NASA TECHNICAL REPORT

NASA TR R-265

NASA TR R-265

STRESSES AT THE TIP OF A LONGITUDINAL CRACK IN A PLATE STRIP

by W. B. Fichter

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1967

Errata

NASA Technical Report R-265

Stresses at the Tip of a Longitudinal Crack in a Plate Strip

W. B. Fichter

August 1967

Pages 20 and 21: Replace figures 10 and 11 with the following figures:

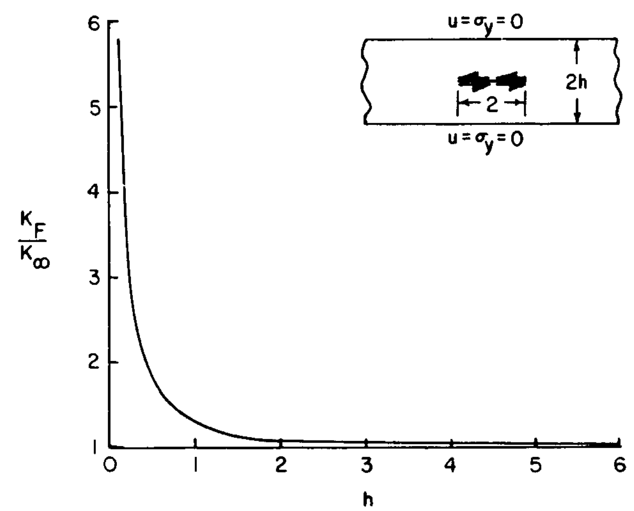

Figure 10.- Variation of K_F/K_∞ with h. Uniform shear stress and zero lateral displacement.

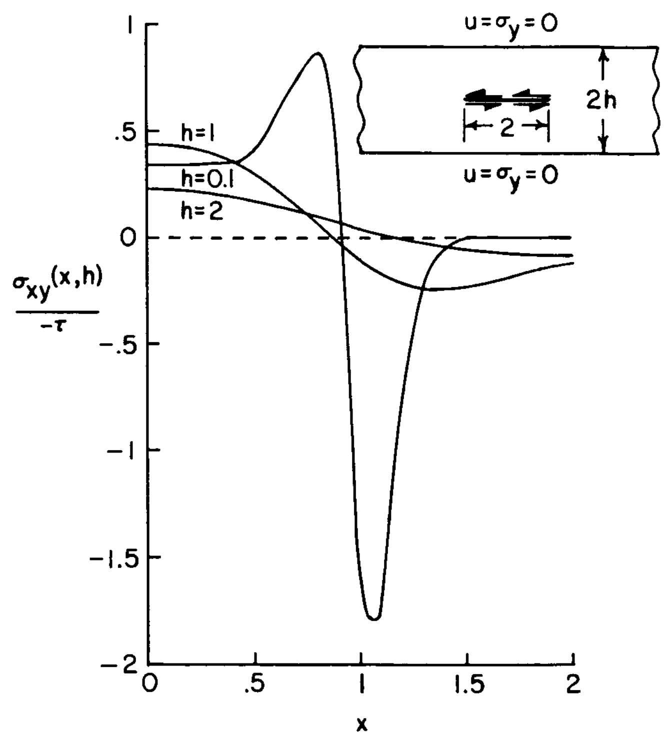

Figure 11.- Shear stress on longitudinal edge. Uniform shear stress and zero lateral displacement.

Page 29, third full paragraph: Replace with the following paragraph:

For the third shear problem, in which tangential displacement of the longitudinal edges is prevented, the stress intensity factor differs appreciably from the infinite-sheet value only for h less than 2. For small h, the trend is similar to that of the other shear problems, though somewhat more pronounced.

Issue Date: July 1986

NASA TR R-265

STRESSES AT THE TIP OF A LONGITUDINAL CRACK IN A PLATE STRIP

By W. B. Fichter

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

STRESSES AT THE TIP OF A LONGITUDINAL CRACK IN A PLATE STRIP*

By W. B. Fichter
Langley Research Center

SUMMARY

The stress field near the tip of a central longitudinal crack in a plate strip is investigated. Several plane elastostatic problems involving opening and in-plane shearing of the crack, as well as one plate bending problem, are analyzed. Stress intensity factors are obtained as a function of the ratio of strip width to crack length. In addition, for several problems some plots of the stresses on the longitudinal edges are presented to illustrate some additional effects of the proximity of the crack to the boundary. Finite strip width is shown to exert a strong influence on stress intensity factors in certain ranges of the ratio of strip width to crack length. Results for the fixed-edge plane problems and the plate bending problem are only slightly influenced by changes in Poisson's ratio. The results should be useful in the design of fracture test specimens and in the analysis of fracture test data.

INTRODUCTION

In the aerospace industry there is concern over the deleterious effects of cracks in metal structures. Many in-service failures of aircraft components have been traced to fatigue cracks. In addition, the use of some high strength alloys has been compromised by crack sensitivity. Such considerations have motivated extensive development of the technology of predicting fracture of materials in the presence of cracks. An important aspect of this technology is the determination of stress fields near cracks and the study of the influence of loadings and boundary conditions on these stress fields.

It is widely known that the linear theory of elasticity predicts a square root singularity in stress at the tip of a crack in a plate when the plate is subjected to loads which cause relative displacement of the opposing crack faces. (See, for example, refs. 1 to 3.) Although this singularity in stress is physically unrealistic, the strength of the stress singularity or some numerical modification of it, usually referred to as the stress

---

*The information presented herein is based in part upon a thesis offered in partial fulfillment of the requirements for the degree of Master of Science, Virginia Polytechnic Institute, Blacksburg, Virginia, June 1966.

intensity factor, furnishes a means by which a few judiciously designed experiments might be utilized to predict the crack sensitivity of a larger class of configurations. As a result, requirements for costly and time-consuming testing programs might be reduced.

Analytical information on stress fields around cracks in doubly infinite plates abounds in the technical literature. (See, for example, refs. 1 to 3, where numerous additional references are cited.) On the other hand, relatively few analyses have been made of cracks in finite plates. More analytical information on the effects of finite in-plane plate dimensions and the range over which these effects are important would be useful in the design of practical structural components.

Almost all analyses which have been performed on finite plates have been confined to cases in which the finite-plate dimension is parallel to the crack direction. The present report contains analyses of an infinitely long plate strip of width $2ah$ containing a central longitudinal crack of length $2a$; that is, the finite-plate dimension is perpendicular to the crack direction. Several plane elastostatic problems and one bending problem are treated. Stress intensity factors are determined as a function of h, the ratio of strip width to crack length, to facilitate an assessment of the effects of finite strip width. The results are normalized with respect to those for a doubly infinite plate under the same loading conditions. For plane problems involving geometrical restraint of the longitudinal edges of the strip, some illustrative plots of stress distributions on the edges are presented.

In all the problems analyzed herein, the external loading has been applied to the crack surfaces in order to make use of well-developed transform methods. However, with the exception of the problems involving concentrated forces, these problems can be easily extended by superposition to problems involving stress-free crack surfaces and uniform stresses or displacements on the longitudinal edges of the strip. The stress intensity factors are unaltered by such extensions.

SYMBOLS

A,B,C,D	unknown coefficients in general solution of transformed biharmonic equation
A_m,A_n	coefficients in series
a	half length of crack
$\tilde{D}$	plate bending stiffness

E	Young's modulus of elasticity
$f(\xi)$	unknown function in dual integral equations
G	modulus of shear rigidity
$G(\xi)$	function in dual integral equations
$g(x)$	function of applied load in dual integral equations
h	nondimensional strip width
K	stress intensity factor
k	arbitrary real positive number in solution of dual integral equations
$L_{m,n}$	coefficients in algebraic equations
M	nondimensional applied bending moment
M_X, M_Y, M_{XY}	nondimensional bending and twisting moments
m,n	integers
P	applied concentrated force
P_m	normalized series coefficients
Q_X, Q_Y	nondimensional shear forces
t	plate thickness
u,v	nondimensional in-plane displacements
w	nondimensional lateral displacement
x,y	nondimensional rectangular Cartesian coordinates
Γ	gamma function

$\epsilon_x, \epsilon_y, \gamma_{xy}$	elastic strains
$\kappa = \dfrac{1 - \mu}{1 + \mu}$	
λ	parameter in dual integral equations
μ	Poisson's ratio
ξ	transform variable
σ	nondimensional applied stress
$\sigma_x, \sigma_y, \sigma_{xy}$	nondimensional stresses
τ	nondimensional applied shear stress
ϕ	nondimensional Airy stress function
∇^4	biharmonic operator, $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$

Subscripts:

F	refers to strip
∞	refers to doubly infinite plate

Primes denote dimensional quantities. Bars denote transformed quantities.

ANALYSIS

The configuration is an infinitely long plate strip of width 2ah with a central longitudinal crack of length 2a. (See fig. 1.) The problems are divided into categories in accordance with the types of loading applied to the crack faces. In the first group of problems, the crack is being either opened or sheared by uniform loads.

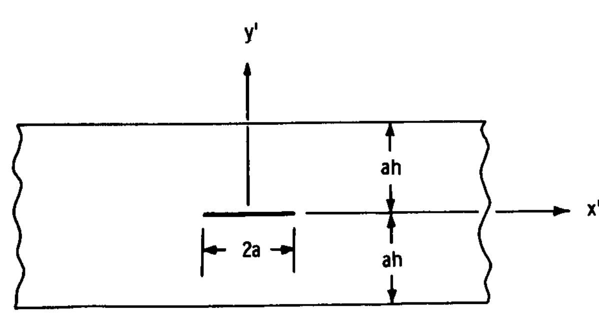

Figure 1.- Plate strip and coordinate system.

The second group involves the opening of the crack due to the application of concentrated loads to the crack faces.

The plane problems and the bending problem are analyzed similarly. All the analyses are based on a method presented by Sneddon in reference 4 for determining the appropriate pair of dual integral equations to be solved for the Fourier transform of the Airy stress function. In the bending problem, the Fourier transform of the lateral displacement w is sought. After the dual integral equations have been obtained, a method given by Tranter in reference 5 is employed to solve the dual integral equations and eventually to obtain the stresses and displacements in the form of infinite series. Here, however, the primary goal is to obtain the stress intensity factors as functions of the ratio of strip width to crack length.

For the plane problems the equations of plane strain are assumed to govern, it being understood that the plane strain solutions can be converted to plane stress solutions by the appropriate substitutions for Young's modulus E and Poisson's ratio μ. (See, for example, ref. 6.)

Under the plane strain assumptions, the following system of equations governs the plane problems:

$$\nabla^4 \phi'(x',y') = 0 \tag{1}$$

where

$$\left.\begin{aligned} \sigma_x' &= \frac{\partial^2 \phi'}{\partial y'^2} \\ \sigma_y' &= \frac{\partial^2 \phi'}{\partial x'^2} \\ \sigma_{xy}' &= -\frac{\partial^2 \phi'}{\partial x' \, \partial y'} \end{aligned}\right\} \tag{2}$$

with the stress-strain relations

$$\left.\begin{aligned} \epsilon_x' &= \frac{1}{E}\left(\sigma_x' - \mu \sigma_y'\right) \\ \epsilon_y' &= \frac{1}{E}\left(\sigma_y' - \mu \sigma_x'\right) \\ \gamma_{xy}' &= \frac{\sigma_{xy}'}{G} \end{aligned}\right\} \tag{3}$$

and the strain-displacement relations

$$\left.\begin{aligned} \epsilon'_x &= \frac{\partial u'}{\partial x'} \\ \epsilon'_y &= \frac{\partial v'}{\partial y'} \\ \gamma'_{xy} &= \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \end{aligned}\right\} \tag{4}$$

The quantities contained in equations (1) to (4) are made nondimensional by the following relationships:

$$\left.\begin{aligned} \phi' &= Ea^2\phi \\ \sigma'_x &= E\sigma_x \\ \sigma'_y &= E\sigma_y \\ \sigma'_{xy} &= E\sigma_{xy} \\ \epsilon'_x &= \epsilon_x \\ \epsilon'_y &= \epsilon_y \\ \gamma'_{xy} &= \gamma_{xy} \\ u' &= au \\ v' &= av \\ x' &= ax \\ y' &= ay \end{aligned}\right\} \tag{5}$$

Equations (1) to (4) then assume the following form:

$$\nabla^4\phi(x,y) = 0 \tag{6}$$

where

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2\phi}{\partial x^2} \\ \sigma_{xy} &= -\frac{\partial^2\phi}{\partial x\,\partial y} \end{aligned}\right\} \tag{7}$$

with

$$\left.\begin{aligned}\epsilon_x &= \sigma_x - \mu\sigma_y \\ \epsilon_y &= \sigma_y - \mu\sigma_x \\ \gamma_{xy} &= \frac{E}{G}\,\sigma_{xy}\end{aligned}\right\} \tag{8}$$

and

$$\left.\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\end{aligned}\right\} \tag{9}$$

In terms of the nondimensional coordinates, the strip is of width $2h$ and contains a crack of length 2.

The fact that the plate is infinite in length strongly suggests the use of infinite integral transforms. With the complete Fourier transform of a function defined by (see, for example, ref. 4)

$$\bar{f}(\xi,y) = \int_{-\infty}^{\infty} f(x,y)\, e^{-i\xi x} dx \tag{10}$$

and inversely

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\xi,y)\, e^{i\xi x} d\xi \tag{11}$$

equation (6) becomes

$$\left(\frac{d^2}{dy^2} - \xi^2\right)^2 \bar{\phi}(\xi,y) = 0 \tag{12}$$

it being assumed that stresses and displacements vanish as $|x| \to \infty$. The general solution to equation (12) may be written as

$$\bar{\phi}(\xi,y) = (A + By)e^{-|\xi|y} + (C + Dy)e^{|\xi|y} \tag{13}$$

where A, B, C, and D are functions of ξ to be determined by application of the boundary conditions. In terms of $\bar{\phi}$, the transformed stresses and displacements are

$$\left.\begin{aligned}\bar{\sigma}_x &= \frac{d^2\bar{\phi}}{dy^2} \\ \bar{\sigma}_y &= -\xi^2\bar{\phi} \\ \bar{\sigma}_{xy} &= -i\xi\frac{d\bar{\phi}}{dy}\end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned}\bar{u} &= \frac{1}{i\xi}\left(\frac{d^2\bar{\phi}}{dy^2} + \mu\xi^2\bar{\phi}\right) \\ \bar{v} &= \frac{1}{\xi^2}\left[\frac{d^3\bar{\phi}}{dy^3} - (2 + \mu)\xi^2\frac{d\bar{\phi}}{dy}\right]\end{aligned}\right\} \tag{15}$$

Because of similarities in the various analyses, two example problems are presented in detail in the text, the remaining problems being discussed only briefly. Detailed analyses of the remaining problems are relegated to appendixes A and B.

The first example problem concerns the opening of the crack by uniform pressure, the edges of the strip being taken to be free of stress. After this detailed treatment, brief discussion is given of other problems which can be analyzed in a similar manner such as uniform normal and shear stress distributions and uniform bending-moment distribution on the crack faces. Detailed analyses of these other problems are given in appendix A.

The second example problem concerns the opening of the crack by symmetrically applied concentrated forces, the edges of the strip being stress free. This problem is followed by brief discussions of other concentrated force problems, detailed analyses of which are given in appendix B.

In the analyses, certain assumptions are made with regard to the permissibility of interchanging the order of summation and integration in some infinite series. These assumptions appear to be justified by the final results.

Crack-Opening Mode – Uniform Pressure

Free longitudinal edges.- In the first problem the edges of the strip are free of external load, and the crack is being opened by a uniform pressure $E\sigma$, where σ is a nondimensional measure of the applied pressure. Because of symmetry, only the upper half of the strip ($0 \leqq y \leqq h$) needs to be considered. The boundary conditions in the dimensionless system are

$$\left.\begin{array}{ll}\sigma_y(x,h) = 0 & \\ \sigma_{xy}(x,h) = 0 & \\ \sigma_{xy}(x,0) = 0 & \\ \sigma_y(x,0) = -\sigma & \left(|x| < 1\right) \\ v(x,0) = 0 & \left(|x| > 1\right)\end{array}\right\} \tag{16}$$

The transformed boundary conditions are

$$\bar{\sigma}_y(\xi,h) = 0$$

$$\bar{\sigma}_{xy}(\xi,h) = 0$$

$$\bar{\sigma}_{xy}(\xi,0) = 0$$

$$-\sigma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\sigma}_y(\xi,0)\, e^{i\xi x} d\xi \qquad \left(|x| < 1\right)$$

$$0 = \int_{-\infty}^{\infty} \bar{v}(\xi,0)\, e^{i\xi x} d\xi \qquad \left(|x| > 1\right)$$

In terms of the transformed stress function $\bar{\phi}$, the boundary conditions are

$$\bar{\phi}(\xi,h) = 0 \tag{17}$$

$$\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \tag{18}$$

$$\frac{d\bar{\phi}}{dy}(\xi,0) = 0 \tag{19}$$

$$2\pi\sigma = \int_{-\infty}^{\infty} \xi^2 \bar{\phi}(\xi,0)\, e^{i\xi x} d\xi \qquad \left(|x| < 1\right) \tag{20a}$$

and, use of equation (19) having been made,

$$0 = \int_{-\infty}^{\infty} \frac{d^3\bar{\phi}}{dy^3}(\xi,0)\, e^{i\xi x}\, \frac{d\xi}{\xi^2} \qquad \left(|x| > 1\right) \tag{20b}$$

Substitution of equation (13) into equations (17) to (19) yields

$$\left.\begin{aligned} A &= \frac{D}{\Delta}\left(e^{2|\xi|h} - 2\xi^2h^2 + 2|\xi|h - 1\right) \\ B &= \frac{|\xi|D}{\Delta}\left(e^{2|\xi|h} + 2|\xi|h - 1\right) \\ C &= \frac{-D}{\Delta}\left(2\xi^2h^2 + 2|\xi|h + 1 - e^{-2|\xi|h}\right) \end{aligned}\right\} \tag{21}$$

where

$$\Delta = |\xi|\left(2|\xi|h + 1 - e^{-2|\xi|h}\right)$$

The direct stresses are certainly even functions of x and so, therefore, is the stress function $\phi(x,y)$. It is then easily shown that $\bar{\phi}(\xi,y)$ is an even function of ξ. This statement further implies that D is an even function of ξ, and allows equations (20) to be modified to

$$\frac{\sigma\pi}{2} = \int_0^{\infty} \frac{\xi^2 D}{\Delta}\left(\cosh 2\xi h - 2\xi^2h^2 - 1\right)\cos \xi x \, d\xi \qquad \left(|x| < 1\right) \tag{22a}$$

and

$$0 = \int_0^{\infty} \frac{\xi D}{\Delta}(\sinh 2\xi h + 2\xi h)\cos \xi x \, d\xi \qquad \left(|x| > 1\right) \tag{22b}$$

Equations (22) are the dual integral equations which must be solved for the unknown function $D(\xi)$.

In order to cast equations (22) in a form suitable for treatment by the method of reference 5, the following identity is used:

$$\cos \xi x = \left(\frac{\pi \xi x}{2}\right)^{1/2} J_{-1/2}(\xi x) \tag{23}$$

and the following substitutions are made:

$$\left.\begin{aligned} f(\xi) &= \frac{\xi^{3/2} D}{\Delta}(\sinh 2\xi h + 2\xi h) \\ G(\xi) &= \xi\left(\frac{\cosh 2\xi h - 2\xi^2h^2 - 1}{\sinh 2\xi h + 2\xi h}\right) \end{aligned}\right\} \tag{24}$$

In equation (23), $J_{-1/2}(\xi x)$ is the Bessel function of the first kind of order -1/2. With these substitutions, equations (22) assume the form

$$g(x) = \int_0^{\infty} G(\xi)\, f(\xi)\, J_{-1/2}(\xi x)\, d\xi \qquad \left(|x| < 1\right) \tag{25a}$$

and

$$0 = \int_0^\infty f(\xi)\, J_{-1/2}(\xi x)\, d\xi \qquad \Big(|x| > 1\Big) \tag{25b}$$

where

$$g(x) = \sigma\left(\frac{\pi}{2}\right)^{1/2} x^{-1/2}$$

and the unknown function is now $f(\xi)$. According to reference 5, equation (25b) is satisfied automatically by

$$f(\xi) = \xi^{1-k} \sum_{m=0}^{\infty} A_m\, J_{2m+k-\frac{1}{2}}(\xi) \tag{26}$$

in which k is required only to be real and positive, and the A_m values are to be determined through the satisfaction of equation (25a). When $g(x)$ is of the form $\lambda x^{-1/2}$, further use of the method of reference 5 converts equation (25a) into the following infinite set of linear algebraic equations in A_m:

$$\left.\begin{aligned} A_0 + \sum_{m=0}^{\infty} L_{m,0} A_m &= \lambda \frac{2^{1-k}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(k - \frac{1}{2}\right)} \\ A_n + \sum_{m=0}^{\infty} L_{m,n} A_m &= 0 \qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{27}$$

where

$$L_{m,n} = (4n + 2k - 1) \int_0^\infty \left[\xi^{2-2k} G(\xi) - 1\right] J_{2m+k-\frac{1}{2}}(\xi)\, J_{2n+k-\frac{1}{2}}(\xi) \frac{d\xi}{\xi} \tag{28}$$

For the present case of uniform pressure in the crack, $\lambda = (\pi/2)^{1/2}\sigma$, and it is convenient to set $k = 3/2$ and $A_n = \frac{\sigma\pi}{2} P_n$. Equations (27) and (28) become

$$\left.\begin{aligned} P_0 + \sum_{m=0}^{\infty} L_{m,0} P_m &= 1 \\ P_n + \sum_{m=0}^{\infty} L_{m,n} P_m &= 0 \qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{29}$$

and

$$L_{m,n} = 2(2n + 1) \int_0^\infty \left(\frac{G(\xi)}{\xi} - 1\right) J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi}$$

which for the present problem becomes

$$L_{m,n} = -2(2n + 1) \int_0^\infty \frac{2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}}{\sinh 2\xi h + 2\xi h}\, J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi} \tag{30}$$

For a specified value of h (>0), a square array of $L_{m,n}$ values is computed from equation (30) by numerical techniques. The order of the array required for accurate determination of the P_n terms (hence, D and eventually $\bar{\phi}(\xi,y)$) is dependent on the assigned value of h. For large values of h (>1) the coefficients in equations (29) constitute a very nearly diagonal matrix. In fact, as h tends toward infinity, the matrix becomes diagonal, and the only nonzero root of equations (29) is $P_0 = 1/2$. In general, then, the P_n values decay more or less rapidly with increasing n, depending on whether h is large or small. Specification of a large value of h dictates the computation of only a few $L_{m,n}$ terms, whereas a small value of h (<1) calls for a large array of $L_{m,n}$ terms.

Once a sufficiently large array of $L_{m,n}$ terms has been computed, the $L_{m,n}$ terms are inserted into a truncated system of equations (29), which is then solved for P_n. In all the problems treated here, more simultaneous equations have been solved than appeared to be necessary for good convergence. The computations have been performed for numerous values of h in the range $0.1 \leqq h \leqq 6$.

On the line containing the crack ($y = 0$),

$$\sigma_y(x,0) = -\frac{1}{\pi} \int_0^\infty \xi^2 \bar{\phi}(\xi,0) \cos \xi x \, d\xi$$

which by virtue of equation (13), equations (21), the first of equations (24), and equation (26), becomes

$$\sigma_y(x,0) = -\sigma \sum_{m=0}^{\infty} P_m \int_0^\infty \frac{\cosh 2\xi h - 2\xi^2h^2 - 1}{\sinh 2\xi h + 2\xi h}\, J_{2m+1}(\xi) \cos \xi x \, d\xi$$

or in a form better suited to present purposes

$$\sigma_y(x,0) = -\sigma \sum_{m=0}^{\infty} P_m \int_0^\infty \left(1 - \frac{2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}}{\sinh 2\xi h + 2\xi h}\right) J_{2m+1}(\xi) \cos \xi x \, d\xi \tag{31}$$

From reference 7, after slight simplification,

$$\left.\begin{aligned}\int_0^\infty J_{2m+1}(\xi)\cos\xi x\,d\xi &= \frac{\cos\left[(2m+1)\sin^{-1}x\right]}{\left(1-x^2\right)^{1/2}} && (x<1)\\ \int_0^\infty J_{2m+1}(\xi)\cos\xi x\,d\xi &= \frac{(-1)^{m+1}}{\left(x^2-1\right)^{1/2}\left[x+\left(x^2-1\right)^{1/2}\right]^{2m+1}} && (x>1)\end{aligned}\right\}\qquad(32)$$

The region of primary interest here is $x > 1$ which contains the area immediately ahead of the crack tip. From equations (31) and (32),

$$\sigma_y(x,0) = \sigma\sum_{m=0}^{\infty} P_m\left\{\frac{(-1)^m}{\left(x^2-1\right)^{1/2}\left[x+\left(x^2-1\right)^{1/2}\right]^{2m+1}}\right.$$

$$\left.+\int_0^\infty \frac{2\xi^2h^2+2\xi h+1-e^{-2\xi h}}{\sinh 2\xi h+2\xi h} J_{2m+1}(\xi)\cos\xi x\,d\xi\right\}\qquad(x>1)\qquad(33)$$

It is apparent from physical considerations that the only singular point of $\sigma_y(x,0)$ is at $x = 1$. It is now convenient to define a (dimensional) stress intensity factor by

$$K' = 2^{1/2}\lim_{x'\to a}\left[(x'-a)^{1/2}\sigma_y'(x',0)\right]$$

or in terms of dimensionless quantities

$$K' = E\sigma(2a)^{1/2}\lim_{x\to 1}\left[(x-1)^{1/2}\frac{\sigma_y}{\sigma}(x,0)\right]$$

where $E\sigma$ is the uniform pressure applied to the crack faces, and the $\sqrt{2}$ has been inserted strictly for convenience. Then the dimensionless stress intensity factor can be defined by

$$K = \frac{K'}{E\sigma a^{1/2}} = 2^{1/2}\lim_{x\to 1}\left[(x-1)^{1/2}\frac{\sigma_y}{\sigma}(x,0)\right]\qquad(34)$$

It is shown in appendix C that the infinite integral in equation (33) is not singular at $x = 1$; therefore, it follows that the stress intensity factor is contained entirely in the first term on the right-hand side of equation (33) because only that term survives the limiting process indicated in equation (34).

Since the dimensionless stress intensity factor for an infinite plate under the same loading is given simply by

$$K_\infty = 1$$

then from equations (33) and (34) the ratio of dimensionless stress intensity factors for the strip and the infinite plate is

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m \tag{35}$$

The ratio K_F/K_∞ may be considered as a multiplying factor on the infinite-sheet stress intensity factor K_∞ to account for the effect of finite strip width. This ratio is plotted in figure 2.

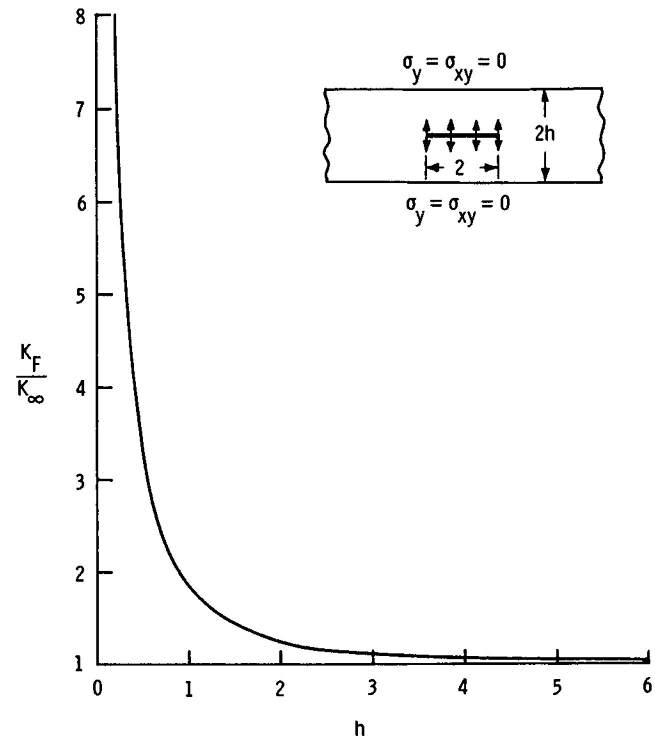

Figure 2.- Variation of K_F/K_∞ with h. Uniform pressure and free longitudinal edge.

This problem can be changed, by superposition of a constant uniaxial stress field $\sigma_y(x,y) = \sigma$, to a physically more realistic problem in which the crack surfaces are free of applied load and the longitudinal edges $y = \pm h$ are subjected to a uniform normal tensile stress. The stress intensity factor is unaffected by this alteration.

Zero normal displacement of longitudinal edges.- This problem differs from the preceding one only in that the condition of zero normal stress on the longitudinal edges is replaced by the condition of zero normal displacement. With symmetry about the longitudinal center line ($y = 0$) accounted for, the boundary conditions on the upper half of the strip are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\sigma \qquad \left(|x| < 1\right)$$

$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

This problem is solved by the same procedure employed in the first problem. The analytical details are presented in appendix A. The ratio of stress intensity factors is again found to be

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

Here, of course, the P_m terms generally differ from those found in the first problem. The ratio K_F/K_∞ is plotted in figure 3 for $0.1 \leqq h \leqq 6$.

Another quantity of some interest is the normal stress distribution on the edge $y = h$. This stress arises from the zero normal displacement boundary condition. In appendix A, it is found to be

$$\sigma_y(x,h) = -2\sigma \sum_{m=0}^{\infty} P_m \int_0^\infty \frac{\sinh \xi h + \xi h \cosh \xi h}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

This stress is plotted in figure 4 for $h = 2$, 1, and 0.1 in the range $0 \leqq x \leqq 2$. It is symmetric about the line $x = 0$.

By superposition, this problem can be converted into one in which the crack faces are free of applied load and the longitudinal edges $y = \pm h$ are uniformly displaced away from the longitudinal center line; that is, in dimensionless terms,

$$v(x,h) = -v(x,-h) = \sigma h$$

The dimensionless normal stress on the edges $y = \pm h$ required to produce this uniform edge displacement is given by

$$\sigma_{y,1} = \sigma + \sigma_y(x,h)$$

This modified version, of course, yields the same stress intensity factor and corresponds to constant-edge-displacement tensile tests if the testing machine grips do not introduce significant shear stresses at the edges. However, if the testing machine grips produce an effectively clamped-edge condition, the results of the following analysis should prove to be more pertinent.

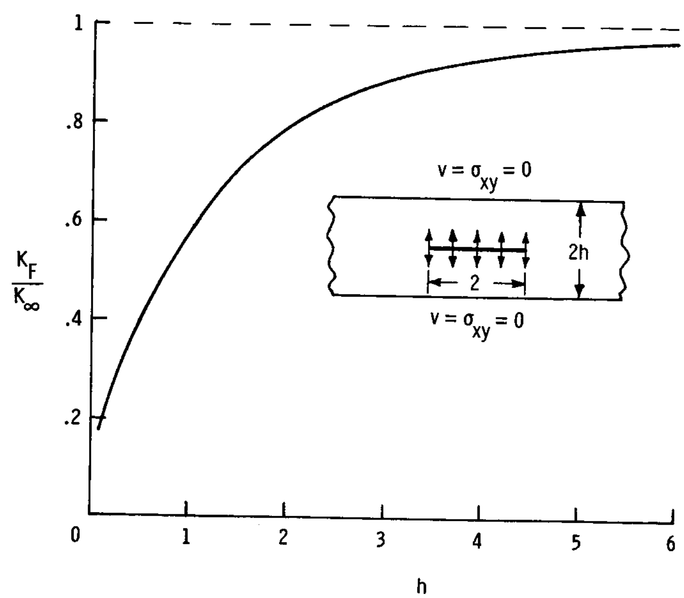

Figure 3.- Variation of K_F/K_∞ with h. Uniform pressure and zero normal displacement of longitudinal edges.

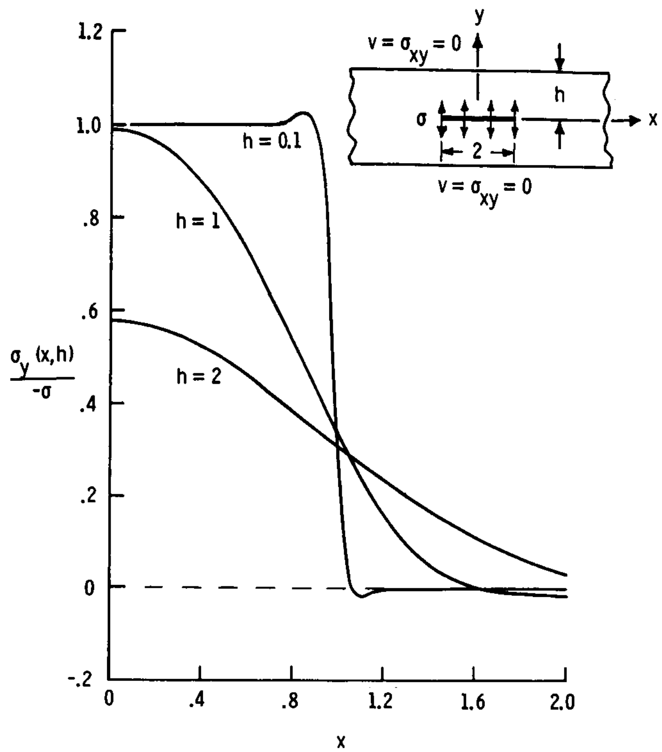

Figure 4.- Normal stress on longitudinal edge. Uniform pressure and zero normal displacement.

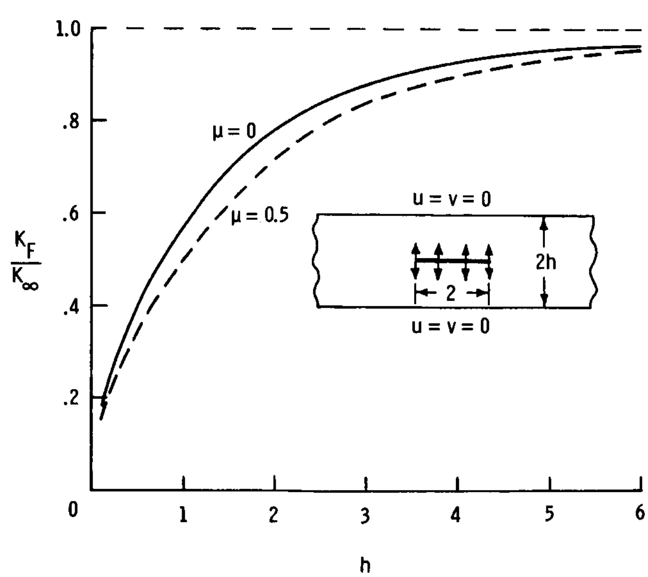

Figure 5.- Variation of K_F/K_∞ with h. Uniform pressure and clamped longitudinal edges.

Clamped longitudinal edges.- For the cracked plate strip with clamped edges and uniform pressure in the crack, the boundary conditions, with symmetry accounted for, are

$$u(x,h) = 0$$
$$v(x,h) = 0$$
$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\sigma \qquad \left(|x| < 1\right)$$
$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

By the procedures employed previously, the ratio of stress intensity factors is again

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

The P_m terms differ from those of the preceding problems and, because of the clamped-edge condition, depend on the value assigned to Poisson's ratio μ. The ratio K_F/K_∞ has been computed for $\mu = 0$ and 0.5, two values which encompass those commonly used for most engineering materials, and for numerous values of h in the range $0.1 \leq h \leq 6$. This ratio is plotted in figure 5.

It might be noted here that these results have been derived under plane strain assumptions. It is easily ascertained, however, that the results for plane stress are bracketed by the plane strain results presented in figure 5.

The normal stress on a longitudinal edge is found to be

$$\sigma_y(x,h) = \frac{-4\sigma}{(1+\mu)^2} \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{2 \cosh \xi h + (1+\mu)\xi h \sinh \xi h}{\frac{3-\mu}{1+\mu} \sinh 2\xi h - 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

This stress has been computed for $\mu = 0$ and 0.5, and is plotted in figure 6 for $h = 4$, 1, and 0.1, in the interval $0 \leqq x \leqq 2$.

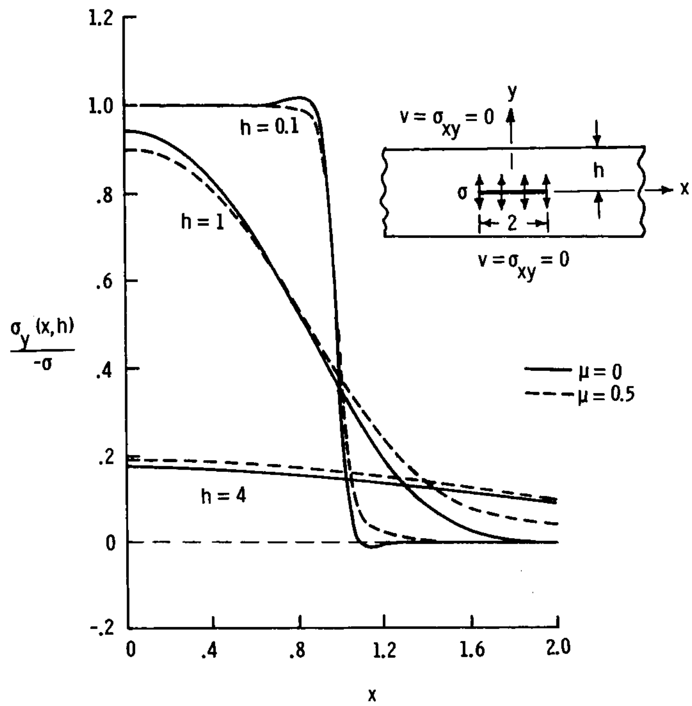

Figure 6.- Normal stress on longitudinal edge. Uniform pressure and clamped longitudinal edges.

Edge Sliding Mode – Uniform Shear

The problems contained in this section are concerned with the sliding of the crack faces on each other in the plane of the plate due to the application of uniform shear stress $E\tau$ to the crack faces. Again from symmetry considerations, it is possible to restrict attention to the upper half of the strip.

Free longitudinal edges.- In this problem the longitudinal edges of the strip are free of external load. The boundary conditions on the upper half of the strip are

$$\sigma_y(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

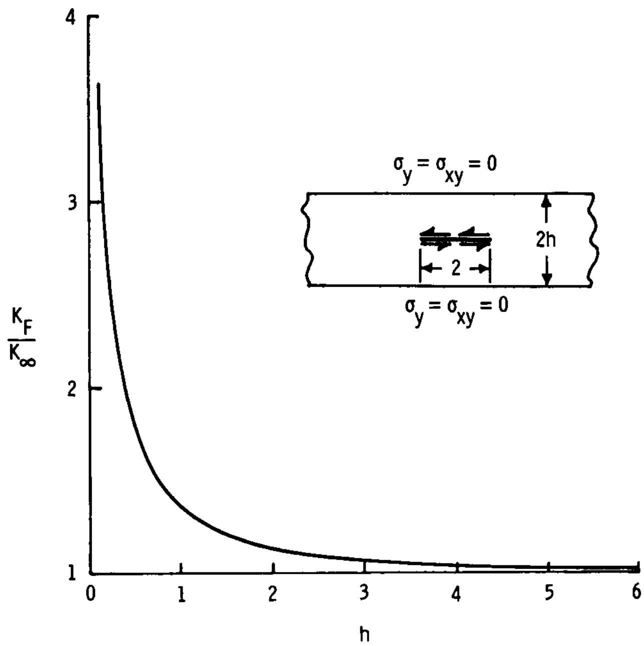

Figure 7.- Variation of K_F/K_∞ with h. Uniform shear stress and free longitudinal edges.

$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad \left(|x| < 1\right)$$

$$u(x,0) = 0 \qquad \left(|x| > 1\right)$$

In these shear problems, it is now the shear stress which is singular at the crack tip. Aside from this fact, however, the analysis of this problem differs little from those which precede it. The ratio of stress intensity factors is determined in appendix A to be

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

This ratio of shear stress intensity factors has been calculated for $0.1 \leqq h \leqq 6$ and is presented in figure 7.

Zero normal displacement of longitudinal edges.- This problem differs from the preceding one in that the longitudinal edges of the strip are restrained against normal displacement. The boundary conditions on the upper half of the strip are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad \left(|x| < 1\right)$$

$$u(x,0) = 0 \qquad \left(|x| > 1\right)$$

Details of this analysis also are contained in appendix A. The ratio of stress intensity factors has the same form as in the preceding problems. It has been computed for $0.1 \leqq h \leqq 6$ and is presented in figure 8.

Restriction of normal displacement of the longitudinal edges gives rise to normal stress there. This stress is found to be

$$\sigma_y(x,h) = \tau \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\xi h \sinh \xi h}{\cosh 2\xi h + 1} J_{2m+1}(\xi) \sin \xi x \, d\xi$$

This normal stress is plotted for $h = 2$, 1, and 0.1 in figure 9, in the range $0 \leqq x \leqq 2$. It is an antisymmetric function of x.

The stress intensity factor for this problem is identical to the stress intensity factor for the problem in which the crack faces are free of stress, and the longitudinal edges are subjected to uniform shear stress while being restrained against normal displacement.

<ins>Zero tangential displacement of longitudinal edges</ins>.- In this problem the longitudinal edges of the strip are restrained against tangential displacement and the crack faces are subjected to uniform shear stress $E\tau$. The boundary conditions on the upper half of the strip are

$$\sigma_y(x,h) = 0$$

$$u(x,h) = 0$$

$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad \left(|x| < 1\right)$$

$$u(x,0) = 0 \qquad \left(|x| > 1\right)$$

For this problem the ratio of stress intensity factors again has the same form. It has been computed for numerous values of h in the range $0.1 \leqq h \leqq 6$. The results are presented in figure 10.

Shear stress is present at the longitudinal edges of the strip, because of restriction of tangential displacement. In appendix A, this stress is found to be

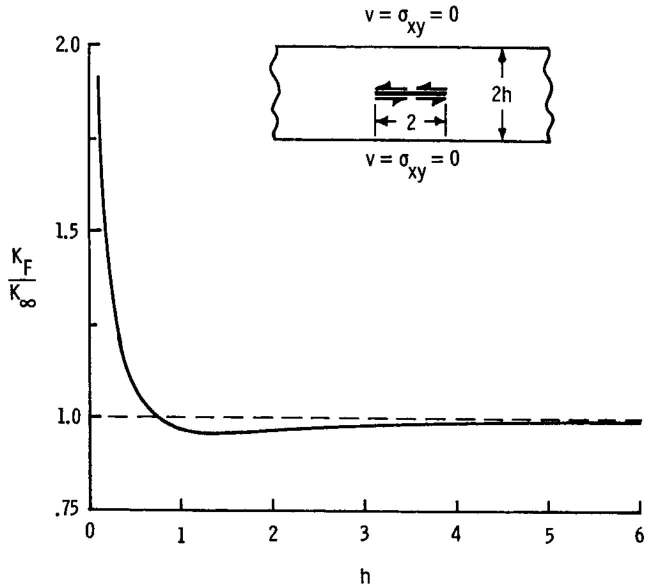

Figure 8.- Variation of K_F/K_∞ with h. Uniform shear stress and zero normal displacement.

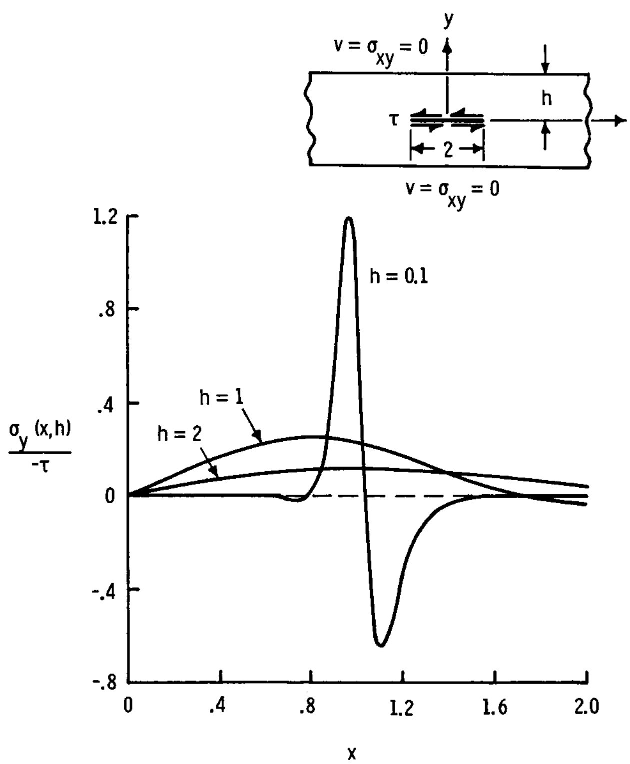

Figure 9.- Normal stress on longitudinal edge. Uniform shear stress and zero normal displacement.

$$\sigma_{xy}(x,h) = 2\tau \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\sinh \xi h - \xi h \cosh \xi h}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

It has been computed for $h = 2$, 1, and 0.1 and is plotted in figure 11 for $0 \leqq x \leqq 2$. It is symmetric in x.

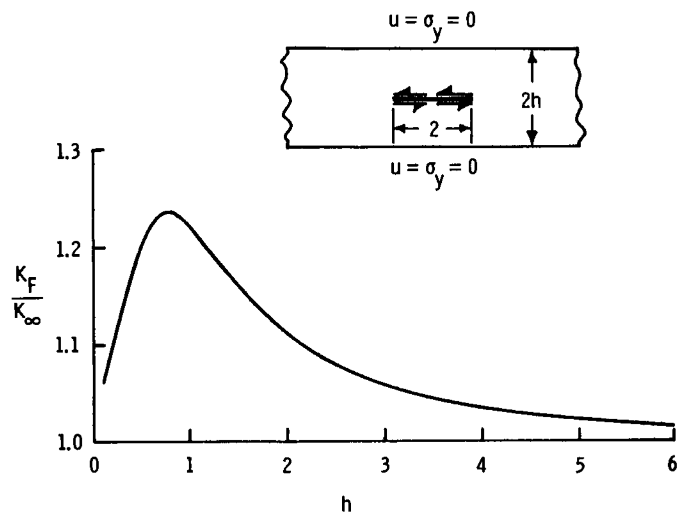

Figure 10.- Variation of K_F/K_∞ with h. Uniform shear stress and zero lateral displacement.

The stress intensity factor for this problem is the same as for the problem in which the crack faces are stress-free and the longitudinal edges are uniformly displaced tangentially by an amount $u(x,\pm h) = \pm \frac{E\tau h}{G}$ and are free of normal stress.

Bending Problem

The lateral bending of a thin strip containing a central longitudinal crack gives rise to infinite moments (hence, infinite bending stresses) at the tips of the crack, according to the classical small-deflection theory of plates. In dimensionless form, the equations governing bending of a thin plate in the absence of lateral load are (see ref. 8)

$$\nabla^4 w(x,y) = 0$$

Moments:

$$M_X = -\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_Y = -\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{XY} = -M_{YX} = (1 - \mu)\frac{\partial^2 w}{\partial x \, \partial y}$$

Shear forces:

$$Q_X = -\frac{\partial}{\partial x} \nabla^2 w$$

$$Q_Y = -\frac{\partial}{\partial y} \nabla^2 w$$

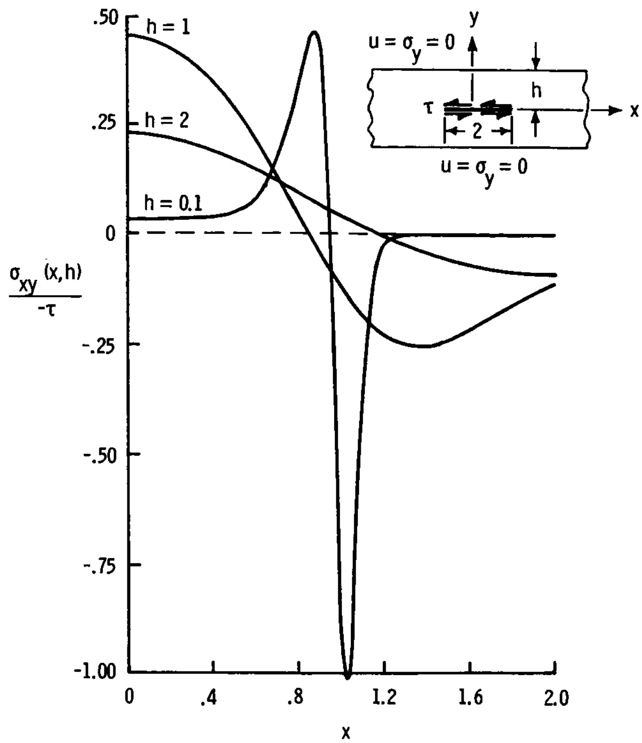

Figure 11.- Shear stress on longitudinal edge. Uniform shear stress and zero lateral displacement.

In the present problem, the longitudinal edges of the strip are simply supported and subjected to a uniform bending moment, designated as $-M$ in the dimensionless system. For purposes of determining the stress intensity factor, a uniform bending moment $M_Y(x,y) = M$ is superimposed on the original system; this process results in uniform bending moments on the crack surfaces and longitudinal edges which are free of bending moment in the y-direction. Attention again is restricted to the upper half of the strip because of symmetry considerations.

The boundary conditions are

$$w(x,h) = 0$$

$$M_Y(x,h) = 0$$

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2}(x,0) + (2 - \mu)\frac{\partial^2 w}{\partial x^2}(x,0)\right] = 0$$

$$M_Y(x,0) = M \qquad \left(|x| < 1\right)$$

$$\frac{\partial w}{\partial y}(x,0) = 0 \qquad \left(|x| > 1\right)$$

Because the bending problem is also governed by the biharmonic equation, the method of analysis employed in the preceding problems is applicable. The quantity of primary interest is now the bending moment $M_Y(x,0)$. In appendix A, it is found to be

$$M_Y(x,0) = -M \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} + \int_0^{\infty} \frac{1 - 2\xi h\left(\frac{1 - \mu}{3 + \mu}\right) + e^{-2\xi h}}{\cosh 2\xi h + 1} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\}$$

The ratio of bending stress intensity factors for the strip and the infinite plate then is

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

Note the dependence of $M_Y(x,0)$ on Poisson's ratio μ. The ratio K_F/K_∞ has been computed for $\mu = 0$ and 0.5 for numerous values of h in the range $0.1 \leqq h \leqq 6$. The results are presented in figure 12.

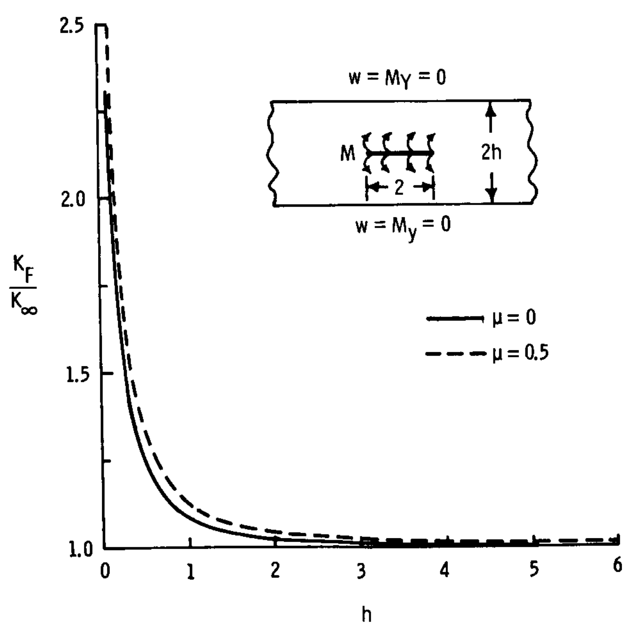

Figure 12.- Variation of K_F/K_∞ with h for bending problem.

Crack Opening Mode –

Concentrated Forces

In the three problems which follow, the crack faces are subjected to equal and opposite compressive concentrated forces at their centers. Because the analysis of this particular class of problems differs somewhat from preceding analyses, the first of these problems is analyzed in some detail, and the analysis of the remaining problems is relegated to appendix B.

Free longitudinal edges.- In this problem the edges of the strip are free of external load and the crack faces are subjected to concentrated forces of (nondimensional) intensity P/Ea. Attention is restricted here to the upper half of the strip by virtue of symmetry properties. The boundary conditions on the upper half of the strip are

$$\sigma_y(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\frac{P}{Ea}\,\delta(x) \qquad (|x| < 1)$$

$$v(x,0) = 0 \qquad (|x| > 1)$$

where $\delta(x)$ is the Dirac delta function. The transformed boundary conditions are

$$\left.\begin{aligned}
&\bar{\phi}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,0) = 0 \\
&\frac{P}{Ea}\,\delta(x) = \frac{1}{\pi}\int_0^\infty \xi^2\bar{\phi}(\xi,0)\cos\,\xi x\,d\xi \qquad (|x| < 1) \\
&0 = \int_0^\infty \frac{d^3\bar{\phi}}{dy^3}(\xi,0)\cos\,\xi x\,\frac{d\xi}{\xi^2} \qquad (|x| > 1)
\end{aligned}\right\} \tag{36}$$

Substitution of the general solution for $\bar{\phi}$ (eq. (13)) into the first three of equations (36) again yields equations (21). The following substitutions are made

$$\cos\,\xi x = \left(\frac{\pi\xi x}{2}\right)^{1/2} J_{-1/2}(\xi x)$$

$$f(\xi) = \frac{\xi^{3/2}D}{\Delta}(\sinh\,2\xi h + 2\xi h)$$

$$G(\xi) = \xi\left(\frac{\cosh\,2\xi h - 2\xi^2h^2 - 1}{\sinh\,2\xi h + 2\xi h}\right)$$

The dual integral equations then assume the form

$$g(x) = \int_0^\infty G(\xi)\,f(\xi)\,J_{-1/2}(\xi x)\,d\xi \qquad (|x| < 1) \tag{37a}$$

and

$$0 = \int_0^\infty f(\xi)\, J_{-1/2}(\xi x)\, d\xi \qquad \left(|x| > 1\right) \tag{37b}$$

where $g(x)$ now has the form $\lambda x^{-1/2}\delta(x)$. Once more it is assumed that

$$f(\xi) = \xi^{-1/2} \sum_{m=0}^{\infty} A_m\, J_{2m+1}(\xi)$$

which automatically satisfies equation (37b). Further application of the method of reference 5 yields a different set of linear algebraic equations

$$P_n + \sum_{m=0}^{\infty} L_{m,n} P_m = 1 \qquad (n = 0, 1, 2, \ldots) \tag{38}$$

where

$$\frac{P}{Ea} P_n = A_n$$

and

$$L_{m,n} = 2(2n + 1) \int_0^\infty \left(\frac{G(\xi)}{\xi} - 1\right) J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi}$$

For $x > 1$, the stress $\sigma_y(x,0)$ is found to be

$$\sigma_y(x,0) = \frac{2P}{\pi Ea} \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} + \int_0^\infty \frac{2\xi^2 h^2 + 2\xi h + 1 - e^{-2\xi h}}{\sinh 2\xi h + 2\xi h} J_{2m+1}(\xi) \cos \xi x\, d\xi \right\} \tag{39}$$

For the problem of concentrated forces it is again convenient to express the effect of strip width in terms of the ratio of stress intensity factors. For the infinite plate with the same loads, the stress $\sigma_{y,\infty}(x,0)$ is given by

$$\sigma_{y,\infty}(x,0) = \frac{2P}{\pi Ea} \sum_{m=0}^{\infty} \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} \tag{40}$$

Then the ratio of stress intensity factors may be written as

$$\frac{K_F}{K_\infty} = \lim_{x \to 1} \frac{\sigma_y(x,0)}{\sigma_{y,\infty}(x,0)} \tag{41}$$

After substitution of equations (39) and (40) into equation (41) and appropriate passage to the limit, the ratio of stress intensity factors is found to be

$$\frac{K_F}{K_\infty} = 1 + 2 \sum_{m=0}^{\infty} (-1)^m \left(P_m - 1\right) \tag{42}$$

This ratio has been computed for $0.1 \leqq h \leqq 6$, and the results are plotted in figure 13.

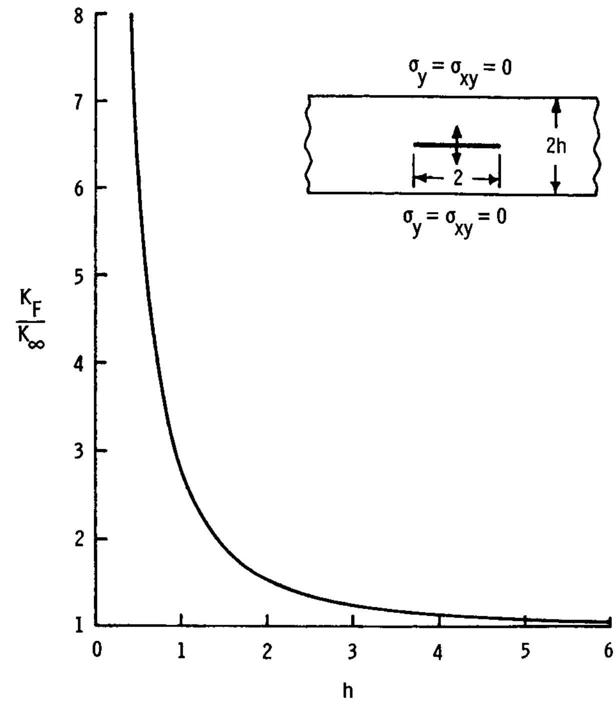

Figure 13.- Variation of K_F/K_∞ with h. Concentrated force and free longitudinal edges.

<u>Zero normal displacement of longitudinal edges</u>.- This problem differs from the preceding one in that the longitudinal edges are restrained against normal displacement. The boundary conditions on the upper half of the strip are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\frac{P}{Ea}\,\delta(x) \qquad \left(|x| < 1\right)$$

$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

The ratio of stress intensity factors for this problem is found in appendix B to have the form of equation (42). The results are presented in figure 14.

The normal stress on the edge $y = h$ also has been obtained. It is

$$\sigma_y(x,h) = -\frac{4P}{\pi Ea} \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\sinh \xi h + \xi h \cosh \xi h}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

This stress is plotted in figure 15 for $h = 4$, 2, and 1.

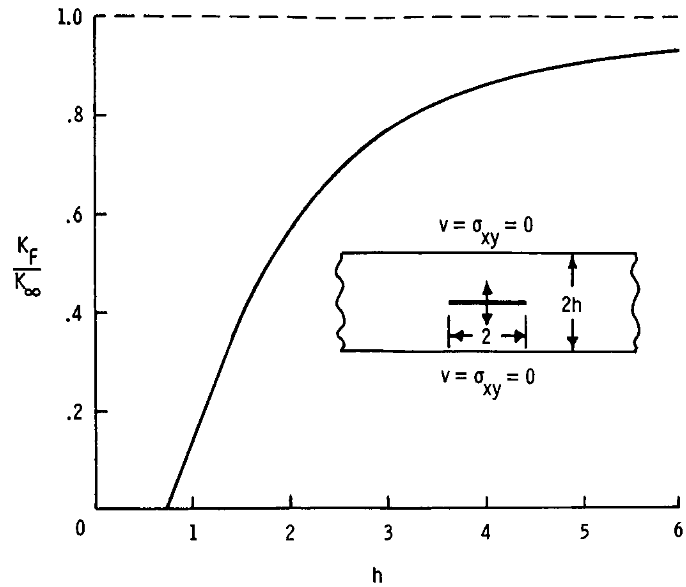

Figure 14.- Variation of K_F/K_∞ with h. Concentrated force and zero normal displacement.

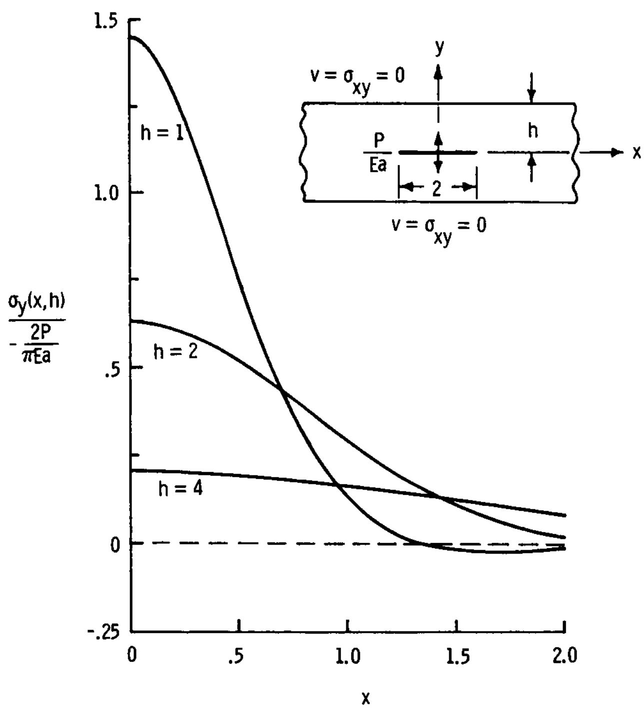

Figure 15.- Normal stress on longitudinal edge. Concentrated force and zero normal displacement.

Clamped longitudinal edges.- In this problem the longitudinal edges are completely restrained against normal and tangential displacements. The boundary conditions on the upper half of the strip are

$$u(x,h) = 0$$
$$v(x,h) = 0$$
$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\frac{P}{Ea}\delta(x) \qquad \left(|x| < 1\right)$$
$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

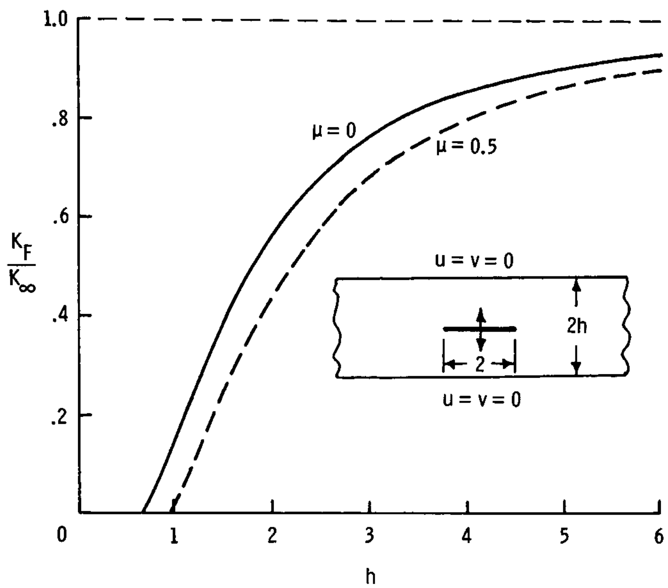

Figure 16.- Variation of K_F/K_∞ with h. Concentrated force and clamped longitudinal edges.

The analytical details of this problem are contained in appendix B. The ratio of stress intensity factors again has the form of equation (42). It has been computed for numerous values of h in the range $0.1 \leqq h \leqq 6$, and the results are presented in figure 16. For this problem, the ratio of stress intensity factors depends on Poisson's ratio. It has been computed for $\mu = 0$ and $\mu = 0.5$.

There are stresses along the longitudinal edges because of the clamped-edge condition. The normal stress on an edge is found in appendix B to be

$$\sigma_y(x,h) = -\frac{4P}{\pi Ea(1+\mu)^2}\sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{2\cosh\xi h + (1+\mu)\xi h \sinh\xi h}{\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h} J_{2m+1}(\xi)\cos\xi x \, d\xi$$

This stress is plotted for h = 4, 2, and 1 in figure 17 for $\mu = 0$ and $\mu = 0.5$.

RESULTS AND DISCUSSION

It is convenient to consider the ratio K_F/K_∞ as a multiplying factor on the infinite-sheet stress intensity factor to account for the finite width of a plate in an actual structure or to account for the finite distance between testing machine grips in laboratory tests of cracked sheet. With regard to the latter application, the results presented should prove useful in the correlation of existing experimental data. These results also should be helpful in the design of test specimens, because they furnish the specimen designer with guidelines for minimizing the perturbing influence on the stress intensity factor of the boundary conditions imposed by the testing machine grips.

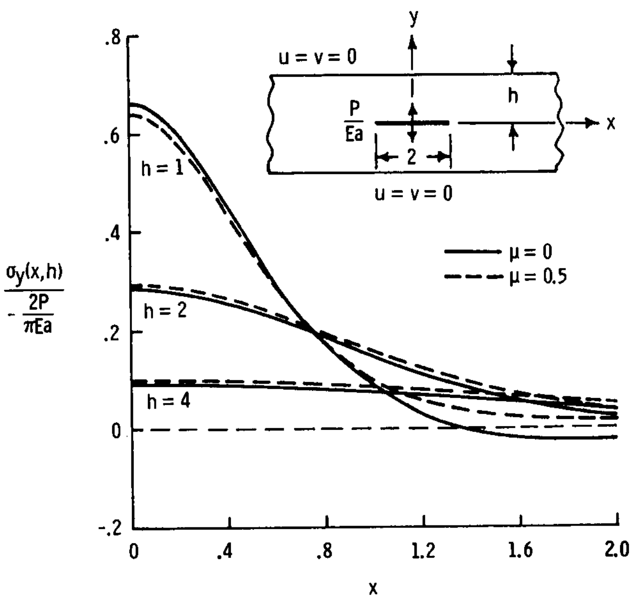

Figure 17.- Normal stress on longitudinal edge. Concentrated force and clamped longitudinal edges.

The K_F/K_∞ curves for the six crack-opening mode problems are presented in figures 2, 3, 5, 13, 14, and 16. For the problems in which the longitudinal edges of the strip are stress-free (see figs. 2 and 13), the ratio of stress intensity factors is seen to be always greater than 1, and to approach 1 rather rapidly for moderately large values of h, the ratio of strip width to crack length. On the other hand, the ratio is always less than 1 for the problems involving restraint of the normal, or normal and tangential, displacements of the edges of the strip. (See figs. 3, 5, 14, and 16.) Again, the ratio is seen to approach 1, although somewhat less rapidly than for free edges, as h becomes large.

Figures 5 and 16 are for problems involving completely fixed longitudinal edges, a set of boundary conditions which causes the stress intensity factor to depend on Poisson's ratio. For these problems, the stress intensity factors have been computed for $\mu = 0$ and $\mu = 0.5$, two values of Poisson's ratio which encompass the values commonly assigned to most engineering materials. The results suggest that Poisson's ratio exerts little influence on stress intensity factors for most practical crack-opening problems.

In the crack-opening mode problems involving restraint of displacements at the longitudinal edges of the strip, normal stresses arise there. These normal stresses have been computed for various values of h and are presented in figures 4, 6, 15, and 17. Figures 4 and 6 are for problems involving uniform pressure in the crack, and figures 15 and 17 are for the concentrated force problems. For larger values of h the normal stresses are rather small and well-behaved, but these stresses and their gradients grow as h becomes small because the stresses due to the applied forces and the crack-tip singularity have less area in which to diffuse and decay.

Figures 6 and 17 are for the problems in which the longitudinal edges of the strip are fixed so that the stresses depend on Poisson's ratio. Much like the stress intensity factors for these problems, the normal stresses are not strongly influenced by changes in Poisson's ratio.

A curious result coming out of the two concentrated force problems which involved restraint of normal displacements at the strip edges can be seen in figures 14 and 16.

As h becomes small, the stress intensity factor goes to zero at a seemingly arbitrary value of h, specifically, in the neighborhood of $h = 0.8$, the precise crossing varying with Poisson's ratio in the case of fixed longitudinal edges. This behavior is due to closure of the crack tips. The K_F/K_∞ curves have been terminated at the crossings because for smaller values of h the problems cease to be only crack problems, and instead become very complicated contact problems which are not expected to yield additional significant information on the stress intensity factors.

The K_F/K_∞ curves for the edge-sliding mode (shear) problems are presented in figures 7, 8, and 10. For the plate with free longitudinal edges (fig. 7), the stress intensity factor increases without limit as h approaches zero, and approaches the infinite-sheet stress intensity factor as h becomes large. This result is similar to those obtained for the crack-opening problems with the same free-edge conditions.

For the second shear problem, in which normal displacement of the longitudinal edges is prevented, the stress intensity factor for the strip (fig. 8) differs little from its infinite-sheet counterpart except for very small values of h, where the problem has lost most of its physical importance. The restriction of normal displacement of the longitudinal edges gives rise to normal stresses there. This normal stress is presented in figure 9 for several values of h. Not unexpectedly, the normal stress is more severe for small values of h.

An unusual result from the third shear problem, in which the tangential displacement of the longitudinal edges is prevented, is apparent in figure 10. The ratio K_F/K_∞ approaches 1 at both ends of the h-axis, and is greater than 1 at all interior points. This result means that there are two values of h corresponding to every admissible value of K_F/K_∞.

In this shear problem, shear stresses develop at the longitudinal edges. This shear stress is presented in figure 11 for several values of h. As in the preceding problems, the strip width exerts a strong influence on the stress at the boundary.

For the one plate bending problem treated here, the K_F/K_∞ curves are presented in figure 12, where it is seen that the ratio of moment (or outer-fiber bending stress) intensity factors depends on Poisson's ratio. Again, the effect of Poisson's ratio on K_F/K_∞ is small. The ratio of stress intensity factors approaches 1 for large h and increases without limit as h approached zero. Finite strip width is seen to have little influence for ratios of strip width to crack length greater than about 2.

For the problems treated herein, finite strip width is seen to be of little significance for ratios of strip width to crack length greater than 6. In fact, for problems involving free longitudinal edges, h greater than about 3 is large enough to allow treatment of the strip as an infinite plate, at least for the purposes of crack-tip stress field investigations.

CONCLUDING REMARKS

Stress intensity factors have been obtained for several problems involving crack-opening or edge sliding modes and, in one case, opening of the crack by bending, for a plate strip containing a central longitudinal crack. The stress intensity factors, normalized with respect to their counterparts for an infinite plate under the same loading, have been presented as functions of the ratio of strip width to crack length for the entire range of practical interest. For problems in which the stresses depend on Poisson's ratio, the results have been obtained for at least two values of Poisson's ratio which encompass the range of primary engineering interest.

In addition, where restrictions of displacements of the longitudinal edges of the strip have given rise to boundary stresses, some sample stress distributions have been presented to illustrate their dependence on the ratio of strip width to crack length.

For the problems treated herein finite strip width is seen to be of little significance for ratios of strip width to crack length greater than 6. In fact, for problems involving free longitudinal edges, h greater than about 3 is large enough to allow treatment of the strip as an infinite plate, at least for the purposes of crack-tip stress field investigations.

The results presented herein should be useful in the conversion of data from tests of cracked sheet-metal specimens to practical information suitable for use by design engineers. The results should also be of assistance in the design of cracked-sheet test specimens to avoid edge effects.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., February 1, 1967,
124-08-06-07-23.

APPENDIX A

PROBLEMS INVOLVING UNIFORM LOADS ON CRACK FACES

With the exception of the first crack-opening problem, which has been given detailed treatment in the body of this report, this appendix contains the bulk of the analytical details of the problems involving uniform loading of the crack faces. The three types of uniform loading considered are: (1) normal stress, (2) shear stress, and (3) bending moment.

Crack Opening Mode – Uniform Pressure

Zero normal displacement of longitudinal edges.- In this problem, the edges of the strip are restrained against normal displacement and the crack faces are being separated by uniform normal stress. Because of symmetry considerations, only the upper half of the strip needs to be considered. The boundary conditions there are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\sigma \qquad \left(|x| < 1\right)$$

$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

In terms of $\bar{\phi}(\xi,y)$ the transformed boundary conditions are

$$\left.\begin{aligned}
&\frac{d^3\bar{\phi}}{dy^3}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,0) = 0 \\
&\sigma = \frac{1}{\pi}\int_0^\infty \xi^2\bar{\phi}(\xi,0)\cos \xi x \, d\xi \qquad \left(|x| < 1\right) \\
&0 = \int_0^\infty \frac{d^3\bar{\phi}}{dy^3}(\xi,0)\cos \xi x \frac{d\xi}{\xi^2} \qquad \left(|x| > 1\right)
\end{aligned}\right\} \quad (A1)$$

APPENDIX A

Substitution of the expression for $\bar{\phi}(\xi,y)$ (eq. (13)) into the first three of equations (A1) yields

$$\left.\begin{aligned} A &= -\frac{D}{\Delta}\left(e^{2|\xi|h} + 2|\xi|h - 1\right) \\ B &= -\frac{|\xi|D}{\Delta}\left(e^{2|\xi|h} - 1\right) \\ C &= -\frac{D}{\Delta}\left(2|\xi|h + 1 - e^{-2|\xi|h}\right) \end{aligned}\right\} \tag{A2}$$

where

$$\Delta = |\xi|\left(1 - e^{-2|\xi|h}\right)$$

Then by use of equations (A2), the identity (23), and the following substitutions:

$$f(\xi) = -\frac{\xi^{3/2}D}{\Delta}(\cosh 2\xi h - 1)$$

$$G(\xi) = \xi\left(\frac{\sinh 2\xi h + 2\xi h}{\cosh 2\xi h - 1}\right)$$

the last two of equations (A1) assume the form

$$\left.\begin{aligned} g(x) &= \int_0^\infty f(\xi)\, G(\xi)\, J_{-1/2}(\xi x)\, d\xi && \left(|x| < 1\right) \\ 0 &= \int_0^\infty f(\xi)\, J_{-1/2}(\xi x)\, d\xi && \left(|x| > 1\right) \end{aligned}\right\} \tag{A3}$$

and again

$$g(x) = \sigma\left(\frac{\pi}{2}\right)^{1/2} x^{-1/2}$$

Therefore, the method of reference 5 may be applied. It is assumed that

$$f(\xi) = \frac{\sigma\pi}{2} \sum_{m=0}^{\infty} P_m J_{2m+1}(\xi)$$

and the following infinite system of linear algebraic equations results:

$$\left.\begin{aligned} P_0 + \sum_{m=0}^{\infty} L_{m,0}P_m &= 1 \\ P_n + \sum_{m=0}^{\infty} L_{m,n}P_m &= 0 \qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{A4}$$

$$L_{m,n} = 2(2n + 1)\int_0^{\infty}\left(\frac{G(\xi)}{\xi} - 1\right)J_{2m+1}(\xi)\, J_{2n+1}(\xi)\,\frac{d\xi}{\xi}$$

After determination of the P_n terms the stress $\sigma_y(x,0)$ can be written

$$\sigma_y(x,0) = \sigma\sum_{m=0}^{\infty} P_m\left\{\frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} - \int_0^{\infty}\frac{1 + 2\xi h - e^{-2\xi h}}{\cosh 2\xi h - 1}J_{2m+1}(\xi)\cos \xi x\, d\xi\right\} \tag{A5}$$

As before, the infinite integral in equation (A5) can be proved convergent at $x = 1$. Consequently, the ratio of stress intensity factors is again found to be

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty}(-1)^m P_m$$

Because of the restriction against normal displacements of the longitudinal edges, there is normal stress on these edges. In terms of the transformed stress function $\bar{\phi}$, this normal stress is

$$\sigma_y(x,h) = -\frac{1}{\pi}\int_0^{\infty}\xi^2\,\bar{\phi}(\xi,h)\cos \xi x\, d\xi$$

which with the proper substitutions becomes

$$\sigma_y(x,h) = -2\sigma\sum_{m=0}^{\infty} P_m\int_0^{\infty}\frac{\sinh \xi h + \xi h \cosh \xi h}{\cosh 2\xi h - 1}J_{2m+1}(\xi)\cos \xi x\, d\xi \tag{A6}$$

This stress distribution is symmetric in x. With a sufficient number of the P_m values already computed for this problem, the computation of $\sigma_y(x,h)$ is merely a matter of numerically performing the integration indicated in equation (A6).

By superposition, this problem can be converted into a problem in which the crack faces are stress-free and the longitudinal edges of the strip are uniformly displaced away from the longitudinal center line; that is, in dimensionless terms,

$$v(x,\pm h) = \pm\sigma h$$

The (dimensionless) normal stress on the edges $y = \pm h$ which is required to produce this uniform edge displacement is given by

$$\sigma_{y,1}(x,\pm h) = \sigma + \sigma_y(x,h)$$

This modified version, of course, yields the same stress intensity factor and corresponds to constant-edge-displacement tensile testing machine experiments if the testing machine grips do not introduce significant shear stresses at the edges. On the other hand, if the testing machine grips do produce a truly clamped-edge condition, the results of the following analysis should prove to be more pertinent.

Clamped longitudinal edges.- For the centrally cracked plate with clamped edges and uniform pressure in the crack, the boundary conditions for the upper half of the strip are

$$\begin{aligned} u(x,h) &= 0 \\ v(x,h) &= 0 \\ \sigma_{xy}(x,0) &= 0 \\ \sigma_y(x,0) &= -\sigma && (|x| < 1) \\ v(x,0) &= 0 && (|x| > 1) \end{aligned}$$

In terms of $\bar{\phi}$, the transformed boundary conditions are

$$\left.\begin{aligned} &\frac{d^2\bar{\phi}}{dy^2}(\xi,h) + \mu\xi^2\bar{\phi}(\xi,h) = 0 \\ &\frac{d^3\bar{\phi}}{dy^3}(\xi,h) - (2 + \mu)\xi^2 \frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\ &\frac{d\bar{\phi}}{dy}(\xi,0) = 0 \\ &\sigma = \frac{1}{\pi}\int_0^\infty \xi^2\bar{\phi}(\xi,0) \cos \xi x \, d\xi \qquad (|x| < 1) \\ &0 = \int_0^\infty \frac{d^3\bar{\phi}}{dy^3}(\xi,0) \cos \xi x \, d\xi \qquad (|x| > 1) \end{aligned}\right\} \tag{A7}$$

Substitution of equation (13) into the first three of equations (A7) gives

$$\left.\begin{aligned}
A &= -\frac{D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{2|\xi|h} + 2\xi^2h^2 - 2|\xi|h + \frac{4\kappa}{1+\mu} + 1\right)\\
B &= -\frac{|\xi|D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{2|\xi|h} - 2|\xi|h + 1\right)\\
C &= -\frac{D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{-2|\xi|h} + 2\xi^2h^2 + 2|\xi|h + \frac{4\kappa}{1+\mu} + 1\right)
\end{aligned}\right\} \tag{A8}$$

where

$$\Delta = |\xi|\left(\frac{3-\mu}{1+\mu}e^{-2|\xi|h} + 2|\xi|h + 1\right)$$

and

$$\kappa = \frac{1-\mu}{1+\mu}$$

By use of equations (A8), the identity (23), and the following substitutions:

$$f(\xi) = -\frac{\xi^{3/2}D}{\Delta}\left(\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h\right)$$

$$G(\xi) = \xi\left(\frac{\frac{3-\mu}{1+\mu}\cosh 2\xi h + 2\xi^2h^2 + \frac{4\kappa}{1+\mu} + 1}{\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h}\right)$$

the last two of equations (A7) again take on the form of equations (A3). Application of the method of reference 5 yields the now-familiar system of equations (A4), which are to be solved for the P_n terms.

After some substitutions and manipulations, the stress $\sigma_y(x,0)$ for $x > 1$ is found to be

$$\sigma_y(x,0) = \sigma\sum_{m=0}^{\infty} P_m\left\{\frac{(-1)^m}{(x^2-1)^{1/2}\left[x + (x^2-1)^{1/2}\right]^{2m+1}}\right.$$

$$\left. - \int_0^{\infty}\frac{1 + \frac{4\kappa}{1+\mu} + 2\xi h + 2\xi^2h^2 + \frac{3-\mu}{1+\mu}e^{-2\xi h}}{\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h}J_{2m+1}(\xi)\cos\xi x\,d\xi\right\}$$

From this equation, it is found that the ratio of stress intensity factors has the form

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

An additional complication in this problem is that the stress intensity factor now depends on Poisson's ratio μ. In order to obtain an envelope of K_F/K_∞ curves, the $L_{m,n}$ terms (hence, the P_m terms and K_F/K_∞) have been computed for $\mu = 0$ and $\mu = 0.5$, two values which encompass those commonly used for engineering materials.

Because of the dependence on Poisson's ratio, the results for plane stress conditions will differ slightly from the present results, which are for plane strain. It is easily shown, however, that the K_F/K_∞ curves presented here bracket the curves for plane stress. In view of the generally small differences between the $\mu = 0$ and $\mu = 0.5$ curves, it is not felt that the additional calculation of corresponding curves for plane stress is necessary.

Again there is normal stress on the edges $y = \pm h$. It is found to be

$$\sigma_y(x,h) = -\frac{4\sigma}{(1+\mu)^2} \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{2 \cosh \xi h + (1+\mu)\xi h \sinh \xi h}{\frac{3-\mu}{1+\mu} \sinh 2\xi h - 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

The required operations are performed numerically.

Edge Sliding Mode – Uniform Shear

The problems contained in this section are concerned with the sliding of the crack faces on each other in the plane of the strip due to the application of opposing uniform shear stresses $E\tau$ to the crack faces. Again, because of symmetry considerations, the investigation can be restricted to the upper half of the strip.

Free longitudinal edges.- In this problem the edges of the strip are stress free. The boundary conditions on the upper half of the strip are

$$\sigma_y(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad (|x| < 1)$$

$$u(x,0) = 0 \qquad (|x| > 1)$$

APPENDIX A

The transformed boundary conditions are

$$\left.\begin{aligned}
&\bar{\phi}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\bar{\phi}(\xi,0) = 0 \\
\tau = -\frac{i}{\pi}\int_0^{\infty} \xi \frac{d\bar{\phi}}{dy}(\xi,0) \cos \xi x \, d\xi \qquad & \left(|x| < 1\right) \\
0 = \int_0^{\infty} \frac{d^2\bar{\phi}}{dy^2}(\xi,0) \cos \xi x \frac{d\xi}{\xi} \qquad & \left(|x| > 1\right)
\end{aligned}\right\} \tag{A9}$$

Substitution of equation (13) into the first three of equations (A9) gives

$$\left.\begin{aligned}
A &= -C = -\frac{2|\xi|h^2 D}{\Delta} \\
B &= -\frac{D}{\Delta}\left(e^{2|\xi|h} - 2|\xi|h - 1\right)
\end{aligned}\right\} \tag{A10}$$

where

$$\Delta = 1 - 2|\xi|h - e^{-2|\xi|h}$$

With the use of equation (23), equations (A10), and the substitutions

$$f(\xi) = -\frac{i\xi^{1/2}D}{\Delta}(\sinh 2\xi h - 2\xi h)$$

$$G(\xi) = \xi\left(\frac{\cosh 2\xi h - 2\xi^2h^2 - 1}{\sinh 2\xi h - 2\xi h}\right)$$

the last two of equations (A9) assume the form of equations (A3), where

$$g(\xi) = \tau\left(\frac{\pi}{2}\right)^{1/2} x^{-1/2}$$

The method of reference 5 is now applied, and again equations of the form (A4) are obtained.

However, for these shear problems, it is now the shear stress $\sigma_{xy}(x,0)$ which is singular at the crack tip. For $x > 1$, it is found to be

$$\sigma_{xy}(x,0) = -\tau \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} + \int_0^{\infty} \frac{1 - 2\xi h + 2\xi^2 h^2 - e^{-2\xi h}}{\sinh 2\xi h - 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\}$$

With the following definition for the shear-stress intensity factor

$$K = 2^{1/2} \lim_{x \to 1}\left[(x - 1)^{1/2} \frac{\sigma_{xy}}{\tau}(x,0)\right]$$

the ratio of stress intensity factors is

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

The problem is equivalent, insofar as stress intensity factors are concerned, to one in which the crack faces are stress free and the longitudinal edges are subjected to uniform shear stress of magnitude $E\tau$.

Zero normal displacement of longitudinal edges.- This problem differs from the preceding one in that the longitudinal edges are restrained against vertical displacement. The boundary conditions are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad \left(|x| < 1\right)$$

$$u(x,0) = 0 \qquad \left(|x| > 1\right)$$

APPENDIX A

The transformed boundary conditions are

$$
\left.
\begin{aligned}
&\frac{d^3\bar{\phi}}{dy^3}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\bar{\phi}(\xi,0) = 0 \\
&\tau = -\frac{i}{\pi}\int_0^\infty \xi \frac{d\bar{\phi}}{dy}(\xi,0)\cos \xi x \, d\xi \qquad \left(|x| < 1\right) \\
&0 = \int_0^\infty \frac{d^2\bar{\phi}}{dy^2}(\xi,0)\cos \xi x \, \frac{d\xi}{\xi} \qquad \left(|x| > 1\right)
\end{aligned}
\right\} \tag{A11}
$$

Substitution of equation (13) into the first three of equations (A11) yields

$$
\left.
\begin{aligned}
A &= -C = \frac{2hD}{\Delta} \\
B &= -\frac{D}{\Delta}\left(e^{2|\xi|h} + 1\right)
\end{aligned}
\right\} \tag{A12}
$$

where

$$\Delta = 1 + e^{-2|\xi|h}$$

With the use of the identity (23), equations (A12), and the following substitutions:

$$f(\xi) = \frac{2i\xi^{1/2}D}{\Delta}(\cosh 2\xi h + 1)$$

$$G(\xi) = \xi\left(\frac{\sinh 2\xi h + 2\xi h}{\cosh 2\xi h + 1}\right)$$

the last two of equations (A11) take on the form of equations (A3). The method of reference 5 is applied and equations (A4) again are obtained. For $x > 1$, the shear stress $\sigma_{xy}(x,0)$ is obtained as

$$\sigma_{xy}(x,0) = -\tau \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{(x^2 - 1)^{1/2}\left[x + (x^2 - 1)^{1/2}\right]^{2m+1}} + \int_0^{\infty} \frac{1 - 2\xi h + e^{-2\xi h}}{\cosh 2\xi h + 1} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\}$$

From this equation, the ratio of stress intensity factors is found to have the same form as in the preceding problems.

The normal stress on the longitudinal edges, caused by the $v(x,\pm h) = 0$ boundary condition, is found to be

$$\sigma_y(x,h) = \tau \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\xi h \sinh \xi h}{\cosh 2\xi h + 1} J_{2m+1}(\xi) \sin \xi x \, d\xi$$

In this case, the normal stress is an antisymmetric function of x.

With respect to the stress intensity factor, this problem is equivalent to one in which the crack faces are stress free, and the edges of the strip are subjected to uniform shear stress while being restrained against normal displacement.

Zero tangential displacement of longitudinal edges.- In this problem the edges of the strip are free of normal stress and restrained against tangential displacements. The boundary conditions are

$$\sigma_y(x,h) = 0$$
$$u(x,h) = 0$$
$$\sigma_y(x,0) = 0$$

$$\sigma_{xy}(x,0) = \tau \qquad \left(|x| < 1\right)$$
$$u(x,0) = 0 \qquad \left(|x| > 1\right)$$

In terms of $\bar{\phi}$, the transformed boundary conditions are

$$\left.\begin{aligned} &\bar{\phi}(\xi,h) = 0 \\ &\frac{d^2\bar{\phi}}{dy^2}(\xi,h) = 0 \\ &\bar{\phi}(\xi,0) = 0 \end{aligned}\right\}$$

(Equations continued on next page)

$$\left.\begin{aligned}\tau &= -\frac{i}{\pi}\int_0^\infty \xi \frac{d\bar{\Phi}}{dy}(\xi,0)\cos \xi x \, d\xi \qquad &(|x| < 1)\\ 0 &= \int_0^\infty \frac{d^2\bar{\phi}}{dy^2}(\xi,0)\cos \xi x \frac{d\xi}{\xi} \qquad &(|x| > 1)\end{aligned}\right\} \tag{A13}$$

Equation (13) substituted into the first three of equations (A13) yields

$$\left.\begin{aligned}A &= -C = \frac{2hD}{\Delta}\\ B &= \frac{D}{\Delta}\left(e^{2|\xi|h} - 1\right)\end{aligned}\right\} \tag{A14}$$

with

$$\Delta = 1 - e^{-2|\xi|h}$$

After substitution of equations (A14) into the last two of equations (A13), use of the identity (23), and use of the following substitutions:

$$f(\xi) = -\frac{i\xi^{1/2}D}{\Delta}(\cosh 2\xi h - 1)$$

$$G(\xi) = \xi\left(\frac{\sinh 2\xi h - 2\xi h}{\cosh 2\xi h - 1}\right)$$

the system of equations (A3) again is obtained. The method of reference 5 is employed to solve these equations for the P_m values, and the stress $\sigma_{xy}(x,0)$ is

$$\sigma_{xy}(x,0) = -\tau \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} - \int_0^\infty \frac{1 - 2\xi h - e^{-2\xi h}}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\}$$

The ratio of stress intensity factors is again found to be

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

APPENDIX A

The shear stress at the edge $y = h$, caused by restriction of the tangential displacement there, is given by

$$\sigma_{xy}(x,h) = 2\tau \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\sinh \xi h - \xi h \cosh \xi h}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

This shear stress is symmetric in x. The stress intensity factor for this problem is the same as for the problem in which the crack faces are stress free, and the edges of the strip are uniformly displaced tangentially by an amount $u(x,\pm h) = \pm \frac{E\tau h}{G}$ and are free of normal stress.

Bending Problem

This problem is concerned with the lateral bending of a thin plate strip containing a central longitudinal crack. In the classical small-deflection theory of plates (see ref. 8), the governing equations for a plate free of lateral load are

$$\tilde{D}\nabla'^4 w'(x',y') = 0$$

where

$$\tilde{D} = \frac{Et^3}{12\left(1 - \mu^2\right)}$$

and w' is the lateral deflection of the plate middle surface, with moments given by

$$M'_X = -\tilde{D}\left(\frac{\partial^2 w'}{\partial x'^2} + \mu \frac{\partial^2 w'}{\partial y'^2}\right)$$

$$M'_Y = -\tilde{D}\left(\frac{\partial^2 w'}{\partial y'^2} + \mu \frac{\partial^2 w'}{\partial x'^2}\right)$$

$$M'_{XY} = -M'_{YX} = \tilde{D}(1 - \mu)\frac{\partial^2 w'}{\partial x' \, \partial y'}$$

and shear forces given by

$$Q'_X = -\tilde{D} \frac{\partial}{\partial x'} \nabla'^2 w'$$

$$Q'_Y = -\tilde{D} \frac{\partial}{\partial y'} \nabla'^2 w'$$

The foregoing equations can be nondimensionalized conveniently in the following manner. Let

$$w' = aw$$
$$x' = ax$$
$$y' = ay$$
$$M'_X = \frac{\tilde{D}}{a} M_X$$
$$M'_Y = \frac{\tilde{D}}{a} M_Y$$
$$M'_{XY} = -M'_{YX} = \frac{\tilde{D}}{a} M_{XY} = -\frac{\tilde{D}}{a} M_{YX}$$
$$Q'_X = \frac{\tilde{D}}{a^2} Q_X$$
$$Q'_Y = \frac{\tilde{D}}{a^2} Q_Y$$

Then the governing equations become

$$\nabla^4 w(x,y) = 0 \tag{A15}$$

where

$$\nabla^4 = a^4 \nabla'^4$$

with moments

$$\left.\begin{aligned} M_X &= -\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \\ M_Y &= -\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \\ M_{XY} &= -M_{YX} = (1 - \mu)\frac{\partial^2 w}{\partial x \, \partial y} \end{aligned}\right\} \tag{A16}$$

and shear forces

$$\left.\begin{aligned} Q_X &= -\frac{\partial}{\partial x} \nabla^2 w \\ Q_Y &= -\frac{\partial}{\partial y} \nabla^2 w \end{aligned}\right\} \tag{A17}$$

In the problem to be analyzed here, the longitudinal edges of the strip are simply supported, and a uniform bending moment $\tilde{D}M/a$ is being applied to the crack surfaces (M is a dimensionless measure of the magnitude of the applied bending moment).

In terms of dimensionless quantities, the boundary conditions are

$$w(x,\pm h) = 0$$

$$M_Y(x,\pm h) = 0$$

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2}(x,0) + (2 - \mu)\frac{\partial^2 w}{\partial x^2}(x,0)\right] = 0$$

$$M_Y(x,0) = M \qquad \left(|x| < 1\right)$$

$$\frac{\partial w}{\partial y}(x,0) = 0 \qquad \left(|x| > 1\right)$$

The first two boundary conditions describe the simple supports. The third derives for $x < 1$ from the fact that the crack faces are free of shear force, and for $x > 1$ from considerations of symmetry. The fourth is due to the applied uniform bending moment ($x < 1$) and to symmetry ($x > 1$).

After Fourier transformation, the governing equations (A15) to (A17) take the form

$$\left(\frac{d^2}{dy^2} - \xi^2\right)^2 \bar{w}(\xi,y) = 0 \tag{A18}$$

with moments

$$\left.\begin{aligned} \bar{M}_X &= \xi^2\bar{w} - \mu\frac{d^2\bar{w}}{dy^2} \\ \bar{M}_Y &= -\frac{d^2\bar{w}}{dy^2} + \mu\xi^2\bar{w} \\ \bar{M}_{XY} &= -\bar{M}_{YX} = -i\xi(1 - \mu)\frac{d\bar{w}}{dy} \end{aligned}\right\} \tag{A19}$$

and shear forces

$$\left.\begin{aligned} \bar{Q}_X &= -i\xi\left(\frac{d^2}{dy^2} - \xi^2\right)\bar{w} \\ \bar{Q}_Y &= -\frac{d}{dy}\left(\frac{d^2}{dy^2} - \xi^2\right)\bar{w} \end{aligned}\right\} \tag{A20}$$

In terms of $\bar{w}(\xi,y)$, the transformed boundary conditions, with symmetry about the longitudinal center line accounted for, are

$$\bar{w}(\xi,h) = 0 \tag{A21}$$

$$\left(\frac{d^2}{dy^2} - \mu\xi^2\right)\overline{w}(\xi,h) = 0 \qquad (A22)$$

$$\frac{d}{dy}\left[\frac{d^2}{dy^2} - (2 - \mu)\xi^2\right]\overline{w}(\xi,0) = 0 \qquad (A23)$$

$$-M\pi = \int_0^\infty \left[\frac{d^2}{dy^2}\,\overline{w}(\xi,0) - \mu\xi^2\overline{w}(\xi,0)\right]\cos\,\xi x\,d\xi \qquad \left(|x| < 1\right) \qquad (A24a)$$

and

$$0 = \int_0^\infty \frac{d\overline{w}}{dy}(\xi,0)\,\cos\,\xi x\,d\xi \qquad \left(|x| > 1\right) \qquad (A24b)$$

The general solution of equation (A18) can be written as

$$\overline{w}(\xi,y) = (A + By)e^{-|\xi|y} + (C + Dy)e^{|\xi|y} \qquad (A25)$$

Substitution of equation (A25) into equations (A21) to (A23) gives

$$\left.\begin{aligned} A &= \frac{D}{\Delta}\left(1 + 2|\xi|h + e^{2|\xi|h}\right) \\ B &= -\frac{|\xi|\kappa D}{\Delta}\left(1 + e^{2|\xi|h}\right) \\ C &= -\frac{D}{\Delta}\left(1 - 2|\xi|h\kappa + e^{-2|\xi|h}\right) \end{aligned}\right\} \qquad (A26)$$

where

$$\Delta = |\xi|\kappa\left(1 + e^{-2|\xi|h}\right)$$

$$\kappa = \frac{1 - \mu}{1 + \mu}$$

Then with equations (A26) and the substitutions

$$\cos\,\xi x = \left(\frac{\pi\xi x}{2}\right)^{1/2} J_{-1/2}(\xi x)$$

$$f(\xi) = -\frac{\xi^{3/2}D}{\Delta}(\cosh\,2\xi h + 1)$$

equations (A24) become

$$\frac{M}{3+\mu}\left(\frac{\pi}{2}\right)^{1/2} x^{-1/2} = \int_0^\infty \xi f(\xi)\left[\frac{\sinh 2\xi h + 2\xi h\left(\frac{1-\mu}{3+\mu}\right)}{\cosh 2\xi h + 1}\right] J_{-1/2}(\xi x)\, d\xi \qquad \left(|x| < 1\right) \tag{A27a}$$

and

$$0 = \int_0^\infty f(\xi) J_{-1/2}(\xi x)\, d\xi \qquad \left(|x| > 1\right) \tag{A27b}$$

Equation (A27b) is satisfied by the assumption

$$f(\xi) = \frac{M}{3+\mu}\, \xi^{-1/2} \sum_{m=0}^{\infty} P_m J_{2m+1}(\xi)$$

and further use of the method of reference 5 yields the familiar equations (A4), where now

$$L_{m,n} = -2(2n+1)\int_0^\infty \frac{1 - 2\xi h\left(\frac{1-\mu}{3+\mu}\right) + e^{-2\xi h}}{\cosh 2\xi h + 1} J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi}$$

The quantity of particular interest here is the moment $M_Y(x,0)$, given by

$$M_Y(x,0) = -\frac{1}{\pi}\int_0^\infty \left[\frac{d^2\bar{w}}{dy^2}(\xi,0) - \mu\xi^2\bar{w}(\xi,0)\right] \cos \xi x\, d\xi$$

With the appropriate substitutions, the moment is found to be

$$M_Y(x,0) = -M \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2-1\right)^{1/2}\left[x + \left(x^2-1\right)^{1/2}\right]^{2m+1}} + \int_0^\infty \frac{1 - 2\xi h\left(\frac{1-\mu}{3+\mu}\right) + e^{-2\xi h}}{\cosh 2\xi h + 1} J_{2m+1}(\xi) \cos \xi x\, d\xi \right\} \qquad (x > 1) \tag{A28}$$

The infinite integral in equation (A28) can be proved convergent at $x = 1$, so that the ratio of the "moment intensity factors" is found to be

$$\frac{K_F}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

which is equivalent to the ratio of the outer-fiber bending stress intensity factors, since the stress is given in terms of the moment by

$$\sigma_y' = \frac{6M_Y'}{t^2}$$

Since the $L_{m,n}$ values depend on Poisson's ratio, so also does the stress intensity factor. It has been computed for four values of μ in the range $0 \leqq \mu \leqq 0.5$. The results are presented in figure 12 where, because of the relative insensitivity of K_F/K_∞ to μ, only the extreme values (for $\mu = 0$ and $\mu = 0.5$) are shown.

APPENDIX B

PROBLEMS INVOLVING CONCENTRATED LOADS ON CRACK FACES

This appendix contains most of the analytical details of the problems concerned with opening of a central crack in a strip by the application of equal and opposite concentrated loads. The problem involving free longitudinal edges has been given detailed treatment in the body of this report. For the sake of brevity, it will be referred to in this appendix, wherever possible, in the discussion of the remaining concentrated load problems.

Crack Opening Mode – Concentrated Forces

Zero normal displacement of longitudinal edges.- In this problem the crack faces are being separated by symmetrically applied concentrated forces of dimensionless magnitude P/Ea. The edges of the strip are free of shear stress and are restrained against normal displacement. The boundary conditions are

$$v(x,h) = 0$$

$$\sigma_{xy}(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\frac{P}{Ea}\,\delta(x) \qquad \left(|x| < 1\right)$$

$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

In terms of $\bar{\phi}$, the transformed boundary conditions are

$$\left.\begin{aligned}
&\frac{d^3\bar{\phi}}{dy^3}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,0) = 0 \\
&\frac{P}{Ea}\,\delta(x) = \frac{1}{\pi}\int_0^\infty \xi^2\bar{\phi}(\xi,0)\cos\xi x\,d\xi \qquad \left(|x| < 1\right) \\
&0 = \int_0^\infty \frac{d^3\bar{\phi}}{dy^3}(\xi,0)\cos\xi x\,\frac{d\xi}{\xi^2} \qquad \left(|x| > 1\right)
\end{aligned}\right\} \qquad \text{(B1)}$$

APPENDIX B

Substitution of equation (13) into the first three of equations (B1) gives

$$\left.\begin{aligned} A &= -\frac{D}{\Delta}\left(e^{2|\xi|h} + 2|\xi|h - 1\right) \\ B &= -\frac{|\xi|D}{\Delta}\left(e^{2|\xi|h} - 1\right) \\ C &= -\frac{D}{\Delta}\left(2|\xi|h + 1 - e^{-2|\xi|h}\right) \end{aligned}\right\} \tag{B2}$$

with

$$\Delta = |\xi|\left(1 - e^{-2|\xi|h}\right)$$

Substitution of the identity (23) and equations (B2) into the last two of equations (B1) yields

$$\left.\begin{aligned} \frac{P}{Ea}\left(\frac{\pi}{2}\right)^{1/2} x^{-1/2}\delta(x) &= \int_0^\infty f(\xi)\, G(\xi)\, J_{-1/2}(\xi x)\, d\xi \qquad &(|x| < 1) \\ 0 &= \int_0^\infty f(\xi)\, J_{-1/2}(\xi x)\, d\xi \qquad &(|x| > 1) \end{aligned}\right\} \tag{B3}$$

In this problem,

$$f(\xi) = -\frac{\xi^{3/2}D}{\Delta}(\cosh 2\xi h - 1)$$

$$G(\xi) = \xi\left(\frac{\sinh 2\xi h + 2\xi h}{\cosh 2\xi h - 1}\right)$$

In accordance with reference 5, it is assumed in equations (B3) that

$$f(\xi) = \frac{P}{Ea}\, \xi^{-1/2} \sum_{m=0}^{\infty} P_m J_{2m+1}(\xi)$$

which satisfies the second of equations (B3) automatically. Further application of the method of reference 5 yields

$$P_n + \sum_{m=0}^{\infty} L_{m,n} P_m = 1 \qquad (n = 0, 1, 2, \ldots) \tag{B4}$$

in which, as usual,

$$L_{m,n} = 2(2n + 1) \int_0^\infty \left[\frac{G(\xi)}{\xi} - 1\right] J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi} \tag{B5}$$

After solution of equations (B4) for a sufficient number of the P_n values, the stress of particular interest is found to be

$$\sigma_y(x,0) = \frac{2P}{\pi Ea} \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{\left(x^2 - 1\right)^{1/2}\left[x + \left(x^2 - 1\right)^{1/2}\right]^{2m+1}} - \int_0^{\infty} \frac{1 + 2\xi h - e^{-2\xi h}}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\} \qquad (x > 1)$$

from which the ratio of stress intensity factors is determined as

$$\frac{K_F}{K_\infty} = 1 + 2 \sum_{m=0}^{\infty} (-1)^m \left(P_m - 1\right) \tag{B6}$$

The normal stress on a longitudinal edge is

$$\sigma_y(x,h) = -\frac{4P}{\pi Ea} \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{\sinh \xi h + \xi h \cosh \xi h}{\cosh 2\xi h - 1} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

It is symmetric in x.

<u>Clamped longitudinal edges</u>.- In this problem, the edges of the strip are completely restrained against normal and tangential displacement. The boundary conditions are

$$u(x,h) = 0$$

$$v(x,h) = 0$$

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_y(x,0) = -\frac{P}{Ea}\delta(x) \qquad \left(|x| < 1\right)$$

$$v(x,0) = 0 \qquad \left(|x| > 1\right)$$

In terms of $\overline{\phi}$, the transformed boundary conditions are

$$\left.\begin{aligned} &\frac{d^2\overline{\phi}}{dy^2}(\xi,h) + \mu\xi^2\overline{\phi}(\xi,h) = 0 \\ &\frac{d^3\overline{\phi}}{dy^3}(\xi,h) - (2 + \mu)\xi^2 \frac{d\overline{\phi}}{dy}(\xi,h) = 0 \\ &\frac{d\overline{\phi}}{dy}(\xi,0) = 0 \end{aligned}\right\}$$

(Equations continued on next page)

$$\left.\begin{aligned}\frac{P}{Ea}\,\delta(x) &= \frac{1}{\pi}\int_0^{\infty} \xi^2\bar{\phi}(\xi,0)\cos \xi x\, d\xi \qquad &(|x| < 1)\\ 0 &= \int_0^{\infty} \frac{d^3\bar{\phi}}{dy^3}(\xi,0)\cos \xi x \frac{d\xi}{\xi^2} \qquad &(|x| > 1)\end{aligned}\right\} \tag{B7}$$

Substitution of equation (13) into the first three of equations (B7) gives

$$\left.\begin{aligned}A &= -\frac{D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{2|\xi|h} + 2\xi^2h^2 - 2|\xi|h + \frac{4\kappa}{1+\mu} + 1\right)\\ B &= -\frac{|\xi|D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{2|\xi|h} - 2|\xi|h + 1\right)\\ C &= -\frac{D}{\Delta}\left(\frac{3-\mu}{1+\mu}e^{-2|\xi|h} + 2\xi^2h^2 + 2|\xi|h + \frac{4\kappa}{1+\mu} + 1\right)\end{aligned}\right\} \tag{B8}$$

where

$$\Delta = |\xi|\left(\frac{3-\mu}{1+\mu}e^{-2|\xi|h} + 2|\xi|h + 1\right)$$

and

$$\kappa = \frac{1-\mu}{1+\mu}$$

By virtue of the identity (23) and equations (B8), the last two of equations (B7) take on the form of equations (B3), in which, for this problem,

$$f(\xi) = \frac{\xi^{3/2}D}{\Delta}\left(\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h\right)$$

$$G(\xi) = \xi\left(\frac{\frac{3-\mu}{1+\mu}\cosh 2\xi h + 2\xi^2h^2 + \frac{4\kappa}{1+\mu} + 1}{\frac{3-\mu}{1+\mu}\sinh 2\xi h - 2\xi h}\right)$$

Further use of the method of reference 5 yields equations (B4) and (B5). The stress of primary interest is

$$\sigma_y(x,0) = \frac{2P}{\pi Ea} \sum_{m=0}^{\infty} P_m \left\{ \frac{(-1)^m}{(x^2 - 1)^{1/2}\left[x + (x^2 - 1)^{1/2}\right]^{2m+1}} \right.$$

$$\left. - \int_0^{\infty} \frac{1 + \frac{4\kappa}{1 + \mu} + 2\xi h + 2\xi^2 h^2 + \frac{3 - \mu}{1 + \mu} e^{-2\xi h}}{\frac{3 - \mu}{1 + \mu} \sinh 2\xi h - 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi \right\} \quad (x > 1)$$

The ratio of stress intensity factors again is given by equation (B6). In this problem, the ratio of stress intensity factors depends on μ because of the clamped-edge conditions. It has been computed for $\mu = 0$ and $\mu = 0.5$.

The normal stress on an edge of the strip is

$$\sigma_y(x,h) = - \frac{4P}{\pi Ea(1 + \mu)^2} \sum_{m=0}^{\infty} P_m \int_0^{\infty} \frac{2 \cosh \xi h + (1 + \mu) \xi h \sinh \xi h}{\frac{3 - \mu}{1 + \mu} \sinh 2\xi h - 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi$$

This stress is symmetric in x.

APPENDIX C

CONVERGENCE PROOF FOR AN INFINITE INTEGRAL

The integral to be investigated for convergence at $x = 1$ is

$$I(1,h) = \int_0^\infty \frac{2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}}{\sinh 2\xi h + 2\xi h} J_{2m+1}(\xi) \cos \xi x \, d\xi \bigg|_{x=1} \qquad (m = 0, 1, 2, \ldots)$$

where h is real and positive. Let the integral be written as

$$I(1,h) = \int_0^\infty H(\xi,h) \, J_{2m+1}(\xi) \cos \xi \, d\xi \tag{C1}$$

The function $H(\xi,h)$ is a positive steadily decreasing function, for all $h > 0$, in the interval $0 < \xi < \infty$. It is well-known that

$$\left| J_{2m+1}(\xi) \right| < 1 \qquad (m = 0, 1, 2, \ldots) \tag{C2}$$

and

$$\left| \cos \xi \right| \leqq 1 \tag{C3}$$

so that

$$\left| J_{2m+1}(\xi) \cos \xi \right| < 1 \tag{C4}$$

Assume that the integral

$$\int_0^\infty H(\xi,h) \, d\xi \tag{C5}$$

exists. If the integral (C5) exists, so does the integral

$$\int_0^\infty H(\xi,h) \left| J_{2m+1}(\xi) \cos \xi \right| d\xi \tag{C6}$$

which follows from a simple comparison test for infinite integrals (see ref. 9, p. 71), by virtue of the inequality (C4). By use of a theorem on absolute convergence of infinite integrals (see ref. 9), it further follows that the integral

$$\int_0^\infty H(\xi,h) \, J_{2m+1}(\xi) \cos \xi \, d\xi$$

exists, if the integral (C6) exists. Therefore, all that remains for proof of convergence of equation (C1) is to prove the convergence of integral (C5).

Now $H(\xi,h)$ can be written as

$$H(\xi,h) = 2e^{-2\xi h}\left(\frac{2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}}{1 + 4\xi he^{-2\xi h} - e^{-4\xi h}}\right)$$

Note that $H(\xi,h) = 1$ at $\xi = 0$. For $\xi > \xi_1$, where ξ_1 is the finite (real) root of

$$\left(4\xi h - e^{-2\xi h}\right)e^{-2\xi h} = 0$$

it is seen that

$$H(\xi,h) < 2e^{-2\xi h}\left(2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}\right)$$

It is then permissible to write, for $\epsilon > 0$,

$$\int_0^\infty H(\xi,h)\,d\xi < \int_0^{\xi_1+\epsilon} H(\xi,h)\,d\xi + 2\int_{\xi_1+\epsilon}^\infty e^{-2\xi h}\left(2\xi^2h^2 + 2\xi h + 1 - e^{-2\xi h}\right)d\xi \tag{C7}$$

Both integrals in the right-hand side of the inequality (C7) are convergent. The first integral is convergent because $H(\xi,h)$ contains no singularities, integrable or otherwise, in the range of integration, and the second integral converges because every term in the integrand is of negative exponential order. Therefore, the integral (C5) exists, and the proof is complete.

REFERENCES

1. Muskhelishvili, N. I. (Radok, J. R. M., trans.): Some Basic Problems of the Mathematical Theory of Elasticity. Third ed., P. Noordhoff, Ltd. (Groningen, The Netherlands), 1953.

2. Sih, G. C.; Paris, P. C.; and Erdogan, F.: Crack-Tip, Stress-Intensity Factors for Plane Extension and Plate Bending Problems. Trans. ASME, Ser. E., J. Appl. Mech., vol. 29, no. 2, June 1962, pp. 306-312.

3. Paris, Paul C.; and Sih, George C.: Stress Analysis of Cracks. Fracture Toughness Testing and Its Applications, Spec. Tech. Publ. No. 381, Am. Soc. Testing Mater. c.1965, pp. 30-83.

4. Sneddon, Ian N.: Fourier Transforms. First ed., McGraw-Hill Book Co., Inc., 1951.

5. Tranter, C. J.: Integral Transforms in Mathematical Physics. Second ed., John Wiley & Sons, Inc., 1956. (Reprinted 1962.)

6. Sokolnikoff, I. S.: Mathematical Theory of Elasticity. Second ed., McGraw-Hill Book Co., Inc., 1956.

7. Magnus, Wilhelm; and Oberhettinger, Fritz (J. Wermer, trans.): Formulas and Theorems for the Special Functions of Mathematical Physics. Chelsea Pub. Co. (New York), 1949.

8. Timoshenko, S.; and Woinowsky-Krieger, S.: Theory of Plates and Shells. Second ed., McGraw-Hill Book Co., Inc., 1959.

9. Whittaker, E. T.; and Watson, G. N.: A Course of Modern Analysis. Fourth ed., Cambridge Univ. Press, 1962.

NASA-Langley, 1967 —— 32 L-5124

8/16/67

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546