ON THE STABILITY OF SOME LINEAR NONAUTONOMOUS SYSTEMS*

by

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The stability of systems described by differential equations with time varying coefficients has been the subject of numerous mathematical studies, see for example [1]; however very limited sucess has been achieved from the practical viewpoint with the exception of the case in which the coefficients are periodic. Recently Kozin [2], Caughey and Gray [3] and Ariaratnam [4] among others have studied the stability of linear systems with stochastic coefficients; in these studies the principal tools used have been Gronwall's inequality and a norm used to reduce the vector differential equation to a scalar equation. Kozin [2] used the so-called taxicab norm, Caughey and Gray [3] used a very special quadratic norm and obtained results superior to those of Kozin. A natural problem within this context is to determine the optimum norm, among a certain class, for a specific problem.

The stability theorems given in [2] and [3] depend on the specific norms used in their proofs. The object of this paper is to extend these theorems so that they are applicable for any quadratic norm. This can be easily done through the use of.well known results on pencils of quadratic forms [5], an application which seems to have been overlooked. The theorem obtained in this manner, and two corollaries, are then applied to the determination of conditions for the stability of second order equations, for which it is possible to obtain the optimum quadratic norm. The stability results obtained in this manner, which as expected represent sufficient but not necessary conditions, constitute a considerable improvement over those presented in [2] and [3], and are believed to be new. The examples are limited to second order systems since problems of this type are often reduced to them.

The notation used here follows that of [2] and [3], and emphasizes the application to stochastic processes. Naturally, the results are equally applicable to deterministic systems which satisfy the condition of Equation (2).

A STABILITY THEOREM

Consider the differential equation

$$\dot{x} = [A + F(t)]x \; , \tag{1}$$

where x is an n vector, A is a constant matrix and $F(t)$ is a matrix whose nonzero elements $f_{ij}(t)$ are stochastic processes, measurable, strictly stationary, and that they satisfy an ergodic property ensuring the equality of time averages and ensemble averages. If G is a measurable, integrable, function defined on $f_{ij}(t)$ then

$$E\{G(f_{ij}(t))\} = E\{G(f_{ij}(0))\} = \lim_{t \to \infty} \frac{1}{t-t_o} \int_{t_o}^{t} G(f_{ij}(\tau))d\tau \qquad (2)$$

exists with probability one. For simplicity let, in (1), $E\{F(t)\} = 0$ and denote by $\lambda_{max}[Q]$ the largest eigenvalue of the matrix Q, Q' the transpose of Q.

<u>THEOREM</u>: If, for some positive definite matrix B and some $\epsilon > 0$

$$E\{\lambda_{max}[A' + F'(t) + B(A+F(t))B^{-1}] \leqq -\epsilon , \qquad (3)$$

then (1) is almost surely asymptotically stable in the large.

<u>Proof</u>: Consider the quadratic (Liapunov) function $V(x) = x'Bx$. Then, along the trajectories of (1), define

$$\lambda(t) = \frac{\dot{V}(x)}{V(x)} = \frac{x'[(A+F)'B + B(A+F)]x}{x'Bx} \quad . \qquad (4)$$

From the extremal properties of pencils of quadratic forms [5] the inequality

$$\lambda_{min}[(A+F)'+B(A+F)B^{-1}] \leqq \lambda(t) \leqq \lambda_{max}[(A+F)'+B(A+F)B^{-1}] \qquad (5)$$

is obtained, where λ_{max} and λ_{min}, being the maximum and minimum eigenvalues of a pencil, are real. It follows from (4) and (5) that

$$V(x(t)) = V(x(t_o))e^{\int_{t_o}^{t}\lambda(\tau)d\tau} = V(x(t_o))e^{(t-t_o)[\frac{1}{t-t_o}\int_{t_o}^{t}\lambda(\tau)d\tau]}, \quad (6)$$

from which it follows that, if $E\{\lambda(t)\} \leqq -\epsilon$ for some $\epsilon > 0$, $V(x(t))$ is bounded and that $V(x(t)) \to 0$ as $t \to \infty$. This is the condition imposed by (3), which proves the result.

It is remarked that a necessary condition for (3) to hold is that the eigenvalues of matrix A have negative real parts. The eigenvalue computation specified by (3) is far simple. It is possible to obtain a result which is easier to compute, but not as sharp.

<u>COROLLARY 1</u>: If, for some positive definite matrix B and some $\epsilon > 0$

$$E\{\lambda_{max}[F'(t)+BF(t)B^{-1}]\} \leqq -\lambda_{max}[A'+BAB^{-1}]-\epsilon \ , \quad (7)$$

then (1) is almost surely asymptotically stable in the large.

<u>Proof</u>: The proof follows immediately from the theorem by noting that

$$\lambda(t) \leqq \lambda_{max}[(A+F)'+B(A+F)B^{-1}] \leqq \lambda_{max}[A'+BAB^{-1}]+\lambda_{max}[F'+BFB^{-1}], \quad (8)$$

from which it follows upon the application of (7), that

$$E\{\lambda(t)\} \leqq \lambda_{max}[A'+BAB^{-1}]+E\{\lambda_{max}[F'(t)+BF(t)B^{-1}]\} \leqq -\epsilon \quad , \tag{9}$$

the desired result.

It is obvious that unless the second inequality in (8) is an equality the stability results obtained will not be as good as those given by the theorem. For computational purposes, it is desirable to further simplify the theorem. For this purpose let (1) be written as

$$\dot{x} = Ax + \sum_{i=1}^{R} f_i(t)C_i \quad , \tag{10}$$

where $R \leqq n^2$, and recall that $E\{f_i(t)\} = 0$.

COROLLARY 2: If, for some positive definite matrix B and some $\epsilon > 0$

$$\sum_{i=1}^{R} \tfrac{1}{2}E\{|f_i(t)|\}(\lambda_{max}[C_i'+BC_iB^{-1}]-\lambda_{min}[C_i'+BC_iB^{-1}]) \leqq -\lambda_{max}[A'+BAB^{-1}]-\epsilon \quad , \tag{11}$$

then (10) is almost surely asymptotically stable in the large.

Proof: In this case equation (4) of the theorem becomes

$$\lambda(t) = \frac{x'(A'B+BA)x}{x'Bx} + \sum_{i=1}^{R} f_i(t)\,\frac{x'(C_i'B+BC_i)x}{x'Bx} \quad . \tag{12}$$

Since $E\{f_i(t)\} = 0$ by assumption, define the two functions

$$f_i^+(t) = \begin{cases} f_i(t) & \text{if} \quad f_i(t) \geqq 0 \\ 0 & \text{if} \quad f_i(t) \leqq 0 \end{cases} ,$$
$$f_i^-(t) = \begin{cases} f_i(t) & \text{if} \quad f_i(t) \leqq 0 \\ 0 & \text{if} \quad f_i(t) \geqq 0 \end{cases} . \tag{13}$$

It then follows that

$$E\{f_i^+(t)\} = -E\{f_i^-(t)\} = \tfrac{1}{2}E\{|f_i(t)|\}, \tag{14}$$

and Equation (12) yields

$$E\{\lambda(t)\} \leqq \lambda_{max}[A + BAB^{-1}] + \sum_{i=1}^{R} \tfrac{1}{2}E\{|f_i(t)|\}(\lambda_{max}[C_i + BC_iB^{-1}] - \lambda_{max}[C_i + BC_iB^{-1}]) , \tag{15}$$

from which, through application of condition (11),

$$E\{\lambda(t)\} \leqq -\epsilon \tag{16}$$

is obtained, proving the corollary.

It is again to be expected that the results obtained from this corollary will seldom be as good as those given by either the Theorem or Corollary 1, since the majorizations used are rougher than the previous ones.

The above theorem and corollaries say nothing regarding how the matrix B should be chosen. If this matrix is chosen, as in [3], as the solution of the matrix equation $A'B+BA = -I$ then the stability condition of the Theorem, Equation (3), becomes

$$E\{\lambda_{max}[-B^{-1}+F'(t)+BF(t)B^{-1}]\} \leqq -\epsilon \ , \tag{3'}$$

Corollary 1 yields the stability condition

$$E\{\lambda_{max}[F'(t)+BF(t)B^{-1}]\} \leqq \frac{1}{\lambda_{max}[B]} -\epsilon \ , \tag{7'}$$

and the condition of Corollary 2 becomes

$$\sum_{i=1}^{R} \tfrac{1}{2}E\{|f_i(t)|\}(\lambda_{max}[C_i'+BC_iB^{-1}]-\lambda_{min}[C_i'+BC_iB^{-1}]) \leqq \frac{1}{\lambda_{max}[B]} -\epsilon \ . \tag{11'}$$

The conditions implied by (7') and (11') are clearly satisfied if we majorize further in these equations by noting that, if $Q(t) = F'(t)+BF(t)B^{-1}$,

$$\lambda_{max}[Q(t)] \leqq \sum_{i,j} |Q_{ij}| \ ,$$

and further that

$$\tfrac{1}{2}(\lambda_{max}[C_i' + BC_iB^{-1}] - \lambda_{min}[C_i' + BC_iB^{-1}]) \leqq |\mu^i|_{max} ,$$

where $|\mu^i|_{max}$ is the largest eigenvalue, in absolute value, of $C_i' + BC_iB^{-1}$. With these majorizations equations (7') and (11') become

$$E\{\sum_{i,j} |Q_{ij}|\} \leqq \frac{1}{\lambda_{max}[B]} - \epsilon \qquad (7'')$$

and

$$\sum_{i=1}^{R} E\{|f_i(t)|\}|\mu^i|_{max} \leqq \frac{1}{\lambda_{max}[B]} - \epsilon , \qquad (11'')$$

the stability conditions given by Caughey and Gray [3].

It is then seen that the use of well known results on pencils of quadratic forms yields stability theorems of time varying systems that include those of [3]. The natural question at this juncture is to demand a theorem which yields the optimal matrix B to be used. Unfortunately, this problem does not appear amenable to analysis, as the third example of the next section indicates. The purpose of the following section is to obtain the optimal matrix B of the Theorem and Corollaries 1 and 2 for the two most common second order equations of type (1). A third second order equation is analyzed to show that an optimal matrix B does not exist; finally an application of the theorem of this section to the study of the stability of a nuclear reactor is shown. The stability results thus obtained are compared with those given in [2] and [3], and indicate that the matrix of Caughey and Gray is, in general, not optimal.

SOME EXAMPLES

<u>EXAMPLE 1:</u> Consider the differential equation

$$\ddot{x} + 2\xi\dot{x} + (1+f(t))x = 0 \ , \tag{17}$$

studied by Kozin [2], Caughey and Gray [3] and Ariaratnam [4]. It is assumed that $E\{f(t)\} = 0$, and the equation is rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix} x + f(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x \tag{18}$$

or, $\dot{x} = Ax + F(t)x$. Consider, for the matrix B, the most general quadratic positive definite form

$$B = \begin{bmatrix} \alpha_1^2 + \alpha_2 & \alpha_1 \\ \alpha_1 & 1 \end{bmatrix} \ , \quad \alpha_2 > 0 \tag{19}$$

where α_1 and α_2 are numbers to be determined.

Simple computations immediately yield that

$$B^{-1} = \frac{1}{\alpha_2} \begin{bmatrix} 1 & -\alpha_1 \\ \alpha_2 & \alpha_1^2 + \alpha_2 \end{bmatrix} \tag{20}$$

and that

$$A' + F'(t) + B(A+F(t))B^{-1} =$$

$$= \frac{1}{\alpha_2}\begin{bmatrix} -\alpha_1(1+f)-\alpha_1^2(\alpha_1-2\xi)-\alpha_1\alpha_2 & \alpha_1^2(1+f)+(\alpha_1^2+\alpha_2)[\alpha_1(\alpha_1-2\xi)+\alpha_2] \\ -(1+f)-\alpha_1(\alpha_1-2\xi) & \alpha_1(1+f)+(\alpha_1-2\xi)(\alpha_1^2+\alpha_2) \end{bmatrix}. \tag{21}$$

The maximum eigenvalue of this expression is computed as

$$\lambda_{max}[A'+F'+B(A+F)B^{-1}] = -2\xi + \sqrt{4(\xi-\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1-f(t)+2\alpha_1(\xi-\alpha_1)]^2} \tag{22}$$

and setting $f = 0$ in this equation

$$\lambda_{max}[A'+BAB^{-1}] = -2\xi + \sqrt{4(\xi-\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+2\alpha_1(\xi-\alpha_1)]^2}. \tag{23}$$

is obtained. Finally,

$$F'(t) + BF(t)B^{-1} = \frac{f(t)}{\alpha_2}\begin{bmatrix} -\alpha_1 & \alpha_1^2 - \alpha_2 \\ -1 & \alpha_1 \end{bmatrix}, \tag{24}$$

from which the eigenvalue expression

$$\lambda_{max}[F'(t) + BF(t)B^{-1}] = \frac{1}{\sqrt{\alpha_2}}\,|f(t)| \tag{25}$$

is immediately computed.

In this particular example, then, the conditions for almost sure asymptotic stability given by the previous section become, for the theorem

$$E\{-2\xi + \sqrt{4(\xi-\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1-f(t)+2\alpha_1(\xi-\alpha_1)]^2}\} \leqq -\epsilon \tag{26}$$

and from either of the two corollaries

$$E\{|f(t)|\}\frac{1}{\sqrt{\alpha_2}} \leqq 2\xi - \sqrt{4(\xi-\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+2\alpha_1(\xi-\alpha_1)]^2} - \epsilon \tag{27}$$

for some α_1, and some $\alpha_2 > 0$ and $\epsilon > 0$. If the stability conditions are desired in terms of $E\{|f(t)|\}$, the optimum values of α_1 and α_2 for equations (26) and (27) coincide and are easily computed as

$$\begin{aligned} \alpha_1 &= \xi, \quad \alpha_2 = 1-\xi^2, \quad \text{if} \quad \xi \leqq \frac{\sqrt{2}}{2}, \\ \alpha_1 &= \xi, \quad \alpha_2 = \xi^2, \quad \text{if} \quad \xi \geqq \frac{\sqrt{2}}{2}, \end{aligned} \tag{28}$$

upon which the stability conditions (26) become

$$\begin{aligned} E\{|f(t)|\} &\leqq 2\xi\sqrt{1-\xi^2} - \epsilon, \qquad \xi \leqq \frac{\sqrt{2}}{2}, \\ E\{|f(t)+1-2\xi^2|\} &\leqq 2\xi^2 - \epsilon, \qquad \xi \geqq \frac{\sqrt{2}}{2}, \end{aligned} \tag{29}$$

while conditions (27) yield

$$
\begin{aligned}
&E\{|f(t)|\} \leq 2\xi\sqrt{1-\xi^2} - \epsilon\,, && \xi \leq \frac{\sqrt{2}}{2}\,,\\
&E\{|f(t)|\} \leq 1 - \epsilon\,, && \xi \geq \frac{\sqrt{2}}{2}\,.
\end{aligned}
\tag{30}
$$

As expected, conditions (29) are weaker than conditions (30); this is strongly emphasized by obtaining stability conditions from (29) and (30) in terms of $E\{f^2(t)\}$ through the use of Schwarz's inequality, remembering that $E\{f(t)\} = 0$. This process yields the stability conditions

$$
\begin{aligned}
&E\{f^2(t)\} \leq 4\xi^2(1-\xi^2) - \epsilon\,, && \xi \leq \frac{\sqrt{2}}{2}\,,\\
&E\{f^2(t)\} \leq 4\xi^2-1 - \epsilon\,, && \xi \geq \frac{\sqrt{2}}{2}\,,
\end{aligned}
\tag{29'}
$$

from (29) and, from (30)

$$
\begin{aligned}
&E\{f^2(t)\} \leq 4\xi^2(1-\xi^2) - \epsilon\,, && \xi \leq \frac{\sqrt{2}}{2}\,,\\
&E\{f^2(t)\} \leq 1 - \epsilon\,, && \xi \geq \frac{\sqrt{2}}{2}\,,
\end{aligned}
\tag{30'}
$$

a much more meager result.

If, at the outset, it is desired to obtain stability conditions as a function of $E\{f^2(t)\}$, then the values

$$\alpha_1 = \xi\,, \qquad \alpha_2 = \xi^2 + 1 \tag{31}$$

are optimal for equation (26) which yields

$$E\{f^2(t)\} \leq 4\xi^2\,. \tag{32}$$

These results are a considerable improvement over those of [2] and [3]. Figure 1 displays these results and those of these two references in a pictorial form. It is of interest to note that either (29') or (32) show that, for almost sure asymptotic stability, it is possible to let $E\{f^2(t)\} \to \infty$ as the damping ξ increases; this result therefore answers a question raised by Mehr and Wang [6] in their discussion of [2].

EXAMPLE 2: As a second example consider the equation

$$\ddot{x} + (2\xi + g(t))\dot{x} + x = 0, \quad E\{g(t)\} = 0\,, \tag{33}$$

which is rewritten in the usual companion form yielding, in the notation of (1),

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\xi \end{bmatrix}, \qquad F(t) = g(t)\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{34}$$

Using again the matrix B given by (19) a simple computation yields

$$\lambda_{max}[A'+F'+B(A+F)B^{-1}] = -2\xi - g(t) +$$

$$+\sqrt{(g(t)+2\xi-2\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+\alpha_1 g(t)+2\alpha_1(\xi-\alpha_1)]^2} \tag{35}$$

and

$$\{\lambda_{max}[F'(t)+BF(t)B^{-1}]-\lambda_{min}[F'(t)+BF(t)B^{-1}]\} =$$

$$= |g(t)| \sqrt{1+\frac{\alpha_1^2}{\alpha_2}} \;. \tag{36}$$

Hence, in this case, the theorem of the previous section yields, for stabilty

$$E\{-2\xi+\sqrt{(g(t)+2\xi-2\alpha_1)^2+\frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+\alpha_1 g(t)+2\alpha_1(\xi-\alpha)]^2} \leqq -\epsilon \;; \tag{37}$$

either of the two corollaries give instead the condition

$$E\{|g(t)|\}\sqrt{1+\frac{\alpha_1^2}{\alpha_2}} \leqq 2\xi - \sqrt{4(\xi-\alpha_1)^2+\frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+2\alpha_1(\xi-\alpha_1)]^2} - \epsilon. \tag{38}$$

A straightforward computation yields, in the case that stability conditions are desired as functions of $E\{|g(t)|\}$, that the optimum values for α_1 and α_2 for equations (37) and (38) coincide and are

$$\alpha_1 = \xi\,, \qquad \alpha_2 = 1-\xi^2\,, \qquad \text{if} \quad \xi^2 \leq \frac{\sqrt{5}-1}{2}\;;$$

$$\alpha_1 = \frac{1}{\sqrt{\xi^2+1}}\,, \qquad \alpha_2 = \frac{\xi^2}{\xi^2+1}\,, \qquad \text{if} \quad \xi^2 \geq \frac{\sqrt{5}-1}{2}\;; \tag{39}$$

and the conditions of stability become, for the theorem, equation (37),

$$E\{|g(t)|\} \leq 2\xi\sqrt{1-\xi^2} - \epsilon\,, \qquad \xi^2 \leq \frac{\sqrt{5}-1}{2}\;;$$

$$E\{|g(t)+2\xi - \frac{2}{\sqrt{\xi^2+1}}|\} \leq 2\xi \frac{\xi^2}{\sqrt{1+\xi^2}} - \epsilon, \quad \xi^2 \geq \frac{\sqrt{5}-1}{2}\;; \tag{40}$$

and for either of the corollaries, equation (38),

$$E\{|g(t)|\} \leq 2\xi\sqrt{1-\xi^2} - \epsilon\,, \qquad \xi^2 \leq \frac{\sqrt{2}-1}{2}\;;$$

$$E\{|g(t)|\} \leq 2\xi[\sqrt{1+\frac{1}{\xi^2}} - 1] - \epsilon, \qquad \xi^2 \geq \frac{\sqrt{5}-1}{2}\;; \tag{41}$$

It is noted that equation (40) gives weaker conditions for stability, since application of Schwarz's inequality to this equation gives the stability conditions

$$E\{g^2(t)\} \leq 4\xi^2(1-\xi^2) - \epsilon \qquad , \quad \xi^2 \leq \frac{\sqrt{5}-1}{2}\;;$$

$$E\{g^2(t)\} \leq 4\,\frac{2\xi - \sqrt{\xi^2+1}}{\sqrt{\xi^2+1}} - \epsilon \qquad , \quad \xi^2 \geq \frac{\sqrt{5}-1}{2}\;. \tag{40'}$$

If stability conditions are desired as a function of $E\{f^2(t)\}$, the optimum values

$$\alpha_1 = \frac{\xi}{1+\xi^2} \ , \qquad \alpha_2 = 1 - \frac{\xi^2}{(1+\xi^2)^2} \tag{42}$$

applied to equation (37) yield, after application of Schwarz's inequality, the stability condition

$$E\{g^2(t)\} \leq \frac{4\xi^2}{1+\xi^2} - \epsilon \tag{43}$$

These stability results are shown in a pictorial representation in Figure 2.

EXAMPLE 3: Consider, in this case, the differential equation

$$\ddot{x} + [2\xi + g(t)]\dot{x} + [1+f(t)]x = 0 \ , \tag{44}$$

a generalization of the two previous differential equations. Using the same matrix B of equation (19) and repeating the computations indicated in the previous examples the following conditions for almost sure stability in the large are obtained: from the theorem

$$E\{-2\xi + \sqrt{(g(t)+2\xi-2\alpha_1)^2 + \frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+\alpha_1 g(t)-f(t)+2\alpha_1(\xi-\alpha_1)]^2}\} \leq -\epsilon \ , \tag{45}$$

from Corollary 1

$$E\{\sqrt{g^2(t)+\frac{1}{\alpha_2}[f(t)-\alpha_1 g(t)]^2} \leqq$$
$$E\{2\xi-\sqrt{4(\xi-\alpha_1)^2+\frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+2\alpha_1(\xi-\alpha_1)^2} - \epsilon , \tag{46}$$

and from Corollary 2

$$E\{|f(t)|\frac{1}{\sqrt{\alpha_2}} + |g(t)|\sqrt{1+\frac{\alpha_1^2}{\alpha_2}}\}$$
$$\leqq +2\xi - \sqrt{4(\xi-\alpha_1)^2+\frac{1}{\alpha_2}[\alpha_2+\alpha_1^2-1+2\alpha_1(\xi-\alpha_1)]^2} . \tag{47}$$

Inspection of these last three equations shows that, in general, unless $f(t)$ and $g(t)$ are related no optimum matrix B exists. Indeed, if stability conditions as a function of $E\{f^2(t)\}$ and $E\{g^2(t)\}$ are desired, equation (45) yields, upon application of the Schwarz's inequality, the condition

$$\alpha_2 E\{g^2(t)\}+[\alpha_1 E\{g^2(t)\}^{\frac{1}{2}}+E\{f^2(t)\}^{\frac{1}{2}}]^2 \leqq 4\alpha_2-[\alpha_2-\xi^2+1+(\xi-\alpha_1)^2]^2-\epsilon, \tag{48}$$

and it is immediately seen that, for fixed values of α_1 and $\alpha_2 > 0$, it is not possible to obtain simultaneously results which coincide with those given by equation (32), in the event that $g(t) \equiv 0$, and with equation (43), if $f(t) \equiv 0$. Hence, the choice of α_1 and α_2 depends on the relative magnitudes of $E\{f^2(t)\}$ and $E\{g^2(t)\}$.

The two extreme choices for α_1 and α_2 are given by equations (31) and (42), in which cases we obtain the stability conditions

$$(\xi^2+1)E\{g^2(t)\}+[\xi E\{g^2(t)\}^{\frac{1}{2}}+E\{f^2(t)\}^{\frac{1}{2}}]^2 \leqq 4\xi^2-\epsilon,$$

$$(1-\frac{\xi^2}{(1+\xi^2)^2})E\{g^2(t)\}+[\frac{\xi}{1+\xi^2}E\{g^2(t)\}^{\frac{1}{2}}+E\{f^2(t)\}^{\frac{1}{2}}]^2 \leqq \frac{4^2}{1+\xi^2}-\epsilon. \tag{49}$$

The first of these equations yields equation (32) if $g(t) = 0$, while the second becomes equation (43) for $f(t) = 0$. Appropriate choices of α_1 and $\alpha_2 > 0$ will give results bounded by these two extremes.

If results are desired as functions of $E\{|f(t)|\}$ and $E\{|g(t)|\}$, equation (47) can be optimized by the values

$$\alpha_1 = \xi \quad , \qquad \alpha_2 = 1-\xi^2 \qquad \text{if} \quad \xi^2 \leqq \frac{\sqrt{5}-1}{2} \quad ,$$

in which case the stability condition becomes

$$E\{|f(t)|+|g(t)|\} \leqq 2\xi\sqrt{1-\xi^2}-\epsilon, \text{ if } \quad \xi^2 \leqq \frac{\sqrt{5}-1}{2}\,. \tag{50}$$

For $\xi^2 \geqq \frac{\sqrt{5}-1}{2}$ it is not possible to optimize simultaneously, and one is again forced to consider the relative magnitudes of $E\{|f(t)|\}$ and $E\{|g(t)|\}$. To obtain extreme values the values for α_1 and α_2 of equations (28) and (39) are used yielding

$$E\{|f| + \sqrt{2}\,\xi|g|\} \leqq 1-\epsilon \quad , \qquad \text{if} \quad \xi \geqq \frac{\sqrt{2}}{2} \; ;$$
$$E\{|f| + |g|\} \leqq 2\xi\sqrt{1+\frac{1}{\xi^2}} - 1 \; , \qquad \text{if} \quad \xi \geqq \frac{\sqrt{5}-1}{2} \; . \tag{51}$$

Again, appropriate choices of α_1 and $\alpha_2 > 0$ will yield results between these extremes.

As indicated previously, the results of this example are rather disappointing since they indicate that an optimum quadratic norm does not exist. On the other hand, it appears that if a differential equation has only one time varying coefficient then the determination of such a norm does not appear amenable to simple analysis.

<u>EXAMPLE 4</u>: <u>An Application</u>. Consider the application of the theorem of the previous section to the study of the stability of the solutions of the differential equations of the kinetics of a simple nuclear reactor problem. A set of differential equations modeling such a system is

$$\dot{n} = \frac{p(t)-\beta}{\ell}\, n + \lambda c \quad ,$$
$$\dot{c} = \frac{\beta}{\ell}\, n - \lambda c \quad , \tag{52}$$

where

c = concentration of total delayed neutron precursors ($c \geqq 0$)

ℓ = neutron effective lifetime ($\ell > 0$)

n = neutron density ($n \geqq 0$)

p(t) = reactivity, a function of time

β = total delayed neutron fraction ($\beta > 0$)

λ = mean decay constant of delayed neutron precursors. ($\lambda > 0$)

This set of equations and its variants have been the subject of numerous studies [7]. In [8], for example, it was proved that if $p(t)$ is sinusoidal, for every frequency of the sinusoid and values of the parameters, the solutions of (52) are unstable.

For notational simplicity, let

$$x_1 = n \ , \ x_2 = c \ , \ a = \frac{1}{\ell} \ , \ b = \frac{\beta}{\ell} \tag{53}$$

and define

$$E\{\frac{p(t)}{\ell}\} = -m \ , \quad f(t) = \frac{p(t)}{\ell} + m \ . \tag{54}$$

Equations (52) then becomes

$$\dot{x} = \begin{bmatrix} -m-b & \lambda \\ b & -\lambda \end{bmatrix} x + f(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \ , \tag{55}$$

in the same form as given by (1). Application of the matrix B given by (19) yields, after some computations and application of Schwarz's inequality, that the theorem of the previous section will predict stability for some $\alpha_1, \alpha_2 > 0$ and $\epsilon > 0$ if

$$E\{f^2(t)\}(\alpha_2+\alpha_1^2) \leqq 4\alpha_2\lambda[m+b+\lambda\alpha_1]-\{b+\lambda(\alpha_1+\alpha_2)-\alpha_1(m+b+\lambda\alpha_1)\}^2-\epsilon. \quad (56)$$

The optimum values of α_1 and α_2 are immediately found to be,

$$\alpha_1 = 0, \qquad \alpha_2 = \frac{b}{\lambda} , \quad (57)$$

upon which (56) becomes

$$E\{f^2(t)\} \leqq 4m\lambda ; \quad (58)$$

or, in the notation of equation (52), the condition for almost sure asymptotic stability in the large becomes

$$E\{p(t)^2\} \leqq E\{p(t)\}^2 - 4\lambda\ell E\{p(t)\}-\epsilon . \quad (59)$$

It is evident from this expression that $E\{p(t)\}$ must be negative for stability. In the specific case that the reactivity varies sinusoidally as

$$p(t) = -m + h \sin \omega t \quad (60)$$

stability condition (59) becomes

$$h^2 \leqq 8\, m\ell\lambda - \epsilon \,, \tag{61}$$

for some $\epsilon > 0$.

CONCLUSIONS

A simple theorem that gives sufficient conditions for the almost sure stability of linear time varying systems has been presented. As the applications of this theorem and its corollaries to examples show, the stability results obtained are quite good and simple to use. The question of determination of the optimum quadratic norm for a system of differential equation with only one time varying coefficient has not been resolved, and remains an open problem.

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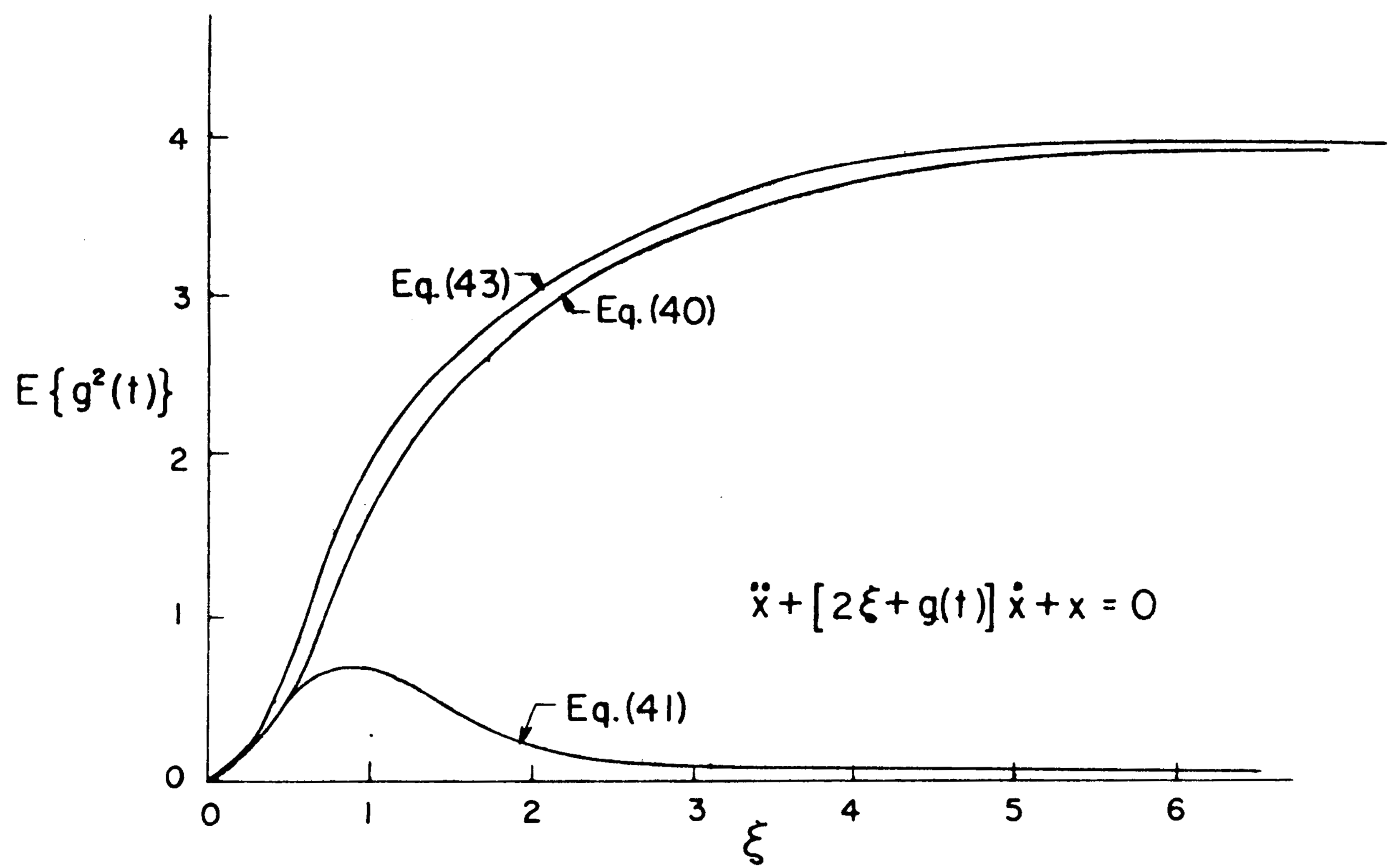
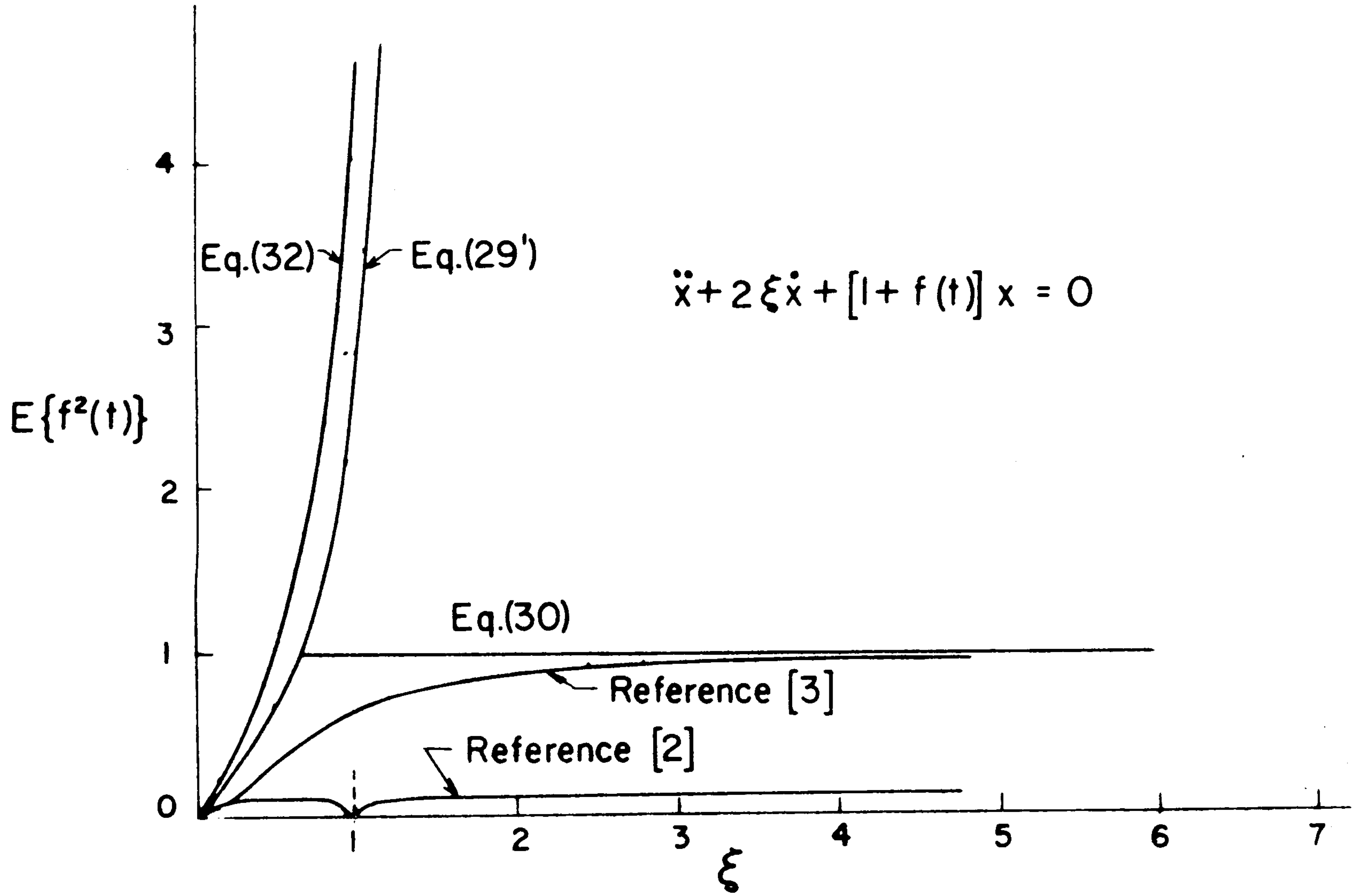
LIST OF FIGURES



E{g²(t)}
Eq.(43)
Eq.(40)
Eq.(41)
ẍ + [2ξ + g(t)] ẋ + x = 0
ξ
0
1
2
3
4
5
6



Eq.(32)
Eq.(29')
$\ddot{x} + 2\xi\dot{x} + [1 + f(t)]\,x = 0$
$E\{f^2(t)\}$
Eq.(30)
Reference [3]
Reference [2]
0
1
2
3
4
5
6
7
ξ

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