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A THEORY OF MANUAL SPACE NAVIGATION

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PREFACE

During the author's appointment as a staff member of the Massachusetts Institute of Technology Instrumentation Laboratory in the Apollo project (1963-1965) the general problem was formulated: could man navigate in space without the assistance of an electronic computer? And if manual navigation was possible what would be the capabilities of the manual navigation system?

It seemed apparent to the author that if this country's space program progressed to a future stage of more or less routine, extended space voyages, voyages with capabilities for on the spot decision, changes of plans, inspections, etc., in short routine voyages with very flexible flight plans, then manual navigation capability to backup and/or augment more sophisticated electronic navigation systems could contribute substantially to mission reliability.

If a manual navigation system were to be of use to future space voyages of flexible flight plan it seemed necessary to liberate the navigation system from dependence on prestored reference trajectories. This was a consideration in the following work.

R. H. Battin (Astronautical Guidance, McGraw Hill, New York, 1964, Chapter 7.1) well summarized the problem of navigating by direct use of celestial fixes. He says "An exact determination of position by methods described in this section (direct use of celestial fixes) has a number of distinct disadvantages. First of all, the resulting algebraic equations to be solved are always nonlinear, which might prove to be a fairly significant obstacle to on-board computation. Second, the method requires simultaneous measurements which are almost certainly impractical. Finally, and perhaps most important of all, no satisfactory method of incorporating redundant measurements to compensate for instrumentation inaccuracies is known." Then Battin goes on to indicate that navigation with respect to a reference trajectory can overcome these disadvantages.

This work overcomes the first and third disadvantages described above by developing navigation equations which though directly using celestial fixes without a reference trajectory, are linear equations which readily incorporate redundant measurements in their orbit determination.

The second disadvantage, the impracticability of simultaneous measurements, has been shown in this work to pose a not insurmountable problem (Chapter VI).

Although the original motivation of the author in deriving these new navigation equations was the development of a manual navigation system, it seems possible that the navigation equations of this work may have application as a new starting point in computerized navigation. This looks particularly promising to the author in light of the freedom of this new approach from reference trajectories.

Another point which is of practical interest, the navigation equations developed in this work do not use time as a measured parameter. This freedom from time has the advantage of liberating space navigation from an accurate long term time standard.

This new approach to space navigation was first discussed in a preliminary report, E-1540, "A Preliminary Study of a Back-up Manual Navigation Scheme", August, 1964, M. I. T. Instrumentation Laboratory, supported by the National Aeronautics and Space Administration through Contract NAS 9-153.

The author would like to thank his group leader at the M. I. T. Instrumentation Laboratory, Mr. John Dahlen, for encouragement during the early phases of the work, and Miss Ann Romer of the Montana State University Physics Department, who did the computer work necessary to produce the perturbation tables in this work.

The author would also like to thank Harold A. Hamer and Margery E. Hannah of the Langley Research Center of N. A. S. A. who gave this work a careful technical reading and found many places where he could improve the manuscript.

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ABSTRACT

This work develops a set of space navigation equations which are exact equations, yet are linear equations. This makes them particularly suited to manual space navigation applications.

The linearity of the equations leads to simple methods for incorporating redundant observations into the orbit parameter estimation equations.

Navigation accuracy and other operational aspects of a proposed manual space navigation system are studied.

CHAPTER I

INTRODUCTION

A. Review of Standard Navigational Methods

For the purposes of this work space navigation is defined as the accurate estimation of the future position of a spacecraft with respect to a dominant central gravitational body (a planet, moon, or the sun, etc.).

Methods exist (see Battin, Astronautical Guidance, McGraw Hill, New York, 1964, Chapter VII) for finding a spacecraft orbit by direct use of several celestial angle measurements taken at a given time. However as Battin points out such direct use of celestial angular data as a basis of a space navigation system has been thought to contain serious disadvantages:

1. Operational difficulty of making simultaneous measurements.

2. Nonlinearity of the equations, and consequently the mathematical complexity of their solution, not amenable to quick solution by small computers or by manual means.

3. Difficulty of incorporating redundant measurements into the navigation equations.

As a result of the above difficulties the standard approach to space navigation has been to define a reference trajectory which obeys

$$d^2\overline{R}_o/dt^2 = \overline{G}(\overline{R}_o) \tag{1.1}$$

where $\overline{G}$ is the total gravitational acceleration and $\overline{R}_o$ is specified uniquely by initial conditions (a position and velocity vector). Due to the complexity of the gravitational field when perturbing bodies are present, the integration of (1.1) to obtain the reference trajectory requires substantial computer effort.

Navigation consists of knowing where the spacecraft is relative to the reference trajectory. Defining

$$\bar{R} = \bar{R}_o + \Delta\bar{R} \tag{1.2}$$

as the true spacecraft position, with $\Delta\bar{R}$ the "off course" vector, a linearized differential equation for $\Delta\bar{R}$ is obtained

$$d^2\Delta\bar{R}/dt^2 = [\Omega]\ \Delta\bar{R} \tag{1.3}$$

where the matrix $[\Omega]$ is given by the gradients of the gravitational potential

$$[\Omega]_{ij} \equiv \partial\ G_j(\bar{R}_o)\ /\ \partial\ R_i \tag{1.4}$$

The matrix $[\Omega]$ is in general a complicated function of time which must be stored by a computer. The validity of (1.3) depends on the "off course" vector being sufficiently small. Large excursions from the reference trajectory require the calculation of a new reference trajectory.

(1.3) is the starting point of practical, computerized navigation. The navigational goal is to estimate the six numbers which give the initial conditions for the "off course" vector. Six celestial measurements at known times give a solution for the "off course" vector for all times. The linearity of (1.3) allows for a simple recursive method for incorporating redundant measurements (Battin, Chapter IX).

B. Manual Space Navigation

The purpose of this work will be to develop a navigation system which is not dependent on electronic equipment. In particular we wish to develop a manual navigation system which does not depend on on on-board computers, pre-calculated reference trajectories, a ground communication link, or a long term accurate time standard.

The manual navigation system must use relatively simple equations which can be solved readily by man without electronic aid. It will be assumed that a space navigator will have a sextant type device to measure celestial angles to an rms accuracy of about .5 arc-minutes. B. A. Lampkin, R. J. Randle "Investigations of a Manual Sextant-Sighting Task in the Ames Midcourse Navigation and Guidance Simulator", Ames

Research Center, Moffett Field, California, N. A. S. A. TN-D-2844, suggest that .4 arc-minutes rms accuracy can be obtained from a hand helded sextant after a reasonable training period.

We also assume that trigonometric and arithmetic tables could be stored on board to assist the navigator in solving any navigation equations. Reasonable weight and bulk restrictions would of course be present.

C. Symbol List

Below is a list of symbols which will be used in various parts of this work.

R	spacecraft distance from central body
θ	spacecraft azimuth from orbit perigee
R_c	radius of central body
e	orbital eccentricity
b	orbit dimensionless semi-latus rectum
n	dimensionless spacecraft distance from central body (measured in units of R_c)
h	angular momentum constant of orbital motion
U	energy constant of orbital motion
G	universal gravitational constant
M	mass of central body
T_o	characteristic time of central body
s	semisubtended angle of central body
$\theta_o(1)$	spacecraft azimuth when star one makes its minimum angle with the central body
$\beta(1)$	spacecraft azimuth measured from $\theta_o(1)$
A,B,C	constant parameters in the one star navigation equation

$\gamma_o(1)$	elevation angle of star one from orbital plane
$\gamma(1)$	star one central body angle
N	total number of navigational observations
N_1	number of observations in cluster one
c_1, c_2, c_3	constants useful in solution of one star navigation equations
A', B', C'	constant parameters in the two star navigation equations for b
c_1', c_2', c_3'	constants useful in solution of two star navigation equations for b
y	navigation variable defined for two star navigation equations for e
A'', B'', C''	constant parameters in the two star navigation equations for e
c_1'', c_2'', c_3''	constants useful in solution of two star navigation equations for e
m_α	general measureables in navigation equations
K_α	general constant parameters in navigation equations
ε_i	measurement error elements
b_i	bias components
Δ	rms angular measurement error
σ	total rms value of error element
ξ_1, ξ_2	useful constants defined in error analysis
ξ_i	eccentric anomaly
$\hat{r}$	unit radial vector
$\hat{p}$	unit vector perpendicular to orbital plane
$\hat{t}$	unit vector normal to both $\hat{r}$ and $\hat{p}$
$\hat{s}$	unit vector toward the navigation star

$D_{x,y,z}$	components of dimensionless "off course" vector
$A_{x,y,z}$	components of perturbing acceleration
G_{xy}	partial derivatives of central body gravitational acceleration components
M_L	mass of perturbing body
L	distance of perturbing body from central body
θ_L	perturbing body azimuth from spacecraft orbital perigee
γ_L	perturbing body elevation angle from spacecraft orbital plane
λ	dimensionless perturbation strength parameter
Q_i	Perturbation correction term to the two star navigation equations for b
Q_i'	Perturbation correction term to the two star navigation equations for e
Λ	useful combination of in-plane "off course" components
$\Delta T(1)$	time difference between measuring central body subtended angle and star one central body angle
η	useful defined function in non-equal-time error analysis
w_α	weighting vectors used in optimum navigation
$[T]$	transformation matrix to a general function of orbital parameters
$(Cf)_{\alpha\alpha'}$	cofactor elements of a matrix
ψ_α	linear combination of weighting vectors w_α
μ_α	linear combination of measureables m_α
$[E]$	error correlation matrix
ϕ_i	coefficients of systematic errors
H	eccentric anomaly for hyperbolic orbits

ρ_i	residues of navigation equation violation for individual observations

Subscripts

per	perigee
ap	apogee
i	refers to the ith observation
α	refers to αth constant parameter
nav	navigation
opt	optimum
o	generally refers to quantity evaluated at $\theta=\theta_o$
L	refers to perturbing body

(throughout this work common subscripts are to be summed over unless otherwise stated)

Other Notation

δx	a small differential of the variable x
[M]	a square matrix
{V}	a column vector
$\langle x \rangle$	statistical average of variable x
$\sum_c(x)$	sum of variable x over observations in cluster c
∂	partial derivative symbol
$\lvert x \rvert$	absolute value of x
$\mathbf{V}$	spatial vector
$\dot{x}$	time derivative of x
∇	spatial gradient vector
$\hat{v}$	unit vector

CHAPTER II

ONE STAR NAVIGATION EQUATIONS

A. Introduction

For a spacecraft in the vicinity of a central body, it is assumed that a sextant-type instrument is capable of measuring the angle between the central body edge and any of several fixed stars. Also the angular size of the central body is assumed measureable.

The angle between the central body edge and other non-fixed bodies (other planets or moons) could also be measured, but the mathematical equations necessary to incorporate such measurements into navigation equations are of such complexity as to rule out these measurements for practical manual navigation systems.

The goal then is to develop navigation equations which use as directly as possible (with a minimum of arithmetical manipulation) the measurement of central body subtended angles and the measurements of central body-star angles.

First the navigation equations for pure Keplerian orbits will be derived. In a later chapter (Chapter V) methods for correcting for perturbations are developed.

B. Review of Useful Properties of Conic Orbits

All orbits in an inverse square acceleration field are described in polar coordinates by

$$R = R_c\ b/(1+e\cos\theta) \tag{2.1}$$

where R is the radial distance of the spacecraft and θ is its azimuth measured from the point of closest approach to the central body (perigee). R_c is the central body radius, b and e are dimensionless constants specifying the shape of the orbit.

Measuring radial distances in units of the central body radius, (2.1) becomes

$$n = b/(1+e\cos\theta) \tag{2.2}$$

with

$$n = R/R_c \tag{2.3}$$

Perigee distance is given by

$$n_{per} = b/(1+e) \tag{2.4}$$

and for closed orbits (ellipses) which have $e < 1$ the apogee is given by

$$n_{ap} = b/(1-e) \tag{2.5}$$

Kepler's second law or the law of conservation of angular momentum states that

$$R^2\, d\theta/dt = h \tag{2.6}$$

with the constant h given by

$$h = \sqrt{GMbR_c} \tag{2.7}$$

G is the universal gravitational constant, M the mass of the central body. The other constant of the motion is the total energy

$$U = -GM(1-e^2)/2bR_c \tag{2.8}$$

The plane of the orbital motion is another constant in time, the unit vector normal to the orbital plane $\hat{p}$ being a constant.

At several points in this work when dealing with time it will be convenient to measure time in units

$$T_o = \sqrt{R_c^{\,3}/GM} \tag{2.9}$$

C. Derivation of One Star Navigation Equations

Using (2.2) and defining

$$1/n = \sin s \tag{2.10}$$

with s the semisubtended angle of the central body as shown in Figure (2.1), the Keplerian orbits can be expressed as

$$b \sin s = 1 + e \cos\theta \tag{2.11}$$

Instead of referring the azimuth angle to perigee another angle $\theta_o(1)$ is introduced, and (2.11) takes the form

$$b \sin s = 1 + e\cos\theta_o(1)\cos(\theta-\theta_o(1)) - e\sin\theta_o(1)\sin(\theta-\theta_o(1)) \tag{2.12}$$

or defining

$$\beta(1) = \theta - \theta_o(1) \tag{2.13}$$

(2.12) becomes

$$b \sin s = 1 + e\cos\theta_o(1)\cos\beta(1) - e\sin\theta_o(1)\sin\beta(1) \tag{2.14}$$

Defining the three constants

$$A = e \cos\theta_o(1) \tag{2.15a}$$
$$B = -e \sin\theta_o(1) \tag{2.15b}$$
$$C = -b \tag{2.15c}$$

(2.14) becomes

$$A \cos\beta(1) + B \sin\beta(1) + C \sin s + 1 = 0 \tag{2.16}$$

(2.16) is still only a disguised form of the original conic equation (2.1). We now show that for an appropiate choice of $\theta_o(1)$ both s and $\beta(1)$ are closely connected with measureable

RELATIONSHIP BETWEEN n AND THE MEASURABLE (semi subtended) ANGLE s

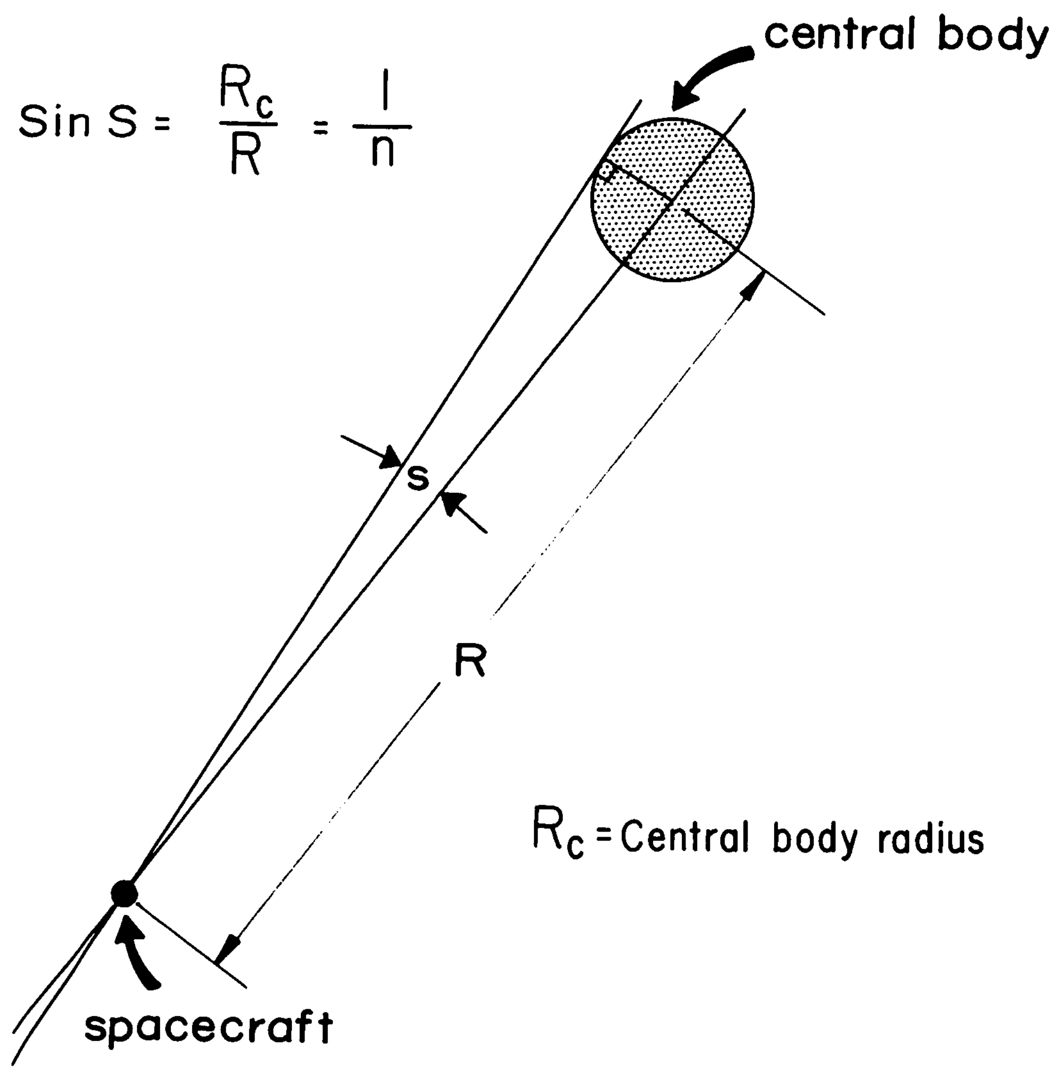

Figure 2.1 The semisubtended angle of the central body, s, is shown to give the range of the spacecraft from the central body by $\sin s = R_c/R = 1/n$.

RELATIONSHIPS AMONG VARIOUS NAVIGATION ANGLES

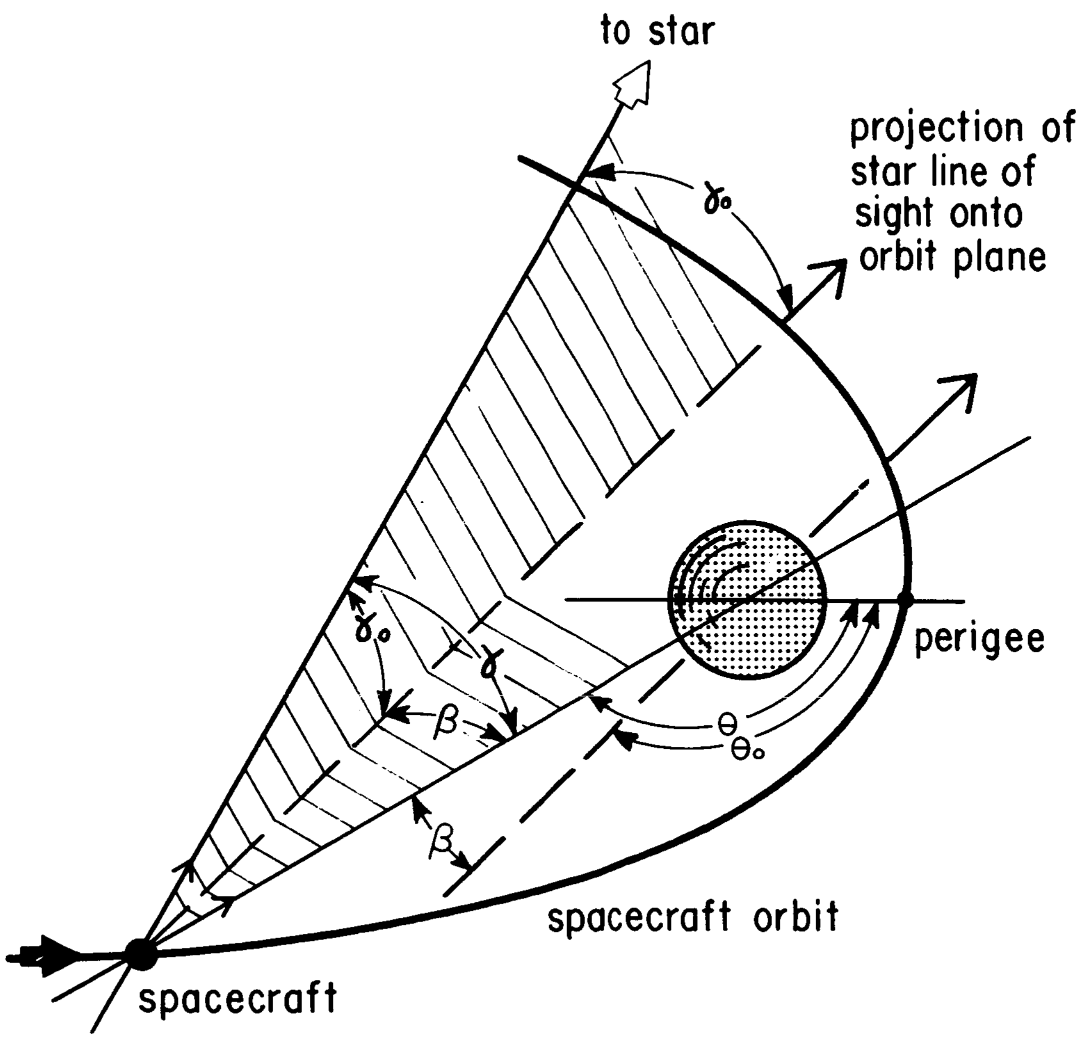

Figure 2.2 The various navigation angles and their relationships are shown. θ = spacecraft azimuth from perigee, θ_o = spacecraft azimuth when star lies directly above (below) the central body, β = spacecraft azimuth measured from θ_o, γ_o = star elevation angle from orbital plane, and γ = star-central body angle.

quantities. The fact that the constants A,B,C appear linearly in (2.16) will be useful in the development of our navigation equations.

From Figure (2.1) it is seen that s is the measureable semisubtended angle of the central body.

$\theta_o(1)$ is now chosen to be that spacecraft azimuth at which the star-central body angle is minimum. This is shown in Figure (2.2). Given a star at azimuth $\theta_o(1)+\pi$ and elevation angle $\gamma_o(1)$ from the spacecraft orbital plane, spherical trigonometry gives the star-central body angle $\gamma(1)$ to be

$$\cos\gamma(1) = \cos\gamma_o(1)\ \cos\beta(1) \tag{2.17}$$

or

$$\cos\beta(1) = \cos\gamma(1)/\cos\gamma_o(1) \tag{2.18}$$

From (2.18) it is seen that one can convert a measurement of $\gamma(1)$ into a value for $\beta(1)$ if $\gamma_o(1)$ is known. $\gamma_o(1)$ can be measured by tracking the star during its passage through the minimum of $\gamma(1)$. This means a star approaching the central body must be initially picked.

Consider three pairs of measured angles $s_1\ \gamma(1)_1$, $s_2\ \gamma(1)_2$, $s_3\ \gamma(1)_3$ available to the navigator.* Then three linear equations for the three unknowns are obtained

$$\begin{aligned} \cos\beta(1)_1 A + \sin\beta(1)_1 B + \sin s_1 C &= -1 \\ \cos\beta(1)_2 A + \sin\beta(1)_2 B + \sin s_2 C &= -1 \\ \cos\beta(1)_3 A + \sin\beta(1)_3 B + \sin s_3 C &= -1 \end{aligned} \tag{2.20}$$

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* The angle $\gamma(1)$ in (2.18) is the angle measured to the central body's center. The actual measured angle will probably be to the central body's edge. However we have simply

$$\gamma(1) = \gamma(1)_m \pm s \tag{2.19}$$

where s is a measured angle. We assume in the rest of the theoretical analysis that $\gamma(1)$ is measured or constructed from the measureables by (2.19).

The solution of (2.20) for A,B,C is straightforward algebra. By using the inverse of equations (2.15) the orbital parameters can be calculated

$$b = -C$$

and (2.21)

$$e^2 = A^2 + B^2$$

A knowledge of e and b gives the shape of the spacecraft orbit and consequently supplies much of the necessary information for future spacecraft maneuvers designed to fulfill some operational goal. For example, safe reentry into the central body atmosphere requires an orbit perigee within a certain reentry "window". Perigee altitude is determined by e and b.

D. Navigation Equations with Redundant Measurements

In general there will be noise error in the measured angles s and $\gamma(1)$. Therefore a redundant number of pairs of measured angles are desirable in order to reduce the final error in estimating e and b.

The linear form of (2.16) allows for a simple incorporation of redundant measurements. Consider N > 3 pairs of measurements s_i and $\gamma(1)_i$. i runs from 1 to N. Divide the measurements into three clusters of N_1, N_2, and N_3 pairs of measured angles. Then simply summing (2.16) over the measurements in each cluster gives

$$\begin{aligned}
\textstyle\sum_1(\cos\beta(1))A + \sum_1(\sin\beta(1))B + \sum_1(\sin s)C &= -N_1 \\
\textstyle\sum_2(\cos\beta(1))A + \sum_2(\sin\beta(1))B + \sum_2(\sin s)C &= -N_2 \qquad (2.22) \\
\textstyle\sum_3(\cos\beta(1))A + \sum_3(\sin\beta(1))B + \sum_3(\sin s)C &= -N_3
\end{aligned}$$

where the notation in (2.22) is as follows

$$\textstyle\sum_1(\cos\beta(1)) \equiv \sum_{i \text{ in cluster } 1} \cos\beta(1)_i$$

and

$$\textstyle\sum_3(\sin s) \equiv \sum_{i \text{ in cluster } 3} \sin s_i \qquad \text{etc.} \qquad (2.23)$$

It is observed that (2.22) is of the exact same form as (2.20) except now the coefficients of A,B,C are sums of measured quantities. This leads to an operational procedure especially suited to manual calculation. As each new measurement of navigation angles is made the appropiate sums in (2.22) are updated. The calculation of A,B,C need only be made once by solving (2.22) after sufficient data has been accumulated.

Below we exhibit the solution of (2.22) for C. A complete discussion of the operational procedures of a practical navigational system will be made in Chapter VII.

$$C = -\frac{c_1 N_1 + c_2 N_2 + c_3 N_3}{c_1 \sum_1(\mathrm{sins}) + c_2 \sum_2(\mathrm{sins}) + c_3 \sum_3(\mathrm{sins})} \tag{2.24}$$

where

$$\begin{aligned} c_1 &= \textstyle\sum_2(\cos\beta(1))\sum_3(\sin\beta(1)) - \sum_3(\cos\beta(1))\sum_2(\sin\beta(1)) \\ c_2 &= \textstyle\sum_3(\cos\beta(1))\sum_1(\sin\beta(1)) - \sum_1(\cos\beta(1))\sum_3(\sin\beta(1)) \\ c_3 &= \textstyle\sum_1(\cos\beta(1))\sum_2(\sin\beta(1)) - \sum_2(\cos\beta(1))\sum_1(\sin\beta(1)) \end{aligned} \tag{2.25}$$

A consideration of (2.18) indicates that each $\gamma(1)$ measurement is converted to a $\cos\beta(1)$ value by dividing by $\cos\gamma_o(1)$. Then a trigonometric table must be used to obtain $\sin\beta(1)$ from $\cos\beta(1)$. In order to free the manual navigation equations from these manipulations a two star navigation procedure is developed in Chapter III which does not require this repeated use of the quantity $\gamma_o(1)$.

CHAPTER III

TWO STAR NAVIGATION EQUATIONS

A. Two Star Navigation Equations for b

Several operational simplifications in the required manual arithmetic of space navigation can be made by measuring two different star-central body angles with each central body subtended angle measurement. Consider (2.12) for two different stars

$$b \sin s = 1 + e \cos\theta_o(1) \cos\beta(1) - e \sin\theta_o(1) \sin\beta(1) \qquad (3.1a)$$

$$b \sin s = 1 + e \cos\theta_o(2) \cos\beta(2) - e \sin\theta_o(2) \sin\beta(2) \qquad (3.1b)$$

where $\beta(1) = \theta - \theta_o(1)$ and $\beta(2) = \theta - \theta_o(2)$.

It would be useful if the $\sin\beta(1)$ and $\sin\beta(2)$ terms could be eliminated from (3.1a,b) since obtaining these terms from the measured angles requires the substantial arithmetical operations described at the end of Chapter II. This elimination can be accomplished. $\beta(1)$ and $\beta(2)$ are related by

$$\sin\beta(1) = \sin\left(\beta(2) - \theta_o(12)\right) = \sin\beta(2)\cos\theta_o(12) - \cos\beta(2)\sin\theta_o(12) \qquad (3.2)$$

where

$$\theta_o(12) = \theta_o(1) - \theta_o(2) \qquad (3.3)$$

is another constant angle. Making the substitution (3.2) for $\sin\beta(1)$ in (3.1a) and subtracting (3.1b) multiplied by the factor $\sin\theta_o(1)\cos\theta_o(12)/\sin\theta_o(2)$ from (3.1a) eliminates all $\sin\beta$ terms, leaving a single equation of the form

$$A'\cos\gamma(1) + B'\cos\gamma(2) + C'\sin s + 1 = 0 \qquad (3.4)$$

where $\cos\beta(1)$ and $\cos\beta(2)$ have been expressed in terms of the measured $\gamma(1)$ and $\gamma(2)$ by use of (2.18). The constants in (3.4)

$$A' = -e \sin\theta_o(2) / \sin\theta_o(12) \cos\gamma_o(1) \quad (3.5a)$$

$$B' = e \sin\theta_o(1) / \sin\theta_o(12) \cos\gamma_o(2) \quad (3.5b)$$

$$C' = -b \quad (3.5c)$$

All of the variables in (3.4) are now closely related to measured quantities. Only a minimum of arithmetical manipulation is now required with the data.

The orbital constant b is still easily extracted from using (3.5c), but it is seen that e is impossible to obtain from A',B' unless the angles in these constants were known very accurately. Another navigation equation will be necessary to obtain e.

Note that knowledge of $\gamma_o(1)$ or $\gamma_o(2)$ is totally unnecessary in order to obtain b.

Following the procedure in one star navigation it is assumed that redundant measurements are grouped into three clusters. Summing (3.4) in each cluster gives

$$\begin{aligned}
\textstyle\sum_1(\cos\gamma(1))A' + \sum_1(\cos\gamma(2))B' + \sum_1(\sin s)C' &= -N_1 \\
\textstyle\sum_2(\cos\gamma(1))A' + \sum_2(\cos\gamma(2))B' + \sum_2(\sin s)C' &= -N_2 \qquad (3.6) \\
\textstyle\sum_3(\cos\gamma(1))A' + \sum_3(\cos\gamma(2))B' + \sum_3(\sin s)C' &= -N_3
\end{aligned}$$

where the notation used above is the same as defined in (2.23). (3.6) need only be solved for C' giving

$$-C' = b = \frac{c_1'N_1 + c_2'N_2 + c_3'N_3}{c_1'\sum_1(\sin s) + c_2'\sum_2(\sin s) + c_3'\sum_3(\sin s)} \quad (3.7)$$

where

$$\begin{aligned}
c_1' &= \textstyle\sum_2(\cos\gamma(1))\sum_3(\cos\gamma(2)) - \sum_3(\cos\gamma(1))\sum_2(\cos\gamma(2)) \\
c_2' &= \textstyle\sum_3(\cos\gamma(1))\sum_1(\cos\gamma(2)) - \sum_1(\cos\gamma(1))\sum_3(\cos\gamma(2)) \qquad (3.8) \\
c_3' &= \textstyle\sum_1(\cos\gamma(1))\sum_2(\cos\gamma(2)) - \sum_2(\cos\gamma(1))\sum_1(\cos\gamma(2))
\end{aligned}$$

B. Navigation Equations For e

Having solved (3.7) for b we can return to the one star navigation equation for e. Define

$$y = 1 - b \sin s \tag{3.9}$$

where it is assumed that b is now known, then (2.12) can be rewritten as

$$e \cos\theta_o \cos\beta + y = e \sin\theta_o \sin\beta \tag{3.10}$$

The label 1 or 2 has been dropped from θ_o and β, as (3.10) can be used for either or both stars used in the two star navigation. Squaring (3.10), using (2.18), and the identity

$$\sin^2\beta = 1 - \cos^2\beta \tag{3.11}$$

a new navigation equation is obtained

$$A'' - B'' \cos^2\gamma + C'' y\cos\gamma - y^2 = 0 \tag{3.12}$$

where

$$A'' = e^2 \sin^2\theta_o \tag{3.13a}$$

$$B'' = e^2 / \cos^2 \gamma_o \tag{3.13b}$$

$$C'' = -2e \cos\theta_o / \cos\gamma_o \tag{3.13c}$$

Solving (3.12) for B'', e can be obtained from (3.13b)

$$e^2 = B'' \cos^2\gamma_o \tag{3.14}$$

if a measurement of γ_o of sufficient accuracy is available. If no measurement of γ_o is available, one can still obtain e by solving for all three constants A'',B'',C''. From (3.13a,b,c) we have

$$\cos^2\gamma_o = A'' / (B'' - (C'')^2/4) \tag{3.15}$$

giving

$$e^2 = A'' B'' / \left(B'' - (C'')^2/4\right) \tag{3.16}$$

Substantially more computational effort is needed to obtain e by use of (3.16) than by using (3.14). But (3.16) allows navigation to become completely independent of a γ_o measurement. In those cases where a reasonable γ_o measurement is available the simpler (3.14) can be used.

Forming three clusters of observations and using (3.12) the solution for B" is

$$B'' = - \frac{c_1'' \sum_1(y^2) + c_2'' \sum_2(y^2) + c_3'' \sum_3(y^2)}{c_1'' \sum_1(\cos^2\ ) + c_2'' \sum_2(\cos^2\ ) + c_3'' \sum_3(\cos^2\ )} \tag{3.17}$$

where

$$c_1'' = N_2\sum_3(y\cos\gamma) - N_3\sum_2(y\cos\gamma) \tag{3.18a}$$

$$c_2'' = N_3\sum_1(y\cos\gamma) - N_1\sum_3(y\cos\gamma) \tag{3.18b}$$

$$c_3'' = N_1\sum_2(y\cos\gamma) - N_2\sum_1(y\cos\gamma) \tag{3.18c}$$

We now possess navigation equations for both e and b which use measured data with only a minimum of arithmetical manipulations. The use of γ_o has been reduced to a minimum. At most a single multiplication by $\cos^2\gamma_o$ is required to obtain e.

CHAPTER IV

NAVIGATION ACCURACY

A. Introduction

In the approach to navigation studied in this work, as well as in other forms of navigation, orbital parameters are estimated from an accumulation of celestial angle measurements. In general these measurements will contain errors due to both inaccuracies in the measuring instruments and observational errors. Measurement errors are of two types; noise errors in which the error in any one measurement is independent of the error in any other measurement, and systematic errors in which the error in the several measurements are correlated with each other. For example, systematic errors can be the result of bent or miscalibrated measuring instruments, biases in the observers, or instrument reading techniques.

The influence of measurement errors on parameter estimation accuracy is now studied and applied to the navigation equations developed in the previous chapters.

B General Theory of Error

The general form of navigation equation employed in this work is of the form

$$(m_\alpha)_i\ K_\alpha + (m_o)_i = 0 \tag{4.1}$$

where K_α are several (k) constant parameters to be estimated, $(m_\alpha)_i$ and $(m_o)_i$ are measureable functions. The common index α is to be summed from 1 to k. The subscript i refers to the ith observation. i runs from 1 to N, N being the total number of observations. Redundancy of observations means that $N > k$. In order to obtain k equations for the k unknowns K_α, the N observations are grouped into k clusters, giving the k equations

$$\textstyle\sum_{\alpha'} (m_\alpha)\ K_\alpha + \sum_{\alpha'} (m_o) = 0 \tag{4.2}$$

for each α' from 1 to k. The notation defined by (2.23) is used in (4.2). In matrix notation (4.2) reads

$$[M]\{K\} = \{V\} \tag{4.3}$$

where the entries of the matrix are given by

$$[M]_{\alpha',\alpha} = \sum_{i \text{ in cluster } \alpha'} (m_\alpha)_i \tag{4.4}$$

and the entries of the vectors are given by

$$\{K\}_\alpha = K_\alpha \tag{4.5}$$

and

$$\{V\}_{\alpha'} = -\sum_{i \text{ in cluster } \alpha'} (m_o)_i \tag{4.6}$$

Taking the differential of (4.3) gives

$$[M]\{\delta K\} = \{\varepsilon\} \tag{4.7}$$

with an error vector defined by

$$\{\varepsilon\} = \{\delta V\} - [M]\{K\} \tag{4.8}$$

Explicitly showing the components of the error vector

$$\{\varepsilon\}_{\alpha'} = -\sum_{i \text{ in cluster } \alpha'} \left(\delta(m_o)_i + \delta(m_\alpha)_i K_\alpha\right) \tag{4.9}$$

The $\delta(m_o)_i$ and $\delta(m_\alpha)_i$ are the measurement errors. Defining the error element for the ith observation

$$\varepsilon_i = -\left(\delta(m_o)_i + \delta(m_\alpha)_i K_\alpha\right) \tag{4.10}$$

(4.9) can be written as

$$\{\varepsilon\}_\alpha = \sum_{i \text{ in cluster } \alpha} \varepsilon_i \tag{4.11}$$

The useful quantitative properties of the error elements are given by the first two moments of their probability distribution. The mean of the error element is denoted by $<\varepsilon_i>$. If the second moment of the errors is given by

$$\begin{aligned} <\varepsilon_i \varepsilon_j> &= \sigma_i^2 \qquad \text{if } i = j \\ &= 0 \qquad \text{if } i \neq j \end{aligned} \tag{4.12}$$

the errors are said to be uncorrelated with σ_i^2 being the mean square error. If

$$<\varepsilon_i \varepsilon_j> = b_i b_j \tag{4.13}$$

the errors are said to be uncorrelated with an unknown systematic error in the measurements of the functional form b_i. In general we will refer to the $<\varepsilon_i \varepsilon_j>$ as the error correlation matrix.

Returning to (4.7) and taking a statistical average

$$<\{\delta K\}> = [M]^{-1}<\{\varepsilon\}> \tag{4.14}$$

where the inverse of the $[M]$ matrix is indicated. If there are no mean measurement errors, the mean estimation errors of the parameters K_α will be zero. Then multiplying (4.14) by its transpose and taking the statistical average yields

$$<\{\delta K\}\{\delta K\}^T> = [M]^{-1}<\{\varepsilon\}\{\varepsilon\}^T>[M]^{-1T} \tag{4.15}$$

In particular the mean square error of any particular parameter is given by

$$<\delta K_\alpha^2> = [M]^{-1}_{\alpha\alpha'}[M]^{-1}_{\alpha''\alpha}<\varepsilon_{\alpha'}\varepsilon_{\alpha''}> \tag{4.16}$$

where the matrix row and column labels are exhibited. The subscripts α' and α'' are summed over. For the case of no systematic errors (4.12) is valid and (4.16) becomes

$$<\delta K_\alpha^2> = [M^{-1}]_{\alpha\alpha'}[M^{-1}]_{\alpha'\alpha} \sum_{i \text{ in cluster } \alpha'} \sigma_i^2 \tag{4.17}$$

α' is summed in (4.17).

A discussion of the optimal use of the measured data and the elimination of systematic errors from measurements is presented in Appendix A and Appendix B.

C. Application to One Star Navigation Equations

The one star navigation equation given by (2.16) has

$$(m_1)_i = \cos\beta_i \tag{4.18a}$$

$$(m_2)_i = \sin\beta_i \tag{4.18b}$$

$$(m_3)_i = \sin s_i \tag{4.18c}$$

$$(m_o)_i = 1 \tag{4.18d}$$

Then (4.10) becomes

$$\varepsilon_i = b\ \delta\sin s_i - e(\cos\theta_o\ \delta\cos\beta_i - \sin\theta_o\ \delta\sin\beta_i) \tag{4.19}$$

(4.19) can be expressed directly in terms of the angular measurement errors $\delta\gamma_i$ and δs_i by use of the relations and definitions in Chapter II.

$$\varepsilon_i = \left(b\cos s_i \pm \frac{e\sin\gamma_i\ \sin\theta_i}{\cos\gamma_o\ \sin\beta_i}\right)\delta s_i$$

$$+ \frac{e\sin\gamma_i\ \sin\theta_i}{\cos\gamma_o\ \sin\beta_i}\ \delta\gamma_i \tag{4.20}$$

In obtaining (4.20) it has been assumed that s_i the semi-subtended angle of the central body was added to (or subtracted from) the star-central body edge angle.

Squaring ε_i, taking the statistical average, and assuming that $\langle\delta s_i^{\,2}\rangle = \langle\delta\gamma_i^{\,2}\rangle = \Delta^2$, and that $\cos s_i \simeq 1$

$$\langle \varepsilon_i^2 \rangle = \sigma_i^2 = \Delta^2 \left(b^2 + 2(e \sin\gamma_i \sin\theta_i/\cos\gamma_o \sin\beta_i)^2 \right.$$

$$\left. \pm 2be \sin\gamma_i \sin\theta_i/\cos\gamma_o \sin\beta_i \right) \qquad (4.21)$$

If the choice is available it is worthwhile to measure to the central body edge which gives the sign in (4.21) which makes σ_i smaller. (4.21) also suggests that γ_o be made as close to zero as possible in order to minimize σ_i. γ_o zero would mean the navigation star is lying in the orbital plane.

In order to make a quantitative estimate of navigation accuracy by using (4.17) a navigation mission schedule is introduced. Consider the spacecraft trajectory of Figure (4.1). At three positions on the spacecraft orbit, $\theta_1,\theta_2,\theta_3$ clusters of N_1,N_2, N_3 observations are made of the navigation angles.

By use of (4.16) the mean square error in the orbital parameter b is given by

$$\langle \delta b^2 \rangle = b^2 \frac{\left(\frac{\sigma_1^2 \sin^2\theta_{23}}{N_1} + \frac{\sigma_2^2 \sin^2{}_{13}}{N_2} + \frac{\sigma_3^2 \sin^2{}_{12}}{N_3} \right)}{\left(\sin\theta_{23} - \sin\theta_{13} + \sin\theta_{12} \right)^2} \qquad (4.22)$$

where $\theta_{12} = \theta_1 - \theta_2$, etc. The derivation of (4.22) is given in Appendix C. σ_1^2 is a typical mean square error element in cluster 1 given by (4.21).

(4.22) can be minimized with respect to the division of N observations into N_1,N_2,N_3, giving the optimum division of a fixed number of observations. Setting

$$\partial \langle \delta b^2 \rangle / \partial N_\alpha = 0 \qquad \text{for } \alpha = 1,2,3 \qquad (4.23)$$

subject to the constraint $N_1 + N_2 + N_3 = N$, we obtain

$$N_1/N = |\sin\theta_{23}|\ \sigma_1 / \text{SUM} \qquad (4.24a)$$

$$N_2/N = |\sin\theta_{13}|\ \sigma_2 / \text{SUM} \qquad (4.24b)$$

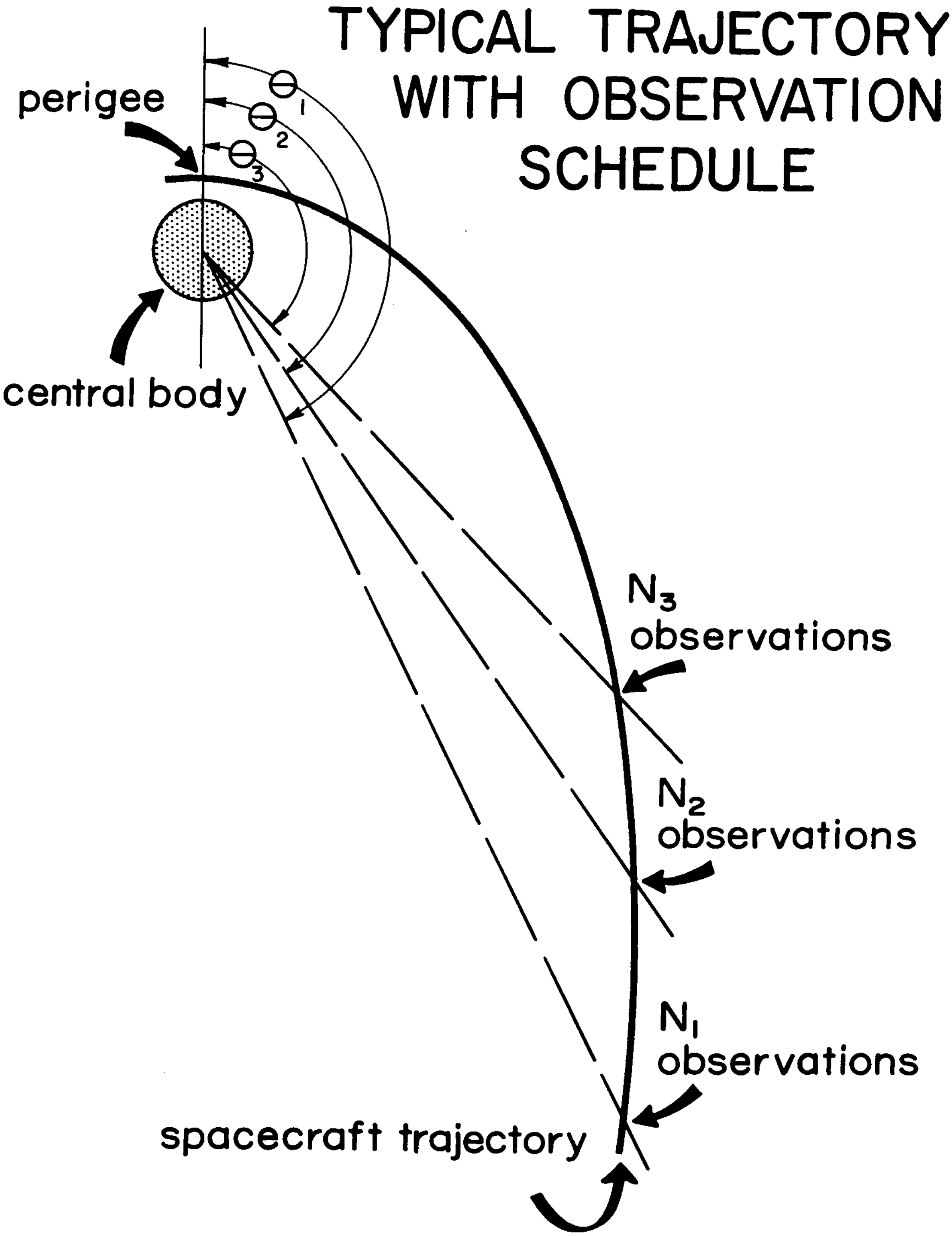

Figure 4.1 A typical mission schedule is shown. Three clusters of observations with N_1, N_2, N_3 observations in each cluster are taken when spacecraft azimuth is about $\theta_1, \theta_2, \theta_3$, repectively.

$$N_3/N = |\sin\theta_{12}|\ \sigma_3\ /\ \mathrm{SUM} \tag{4.24c}$$

where

$$\mathrm{SUM} = |\sin\theta_{23}|\ \sigma_1 + |\sin\theta_{13}|\ \sigma_2 + |\sin\theta_{12}|\ \sigma_3 \tag{4.25}$$

This gives a minimized error of

$$<\delta b^2> = (b\ \mathrm{SUM})^2\ /\ N\ (\sin\theta_{23} - \sin\theta_{13} + \sin\theta_{12})^2 \tag{4.26}$$

We next minimize (4.26) with respect to θ_2 for fixed θ_1 and θ_3, that is we seek the location of the middle cluster of observations which minimizes the error in b. Setting

$$\partial <\delta b^2> / \partial\ \theta_2 = 0 \tag{4.27}$$

yields

$$\sigma_3\ \cos\theta_{21} - \sigma_1\ \cos\theta_{32} = \sqrt{N}\ \delta b_{rms}(\cos\theta_{21} - \cos\theta_{32})/b \tag{4.28}$$

where rms indicates root mean square

$$\delta b_{rms} = \sqrt{<\delta b^2>} \tag{4.29}$$

For $\sigma_1 = \sigma_3$ (4.28) gives

$$\theta_2 = (\theta_1 + \theta_3)/2 \tag{4.30}$$

The intermediate cluster of observations should be midway, measuring in terms of orbital azimuth, between the first and last cluster of observations.

If the result (4.30) is used in (4.24a,b,c) and if the angle $\theta_3 - \theta_1$ is small which it usually is, then

$$N_1 = N_3 = N_2/2 = N/4 \tag{4.31}$$

for the best accuracy of b prediction.

Calling $|\theta_3 - \theta_1| = \theta_{nav}$ the angle through which navigation takes place, (4.26) takes the simple form for optimum navigation (navigation fulfilling (4.30) and (4.31))

$$\delta b_{rms} = \frac{b}{\sqrt{N}} \sigma \frac{\sin\theta_{nav} + 2\sin(\theta_{nav}/2)}{-\sin\theta_{nav} + 2\sin(\theta_{nav}/2)} \quad (4.32)$$

By use of the trigonometric multiple angle relations

$$\sin\theta_{nav} = 2\sin(\theta_{nav}/2)\cos(\theta_{nav}/2) \quad (4.33a)$$

and

$$1 - \cos(\theta_{nav}/2) = 2\sin^2(\theta_{nav}/4) \quad (4.33b)$$

(4.32) becomes

$$\delta b_{rms} = \frac{b\,\sigma}{\sqrt{N}} \cot^2(\theta_{nav}/4) \quad (4.34)$$

For most applications $\theta_{nav}/4$ is a small angle so (4.34) is well approximated by

$$\delta b_{rms} = \frac{16\,b}{\sqrt{N}} \frac{\sigma}{\theta^2_{nav}} \quad (4.35)$$

In most all cases (4.21) is approximated well by

$$\sigma \simeq b\,\Delta \quad (4.36)$$

If we consider an eccentric orbit around earth with perigee equal to about an earth radii, then $e \simeq 1$, $b \simeq 2$. Assuming $N = 49$ observations and a measurement rms error of $\Delta = .5$ arc-minutes, (4.35) can be converted to perigee rms estimation error, obtaining

$$(\text{Perigee Error})_{rms} = 2.5 / \theta^2_{nav} \quad \text{miles} \quad (4.37)$$

(4.37) is plotted in Figure (4.2) for various combinations of n_{min} and n_{max}, the ranges of the spacecraft position during navigation. Figure (4.2) can be scaled to any other rms measuring error and number of observations by use of (4.35).

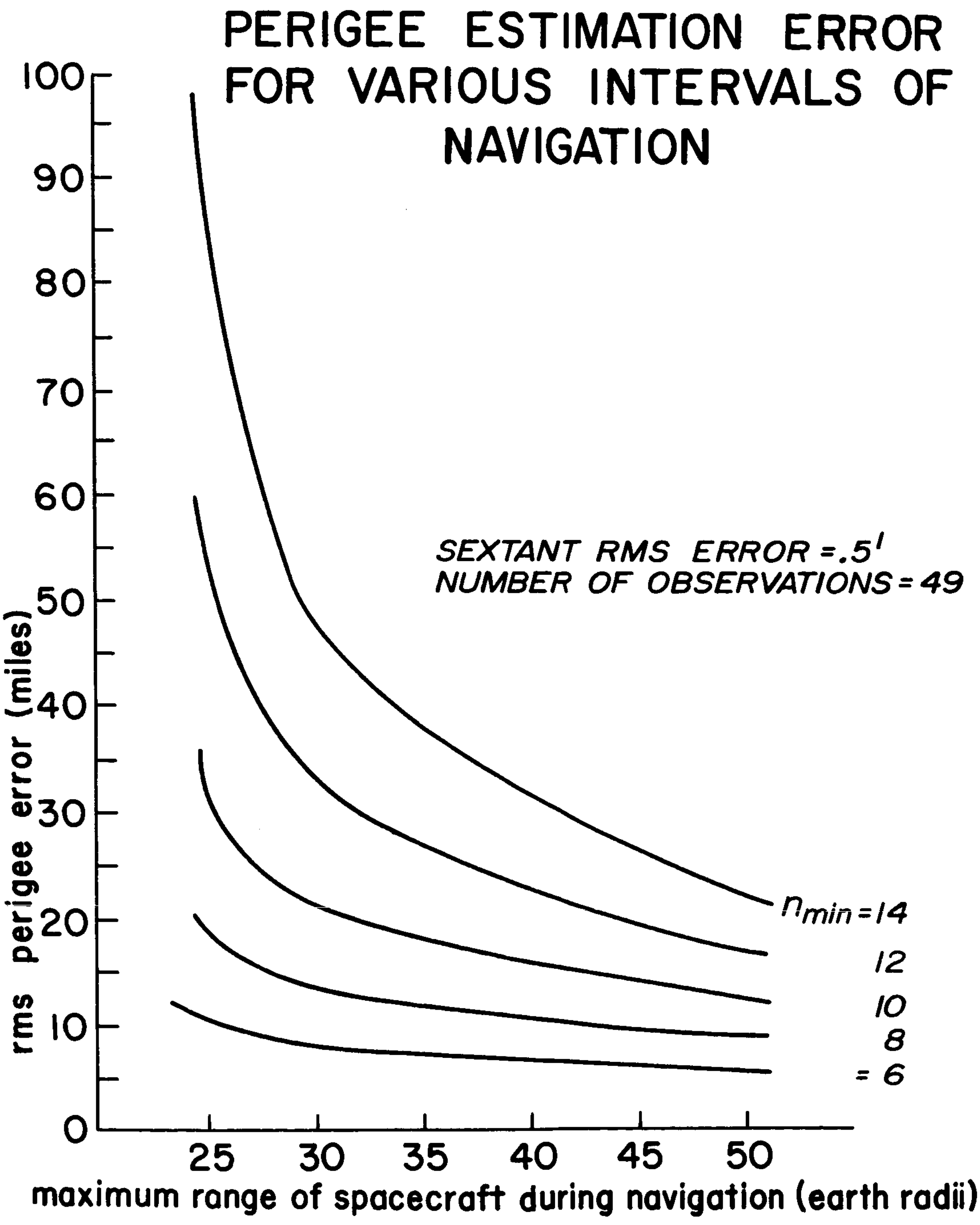

Figure 4.2 Perigee prediction accuracy is given for an eccentric orbit with perigee about one earth radii. It is assumed that good navigation stars are used, that measurement accuracy is Δ = .5 arc-minutes, and that a total of N = 49 observations were made. n_{min} and n_{max} are minimum and maximum spacecraft range during navigation.

There is also an error associated with the estimation of e, the orbital eccentricity. It can be shown that navigational errors at perigee due to e uncertainty are of an order of magnitude smaller than those due to b uncertainty. So the error estimates of (4.37) represent essentially all the navigation error to be expected.

Physically it is straightforward to see why uncertainty in b contributes an order of magnitude greater error in perigee than does uncertainty in e. From

$$n = b / (1 + e \cos\theta) \tag{4.38}$$

the sensitivity of b and e to an error in θ gives

$$\partial b/\partial \theta = - e\, n \sin\theta \tag{4.39}$$

and

$$\partial e/\partial \theta = e \tan\theta \tag{4.40}$$

The ratio of the errors

$$\delta b/\delta e = - n / \cos\theta \tag{4.41}$$

shows that the b error is an order of magnitude larger than the e error. The error in perigee

$$\delta n_{per} = \delta(b/(1+e)) \tag{4.42}$$

is then dominated by the b error.

D. Sensitivity of One Star Navigation to γ_o Error

In one star navigation equations $\gamma_o(1)$ must be measured and used to convert measured quantities to more useful quantities which appear in the navigation equation. An error in measuring $\gamma_o(1)$ then will produce a systematic error in every converted measurement. First we note that s_i measurements are uneffected by the $\gamma_o(1)$ error. From (2.18) however we have

$$\partial \cos\beta(1)/\partial \gamma_o(1) = \cos\beta(1) \tan\gamma_o(1) \tag{4.43}$$

and it is then straightforward to obtain

$$-\partial \sin\beta(1)/\partial \gamma_o(1) = \cos\beta(1) \cot\beta(1) \tan\gamma_o(1) \tag{4.44}$$

Combining (4.43) and (4.44) into (4.19) gives

$$\partial \varepsilon/\partial \gamma_o(1) = -\frac{e \tan\gamma_o(1) \sin\theta_i \cos\beta(1)_i}{\sin\beta(1)_i} \tag{4.45}$$

Adapting (4.14) to the purposes of this section

$$\partial K_\alpha/\partial \gamma_o(1) = [M]^{-1}_{\alpha\alpha'} \partial \varepsilon_{\alpha'}/\partial \gamma_o(1) \tag{4.46}$$

Evaluating (4.46) for K = C = -b using the equations derived in Appendix C gives

$$\partial b/\partial \gamma_o(1) = -\, be \tan\gamma_o(1) / \sin^2\left(\theta_{nav}/4\right) \times \left(\sin\theta_1 \cot\beta(1)_1 + \sin\theta_3 \cot\beta(1)_3 - 2\cos\left(\theta_{nav}/2\right) \sin\theta_2 \cot\beta(1)_2\right) \tag{4.47}$$

For most applications $\beta(1)_1$ is smaller than $\beta(1)_2$ or $\beta(1)_3$ and dominates (4.47). Then (4.47) becomes approximately

$$\partial b/\partial \gamma_o(1) = -16be\tan\gamma_o(1) \sin\theta_1 \cot\beta(1)_1 / \theta^2_{nav} \tag{4.48}$$

where we have assumed $\theta_{nav}/4$ a small angle. (4.48) goes as $\tan\gamma_o(1)$ so it suggests picking stars in the orbital plane to keep the sensitivity of b to $\gamma_o(1)$ error small. Even for $\gamma_o(1)$ less than 10^o (4.48) gives substantial sensitivity of b to $\gamma_o(1)$ error, giving us another argument for going to two star navigation equations.

E. Application to Two Star Navigation Equations

The two star navigation equations for b given by (3.4) have

$$(m_1)_i = \cos\gamma(1)_i \tag{4.49a}$$

$$(m_2)_i = \cos\gamma(2)_i \tag{4.49b}$$

$$(m_3)_i = \sin s_i \tag{4.49c}$$

$$(m_o)_i = 1 \tag{4.49d}$$

Using the definitions of the constants A',B',C' given by (3.5 a,b,c) the error element is

$$\varepsilon_i = b\,\delta \sin s_i + \frac{e}{\sin\theta_o(12)}\left(\frac{\sin\theta_o(1)\ \delta\cos\gamma(2)_i}{\cos\gamma_o(2)} - \frac{\sin\theta_o(2)\ \delta\cos\gamma(1)_i}{\cos\gamma_o(1)}\right) \tag{4.50}$$

Expressing (4.50) in terms of the measured angle errors, squaring, and taking the statistical average yields

$$<\varepsilon_i^{\,2}> = \Delta^2\ (\ \xi_1^{\,2} + \xi_2^{\,2} + (b \pm \xi_1 \pm \xi_2)^2\ ) \tag{4.51}$$

where

$$\xi_1 = e\ \sin\theta_o(2)\ \sin\gamma(1)_i\ /\ \sin\theta_o(12)\ \cos\gamma_o(1)$$

and

$$\xi_2 = e\ \sin\theta_o(1)\ \sin\gamma(2)_i\ /\ \sin\theta_o(12)\ \cos\gamma_o(2)$$

We have made the approximations $\cos s_i \simeq 1$, $<\delta s_i^{\,2}> \simeq <\delta\gamma(1)_i^{\,2}> \simeq <\delta\gamma(2)_i^{\,2}> \simeq \Delta^2$.

Again as in one star navigation it is worthwhile to measure angles to the central body edge which gives the sign in (4.51) making $<\varepsilon_i^{\,2}>$ the smallest.

(4.51) also suggests that both $\gamma_o(1)$ and $\gamma_o(2)$ should be kept small as possible. Also $\theta_o(12)$ is to be kept as close to $\pm 90^\circ$ as possible. In other words the two stars used in navigation should lie in the orbital plane but perpendicular to each other. If these conditions are fulfilled (4.51) gives to good approximation

$$<\varepsilon_i^{\,2}> = b^2\ \Delta^2 \tag{4.52}$$

As shown in Appendix C the error equation for b is similar in one star and two star navigation. This leads to

$$b_{rms} = 16\ b^2\ \Delta\ /\ \sqrt{N}\ \theta_{nav}{}^2 \tag{4.53}$$

being valid in both cases. Therefore the curves plotted in Figure (4.2) are useful for both one and two star navigation.

As in one star navigation, in two star navigation errors in e are an order of magnitude smaller than errors in b.

F. Sensitivity of Two Star Navigation to γ_o Error

Two star navigation was developed to simplify the navigation equations' dependence on γ_o, the star elevation angle measurement. As was derived in Chapter III the estimate of the orbital parameter b was made independent of γ_o. The dependence of the parameter e on γ_o was simply that expressed in (3.14)

$$e^2 = B''\ \cos^2\gamma_o \tag{4.54}$$

Taking a derivative of (4.54) yields

$$\partial\ e/\partial\ \gamma_o = -\ e\ \tan\gamma_o \tag{4.55}$$

In terms of perigee sensitivity to γ_o

$$\partial\ \text{Perigee}/\partial\ \gamma_o = R_c\ n_{per}\ \frac{e}{1 + e}\ \tan\gamma_o \tag{4.56}$$

For eccentric orbits ($e \simeq 1$) around earth with a perigee of about $n_{per} \simeq 1$, (4.56) gives

$$\partial\ \text{Perigee}/\partial\ \gamma_o = \frac{1}{2}\ \tan\gamma_o \quad \text{miles/arc-minute} \tag{4.57}$$

If γ_o is small this is a quite acceptable error, being a smaller contributor to navigation error than noise error. (4.57) is at least an order of magnitude improvement over the one star navigation sensitivity given by (4.48).

CHAPTER V

GRAVITATIONAL PERTURBATIONS

A. The Linearized Perturbation Equations

For many space missions in which navigation is required the spacecraft will not be moving in a perfect Keplerian orbit. Other gravitational bodies will be present to perturb the spacecraft orbit. The navigation equation for perfect Keplerian orbits developed in the previous chapters must be modified to include perturbations.

Let $\overline{R}_o(t)$ be a conic orbit which would be the true spacecraft orbit in the absence of perturbations. Let $\overline{A}$ be the perturbing acceleration, $\overline{G}$ the dominant central body acceleration. $\Delta\overline{R}$ is an "off course" vector; the true trajectory is given by

$$\overline{R}(t) = \overline{R}_o(t) + \Delta\overline{R}(t) \tag{5.1}$$

The equation of motion for $\overline{R}(t)$ is given by

$$\ddot{\overline{R}}(t) = \ddot{\overline{R}}_o(t) + \Delta\ddot{\overline{R}}(t) = \overline{G}(\overline{R}) + \overline{A}(\overline{R}) \tag{5.2}$$

Linearizing (5.2) for small $\overline{R}(t)$

$$\Delta\ddot{\overline{R}} = \overline{A}(\overline{R}_o) + \Delta\overline{R}\cdot\overline{\nabla G}(\overline{R}_o) \tag{5.3}$$

where we have eliminated $\ddot{\overline{R}}_o$ by the definition of the unperturbed orbit

$$\ddot{\overline{R}}_o = \overline{G}(\overline{R}_o) \tag{5.4}$$

Since our navigation equations do not employ time as a variable (5.3) must be reexpressed in terms of θ, azimuth angle, as the independent variable. To transform (5.4) we use

$$\frac{d}{dt} = \dot{\theta}\ \frac{d}{d\theta} \tag{5.5a}$$

and

$$\frac{d^2}{dt^2} = (\dot{\theta})^2 \frac{d^2}{d\theta^2} + \ddot{\theta} \frac{d}{d\theta} \tag{5.5b}$$

$\dot{\theta}$ can be expressed in terms of θ by use of the conservation of angular momentum (2.6)

$$\dot{\theta} = h\ (1 + e\cos\theta)^2 /\ R_c^{\ 2}\ b^2 \tag{5.6}$$

It is convenient to make the substitution

$$\Delta\bar{R} = \frac{R_c}{1 + e\cos\theta}\ \bar{D} \tag{5.7}$$

where $\bar{D}$ is a new dimensionless "off course" variable. (5.3) then takes the form

$$d^2D_i/d\theta^2 + \left(e\cos\theta/(1+e\cos\theta) - b^4R_c^{\ 3}G_{ii}/h^2(1+e\cos\theta)^4\right)D_i$$
$$= b^4R_c^{\ 3}\ A_i/h^2(1+e\cos\theta)^3 + b^4R_c^{\ 3}\ D_jG_{ji}/h^2(1+e\cos\theta)^4 \tag{5.8}$$

for each $i = x,y,z$. The common subscript j is summed in (5.8) The G_{ij} are the gradients of the central body acceleration. To the linear order we are doing this problem they are given by

$$G_{xx} = -\ GM(1 - 3x^2/R^2)/R^3 \tag{5.9a}$$

$$G_{yy} = -\ GM(1 - 3y^2/R^2)/R^3 \tag{5.9b}$$

$$G_{zz} = -\ GM/R^3 \tag{5.9c}$$

$$G_{xy} = G_{yx} = 3GM\ xy/R^5 \tag{5.9d}$$

and

$$G_{xz} = G_{zx} = G_{yz} = G_{zy} = 0 \tag{5.9e}$$

The xy plane is defined as the spacecraft orbital plane with the perigee along the positive y axis with θadvancing clockwise. Using (5.9a-e) the three equations of (5.8) become

$$d^2D_x/d\theta^2 = -\ \left(1 - 3\sin^2\theta/(1+e\cos\theta)\right)\ D_x$$
$$+\ 3\sin\theta\cos\theta\ D_y\ /(1+e\cos\theta) + b^4R_c^{\ 3}\ A_x/h^2(1+e\cos\theta)^3 \tag{5.10a}$$

$$d^2D_y/d\theta^2 = -\left(1 - 3\cos^2\theta/(1+e\cos\ )\right) D_y \tag{5.10b}$$

$$+ 3\sin\theta\cos\theta\ D_x/(1+e\cos\theta) + b^4R_c{}^3\ A_y/h^2(1+e\cos\theta)^3$$

and

$$d^2D_z/d\theta^2 = -D_z + b^4R_c{}^3\ A_z/h^2(1+e\cos\theta)^3 \tag{5.10c}$$

It is useful to treat the perturbing gravitational field in terms of its leading multipole term. This allows us to factor out most constants as scale factors in the perturbation problem, leaving the perturbation differential equation free of almost all constants of the problem.

A perturbing body of mass M_L located at $\bar{L}$ from the central body gives a dominant quadropole perturbing acceleration

$$\bar{A}(\bar{R}) = GM_L\ (3\bar{R}\cdot\bar{L}\ \bar{L}/L^5 - \bar{R}/L^3) \tag{5.11}$$

It is convenient to define the dimensionless strength parameter

$$\lambda = \frac{M_L}{M}\left(\frac{R_c}{L}\right)^3 b^4 \tag{5.12}$$

If the position of the perturbation is expressed in polar coordinates-- L (radial distance), θ_L (azimuth from orbital perigee), γ_L (elevation angle from orbital plane), then the inhomogeneous terms in (5.10a,b,c) (call them $A'_{x,y,z}$) are

$$A'_x = \lambda\left[3\cos^2\gamma_L\ \sin\theta_L\ \cos(\theta-\theta_L) - \sin\theta\right]/(1+e\cos\theta)^4 \tag{5.13a}$$

$$A'_y = \lambda\left[3\cos^2\gamma_L\ \cos\theta_L\ \cos(\theta-\theta_L) - \cos\theta\right]/(1+e\cos\theta)^4 \tag{5.13b}$$

$$A'_z = 3\lambda\ \cos\gamma_L\ \sin\gamma_L\ \cos(\theta-\theta_L)\ /\ (1+e\cos\theta)^4 \tag{5.13c}$$

By use of the identity

$$\cos(\theta-\theta_L) = \cos\theta_L\ \cos\theta + \sin\theta_L\ \sin\theta \tag{5.14}$$

we can include all position dependence of the perturbation in the strength factors. (5.13a,b,c) then read

$$A'_x = \left(\tfrac{3}{2}\lambda\cos^2\gamma_L\sin 2\theta_L\right)\cdot\cos\theta/(1+e\cos\theta)^4 \qquad (5.15a)$$
$$+\ \lambda(3\cos^2\gamma_L\ \sin^2\theta_L - 1)\cdot\sin\theta/(1+e\cos\theta)^4$$

$$A'_y = \left(\tfrac{3}{2}\lambda\cos^2\gamma_L\ \sin 2\theta_L\right)\cdot\sin\theta/(1+e\cos\theta)^4 \qquad (5.15b)$$
$$+\ \lambda(3\cos^2\gamma_L\ \cos^2\theta_L - 1)\cdot\cos\theta/(1+e\cos\theta)^4$$

$$A'_z = \left(\tfrac{3}{2}\lambda\sin 2\gamma_L\cos\theta_L\right)\cdot\cos\theta/(1+e\cos\theta)^4 \qquad (5.15c)$$
$$\left(\tfrac{3}{2}\lambda\sin 2\gamma_L\sin\theta_L\right)\cdot\sin\theta/(1+e\cos\theta)^4$$

The significant thing to notice about (5.10a,b,c) with the inhomogeneous terms expressed as (5.15a,b,c) is that the differential equations have only the parameter e left in them. All other parameters including b and the strength and location of the perturbing body have been included in scale factors of the inhomogeneous terms. This simplifies the presentation of necessary perturbation tables or graphs to the navigator.

B. Effect of Perturbations on the Navigation Equations

If the spacecraft deviates from a conic trajectory by an amount $\Delta\bar{R}(\theta)$ then the measurements s and γ will be systematically altered from their values as measured from the hypothetical unperturbed conic orbit.

Let the conic orbit which best fits the perturbed orbit at perigee be defined as the reference conic orbit. Relative to this orbit (which provides the proper orbital information for safe reentry into a planetary atmosphere) the spacecraft is "off course" by $\Delta\bar{R}(\theta)$. At a given azimuth θ this will lead to an altered semisubtended angle s_i

$$\delta\sin s_i = -\ \delta n_i/n^2 = -\ \sin^2 s_i\ \left[\Delta\bar{R}\cdot\hat{r} - \frac{1}{n}\frac{dn}{d\theta}\ \Delta\bar{R}\cdot\hat{t}\right]/R_c \qquad (5.16)$$

where $\hat{r}$ is a unit radial vector, $\hat{t}$ a unit vector in the orbital plane perpendicular to the radial vector. This is seen in Figure (5.1). The first term in brackets in (5.16) is simply the "off course" vector projected along the radial distance. The second term is necessary because if the "off course" vector has a component in the tangential direction, adjustment must be made for the fact that the reference conic is changing its radial distance with θ. In other words if

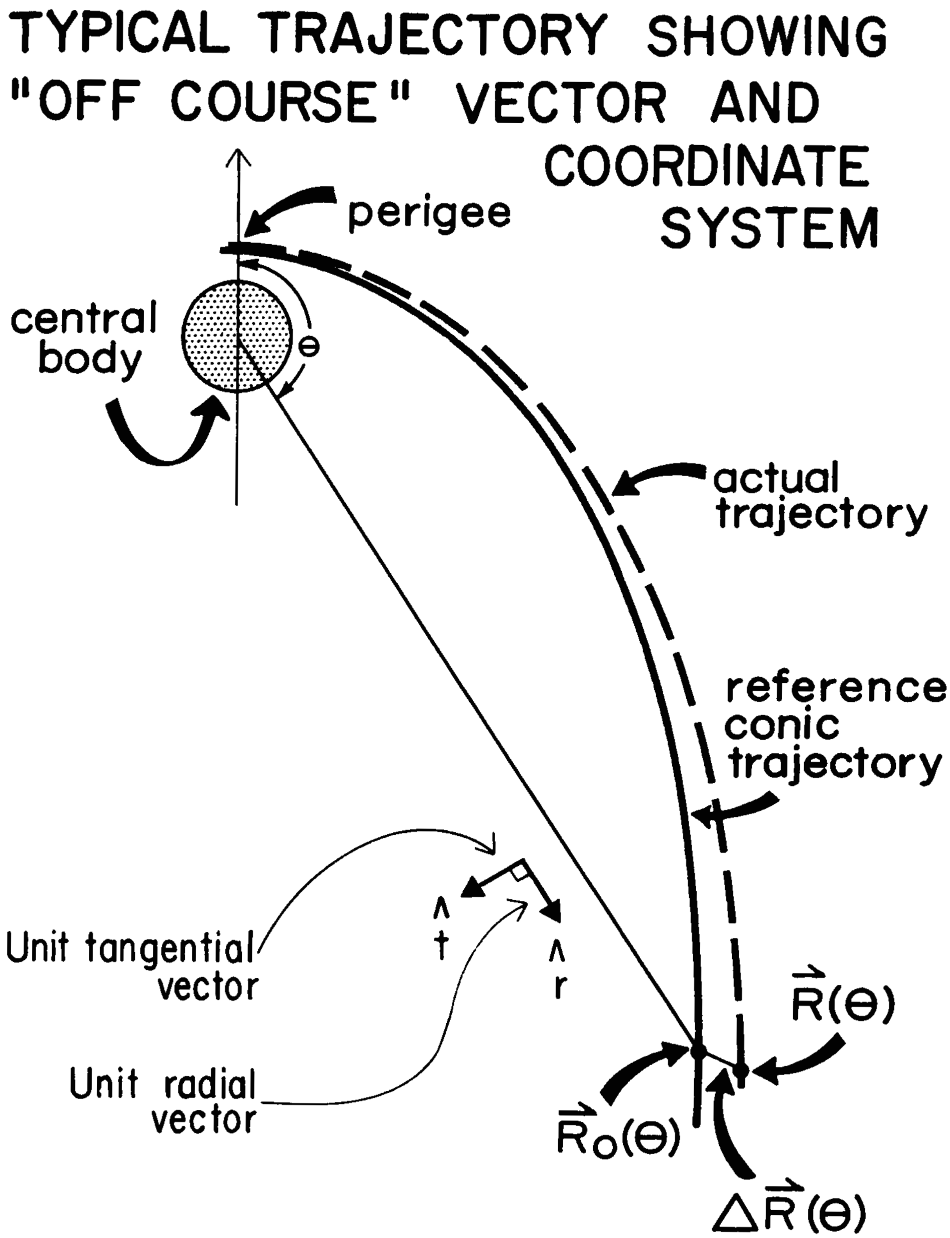

Figure 5.1 The reference conic trajectory and the actual trajectory due to perturbations are shown, the difference $\Delta\bar{R}$ is the "off course" vector. In the orbital plane we have defined two unit vectors, $\hat{r}$ a radial vector, and $\hat{t}$.

$$\Delta\overline{R}\cdot\hat{r}/\Delta\overline{R}\cdot\hat{t} = \frac{1}{n}\frac{dn}{d\theta} \tag{5.17}$$

were to hold (5.16) indicates no change in s. But (5.17) is just the condition for the "off course" vector to lie along the reference trajectory.

In terms of the dimensionless $D(\theta)$ defined by (5.7) (5.16) becomes

$$\delta \sin s_i = -\left\{\sin\theta_i\ D(\theta_i)_x + (e+\cos\theta_i)D(\theta_i)_y\right\} / b^2 \tag{5.18}$$

From (5.18) it is seen that only in-plane perturbations effect the subtended angle measurement.

Out-of-plane perturbations effect the γ measurement however. Starting from

$$\cos\gamma_i = -\hat{r}\cdot\hat{s} \tag{5.19}$$

where $\hat{s}$ is a unit vector toward the navigation star, then

$$\delta\cos\gamma_i = -\delta\hat{r}_i\cdot\hat{s} \tag{5.20}$$

Defining the unit vector $\hat{p}$ perpendicular to the orbital plane we have

$$\delta\hat{r} = \Delta\overline{R}\cdot\hat{p}\ \hat{p} \quad / R_c n \tag{5.21}$$

Therefore from (5.20)

$$\delta\cos\gamma_i = -\Delta\overline{R}\cdot\hat{p}\ \hat{p}\cdot\hat{s} / R_c n \tag{5.22}$$

(5.22) expressed in terms of the dimensionless "off course" vector, using (2.2), becomes

$$\delta\cos\gamma_i = -D_z \sin\gamma_o / b \tag{5.23}$$

So it is seen that γ measurements are only altered by out-of-plane perturbations. As the star elevation angle γ_o goes to zero (5.23) indicates that γ_i is insensitive to developing an "off course" error.

Consider now the two star navigation equation for b given by (3.4). It will be modified by the systematic perturbation to read

$$A' \cos\gamma(1)_i + B' \cos\gamma(2)_i + C' \sin s_i + 1 + Q_i = 0 \qquad (5.24)$$

where

$$Q_i = -\left(C' \delta \sin s_i + A' \delta\cos\gamma(1)_i + B' \delta\cos\gamma(2)_i\right) \qquad (5.25)$$

A',B',C' are the orbital parameters for the reference conic trajectory. The factor Q_i corrects the actual measured angles on the perturbed orbit to their values if measured from the reference conic orbit.

The correction to the navigation equation (5.24) leads to the correction for the evaluation of b. (3.7) should simply be modified by the substitutions

$$N_{1,2,3} \rightarrow N_{1,2,3} + \sum(Q)_{1,2,3} \qquad (5.26)$$

where the notation of (2.23) is indicated. The operational form of (5.26) is

$$Q_i = -\frac{1}{b}\left(\sin\theta\, D_x + (e+\cos\theta)\, D_y\right)_i \qquad (5.27)$$

$$+ \frac{e\, D_{zi}}{b \sin\theta_o(12)} \left(\sin\theta_o(2)\tan\gamma_o(1) - \sin\theta_o(1)\tan\gamma_o(2)\right)$$

In the same manner corrections to the equation for B'' (3.17) used to obtain e are made by the substitution in (3.17) of

$$\sum(y^2)_{1,2,3} \rightarrow \sum(y^2)_{1,2,3} + \sum(Q')_{1,2,3} \qquad (5.28)$$

where the notation of (2.23) is indicated. In (5.28) the quantity Q' is given by

$$Q_i' = \frac{2e \sin{}_o}{b} \begin{pmatrix} \left(\sin\theta_i\, D_{xi} + (e+\cos\theta)\, D_{yi}\right)\sin\beta_i \\ - e \tan\gamma_o \sin\theta_i\, D_{zi} \end{pmatrix} \qquad (5.29)$$

From (5.23) one sees that the change in γ_o due to perturbations is

$$\delta\gamma_o = D_{zo} / b \qquad (5.30)$$

where the o subscript indicates the value of D_z at $\theta = \theta_o$. The correction of e can then be made by using (3.14)

$$\delta e = - e D_{zo} \tan\gamma_o / b \qquad (5.31)$$

In-plane perturbations always appear in the combination

$$\Lambda(\theta) = \sin\theta \, D_x(\theta) + (e+\cos\theta) \, D_y(\theta) \qquad (5.32)$$

The two functions $\Lambda(\theta)$ and $D_z(\theta)$ have been computed by integrating (5.10a,b,c) for various values of e and for the several inhomogeneous terms given in (5.15a,b,c) with unit strength coefficient. Examination of (5.15a,b,c) indicates that there are five independent inhomogeneous terms which will produce the most general perturbation with arbitrary strength and location and for any value of the orbital parameter b. Three of the inhomogeneous terms are required for the most general $\Lambda(\theta)$ and two for the most general $D_z(\theta)$.

Figures (5.2-5.6) give these integrated functions. The navigator would scale each of these sets of curves by the strength coefficient appropiate to his particular perturbation problem.

The curves of Λ and D_z have been plotted as a function of the variable $1+e\cos\theta$ instead of the variable θ. By (2.11) it is seen that this new variable is more convenient and more closely related to a measureable, s. This aspect of the problem is discussed more fully in Chapter VII.

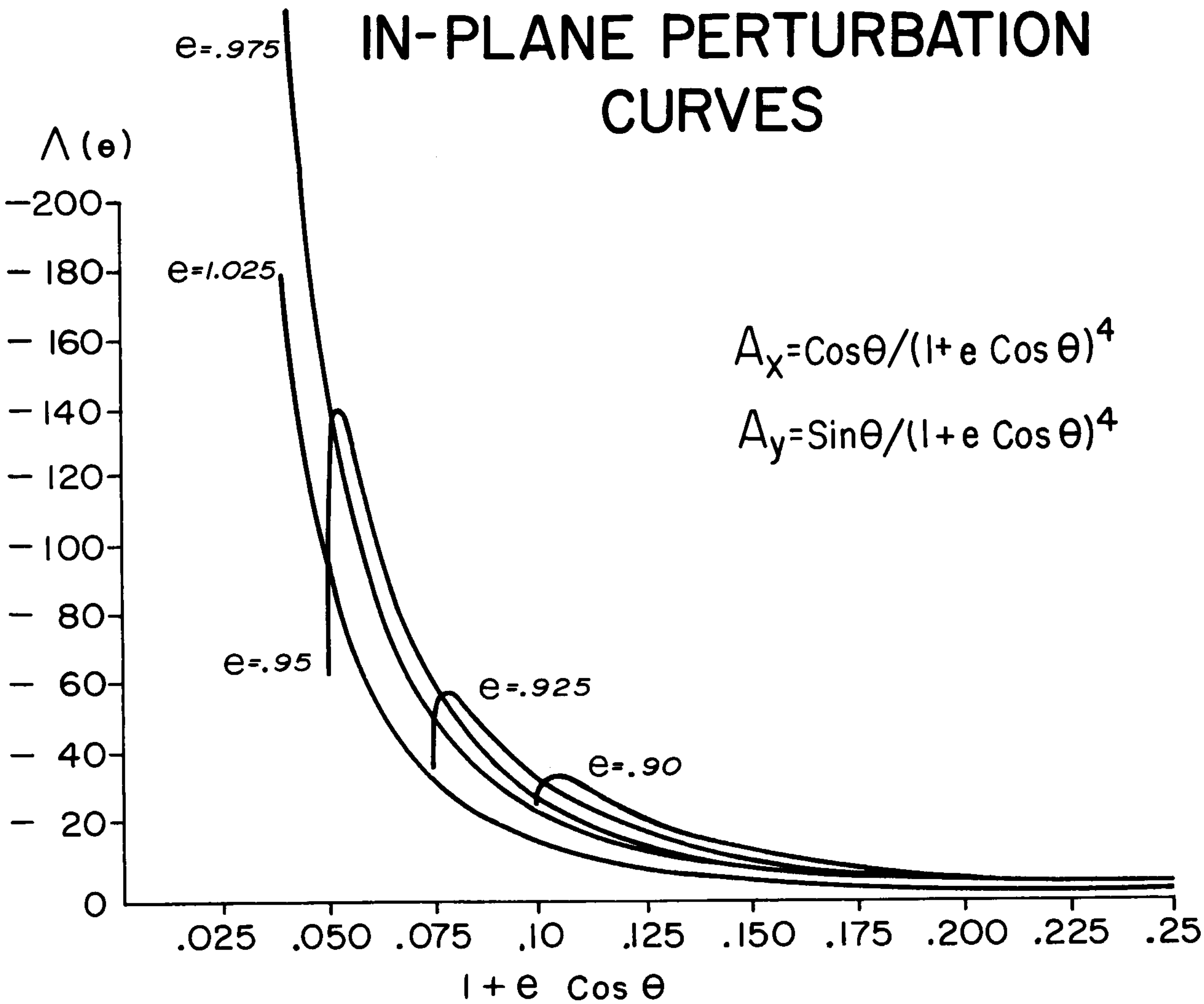

Figure 5.2 The perturbation function $\Lambda = \sin\theta\, D_x + (e+\cos\theta) D_y$ is plotted as a function of the variable $1+e\cos\theta$, for various values of e, the orbital eccentricity. The inhomogeneous acceleration values are $A_x = \cos\theta/(1+e\cos\theta)^4$, $A_y = \sin\theta/(1+e\cos\theta)^4$. The curves were obtained by computer integration of (5.10a,b).

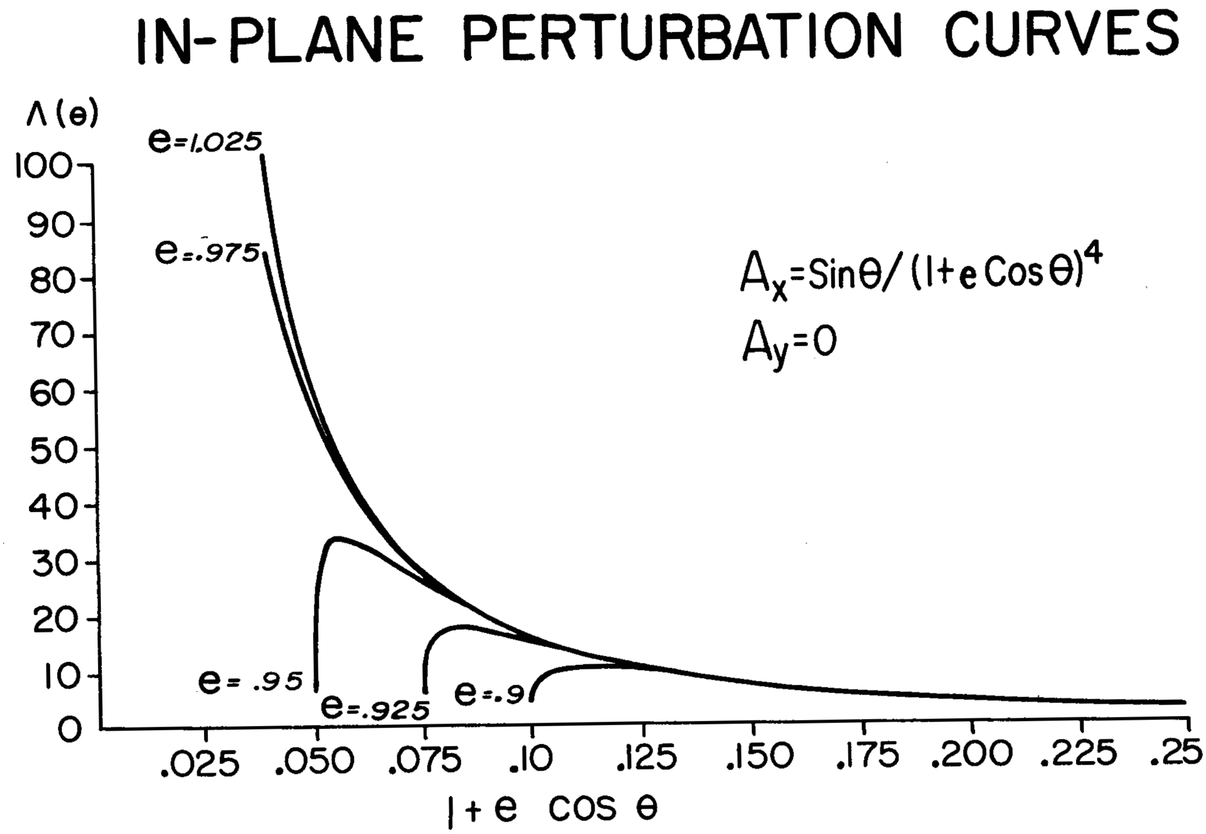

Figure 5.3 The perturbation function $\Lambda = \sin\theta\ D_x + (e+\cos\theta)D_y$ is plotted as a function of the variable $1+e\cos\theta$, for various values of e, orbital eccentricity. The inhomogeneous acceleration values are $A_x = \sin\theta/(1+e\cos\theta)^4$, $A_y = 0$. The curves were obtained by computer integration of (5.10a,b).

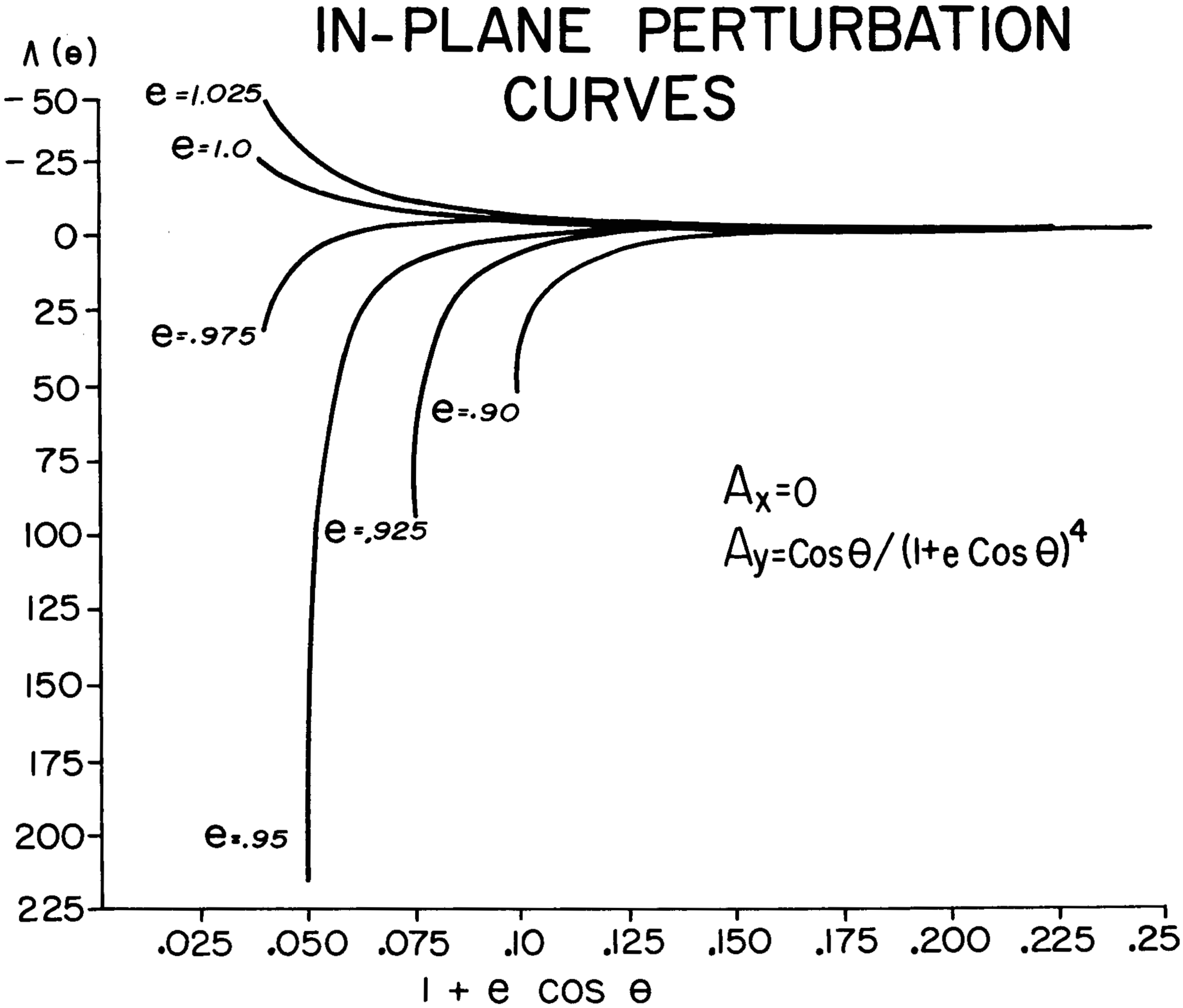

Figure 5.4 The perturbation function $\Lambda = \sin\theta\, D_x + (e+\cos\theta)D_y$ is plotted as a function of the variable $1+e\cos\theta$, for various values of e, orbital eccentricity. The inhomogeneous acceleration values are $A_x = 0$, $A_y = \cos\theta/(1+e\cos\theta)^4$. The curves were obtained by computer integration of (5.10a,b).

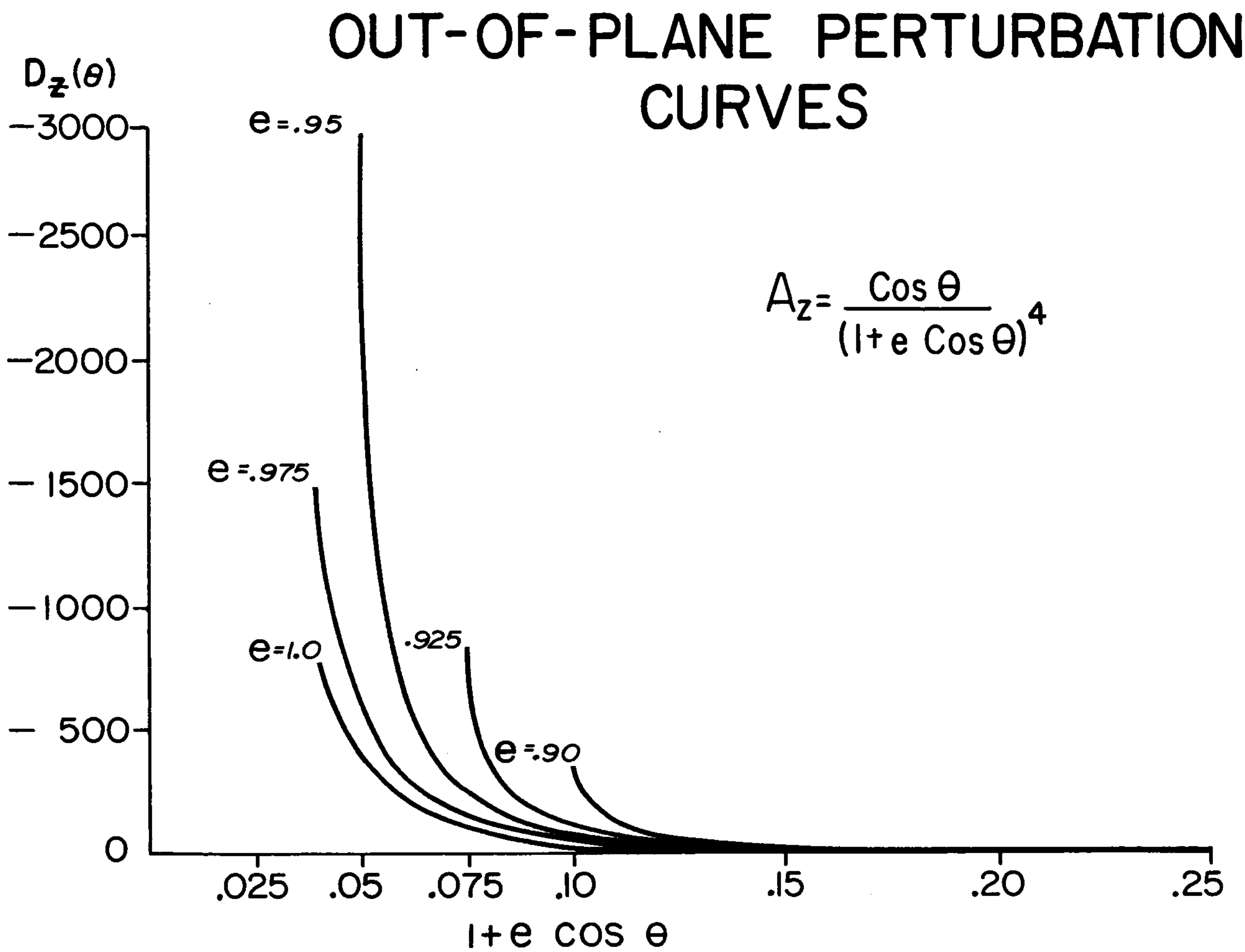

Figure 5.5 The perturbation function D_z is plotted as a function of the variable $1+e\cos\theta$, for various values of e, orbital eccentricity. The inhomogeneous acceleration value is $A_z = \cos\theta/(1+e\cos\theta)^4$. The curves were obtained by computer integration of (5.10c).

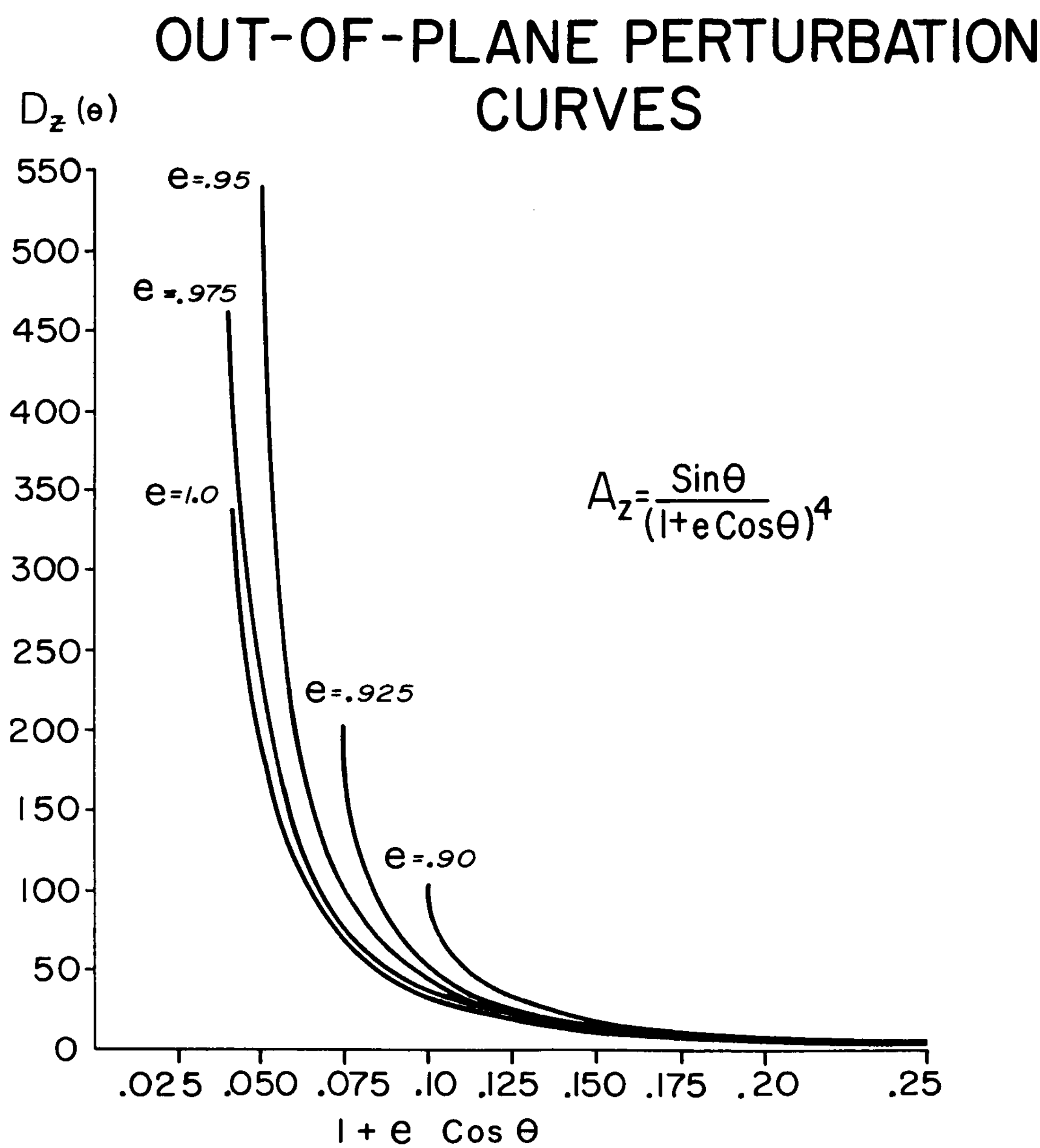

Figure 5.6 The perturbation function D_z is plotted as a function of the variable 1+ecos , for various values of e, orbital eccentricity. The inhomogeneous acceleration value is A_z = sin /(1+ecos)^4. The curves were obtained by computer integration of (5.10c).

CHAPTER VI

ELIMINATION OF NON EQUAL TIME ERRORS

A. Derivation of Non-Equal-Time Errors

The navigation equations derived throughout this work assume that the measured quantities in each observation are measured at the same time. We are not saying that the several observations in each cluster must be made at the same time; only that for each i the angles s_i, $\gamma(1)_i$, $\gamma(2)_i$ in two star navigation are made at the same time.

Operationally these measurements must be made at slightly different times. We now examine methods to eliminate any navigation errors due to these non-equal-times of measurement.

Consider the two star navigation equations. Let the s_i measurement time define the ith measurement time. Let $\Delta T(1)_i$ and $\Delta T(2)_i$ be the time differences between the s_i measurement and the $\gamma(1)_i$ and $\gamma(2)_i$ measurements, respectively. Starting from

$$\cos\gamma(1)_i = \cos\left(\gamma_e(1) \pm s\right)_i = \cos\gamma_o(1)\ \cos\beta(1)_i \tag{6.1}$$

where $\gamma_e(1)$ is the angle between the star and the central body edge, and noting that $\beta(1) = \theta - \theta_o(1)$, and assuming that the $\Delta T(1,2)$ are small times, then

$$\begin{aligned} \delta\cos\gamma(1)_i = &- \cos\gamma_o(1)\ \sin\beta(1)_i\ \delta\theta(1)_i \\ &\pm \sin\gamma(1)_i\ \delta s(1)_i \end{aligned} \tag{6.2}$$

But from (2.11)

$$\delta s(1)_i = - \frac{e\ \sin\theta_i}{b\ \cos s_i}\ \delta\theta(1)_i \tag{6.3}$$

Therefore

$$\begin{aligned} \delta\cos\gamma(1)_i &= - \left(\cos\gamma_o(1)\sin\beta(1)_i \pm \frac{e\sin\gamma(1)_i\sin\theta_i}{b\ \cos s_i}\right)\delta\theta(1)_i \\ &\equiv \eta(1)_i\ \delta\theta(1)_i \end{aligned} \tag{6.4}$$

The azimuth angle change can be related to the non-equal-time of the measurements by use of the equation giving the angular velocity of the spacecraft (5.6).

$$\delta\theta(1,2)_i = \sqrt{b}\ \sin^2 s_i\ \Delta T(1,2)_i\ /\ T_o \qquad (6.5)$$

with T_o given by (2.9). Using the two star navigation equation (3.4) we obtain the total non-equal-time error element

$$\varepsilon_i = \frac{\sqrt{b}}{T_o}\left(A'\eta(1)_i\ \Delta T(1)_i + B'\eta(2)_i\ \Delta T(2)_i\right)\ \sin^2 s_i \qquad (6.6)$$

The $\eta(1,2)$ factor changes slowly during the observational period, changing essentially as azimuth angle changes. The $\sin s_i$ factor however changes rapidly during a sequence of observations as $\sin^2 s_i$ is proportional to the spacecraft range to the inverse square power $(1/n^2)$. It is expected that the errors given by (6.6) will be dominated by their contribution in cluster 3 (see Figure (4.1)) when the spacecraft is closest to the central body and the factor $1/n$ is largest.

The estimation error in b produced by (6.6) is then given by applying (4.14) giving

$$\delta b = \frac{c_3'\sum_3(\varepsilon)}{c_1'\sum_1(\text{sins}) + c_2'\sum_2(\text{sins}) + c_3'\sum_3(\text{sins})} \qquad (6.7)$$

where the constants c' are given by (3.8a,b,c). The notation in (6.7) is that defined in (2.23).

If we randomize the time ordering of the measurements between the order $\gamma(1),s,\gamma(2)$ and $\gamma(2),s,\gamma(1)$ the error elements have the statistical properties

$$\langle\varepsilon_i\rangle = 0 \qquad (6.8)$$

and

$$\langle\varepsilon_i\varepsilon_j\rangle = b\ \sin^4 s_i\left(A'\eta(1)_i - B'\eta(2)_i\right)^2\ \frac{\langle\Delta T^2\rangle}{T_o^2} \qquad (6.9)$$

if i = j and zero otherwise. This leads to a mean square estimation error for b of

$$<\delta b^2> = \left[\frac{c_3' N_3}{c_1'\sum_1(\text{sins}) + c_2'\sum_2(\text{sins}) + c_3'\sum_3(\text{sins})}\right]^2 \text{ times}$$

$$\frac{\left(A'\eta(1) - B'\eta(2)\right)^2}{N_3} b \sin^4 s \frac{<\Delta T^2>}{T_o^2} \tag{6.10}$$

with (6.10) evaluated in cluster 3. Using Appendix C (6.10) gives to good approximation

$$\delta b_{rms} = \frac{16 \sin^2 s}{\sqrt{N_3}\ \theta^2_{nav}} \Delta T_{rms} / T_o \tag{6.11}$$

for optimally selected stars as discussed in Chapter IV and V. Using ΔT_{rms} = 1 minute, T_o = 780 seconds appropiate to earth, (6.11) yields a perigee error for eccentric orbits ($e \approx 1$) of

$$\delta \text{Perigee}_{rms} = 2200 \sin^2 s / \sqrt{N_3}\ \theta^2_{nav} \quad \text{miles} \tag{6.12}$$

(6.12) is plotted in Figure (6.1) for several values of n_{min} and n_{max}, with N_3 = 16 observations. The perigee errors shown are unacceptable for most navigational purposes. Better methods must be examined for handling non-equal-time errors.

B. A Non-Equal-Time Error Nulling Procedure

It was seen that $\eta(1,2)$ are slowly varying factors and $\sin^2 s$ the fast varying factor in (6.6). This leads to a method for nulling the non-equal-time errors.

First the order of the measurements $\gamma(1),s,\gamma(2)$ and $\gamma(2),s,\gamma(1)$ are alternated. The times $\Delta T(1,2)$ are recorded to some rough precision (± a couple seconds). The sums

$$\sum_{i \text{ in cluster } 3} \left(\sin^2 s_i \ \Delta T(1,2)_i\right) \tag{6.13}$$

are kept for each star by the navigator. The $\Delta T(1,2)$ will be alternating in sign in the sum (6.13) so (6.13) will not continuously grow larger with growing N_3.

But after the observations in the cluster have been obtained and the two sums indicated by (6.13) computed each sum will in general have a residual non zero value. At this

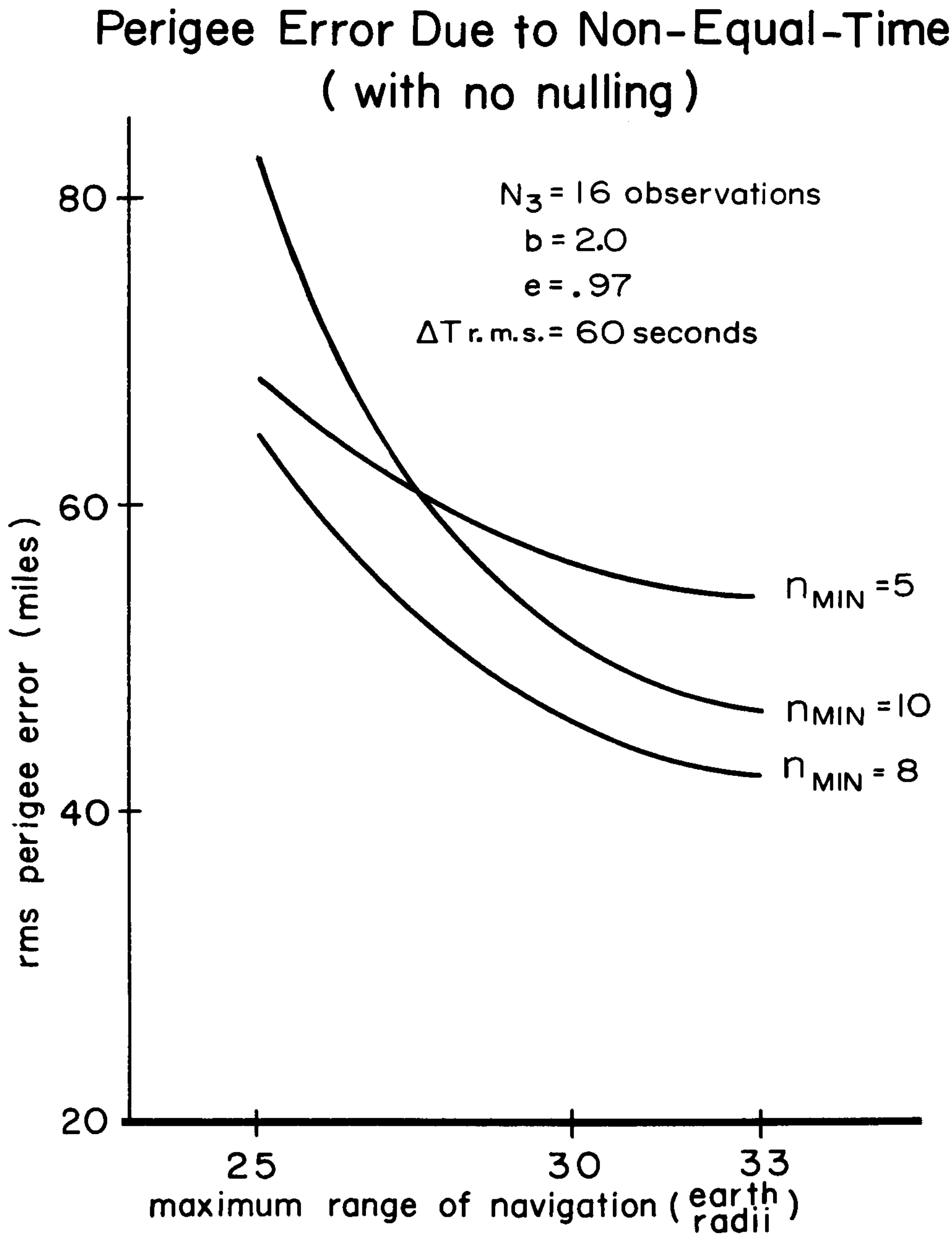

Figure 6.1 Error in perigee estimation due to uncorrected non-equal-time of measurements are shown for various values of n_{min} and n_{max}, spacecraft minimum and maximum range during navigation.

point the navigator has several choices. He can make one more observation adjusting the $\Delta T(1)$ and $\Delta T(2)$ so as to null the sums given by (6.13). Or he can go back through the cluster of observations and delete the one which best nulls (6.13).

By nulling the sums given by (6.13) the navigator essentially makes zero the numerator of (6.7) which gives the non-equal-time error in b.

CHAPTER VII

OPERATIONAL ASPECTS OF THE NAVIGATION

A. Introduction

This chapter will give a brief description of the actual operational steps which would be required of a space navigator using the navigational equations developed in this work. The equations which are to be employed in this chapter have all been derived and analyzed in previous chapters. It is the purpose here to use them all together in a navigation problem so that the reader can get a view of the operational procedures free of the theoretical work of previous chapters.

It should be pointed out that the purposed charts, tables, arithmetic steps, etc., discussed and used in this chapter cannot be considered as already optimally designed. Their optimal design must wait upon the results of actual man-system tests and experiments in which the more efficient of alternative operational procedures are found.

B. The Measurements

It is assumed for the purposes of this chapter that two star navigation is being used. The mission is the return of a spacecraft to earth from the lunar vicinity. The pertinent parameters of the problem are given below:

$$\text{Orbit} \quad \begin{Bmatrix} e \simeq .97 \\ b \simeq 2 \end{Bmatrix} \tag{7.1}$$

$$\text{Navigation Stars} \quad \begin{bmatrix} \gamma_o(1) = 10^0, \ \theta_o(1) = 167^0 \\ \gamma_o(2) = 15^0, \ \theta_o(2) = 250^0 \end{bmatrix} \tag{7.2}$$

$$\text{Lunar Perturbation} \quad \gamma_L = 10^0, \ \theta_L = 190^0 \tag{7.3}$$

The two navigation stars have been properly selected to minimize navigational error. They fulfill the conditions

1. Star line of sights lie close to the orbital plane.

2. Star line of sights are approximately perpendicular to each other.

3. One star line of sight has an azimuth such that γ_o can be measured for that star.

Note that the stars used for navigation can be any stars which fulfill the above conditions. The stars do not have to be identified with pre existing tables.

When at some distance from earth, $n_{max} \simeq 30$ earth radii, the navigator takes the first cluster of celestial measurements- $\gamma(1)_i, s_i, \gamma(2)_i$. $\gamma(1)_i$ is the angle between star one and the earth edge, $\gamma(2)_i$ is the star two to earth edge angle, and s_i is the semisubtended angle of the earth. The cluster of observations are taken about five or ten minutes apart. But it is important that the three angle measurements in each observation be taken as close together in time as possible. In each observation the order of the three measurements is reversed from the order in the previous observation.

At a later time when the spacecraft is closer to earth another cluster of measurements are taken. Finally when the spacecraft reaches $n_{min} \simeq 6$ earth radii, the third and final cluster of measurements are taken. (4.30) and (4.31) indicate that optimal navigation accuracy is obtained when

$$\theta_2 = (\theta_1 + \theta_3)/2 \tag{7.4}$$

and

$$N_1 = N_3 = N_2/2 = N/4 \tag{7.5}$$

and therefore the navigator endeavors to approximately fulfill these two conditions. For this example navigation problem the total number of observations N = 8.

C. The Data Tables

Figure (7.1) is a possible format for producing the necessary navigation sums. Note that all sums are simply updated upon making each new measurement. The first three columns are the measured angles. The other columns are generated from the data in the first three columns by either addition or by use of trigonometric tables, as indicated.

Figure (7.2) contains additional data and sums which would be needed for perturbation corrections and for the nulling of non-equal-time errors (see Chapter VI) if necessary. The perturbation data is taken from Figures (5.2-5.6).

Because the lunar perturbations are quite small until the spacecraft position is far from earth, in most all navigational cases like this navigation problem, only cluster one perturbation data need be obtained if any at all.

On the other hand azimuth rate of change is growing rapidly as the spacecraft approaches earth, so non-equal-time nulling data need only be obtained for cluster three in most cases.

D. The Navigation Calculations

All of the arithmetical calculations needed to estimate the orbit parameters can now be made using the sums of measured data generated in Table (7.1) and Table (7.2).

Table (7.3) shows the calculations needed to estimate b.

With b estimated and using a measured value for γ_o, Table (7.4) gives the sequence of calculations necessary to estimate the orbital parameter e. (If γ_o cannot be measured an alternative method for estimating γ_o is given at the end of Chapter III).

Actually the perturbation data of Table (7.2) cannot be completed until at least a rough estimate of b and e are made. The ordinate of Figures (5.2-5.6) is the quantity $1+e\cos\theta = b\sin s$ so requires the measured angle s and the orbital parameter b. Also the curves of Figures (5.2-5.6) form a one parameter family of curves with e being the parameter, hence an estimate of e must be made before one can read out the perturbation data.

Table (7.5) shows the calculations used to obtain the perturbation correction to b.

A table for perturbation corrections to e could also be produced using (5.28) through (5.31). But such corrections are an order of magnitude smaller than b corrections. Except in extraordinary circumstances e corrections can be neglected.

TABLE 7.1

MEASUREMENT DATA

s	$\gamma(1)$	$\gamma(2)$	$s+\gamma(1)$	$s+\gamma(2)$	$\sin s$	$\cos s+\gamma(1)$	$\cos s+\gamma(2)$		$2s+\gamma(1)$	$\sin^2 s$	$\sin^2 s+\gamma(1)$	$\sin\gamma(1)$	$\sin 2s+\gamma(1)$
.0315	.1465	1.4550	.1780	1.4865	.0315	.9841	.0842	1	.2095	.0010	.0314	.1459	.2080
///////	///////	///////	///////	///////	.0315	.9841	.0842		///////	.0010	.0314	.1459	.2080
.0327	.1472	1.4623	.1799	1.4590	.0326	.9839	.0758	2 = N_1	.2125	.0011	.0320	.1467	.2109
///////	///////	///////	///////	///////	.0641	1.9681	.1600		///////	.0021	.0634	.2926	.4189
.0739	.2469	1.6401	.3208	1.7140	.0738	.9490	-.1427	1	.3946	.0054	.0994	.2444	.3845
///////	///////	///////	///////	///////	.0738	.9490	-.1427		///////	.0054	.0994	.2444	.3845
.0780	.2572	1.6529	.3355	1.7309	.0779	.9442	-.1594	2	.4134	.0061	.1084	.2547	.4018
///////	///////	///////	///////	///////	.1517	1.8932	-.3021		///////	.0115	.2078	.4991	.7863
.0801	.2629	1.6592	.3429	1.7393	.0800	.9418	-.1677	3	.4230	.0064	.1131	.2598	.4105
///////	///////	///////	///////	///////	.2317	2.8350	-.4698		///////	.0179	.3209	.7589	1.1968
.0822	.2682	1.6655	.3504	1.7477	.0821	.9392	-.1760	4 = N_2	.4327	.0067	.1179	.2650	.4193
///////	///////	///////	///////	///////	.3138	3.7742	-.6458		///////	.0246	.4388	1.0239	1.6161
.1547	.4194	1.8282	.5741	1.9829	.1541	.8397	-.4005	1	.7288	.0237	.2949	.4072	.6660
///////	///////	///////	///////	///////	.1541	.8397	-.4005		///////	.0237	.2949	.4072	.6660
.1577	.4246	1.8336	.5824	1.9913	.1571	.8352	-.4082	2 = N_3	.7401	.0247	.3025	.4120	.6744
///////	///////	///////	///////	///////	.3112	1.6749	-.8087		///////	.0484	.5974	.8192	1.3404
					S_A	S_B	S_C			S_D	S_E	S_F	S_G

Table 7.1 An example of an eight observation data table is shown. The measured data is given in the first three columns. With the aid of tables the other columns are filled. Running sums are kept of the quantities S_A through S_G for each cluster of observations. These sums are used later in the solution for the orbital parameters.

TABLE 7.2

NON-EQUAL-TIME NULLING DATA

$\Delta T(1)$	$\Delta T(2)$	$\sin^2 s$	$\sin^2 s\ \Delta T(1)$	$\sin^2 s\ \Delta T(2)$
58	-63	.0237	1.37	-1.49
////////	////////	////////	1.37	-1.49
-61	55	.0247	-1.50	1.36
////////	////////	////////	-0.13	0.13

LUNAR PERTURBATION CORRECTION DATA

b sins	D_{Z1}	D_{Z2}	Λ_1	Λ_2	Λ_3
.0629	-300	140	-85	35	0
////////	-300	140	-85	35	0
.0651	-250	130	-80	33	0
////////	-550	270	-165	68	0
	S_{Z1}	S_{Z2}	$S_{\Lambda 1}$	$S_{\Lambda 2}$	$S_{\Lambda 3}$

Table 7.2 The tables shown give the additional data necessary to make the non-equal-time nulling and to make the perturbation corrections to the orbital parameters. The quantity b sins is the ordinate variable of the perturbation curves of Figures (5.2-5.6).

TABLE 7.3

ESTIMATION EQUATIONS FOR b

$$c_1' = S_{B2}\, S_{C3} - S_{B3}\, S_{C2}$$

$$= (3.7742)(-.8087)-(1.6749)(-.6458) = -1.9705$$

$$c_2' = S_{B3}\, S_{C1} - S_{B1}\, S_{C3}$$

$$= (1.6749)(.1600)-(1.9681)(-.8087) = 1.8596$$

$$c_3' = S_{B1}\, S_{C2} - S_{B2}\, S_{C1}$$

$$= (1.9681)(-.6458)-(3.7742)(.1600) = -1.8749$$

$$b = \frac{c_1'N_1 + c_2'N_2 + c_3'N_3}{c_1'S_{A1} + c_2'S_{A2} + c_3'S_{A3}} = \frac{\text{NUM}}{\text{DEN}}$$

$$= \frac{(-1.9705)(2)+(1.8596)(4)+(-1.8749)(2)}{(-1.9705)(.0641)+(1.8596)(.3138)+(-1.8749)(.3112)}$$

$$= \frac{-.2524}{-.1263} = 1.9984$$

Table 7.3 The arithmetical steps used in estimating the orbital parameter b are shown. First c_1', c_2', c_3' are computed from sums in Table 7.1. Then b is computed from the expression shown above. The text equations used in this table are (3.7) and (3.8).

TABLE 7.4

ESTIMATION EQUATIONS FOR e

$$S_{H1} = S_{B1} + b(S_{F1} - S_{G1})/2$$
$$= (1.9681)+(.9992)(.2926-.4189) = 1.8419$$

$$S_{H2} = S_{B2} + b(S_{F2} - S_{G2})/2$$
$$= (3.7742)+(.9992)(1.0239-1.6161) = 3.1825$$

$$S_{H3} = S_{B3} + b(S_{F3} - S_{G3})/2$$
$$= (1.6749)+(.9992)(.8192-1.3404) = 1.1541$$

$$S_{J1} = -S_{A1} + bS_{D1}/2$$
$$= (.9992)(.0021)-(.0641) = -.0620$$

$$S_{J2} = -S_{A2} + bS_{D2}/2$$
$$= (.9992)(.0246)-(.3138) = -.2892$$

$$S_{J3} = -S_{A3} + bS_{D3}/2$$
$$= (.9992)(.0484)-(.3112) = -.2628$$

$$c_1'' = N_2\, S_{H3} - N_3\, S_{H2}$$
$$= (4)(1.1541)-(2)(3.1825) = -1.7486$$

$$c_2'' = N_3\, S_{H1} - N_1\, S_{H3}$$
$$= (2)(1.8419)-(2)(1.1541) = 1.3756$$

$$c_3'' = N_1\, S_{H2} - N_2\, S_{H1}$$
$$= (2)(3.1825)-(4)(1.8419) = -1.0026$$

$$W = \frac{c_1''S_{J1} + c_2 S_{J2} + c_3''S_{J3}}{c_1''S_{E1} + c_2 S_{E2} + c_3''S_{E3}} = \frac{\text{NUM'}}{\text{DEN'}}$$

$$W = \frac{(-1.7486)(-.0620)+(1.3756)(-.2892)+(-1.0026)(-.2628)}{(-1.7486)(.0634)=(1.3756)(.4388)+(-1.0026)(.5974)}$$
$$= \frac{-.0259}{-.1063} = .2437$$

$\gamma_o = 10^o$ (measured) $\cos\gamma_o = .9848$ (from table)

$$e^2 = 2\ b\ W \cos^2\gamma_o$$
$$= (2)(1.9984)(.2437)(.9848)^2 = .9444$$

$e = .9718$ (from √ table)

Table 7.4 The equations necessary to estimate e are shown. First several new sums of data are obtained from sums generated in Table 7.1. These are used to obtain the coefficients c_1'', c_2'', c_3''. A quantity W is computed, which along with a measured value of γ_o gives an expression for e^2. A square root table then gives e. The text equations used in this table are (3.14), (3.17) and (3.18a,b,c).

TABLE 7.5

PERTURBATION EQUATIONS FOR b

$\theta_L \simeq 190^\circ$ (estimated) $\qquad \gamma_L \simeq 10^\circ$ (estimated)

$$K_{\Lambda 1} = 3\cos^2\gamma_L \cos\theta_L \sin\theta_L \simeq .5$$

$$K_{\Lambda 2} = 3\cos^2\gamma_L \sin^2\theta_L - 1 \simeq -.9$$

$$K_{\Lambda 3} = 3\cos^2\gamma_L \cos^2\theta_L - 1 \simeq 2.0$$

$$K_{Z1} = 3\cos\gamma_L \sin\gamma_L \cos\theta_L \simeq -.5$$

$$K_{Z2} = 3\cos\gamma_L \sin\gamma_L \sin\theta_L \simeq -.1$$

$$OP = \frac{e\left[\sin\theta_o(2)\tan\gamma_o(1) - \sin\theta_o(1)\tan\gamma_o(2)\right]}{\sin\theta_o(12)} \simeq .16$$

$\lambda = 5.74\ 10^{-8}$ (stored constant for earth-moon system)

$$S_K = -\lambda b^3\left[K_{\Lambda 1}S_{\Lambda 1} + K_{\Lambda 2}S_{\Lambda 2} + K_{\Lambda 3}S_{\Lambda 3} - OP\left(K_{Z1}S_{Z1} + K_{Z2}S_{Z2}\right)\right]$$

$$\simeq (5.74)(8)\ 10^{-8}$$

$$\cdot\left[(.5)(-165)+(-.9)(68)-(-.5)(.16)(-550) - (-.1)(.16)(270)\right]$$

$$\simeq -.84\ 10^{-4}$$

$$\delta b = c_1'\ S_K / DEN = \frac{(-1.97)(-.84)\ 10^{-4}}{(-.1263)} = -.0014$$

Table 7.5 The perturbative correction to b is made above. From rough measurements of the moon's position the five position coeffients are computed. Then OP, an out-of-plane constant, is computed. Finally S_K, the total perturbation sum is computed. δ b is then computed from an expression containing c_1' and DEN which were previously computed in Table 7.3. Text equations used in this table are (5.15a,b,c) and (5.26).

CHAPTER VIII

SUMMARY

A. The Navigation Equations

The requirement that navigation equations be linear is a strong one. Linear equations admit of straightforward arithmetic solutions, while nonlinear equations almost always present an impossible task for quick manual solution.

In Chapters II and III the first accomplishment was to transform Keplerian orbit equations (conic equations) into a form where all the unknown parameters to be estimated appeared in a linear fashion. Note that we have obtained exact equations with the unknowns appearing in a linear fashion, not linearized equations valid only for sufficiently small deviations from a reference trajectory. Our navigator needs no reference trajectory to start his navigating from.

The linearity of the navigation equations also leads to a straightforward operationally simple method of incorporating redundant measurements into the navigation equations. Individual pieces of measured data simply become sums of measured data.

After linear equations were obtained in Chapter II the variables in the equations (the quantities which vary from measurement to measurement along the spacecraft trajectory) were manipulated in order to find other variables which were closer to the directly measured data. "Closer to the directly measured data" means that less computational steps are needed to convert the measured angles into the variables which appear in the navigation equations.

The chief reason for going from the one star navigation (Chapter II) to two star navigation (Chapter III) was that the variables in the two star navigation equations could be brought much closer to the measured data, thus eliminating several calculational steps for the navigator.

The final navigation equations which are developed are independent of the time variable. This liberates the system from dependence on an accurate clock.

B. Navigation Accuracy

In order to obtain an analytical estimate of operational navigation accuracy with our navigation equations, an error analysis was made in Chapter IV. Assuming that all the measurement errors were small quantities, a first differential of the navigation equations was made. Then using the statistical properties of the measurement errors, equations were obtained which gave the expected statistical properties of orbit parameter navigation errors.

These formulas are exceedingly useful because the dependence of navigation accuracy on all of the adjustable parameters of the navigation problem can be explicitly calculated and studied analytically. From this knowledge operational rules can be made which will guarantee near optimum navigation accuracy.

The error equations in Chapter IV have been derived in the most general case. But whenever quantitative estimates were made, the particular case of an eccentric orbit around earth with orbit perigee equal to about one earth radii was used as the illustrative example. These particular estimates are valid for the typical return trajectory from the lunar vicinity to earth.

Listed below are several of the conclusions that can be made concerning navigation accuracy which come out of the work on Chapter IV.

1. Navigation accuracy goes approximately as $1/\theta^2_{nav}$ where θ_{nav} is the total spacecraft orbit azimuth change during the navigational period.

2. Error in estimating the orbit parameter b is an order of magnitude larger than the error in e.

3. Contrary to many comments in the literature, it has been found that the planet subtended angle measurement and the star planet angle measurement contribute about the same size error to b or e estimation. Often it is stated that planet subtended angle is a poor measurement when far from the planet. This is only true if estimation of present spacecraft position is desired. This is not true if it is orbital parameters e and b which are to be estimated.

4. Optimal accuracy is obtained when a cluster of measurements is performed at each extreme azimuth limit of the spacecraft

trajectory with a third cluster of observations performed at an azimuth bisecting the minimum and maximum azimuths. For a fixed total number of measurements about half of them should be performed in the middle cluster, a quarter each in the other clusters.

5. In two star navigation the stars should lie as much in the spacecraft orbital plane as possible and perpendicular to each other.

C. Gravitational Perturbations

The basic navigation equations developed in Chapters II and III are valid for perfect Keplerian (conic) orbits. In many navigation problems there will be one or more perturbing gravitational bodies present which will cause orbit deviations from the perfect conic orbits.

In order to treat such corrections within the simple linear equations developed in this work we have restricted the perturbation study to cases where the perturbations are small, that is the spacecraft is substantially within the sphere of influence of the central body.

If a spacecraft is slightly off course from the conic orbit the measured angles will be systematically in slight error from what their values would be if moving on the conic trajectory. In other words, one can view the gravitational perturbation as causing a systematic error in the measured data. By applying opposite signed systematic errors to the actual measured data one generates the data which would be obtained if orbiting on a perfect conic trajectory. The equations of Chapter IV dealing with known systematic errors can then be applied to obtaining corrections to the orbit parameters which are being estimated.

The differential equations for the deviations of an orbit from a conic trajectory due to a perturbation are linearized in Chapter V, and then the perturbing acceleration is expanded in a multipole expansion with only the leading multipole retained. This results in the simplification that the final perturbation table is reduced to dependence on just one parameter, the orbital eccentricity e. All the other parameters of the problem like the strength and location of the perturbing body and the orbital parameter b are factored out into coefficients which just scale the size of the perturbations.

We have produced a general perturbation table which will handle estimation corrections due to a perturbing body of arbitrary (but weak) mass and location.

D. The Non Equal Time Problem

One of the difficulties with direct navigation from celestial fixes is the necessity of simultaneous measurements. In Chapter VI this problem is investigated for the navigation equations developed in this work. Due to the fact that sums of data from clusters of observations are the final numbers used in the navigation equations, it becomes possible to quench the errors due to individual non-equal-times of measurement.

E. Operational Problems and Aspects of Manual Navigation

In Chapter VII a model navigation problem is gone through step by step in order to give the reader a unified view of the manual navigation system in operation. It remains for actual man-system experiments and tests to decide on optional operational procedures, and to also test the actual feasibility of this approach to manual navigation.

F. The Appendices

Several different problems and derivations which explore topics of direct relevance to this approach to space navigation but which are not required to follow the main development of this work are included in appendices.

In A equations for the optimum use of each measurement are derived. The optimum navigation equations require substantially more sums of data to be kept. In the limit that observations are condensed into three compact clusters, it is straightforward to show that the equations of Chapter II and III are optimum.

In B we obtain the important result that systematic bias errors in a man-machine system can be filtered out of parameter estimations. If the filtering techniques derived here are not feasible for manual navigation because of the additional computation required, these techniques can still be adopted in computerized navigational systems in order to protect navigation estimates from hidden systematic errors.

In C we derive expressions for useful cofactors and matrix determinants which are needed in the analytical expressions for navigation error.

In D the necessary calculations for estimating the time to perigee are derived. These expressions would be useful for spacecrafts on a reentry trajectory.

Appendix E presents the necessary equations for a navigator to make an a posteriori calculation of his navigational accuracy. In most cases the considerable additional calculational effort probably makes such an a posteriori estimate impractical except in situations where unlimited time is available to the navigator.

APPENDIX A

OPTIMUM USE OF NAVIGATION MEASUREMENTS

In the body of this work we developed navigation equations from redundant observations. This involved grouping the observations into three clusters. It is the purpose of this appendix to derive the optimum linear navigation equations and compare them with the equations derived for operational use.

Consider the general navigation equation

$$(m_\alpha)_i \; K_\alpha + (m_o)_i = 0 \tag{A.1}$$

for each i (the observation label) from 1 to N. $\alpha$is summed over the constant parameters from 1 to k, there being k parameters in the linear equation. Instead of clustering the N equations (A.1) into k groups as was done earlier in this work, we assume k weighting vectors $(w_\alpha)_i$ where α goes from 1 to k. We can obtain the desire k equations for the K_α by dotting (A.1) with each weighting vector

$$w_{\alpha'} \cdot m_\alpha \; K_\alpha = - \; w_{\alpha'} \cdot m_o \tag{A.2}$$

where the dot product notation used is

$$w_{\alpha'} \cdot m_\alpha \equiv \sum_{i=1}^{N} (w_{\alpha'})_i (m_\alpha)_i \tag{A.3}$$

The problem is to pick the $(w_\alpha)_i$ so as to minimize the rms error in the estimation of the constant parameters K_α. Taking a differential of (A.2) we obtain

$$w_{\alpha'} \cdot m_\alpha \; \delta K_\alpha = - \; w_{\alpha'} \cdot \varepsilon \tag{A.4}$$

where

$$\varepsilon_i = \delta (m_o)_i + \delta (m_\alpha)_i \; K_\alpha \tag{A.5}$$

Consider a linear transformation on the parameter error components

$$\delta K'_{\alpha'} = T_{\alpha' \alpha} \; \delta K_\alpha \tag{A.6}$$

$K'_{\alpha'}$ could represent any function of the orbital parameters K_α with the elements of the transformation matrix given by

$$T_{\alpha'\alpha} = \partial\ K'_{\alpha'}\ /\ \partial\ K_\alpha \tag{A.7}$$

We only require that the transformation (A.7) have an inverse. Going to matrix notation (A.4) becomes

$$[M][T]^{-1}\{\delta K'\} = -\ \{w \cdot \varepsilon\} \tag{A.8}$$

where

$$[M]_{\alpha'\alpha} \equiv w_{\alpha'} \cdot m_\alpha \tag{A.9a}$$

and

$$\{w \cdot \varepsilon\}_\alpha \equiv w_\alpha \cdot \varepsilon \tag{A.9b}$$

Defining the new matrix

$$[N] = [M][T]^{-1} \tag{A.10}$$

(A.8) can be formally solved to yield

$$\{\delta K'\} = [N]^{-1}\{w \cdot \varepsilon\} \tag{A.11}$$

Letting

$$(Cf)_{\alpha'\alpha} = \text{cofactor of } [N]_{\alpha'\alpha} \tag{A.12}$$

the solution of (A.11) for any particular $\delta K'_{\alpha'}$ is

$$\delta K'_{\alpha'} = -\sum_{\alpha=1}^{k}\left[(Cf)_{\alpha\alpha'}\{w \cdot \varepsilon\}_\alpha\right]\ /\ \sum_{\alpha=1}^{k}\left[(Cf)_{\alpha\alpha'}[N]_{\alpha\alpha'}\right] \tag{A.13}$$

But by (A.10) and (A.9) we have finally

$$\delta K'_{\alpha'} = -\ \Psi_{\alpha'} \cdot \varepsilon\ /\ \Psi_{\alpha'} \cdot \mu_{\alpha'} \tag{A.14}$$

where we have defined the quantities

$$\sum_{\alpha=1}^{k} (Cf)_{\alpha\alpha'}(w_\alpha)_i = (\Psi_{\alpha'})_i \tag{A.15}$$

and

$$\sum_{\alpha''=1}^{k} (m_{\alpha''})_i [T]^{-1}_{\alpha''\alpha'} = (\mu_{\alpha'})_i \tag{A.16}$$

Squaring (A.14) and taking a statistical average yields

$$<(\delta K'_{\alpha'})^2> = \frac{\sum\limits_{i,j=1}^{N} (\Psi_{\alpha'})_i (\Psi_{\alpha'})_j <\varepsilon_i \varepsilon_j>}{\left(\sum\limits_{i=1}^{N} (\Psi_{\alpha'})_i (\mu_{\alpha'})_i\right)^2} \tag{A.17}$$

Minimizing (A.17) with respect to $(\Psi_{\alpha'})_i$ yields

$$\sum_{j=1}^{N} <\varepsilon_i \varepsilon_j> (\Psi_{\alpha'})_j = \text{constant} \cdot (\mu_{\alpha'})_i \tag{A.18}$$

or defining the inverse error correlation matrix $[E]^{-1}$ by

$$[E]^{-1}[E] = [1] \tag{A.19}$$

where*

$$[E]_{ij} = <\varepsilon_i \varepsilon_j> \tag{A.20}$$

(A.18) can be solved for $(\Psi_{\alpha'})_i$

$$(\Psi_{\alpha'})_i = \sum_{j=1}^{N} [E]^{-1}_{ij} (\mu_{\alpha'})_j \tag{A.21}$$

From (A.15) it is seen that $\Psi_{\alpha'}$ is a linear combination of the weighting vectors w_α. From (A.16) it is seen that $\mu_{\alpha'}$ is a linear combination of the measureables m_α. But the original

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* Note that the error correlation matrix is an N x N matrix, the labels i,j referring to observations. In other words the error matrix is in a different space from the k x k matrices previously introduced in this work.

navigation equations are invariant under taking any linear combination of themselves, hence (A.21) is equivalent to setting

$$(w_{\alpha'})_i = \sum [E]^{-1}_{ij}(m_{\alpha'})_j \tag{A.22}$$

for the optimum weighting factors. This result is independent of the transformation matrix [T], hence (A.22) minimizes the rms size of any linear combination of the parameter errors.

Inserting (A.22) in to (A.3) gives the optimum navigation equations

$$[M_{opt}]\{K\} = \{v_{opt}\} \tag{A.23}$$

with

$$[M_{opt}]_{\alpha'\alpha} = \sum_{i,j=1}^{N} [E]^{-1}_{ij}(m_{\alpha'})_j(m_\alpha)_i \tag{A.24}$$

and

$$\{v_{opt}\}_{\alpha'} = -\sum_{i,j=1}^{N} [E]^{-1}_{ij}(m_{\alpha'})_j(m_o)_i \tag{A.25}$$

For the case of constant uncorrelated errors we have

$$\begin{aligned}[E]_{ij} &= \sigma^2 \qquad && i = j \\ &= 0 && i \neq j\end{aligned} \tag{A.26}$$

and (A.23) takes the particularly simple and symmetric form

$$m_{\alpha'} \cdot m_\alpha \; K_\alpha = - m_{\alpha'} \cdot m_o \tag{A.27}$$

One disadvantage of using the optimum navigation equation (A.27) in which the weighting factors are determined by the measureables is that we produce mean errors in K_α even though the measurement errors have zero mean value. To understand this take a differential of (A.27)

$$m_{\alpha'} \cdot m_\alpha \; \delta K_\alpha = - m_{\alpha'} \cdot \varepsilon - \delta m_{\alpha'} \cdot \varepsilon \tag{A.28}$$

where ε_i is given by (A.5). Taking the statistical average of (A.28) gives

$$m_{\alpha'} \cdot m_\alpha \langle \delta K_\alpha \rangle = - \langle \delta m_{\alpha'} \cdot \varepsilon \rangle \neq 0 \tag{A.29}$$

which does not vanish even when $<\delta m_\alpha> = 0$. The left and right side of (A.29) grow linearly with N, so $<\delta K_\alpha>$ does not diminish with the accumulation of additional observations.

The operational disadvantage of the optimum equations (A.27) is seen by noting that the required sums of measureables in (A.27) includes all products among the measureables. In the case of k = 3, (A.27) requires twelve different sums of measureables to be kept, while we have seen that by clustering observations we required only three different sums of measureables.

It is straightforward to show that in the limit as the observations in each cluster are made at times near each other, the simple navigation equations of Chapter II and III approach the optimum equations. For practical observation schedules the simple navigation equations are within 5 or 10% of giving optimum rms parameter errors.

APPENDIX B

ELIMINATION OF SYSTEMATIC MEASUREMENT ERROR FROM NAVIGATION MEASUREMENTS

In Chapter IV it was seen that systematic measurement errors led to systematic navigation errors which do not decrease in size with additional data as $1/\sqrt{N}$. In the long run, then, systematic errors could determine the accuracy of navigation.

We present here an outline of a procedure which can eliminate the systematic errors from the measurements without a priori knowing the magnitude of the systematic errors.

Consider the two star navigation equation (3.4)

$$A' \cos\gamma(1)_i + B' \cos\gamma(2)_i + C' \sin s_i + 1 = 0 \tag{B.1}$$

Suppose that a combination of man and instrument biases led to the possible systematic error in the measurement of the two star-central body angles, $\delta\gamma(1)_i = \delta\gamma(2)_i = \delta\gamma$. Then (B.1) would become

$$A' \cos\gamma(1)_i + B' \cos\gamma(2)_i + C' \sin s_i + 1 + \phi_i\delta\gamma = 0 \tag{B.2}$$

where the additional term ϕ_i is the systematic error correction coeffient

$$\phi_i = A' \sin\gamma(1)_i + B' \sin\gamma(2)_i \tag{B.3}$$

The unknown bias $\delta\gamma$ can now be considered a new parameter to be estimated by the navigation equations. Then we must divide the observations into four clusters to obtain four equations. The fourth equation can formally be solved for giving

$$\delta\gamma = -\left[N_4 + A'\textstyle\sum_4(\cos\gamma(1)) + B'\sum_4(\cos\gamma(2)) + C'\sum_4(\sin s)\right]/\textstyle\sum_4(\phi) \tag{B.4}$$

The $\delta\gamma$ obtained from (B.4) can now be inserted into the other three cluster equations giving three modified equations for the three orbital parameter unknowns, A', B', C'. The navigation equations of Chapter III are then simply modified by the substitution

$$\sum_k(v) \rightarrow \sum_k(v) - \sum_4(v) \sum_k(\phi) / \sum_4(\phi) \qquad (B.5)$$

where k represents the kth cluster (k=1,2,3) and v represents any variable which is formed into sums. The substitutions indicated by (B.5) represent a "bias filter", they filter out of the data any systematic error of the form given by (B.2).

This technique can be generalized to filter out several systematic errors at once, the cost being increased complexity of the navigation equations and an increase in the noise error of the parameters.

The bias filter (B.5) would also result by assuming an error correlation matrix

$$[E]_{ij} = \langle \varepsilon_i \varepsilon_j \rangle = \phi_i \phi_j \qquad (B.6)$$

and using the optimum navigation equations given by (A.23).

APPENDIX C

DERIVATION OF RMS ERROR FORMULAS

In order to obtain analytical expressions for orbital prediction error, several expressions in the body of this work must be simplified. First in one star navigation the denominator of (2.24) is analyzed.

Assuming all the measurements in each cluster to be made approximately at the same time, (2.25a,b,c) become

$$c_1 = N_2N_3 \sin\left(\beta(1)_3-\beta(1)_2\right) = N_2N_3 \sin(\theta_3-\theta_2) \tag{C.1}$$

$$c_2 = N_1N_3 \sin\left(\beta(1)_1-\beta(1)_3\right) = N_1N_3 \sin(\theta_1-\theta_3) \tag{C.2}$$

$$c_3 = N_2N_1 \sin\left(\beta(1)_2-\beta(1)_1\right) = N_2N_1 \sin(\theta_2-\theta_1) \tag{C.3}$$

Expressing

$$\sin s = (1+e\cos\theta)/b \tag{C.4}$$

the $\cos\theta$ terms cancel when summed over the three terms in the denominator of (2.24) leaving the denominator to be

$$DEN_1 = N_1N_2N_3 \left(\sin(\theta_3-\theta_2)+\sin(\theta_1-\theta_3)+\sin(\theta_2-\theta_1)\right)/b \tag{C.5}$$

For the error in b (4.17) becomes

$$\langle\delta b^2\rangle = \sum_{\alpha=1}^{3} [M]^{-1}_{\alpha 3}[M]^{-1}_{3\alpha} \sum\nolimits_\alpha \sigma_i^{\,2} \tag{C.6}$$

The elements of the inverse matrices are constructed from cofactors of the original matrix divided by the determinant of the matrices. This yields

$$[M]^{-1}_{13}[M]^{-1}_{31} = N_2{}^2N_3{}^2 \sin^2(\theta_3-\theta_2) \;/\; DEN_1{}^2 \tag{C.7}$$

$$[M]^{-1}_{23}[M]^{-1}_{32} = N_1{}^2N_3{}^2 \sin^2(\theta_3-\theta_1) \;/\; DEN_1{}^2 \tag{C.8}$$

and

$$[M]^{-1}_{33}[M]^{-1}_{33} = N_1{}^2N_2{}^2 \sin^2(\theta_1-\theta_2) \;/\; DEN_1{}^2 \tag{C.9}$$

Using the above derived results (C.6) then becomes (4.22) of the body of this work.

Similarly, we consider the denominator of the two star navigation equations (3.7). (3.8a,b,c) are given by

$$c_1' = N_2N_3 \cos\gamma_o(1)\cos\gamma_o(2)\left[\cos\beta(1)_2\cos\beta(2)_3-\cos\beta(1)_3\cos\beta(2)_2\right] \tag{C.10}$$

$$c_2' = N_1N_3 \cos\gamma_o(1)\cos\gamma_o(2)\left[\cos\beta(1)_3\cos\beta(2)_1-\cos\beta(1)_1\cos\beta(2)_3\right] \tag{C.11}$$

$$c_3' = N_1N_2 \cos\gamma_o(1)\cos\gamma_o(2)\left[\cos\beta(1)_1\cos\beta(2)_2-\cos\beta(1)_2\cos\beta(2)_1\right] \tag{C.12}$$

Again using (C.4) and canceling the terms from the $\cos\theta$ part we get for the entire two star navigation denominator

$$DEN_2 = N_1N_2N_3\cos\gamma_o(1)\ \cos\gamma_o(2)\ /\ b\ \cdot$$
$$\left[\cos\beta(1)_1\left(\cos\beta(2)_2-\cos\beta(2)_3\right) + \cos\beta(1)_2\left(\cos\beta(2)_3-\cos\beta(2)_1\right)\right.$$
$$\left. + \cos\beta(1)_3\left(\cos\beta(2)_1-\cos\beta(2)_2\right)\right] \tag{C.13}$$

If we use the optimum location of θ_2 as given by (4.30) then several trionometric manipulations give

$$DEN_2 = 8N_1N_2N_3\ \cos\gamma_o(1)\ \cos\gamma_o(2)\ \sin\theta_o(12)\ \cos\theta_{31}/4$$
$$\cdot\ \sin^3\theta_{31}/4\ /b \tag{C.14}$$

where $\theta_o(12) = \theta_o(1)-\theta_o(2)$ and $\theta_{31} = \theta_3-\theta_1$. If the optimum θ_2 as given by (4.30) is used in (C.5) we also obtain

$$DEN_1 = 8N_1N_2N_3\ \cos\theta_{31}/4\ \sin^3\theta_{31}/4\ /b \tag{C.15}$$

If good navigation stars are selected which fulfill the conditions

$$\cos\gamma_o(1) \simeq \cos\gamma_o(2) \simeq \sin\theta_o(12) \simeq 1 \tag{C.16}$$

it is seen that

$$DEN_2 \simeq DEN_1 \tag{C.17}$$

If (C.16) is fulfilled then $c_{1,2,3}' = c_{1,2,3}$ and it can be concluded that one star and two star navigation accuracy in estimating b will be the same if the $b\ \delta sins$ term dominates the error elements which is generally the case.

APPENDIX D

TIME TO PERIGEE

It would be desirable for the navigator to be able to estimate the time remaining until the spacecraft will arrive at orbital perigee. Preparation for reentry could then be allotted the necessary time, or other maneuvers which are desirable to perform at perigee could be made at the proper time.

For an elliptical orbit the time to perigee is given in terms of the eccentric anomaly ξ*

$$\cos\xi = (e+\cos\theta)/(1+e\cos\theta) \tag{D.1}$$

or

$$\tan\xi/2 = \sqrt{(1-e)/(1+e)}\ \tan\theta/2 \tag{D.2}$$

The time to perigee T_{per} is then given by

$$T_{per} = T_o \left[b/(1-e^2)\right]^{3/2} (\xi - e\sin\xi) \tag{D.3}$$

where T_o has been defined in (2.9).

For hyperbolic orbits a slightly modified equation replaces (D.3)

$$T_{per} = T_o \left[b/(e^2-1)\right]^{3/2} (e\sinh H - H) \tag{D.4}$$

where

$$\tanh H/2 = \sqrt{(e-1)/(e+1)}\ \tanh\theta/2 \tag{D.5}$$

It is more useful to express ξ and H in terms of the measureable sins. From the trigonometric identity

$$\tan\theta/2 = \sqrt{(1-\cos\theta)/(1+\cos\theta)} \tag{D.6}$$

(D.2) gives

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* P. Van DeKamp, Elements of Astromechanics, (Freeman, San Francisco, 1964) Chapter 4

$$\tan\xi/2 = \sqrt{(1-e)(e-bsins+1)/(1+e)(e+bsins-1)} \qquad (D.7)$$

For hyperbolic orbits the direct substitution

$$\theta = \text{arc-cos}\{(bsins-1)/e\} \qquad (D.8)$$

must be used to express H in terms of sins.

The accuracy of time to perigee prediction can now be calculated. From (2.6)

$$R^2 \, d\theta/dt = h \qquad (D.9)$$

which gives

$$\delta T_{per} = R^2 \, \partial\theta/\partial sins \quad \delta sins \quad /h \qquad (D.10)$$

But

$$1+e\cos\theta = b \; sins \qquad (D.11)$$

so finally

$$\delta T_{per} = - \sqrt{b} \; T_o \; \delta sins \; / \; e \sin\theta \; \sin^2 s \qquad (D.12)$$

For an eccentric orbit with perigee of about one central body radii ($e \simeq 1$, $b \simeq 2$), setting sins $\equiv 1/n \simeq 1/6$, and δsins $\simeq$ 1 arc-minute

$$\delta T_{per} \simeq -13 \text{ seconds} \qquad (D.13)$$

for the case of earth.

APPENDIX E

A POSTERIORI DETERMINATION OF NAVIGATION ACCURACY

Consider the general navigation equation (4.1)

$$(m_\alpha)_i K_\alpha + (m_o)_i + Q_i = 0 \tag{E.1}$$

including the perturbation corrections Q_i. The navigation equations demand that (E.1) be valid only for the sum of observations in the cluster. For each i (E.1) should actually give a residue

$$\rho_i = (m_\alpha)_i \delta K_\alpha + \varepsilon_i \tag{E.2}$$

with ε_i the error element given by (4.10). Squaring (E.2), summing over i, and taking a statistical average yields

$$\sum_{i=1}^{N} \langle \rho_i^2 \rangle = \sum_{i=1}^{N} \langle \varepsilon_i^2 \rangle + \sum_{i=1}^{N} (m_\alpha)_i (m_{\alpha'})_i \langle \delta K_\alpha \delta K_{\alpha'} \rangle + 2 \sum_{i=1}^{N} (m_\alpha)_i \langle \varepsilon_i \delta K_\alpha \rangle \tag{E.3}$$

Only the first term in (E.3) grows as N the total number of observations. Because $\langle \delta K_\alpha \rangle$ goes as $1/\sqrt{N}$ the other terms become negligible compared with the leading term as N gets large. Therefore for large N (E.3) becomes

$$\sum_{i=1}^{N} \langle \rho_i^2 \rangle \simeq \sum_{i=1}^{N} \langle \varepsilon_i^2 \rangle \tag{E.4}$$

The navigator then has an operational way to estimate the size of his observational noise errors

$$\sigma^2 \simeq \frac{1}{N} \sum_{i=1}^{N} \left((m_\alpha)_i K_\alpha + (m_o)_i + Q_i \right)^2 \tag{E.5}$$

where the measured $(m_\alpha)_i$ and estimated K_α are used in (E.5).

Note that substantially more sums of measured data are necessary to produce the a posteriori error estimate.

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