are presented in Figures 3-12, 3-13, and 3-14. Figure 3-12 shows that the Type One (linear) basis vector breaks down completely as far as yielding a satisfactory plant description for control at the T-h points considered. This is significant when compared with the Type One basis vector T-h planes of the previous two ϵ values. The free response of Figure 3-1 indicates that the ϵ = 5.0 Van der Pol plant is considerably more non-linear than the other two and the linear basis vector description is not satisfactory.

Figure 3-13 shows that all five T-h points also yielded unsatisfactory performance using the Type Two basis vector and Figure 3-14 shows that only the lower two T-h points yielded satisfactory control performance using the Type Three basis vector. For the T-h points considered, the ϵ = 5.0 Van der Pol plant appears to be a limiting case as far as control is concerned. A more practical limit would appear to be a Van der Pol plant with an ϵ value somewhere between 1.0 and 5.0.

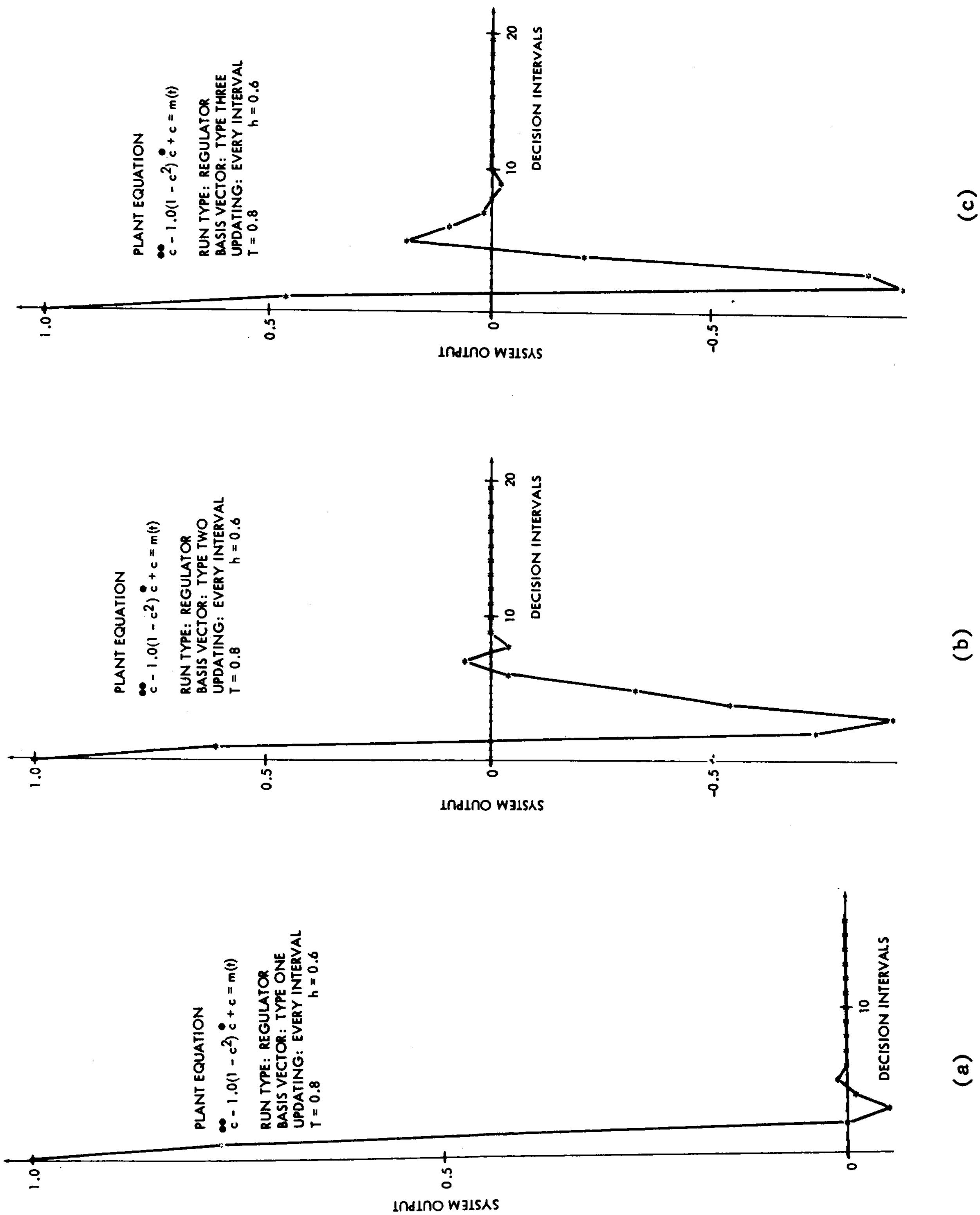


FIGURE 3-11 REGULATOR RESPONSE OF A VAN DER POL PLANT

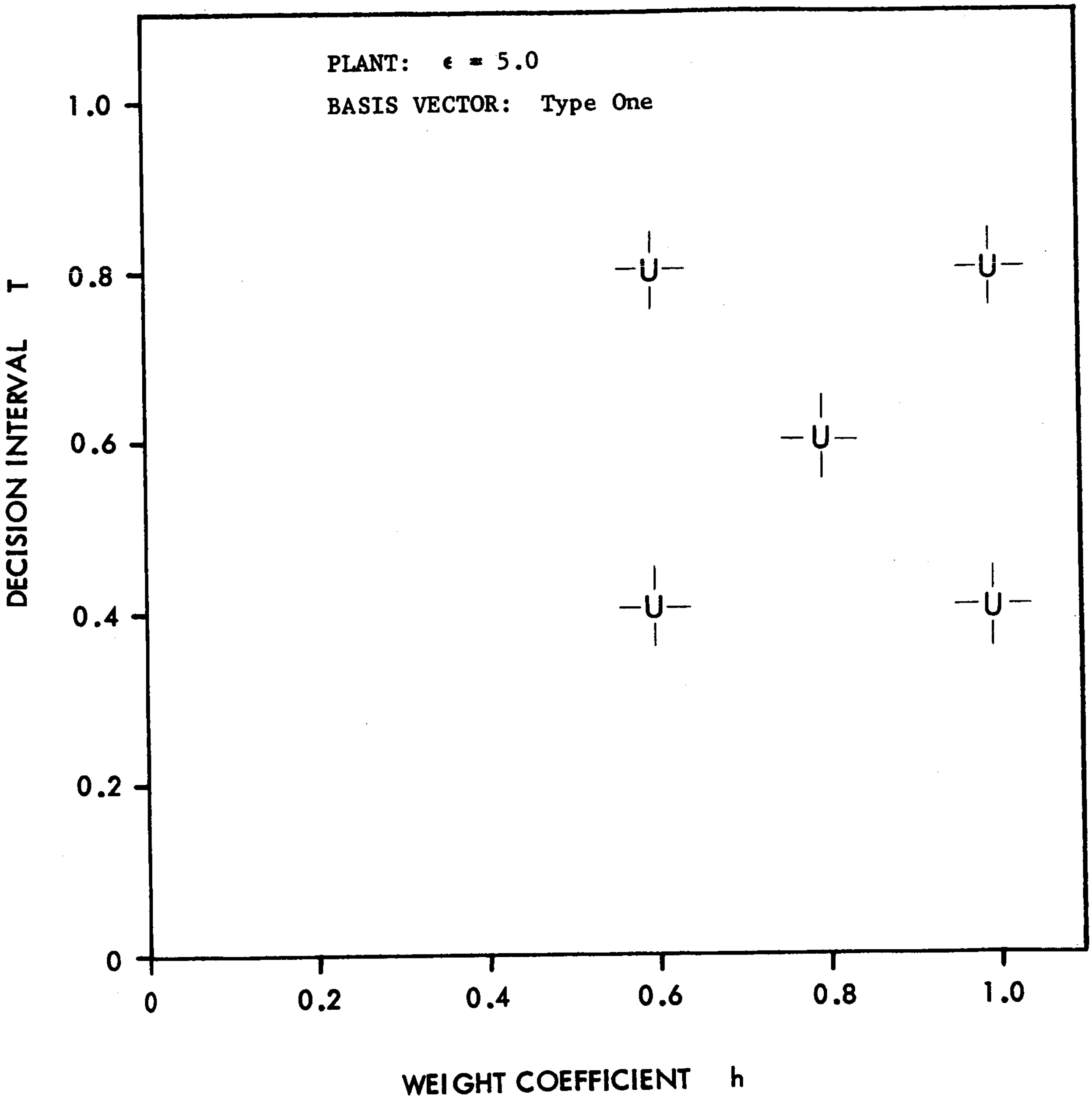


FIGURE 3-12 T-h PLANE OF THE TYPE ONE BASIS VECTOR

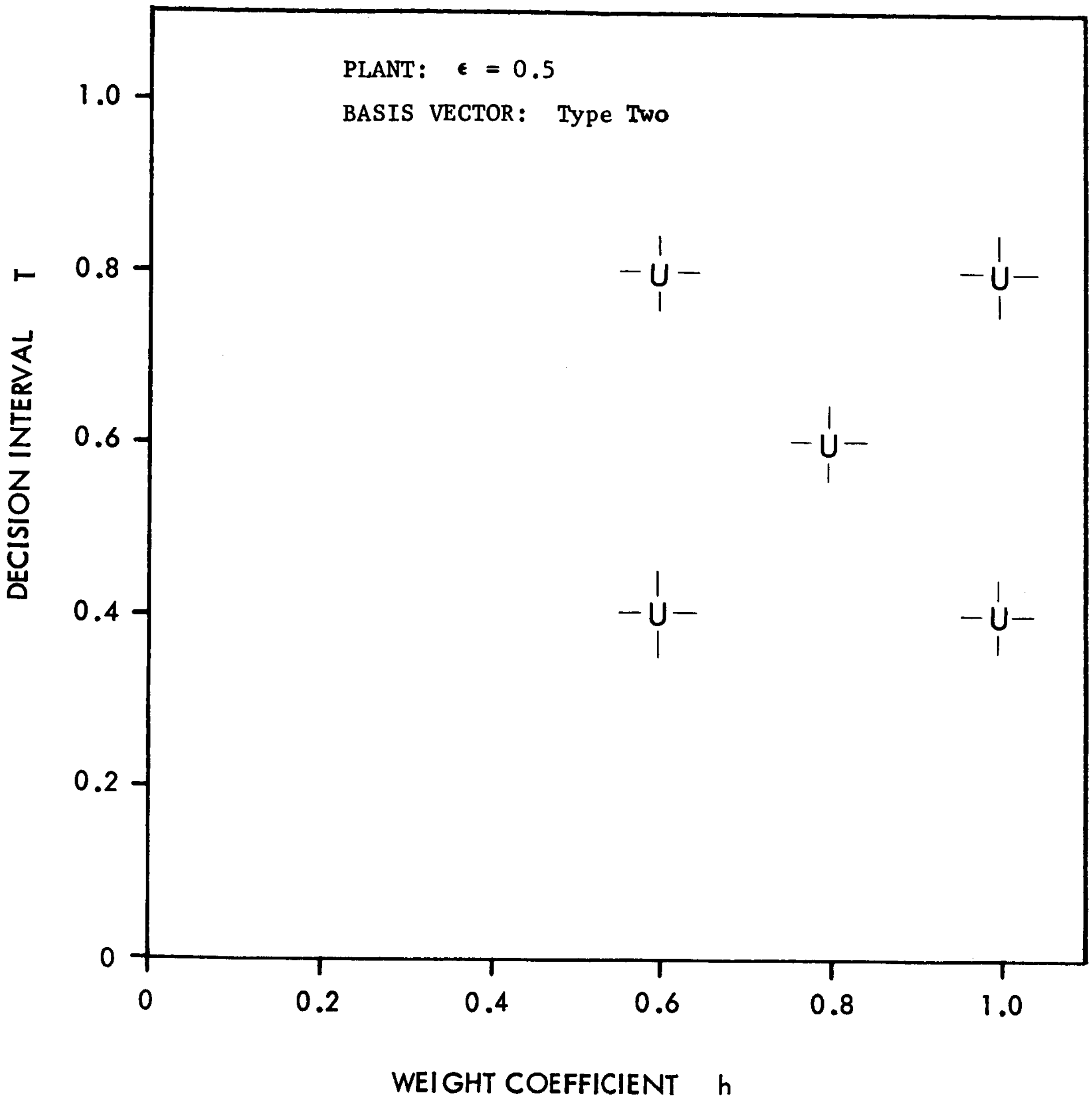


FIGURE 3-13 T-h PLANE OF THE TYPE TWO BASIS VECTOR

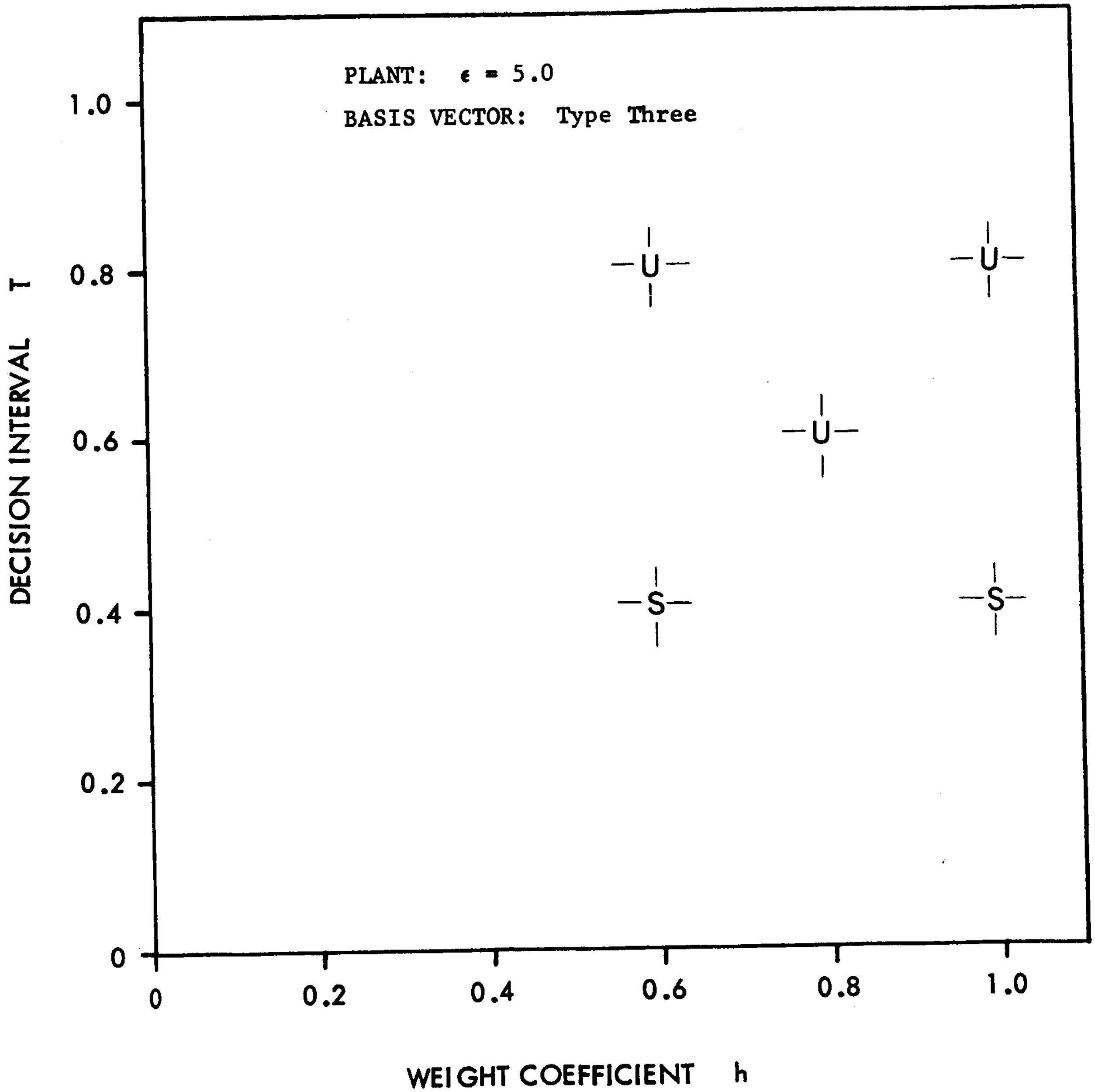


FIGURE 3-14 T-h PLANE OF THE TYPE THREE BASIS VECTOR

TRAJECTORY RESULTS

The control performance for the Van der Pol plants was investigated for the case where the desired output is a sine wave. The other elements of $\underline{r}(t)$ are the corresponding derivatives of the output. Control simulations were made using all three of the basis vector types. In all experiments the initial state of the system was $x = 1.0$ and $\dot{x} = 1.0$ and new data was shifted into the interpolation matrices every interval. The updating monitor was utilized in the Type One (linear) basis vector control simulations with $\delta_i = 0.05$ for all i. The use of the updating monitor is a direct extension of the procedure developed for the linear time-varying systems discussed in Section 2 and the experimental procedure used for updating in the non-linear experiments is identical.

Trajectory Results For ϵ Of 0.2. - Typical trajectory results for the $\epsilon = 0.2$ Van der Pol plant are presented in Figures 3-15, 3-16, and 3-17 for Type One, Type Two, and Type Three basis vectors respectively. The T-h point is the same ($T = 0.4$, $h = 0.6$) in all three figures. Various degrees of tracking difficulty are apparent for the different basis vector descriptions.

Figures 3-18 and 3-19 show the trajectory response at two other T-h points using the linear basis vector. The tracking of Figure 3-18 is somewhat degraded over that of Figure 3-15 for the same h value but a decision interval one-half the length. The higher h value of Figure 3-19 shows the effect of moving out in the T-h plane holding the length of the decision interval constant. The trends shown in T and h for the linear basis vector are also typical of those obtained using either of the two non-linear basis vector descriptions.

Figure 3-19 demonstrates the interesting result that even though the regulator response at the T-h point was satisfactory (Figure 3-2) the trajectory response appears to be diverging at the end of the run.

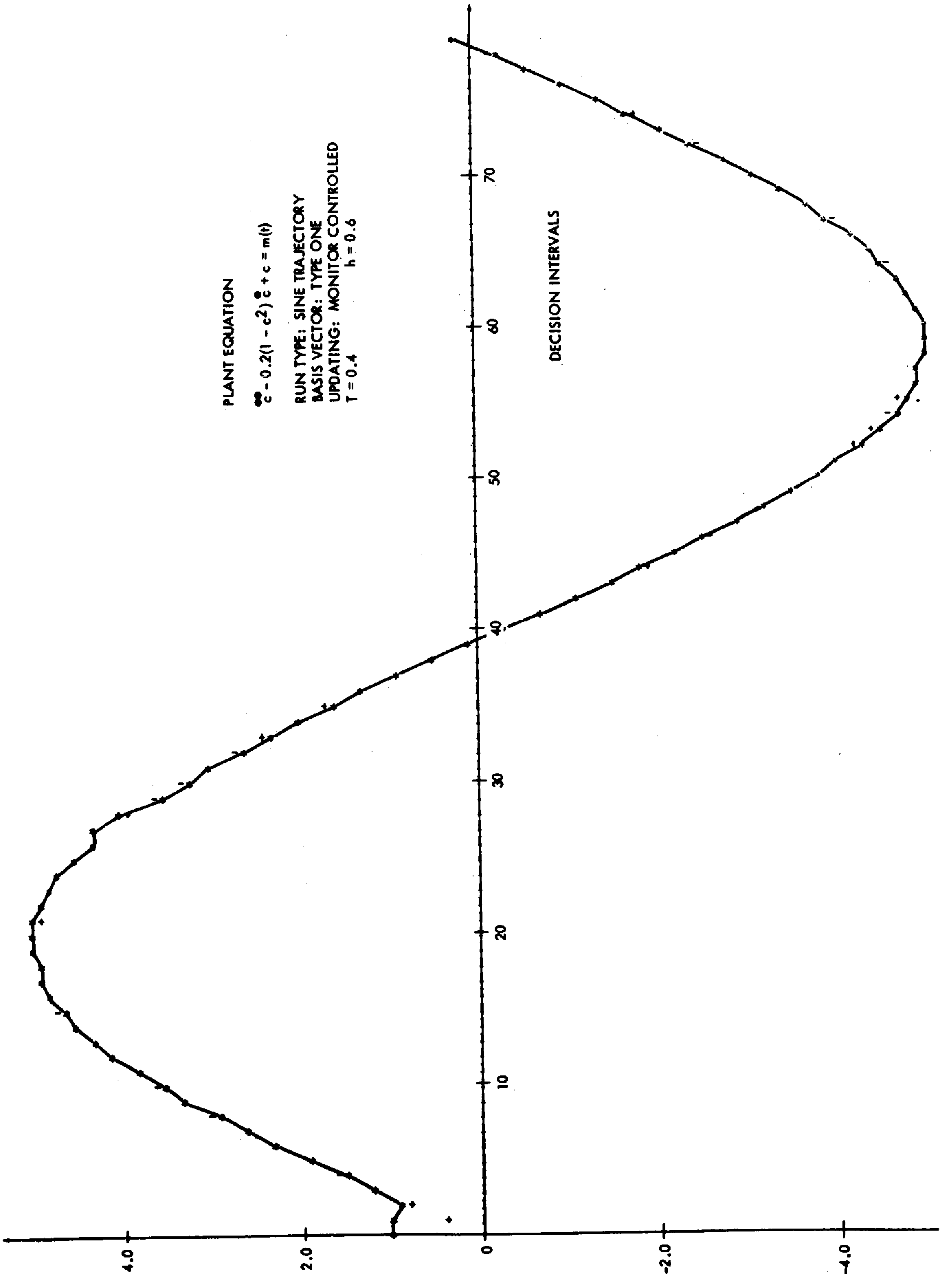


FIGURE 3-15 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE ONE BASIS VECTOR

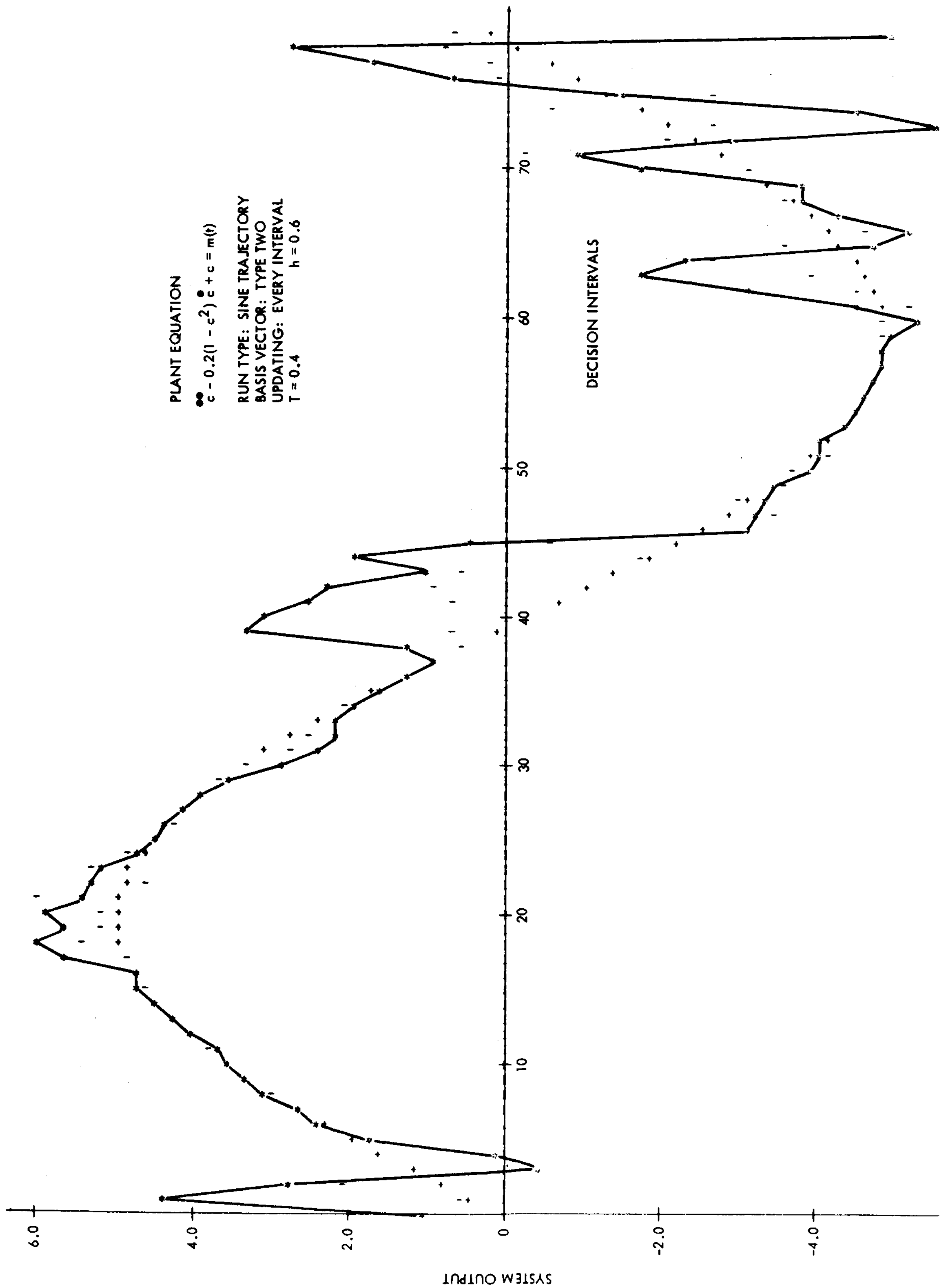


FIGURE 3-16 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE TWO BASIS VECTOR

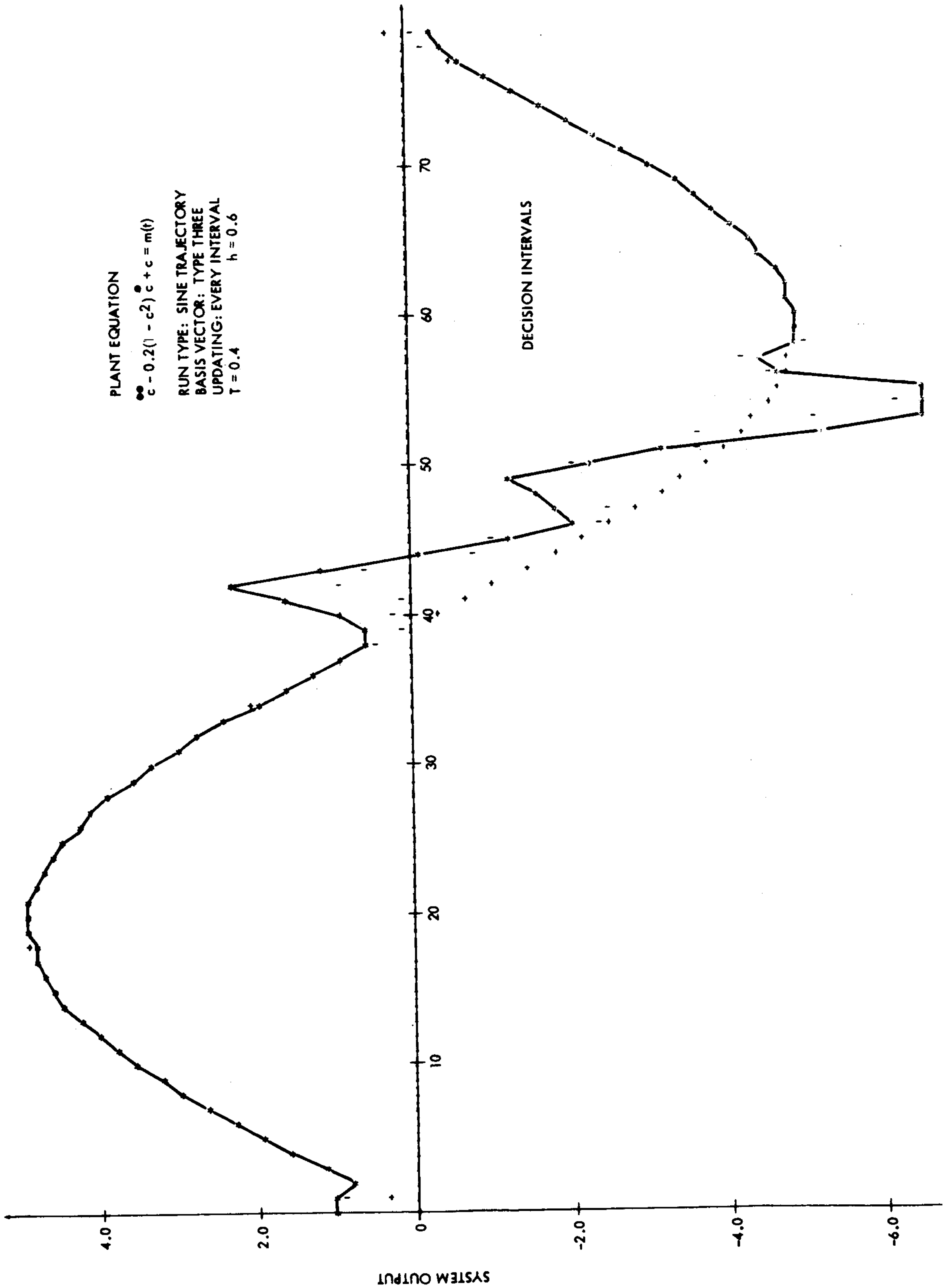


FIGURE 3-17 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE THREE BASIS VECTOR

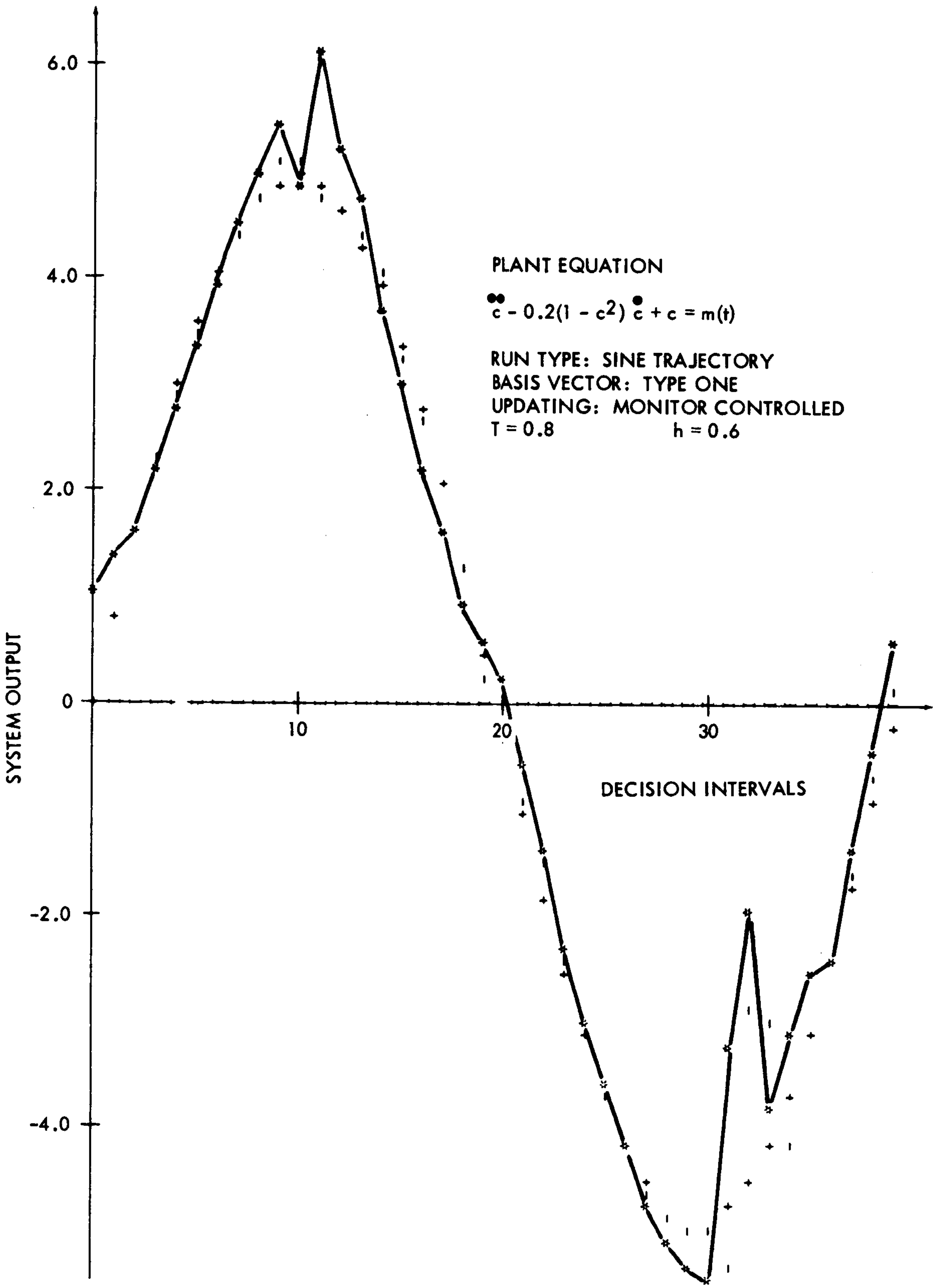


FIGURE 3-18 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE ONE BASIS VECTOR

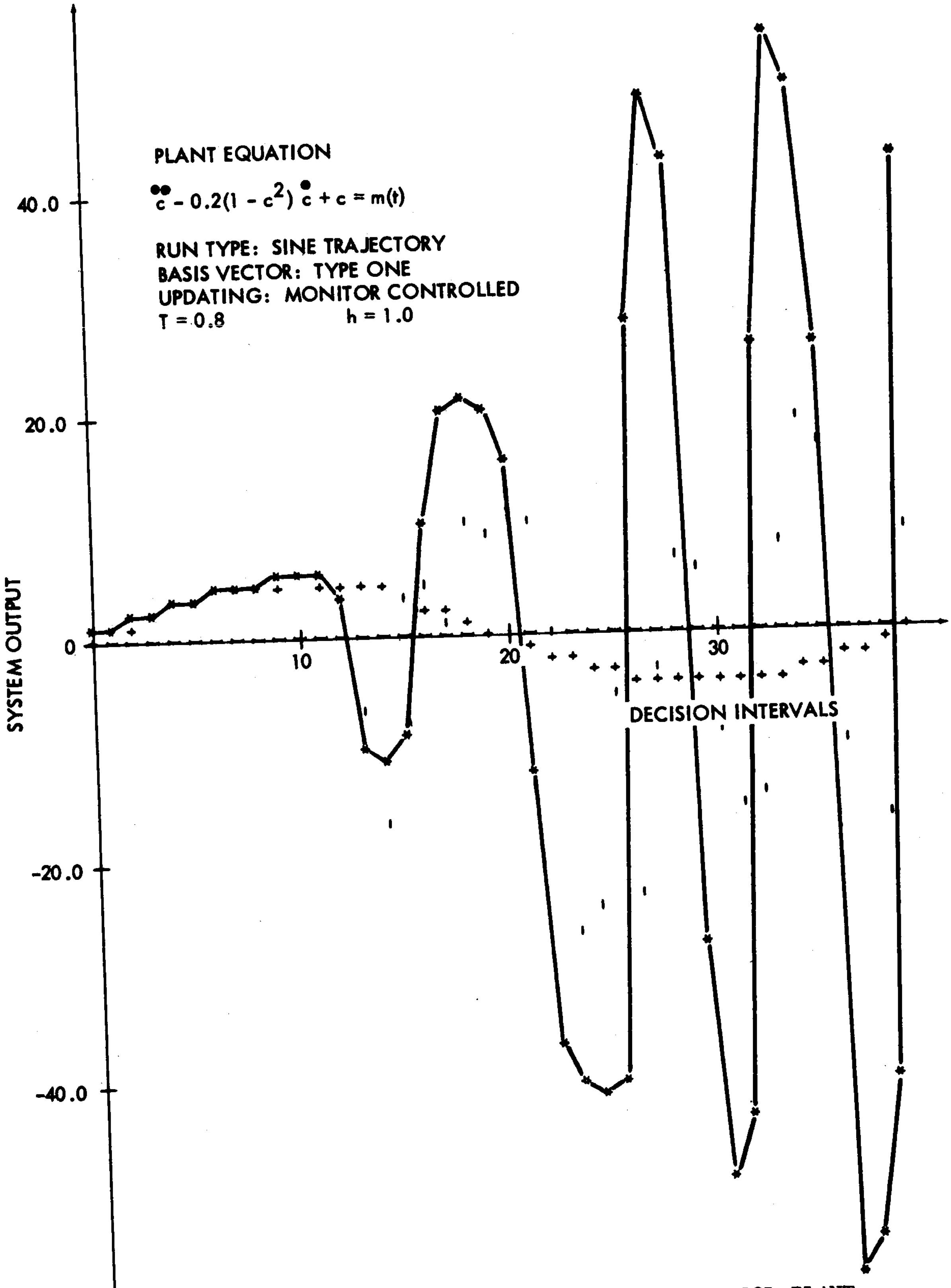


FIGURE 3-19 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE ONE BASIS VECTOR

Trajectory Results For $\epsilon = 1.0$. - Typical trajectory results for the $\epsilon = 1.0$ Van der Pol plant are presented in Figures 3-20, 3-21, and 3-22 for Type One, Type Two, and Type Three basis vectors respectively. In all three cases bad plant estimation occurs somewhere during the run. The tracking response might be classified as adequate for the Type One and Type Three basis vectors, however, the control using the Type Two basis vector is inadequate at the end of the run.

Trajectory Results For $\epsilon = 5.0$. - A few trajectory control simulations were made for the $\epsilon = 5.0$ Van der Pol plant with rather discouraging results. This might be expected from the T-h plane plots where only two satisfactory regulator points existed for the Type Three basis vector and none existed for the other two basis vectors.

3.3 SUMMARY OF EXPERIMENTAL RESULTS

The experimental results presented in paragraph 3.2 show that two of the Van der Pol plants ($\epsilon = 0.2$ and $\epsilon = 1.0$) could be controlled with some degree of success. The control of a third Van der Pol plant ($\epsilon = 5.0$), the most non-linear of the three, would have to be graded as unsatisfactory. In all cases a somewhat limited region of the T-h plane was investigated so that conclusions must be confined to this area. It is possible that more satisfactory performance might be obtained in other parts of the T-h plane but to expect this would require a great deal of optimism. Although the non-linear plants used in the study were rather limited in number, the general feasibility of the linear and non-linear control algorithms has been demonstrated. The results for the three types of basis vectors are considered in the following paragraphs.

LINEAR CONTROL POLICY RESULTS

The linear basis vector defined by equation 3-8 of paragraph 3.1 was used in both regulator and trajectory control simulation experiments. The experimental results using the linear basis vector appear to be the most positive of the basis vectors studied except for the $\epsilon = 5.0$ Van der Pol

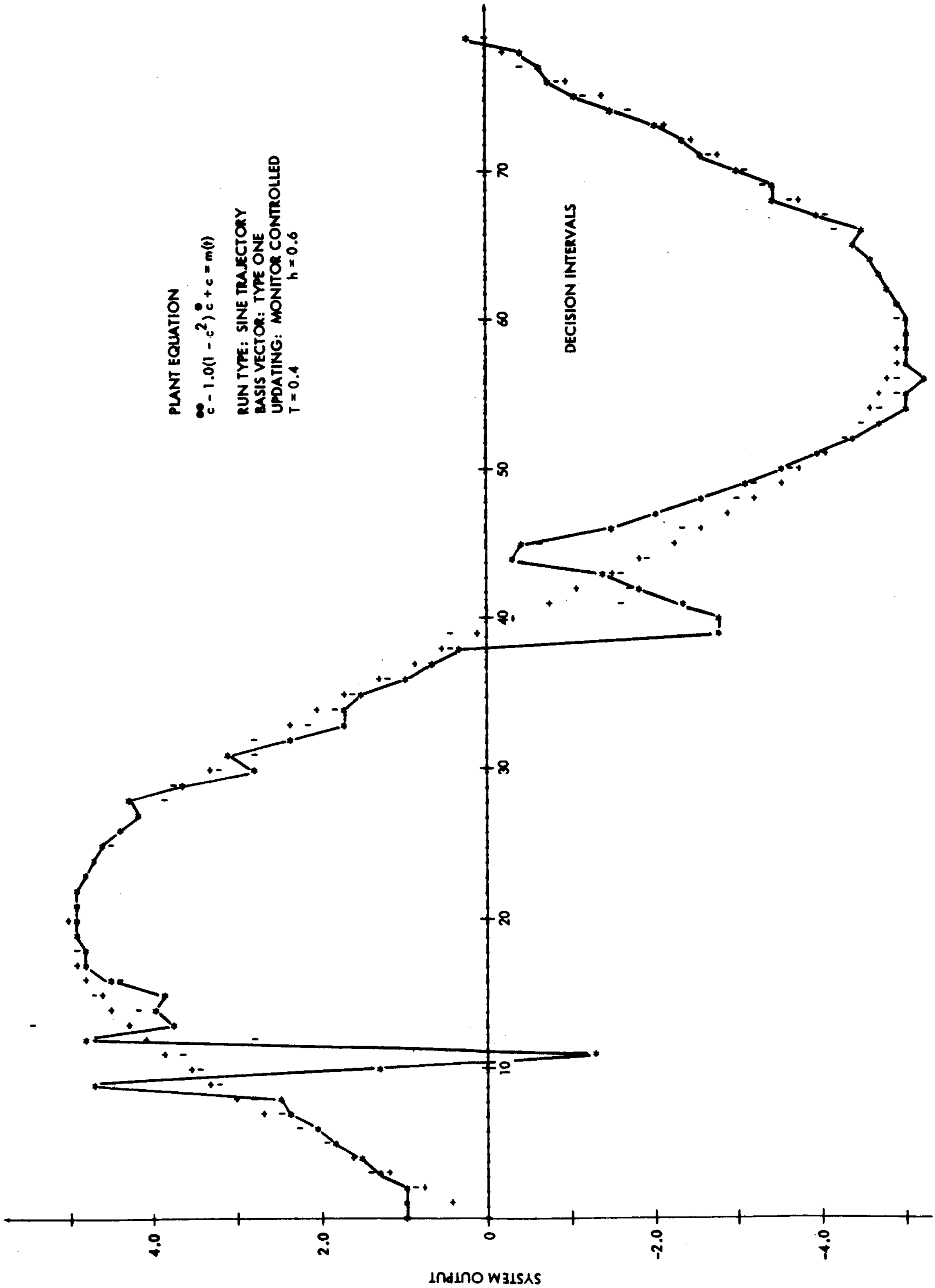


FIGURE 3-20 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE ONE BASIS VECTOR

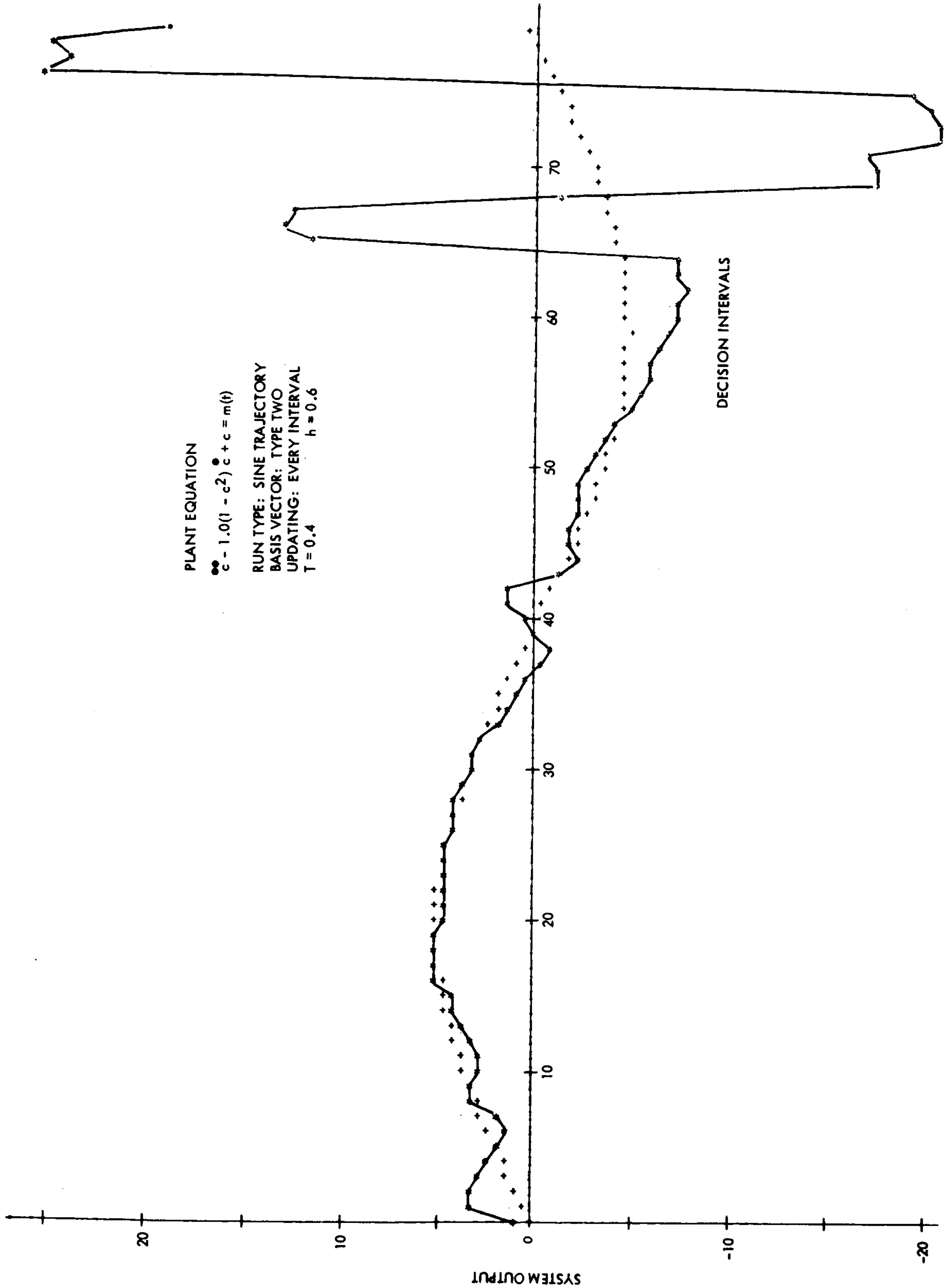


FIGURE 3-21 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE TWO BASIS VECTOR

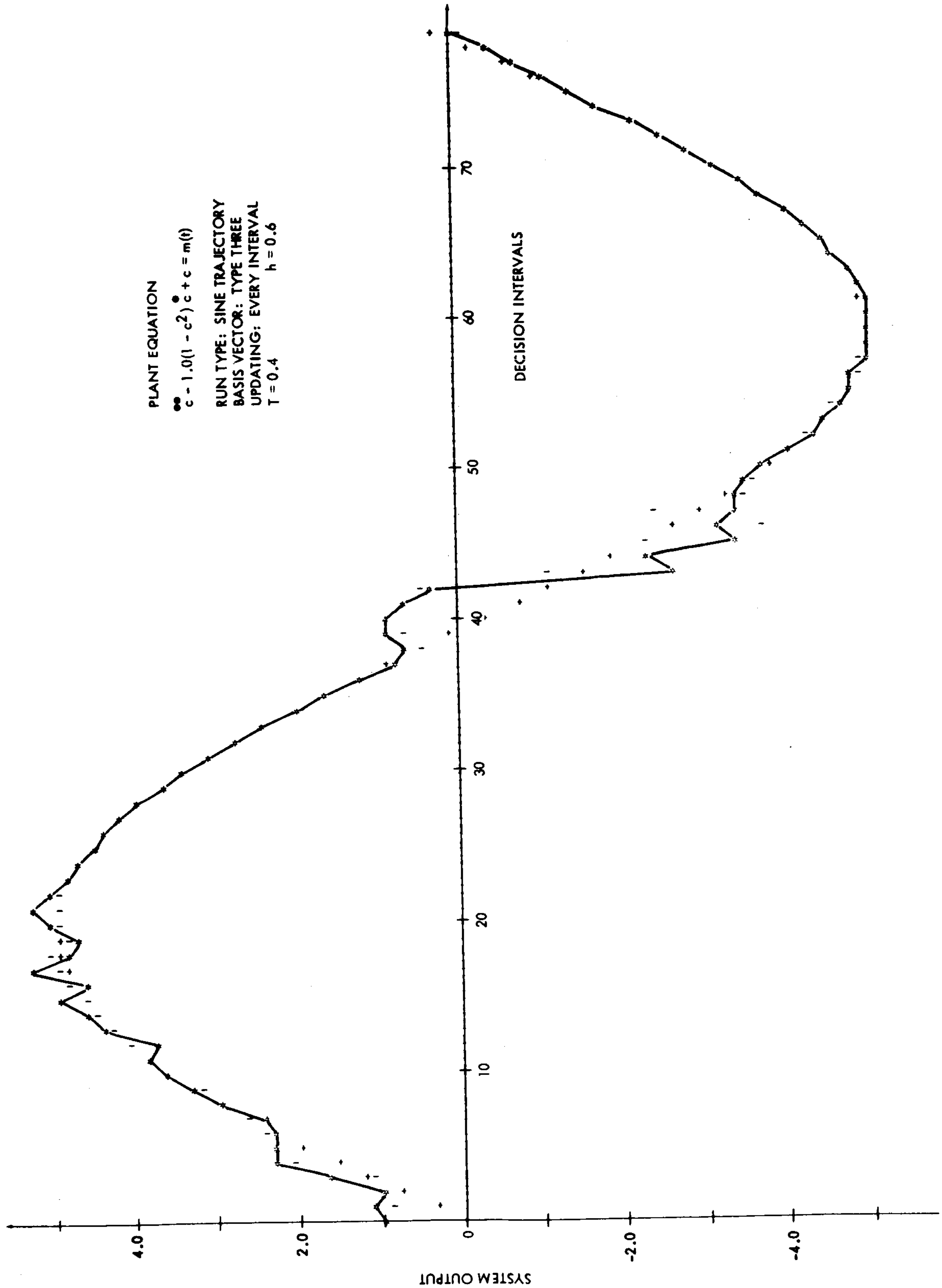


FIGURE 3-22 TRAJECTORY RESPONSE OF A VAN DER POL PLANT - TYPE THREE BASIS VECTOR

plant. The regulator control simulations at all five T-h points investigated demonstrated satisfactory performance in the cases of the $\epsilon = 0.2$ and $\epsilon = 1.0$ Van der Pol plants. The regulator response of the $\epsilon = 5.0$ Van der Pol plant was unsatisfactory at all five T-h points investigated. The interpolation estimates of the plant matrices were updated every interval during the regulator runs but even under these conditions the linear interpolation estimation procedure "broke down" when the plant became fairly non-linear. This illustrates that the limit of the description obtained using the linear basis vector exists for an ϵ value somewhere between 1.0 and 5.0.

The trajectory control simulations using a sine wave as the desired output demonstrated various degrees of tracking capability depending upon the plant and the particular T-h point. The updating monitor described in paragraph 2.1 and used in the experiments with time-varying plants was also used in the trajectory control simulations of the Van der Pol plants with the linear basis vector. The monitor dictated fairly frequent updating with the $\epsilon = 0.2$ Van der Pol plant and updating almost every interval with the $\epsilon = 1.0$ plant. No trajectory control simulations were performed for the $\epsilon = 5.0$ Van der Pol plant because of the negative results of the regulator runs.

The best tracking in the trajectory experiments occurred at the lower T-h points in the plane. Typically the best T-h point was $T = 0.4$, $h = 0.6$ with the T-h point of $T = 0.4$, $h = 1.0$ providing a slightly more sluggish response.

NON-LINEAR CONTROL POLICY RESULTS

Two non-linear basis vector descriptions were studied as defined by equations 3-9 and 3-10 of paragraph 3.1. The difference between the two was that one contained the cross-products of the state vector elements (referred to as the Type Three basis vector in paragraph 3.2) and the other did not (referred to as the Type Two basis vector).

The regulator control simulations at the five T-h points studied demonstrated varying degrees of success of the control policy with both non-linear basis vectors. In no case did all five points yield satisfactory performance for any one plant using either of the basis vector descriptions. There were four satisfactory T-h points for the $\epsilon = 0.2$ Van der Pol plant using each of the two basis vectors although not the same four. The $\epsilon = 1.0$ Van der Pol plant yielded four satisfactory T-h points using the Type Two basis vector and three using the Type Three basis vector. Only two T-h points (the lower two in the plane) gave satisfactory regulator performance using the Type Three basis vector for control of the $\epsilon = 5.0$ Van der Pol plant and none of the five T-h points yielded satisfactory performance using the Type Two basis vector.

The trajectory control simulations using a sine wave as the desired output demonstrated feasibility of the non-linear control method although in most cases the performance was no better than that obtained using the linear (Type One) basis vector and in many was not as good. As for the linear basis vector, the most satisfactory tracking occurred at the T-h point T = 0.4, h = 0.6 for both basis vectors. The Type Three basis vector which includes the state variable cross terms gave slightly better results. Based upon the more positive results obtained at small T values, trajectory runs using the $\epsilon = 1.0$ Van der Pol plant were made at the T-h point T = 0.2, h = 0.8 to see if the trend persisted. The results using the Type Two basis vector were negative. While the tracking using the Type Three basis vector exhibited extremely good initial tracking, the matrix of basis vectors very quickly became ill conditioned preventing successful updates. The cause of this may be due to the close proximity of the data points caused by a small decision interval and the size of the matrix (9 x 9).

SECTION 4

DATA TRUNCATION STUDY

4.1 APPROACH

The experimentation reported in the previous sections and the majority of the past DACS research (reference 1) has utilized the Interpolation Prediction method to obtain estimates of the plant matrices. The application of the Interpolation Prediction method to either linear or non-linear plants involves the use of measured (or estimated) output state variables. Before presenting the specific form of data truncation used in this study, a discussion of the reason for such an investigation is pertinent.

INTERPOLATION PREDICTION REVIEW

In review it may be remembered that Gorman and Zaborszky (reference 8) demonstrated the usefulness of interpolation as a simple way of selecting a continuous functional, $\tilde{x}\left[u, \underline{\eta}\right](t)$, that coincides with the system functional at measured data points when no information is available regarding the dynamic relations of the plant. Also, Ostfield (reference 9) considered in more detail the particular type of interpolation procedure which applies to the control algorithm utilized in the DACS control policy. Because the control inputs are constant over decision intervals T seconds in length, it is convenient to have the set of measured data points take the form of the initial conditions $\underline{\eta}_m$ at the beginning of M decision intervals, the control inputs, u_m (constants), during the intervals, and the outputs x_m at the end of the intervals.

The method of solution for the approximating functional takes the form of solving a determinant equation which in turn yields the following solution for the interpolation estimate of the state at t = (k+1)T:

$$\underline{\tilde{x}}((k+1)T) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k,\underline{\eta}_k) \tag{4-1}$$

where for the linear stationary case the basis vector is defined by:

$$\underline{\phi}'(u_i\underline{\eta}_i) = \underbrace{x'(iT)\ u_i} \tag{4-2}$$

The matrix of basis vectors, $\underline{\Phi}$, consists of an appropriate set of basis vectors, $\underline{\phi}_i$'s, which need not be consecutive, and ${}_D\underline{X}$ consists of a corresponding set of state variable measurements.

The partitioned matrix, $\underline{B}$ (of assumed order p), is defined as:

$$\underline{B} = {}_D\underline{X}\,\underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \;\middle|\; \underline{\phi}_1\right] \tag{4-3}$$

where $\underline{\Theta}_1$ is a pth order square matrix and $\underline{\phi}_1$ is a pth order vector. This form yields as an estimate of state at (k+1)T:

$$\underline{\tilde{x}}((k+1)T) = \underline{\Theta}_1\ \underline{x}(kT) + \underline{\phi}_1\ u_k \tag{4-4}$$

where $\underline{\Theta}_1$ and $\underline{\phi}_1$ are the interpolation estimates of the plant matrices. An equation presented for review is the associated control policy equation.

$$u_k = \frac{\underline{\phi}_1'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{\Theta}_1\ \underline{x}(kT)\right]}{\underline{\phi}_1'\ \underline{K}\ \underline{\phi}_1} \tag{4-5}$$

Based on the above equational review, the control performance is dependent on the validity of the Interpolation Prediction estimates of the plant matrices. However, these estimates, $\underline{\Theta}_1$ and $\underline{\phi}_1$, are in turn directly effected by the measurement accuracy of the output state variables. Therefore, any form of measurement accuracy investigation is certainly of considerable interest. A particular one (data truncation) was selected since it seemed a logical and convenient starting point for such an investigation. More elaborate study was beyond the scope of the study.

METHOD OF TRUNCATION

As was noted above, only truncation of measured data was considered in this limited depth study of measurement accuracy requirements for the Interpolation Prediction method. The particular form considered truncation of significant digits of each measured value. For example, an exact measurement such as 109.86541 would become 109.86500 when truncated to six significant figures and 109.00000 when truncated to three significant figures. It may be noted that round off was not considered in the type of truncation employed in this study.

Also of importance is the number of elements of the state vector which are truncated and the manner of such truncation. Referring to equation 4-3, it can be seen that the estimates of the plant matrices depend on the ${}_{D}\underline{X}$ and $\underline{\Phi}$ matrices. As was noted then, the $\underline{\Phi}$ matrix consists of the output state measurements and control forces, where as the ${}_{D}\underline{X}$ matrix consists only of output state measurements. Throughout this study the control inputs were assumed known to the actual computer accuracy (i.e. untruncated or eight place accuracy). The output state variable measurements included in both the ${}_{D}\underline{X}$ and $\underline{\Phi}$ matrices were always truncated in one of the following manners:

Type One: The entire output state vector (i.e. the output and all the derivatives) was truncated to the same number of significant figures.

Type Two: The output was considered to be known to eight place accuracy, but the derivatives were all truncated to some lower number of significant figures.

4.2 EXPERIMENTAL STUDIES

This section presents the experimental response characteristics of a selected set of linear stationary plants controlled by the DACS control policy for the above two types of output state vector truncation with various significant figures. The set of plants consisted of eight third and fourth order pole configuration transfer functions. Each order was equally divided between plants with and without an integration (i.e. a pole at the orgin). Throughout this investigation only one stable T-h combination was considered for each plant (T = 0.6, h = 0.8). Also, new data was shifted into the matrix of basis vectors every interval, and new Interpolation Prediction estimates of the plant were recalculated every fifth interval. The desired output was a sine trajectory, and the initial condition on each element of the state vector was +10 units.

The specific objective of this experimental program was to determine the effect of measurement data truncation on the control performance of the control system.

Type One Results. - The results of this investigation for both third and fourth order plants are typified by Figures 4-1 through 4-4. Figure 4-1 shows the untruncated (full measurement accuracy) control performance for one of the fourth order systems. The following three figures (4-2, 4-3, and 4-4) show the control performance for the same system, but with data measurements truncated at six, five, and four significant figures respectively. This same system was also investigated for data measurements truncated at three significant figures. In this case the controlled output quickly diverged from the desired output trajectory.

Type Two Results. - The results of this investigation for both third and fourth order plants are typified by Figures 4-5, 4-6, and 4-7. Figure 4-1 shows the control performance for the untruncated case. Control performance with the measurements truncated at six, five, and four significant figures are presented in Figures 4-5, 4-6 and 4-7. The controlled output diverged for the case where the data measurements were truncated at three significant figures.

4.3 SUMMARY OF EXPERIMENTAL RESULTS

The experimental results indicate that either Type One or Type Two state vector truncation at six significant figures provides equivalent control to that of the untruncated case. Also, that both types completely fail to provide any degree of control for truncation at three significant figures.

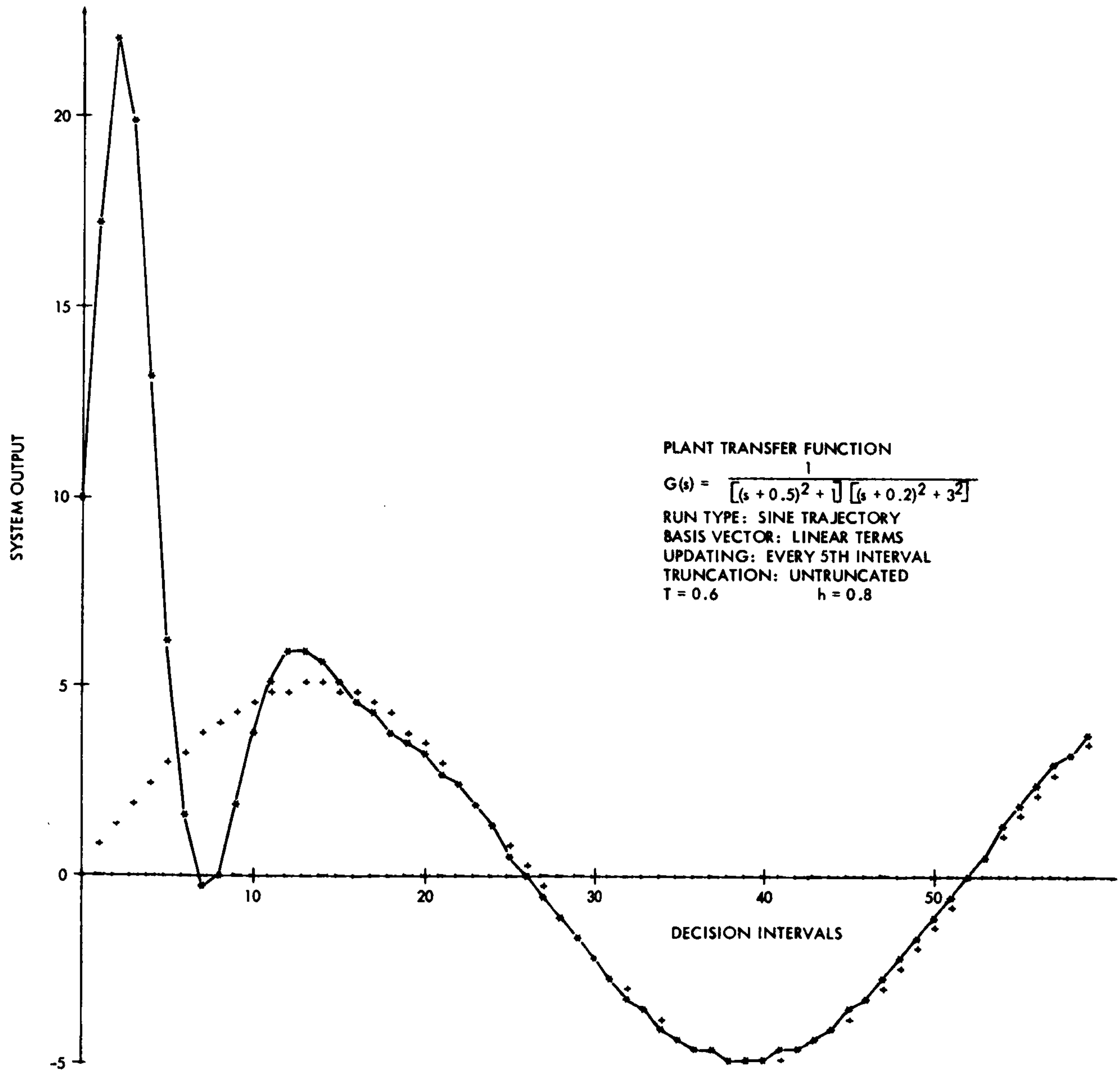


FIGURE 4-1 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - UNTRUNCATED

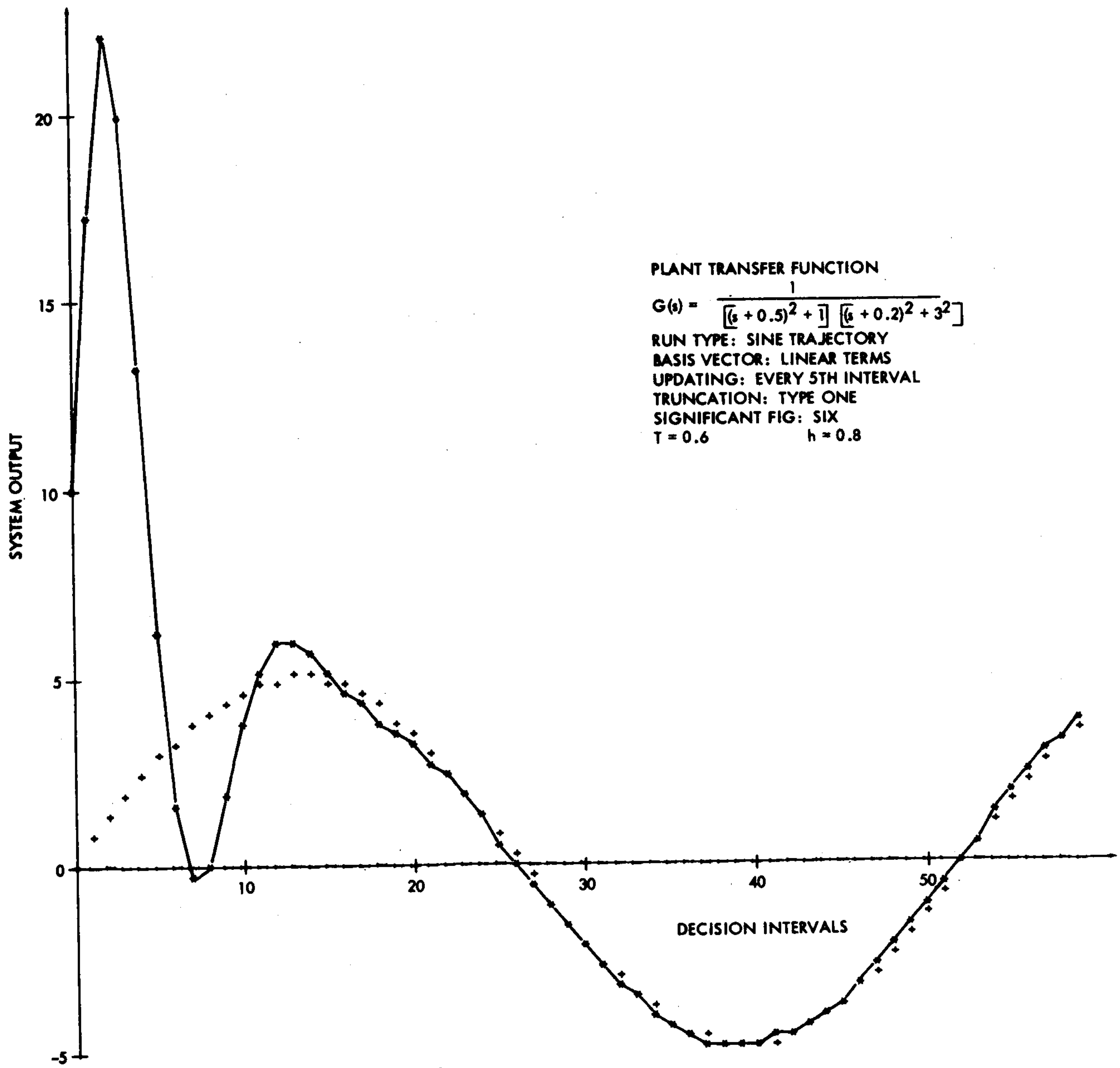


FIGURE 4-2 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - SIX SIGNIFICANT FIGURES

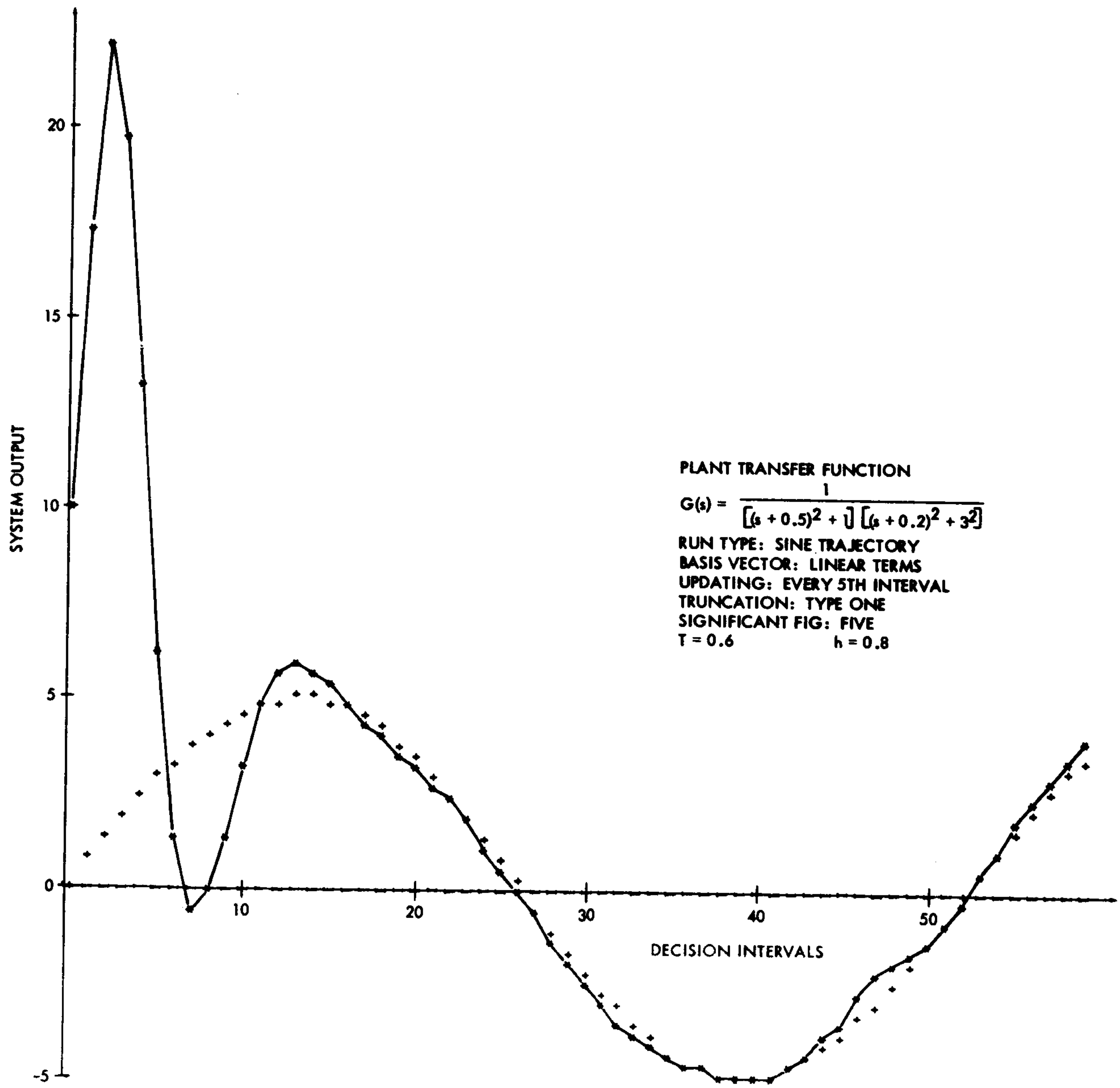


FIGURE 4-3 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - FIVE SIGNIFICANT FIGURES

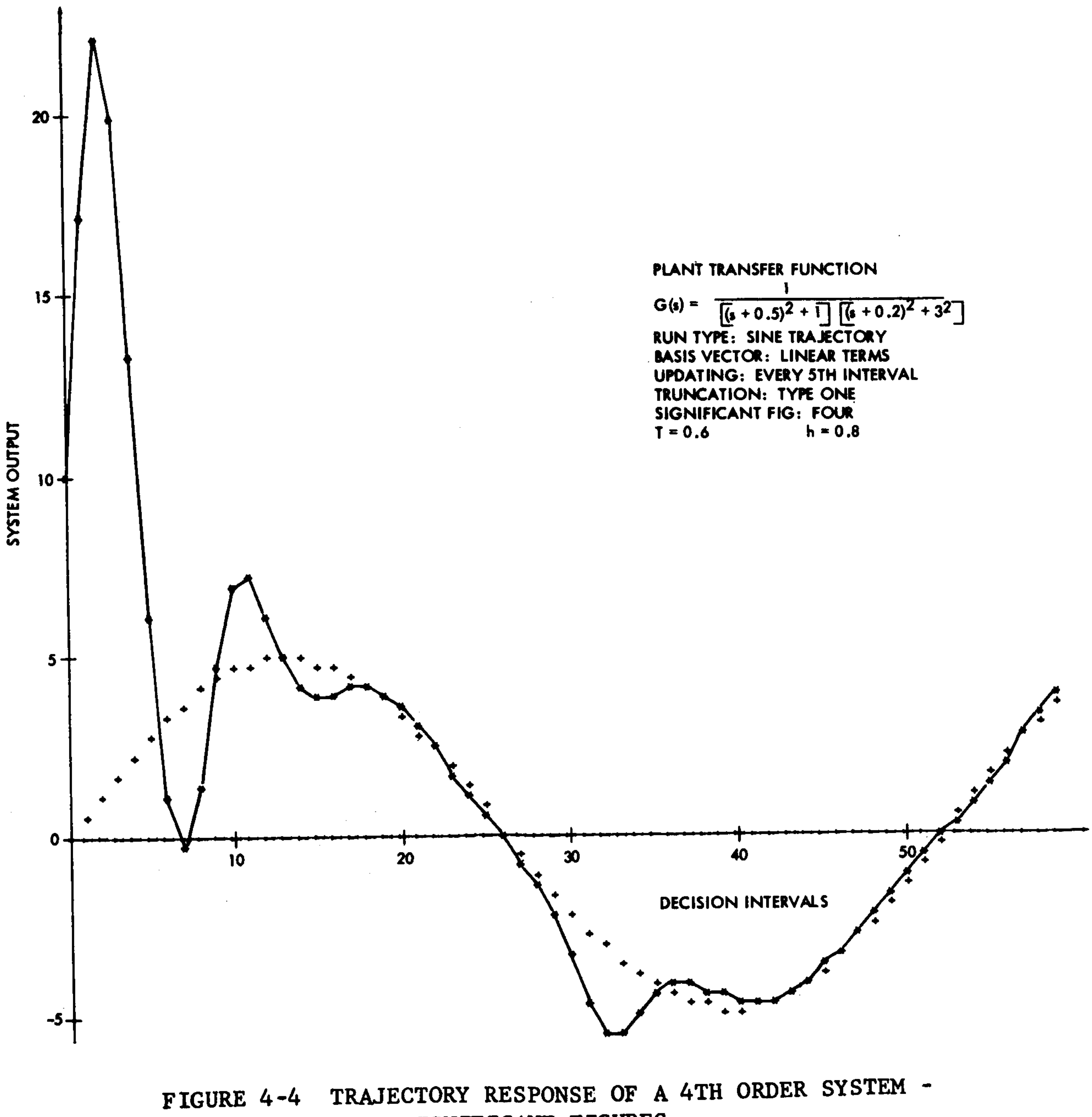


FIGURE 4-4 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - FOUR SIGNIFICANT FIGURES

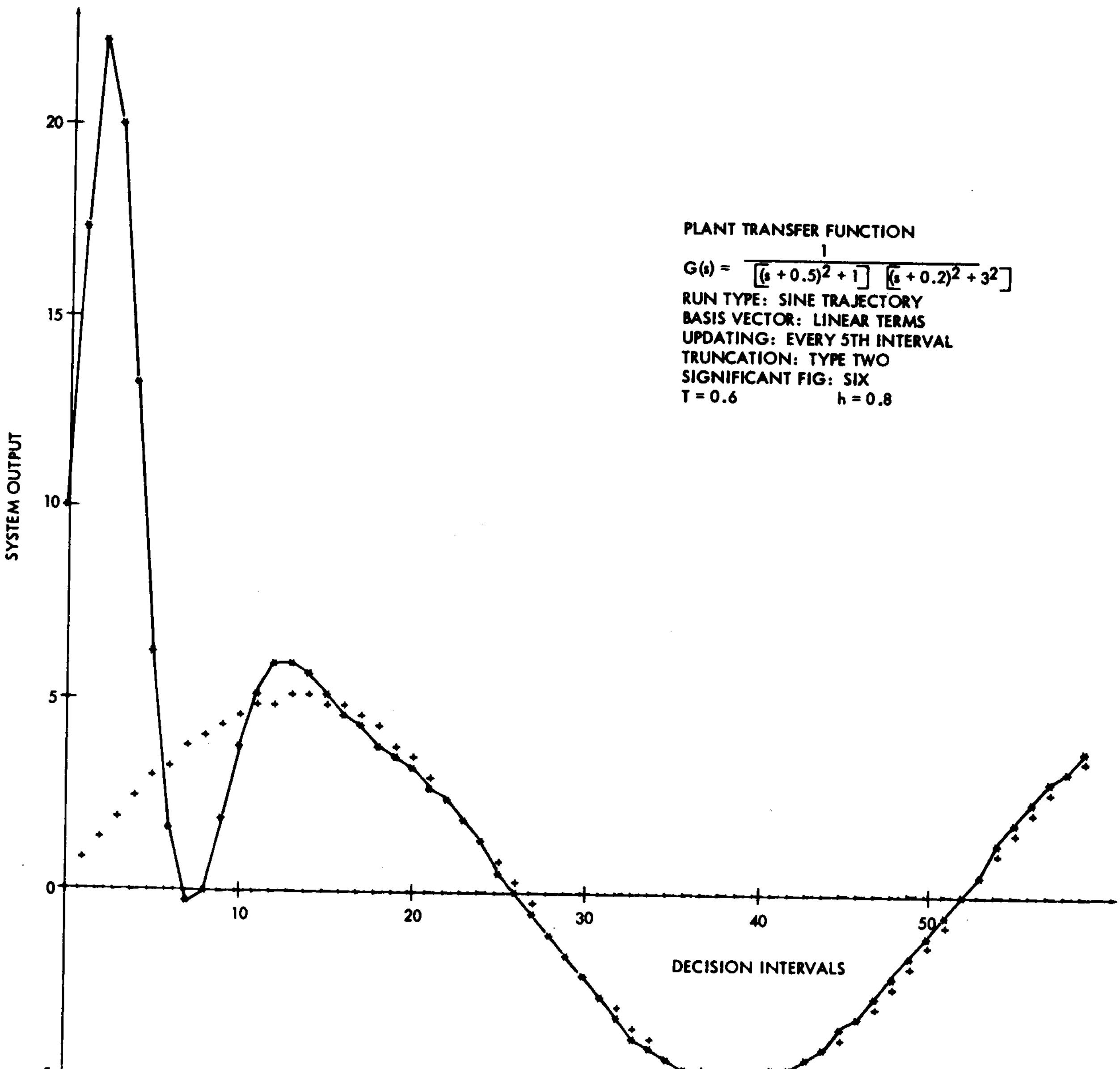


FIGURE 4-5 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - SIX SIGNIFICANT FIGURES

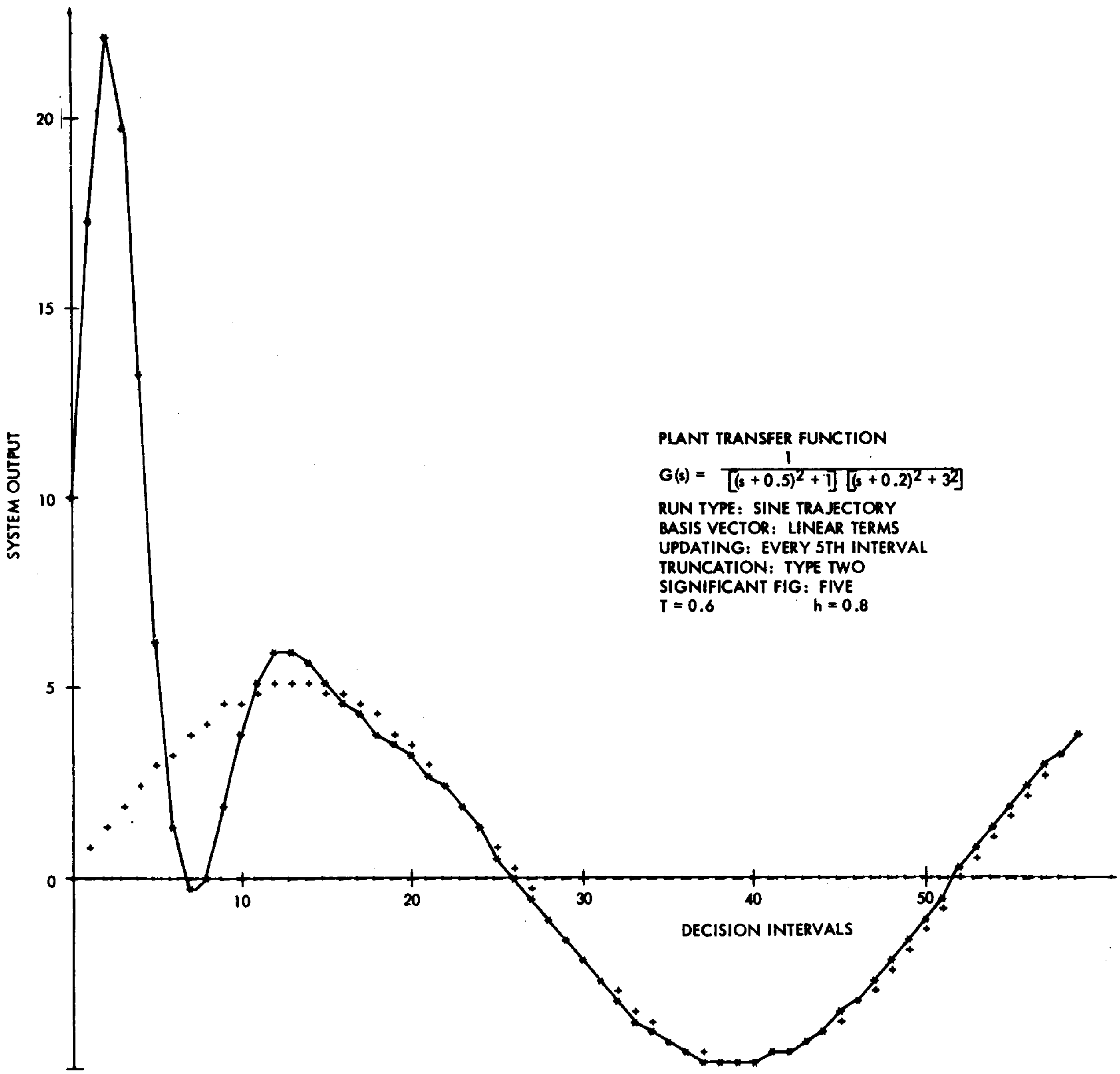


FIGURE 4-6 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - FIVE SIGNIFICANT FIGURES

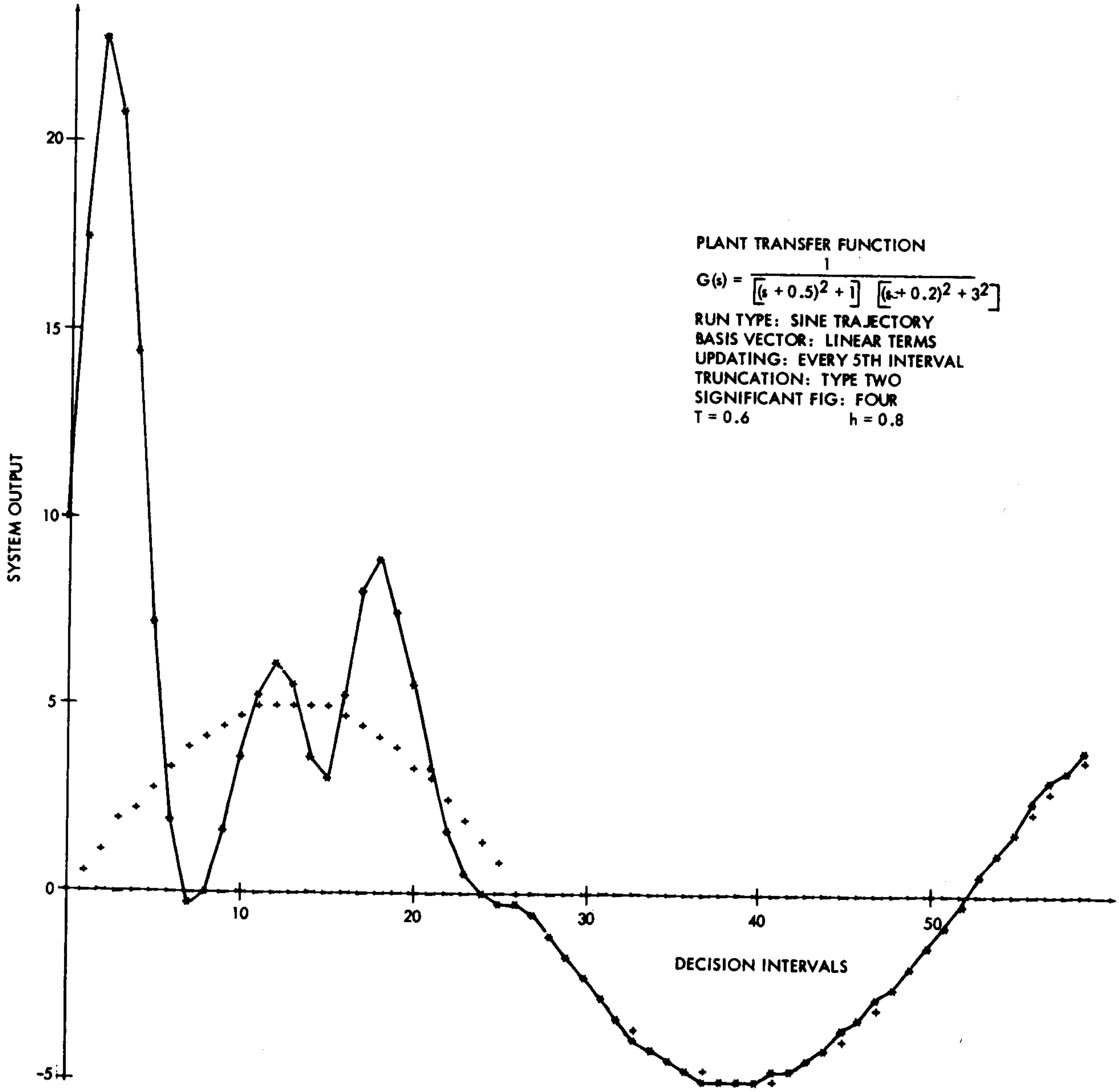


FIGURE 4-7 TRAJECTORY RESPONSE OF A 4TH ORDER SYSTEM - FOUR SIGNIFICANT FIGURES

However, at five and four significant figure truncation the picture for Type One is somewhat different from that of Type Two. For Type One with truncation at five significant figures the control is somewhat deteriorated from that of the untruncated control performance. However, Type Two truncation at five significant figures did not result in any control performance deterioration. In fact, the control is about the same as that of the untruncated case. Both types of state vector truncation at four significant figures provided about the same control performance. Poor tracking is noted in both cases as well as occurrences of poor Interpolation Prediction estimates of the plant matrices. However, with fixed updating the estimates of the plant matrices were corrected at a later time which is quite noticeable in the figures.

In summary, the experimental results show that in the worst case condition of no data smoothing at least six significant figure data measurements are required for control performance equivalent to that of the untruncated case. Also, from the Type Two investigation it may be concluded that with good measurement of the output, the higher derivatives need only be known to five significant figures for control similar to that of the untruncated case. Also, of interest is the fact that control performance could usually be rated as fair even at four significant figure truncation.

SECTION 5

RECOMMENDATIONS FOR FURTHER STUDY

The theoretical and experimental research presented in this report is to a great degree inseparably intertwined with that of reference 1. In most cases the experimental studies are extensions of areas covered briefly in the previous research and listed as primary recommendations for further study. Throughout the course of these investigations certain areas noteworthy of further study have been recognized and are summarized in the paragraphs which follow. The primacy of an ultimate practical application of the concepts developed in this research effort serves as a guide to the recommendations made here. They embody areas which require more study before practical applications may be meaningfully pursued. The recommendations are:

Definition of a more specific area of possible practical application of the control methods.

Extension of the investigation in the area of measurement requirements on both a theoretical and experimental basis.

Investigation of possible learning and pattern recognition techniques with regard to obtaining the T-h parameters for best possible control performance.

Investigation of data processing and computing techniques so as to simplify the numerical computations. An example of this is the recursive method of matrix inversion presented in Appendix D.

Investigation of possible methods for avoiding the use of inaccurate estimates of the plant dynamics obtained using the updating technique.

Each of these areas is considered in more detail in the paragraphs which follow.

The definition of a more selected area of possible practical application is perhaps one of the most important recommendations. The research effort has been conducted on as general a basis as possible to this point. It has repeatedly been necessary to arbitrarily fix some of the general problem variants while studying others allowed to remain free. An example of this is the type of plant time variations investigated. In this study sinusoidal time variation and, to a lesser extent, linear variation of one coefficient of the plant differential equation at a time was studied. It became increasingly evident during the study of linear time-varying systems and non-linear systems that the number of practical variants proliferates so greatly that a completely general research effort in these areas is impractical.

The general problem of data measurement requirements is of prime importance as far as practical application of the control method. A "first look" at the problem is presented in Section 4 where limiting the number of significant figures in the data was studied. Investigation into the area of data smoothing and filtering is necessary before practical limits may be set. Consideration of more intervals of data with the use of puesdoinverse techniques offers promise.

An investigation in the area of learning in the sense of adjustment of the T-h system parameters to provide the best possible control is important from the point of view of the adaptiveness of the control method. In many practical situations the plant dynamics could be expected to vary over relatively wide ranges. Some criterion for adjustment of the control system parameters, T and h, would be necessary to maintain adequate, preferable best, control of the system performance.

The recursive technique presented in Appendix D for obtaining the inverse of the type of matrix inherent in the interpolative procedure offers great promise but is experimentally untested. This and other numerical techniques should be investigated with the goal of minimizing the computing load on the central control computer.

A criterion for when updating of the interpolation estimates of the plant matrices should occur has been developed and studied. The experimental studies of the criterion produced promising results although the area is by no means exhausted. A companion criterion which determines when to use the updated estimate is needed. This criterion requirement is related to two problem areas. The first is the case where the matrix of basis vectors becomes ill conditioned and the numerical matric inversion techniques produce inaccurate results. In the present study this situation was handled by premultiplying the matrix of basis vectors by its computed inverse and the resultant matrix was compared to the unity matrix. This procedure proved to be sufficient to avoid inaccurate interpolation estimates due to ill conditioning. The second problem area is somewhat more obscure and is not recognizable by the unity matrix comparison. It was observed that in many instances the updated interpolation estimates resulted in poor prediction even though the matrix of basis vectors was invertable. The problem which remains to be solved centers about recognizing when the more current data contained in the interpolation matrices will yield a better estimate of the plant dynamics than the one currently being used. In many instances it was observed that better performance would have resulted if the current estimate was retained until a more accurate update was possible.

APPENDIX A

DERIVATION OF THE FIRST ORDER VOLTERRA SERIES WORKING EQUATIONS

The purpose of this appendix is to present the equational development leading to the R = 1 Volterra series working equations. The development presented here is somewhat isolated from the remainder of the text as all other sections and appendices are more related to the interpolation procedure. The R = 1 Volterra series approximation is an alternate way to estimate the dynamics of linear stationary and linear time-varying plants.

The Volterra series procedure has been thoroughly documented (reference 10) and the detailed development of the R = 2 Volterra series working equations appears in reference 1. The reader is referred to both of these documents as the underlying equational development leading up to the working equations will not be presented here.

In the way of a brief review, it is assumed that the state of a system for $t \geq nT$ may be written as:

$$\underline{x}(t) = \underline{x}_n(t) + \underline{A}_n \, \underline{u}_n \tag{A-1}$$

where:

$$\underline{x}_n = \underline{y}(t) + \sum_{k=0}^{n-1} \underline{A}_k(t) \, \underline{u}_k \tag{A-2}$$

The plant may be non-linear and time-varying but a basic assumption in equations A-1 and A-2 is that it can be approximated over finite time intervals by a finite functional polynomial of the form:

$$x(t) = y(t) + \sum_{j=1}^{J} \int_0^t \cdots \int_0^t h_j(t, \tau_1, \ \ldots \ , \tau_j) \, u(\tau_1) \ \ldots \ u(\tau_j) \, d\tau_1 \ \ldots \ d\tau_j \tag{A-3}$$

where h_j are the kernels of the functional polynomial fit.

It is further assumed that the output state, $\underline{x}(t)$, may be written in the form:

$$\underline{x}(t) = \sum_{e=0}^{P} g_{ke} \; t^e \qquad kT \leq t < (k+1)T \qquad \text{(A-4)}$$

This is simply a Taylor series expansion about some convenient point (ideally $P = \infty$) of the system output and its derivatives. The g_{ke} coefficients are linear combinations of the $A_{\langle n \rangle ps\pm}$ and the y_p coefficients (reference 10).

If it is assumed that the signal, $\underline{x}(t)$, can be measured exactly, then a definite set of g_{ke} can be established for each interval $kT \leq t < (k+1)T$. When these are equated to the expressions obtained from expressions which lead to the development of equations A-1 and A-2 from the general Volterra series, equation A-3, a set of simultaneous linear equations results which uniquely determines the $A_{\langle n \rangle ps\pm}$ and y_p coefficients.

Specifically, for the first term in g_{ke}:

$$g_{ke} = y_e + \sum_{s=0}^{S} \sum_{p=0}^{P} \binom{p}{e} \sum_{\langle n \rangle} m_{<n>} \bullet \qquad \text{(A-5)}$$

$$\left[\sum_{h=n-k+1}^{n} \left\{ U_{<n-h>} A_{<n>ps+} (-hT)^s \left[-(n-h)T \right]^{p-e} \right\} + U_{<k>} A_{<n>ps-} \left[-(n-k)T \right]^s (-kT)^{p-e} \right]$$

$k = 0, 1, 2, \; \ldots \; , n$

$e = 0, 1, 2, \; \ldots \; , P$

This will yield a sufficient number of equations if:

$$n = 2\,\mu(S+1) + 1 \tag{A-6}$$

where μ is the number of $\langle n\rangle$ sets considered significant and the determination of which is desired.

A.1 THE R = 1 CASE SIMPLIFICATIONS

The R = 1, or linear case, allows considerable simplification of the general equations. First of all, only one $\langle n\rangle$ set is significant due to the linearity of the approximation when R = 1. The $A_{\langle k\rangle}(t)$ terms reduce to one type, namely $A_k(t)$. The number, μ, is therefore automatically one and the linear version of the g_{ke} equation A-5 is given by:

$$g_{ke} = y_e + \sum_{s=o}^{S} \sum_{p=o}^{P} \binom{p}{e} \Big\{ u_{k-1} A_{(k-1)ps+} \left[-(n-k+1)\right]^s \left[-(k-1)T\right]^{p-e} + u_k\, A_{kps-} \left[-(n-k)\right]^s (-kT)^{p-e} \Big\} \tag{A-7}$$

$k = 0,1,2, \ldots, n$

$e = 0,1,2, \ldots, P$

In order to make the R = 1 case more tractable to implementation, some practical limits must be imposed. These take the form of specific values for P, the upper limit of truncated Taylor series approximations of $y(t)$ and $A_k(t)$, and S, the series expansions of the Taylor series coefficients y_p and $A_{kp\pm}$ which accounts for time variation of the plant. A linear approximation of the time variation of the plant should be sufficient in most cases (S = 1) and in many cases S = 0 may give sufficient accuracy if the plant is relatively slowly time varying.

The definition of a particular R = 1 case takes the form of specifying:

- P - The point at which the Taylor series for $y(t)$ and A_k are truncated.
- S - The degree of the polynomial fit accounting for plant time variation.

EXAMPLE CASE ONE

In this case, the $R = 1$ working equations are developed with the following assumptions:

$$P = 2$$
$$S = 0$$

Under the assumptions, equation A-7 becomes:

$$g_{ke} = y_e + \sum_{p=0}^{2} \binom{p}{e} \Big\{ u_{k-1} A_{(k-1)p+} \left[-(k-1)T\right]^{p-e} + u_k A_{kp-} (-kT)^{p-e} \Big\} \quad \text{(A-8)}$$

The number of past intervals of data which are necessary to determine the coefficients is given by equation A-6 to be three.

The coefficients for which values are needed during the interval $nT \leq t < (n+1)T$ are:

$$\left.\begin{matrix} A_{np-} \\ A_{np+} \\ y_p \end{matrix}\right\} \quad p = 0,1,2 \quad \text{(A-9)}$$

which gives a total of nine unknowns. A set of nine equations of the form of equation A-8 must be formulated in order to evaluate these coefficients. To obtain the nine equations the g_{ke} coefficients are measured in the form of equation A-4 where the expansion point is assumed to be absorbed in the coefficient. These measurements are made during three intervals of the immediate past and the equations will be formed by equating the measured g_{ke} coefficients to the unknown coefficients through equation A-7.

In this example, these equations for the interval $kT \leq t < (k+1)T$ take the form shown in equations A-10, A-11, and A-12 where, for the sake of notational brevity, $q = k-1$.

$$g_{ko} = y_0 + A_{q(0+)} u_q + A_{q(1+)} u_q(-qT) + A_{q(2+)} u_q (-qT)^2$$
$$+ A_{k(0-)} u_k + A_{k(1-)} u_k (-kT) + A_{k(2-)} u_k (-kT)^2 \qquad \text{(A-10)}$$

$$g_{k1} = y_1 + A_{q(1+)} u_q + 2A_{q(2+)} u_q (-qT) + A_{k(1-)} u_k$$
$$+ 2A_{k(2-)} u_k (-kT) \qquad \text{(A-11)}$$

$$g_{k2} = y_2 + A_{q(2+)} u_q + A_{k(2-)} u_k \qquad \text{(A-12)}$$

A total of nine equations is obtained if k assumes three values corresponding to three intervals of the immediate past. The corresponding $A_{kp\pm}$ coefficients of the different intervals may be set equal to each other and equated to $A_{np\pm}$ which gives a total of nine equations and nine unknowns. In matric form the set of nine equations can be expressed compactly as:

$$\underline{M}\, \underline{\alpha} = \underline{g} \qquad \text{(A-13)}$$

where:

$$\underline{g}' = \lfloor g_{k_1 0}\ g_{k_1 1}\ g_{k_1 2} \cdot \cdot \cdot \cdot\ g_{k_3 0}\ g_{k_3 1}\ g_{k_3 2} \rfloor \qquad \text{(A-14)}$$

and:

$$\underline{\alpha}' = \lfloor y_0\ y_1\ y_2\ A_{n(0+)}\ A_{n(1+)}\ A_{n(2+)}\ A_{n(0-)}\ A_{n(1-)}\ A_{n(2-)} \rfloor \qquad \text{(A-15)}$$

The matrix $\underline{M}$ is a 9 x 9 square matrix consisting of the known constants and control forces of equations A-10, A-11, and A-12. It is possible to partition $\underline{M}$ in such a way that inversion of the full matrix is not required. Instead, a partial solution is obtained from the equation resulting from the partitioned matrix $\underline{M}$:

$$\underline{\hat{M}}\, \underline{\hat{\alpha}}_1 = \underline{\hat{g}} \qquad \text{(A-16)}$$

where:

$$\underline{\hat{g}}' = \lfloor g_{k_1 2}\ g_{k_2 2}\ g_{k_3 2} \rfloor \qquad \text{(A-17)}$$

$$\underline{\hat{\alpha}}' = \lfloor y_2\ A_{n(2+)}\ A_{n(2-)} \rfloor \qquad \text{(A-18)}$$

and $\underline{\hat{M}}$ is a 3 x 3 matrix given by:

$$\underline{\hat{M}} = \begin{bmatrix} 1 & u_{k_1-1} & u_{k_1} \\ 1 & u_{k_1} & u_{k_2} \\ 1 & u_{k_2} & u_{k_3} \end{bmatrix} \tag{A-19}$$

By inverting $\underline{\hat{M}}$, a solution is obtained for the unknowns contained in $\underline{\hat{\alpha}}_1$.

$$\underline{\hat{\alpha}}_1 = \underline{\hat{M}}^{-1} \underline{\hat{g}} \tag{A-20}$$

Having found values for three of the unknowns, a solution for another three is obtained in the form:

$$\underline{\hat{\alpha}}_2 = \underline{\hat{M}}^{-1} \underline{\beta}_1 \tag{A-21}$$

where $\underline{\hat{M}}$ is the same matrix as given in equation A-19. The quantity $\underline{\hat{\alpha}}_2$ is defined as:

$$\underline{\hat{\alpha}}_2' = \lfloor y_1 \; A_{n(1+)} \; A_{n(1-)} \rfloor \tag{A-22}$$

and $\underline{\beta}_1$ is a vector function of the $g_{k_i 1}$ and the solution for the quantities contained in $\underline{\hat{\alpha}}_1$.

Similarly, a solution is obtained for the remaining unknowns:

$$\underline{\hat{\alpha}}_3 = \underline{\hat{M}}^{-1} \underline{\beta}_2 \tag{A-23}$$

where $\underline{\hat{M}}$ is again defined by equation A-19 and:

$$\underline{\hat{\alpha}}_3 = \lfloor y_o \; A_{n(0+)} \; A_{n(0-)} \rfloor \tag{A-24}$$

The vector $\underline{\beta}_2$ is a function of the $g_{k_i 0}$ and the previously determined unknowns in $\underline{\hat{\alpha}}_1$ and $\underline{\hat{\alpha}}_2$.

Assuming a second order system, the output state is given by:

$$x(t) = \sum_{e=0}^{2} g_{ne}\, t^{e}$$

$$\dot{x}(t) = \sum_{e=0}^{2} e\, g_{ne}\, t^{e-1} \qquad nT \leq t < (n+1)T \qquad \text{(A-25)}$$

where the g_{ne} coefficients are obtained by substituting the solutions for the y_p and $A_{kp\pm}$ coefficients along with the control forces u_n and u_{n-1} into equations A-10, A-11, and A-12.

The state vector, $\underline{x}(t)$, may be approximated during the interval $nT \leq t < (n+1)T$ by an equation of the form:

$$\begin{bmatrix} x(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_{n1}(t) \\ \dot{x}_{n2}(t) \end{bmatrix} + \begin{bmatrix} a_{n1} \\ a_{n2} \end{bmatrix} u_n \qquad \text{(A-26)}$$

where equation A-26 follows from equation A-25 by proper division of the terms contained in the solutions for the g_{ne} coefficients.

The matrix $\hat{\underline{M}}$ must be updated every interval by shifting the most recent control forces into the last row of $\hat{\underline{M}}$ (equation A-19) and shifting the other rows in the matrix up one thereby shifting the first row of 'oldest' control forces out of the matrix.

EXAMPLE CASE TWO

In this case, the degree of the Taylor series expansions is raised by one. The case is specified by:

$P = 3$

$S = 0$

Under these assumptions equation A-7 becomes:

$$g_{ke} = y_e + \sum_{p=0}^{3} \binom{p}{e} \Big\{ u_{k-1} A_{(k-1)p+} \left[-(k-1)T\right]^{p-e} + u_k A_{kp-} (-kT)^{p-e} \Big\} \tag{A-27}$$

The number of past intervals of data required is given by equation A-6 to be three.

The unknown coefficients to be determined are:

$$\left.\begin{array}{l} A_{np+} \\ A_{np-} \\ y_p \end{array}\right\} \quad p = 0,1,2,3 \tag{A-28}$$

where in this case the number of unknowns is 12. The solution proceeds in a manner identical to that of the first example case except for the presence of terms corresponding to $p = 3$. The matrix $\underline{\hat{M}}$ is identical to that defined by equation A-19 however a $\underline{\beta}_3$ vector exists and four matrix multiplications of the form of equations A-20, A-21, and A-23 of example case one are necessary to solve for all of the unknown y_p and $A_{kp\pm}$ coefficients.

EXAMPLE CASE THREE

In this case a time-varying plant is assumed.

$P = 2$

$S = 1$

Under these assumptions, equation A-7 becomes:

$$g_{ke} = y_e + \sum_{s=0}^{1} \sum_{p=0}^{2} \binom{p}{e} \Big\{ u_{k-1} A_{(k-1)ps+} \left[-(n-k+1)T\right]^{s} \left[-(k-1)T\right]^{p-e} + u_k A_{kps-} \left[-(n-k)T\right]^{s} (-kT)^{p-e} \Big\} \tag{A-29}$$

The past intervals of data necessary to determine the coefficients is given by equation A-6 to be five. The unknown coefficients to be determined are:

$$\left.\begin{array}{l} A_{nps+} \\ A_{nps-} \\ y_p \end{array}\right\} \quad \begin{array}{l} p = 0,1,2 \\ s = 0,1 \end{array} \qquad \text{(A-30)}$$

where the number of unknowns in this case is 15. The solution proceeds in a manner very similar to that of example case one except that there are now two A_{kp} terms on the right-hand sides of equations equivalent to equations A-10, A-11, and A-12 of example case one because s takes on values of 0 and 1 whereas only the s = 0 terms were considered before. The matrix $\hat{\underline{M}}$ in this case will be 5 x 5 corresponding to the five unknowns for each value of p. Once $\hat{\underline{M}}$ is inverted, it can be used repeatedly in the three matric multiplications needed to solve for all 15 unknowns. Once the solutions for the g_{ne} coefficients are obtained, the solutions for $\underline{x}(t)$ may be placed in a form identical to equation A-26.

A.2 CONCLUSIONS

The R = 1 Volterra series equations are obtained in a procedure very similar to that used for the R = 2 working equations (reference 1). The corresponding simplifications that occur in reducing the assumed order of the Volterra series are reflected in simpler equations in the R = 1 case. The matrix which must be inverted is again a function of the control forces and the size of the required matric inversion is also less than the number of unknowns.

A summary of some important equations is given below:

1. The number of intervals of data required to identify the unknown coefficients:

$$n = 2(S+1) + 1 \qquad \text{(A-31)}$$

2. The number of unknowns:

$$\text{unknowns} = (P+1)\left\{2(S+1) + 1\right\} \tag{A-32}$$

3. The size of the matrix which must be inverted to identify the unknown coefficients:

$$\text{size} = 2(S+1) + 1 \tag{A-33}$$

4. The number of matric multiplications which must be performed to identify the unknown coefficients:

$$\text{Multiplications} = P + 1 \tag{A-34}$$

Note that the size of the matrix which must be inverted is equal to the number of intervals of data required and is independent of the order of the Taylor series approximating the $A_k(t)$ terms except as it is reflected through the magnitude of S. The order of the Taylor series determines the number of matric multiplications which must be made.

APPENDIX B

A SELECTED SET OF LINEAR TIME-VARYING PLANTS

The set of linear time-varying plants (equations) documented in this appendix numbers thirth-five and represents the plants which were used for the majority of all the experimental investigations. The plants range from third through fifth order. Also, as was previously noted, this set is restricted to plants which are not sensitive to derivatives of the input (See Section 2).

Upon inspection of these plants several obvious observations are:

> All plants contain only one (1) time-varying coefficient.
>
> All plants have this one coefficient vary as a sine function of time.
>
> In all cases, except for the plants with time-varying gain, two (2) speeds of time variation are considered for each plant configuration.

Also, it may be noted that the majority of the plants considered are of fourth order, and that a time-varying gain plant of each order is included in the set. This set of plants was selected since the experimental funds were not unlimited, and so it was necessary to limit the spectrum of plants to those which are somewhat restricted by the above three observations.

B.1 THIRD ORDER PLANTS

(3-1) $\dddot{c}+(0.6 + 0.3 \sin 0.125t)\, \ddot{c} + \dot{c} = m(t)$

(3-2) $\dddot{c}+(0.6 + 0.3 \sin 0.25t)\, \ddot{c} + \dot{c} = m(t)$

(3-3) $\dddot{c}+0.6\ddot{c} + (1+0.5 \sin 0.125t)\, \dot{c} = m(t)$

(3-4) $\dddot{c}+0.6\ddot{c} + (1+0.5 \sin 0.25t)\, \dot{c} = m(t)$

(3-5) $\dddot{c}+(1.6+0.4 \sin 0.125t)\,\ddot{c} + 1.6\dot{c} + c = m(t)$

(3-6) $\dddot{c}+(1.6+0.4 \sin 0.25t)\,\ddot{c} + 1.6\dot{c} + c = m(t)$

(3-7) $\dddot{c}+1.6\ddot{c} + (1.6+0.4 \sin 0.125t)\,\dot{c} + c = m(t)$

(3-8) $\dddot{c}+1.6\ddot{c} + (1.6+0.4 \sin 0.25t)\,\dot{c} + c = m(t)$

(3-9) $\dddot{c}+1.6\ddot{c} + 1.6\dot{c} + (1+0.25 \sin 0.125t)\,c = m(t)$

(3-10) $\dddot{c}+1.6\ddot{c} + 1.6\dot{c} + (1+0.25 \sin 0.25t)\,c = m(t)$

B.2 FOURTH ORDER PLANTS

(4-1) $c^{(4)} + (1.6+0.4 \sin 0.125t)\,\dddot{c} + 1.6\ddot{c} + \dot{c} = m(t)$

(4-2) $c^{(4)} + (1.6+0.4 \sin 0.25t)\,\dddot{c} + 1.6\ddot{c} + \dot{c} = m(t)$

(4-3) $c^{(4)} + 1.6\dddot{c} + (1.6+0.4 \sin 0.125t)\,\ddot{c} + \dot{c} = m(t)$

(4-4) $c^{(4)} + 1.6\dddot{c} + (1.6+0.4 \sin 0.25t)\,\ddot{c} + \dot{c} = m(t)$

(4-5) $c^{(4)} + 1.6\dddot{c} + 1.6\ddot{c} + (1+0.25 \sin 0.125t)\,\dot{c} = m(t)$

(4-6) $c^{(4)} + 1.6\dddot{c} + 1.6\ddot{c} + (1+0.25 \sin 0.25t)\,\dot{c} = m(t)$

(4-7) $c^{(4)} + (3+0.75 \sin 0.125t)\,\dddot{c} + 13.25\ddot{c} + 11.25\dot{c} + 12.5c = m(t)$

(4-8) $c^{(4)} + (3+0.75 \sin 0.25t)\,\dddot{c} + 13.25\ddot{c} + 11.25\dot{c} + 12.5c = m(t)$

(4-9) $c^{(4)} + 3\dddot{c} + (13.25 + 3.3125 \sin 0.125t)\,\ddot{c} + 11.25\dot{c} + 12.5c = m(t)$

(4-10) $c^{(4)} + 3\dddot{c} + (13.25 + 3.3125 \sin 0.25t)\,\ddot{c} + 11.25\dot{c} + 12.5c = m(t)$

(4-11) $c^{(4)} + 5\dddot{c} + 8.25\ddot{c} + 8\dot{c} + (3.75 + 1.875 \sin 0.125t)\,c = m(t)$

(4-12) $c^{(4)} + 5\dddot{c} + 8.25\ddot{c} + 8\dot{c} + (3.75 + 1.875 \sin 0.25t)\,c = m(t)$

(4-13) $c^{(4)} + 5\dddot{c} + 8.25\ddot{c} + (8+2 \sin 0.125t)\,\dot{c} + 3.75c = m(t)$

(4-14) $c^{(4)} + 5\dddot{c} + 8.25\ddot{c} + (8+2 \sin 0.25t)\,\dot{c} + 3.75c = m(t)$

(4-15) $c^{(4)} + 5\dddot{c} + (8.25 + 2.0625 \sin 0.125t)\,\ddot{c} + 8\dot{c} + 3.75c = m(t)$

(4-16) $c^{(4)} + 5\dddot{c} + (8.25 + 2.0625 \sin 0.25t)\,\ddot{c} + 8\dot{c} + 3.75c = m(t)$

B.3 FIFTH ORDER PLANTS

(5-1) $\overset{(5)}{c} + 5\overset{(4)}{c} + 8.25\dddot{c} + 8\ddot{c} + (3.75 + 1.875 \sin 0.125t)\,\dot{c} = m(t)$

(5-2) $\overset{(5)}{c} + 5\overset{(4)}{c} + 8.25\dddot{c} + 8\ddot{c} + (3.75 + 1.875 \sin 0.25t)\,\dot{c} = m(t)$

(5-3) $\overset{(5)}{c} + 5\overset{(4)}{c} + 8.25\dddot{c} + (8+2 \sin 0.125t)\,\ddot{c} + 3.75\,\dot{c} = m(t)$

(5-4) $\overset{(5)}{c} + 5\overset{(4)}{c} + 8.25\dddot{c} + (8+2 \sin 0.25t)\,\ddot{c} + 3.75\dot{c} = m(t)$

(5-5) $\overset{(5)}{c} + (3 + 1.5 \sin 0.125t)\overset{(4)}{c} + 13.25\dddot{c} + 11.25\ddot{c} + 12.5\dot{c} = m(t)$

(5-6) $\overset{(5)}{c} + (3+1.5 \sin 0.25t)\overset{(4)}{c} + 13.25\dddot{c} + 11.25\ddot{c} + 12.5\dot{c} = m(t)$

B.4 TIME-VARYING GAIN PLANTS

(3-11) $\dddot{c} + \ddot{c} + 1.25\dot{c} = [1. + 0.5 \sin 0.25t]\ m(t)$

(4-17) $\overset{(4)}{c} + 2\dddot{c} + 2.25\ddot{c} + 1.25\dot{c} = [1 + 0.5 \sin 0.25t]\ m(t)$

(5-7) $\overset{(5)}{c} + 7\overset{(4)}{c} + 12.25\dddot{c} + 12.5\ddot{c} + 6.25\dot{c} = [1 + 0.5 \sin 0.25t]\ m(t)$

B.5 A SUBSET OF LINEAR TIME-VARYING PLANTS

The following subset of third order plants was mainly used for the experimentation presented under the 'STATIONARY VS TIME-VARYING BASIS VECTOR RESULTS' heading of paragraph 2.2. These plants again have only one time-varying coefficient. However, only linear time variation of the coefficient is considered by this set of plants. The two basic configurations considered are:

(1) $\dddot{c} + (0.4 + at)\,\ddot{c} + \dot{c} = m(t)$

(2) $\dddot{c} + 1.6\ddot{c} + 1.6\dot{c} + (1+at)\ c = m(t)$

where the values of "a" considered in both cases are as follows: 0.02, 0.04, 0.08, 0.12, 0.16, 0.2, 0.3, 0.5, and 1.0.

APPENDIX C

INCLUSION OF TIME VARIATION IN THE INTERPOLATION PROCEDURE

The usefulness of interpolation has been demonstrated (reference 8) and documented (reference 1) as a particularly simple way of selecting a continuous functional that coincides with the actual system functional at measured data points. The data points take the form of a set of measured values for the system input u_m, initial conditions $\underline{\eta}_m$, and the system output x_m, $m = 1, 2, \ldots, M$. The set of measured items may include a variety of quantities which are assumed to affect the dynamics of the plant. The specific concern of this appendix is the inclusion of time in such a way as to account for the time dependent nature of the dynamics of time-varying plants.

A general development of the interpolation equations applicable to this study appears in reference 1. They are reviewed here for the sake of continuity.

C.1 THE BASIC INTERPOLATION EQUATIONS

Because the control inputs to the plant are constant over decision intervals T seconds in length, it is convenient to have the set of measured values take the form of the initial conditions, $\underline{\eta}_m$, at the beginning of the decision intervals, the control inputs, u_m (constants), applied during the decision intervals, and the outputs, x_m, at the end of the intervals.

The method of solution for the approximating functional, $\tilde{x}(u, \underline{\eta})$, takes the form of solving the determinant equation:

$$\text{Det}\begin{bmatrix} \tilde{x}(u,\underline{\eta}) & x_1(u_1,\underline{\eta}_1) & \cdot & \cdot & \cdot & x_M(u_M,\underline{\eta}_M) \\ \underline{\phi}(u,\underline{\eta}) & \underline{\phi}_1(u_1,\underline{\eta}_1) & \cdot & \cdot & \cdot & \underline{\phi}_M(u_M,\underline{\eta}_M) \end{bmatrix} = 0 \qquad \text{(C-1)}$$

where $x_1, x_2, \ldots, x_M$ are the measured outputs of this system at the end of decision intervals. $\underline{\phi}(u,\underline{\eta})$ is a basis vector of M linearly independent analytic base functionals $\left[u, \underline{\eta}\right]$ upon which the system dynamics is assumed to depend, and $\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_M$ are the basis vector evaluated at the measured data points, $u_1\underline{\eta}_1, u_2\underline{\eta}_2, \ldots, u_M\,\underline{\eta}_M$.

Equation C-1 may be expanded in terms of minors of the first column, yielding as a solution for $\tilde{x}(u,\underline{\eta})$:

$$\tilde{x}(u,\underline{\eta}) = {}_D\underline{x}'\ \underline{\Phi}^{-1}\ \underline{\phi}(u,\underline{\eta}) \qquad \text{(C-2)}$$

where ${}_D\underline{x}$ and $\underline{\Phi}$ are defined by equations C-3 and C-4:

$$ {}_D\underline{x}' = \left\lfloor x_1\ x_2\ \cdot\ \cdot\ \cdot\ x_M \right\rfloor \qquad \text{(C-3)}$$

$$\underline{\Phi} = \left[\underline{\phi}_1(u_1,\underline{\eta}_1) \quad \underline{\phi}_2(u_2,\underline{\eta}_2) \quad \cdot\ \cdot\ \cdot\ \cdot \quad \underline{\phi}_M(u_M,\underline{\eta}_M)\right] \qquad \text{(C-4)}$$

An approximating functional for each of the elements of the output state vector elements may be obtained by replacing each of the measured output states x_m in equation C-1 by the measured state variable elements $\overset{i}{x}_m$ yielding as a solution for the functional approximation of the ith state variable:

$$\overset{i}{\tilde{x}}(u,\underline{\eta}) = {}_D\overset{i}{\underline{x}}'\ \underline{\Phi}^{-1}\ \underline{\phi}(u,\underline{\eta}) \qquad \text{(C-5)}$$

where $\underline{\Phi}$ is identical to that defined by equation C-4 and ${}_D\overset{i}{\underline{x}}'$ is defined by equation C-6:

$$ {}_D\overset{i}{\underline{x}}' = \left\lfloor \overset{i}{x}_1, \overset{i}{x}_2, \ldots, \overset{i}{x}_M \right\rfloor \qquad \text{(C-6)}$$

The solution for the total state vector or whatever part of it is desired or measureable is conveniently combined into the single equation

$$\underline{\tilde{X}}(u,\underline{\eta}) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u,\underline{\eta}) \tag{C-7}$$

where ${}_D\underline{X}$ is the rectangular matrix:

$${}_D\underline{X} = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_M \\ \dot{x}_1 & \dot{x}_2 & \cdot & \cdot & \cdot & \dot{x}_M \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overset{(p)}{x_1} & \overset{(p)}{x_2} & \cdot & \cdot & \cdot & \overset{(p)}{x_M} \end{bmatrix} \tag{C-8}$$

and p is the number of elements of the system state vector for which estimates are obtained. The matrix of basis vectors, $\underline{\Phi}$, is defined by equation C-4. Due to the discrete nature of the interpolation equation, it is convenient to rewrite equation C-7 in the form:

$$\underline{\tilde{X}}((k+1)T) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k,\underline{\eta}_k) \tag{C-9}$$

where the estimate of the state at the end of the kth decision interval, $\underline{\tilde{X}}((k+1)T)$, is given in terms of ${}_D\underline{X}$, $\underline{\Phi}$, and the basis vector evaluated at the beginning of the interval, $\underline{\phi}(u_k,\underline{\eta}_k)$.

The number of measurements required to determine $\underline{\Phi}$ and ${}_D\underline{X}$ is equal to the dimension of the basis vector $\underline{\phi}(u,\underline{\eta})$. Thus, if the dimension of $\underline{\phi}$ is M then a total of M decision intervals are required to specify the interpolation estimate.

The data contained in ${}_D\underline{X}$ and $\underline{\Phi}$ may be changed from time to time by shifting a new column of data in and dropping one of the columns corresponding to older data out of the matrices. This process is conveniently mechanized by shifting the new columns of data in one end of the matrices and dropping the first or last column of data out depending upon which end of the new data is shifted into. The columns of data in the matrices shift from right to left (or vice-versa) as more new data is sequentially shifted in.

C.2 TIME VARIATION USING CONSTANT T

One method of accounting for plant time variations is to include the length of the decision interval, T, in the interpolation basis vector. This technique followed directly from the description of stationary plants where the basis vector was defined by:

$$\underline{\phi}'(u,\underline{\eta}) = \underbrace{\underline{x}'(t)\ u}\tag{C-10}$$

With reference to equation C-5, the interpolation estimate of the ith state vector element is given by equation C-11:

$$\overset{i}{x}((k+1)T) = \sum_{j=1}^{p} a_{ij}\ x_j(kT) + b_i\ u_k\tag{C-11}$$

where the a_{ij} and b_i coefficients are given by the product $_D\underline{x}'\ \underline{\Phi}^{-1}$. To account for time variation of the plant, a linear term in t may be added to equation C-11 so that the interpolation estimate of the ith state vector element would be given by:

$$\overset{i}{x}((k+1)T) = \sum_{j=1}^{p} a_{ij}\ x_j(kT) + b_i\ u_k + c_i\ T\tag{C-12}$$

where in this case a_{ij}, b_i and c_i are given by the product $_D\underline{x}'\ \underline{\Phi}^{-1}$. The inherent time reference is the time at which the initial state of the system, $\underline{x}(kT)$, occurs so that the time at which the final state occurs is always T seconds later. Using this estimate for successive decision intervals reestablishes the time base each time to that of the initial conditions. The basis vector for this interpolation estimate is:

$$\underline{\phi}'(u_i,\underline{\eta}_i) = \underbrace{\underline{x}'(iT)\ u_i\ T}\tag{C-13}$$

This method of accounting for time variations is presented in reference 1 along with experimental results.

C.3 TIME VARIATION USING RUNNING TIME

A second method of accounting for time variations utilizes a running time base. This method follows more directly from the plant dynamical equations than the first method.

Consider the vector differential equation of the plant as given by equation 2-2 of paragraph 2.1 of the text:

$$\dot{\underline{x}}(t) = \underline{H}(t)\ \underline{x}(t) + \underline{g}(t)\ u(t) \tag{C-14}$$

and its general continuous solution:

$$\underline{x}(t) = \underline{F}(t,t_o)\ \underline{x}(t_o) + \int_{t_o}^{t} F(t,\tau)\ \underline{g}(\tau)\ u(\tau)\ d\tau \tag{C-15}$$

If the control input, u(t), is considered to be a constant equation C-15 may be rewritten in the form:

$$\underline{x}(t) = \underline{F}(t,t_o)\ \underline{x}(t_o) + \underline{\lambda}(t,t_o)\ u_c \tag{C-16}$$

were equation C-16 is valid as long as u(t) remains constant at the value u_c. Equation C-16 is therefore valid for any state which occurs during one decision interval.

With reference to equation C-16, the state $\underline{x}(t)$ may be written as a vector function:

$$\underline{x}(t) = \underline{G}(\underline{x}(t_o),\ u_c,\ t) \tag{C-17}$$

where t is included as an explicit argument of $\underline{G}$ when the system is time-varying. Forming the total differential of $\underline{x}(t)$:

$$d\underline{x}(t) = \frac{\partial \underline{G}}{\partial \underline{x}(t_o)}\ d\underline{x}(t_o) + \frac{\partial \underline{G}}{\partial u_c}\ du_c + \frac{\partial \underline{G}}{\partial t}\ dt \tag{C-18}$$

If the plant is sufficiently slowly time-varying the partial derivatives are approximately constant and the integral of equation C-18 may be approximated by:

$$\tilde{\underline{x}}(t) = \underline{\theta}_1(t-t_o)\ \underline{x}(t_o) + \underline{\phi}_1(t-t_o)\ u_c + \underline{\phi}_2\ t \tag{C-19}$$

Equation C-19 may be converted to a discrete form by setting $t - t_o = T$. The form of the discrete equation for the interval $kT \leq t < (k+1)T$ would then be:

$$\underline{\tilde{x}}((k+1)T) = \underline{\Theta}_1(T)\ \underline{x}(kT) + \underline{\phi}_1(T)\ u_k + \underline{\phi}_2\ (k+1)T \tag{C-20}$$

where the matrices $\underline{\Theta}_1$, $\underline{\phi}_1(T)$, and $\underline{\phi}_2(T)$ are evaluated for the constant decision interval length of T seconds. Equation C-20 is approximately equivalent to the discrete form of equation C-16 for short intervals of time the length of which will depend upon the rate of time variation of the plant.

The interpolation estimate of equation C-20 may be obtained by using a basis vector of the form:

$$\underline{\phi}'(u_k, \underline{\eta}_k) = \underline{\underline{x}'(kT)\ u_k\ (k+1)T} \tag{C-21}$$

where the time origin to which (k+1)T is reference is arbitrary. The interpolation matrices may be built up using equation C-21 as the basis vector. The dimension of the interpolation matrices is increased by only one over that of the stationary basis vector.

C.4 TIME VARIATION USING A TRUNCATED TAYLOR EXPANSION

A third method of accounting for time variations of the plant is obtained by a direct Taylor expansion of the matric elements of the discrete state equation.

The discrete state equation of a time-varying plant is given in equation 2-8 of paragraph 2.1 to be:

$$\underline{x}((k+1)T) = \underline{F}((k+1)T, kT)\ \underline{x}(kT) + \underline{\lambda}((k+1)T, kT)\ u_k \tag{C-22}$$

Each of the elements of the $\underline{F}$ matrix and the $\underline{\lambda}$ vector may be expanded in a Taylor series about some time reference $t = t_o$

$$\alpha_i(t) = \gamma_o + \gamma_1(t - t_o) + \ldots \tag{C-23}$$

For sufficiently slow time variations the Taylor series of each element may be truncated after two terms without prohibitive loss of accuracy so that equation C-22 may be approximated by:

$$\underline{\tilde{x}}((k+1)T) = \left\{ \underline{F}_o + \underline{F}_1 \ (k+1)T \right\} \underline{x}(kT) + \left\{ \underline{\lambda}_o + \underline{\lambda}_1 \ (k+1)T \right\} u_k \qquad \text{(C-24)}$$

An appropriate interpolation estimate of equation C-24 would be:

$$\underline{\tilde{x}}((k+1)T) = \underline{\Theta}_1 \underline{x}(kT) + \underline{\Theta}_2 \{(k+1)T\} \underline{x}(kT) + \underline{\phi}_1 \ u_k + \underline{\phi}_2 \{(k+1)T\} u_k \quad \text{(C-25)}$$

The corresponding basis vector to this interpolation procedure is:

$$\underline{\phi}'(u_k, \underline{\eta}_k) = \left\lfloor \underline{x}'(kT) \quad \{(k+1)T\} \underline{x}(kT) \quad u_k \quad \{(k+1)T\} u_k \right\rfloor \qquad \text{(C-26)}$$

The interpolation estimate of this method is given by:

$$\underline{\tilde{x}}((k+1)T) = {}_D\underline{X} \, \underline{\Phi}^{-1} \, \underline{\phi}(u_k, \underline{\eta}_k) \qquad \text{(C-27)}$$

where the $\underline{B}$ matrix may be factored into four submatrices:

$$\underline{B} = {}_D\underline{X} \, \underline{\Phi}^{-1} = \left[\underline{\Theta}_1 \; \vdots \; \underline{\Theta}_2 \; \vdots \; \underline{\phi}_1 \; \vdots \; \underline{\phi}_2 \right] \qquad \text{(C-28)}$$

which correspond to the matrices of equation C-25.

This method should be the most exact of the three presented in this appendix , however, the size of the matrix of basis vectors is substantially larger. If the assumed order of $\underline{x}(t)$ is p, then the $\underline{\Phi}$ matrix of either of the first two methods is p+2. The order of the $\underline{\Phi}$ matrix in this last case is 2(p+1).

APPENDIX D

A RECURSIVE METHOD FOR INVERSION OF CERTAIN MATRICES

Much of the previous DACS research (reference 1) and all that documented by this report utilized the Interpolation Prediction method to obtain estimates of the plant matrices. Associated with this method regardless if applied to linear or non-linear systems is the inversion of a certain matrix. In our research it has been referred to as the matrix of basis vectors, and is formed from state variable measurements at past decision intervals. Under the assumption of no plant knowledge with respect to time variability, it is necessary to frequently update the interpolation estimates of the plant. This, in turn, requires frequent inversion of the matrix of basis vectors compiled over different time intervals. For ease and speed of computation a recursive method for obtaining the inversion of this matrix at various times is definitely a desirable goal. Such a recursive procedure (reference 14) is presented in the following paragraphs.

D.1 STATEMENT OF THE PROBLEM

Given a square matrix, $\underline{A}_1$ and its inverse, $\underline{B}_1$, it is required to obtain the inverse of $\underline{A}^*$. The square matrix $\underline{A}^*$ is of the same order as $\underline{A}_1$, but differs from $\underline{A}_1$ with respect to only one column. The required inverse is $[\underline{A}^*]^{-1}$ which is desired to be computed in a recursive manner without actually having to invert $\underline{A}^*$.

D.2 DEVELOPMENT OF THE PROCEDURE

In order to develop the recursive procedure the matrices $\underline{A}$ and $\underline{B}$ are assumed known. The matrix $\underline{A}$ is composed of measurements taken over a number of discrete decision intervals of time and the matrix $\underline{B}$, is defined by:

$$\underline{B} = \underline{A}^{-1} \tag{D-1}$$

Therefore, the matrix, $\underline{A}$, may be written as:

$$\underline{A} = \begin{bmatrix} \underline{c}_1 \ \underline{c}_2 \ \cdots \ \underline{c}_i \ \cdots \ \underline{c}_n \end{bmatrix} \tag{D-2}$$

where $\underline{c}_i$ refers to a column whose elements are formed by state variable measurements taken at $t = T_i$. The matrix, $\underline{B}$, may also be written as:

$$\underline{B} = \underline{A}^{-1} = \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \vdots \\ \underline{r}_n \end{bmatrix} \tag{D-3}$$

At $t = T_{n+1}$ a new column of measurements, $\underline{c}_{n+1}$, is available and is used to define $\underline{A}^*$.

$$\underline{A}^* = \begin{bmatrix} \underline{c}_2 \ \underline{c}_3 \ \cdots \ \underline{c}_n \ \underline{c}_{n+1} \end{bmatrix} \tag{D-4}$$

It should be noted that $\underline{A}^*$ contains the most recent system information, and differs from $\underline{A}$ by only one column. Thus, it is obvious that $\underline{A}$ and $\underline{A}^*$ contain (n-1) identical columns.

At this point a matrix $\underline{A}_1$ is defined as:

$$\underline{A}_1 = \begin{bmatrix} \underline{c}_2 \ \underline{c}_3 \ \cdots \ \underline{c}_n \ \underline{c}_1 \end{bmatrix} \tag{D-5}$$

Note that $\underline{A}_1$ has the same columns of $\underline{A}$, but interchanged in a particular manner.

Using the above definitions, it can be shown that:

$$\underline{A}_1^{-1} = \underline{B}_1 = \begin{bmatrix} \underline{r}_2 \\ \underline{r}_3 \\ \vdots \\ \underline{r}_n \\ \underline{r}_1 \end{bmatrix} \tag{D-6}$$

The interesting point here is that $\underline{B}_1$ has the same rows of $\underline{B}$, but interchanged in the same manner as are the columns of $\underline{A}$ to obtain $\underline{A}_1$.

Since $\underline{A}_1$ and $\underline{A}^*$ differ only in the last column, it is possible to define $\underline{A}^*$ as:

$$\underline{A}^* = \left[\underline{A}_1 + \underline{D}\right] \tag{D-7}$$

where

$$\underline{D} = \begin{bmatrix} 0 & 0 & . & . & . & 0 & d_1 \\ 0 & 0 & . & . & . & 0 & d_2 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & . & . & . & 0 & d_n \end{bmatrix} \tag{D-8}$$

and the last column, $\underline{d}$, of $\underline{D}$ is defined by:

$$\underline{d} = \left[\underline{c}_{n+1} - \underline{c}_1\right] \tag{D-9}$$

At this point, the inverse of $\underline{A}^*$ may be written as:

$$\left[\underline{A}^*\right]^{-1} = \left[\underline{A}_1 + \underline{D}\right]^{-1} = \left[\underline{I} + \underline{B}_1\ \underline{D}\right]^{-1} \underline{B}_1 \tag{D-10}$$

where $\underline{I}$ is an identity matrix of order n.

A new matrix $\underline{G}$ is now defined as:

$$\underline{G} = \underline{B}_1\ \underline{D} \tag{D-11}$$

This matrix can be easily written in the following form:

$$\underline{G} = \begin{bmatrix} 0 & 0 & . & . & . & 0 & g_1 \\ 0 & 0 & . & . & . & 0 & g_2 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & . & . & . & 0 & g_n \end{bmatrix} \tag{D-12}$$

where

$$g_k = \sum_{j=1}^{n} b_{kj}\ d_j \tag{D-13}$$

Forming the $\left[\underline{I} + \underline{B}_1 \, \underline{D}\right]$ portion of equation D-10 yields:

$$\left[\underline{I+G}\right] = \left[\begin{array}{cccccc|c} 1 & 0 & . & . & . & 0 & 0 & g_1 \\ 0 & 1 & . & . & . & 0 & 0 & g_2 \\ \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & & & & 0 & 1 & g_{n-1} \\ \hline 0 & 0 & . & . & . & 0 & 0 & (1+g_n) \end{array}\right] \qquad \text{(D-14)}$$

It can be readily shown that $\left[\underline{I+G}\right]^{-1}$ exists if $g_n \neq -1$, and has the same form as $\left[\underline{I+G}\right]$. That is:

$$\left[\underline{I+G}\right]^{-1} = \underline{Q} = \begin{bmatrix} 1 & 0 & . & . & . & 0 & 0 & \frac{-g_1}{(1+g_n)} \\ 0 & 1 & . & . & . & 0 & 0 & \frac{-g_2}{(1+g_n)} \\ \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & . & . & . & 0 & 1 & \frac{-g_{n-1}}{(1+g_n)} \\ 0 & 0 & . & . & . & 0 & 0 & \frac{1}{(1+g_n)} \end{bmatrix} \qquad \text{(D-15)}$$

And so $\underline{Q}$ has the following form:

$$\underline{Q} = \begin{bmatrix} \underline{I}_{n-1} & \underline{q}_{12} \\ \underline{0} & q_{22} \end{bmatrix} \qquad \text{(D-16)}$$

where $\underline{I}_{n-1}$ is an identity matrix of order (n-1).

Therefore, the matrix $\underline{Q}$ is very readily determined by only the computation of the elements of the last column. This in turn leads to the formulation of the desired inverse recursive relation.

$$\left[\underline{A}^*\right]^{-1} = \underline{Q}\,\underline{B}_1 = \underline{Q}\,\underline{A}_1^{-1} \qquad \text{(D-17)}$$

By proper partitioning of $\underline{B}_1$ it is possible to write:

$$\left[\underline{A}^*\right]^{-1} = \begin{bmatrix} \underline{I}_{n-1} & \underline{q}_{12} \\ \underline{0} & q_{22} \end{bmatrix} \begin{bmatrix} \underline{B}_{11} & \underline{b}_{12} \\ \underline{b}_{21} & b_{22} \end{bmatrix} \qquad \text{(D-18)}$$

and finally as

$$\left[\underline{A}^*\right]^{-1} = \underline{B}_1 + \begin{bmatrix} \underline{q}_{12}\ \underline{b}_{21} & \underline{q}_{12}\ b_{22} \\ (q_{22}-1)\underline{b}_{21} & (q_{22}-1)b_{22} \end{bmatrix} \qquad \text{(D-19)}$$

This equation is an alternate form the recursive relation of equation D-17.

In conclusion it is worthwhile to note that the recursive method just presented can be extended to the inversion of an arbitrary matrix starting from an identity matrix.

D.3 EXAMPLE

This recursive method was programmed for the GE-235 digital computer and used to test the inverse of various arbitrary matrices. Approximately twenty tests were made for third through fifth order matrices. These test results were very good, and a typical example is given below.

The following matrices were known:

$$\underline{A}_1 = \begin{bmatrix} 2.15415\ \text{E-2} & .034887 & -2.04179\ \text{E-2} & 9.12530\ \text{E-5} & -1.58968\ \text{E-4} \\ .204188 & -9.19926\ \text{E-2} & 9.57817\ \text{E-3} & 5.71688\ \text{E-3} & -1.55386\ \text{E-3} \\ -.386151 & 2.09739\ \text{E-3} & .064822 & -2.12117\ \text{E-2} & 2.28264\ \text{E-3} \\ -.380493 & .394947 & -.126053 & -4.16931\ \text{E-2} & 5.75911\ \text{E-3} \\ .776518 & .529451 & -.581688 & -6.83791\ \text{E-2} & -2.91468\ \text{E-3} \end{bmatrix}$$

$$\underline{B}_1 = \begin{bmatrix} 8.46194 & 13.0074 & -.429154 & 3.46684 & -.881939 \\ 75.3261 & 20.1942 & 6.13629 & 3.84968 & -2.46193 \\ 112.8 & 49.4594 & 13.9172 & 8.34951 & -5.12259 \\ -144.517 & -107.3 & -53.18 & -11.8844 & -4.50091\ \text{E-2} \\ -3183.93 & -219.75 & -529.548 & 235.398 & -1.88191 \end{bmatrix}$$

$$\underline{A}^* = \begin{bmatrix} 2.15415 \text{ E-2} & .034887 & -2.04179 \text{ E-2} & 9.12530 \text{ E-5} & -3.27030 \text{ E-4} \\ .204188 & -9.19926 \text{ E-2} & 9.57817 \text{ E-3} & 5.71688 \text{ E-3} & 7.30297 \text{ E-4} \\ -.386151 & 2.09739 \text{ E-3} & .064822 & -2.12117 \text{ E-2} & 1.43941 \text{ E-4} \\ -.380493 & .394947 & -.126053 & -4.16931 \text{ E-2} & -3.24154 \text{ E-3} \\ .776418 & .529451 & -.581688 & -6.83791 \text{ E-2} & -5.04379 \text{ E-3} \end{bmatrix}$$

The goal was to obtain the $[A^*]^{-1}$ by the standard inversion methods and by the recursive method. The actual inverse is denoted by $[A^*]^{-1}_{a}$, and that obtained by the recursive method, $[A^*]^{-1}_{r}$. The resulting inverses were:

$$[\underline{A}^*]^{-1}_{a} = \begin{bmatrix} 1.00469 & 12.4927 & -1.66944 & 4.01817 & -.886346 \\ -490.808 & -18.8795 & -88.0226 & 45.7057 & -2.79655 \\ 113.19 & 49.4863 & 13.9821 & 8.32066 & -5.12236 \\ -144.747 & -107.316 & -53.2183 & -11.8673 & -4.51452 \text{ E-2} \\ -62457.5 & -4310.72 & -10387.9 & 4617.67 & -36.9165 \end{bmatrix}$$

and

$$[A^*]^{-1}_{r} = \begin{bmatrix} 1.00468 & 12.4927 & -1.66944 & 4.01817 & -.886346 \\ -490.808 & -18.8795 & -88.0226 & 45.7057 & -2.79655 \\ 113.19 & 49.4863 & 13.9821 & 8.32067 & -5.12236 \\ -144.747 & -107.316 & -53.2183 & -11.8673 & -4.51452 \text{ E-2} \\ -62457.5 & -4310.72 & -10387.9 & 4617.67 & -36.9165 \end{bmatrix}$$

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