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THE USE OF RESONANT LIBRATING ORBITS IN SATELLITE GEODESY

by C. A. Wagner

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

A satellite for which the orbital rate is commensurate (or rational) with the earth's rotation rate will describe a constant ground-track over a whole number of sidereal days. On such a satellite, the small orbit-averaged, along-track force from certain longitude harmonics of gravity can build up over many commensurate (synodic) periods to produce appreciable changes in the semimajor axis of the orbit and longitude placement of the ground-track. The libratory nature of these changes and the specific harmonics which can cause them for any resonance case are discussed with reference to a circular orbit. General formulas are presented for these librations, with typical errors determined by numerical integration. Actual orbit data from three 24 hour satellites are used in connection with these formulae to determine the resonant earth gravity sectorial harmonics through third order. These are found to be $10^6 \bar{C}_{22} = 2.42 \pm 0.03$, $10^6 \bar{S}_{22} = -1.44 \pm 0.03$, $10^6 \bar{C}_{33} = 0.32 \pm 0.11$, and $10^6 \bar{S}_{33} = 1.18 \pm 0.11$. It is shown that a number of subsynchronous circular resonant orbits offer similar promise in easily and uniquely discriminating other low and high-order longitude Harmonics, including H_{32}, H_{43}, H_{54}, H_{55}, H_{76}, and H_{98}.

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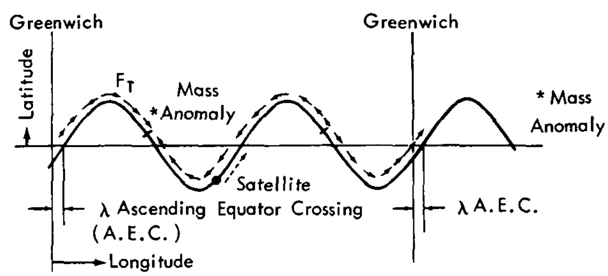
INTRODUCTION

Since the beginning of the space age, synchronous (earth-stationary) high altitude satellites have been the object of intense investigation because of the unique advantages they offer for world-wide communications. More recently, lower altitude satellites with periodically repetitive ground-tracks have also been proposed for convenient communications applications. It has been known for some time that all satellites with periods that are commensurate with the earth's rotation period can suffer long-term changes in their elements, because of longitudinal variations in the earth's gravity field (Reference 1). This is true since the satellite with a commensurate period and viewed in a system of coordinates rotating with the earth, maintains a fixed relationship to the underlying longitude gravity field (Figure 1). Resonant orbits as they are used here are synonomous with commensurate or constant ground track orbits, that is, those whose orbital rates arė rational with respect to the earth's rotation rate.



r orbits in n' sidereal days, to repetition of ground track.

The vectors above show the magnitude and direction of along track forces, F_T, due to the mass anomaly.

Figure 1—Ground-track of a resonant circular orbit.

Recent researches into these perturbations have been made by Blitzer, Gedeon and Yiounulis in the United States (References 2, 3, and 4); A. H. Cook and R. R. Allan in England (References 1 and 5); Morando in France (Reference 6); Sehnal in Czechoslavakia (Reference 7); and Batrakof in Russia (Reference 8). I will treat the nature of these orbit changes, in particular that of the semimajor axis, from a very elementary orbit averaging viewpoint. Restricting our analysis to circular orbits permits easy derivation of integrable libration equations for the evolution of these orbits under a restricted spectrum of longitude gravity harmonics. We will illustrate the power of these amplified effects for satellite geodesy by solving (with apparent uniqueness) for two low order sectorial harmonics of the earth's field from observed ground track data on three 24 hour satellites. In addition we will show that many other specific harmonics may be as easily discriminated from similar observations on other constant ground track orbits.

Assume that, because of a gravity anomaly, there is net change in an element over the period of commensurability. Then, over many commensurate or synodic periods, no matter how small the net change is, a large change in that element can be generated as long as the geographic configuration of the orbit is relatively unchanged. The large, long term change is merely the sum of the many small net periodic changes.

This paper only concerns the long term effects of longitude gravity harmonics on the semimajor axis of circular orbits with constant, or nearly constant, ground-track. For a circular orbit, without inclination restriction, it is possible to derive a very good approximation of the secular change of the semimajor axis by straightforward orbit-averaging of the along-track-disturbing forces. The only component of a general disturbing force which can effect a change in the semimajor axis of an orbit is the along-track component F_T. This is true because the instantaneous semimajor axis of an orbit is a function only of its total energy (kinetic and two-body gravitational potential). The only force which does work on, and thus effects the total energy of the satellite, is the along-track force.

On a constant ground-track, or resonant orbit, the along-track forces on the satellite due to a mass anomaly are seen to vary periodically, in step with the predominant orbit motion (Figure 1). Part of this action is analogous to familiar mechanical resonance and can be imagined to lead to a buildup in the eccentricity of the orbit (for example, over a long period of time). But unlike the usual mechanical resonance case, the orbit average of the along-track forces because of the anomaly will not in general be zero. If the orbit-averaged along-track force is not zero, then the orbit energy, or semimajor axis will change over the resonant period. A change in the mean motion implies a shift or drift rate in the geographic longitude position of the track as the exact commensurability with the earth's rotation is destroyed. However, a feedback mechanism operates here with respect to the longitude position of the orbit. It is evident that the orbit-averaged force will depend strongly on the longitudinal location of the track with respect to the disturbing gravity anomalies. Over many orbits, the track may shift to a location where the averaged force reverses direction; then the action will be reversed. The ground-track may eventually come to rest and be driven in the opposite direction.

At the same time as the ground-track goes through its long-term oscillation, the orbit oscillates in unison, as the semimajor axis slowly grows and shrinks. In fact, both of these special actions in response to the longitude-dependent gravity forces of the earth are parts of the same phenomenon, called resonant libration. The proper analogy of this action is the circular pendulum. For the restricted case of circular orbits and with the anomalies defined in terms of an infinite series of spherical harmonics, it will be shown easily that the defining equation for the evolution of the nearly-constant ground-track orbit due to each individual relevant harmonic is just that of a circular pendulum.

As a consequence of the required resonance condition in terms of the longitude harmonics of gravity, it will be seen that only a very few of these harmonics can cause this long-term libration on a given constant ground-track circular orbit. This fact will prove to be vital to the special usefulness of these resonant orbits as geodetic tools.

RESONANT ORBIT EVOLUTION

In order to keep the analysis here as simple as possible without losing the essence of the problem, we will temporarily ignore the oblate character of the earth. So long as the orbit is reasonably circular, oblateness merely effects the semimajor axis necessary for commensurability. The assumption is made here that the mean motion is entirely due to a spherical earth. The evolution of the orbit under the continually acting residual along-track forces is easily calculated through the consequent orbit energy and period changes.

In the absence of perturbations, an orbit has a repeating ground-track if, and only if, the ratio of the satellite's mean motion ω_s to the earth's rotation rate ω_e is a rational fraction. This fraction can be expressed as

$$\frac{\omega_s}{\omega_e} = \frac{r}{n'} , \tag{1}$$

where r is the whole number of orbits the satellite makes in n' sidereal days, the synodic or ground-track repetition period.

The two-body energy E of a satellite and the period T of its orbit are given by

$$E = -\frac{\mu}{2a} , \tag{2a}$$

and

$$T = \frac{2\pi a^{3/2}}{\mu^{1/2}} , \tag{2b}$$

where a is the semimajor axis of the orbit and μ is the earth's Gaussian gravitational constant. The first order differences of E and T with respect to the semimajor axis are

$$\Delta E = \frac{\mu \, \Delta a}{2a^2} , \tag{3a}$$

and

$$\Delta T = \frac{3\pi a^{1/2} \Delta a}{\mu^{1/2}} . \tag{3b}$$

The work ΔE done on the circular orbit satellite by the along-track force, averaged over the repetition period n', is

$$\Delta E = 2\pi \, a r \bar{F}_T , \tag{4a}$$

where

$$\bar{F}_T = \frac{1}{n'} \int_0^{n'} F_T \, dt , \tag{4b}$$

and F_T is the along-track component of the perturbation force F. Equating the expressions for ΔE in Equations 3a and 4a yields Δa, the change in the semimajor axis each r orbits due to the orbit averaged force:

$$\Delta a = \frac{4\pi\, a^3\, r\, \overline{F}_T}{\mu} . \tag{5}$$

If the expression Δa (Equation 5) is substituted into Equation 3b, the change of period each r orbits due to $\overline{F}_T$, ΔT, is

$$\Delta T = \frac{12\pi^2\, a^{7/2}\, r\, \overline{F}_T}{\mu^{3/2}} . \tag{6}$$

It can be easily shown (See Reference 9 for example) that the drift rate $\dot{\lambda}$ of the ground track due to a small period change ΔT, from commensurate conditions, after n' days, is

$$\dot{\lambda} = -\,2\pi n' \frac{\Delta T}{T} \text{ radians}/n' \text{ days} . \tag{7}$$

From Equations 2b, 6, and 7, the effect of $\overline{F}_T$ is to cause a drift rate after n' days and r orbits from commensurate conditions, of

$$\dot{\lambda} = \frac{-12\pi^2\, r\, n'\, \overline{F}_T}{\mu/a^2} \text{ radians}/n' \text{ days} . \tag{8}$$

Since this initial orbit change results in only a small deviation from the original ground-track, the disturbing forces will be virtually the same over the next r orbits. Thus, the drift rate under this disturbance increases linearly from the initial rate by the addition of an increment of the drift rate (Equation 8) each r orbits or n' days. The acceleration of the ground-track is thus the same as the drift rate at n' days from exact resonance divided by n' days, or as shown in Equation 9a.

$$\ddot{\lambda} = \frac{-12\pi^2\, r\, n'\, \overline{F}_T}{\left(\mu/a^2\right)} , \quad \text{radians}/(n' \text{ days})^2$$

$$= -\,12\pi^2\, (r/n')\, \overline{F_T'} , \quad \text{radians/sid. day}^2 , \tag{9a}$$

where the primed disturbing force (actually an acceleration or force-per-unit-mass) is nondimensional and is used to simplify the result, and is given by:

$$F' = \frac{F}{\left(\mu/a^2\right)} . \tag{9b}$$

The coupled orbit expansion rate $\dot{a}$ under $\overline{F}_T$ is just the semimajor axis change (Equation 5) re-written as a differential equation

$$\dot{a} = 4\pi\, a\, (r/n')\, \overline{F_T'} , \tag{10}$$

where $\dot{a}$ is in a units per sidereal day.

THE DISTURBING POTENTIAL

The derivation of the orbit evolution equations has been elementary. Equation 10 is merely the orbit averaged statement of the classic Lagrange equation for the variation of the semimajor axis. The difficult task is to solve for the orbit averaged perturbation force. This is primarily a function of the inclination of the orbit and the longitudinal orientation of the ground-track.

To solve this problem mathematically it is first necessary to obtain an adequate representation of the longitude gravity anomalies. For this purpose, it is convenient to use the spherical harmonic series representation of the gravity potential. This complete earth potential on the circular resonant orbit satellite is expressed as

$$U_e = \frac{\mu}{a}\left[1 - \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R_0}{a}\right)^n P_n^m(\sin\phi)\, J_{nm} \cos m\left(\lambda - \lambda_{nm}\right)\right], \tag{11}$$

where

R_0 = mean equatorial radius of the earth,

a = semimajor axis of orbit,

ϕ = latitude (geographic),

λ = longitude (geographic), and

P_n^m = associated Legendre polynomial (see Appendix).

The longitude anomalies give rise to those terms where the subscript m, the longitude frequency, is not zero. In the following text, we will use the shorthand H_{nm} to refer to the gravity Harmonic of order n and longitude frequency m. It is possible to show rigorously that for a circular orbit, the orbit-averaged along-track force due to the longitude independent ($m = 0$) zonal harmonics, is zero (Reference 9).

To compute the average along-track force due to the longitudinal harmonics, it is most convenient to write the potential U_{nm} in terms of the Keplerian elements (a, I, θ, Ω) of the circular orbit (Equation 12a; Figure 2). This simplifies the evaluation, since the along-track direction for this orbit is in the same direction as the mean motion, described by the nodal argument θ. Also, use of the Lagrange equations

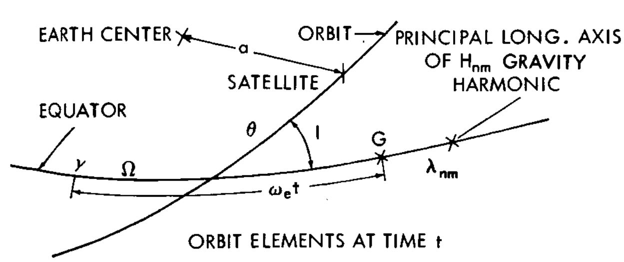


Figure 2—Keplerian coordinates of the circular resonant orbit.

requires that the disturbing potential be in terms of Keplerian elements. Thus

$$U_{nm} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \sum_{p=0}^{n} R\left\{F_{nmp}(I) \exp i \left[(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm})\right]\right\} , \tag{12a}$$

where $i = \sqrt{-1}$, $R\{\ \}$ = real part of $\{\ \}$, $m \neq 0$, and

$$F_{nmp}(I) = \frac{i^{n-m}(n+m)!}{2^n p!(n-p)!} \sum_{K=0}^{n-m} (-1)^k \binom{2n-2p}{K} \binom{2p}{n-m-K} \cos^{3n-m-2p-2K}\left(\frac{I}{2}\right) \cdot \sin^{m-n+2p+2K}\left(\frac{I}{2}\right) \cdot \tag{12b}$$

Apparently, this type development of the longitude harmonics was first used by W. M. Kaula in 1961 (Reference 10). The specific form above, especially the inclination function $F(I)$, and the development below, most closely follows the development by R. R. Allan in 1965.*

The along-track force on the circular orbit satellite is the partial derivative of the potential U_{nm} with respect to the nodal argument theta (Reference 9). The simple orbit-averaging of this derivative provides all the necessary insight into the requirements for and conditions of librational resonance of the circular orbit. From Reference 9, the orbit averaged force $\overline{F_T}'$ is given in terms of the orbit-averaged derivative by

$$\overline{F_T'} = \frac{a}{\mu} \frac{\overline{\partial U_{nm}}}{\partial \theta} . \tag{13}$$

The partial derivative of the potential U_{nm} (Equation 12a) with respect to θ, in terms of real phased harmonics, with θ and time as the only variables, is

$$\frac{\partial U_{nm}}{\partial \theta} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \sum_{p=0}^{n} F'_{nmp}(I)(n-2p) \cos\left[(n-2p)\theta + m(\Omega - \omega_e t - \lambda_{nm}) + \delta_{nm}\right] , \tag{14a}$$

where

$$\delta_{nm} = \frac{\pi}{2}(n-m+1) , \tag{14b}$$

and

$$F'_{nmp}(I) = \frac{F_{nmp}(I)}{i^{n-m}} . \tag{14c}$$

*Allan, R. R., "Resonance Effects for Satellites with Nominally Constant Ground-Track," R. A. E. Internal Document (Private Communication), Royal Aircraft Establishment; Farnborough, England, August 1965.

The orbit-averaged derivative, as a series of $n + 1$ integrals of simple phased harmonics in the time and θ for each longitude harmonic, is

$$\frac{\overline{\partial U_{nm}}}{\partial \theta} = \frac{1}{n'} \int_0^{n'} \frac{\partial U_{nm}}{\partial \theta} \, dt$$

$$= -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \int_0^{n'} \left\{ \sum_{p=0}^{n} \left(\frac{1}{n'}\right) F'_{nmp}(I) \cdot (n-2p) \cos\left[(n-2p)\,\theta + m\left(\Omega - \omega_e t - \lambda_{nm}\right) + \delta_{nm}\right] \right\} dt \,, \tag{15}$$

The nodal argument can be converted to time through the mean motion and the argument at epoch (θ_0, when $t = 0$):

$$\theta = \theta_0 + \omega_s t \,. \tag{16a}$$

Let

$$C_{1_p} = (n-2p)\,\theta_0 + m\left(\Omega - \lambda_{nm}\right) + \delta_{nm} \,. \tag{16b}$$

and

$$C_{2_p} = \omega_s (n-2p) - m\,\omega_e \,, \tag{16c}$$

where the phase constants C_{1_p} are functions of the longitudinal position of the orbit. Substitution of the expressions in Equations 16 into Equation 15 yields

$$\frac{\overline{\partial U_{nm}}}{\partial \theta} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \int_0^{n'} \left[\sum_{p=0}^{n} \left(\frac{1}{n'}\right) F'_{nmp}(I)\,(n-2p) \cos\left(C_{1_p} + C_{2_p} t\right) \right] dt \,. \tag{17}$$

The frequency constants C_{2_p} yield the critical resonance requirements. Integrating the $n + 1$ harmonic terms in Equation 17 over n' days, for the orbit-averaged force, gives

$$\frac{\overline{\partial U_{nm}}}{\partial \theta} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \sum_{p=0}^{n} \left(\frac{1}{n' C_{2_p}}\right) \cdot (n-2p)\, F'_{nmp}(I) \left[\sin\left(C_{1_p} + n' C_{2_p}\right) - \sin C_{1_p}\right] \,. \tag{18}$$

If the frequencies C_{2_p} are not zero (Equation 19), but are arbitrary, then we will have a zero result for each integral over a time interval $(n')^*$ not greater than one day for $r > n'$; and n'/r days for $r < n'$. In this case, the long-term effects on the mean motion due to that longitude harmonic term can be expected to be negligible, at least over the maximum such averaging interval.

The condition for zero orbit-averaged force is

$$(n')^* C_{2_p} = \pm 2\pi, \qquad \pm 4\pi, \cdots \quad . \tag{19}$$

However, when the frequency constant is zero, (Equation 20), each term of the averaged force is in an indeterminate form for any averaging period.

The condition for resonance is

$$n' C_{2_p} = 0 . \tag{20}$$

By L'Hospital's Rule, this form can be evaluated with the nonzero result

$$\lim_{n' C_{2_p} \to 0} \left(\frac{1}{n' C_{2_p}} \right) \left[\sin \left(C_{1_p} + n' C_{2_p} \right) - \sin C_{1_p} \right] = \cos C_{1_p} \quad . \tag{21}$$

The implication of the nonzero averaged force, or resonance, condition, from Equations 20 and 16c is:

$$C_{2_p} = \omega_s (n - 2p) - m\omega_e = 0 , \tag{22a}$$

or

$$\frac{\omega_s}{\omega_e} = \frac{m}{n - 2p} = \frac{r}{n'} , \quad \text{a rational fraction} . \tag{22b}$$

Hence, as expected, the nonzero averaged force occurs only for commensurate, constant ground-track orbits.

As stated previously, with the rational fraction r/n' reduced to lowest terms, n' will be the number of sidereal days over which the ground-track of the satellite will repeat (after r orbits). In addition, this single resonance condition (Equation 22) tells what harmonics n, m will be active on a given commensurate orbit specified by r and n' (Equation 23).

On given resonant orbit (r/n'), only those harmonics H_{nm} are active for which

$$m = \frac{r}{n'} (n - 2p) ; \qquad p = 0, 1, 2, \cdots \quad . \tag{23}$$

Furthermore, for a given resonant orbit r/n' and active harmonics n, m, only one of the $n + 1$ force integrals is relevant (nonzero over the averaging time n'); that integral is the one whose p subscript also satisfies the single resonance criterion (Equation 22), where

$$p = \frac{n - m(n'/r)}{2} = 0,\ 1,\ 2,\ \cdots\ . \tag{24}$$

For the determined r, n', n, m, and p of the resonant orbit-harmonic, $n'C_{2_p} = 0$. The single nonzero force integral is (from the result of Equation 21 in Equation 18):

$$\frac{\partial \overline{U_{nm}}}{\partial \theta} = -\frac{\mu}{a} J_{nm} \left(\frac{R_0}{a}\right)^n \frac{mn'}{r} F_{nmp}(I) \cos \overbrace{\left[\frac{mn'}{r}\,\theta_0 + m(\Omega - \lambda_{nm}) + \delta_{nm}\right]}^{C_{1_p}}, \tag{25}$$

where

$$n - 2p = \frac{mn'}{r}\ .$$

ORBIT EVOLUTION EQUATIONS

Let λ be the geographical longitude of the ascending equator crossing just prior to the epoch (Figure 3). Then, assuming Ω and ω_s constant, since, for the resonant orbit $\omega_e t' = \theta_0 (n'/r)$:

$$\lambda = \frac{n'}{r}\,\theta_0 + \Omega\ . \tag{26}$$

It is recalled that, in our formulation of the potential (Figure 2), epoch is the moment when the Greenwich meridian crosses the vernal equinox.

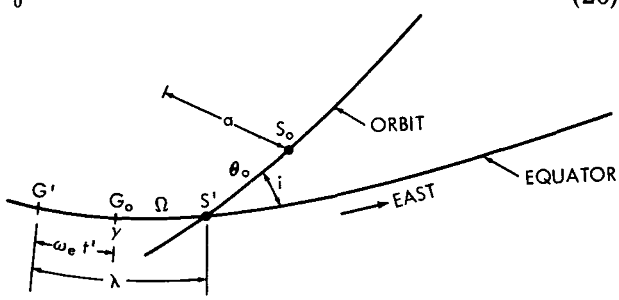


Figure 3—Orbital parameters at t = 0.

From Equations 13, 25, and 26 we finally have the averaged force $\overline{F}_T'$ for each relevant harmonic as a function of the inclination and longitudinal position of the ground-track, and other variables (Equation 27). The relationship between the averaged force and the acceleration of the ground-track having already been established (Equation 9), the desired relationship of this acceleration to the longitudinal position of the nearly constant track is given in Equation 28, which is a pendulum equation for each relevant harmonic. Thus:

$$\overline{F_T'} = -\sum_{n,m\ \mathrm{relevant}} J_{nm} \left(\frac{R_0}{a}\right)^n m \left(\frac{n'}{r}\right) F_{nmp}'(I) \cos\left[m(\lambda - \lambda_{nm}) + \delta_{nm}\right]\ ; \tag{27}$$

$$\ddot{\lambda} = 12\pi^2 \sum_{n,m \text{ relevant}} J_{nm} \left(\frac{R_0}{a}\right)^n m F'_{nmp}(I) \cos\left[m(\lambda - \lambda_{nm}) + \delta_{nm}\right] \text{ radians/sid. day}^2 . \tag{28}$$

The coupled long-term semimajor axis variation under $\bar{F}_T{}'$ is found by substituting $\overline{F_T{}'}$ from Equation 27 into Equation 10. More generally, if $\ell = 1$ refers the track to the longitude of the descending equator crossing, and $\ell = 2$ refers it to the ascending equator crossing, the parallel long-term motion of all the equator crossings can be derived from

$$\ddot{\lambda}_\ell = 12\pi^2 \sum_{n,m \text{ relevant}} J_{nm} \left(\frac{R_0}{a}\right)^n m F'_{nmp}(i) \cos\left[m(\lambda_\ell - \lambda_{nm}) + \delta'(n, m, \ell)\right] \text{ radians/sid. days}^2 , \tag{29a}$$

where

$$\begin{aligned} \delta'(n, m, \ell) &= \delta_{nm} + \pi\ell(n - m) \\ &= \frac{\pi}{2}\left[(n - m)(2\ell + 1) + 1\right] . \end{aligned} \tag{29b}$$

The coupled semimajor axis variation with respect to either node is then

$$\dot{a} = -4\pi a \sum_{n,m \text{ relevant}} J_{nm} \left(\frac{R_0}{a}\right)^n m F'_{nmp}(i) \cos\left[m(\lambda_\ell - \lambda_{nm}) + \delta'(n, m, \ell)\right] \text{ (a units)/sidereal day} . \tag{30}$$

While the longitude regime equations (Equations 29a and 30, see also Reference 2) apply only under commensurate, circular-orbit conditions, they have been shown to give a very good approximation to the resonant orbit evolution for moderate equator crossing drift rates and orbit eccentricities not exceeding about 0.01 (Reference 11, 12).

In reference to present knowledge of the earth's gravity field, the compound "pendulum drift" equation and its coupled semimajor axis variation appear to describe very closely the long-term regime of all near-circular, near-resonant orbit satellites under the action of the longitude harmonics of gravity. This is true even for longitude drift rates well into the circulating-pendulum or world circulation regimes. The numerical studies in References 9, 11, and 12 show that longitude accelerations estimated by Equation 29a should be accurate to within 1 percent of maximum accelerations for any resonant regime. This accuracy should hold so long as the drift rate is not substantially greater than the maximum permitted for libration (Table 1). The eccentricity of the orbit should also be less than 0.01 to assure this desired accuracy.

Table 1

Libratory Characteristics of Geodetically Promising Resonant Circular Orbits.*

Orbit period (Sid. hours)	Approximate semimajor axis (earth radii)	Clearly dominant resonant harmonic (H_{nm})	Inclination (degrees)	$\ddot{\lambda}_{max}$ (10^{-5} rad./day ^{2})	$\dot{\lambda}_{max}$ (deg./day)	T_{min} (years)	$T_{min}/4$ (days)	Maximum libration excursion $(360/m)^\circ$ (degrees)
3.00	1.653	H_{98}	60	0.0752	0.035	7.01	640	45
4.00	2.002	H_{76}	60	0.0753	0.041	3.11	740	60
4.80	2.261	H_{55}	60	0.2462	0.080	4.90	447	72
6.00	2.623	H_{54}	60	0.1025	0.058	8.43	775	90
8.00	3.178	H_{33}	60	1.086	0.22	3.02	275	120
8.00	3.178	H_{33}	30	0.4502	0.14	4.69	428	120
12.00	4.164	H_{32}	30	0.3481	0.15	6.52	595	180
14.40	4.702	H_{55}	30	0.0060	0.0125	31.4	2860	72
14.40	4.702	H_{55}	60	0.0096	0.0158	24.8	2270	72
16.00	5.044	H_{43}	30	0.0342	0.038	17.0	1550	120
24.00†	6.61	H_{22}	0	2.92	0.438	2.25	205	180

*All libratory data, except that for the 24-hour satellite, is derived from a planning curve derived by W. M. Kaula (Reference 15), (Figure 4).
†From actual data (Reference 11).

LIBRATORY CHARACTERISTICS OF RESONANT ORBIT SATELLITES

From Equation 29a, we can see that if the inclination and semimajor axis undergo only small changes, the long-term longitudinal drift of a nearly-constant ground-track orbit because of any relevant harmonic H_{nm} will be analogous to that of a circular pendulum. In fact, many integrations of the complete equations of motion have confirmed this assumption for a wide variety of resonant and near-resonant orbits (References 13, 14, 5). As such, the drift as a function of time (or, more precisely, the time as a function of drift) can be given exactly only in terms of an elliptic integral (and then only for a single harmonic effect), (References 13, 5). However, the drift rate regime and the critical or equilibrium points of the pendulum drift can be found directly from Equation 29a. Additional information about the libratory drift will be the minimum oscillation period (about the stable equilibrium points) and the maximum drift rate possible for "capture" in the libratory regime due to any relevant harmonic. This information also is easily found from Equation 29a.

The standard pendulum equation for the drift due to H_{nm} will be taken as:

$$\ddot{\lambda} + K_{nm}^2 \sin m(\lambda - \lambda_e) = 0 , \tag{31}$$

where, for convenience, λ refers to the ascending equator crossing longitude; λ_e is a position of *stable* equilibrium $(\lambda_{e,s})$ if $K_{nm}^2 > 0$, and unstable equilibrium $(\lambda_{e,u})$ if $K_{nm}^2 < 0$. These positions are separated by

$$\lambda_{e,s} - \lambda_{e,u} = \pm \frac{\pi}{m} . \tag{32}$$

Equation 29a can be put in the form of Equation 31 as

$$\ddot{\lambda} = K_{nm}^2 \cos\left[m(\lambda - \lambda_{nm}) + m(\lambda_{nm} - \lambda_e) + \frac{\pi}{2}\right] , \tag{33}$$

where

$$K_{nm}^2 = 12\pi^2 J_{nm} \left(\frac{R_0}{a}\right)^n m F'_{nmp}(i) \quad \text{rad./sid. day}^2 . \tag{34}$$

Since $J_{nm} \leq 0$, the positions of *unstable* equilibrium are given from Equations 33 and 29 by

$$m(\lambda_{nm} - \lambda_{e,u}) + \frac{\pi}{2} = \delta'(n, m, 2) , \quad \text{for} \quad F'_{nmp} > 0 ,$$

or, (using Equation 29b with $\ell = 2$ and the fact that $n - m$ is an integer or zero),

$$\lambda_{e,u} = \lambda_{nm} - \frac{\pi}{2m}(n - m) . \tag{35}$$

Similarly, those positions of *stable* equilibrium for $F'_{nmp} > 0$ are given from Equations 35 and 32 by

$$\lambda_{e,s} = \frac{\pi}{2m}\left[2 - (n - m)\right] + \lambda_{nm} . \tag{36}$$

It may be noted that for low-inclination satellites ($i \lessdot 30°$), F'_{nmp} is always positive, and for these orbits, Equations 35 and 36 will be the equilibrium criteria. For example, for the 24-hour satellite, the relevant H_{nm} are all those for which $n - m$ is even (see Equation 23), or $n - m = 2K$; $K = 0, 1, 2, \cdots$. Thus, for the low-inclination, 24-hour satellite, the unstable equilibrium ascending equator crossing longitudes are at

$$\lambda_{e,u} = \lambda_{nm} - \frac{\pi K}{m} , \tag{37}$$

and the stable equilibrium longitudes for low-inclination, 24-hour satellites are at

$$\lambda_{e,s} = \lambda_{nm} + \left(\frac{\pi}{m}\right)(1 - K) , \tag{38a}$$

where

$$K = \frac{n - m}{2} = 0, 1, 2, \cdots . \tag{38b}$$

The situation is just the reverse for inclinations (at medium and high inclinations, for example) where $F'_{nmp} < 0$. In these cases, the *stable* equilibrium longitudes are given by Equation 35, and the unstable by Equation 36. Thus,

$$\lambda_{e,s} = \lambda_{nm} - \frac{\pi}{2m}(n-m) , \tag{39}$$

and

$$\lambda_{e,u} = \lambda_{nm} + \frac{\pi}{2m}\left[2-(n-m)\right] , \tag{40}$$

for resonant orbits where $F'_{nmp} < 0$.

Since the unstable and stable positions are equally spaced around the globe and separated by π/m radians, there will be a total of $2m$ of these around the equator. There will thus be m unstable longitudes and m stable longitudes for each relevant H_{nm}. The maximum libratory excursion of the ground-track will thus be $360°/m$.

Proceeding with the integration of the pendulum equation (Equation 31), the variables $(\dot{\lambda})^2$ and λ can be separated by means of the differential reduction

$$\ddot{\lambda} = \frac{d(\dot{\lambda})^2}{2\dot{\lambda}\,dt} = \frac{d(\dot{\lambda})^2}{2\,d\lambda} . \tag{41}$$

First Equation 31 is rewritten to apply to $\lambda_e = \lambda_{e,s}$:

$$\ddot{\lambda} + \left|K^2_{nm}\right| \sin m\left(\lambda - \lambda_{e,s}\right) = 0 . \tag{42}$$

Substitution of Equation 41 into Equation 42 separates variables:

$$d(\dot{\lambda})^2 = -2\left|K^2_{nm}\right| \sin m\left(\lambda - \lambda_{e,s}\right) d\lambda . \tag{43}$$

The integral of Equation 43 is the energy, or first integral, of the pendulum equation:

$$(\dot{\lambda})^2 = C_0 + \frac{2\left|K^2_{nm}\right|}{m} \cos m\left(\lambda - \lambda_{e,s}\right) . \tag{44}$$

If the initial condition $\dot{\lambda} = \dot{\lambda}_0$, at $\lambda = \lambda_0$, is given, then Equation 44 may be written as

$$(\dot{\lambda})^2 = \left(\dot{\lambda}_0\right)^2 + \frac{2\left|K^2_{nm}\right|}{m} \left[\cos m\left(\lambda - \lambda_{e,s}\right) - \cos m\left(\lambda_0 - \lambda_{e,s}\right)\right] . \tag{45a}$$

From Equation 45a can be found the maximum drift rate change possible in a resonant orbit due to each relevant harmonic. This will occur with nearly exact resonance ($\dot{\lambda}_0 \doteq 0$) momentarily over a position of unstable equilibrium. Then the drift rate passing the stable point (π/m radians away from $\lambda_{e,u}$) will be the maximum possible libratory drift rate. For $\dot{\lambda}_0 = 0$, $\lambda_0 = \lambda_{e,u} = \lambda_{e,s} - \pi/m$, and Equation 45a reduces to

$$(\dot{\lambda})^2 = \frac{2\left|K_{nm}^2\right|}{m}\left[\cos m\left(\lambda - \lambda_{e,s}\right) + 1\right] . \tag{45b}$$

The maximum libratory drift rate (at $\lambda = \lambda_{e,s}$) will then be

$$(\dot{\lambda})_{MAX,nm} = 2\sqrt{\frac{\left|K_{nm}^2\right|}{m}} . \tag{46}$$

Since the maximum acceleration in the H_{nm} libratory regime, from Equation 42, is $\left|\ddot{\lambda}_{MAX,nm}\right| = \left|K_{nm}^2\right|$, Equation 46 can be written as

$$(\dot{\lambda})_{MAX,nm} = 2\sqrt{\frac{\left|\ddot{\lambda}_{MAX,nm}\right|}{m}} . \tag{47}$$

Another parameter of interest with respect to the regime of the resonant satellite is the minimum libration period. This is the period of small angle "pendulum" oscillations of the ground-track about the *stable* equilibrium points defined in Equations 36 and 39. If $\lambda - \lambda_{e,s}$ is always sufficiently small (as in a simple pendulum oscillation), $\sin m\left(\lambda - \lambda_{e,s}\right)$ can be replaced by $m\left(\lambda - \lambda_{e,s}\right)$, and Equation 42 becomes a simple harmonic equation

$$\ddot{\lambda} + \left|K_{nm}^2\right| m\left(\lambda - \lambda_{e,s}\right) = 0 . \tag{48}$$

From Equation 48 we can find the small oscillations or minimum period libration about the stable equilibrium point. This minimum period is

$$T_{MIN,nm} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{m\left|K_{nm}^2\right|}} = \frac{2\pi}{\sqrt{m\left|\ddot{\lambda}_{nm,MAX}\right|}} . \tag{49}$$

Table 1 gives these three libratory characteristics, $\ddot{\lambda}_{MAX}$, $(\dot{\lambda})_{MAX}$, and T_{MIN}, for the most geodetically promising (underlined) orbits (see the next section). A comparative case, from actual data on 24-hour satellites, is provided as the last example in Table 1. The scale of these librations can best be appreciated by a study of the data in the last two columns of this table. The full librations of the ground-tracks have periods of years or tens of years, and amplitudes of tens of degrees.

The complete analytic treatment of the drift regime due to a single resonant harmonic (for a circular orbit) can be implemented by solving Equation 45 in terms of elliptic integrals. This will not be done here. The form of the general solution has been shown previously, (Reference 13), as has the specific solution for the long period of libration of one day resonant orbits.* As in the case of the circular pendulum, the actual libration period is strongly dependent on the librational amplitude. In addition, Gedeon et al.* has also obtained analytic solutions for the 1-day librational resonances of an eccentric orbit due to single harmonics.

We note only that, since the basic averaged equation of librational motion, Equation 29a, is nonlinear, the complete solution of the quasi-libration due to the total, infinite set of resonant harmonics, cannot be a simple superposition of the individual harmonic solutions. In fact, there is no second integral to Equation 29a which is in a simple closed form. However, the first, or energy, integral to Equation 29a can be written down analogously to Equation 45a. In terms of ℓ = 2, or the longitudes of the ascending equator crossing, Equation 29a may be rewritten with the differential reduction (Equation 41), as

$$d(\dot{\lambda})^2 = 2\, d\lambda \sum_{n,m \text{ relevant}} K_{nm}^2 \cos\left[m\left(\lambda - \lambda_{nm}\right) + \delta'(n, m, 2)\right] . \tag{50}$$

The integral of Equation 50 in terms of the initial conditions $\dot{\lambda} = \dot{\lambda}_0$ at $\lambda = \lambda_0 \, (t = 0)$, is

$$(\dot{\lambda})^2 = \left(\dot{\lambda}_0\right)^2 + 2 \sum_{n,m \text{ rel.}} \frac{K_{nm}^2}{m} \left\{\sin\left[m\left(\lambda - \lambda_{nm}\right) + \delta'(n, m, 2)\right] - \sin\left[m\left(\lambda_0 - \lambda_{nm}\right) + \delta'(n, m, 2)\right]\right\} . \tag{51}$$

Then, formally, variables in Equation 51 can be separated to find the time as a function of drift:

$$t = \int_{\lambda_0}^{\lambda} d\lambda \left\{\left(\dot{\lambda}_0\right)^2 + 2 \sum_{n,m \text{ rel.}} \frac{K_{nm}^2}{m} \left[\sin\left(m\left[\lambda - \lambda_{nm}\right] + \delta'[n, m, 2]\right) - \sin\left(m\left[\lambda_0 - \lambda_{nm}\right] + \delta'[n, m, 2]\right)\right]\right\}^{-1/2} . \tag{52}$$

In all of this simple derivation, the elements of the orbit have been assumed to be essentially constant (in particular K_{nm}), not only over the resonance period n', but over the long librational time t. While this will undoubtedly be a very good assumption for the semimajor axis (which in most cases will suffer small relative changes over long periods of time), it may not be so with regard to the inclination. Because of the gravitational action of the sun and moon and the earth zonal gravity, the orbit plane of a satellite (otherwise unperturbed) will precess about a plane between the ecliptic and the equator, causing a long-period sinusoidal-like change in the satellite's inclination (Reference 16). The period of this inclination change (with an amplitude of the order of about 10 degrees) can be as low as 10 years for close-earth satellites. However, for the close-earth satellites ($a_* \lesssim 4$, for example) zonal gravity equatorial precession predominates, so that the

*Gedeon, G. S., et. al., "Orbital Resonance Caused by Tesseral Harmonics," TRW Systems Internal Document (Private Communication); Redondo Beach, California, (Sept., 1966).

total inclination change is small. Thus, it will probably be found that only the relatively weak, far-earth resonances can be disturbed seriously by these plane precession effects.

SENSITIVITY AND DISCRIMINATION OF LONG-TERM HARMONIC EFFECTS ON RESONANT ORBIT SATELLITES

For any given commensurate orbit, only a limited number of harmonics cause these resonant librations (Equation 23). On all but 12- and 24-hour satellites having resonant orbits at or below synchronous, only two harmonics of low order n are active, and the lowest is separated from the next highest by two orders of n. Since the strength of gravity harmonics declines as the inverse n^{th} power of the distance from the mass center of the earth, it is reasonable to assume that for many of the resonant orbits, especially the more distant ones, only the single lowest-order longitude harmonic will dominate the libratory regime.

If this assumption is valid, then only two well-defined and separated measurements of the longitude acceleration of the nearly-constant ground-track will be necessary to define the amplitude and phase angle of these dominant harmonics. To find those resonant orbits which appear likely to show the strongest librations and to have the most favorable damping effect on higher order harmonics, a curve for the decline in the strength of the harmonics has been used, which is derived from a recent study of satellite data by W. M. Kaula (Reference 15 and Figure 4). This study reports so called normalized harmonic coefficients. The relative physical significance of the gravity harmonics may be found by a direct comparison of these normalized coefficients. It can be shown (Reference 17) that the normalized harmonics, $\overline{J_{nm}}\, P_n^m(\sin\phi)\cos m(\lambda - \lambda_{nm})$, all have a mean square amplitude of unity over the surface of the geoid (actually a sphere of unit radius) if $\overline{J_{nm}} \equiv -1$. The relation of normalized to unnormalized coefficients is shown in Figure 4. The individual gravity coefficients which affect the resonant orbits, have uncertainties which encompass the curve in Figure 4 in almost all cases. This curve apparently reflects the fact that the strongest longitude gravity anomalies are rather deep in the mantle of the earth. It has been accepted provisionally only for planning purposes. Using it, a table of dominant and subdominant, or next strongest, resonant-acceleration effects on a large number of subsynchronous orbits has been calculated from the longitude regime equation (Table 2). The

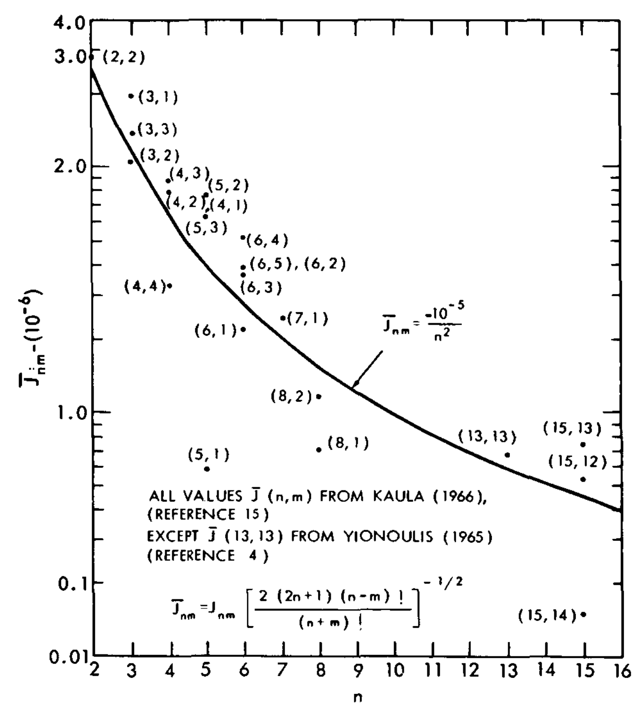


Figure 4—$\overline{J}_{nm}$ as a function of n from recent satellite studies.

Table 2

Long Term Geographic Accelerations on Longitude Gravity Resonant Orbits.[1]

Synodic period: n' (sid. days)	Orbit[2] period: 24 n'/r (sid. hours)	Dominant resonant harmonic (D, H_{nm})	Subdominant resonant harmonic (SD, H_{nm})	Other resonant harmonics (H_{nm})	$\ddot{\lambda}_{max}$ (D, H_{nm}) at i = 30° (10^{-5} rad./sid. day^2)	$\ddot{\lambda}_{max}$ (SD, H_{nm}) at i = 30° (10^{-5} rad./s.d.^{2})	$\ddot{\lambda}_M$(D) at i = 60° (10^{-5} r./s.d.^{2})	$\ddot{\lambda}_M$(SD) at i = 60° (10^{-5} r./s.d.^{2})	$\ddot{\lambda}_M$(D) at i = 90° (10^{-5} r./s.d.^{2})	$\ddot{\lambda}_M$(SD) at i = 90° (10^{-5} r./s.d.^{2})	Orbits per synodic period: r	Approximate[3] semimajor axis a_* (E.R.)
1	12.00	H_{32}	H_{44}	$H_{52}, H_{64}\cdots$	0.3481	0.0476 ✓	0.1516 ✓	0.0922 ✓	-0.2335 ✓	0.0546 ✓	2	4.164
1	8.00	H_{33}	H_{53}	$H_{66}, H_{73}\cdots$	0.4502	0.0336 ✓	1.086	-0.0040	0.9650	-0.0143 ✓	3	3.178
1	6.00	H_{54}	H_{74}	$H_{88}, H_{94}\cdots$	0.0545 ✓	0.0075 ✓	0.1025	-0.0046	-0.0702	0.0049	4	2.623
1	4.80	H_{55}	H_{75}	$H_{95}, H_{10,10}\cdots$	0.0340 ✓	0.0137 ✓	0.2462	0.0115 ✓	0.2918	-0.0144 ✓	5	2.261
1	4.00	H_{76}	H_{96}	$H_{11,6}, H_{12,12}\cdots$	0.0122 ✓	0.0056 ✓	0.0753	-0.0017	-0.0412 ✓	0.0062 ✓	6	2.002
1	3.43	H_{77}	H_{97}	$H_{11,7}, H_{13,7}\cdots$	0.0054 ✓	0.0060 ✓	0.1170 ✓	0.0238 ✓	0.1848	-0.0185 ✓	7	1.806
1	3.00	H_{98}	$H_{11,8}$	$H_{13,8}, H_{15,8}\cdots$	0.0039	0.0040	0.0752	0.0048	-0.0392 ✓	0.0100 ✓	8	1.653
1	2.67	H_{99}	$H_{11,9}$	$H_{13,9}, H_{15,9}\cdots$	0.0014	0.0032	0.0900 ✓	0.0453 ✓	0.1897 ✓	-0.0309 ✓	9	1.528
1	2.40	$H_{11,10}$	$H_{13,10}$	$H_{15,10}, H_{17,10}\cdots$	0.0017	0.0032	0.0992 ✓	0.0232 ✓	-0.0536 ✓	0.0202 ✓	10	1.424
1	2.18	$H_{11,11}$	$H_{13,11}$	$H_{15,11}, H_{17,11}\cdots$	0.0005	0.0021	0.0990 ✓	0.0962 ✓	0.2782 ✓	-0.0657 ✓	11	1.336
1	2.00	$H_{13,12}$	$H_{15,12}$	$H_{17,12}, H_{19,12}\cdots$	0.0009	0.0028	0.1661 ✓	0.0833 ✓	-0.0980 ✓	0.0503 ✓	12	1.261
1	1.846	$H_{13,13}$	$H_{15,13}$	$H_{17,13}\cdots$	0.0002	0.0016	0.1445 ✓	0.2358 ✓	0.5414 ✓	-0.1720 ✓	13	1.196
1	1.714	$H_{15,14}$	$H_{17,14}$	$H_{19,14}\cdots$	0.0006	0.0029	0.3422 ✓	0.3009 ✓	-0.2277 ✓	0.1509 ✓	14	1.138
1	1.600	$H_{15,15}$	$H_{17,15}$	$H_{19,15}\cdots$	0.0002	0.0014	0.2667 ✓	0.6662 ✓	1.332 ✓	-0.5410 ✓	15	1.087
1	1.500	$H_{17,16}$	$H_{19,16}$	$H_{21,16}\cdots$	0.0005	0.0004	0.8447 ✓	0.1493 ✓	-0.6495 ✓	0.0686 ✓	16	1.041
2	16.00	H_{43}	H_{63}	$H_{66}, H_{83}\cdots$	0.0343	0.0003	0.0000	-0.0002	-0.0269	0.0003	3	5.044
2	9.60	H_{65}	H_{85}	$H_{10,5}, H_{10,10}\cdots$	0.0035	0.0002	0.0037	-0.0002	-0.0050	0.0002	5	3.588
2	6.86	H_{87}	$H_{10,7}$	$H_{12,7}, H_{14,7}\cdots$	0.0004	0.0001	0.0018	-0.0001	-0.0016	0.0001	7	2.867
2	5.33	$H_{10,9}$	$H_{12,9}$	$H_{14,9}, H_{16,9}\cdots$	0.0001	0.0000	0.0009	-0.0000	-0.0008	0.0001	9	2.425
3	18.00	H_{54}	H_{74}	$H_{88}, H_{94}\cdots$	0.0039	0.0000	-0.0013	-0.0000	-0.0027	0.0000	4	5.456
3	14.40	H_{55}	H_{75}	$H_{95}, H_{10,10}\cdots$	0.0061	0.0002	0.0095	-0.0001	0.0037	0.0001	5	4.702
3	10.29	H_{77}	H_{97}	$H_{11,7}\cdots$	0.0003	0.0000	0.0012	-0.0000	0.0007	0.0000	7	

NOTES:

(1) Based on maximum effects for a given inclination, from Equation 29a and the $\overline{J_{nm}}$ curve of Figure 4.

(2) Nodal period.

(3) Assumes the nodal period is equal to the Keplerian period.

✓: Checked effects are reasonably strong, but use of such single orbits probably will not permit unambiguous discrimination of these harmonics. Reasonable strength for these effects are based on the tracking record of "Early Bird" (Reference 11). [sensing 0.01×10^{-5} rad./day^2 in 60 days] maximum harmonic accelerations in these orbits occur when the equator crossings of the constant ground-track are midway between one of the m stable longitudes and one of the adjacent m unstable longitudes due to a given harmonic. The accelerations are strongly dependent on inclination, through the sinusoidal inclination functions in the regime Equation 29a. In Table 2, those longitude harmonic accelerations which seem to offer the best chance to be quickly and unambiguously discriminated have been underlined. By best chance, we mean they appear to be resolvable in a few month's time, and of an order of magnitude stronger than the next strongest harmonic effect. This criterion for quick resolution of these effects is based on the tracking of "Early Bird" (Reference 11), which resolved an acceleration of 0.01×10^{-5} rad/day^2 in 60 days.

For 30-degree inclined satellites, it seems that the resonant orbits of 8, 12, 14.4 and 16 hours are especially suited for discrimination of the harmonics H_{33}, H_{32}, H_{55} and H_{43}. At higher inclinations and lower altitudes, the higher-order longitude effects are stronger and stand out more clearly. For example, orbits of 3, 4, 4.8, and 6 hours with about a 60-degree inclination appear favorable for discrimination of the harmonics H_{98}, H_{76}, H_{55} and H_{54}.

The observations necessary to detect these long-term amplified gravity effects are for the most part elementary and easily implemented. The orbit at the start of a free-drift period must be reasonably circular and near-stationary. The inclination should also be known to within a few tenth's of a degree. During the long free-drift periods, precisely observed longitudes of all the equator crossings is sufficient for a harmonic determination from the simple drift regime equation.

DETERMINATION OF H_{22} AND H_{33} FROM 24-HOUR SATELLITE DATA

In the case of the 24-hour satellites, numerical integration has shown (Reference 11) that additional gravity perturbations to the regime equation (Equation 29a) are within the standard deviations of the nodal acceleration observations presented in Table 3. The record consists of 3 years of reduced equator crossing accelerations in 13 longitudinally separated free-drift arcs for Syncom 2 and 3 and the Early Bird satellite, all of which have nearly circular orbits. The gravity coefficients C_{nm}, S_{nm} in Table 3 are defined from the J_{nm}, λ_{nm} of Equation 11 as:

$$C_{nm} = -J_{nm} \cos m\lambda_{nm} ,$$

and

$$S_{nm} = -J_{nm} \sin m\lambda_{nm} ,$$

to conform to another common way of expressing the potential as a fully linear series of harmonic functions. This representation, of less use physically, has the advantage of allowing a fully linear separation of harmonic effects convenient to the analysis and reduction of real data. The normalized longitude coefficients $\overline{C_{nm}}, \overline{S_{nm}}$ are defined from C_{nm}, S_{nm} by

$$\overline{C_{nm}}, \overline{S_{nm}} = C_{nm}, S_{nm} \left[\frac{(n-m)!\,(2n+1)2}{(n+m)!}\right]^{-1/2} ,$$

which yields

$$\overline{J_{nm}} = -\sqrt{\overline{C}_{nm}^2 + \overline{S}_{nm}^2} .$$

Table 3

Long Term Acceleration Data and Longitude Harmonics from 24 Hour Satellites.

Satellite	Arc	a_* (E.R.)	I (deg's)	λ (deg's)	$\ddot{\lambda}$ (10^{-5} rad./day^2)	$\sigma(\ddot{\lambda})$ (10^{-5} rad./day^2)	C_{22} (10^{-6})	S_{22} (10^{-6})	C_{33} (10^{-6})	S_{33} (10^{-6})	S 10^{-7} rad./day^2
Syncom 2	1	6.611	33.02	-55.22	-2.253	0.0500	(1.57 ± 0.06)	(-0.98 ± 0.06)			(21.4) *
Syncom 2	2	6.612	32.83	-60.94	-2.291	0.0572	1.56 ± 0.02	-0.93 ± 0.02	0.044 ± 0.015	0.165 ± 0.015	4.54 †
Syncom 2	4	6.620	32.58	-140.00	2.138	0.0842					
Syncom 2	5A	6.617	32.40	161.00	-0.199	0.0661	$\bar{C}_{22}$	$\bar{S}_{22}$	$\bar{C}_{33}$	$\bar{S}_{33}$	
Syncom 2	5-18	6.618	32.15	104.5	-2.278	0.0656	(10^{-6})	(10^{-6})	(10^{-6})	(10^{-6})	
Syncom 3	6	6.611	0.11	178.71	1.707	0.0591	2.42 ± 0.03	-1.44 ± 0.03	0.32 ± 0.11	1.18 ± 0.11	†
Syncom 3	7	6.612	0.27	176.80	1.550	0.175	(Recent data from lower altitude satellites)				
Syncom 2	8	6.611	31.87	66.12	0.950	0.0616	2.49	-1.40	0.41	1.36 ①	
Early Bird	9	6.611	0.20	-28.70	-1.441	0.0500	2.45	-1.52	0.58	1.62 ②	
Syncom 3	10	6.611	0.00	172.75	1.072	0.0934					
Syncom 2	11-1	6.61	31.4	69.0	0.676	0.0500	① From camera data (due to W. M. Kaula) – Private communication, W. M. Kaula (1966)				
Syncom 2	11-2	6.61	31.4	77.6	-0.152	0.0500					
Early Bird	12-1	6.611	0.43	28.61	-1.430	0.0500	② From Doppler Data (due to R. J. Anderle) – Private communication, W. M. Kaula (1966)				
Early Bird	12-2	6.612	0.55	31.24	-1.597	0.0531					
Early Bird	12-3	6.61	0.6	35.91	-1.901	0.0500					
Early Bird	12-4	6.61	0.74	36.20	-1.961	0.0500					
Syncom 3	13-1	6.61	0.53	169.1	0.831	0.0500	S = Standard error of least squares fit				
Syncom 3	13-2	6.61	0.53	168.29	0.699	0.0500	*First Solution †Second Solution				

*First Solution
†Second Solution

The harmonics H_{22} and H_{33} reported in the right hand columns of Table 3 were derived by a weighted least-squares-solution of Equation 29a using the orbit data in the left columns of the table.

The 24-hour satellite is in librational-resonance with longitude harmonics of all orders, the only restriction being that the difference between the order n and the longitude frequency m must be even. Nevertheless, the altitude of the satellite is so high that the effects of the third order harmonics are considerably damped with respect to that of the second order (H_{22}). This is the harmonic which reflects the earth's elliptical equator.

The fairly extensive data in Table 3 was examined for sensitivity, first to the second order harmonic alone (results in parentheses in Table 3, and secondly to a combination of 2nd and 3rd order harmonics (results directly below those in parentheses). In the second solution, only small correlation was found between the effects of 2nd and 3rd order, and only minor changes from the 2nd order coefficients of the first solution. This and additional analysis which include other relevant harmonics, appear to show that the sectorial harmonics of second and third order, H_{22} and H_{33} have been determined by these data alone to an absolute precision within the bounds shown in the second solution of Table 3. Comparison of these results with recent data from lower-altitude satellites shows reasonable agreement, except perhaps with respect to S_{33}.

The data tests for sensitivity to H_{31}, H_{42} and H_{44} did not produce a significant change in the results for H_{22} and H_{33} or an improvement in their standard errors. It is concluded that the 24-hour data are not accurate enough yet or sufficiently widespread in longitude to allow a good determination of these other resonant harmonics through 4th order.

DISCUSSION

In mechanical systems, resonant phenomena occur when the periodicity of applied loads is close to the natural (unforced) periodicities of the system. Resonance in this sense is characterized by greatly amplified vibrations of the system. Viewed internally, the applied loads are always in step, or in the same direction as the moment of the internal parts upon which they act, and so continually increase the vibrational energy of those parts. But viewed externally, the loads are purely sinusoidal, and so the mean translational energy of the center of mass of the system is unchanged. In the case of the satellite whose orbit yields a constant ground-track with respect to certain harmonics, the applied loads are not purely sinusoidal, but contain a bias component which drives the long-period libration. In the case of the constant ground-track orbit, resonance implies a change in the translational, or orbit energy of the satellite. The vibrational, or internal energy of the orbit can perhaps best be characterized by the eccentricity. In an eccentric orbit, the satellite can be imagined as vibrating continually about its mean anomaly and semimajor axis, or mean radius. This vibrational aspect of the orbit has been ignored by orbit-averaging the disturbing force. This is done because we are here interested only in long-term orbit energy changes. In terms of the mechanical resonance analogy, the full effect would be equivalent to a mass-spring assembly on wheels, vibrating and translating under the action of a biased, sinusoidally varying, external force. This paper has been concerned only with the steady, translational part of the complete motion. In celestial mechanics, these changes are called secular, or effects which increase in time without apparent limit. In the case of commensurate orbits, the bias of the disturbing force will also change sinusoidally over a long period of time. This is the phenomenon called libration, having as its proper analogy the circular pendulum. (It should be noted again that the resonant orbits have been specified to be near circular.)

The "resonance" phenomenon that has been investigated by the orbit-averaging method here, might better be called secular or librational-circular pendulum effects. (In particular, we have really dealt only with the long term changes in the semimajor axis.) But the "in-step action" of the longitude harmonics on these commensurate orbits gives rise also to true vibrational reasonances (in radius and anomaly), following the mechanical analogy (References 2, 18, and 19). Therefore, though the effect here analyzed is not commonly thought of as a resonance effect, it does arise in general for those orbits which also show true resonance effects; thus these orbits may be called resonant justifiably. When a "resonant gravity harmonic" is referred to in this paper, the word resonant is used only in the sense of being capable of producing the amplified changes of libration which are also properties of the resonant orbits.

It has been pointed out that the true vibrational resonances in a commensurate orbit imply a buildup, or amplification, of the orbit eccentricity. This amplification would be without bound if true commensurability could be maintained. Of course it cannot, since the resonant orbit generally librates. In fact, as might be inferred from its dependence on the direction of the disturbing force, the eccentricity in a resonant orbit appears to go through the same long-period libration that the semimajor axis does. However, these true resonances of eccentricity (called "Dynamic Resonance" by Blitzer (Reference 19)) may not be as useful geodetically, since many more of the longitude harmonics H_{nm} appear to contribute to them.

The pendulum-like changes of the semimajor axis (describable from a separated solution of the coupled Equations 29 and 30) (References 13, 14), have a periodicity of years only for near-commensurate conditions. These are the librating orbits first discussed in Reference 20 for the 24-hour case. In cases where the ground-track moves at a considerable rate, the basic circular pendulum Equations 29 and 30, with small modification (along the lines outlined in Reference 2), may still serve as the model of the evolution. In these cases of fast global circulation of the ground-track, evaluation of Equations 29 and 30 shows that the crossing rate and semimajor axis oscillate with a more rapid frequency but much diminished amplitude as compared to a libration case. For example, in a fast circulation case, the period is typically of the order of days, and the amplitude may be of the order of hundreds of meters (for the semimajor axis change). In full libration cases, well away from the equilibrium positions, the amplitudes of the semimajor axis change may be of the order of tens of kilometers (Reference 13). The fast world-circulation regime (far from resonance) is actually a circulating pendulum phenomenon. The reduction in amplitude of the pendulum effect in a fast circulation regime is due to the rapidity with which the orbit-averaged bias force averages out over a global circulation.

In spite of the small amplitudes, circulating pendulum resonance has been used to good effect in discriminating a number of very high order coefficients in the earth's field from a dense global tracking of the Navy's Doppler-Transit Satellites (Reference 4).

The limitations of the model used here to derive the key orbit evolution Equations 29 and 30 should be re-emphasized. Though strictly speaking, these equations apply only to circular orbits of exact resonant period, they have been shown, by numerous examples (Reference 11) and in a rigorous way to be applicable with only slight correction or modification to a much wider class of satellites. This class includes those orbits with drift rates considerably exceeding the maximum permissible for libration as well as for noncircular orbits with eccentricities no higher than about 0.01. The equations, furthermore, give only the gross orbit-averaged effects due to the earth's longitude gravity field. But over a period of the order of months, it may be presumed that, well away from any strong influence of the earth's atmosphere or the moon's gravity, these equations will describe the dominating long-term effect on the geographic configuration of all near-resonant, near-circular orbits, with the exception of inclination change due to the sun and moon. The extension of these regime equations to the class of eccentric resonant orbits is indicated by Gedeon, et. al. (See note at the bottom of page 15)

SUMMARY AND CONCLUSIONS

The long-term geographic evolution of circular-orbit satellite ground-tracks which are originally near stationary has been found. The equation governing the motion of the near-stationary ground-track is essentially that of a circular pendulum for each relevant earth longitude harmonic. These relevant harmonics form a tenuous but infinite subset of the infinite set of longitude harmonics.

For orbit periods very close to resonant, or rational with respect to the earth's rotation rate, the sinusoidal ground-track librates with a maximum excursion of $360°/m$, due to an individual relevant (resonant) earth longitude harmonic. The complete period of libration is two to ten years for the strongest resonances and depends strongly on the libration amplitude and inclination.

The libratory-like evolution equations can be extended to far-from-stationary conditions to cover the smaller, but detectable, drift oscillations of higher frequency, called circulating pendulum resonance.

Circular resonant orbits of periods 3, 4, 4.8, 6, 8, 12, 14.4, and 16 hours appear to be particularly suited to discriminating unambiguously the longitude harmonics H_{98}, H_{76}, H_{55}, H_{54}, H_{33}, H_{32}, H_{55}, and H_{43} respectively, in a reasonably short period of time. As of 1966, the 24-hour resonant orbit satellites have provided the best discrimination of the two leading sectorial harmonics H_{22} and H_{33}.

The orbit determination for the resonant satellite used as a gravity probe would not need to be elaborate. It would have to insure that a reasonably circular orbit has been achieved. The inclination must also be determined to about 0.1 degree accuracy. During free drift periods, the longitudes of all the equator crossings are sufficient data for a rapid harmonic determination.

In conclusion, while the long-term librational resonances of the commensurate orbit satellites will be an annoying problem to the operators of communications satellites, they provide a powerful and unique means to solve many of the mysteries of the earth's complex gravity field.

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Appendix

Symbols

a	The semimajor axis of the satellite's orbit (a_* is the semimajor axis in units of earth radii).
C_{1_p}, C_{2_p}	Phase and frequency constants of the disturbing function.
C_{nm}, S_{nm}	Non-normalized cosine and sine gravity harmonic coefficients.
(D,), (SD,)	Dominant, subdominant.
E	The total energy (kinetic and two-body gravitational potential) of a satellite.
exp[ ]	$e^{[\ ]}$, $e = 2.718...$
F	A perturbation force (when superscripted with a prime a nondimensional force). When subscripted nmp, an inclination force function.
$F_{nmp}(I)$, F'_{nmp}	An inclination force function, when superscripted with a prime, the real value of the force function.
F_T	The tangential (along-track) component of the perturbation force F.
G	Location of the Greenwich meridian.
H_{nm}	Signifying the gravitational harmonic term of order n and power m.
I	Inclination of the satellite's orbit.
i	$\sqrt{-1}$
$\overline{J_{nm}}$, $\overline{C_{nm}}$, $\overline{S_{nm}}$	The normalized amplitude, cosine coefficient and sine coefficient of the gravity harmonic H_{nm}.
J_{nm}, λ_{nm}	The amplitude and phase of the non-normalized gravity harmonic H_{nm}.
K	A positive integer or zero.
K^2_{nm}	An inclination-dependent harmonic forcing function usually assumed constant over a libration period.
ℓ	As a subscript of in $\delta'(n, m, \ell)$; $\ell = 1$ or odd refers to a descending equator crossing, $\ell = 2$ or even refers to an ascending equator crossing.

n' The synodic period (in sidereal days) of the resonant orbit satellite, an integer. The number of integral days for the stationary ground track to begin to repeat itself.

$P_n^m(x)$ The associated Legendre polynomial of order n and degree m:

$$P_n^m(x) = \left(1 - x^2\right)^{m/2} \frac{d^m P_n(x)}{dx^m},$$

where

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} \left\{ x^n - \frac{n(n-1)\, x^{n-2}}{\left(2^1\right)(1!)(2n-1)} + \frac{n(n-1)(n-2)(n-3)\, x^{n-4}}{\left(2^2\right)(2!)(2n-1)(2n-3)} - \cdots \right\}.$$

p A positive integer, or zero.

r Number of orbits in the synodic period of the resonant satellite (n' sidereal days); an integer.

R_0 The mean equatorial radius of the earth.

$R\{\ \}$ The real part of $\{\ \}$.

S Position of the satellite (S_0 is its position at the start of the dynamics, and S' at the reference equator crossing).

T The period of the satellite's orbit.

t Time from the beginning of the dynamics or some arbitrary zero.

T_{MIN} The minimum libration period of a resonant orbit.

U_e The gravitational potential function of the earth.

γ The location of the vernal equinox.

Δ A small, but finite, change.

δ_{nm}, $\delta'(n, m, \ell)$ Phase angles of the disturbing function.

θ Argument of the ascending node, central angle in the orbit from the ascending node to the satellite.

λ Generally, geographic longitude of the ascending equator crossing of the satellite. Can also refer to the geographic longitude of the descending equator crossing.

λ_e; $\lambda_{e,s}$; $\lambda_{e,u}$ An equilibrium longitude; stable, and unstable.

μ The earth's Gaussian gravitational constant.

σ Standard deviation (or estimate of it).

ϕ Geographic latitude of the satellite.

Ω The Keplerian element: Right Ascension of the Ascending Node.

ω A frequency or angular rate.

ω_e The earth's inertial rotation rate.

ω_s Orbital revolution rate of the satellite.

$\overline{(\ )}$ The r orbit average of the quantity ( ), except in reference to $\overline{C_{nm}}$, $\overline{S_{nm}}$, $\overline{J_{nm}}$.

$\dot{(\ )}$ d( )/dt , with the time increment in units of sidereal days.

$\ddot{(\ )}$ $d^2(\ )/dt^2$, with the time increment in units of sidereal days.