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SMITHSONIAN INSTITUTION
ASTROPHYSICAL OBSERVATORY

Research in Space Science

SPECIAL REPORT

Number 229

ON VON ZEIPEL'S METHOD IN GENERAL PLANETARY THEORY

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November 30, 1966

CAMBRIDGE, MASSACHUSETTS, 02138

SAO Special Report No. 229

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ABSTRACT

We eliminate, by Von Zeipel's method, the short-period terms of a first-order general planetary theory. The disturbed planet is referred to the orbital plane of the disturbing planet and the longitudes are counted from the longitude of the ascending node of the disturbed planet.

The short-period terms arising from the indirect part F_{1i} of the disturbing function and those arising from its principal part F_{1p} are eliminated separately. The first terms are calculated using Newcomb operators, the second using Newcomb operators and Laplace coefficients. The powers of eccentricities and mutual inclination higher than the third are neglected.

This elimination reduces the system of canonical equations whose Hamiltonian is F_{1i} to a system whose Hamiltonian F'_{1i} is identically equal to zero. It reduces the system of canonical equations whose Hamiltonian is F_{1p} to a system whose Hamiltonian F'_{1p} is the sum of four secular terms and one long-period term.

A numerical check of our formulas has been carried out in the two particular cases of Jupiter perturbed by Saturn, and Mars perturbed by the Earth.

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ON VON ZEIPEL'S METHOD IN GENERAL PLANETARY THEORY[1]

Jean Meffroy[2]

1. INTRODUCTION

1.1 The purpose of this paper, which develops two previous notes (Meffroy, 1965a, b), is to eliminate, by von Zeipel's method, the short-period terms of a first-order general planetary theory. This elimination is the first step in constructing, by von Zeipel's method, a first-order general planetary theory; the second step deals with the elimination of the long-period terms and that of the secular terms. Von Zeipel's method is an improvement on Delaunay's (1860, 1867), in which the elimination of the short-period terms and that of the long-period terms required a number of subsidiary steps, each step eliminating only one term. It deals with the intensive use of two determining functions the first one eliminates all the short-period terms and transforms the system of canonical equations into a system whose Hamiltonian no longer depends on the mean longitudes; the second one eliminates all the long-period terms and transforms the system of canonical equations whose Hamiltonian is independent of the mean longitude into a system whose Hamiltonian no longer depends on the angular variables. This latter system is solved: its linear variables are constants, and its angular variables are linear functions of time.

1.2 It is in his masterful study of the motion of minor planets perturbed by both Jupiter and the Sun that von Zeipel (1916a, 1916b, 1917) introduced, for the first time, his concept of a determining function. This concept

[1] This work was supported in part by grant NsG 87-60 from the National Aeronautics and Space Administration.

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was elaborated by Brouwer (1959a), who improved on the method used by Delaunay in his lunar theory. Von Zeipel's method was subsequently applied to artificial-satellite theory by Brouwer (1959b), Garfinkel (1959), Hori (1960, 1961), Brouwer and Hori (1961), Kozai (1962), Morando (1962, 1963), Kovalevsky (1964), and Oesterwinter (1965). It was applied by Hori (1963) to the main problem of lunar theory and by Marsden (1966) to the theory of the Galilean satellites of Jupiter.

1.3 In the theory of minor planets, in lunar theory, and in artificial-satellite theory, we deal with only one set of canonical equations, and the determining function that eliminates the short-period terms depends upon six variables. In general planetary theory, we deal with 2, 3, 4, ... sets of canonical equations instead of one, and the determining function that eliminates the short-period terms depends upon 12, 18, 24, ... variables instead of 6 according as the number of the disturbing planets is 1, 2, 3, ... On the other hand, the difficulty of introducing von Zeipel's method in general planetary theory does not lie in the relatively large number of terms we have to consider in the powers of the masses, as we do in lunar theory, but in the expression of the disturbing function itself. This expression is rather simple in lunar theory, but is much more intricate in general planetary theory, where it is divided into two parts — the indirect and the principal parts. The determining function is thus the sum of two terms, one of them arising from the indirect part and the other from the main part. Each of these two terms is checked separately.

1.4 We consider only one disturbing planet and we neglect the powers of the eccentricities and mutual inclination higher than the third. In a first-order theory in which the products of the mass of the disturbing planet and its heliocentric rectangular coordinates are used as corrections to the heliocentric rectangular coordinates of the disturbed planet, we do not need to calculate the indirect part of the disturbing function (Brouwer and Clemence, 1961). Nevertheless, we shall take it into account in order later to eliminate the long-period terms and to build, through von Zeipel's method, a second-order theory.

2. NOTATIONS AND PRELIMINARY CALCULATIONS

We shall adopt the following notations:

S	= Sun,
P_1	= disturbed planet referred to S,
G	= center of mass of S and P_1,
P_2	= disturbing planet referred to G,
r_1	= distance SP_1,
r_2	= distance GP_2,
θ	= angle of vectors $\overrightarrow{SP_1}$ and $\overrightarrow{GP_2}$,
r_{02}	= distance SP_2,
r_{12}	= distance $P_1 P_2$,
a_1	= semimajor axis of osculating ellipse of P_1,
a_2	= semimajor axis of osculating ellipse of P_2,
m_0	= mass of S,
σ	= small parameter of the order of the masses of P_1 and P_2,
β_1, β_2	= finite numerical coefficients,
$\beta_1 \sigma$	= mass of P_1,
$\beta_2 \sigma$	= mass of P_2,
k^2	= constant of gravitation.

The Hamiltonian of the system of canonical equations of P_1 and P_2 is

$$F = \frac{k^2 m_0 \beta_1}{2a_1} + \frac{k^2 m_0 \beta_2}{2a_2} + k^2 m_0 \beta_2 \left(\frac{1}{r_{02}} - \frac{1}{r_2} \right) + \frac{\sigma k^2 \beta_1 \beta_2}{r_{12}} \quad . \tag{1}$$

The Delaunay variables L_1 and L_2 associated with a_1 and a_2 are

$$L_1 = \beta_1 \left(\frac{k^2 m_0^2}{m_0 + \beta_1 \sigma} a_1 \right)^{1/2} ,$$

$$L_2 = \beta_2 \left[\frac{k^2 m_0 (m_0 + \beta_1 \sigma)}{m_0 + (\beta_1 + \beta_2)\sigma} a_2 \right]^{1/2} .$$

Equation (1) then becomes

$$F = \frac{k^4 m_0^2 \beta_1^3}{2L_1^2} \frac{1}{1 + \beta_1 \sigma / m_0} + \frac{k^4 m_0^2 \beta_2^3}{2L_2^2} \frac{1 + \beta_1 \sigma / m_0}{1 + (\beta_1 + \beta_2)\sigma / m_0}$$

$$+ \frac{k^2 m_0 \beta_2}{r_2} \left\{ \frac{1}{\left[1 - 2 \frac{\beta_1 \sigma}{m_0 + \beta_1 \sigma} \left(- \frac{r_1}{r_2} \right) \cos\theta + \left(\frac{\beta_1 \sigma}{m_0 + \beta_1 \sigma} \right)^2 \frac{r_1^2}{r_2^2} \right]^{1/2}} - 1 \right\}$$

$$+ \frac{\sigma k^2 \beta_1 \beta_2}{r_2} \frac{1}{\left[1 - 2 \frac{m_0}{m_0 + \beta_1 \sigma} \frac{r_1}{r_2} \cos\theta + \left(\frac{m_0}{m_0 + \beta_1 \sigma} \right)^2 \frac{r_1^2}{r_2^2} \right]^{1/2}} . \qquad (2)$$

We assume that $(r_1 / r_2) < 1$ and we develop equation (2) into a Taylor series of the powers of the small parameter σ according to the formula

$$F(\sigma) = F(0) + \sigma F'(0) + \frac{\sigma^2}{2} F''(0) + \ldots \quad . \qquad (3)$$

Such an assumption is always realized since the distance between S and P_1 is very small in comparison to the distance between G and P_2 in the in the applications of general planetary theory.

Since we are considering a first-order theory, we reduce (3) to its first two terms $F(0) + \sigma F'(0)$ and we put $F(0) = F_0$, $\sigma F'(0) = F_1$. Equations (2) and (3) show that

$$F_0 = \frac{k^4 m_0^2 \beta_1^3}{2L_1^2} + \frac{k^4 m_0^2 \beta_2^3}{2L_2^2} ,$$

$$F_1 = \sigma \left\{ - \frac{k^4 m_0 \beta_1^4}{2L_1^2} - \frac{k^4 m_0 \beta_2^4}{2L_2^2} + k^2 \beta_1 \beta_2 \left[- \frac{r_1}{r_2^2} \cos\theta + \frac{1}{r_2 \left(1 - 2\frac{r_1}{r_2}\cos\theta + \frac{r_1^2}{r_2^2}\right)^{1/2}} \right] \right\} . \quad (4)$$

Therefore F_1 is the sum of the three terms

$$\sigma \left(- \frac{k^4 m_0 \beta_1^4}{2L_1^2} - \frac{k^4 m_0 \beta_2^4}{2L_2^2} \right) , \quad -\sigma k^2 \beta_1 \beta_2 \frac{r_1}{r_2^2} \cos\theta ,$$

$$\frac{\sigma k^2 \beta_1 \beta_2}{r_2 \left(1 - 2\frac{r_1}{r_2}\cos\theta + \frac{r_1^2}{r_2^2}\right)^{1/2}} .$$

The first term is of the same form as that of F_0, the second one is the indirect part of the disturbing function and we call it F_{1i}, the third one is the principal part of the disturbing function and we call it F_{1p}. Short-period terms of the first order appear only in F_{1i} and in F_{1p}. We calculate F_{1i} by means of Newcomb operators, and F_{1p} by means of Newcomb operators and Laplace coefficients. A calculation of F_{1i} by means of Newcomb operators may seem unneccessary but the use of Newcomb operaotrs will be more imperative when we shall go, later on, into the second-order theory and in order to unify our work, we introduce them systematically in the first-order theory. We choose the orbital plane of P_2 as a plane of reference and we count the longitudes from the longitude of the ascending node of P_1 (Brown and Shook, 1933; see especially chapter 7, p. 179). We shall use the following notation:

ℓ_1 = mean longitude,
g_1 = longitude of the perihelia of P_1,
ℓ_2 = mean longitude,
g_2 = longitude of the perihelia of P_2,
e_1 = eccentricity of the orbital ellipse of P_1,
e_2 = eccentricity of the orbital ellipse of P_2,
I = inclination of the orbital plane of P_1 on the orbital plane of P_2,
τ = $\sin I/2$,
α = a_1/a_2 .

If we neglect powers of e_1, e_2, and τ higher than the third, the expression for F_{1i} is

$$F_{1i} = -\frac{\sigma k^2 \beta_1 \beta_2 \alpha}{a_2}\Bigg[\tau^2 \cos(\ell_1 + \ell_2 + g_1 + g_2)$$

$$+ \left(1 - \tau^2 - \frac{1}{2} e_1^2 - \frac{1}{2} e_2^2\right) \cos(\ell_1 - \ell_2 + g_1 - g_2)$$

$$+ \frac{1}{2} e_1 \tau^2 \cos(2\ell_1 + \ell_2 + g_1 + g_2)$$

$$+ \left(\frac{1}{2} e_1 - \frac{1}{2} e_1 \tau^2 - \frac{3}{8} e_1^3 - \frac{1}{4} e_1 e_2^2\right) \cos(2\ell_1 - \ell_2 + g_1 - g_2)$$

$$+ \left(-\frac{3}{2} e_1 + \frac{3}{2} e_1 \tau^2 + \frac{3}{4} e_1 e_2^2\right) \cos(\ell_2 - g_1 + g_2)$$

$$+ \left(-\frac{3}{2} e_1 \tau^2\right) \cos(\ell_2 + g_1 + g_2)$$

$$+ 2 e_2 \tau^2 \cos(\ell_1 + 2\ell_2 + g_1 + g_2)$$

$$+\left(2e_2 - 2e_2\tau^2 - e_1^2 e_2\right)\cos(-\ell_1 + 2\ell_2 - g_1 + g_2)$$

$$+\frac{3}{8}e_1^2\cos(3\ell_1 - \ell_2 + g_1 - g_2)$$

$$+\frac{1}{8}e_1^2\cos(\ell_1 + \ell_2 - g_1 + g_2)$$

$$+(-3e_1e_2)\cos(2\ell_2 - g_1 + g_2)$$

$$+e_1e_2\cos(2\ell_1 - 2\ell_2 + g_1 - g_2)$$

$$+\frac{1}{8}e_2^2\cos(\ell_1 + \ell_2 + g_1 - g_2)$$

$$+\frac{27}{8}e_2^2\cos(\ell_1 - 3\ell_2 + g_1 - g_2)$$

$$+\frac{1}{3}e_1^3\cos(4\ell_1 - \ell_2 + g_1 - g_2)$$

$$+\frac{1}{24}e_1^3\cos(2\ell_1 + \ell_2 - g_1 + g_2)$$

$$+\frac{1}{4}e_1^2e_2\cos(\ell_1 + 2\ell_2 - g_1 + g_2)$$

$$+\frac{3}{4}e_1^2e_2\cos(3\ell_1 - 2\ell_2 + g_1 - g_2)$$

$$+\frac{1}{16}e_1e_2^2\cos(2\ell_1 + \ell_2 + g_1 - g_2)$$

$$+\left(-\frac{81}{16}e_1e_2^2\right)\cos(3\ell_2 - g_1 + g_2)$$

$$+\frac{27}{16}e_1e_2^2\cos(2\ell_1 - 3\ell_2 + g_1 - g_2)$$

$$+\left(-\frac{3}{16}e_1e_2^2\right)\cos(-\ell_2 - g_1 + g_2)$$

$$+\frac{1}{6}e_2^3\cos(\ell_1 + 2\ell_2 + g_1 - g_2)$$

$$\left.+\frac{16}{3}e_2^3\cos(-\ell_1 + 4\ell_2 - g_1 + g_2)\right] , \qquad (5)$$

and that for F_{1p} is

$$
\begin{aligned}
F_{1p} = \frac{\sigma k^2 \beta_1 \beta_2}{a_2} \Bigg(& \frac{1}{2} b_{1/2}^{(0)} \\
& + e_1^2 \left(\frac{1}{8} D + \frac{1}{8} D^2 \right) b_{1/2}^{(0)} \\
& + e_2^2 \left(\frac{1}{8} D + \frac{1}{8} D^2 \right) b_{1/2}^{(0)} \\
& + \tau^2 \left(- \frac{\alpha}{2} \right) b_{3/2}^{(1)} \\
& + \Bigg[b_{1/2}^{(1)} \\
& + e_1^2 \left(- 1 + \frac{1}{4} D + \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \\
& + e_2^2 \left(- 1 + \frac{1}{4} D + \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \\
& + \tau^2 \left(- \frac{\alpha}{2} b_{3/2}^{(0)} - \frac{\alpha}{2} b_{3/2}^{(2)} \right) \Bigg] \cos \left(- \ell_1 + \ell_2 - g_1 + g_2 \right) \\
& + \Bigg\{ e_1 \left(- 1 - \frac{1}{2} D \right) b_{1/2}^{(1)} \\
& + e_1 \tau^2 \left[\left(\frac{3\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(0)} + \left(\frac{3\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(2)} \right] \\
& + e_1^3 \left(\frac{1}{8} D - \frac{1}{16} D^2 - \frac{1}{16} D^3 \right) b_{1/2}^{(1)} \\
& + e_1 e_2^2 \left(1 + \frac{1}{4} D - \frac{3}{8} D^2 - \frac{1}{8} D^3 \right) b_{1/2}^{(1)} \Bigg\} \cos \left(\ell_2 - g_1 + g_2 \right)
\end{aligned}
$$

$$+\left[e_1\left(-\frac{1}{2}D\right)b_{1/2}^{(0)}\right.$$

$$+e_1\tau^2\left(\frac{\alpha}{2}+\frac{\alpha}{2}D\right)b_{3/2}^{(1)}$$

$$+e_1^3\left(\frac{3}{16}D+\frac{1}{16}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(0)}$$

$$\left.+e_1e_2^2\left(-\frac{1}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right]\cos\ell_1$$

$$+\left\{e_1\left(1-\frac{1}{2}D\right)b_{1/2}^{(1)}\right.$$

$$+e_1\tau^2\left[\left(-\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(0)}+\left(-\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(2)}\right]$$

$$+e_1^3\left(-\frac{5}{4}+\frac{3}{4}D+\frac{3}{16}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(1)}$$

$$\left.+e_1e_2^2\left(-1+\frac{3}{4}D+\frac{1}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right\}\cos(2\ell_1-\ell_2+g_1-g_2)$$

$$+\left\{e_2\left(\frac{3}{2}+\frac{1}{2}D\right)b_{1/2}^{(1)}\right.$$

$$+e_2\tau^2\left[\left(-\alpha-\frac{\alpha}{4}D\right)b_{3/2}^{(0)}+\left(-\alpha-\frac{\alpha}{4}D\right)b_{3/2}^{(2)}\right]$$

$$+e_1^2e_2\left(-\frac{3}{2}-\frac{1}{8}D+\frac{1}{2}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(1)}$$

$$\left.+e_2^3\left(-\frac{7}{4}-\frac{3}{16}D+\frac{3}{8}D^2+\frac{1}{16}D^3\right)b_{1/2}^{(1)}\right\}\cos(-\ell_1+2\ell_2-g_1+g_2)$$

$$+\left[\left(\frac{1}{2}+\frac{1}{2}D\right)b_{1/2}^{(0)}\right.$$

$$+e_2\tau^2\left(-\alpha-\frac{\alpha}{2}D\right)b_{3/2}^{(1)}$$

$$+e_1^2e_2\left(\frac{1}{8}D+\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(0)}$$

$$\left.+e_2^3\left(-\frac{1}{16}+\frac{1}{8}D+\frac{1}{4}D^2+\frac{1}{16}D^3\right)b_{1/2}^{(0)}\right]\cos\ell_2$$

$$+ \left[e_2 \left(-\frac{1}{2} + \frac{1}{2} D \right) b_{1/2}^{(1)} \right.$$

$$+ e_2 \tau^2 \left(-\frac{\alpha}{4} D\, b_{3/2}^{(0)} - \frac{\alpha}{4} D\, b_{3/2}^{(2)} \right)$$

$$+ e_1^2 e_2 \left(\frac{1}{2} - \frac{5}{8} D + \frac{1}{8} D^3 \right) b_{1/2}^{(1)}$$

$$\left. + e_2^3 \left(-\frac{1}{8} - \frac{1}{16} D + \frac{1}{8} D^2 + \frac{1}{16} D^3 \right) b_{1/2}^{(1)} \right] \cos (\ell_1 + g_1 - g_2)$$

$$+ e_1^2 \left(-\frac{1}{8} + \frac{1}{8} D + \frac{1}{8} D^2 \right) b_{1/2}^{(1)} \cos (\ell_1 + \ell_2 - g_1 + g_2)$$

$$+ e_1^2 \left(-\frac{3}{8} D + \frac{1}{8} D^2 \right) b_{1/2}^{(0)} \cos 2\ell_1$$

$$+ e_1^2 \left(\frac{9}{8} - \frac{7}{8} D + \frac{1}{8} D^2 \right) b_{1/2}^{(1)} \cos (3\ell_1 - \ell_2 + g_1 - g_2)$$

$$+ e_1 e_2 \left(-\frac{3}{2} - \frac{5}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \cos (2\ell_2 - g_1 + g_2)$$

$$+ e_1 e_2 \left(-\frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(0)} \cos (\ell_1 + \ell_2)$$

$$+ e_1 e_2 \left(-\frac{1}{2} + \frac{3}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \cos (2\ell_1 + g_1 - g_2)$$

$$+ e_1 e_2 \left(\frac{1}{2} - \frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \cos (- g_1 + g_2)$$

$$+ e_1 e_2 \left(-\frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(0)} \cos (\ell_1 - \ell_2)$$

$$+ e_1 e_2 \left(\frac{3}{2} - \frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(1)} \cos (2\ell_1 - 2\ell_2 + g_1 - g_2)$$

$$+ \ e_2^2\left(\frac{17}{8} + \frac{9}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \cos\,(-\ell_1 + 3\ell_2 - g_1 + g_2)$$

$$+ \ e_2^2\left(\frac{1}{2} + \frac{5}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(0)} \cos\, 2\ell_2$$

$$+ \ e_2^2\left(-\frac{1}{8} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \cos\,(\ell_1 + \ell_2 + g_1 - g_2)$$

$$+ \ \tau^2 \frac{a}{2} b_{3/2}^{(1)} \cos\,(-2\ell_1 - 2g_1)$$

$$+ \ \tau^2 \frac{a}{2} b_{3/2}^{(0)} \cos\,(-\ell_1 - \ell_2 - g_1 - g_2)$$

$$+ \ \tau^2 \frac{a}{2} b_{3/2}^{(1)} \cos\,(-2\ell_2 - 2g_2)$$

$$+ \ e_1\tau^2\left(-\frac{5a}{4} - \frac{a}{4} D\right) b_{3/2}^{(1)} \cos\,(-\ell_1 - 2g_1)$$

$$+ \ e_1\tau^2\left(-\frac{3a}{4} - \frac{a}{4} D\right) b_{3/2}^{(0)} \cos\,(-\ell_2 - g_1 - g_2)$$

$$+ \ e_1\tau^2\left(-\frac{a}{4} - \frac{a}{4} D\right) b_{3/2}^{(1)} \cos\,(\ell_1 - 2\ell_2 - 2g_2)$$

$$+ \ e_1\tau^2\left(-\frac{a}{4} - \frac{a}{4} D\right) b_{3/2}^{(1)} \cos\,(\ell_1 + 2\ell_2 + 2g_2)$$

$$+ \ e_1\tau^2\left(\frac{a}{4} - \frac{a}{4} D\right) b_{3/2}^{(0)} \cos\,(2\ell_1 + \ell_2 + g_1 + g_2)$$

$$+ \ e_1\tau^2\left(\frac{3a}{4} - \frac{a}{4} D\right) b_{3/2}^{(1)} \cos\,(3\ell_1 + 2g_1)$$

$$+ \ e_2\tau^2\left(\frac{a}{2} + \frac{a}{4} D\right) b_{3/2}^{(1)} \cos\,(-2\ell_1 + \ell_2 - 2g_1)$$

$$+ e_2 \tau^2 \frac{a}{4} D\, b_{3/2}^{(0)} \cos(-\ell_1 - g_1 - g_2)$$

$$+ e_2 \tau^2 \left(-\frac{a}{2} + \frac{a}{4} D\right) b_{3/2}^{(1)} \cos(-\ell_2 - 2g_2)$$

$$+ e_2 \tau^2 \left(\frac{3a}{2} + \frac{a}{4} D\right) b_{3/2}^{(1)} \cos(3\ell_2 + 2g_2)$$

$$+ e_2 \tau^2 \left(a + \frac{a}{4} D\right) b_{3/2}^{(0)} \cos(\ell_1 + 2\ell_2 + g_1 + g_2)$$

$$+ e_2 \tau^2 \left(\frac{a}{2} + \frac{a}{4} D\right) b_{3/2}^{(1)} \cos(2\ell_1 + \ell_2 + 2g_1)$$

$$+ e_1^3 \left(-\frac{1}{12} + \frac{1}{12} D + \frac{1}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(1)} \cos(2\ell_1 + \ell_2 - g_1 + g_2)$$

$$+ e_1^3 \left(-\frac{17}{48} D + \frac{3}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(0)} \cos 3\ell_1$$

$$+ e_1^3 \left(\frac{4}{3} - \frac{31}{24} D + \frac{5}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(1)} \cos(4\ell_1 - \ell_2 + g_1 - g_2)$$

$$+ e_1^2 e_2 \left(-\frac{3}{16} + \frac{1}{8} D + \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos(\ell_1 + 2\ell_2 - g_1 + g_2)$$

$$+ e_1^2 e_2 \left(-\frac{3}{16} D - \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \cos(2\ell_1 + \ell_2)$$

$$+ e_1^2 e_2 \left(-\frac{9}{16} + D - \frac{1}{2} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos(3\ell_1 + g_1 - g_2)$$

$$+ e_1^2 e_2 \left(\frac{1}{16} - \frac{1}{8} D + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos(\ell_1 - g_1 + g_2)$$

$$+ e_1^2 e_2 \left(-\frac{3}{16} D - \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \cos(2\ell_1 - \ell_2)$$

$$
\begin{aligned}
&+ e_1^2 e_2\left(\frac{27}{16} - \frac{3}{4} D - \frac{1}{4} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos (3\ell_1 - 2\ell_2 + g_1 - g_2) \\
&+ e_1 e_2^2\left(- \frac{17}{8} - \frac{35}{16} D - \frac{11}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos (3\ell_2 - g_1 + g_2) \\
&+ e_1 e_2^2\left(- \frac{1}{4} D - \frac{5}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \cos (\ell_1 + 2\ell_2) \\
&+ e_1 e_2^2\left(- \frac{1}{8} + \frac{3}{16} D + \frac{1}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos (2\ell_1 + \ell_2 + g_1 - g_2) \\
&+ e_1 e_2^2 \ \frac{1}{8} - \frac{1}{16} D - \frac{3}{16} D^2 - \frac{1}{16} D^3 \Big) b_{1/2}^{(1)} \cos (- \ell_2 - g_1 + g_2) \\
&+ e_1 e_2^2\left(- \frac{1}{4} D - \frac{5}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \cos (\ell_1 - 2\ell_2) \\
&+ e_1 e_2^2\left(\frac{17}{8} + \frac{1}{16} D - \frac{7}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \cos (2\ell_1 - 3\ell_2 + g_1 - g_2) \\
&+ e_2^3 \ \frac{71}{24} + \frac{95}{48} D + \frac{3}{8} D^2 + \frac{1}{48} D^3 \Big) b_{1/2}^{(1)} \cos (- \ell_1 + 4\ell_2 - g_1 + g_2) \\
&+ e_2^3\left(\frac{27}{48} + \frac{19}{24} D + \frac{1}{4} D^2 + \frac{1}{48} D^3\right) b_{1/2}^{(0)} \cos 3\ell_2 \\
&+ e_2^3\left(- \frac{1}{12} + \frac{5}{48} D + \frac{1}{8} D^2 + \frac{1}{48} D^3\right) b_{1/2}^{(1)} \cos (\ell_1 + 2\ell_2 + g_1 - g_2) \Bigg) , \qquad (6)
\end{aligned}
$$

where D is the operator $a(d/da)$, and $b_{1/2}^{(0)}$, $b_{1/2}^{(1)}$, $b_{3/2}^{(0)}$, $b_{3/2}^{(1)}$, $b_{3/2}^{(2)}$ are Laplace coefficients; F_{1i} is a sum of 24 terms, and F_{1p} is a sum of 53 terms. The first 20 and the last 18 terms of F_{1p} are terms of class zero in the Newcomb sense; the other terms of F_{1p} are terms of class one in the Newcomb sense.

3. VON ZEIPEL'S METHOD

We start from the set of Delaunay's variables $L_i, G_i, H_i, \ell_i, g_i, h_i$ $(i = 1, 2)$ in which the index 1 refers to P_1 and the index 2 refers to P_2. We operate the canonical change of variables

$$L_1, L_2, G_1, G_2, H_1, H_2, \ell_1, \ell_2, g_1, g_2, h_1, h_2$$
$$\downarrow$$
$$L'_1, L'_2, G'_1, G'_2, H'_1, H'_2, \ell'_1, \ell'_2, g'_1, g'_2, h'_1, h'_2 \ .$$

We assume that the new Hamiltonian F' depends no longer on the mean anomalies ℓ'_1, ℓ'_2 and that it is connected to the old Hamiltonian F through the equality:

$$\begin{aligned} &F(L_1, L_2, G_1, G_2, H_1, H_2, \ell_1, \ell_2, g_1, g_2, h_1, h_2) \\ &= F'(L'_1, L'_2, G'_1, G'_2, H'_1, H'_2, g'_1, g'_2, h'_1, h'_2) \ . \end{aligned} \tag{7}$$

We call $S(L'_1, L'_2, G'_1, G'_2, H'_1, H'_2, \ell_1, \ell_2, g_1, g_2, h_1, h_2)$ the determining function that eliminates the short-period terms, and we put

$$\begin{aligned} &\frac{\partial S}{\partial L'_i} = \ell'_i, \ \frac{\partial S}{\partial G'_i} = g'_i, \ \frac{\partial S}{\partial H'_i} = h'_i, \\ &\frac{\partial S}{\partial \ell_i} = L_i, \ \frac{\partial S}{\partial g_i} = G_i, \ \frac{\partial S}{\partial h_i} = H_i \ (i = 1, 2) \ . \end{aligned} \tag{8}$$

We put $S = S_0 + S_1 + S_2 + \ldots$, where S_0, S_1, S_2, ... are, respectively, of degrees 0, 1, 2, ... with respect to σ, and we assume that

$$S_0 = L'_1\ell_1 + L'_2\ell_2 + G'_1 g_1 + G'_2 g_2 + H'_1 h_1 + H'_2 h_2 \ . \tag{9}$$

Equations (7), (8), and (9) show that

$$F\left(L_1' + \frac{\partial S_1}{\partial \ell_1},\ L_2' + \frac{\partial S_1}{\partial \ell_2},\ G_1' + \frac{\partial S_1}{\partial g_1},\ G_2' + \frac{\partial S_1}{\partial g_2},\right.$$
$$\left. H_1' + \frac{\partial S_1}{\partial h_1},\ H_2' + \frac{\partial S_1}{\partial h_2},\ \ell_1,\ \ell_2,\ g_1,\ g_2,\ h_1,\ h_2\right)$$
$$= F'\left(L_1', L_2', G_1', G_2', H_1', H_2',\ g_1 + \frac{\partial S_1}{\partial G_1'},\ g_2 + \frac{\partial S_1}{\partial G_2'},\right.$$
$$\left. h_1 + \frac{\partial S_1}{\partial H_1'},\ h_2 + \frac{\partial S_1}{\partial H_2'}\right) . \tag{10}$$

If we develop the first member of (10) in a Taylor series according to the powers of $\partial S_1/\partial \ell_1, \ldots , \partial S_1/\partial h_2$, the second member of (10) in a Taylor series according to the powers of $\partial S_1/\partial G_1', \ldots , \partial S_1/\partial H_2'$, and if we restrict these two developments to their terms of degrees 0 and 1, then we obtain

$$F(L_1', L_2', G_1', G_2', H_1', H_2', \ell_1, \ell_2, g_1, g_2, h_1, h_2)$$
$$+ \frac{\partial S_1}{\partial \ell_1}\frac{\partial F}{\partial L_1'} + \frac{\partial S_1}{\partial \ell_2}\frac{\partial F}{\partial L_2'} + \frac{\partial S_1}{\partial g_1}\frac{\partial F}{\partial G_1'} + \frac{\partial S_1}{\partial g_2}\frac{\partial F}{\partial G_2'} + \frac{\partial S_1}{\partial h_1}\frac{\partial F}{\partial H_1'}$$
$$+ \frac{\partial S_1}{\partial h_2}\frac{\partial F}{\partial H_2'} = F'(L_1', L_2', G_1', G_2', H_1', H_2', g_1, g_2, h_1, h_2)$$
$$+ \frac{\partial S_1}{\partial G_1'}\frac{\partial F'}{\partial g_1} + \frac{\partial S_1}{\partial G_2'}\frac{\partial F'}{\partial g_2} + \frac{\partial S_1}{\partial H_1'}\frac{\partial F'}{\partial h_1} + \frac{\partial S_1}{\partial H_2'}\frac{\partial F'}{\partial h_2} \quad ; \tag{11}$$

F_0 being the set of the terms of F and F_0' the set of the terms of F' of degree zero with respect to σ, F_1 the set of the terms of F and F_1' the set of the terms of F' of degree one with respect to σ. We replace in (11) F by $F_0 + F_1$, F' by $F_0' + F_1'$; and we equate in the two members of (11) the terms of degree zero and the terms of degree one with respect to σ. Then (11) splits into the two equations

$$F_0 = F_0' \quad , \tag{12}$$

$$F_1 + \frac{\partial S_1}{\partial \ell_1}\frac{\partial F_0}{\partial L_1'} + \frac{\partial S_1}{\partial \ell_2}\frac{\partial F_0}{\partial L_2'} = F_1' \quad . \tag{13}$$

We put $F_1 = F_1^* + \overline{F}_1 + \hat{F}_1$, $F_1' = \overline{F}_1' + \hat{F}_1'$, where F_1^* is the set of the short-period terms of F_1, $\overline{F}_1$ the set of the long-period terms of F_1, $\hat{F}_1$ the set of the secular terms of F_1, $\overline{F}_1'$ the set of the long-period terms of F_1', and $\hat{F}_1'$ the set of the secular terms of F_1'. Then (13) becomes

$$F_1^* + \overline{F}_1 + \hat{F}_1 + \frac{\partial S_1}{\partial \ell_1}\frac{\partial F_0}{\partial L_1'} + \frac{\partial S_1}{\partial \ell_2}\frac{\partial F_0}{\partial L_2'} = \overline{F}_1' + \hat{F}_1' \quad . \tag{14}$$

Equating in the two members of (14) the short-period terms, the long-period terms, and the secular terms, we have

$$F_1^* + \frac{\partial S_1}{\partial \ell_1}\frac{\partial F_0}{\partial L_1'} + \frac{\partial S_1}{\partial \ell_2}\frac{\partial F_0}{\partial L_2'} = 0, \qquad \overline{F}_1 = \overline{F}_1', \qquad \hat{F}_1 = \hat{F}_1' \quad . \tag{15}$$

Let us put $S_1 = S_{1i} + S_{1p}$, where S_{1i} is the set of the short-period terms of S_1 arising from F_{1i}, and S_{1p} the set of the short-period terms of S_1 arising from F_{1p}. The first of equations (15) breaks up into the following two linear partial differential equations of the first order:

$$F_{1i}^* + \frac{\partial S_{1i}}{\partial \ell_1}\frac{\partial F_0}{\partial L_1'} + \frac{\partial S_{1i}}{\partial \ell_2}\frac{\partial F_0}{\partial L_2'} = 0 \quad , \tag{16}$$

$$F_{1p}^* + \frac{\partial S_{1p}}{\partial \ell_1}\frac{\partial F_0}{\partial L_1'} + \frac{\partial S_{1p}}{\partial \ell_2}\frac{\partial F_0}{\partial L_2'} = 0 \quad . \tag{17}$$

On the other hand, the equation $F'_1 = \overline{F}'_1 + \hat{F}'_1$ and the last two equations (15) show that $F'_1 = \overline{F}_1 + \hat{F}_1$. Equations (5), (6), and this last equation show that

$$F'_{1i} \equiv 0 \quad , \tag{18}$$

$$\begin{aligned} F'_{1p} = \frac{\sigma k \beta_1 \beta_2}{a'_2} \Bigg[& \frac{1}{2} b^{(0)}_{1/2} + {e'_1}^2 \left(\frac{1}{8} D + \frac{1}{8} D^2 \right) b^{(0)}_{1/2} \\ & + {e'_2}^2 \left(\frac{1}{8} D + \frac{1}{8} D^2 \right) b^{(0)}_{1/2} + {\tau'}^2 \left(- \frac{\alpha}{2} \right) b^{(1)}_{3/2} \\ & + e'_1 e'_2 \left(\frac{1}{2} - \frac{1}{4} D - \frac{1}{4} D^2 \right) b^{(1)}_{1/2} \cos \left(- g'_1 + g'_2 \right) \Bigg] \quad . \end{aligned} \tag{19}$$

4. ELIMINATION OF THE SHORT-PERIOD TERMS ARISING FROM F_{1p}

4.1 We see from (6) that F^*_{1p} is the sum of 51 terms. To each of these terms there is a corresponding equation (17), which may be written

$$A \frac{\partial S_{1p}}{\partial \ell_1} + B \frac{\partial S_{1p}}{\partial \ell_2} = C \cos (p\ell_1 + q\ell_2 + yg_1 + zg_2) \quad , \qquad (20)$$

where A, B, C are quantities independent of ℓ_1 and ℓ_2 and are respectively equal to

$$- \frac{k^4 m_0^2 \beta_1^3}{L_1'^3}, \qquad - \frac{k^4 m_0^2 \beta_2^3}{L_2'^3}, \qquad - \frac{\sigma k^2 \beta_1 \beta_2}{a_2'} f(\alpha)\, g(e_1', e_2', \tau'),$$

and where p, q, y, z are relative integers.

We consider the particular solution of (20),

$$S_{1p} = \frac{C}{Ap + Bq} \sin (p\ell_1 + q\ell_2 + yg_1 + zg_2) \quad , \qquad (21)$$

with $Ap + Bq \neq 0$. Since we restrict ourselves to a first-order theory, we have

$$L_1' \sim k\beta_1 \sqrt{m_0 a_1'} \, , \qquad L_2' \sim k\beta_2 \sqrt{m_0 a_2'} \, ,$$

whence

$$- \frac{k^4 m_0^2 \beta_1^3}{L_1'^3} \sim - \frac{k m_0^{1/2}}{a_1'^{3/2}} , \qquad - \frac{k^4 m_0^2 \beta_2^3}{L_2'^3} \sim - \frac{k m_0^{1/2}}{a_2'^{3/2}} ,$$

and (21) becomes

$$S_{1p} = \frac{\sigma k \beta_1 \beta_2}{m_0^{1/2}} a_1'^{1/2} \alpha \frac{f(\alpha)\ g(e_1', e_2', \tau')}{p + q\alpha^{3/2}} \sin(p\ell_1 + q\ell_2 + yg_1 + zg_2) \quad . \; (22)$$

Each solution (22) is characterized by the six quantities $f(\alpha)$, $g(e_1', e_2', \tau')$, p, q, y, z. We calculate the 51 solutions thus obtained. The sum of these 51 solutions is S_{1p}. We have:

$$\begin{aligned}
S_{1p} = \frac{\sigma k \beta_1 \beta_2}{m_0^{1/2}} a_1'^{1/2} \alpha \Bigg(\Bigg[& b_{1/2}^{(1)} \\
& + e_1'^2 \left(-1 + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)} \\
& + e_2'^2 \left(-1 + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)} \\
& + \tau'^2 \left(-\frac{\alpha}{2} b_{3/2}^{(0)} - \frac{\alpha}{2} b_{3/2}^{(2)}\right)\Bigg] \frac{\sin(-\ell_1 + \ell_2 - g_1 + g_2)}{-1 + \alpha^{3/2}} \\
& + \Bigg\{ e_1' \left(-1 - \frac{1}{2} D\right) b_{1/2}^{(1)} \\
& + e_1' \tau'^2 \left[\left(\frac{3\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(\frac{3\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right] \\
& + e_1'^3 \left(\frac{1}{8} D - \frac{1}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \\
& + e_1' e_2'^2 \left(1 + \frac{1}{4} D - \frac{3}{8} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \Bigg\} \frac{\sin(\ell_2 - g_1 + g_2)}{\alpha^{3/2}} \\
& + \Bigg[e_1' \left(-\frac{1}{2} D\, b_{1/2}^{(0)}\right) \\
& + e_1' \tau'^2 \left(\frac{\alpha}{2} + \frac{\alpha}{2} D\right) b_{3/2}^{(1)} \\
& + e_1'^3 \left(\frac{3}{16} D + \frac{1}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)}
\end{aligned}$$

$$+ e_1' e_2'^2 \left(- \frac{1}{8} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(0)}\Big] \sin \ell_1$$

$$+ \Big\{ e_1' \left(1 - \frac{1}{2} D\right) b_{1/2}^{(1)}$$

$$+ e_1' \tau'^2 \left[\left(- \frac{\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(- \frac{\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]$$

$$+ e_1'^3 \left(- \frac{5}{4} + \frac{3}{4} D + \frac{3}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)}$$

$$+ e_1' e_2'^2 \left(- 1 + \frac{3}{4} D + \frac{1}{8} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \Big\} \frac{\sin (2\ell_1 - \ell_2 + g_2 - g_2)}{2 - \alpha^{3/2}}$$

$$+ \Big\{ e_2' \left(\frac{3}{2} + \frac{1}{2} D\right) b_{1/2}^{(1)}$$

$$+ e_2' \tau'^2 \left[\left(- \alpha - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(- \alpha - \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]$$

$$+ e_1'^2 e_2' \left(- \frac{3}{2} - \frac{1}{8} D + \frac{1}{2} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)}$$

$$+ e_2'^3 \left(- \frac{7}{4} - \frac{3}{16} D + \frac{3}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \Big\} \frac{\sin (- \ell_1 + 2\ell_2 - g_1 + g_2)}{- 1 + 2\alpha^{3/2}}$$

$$+ \Big[e_2' \left(\frac{1}{2} + \frac{1}{2} D\right) b_{1/2}^{(0)}$$

$$+ e_2' \tau'^2 \left(- \alpha - \frac{\alpha}{2} D\right) b_{3/2}^{(1)}$$

$$+ e_1'^2 e_2' \left(\frac{1}{8} D + \frac{1}{4} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(0)}$$

$$+ e_2'^3 \left(- \frac{1}{16} + \frac{1}{8} D + \frac{1}{4} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \Big] \frac{\sin \ell_2}{\alpha^{3/2}}$$

$$+\Bigg[e_2' \left(-\frac{1}{2} + \frac{1}{2} D\right) b_{1/2}^{(1)}$$

$$+ e_2' \tau'^2 \left(-\frac{\alpha}{4} D\, b_{3/2}^{(0)} - \frac{\alpha}{4} D\, b_{3/2}^{(2)}\right)$$

$$+ e_1'^2 e_2' \left(\frac{1}{2} - \frac{5}{8} D + \frac{1}{8} D^3\right) b_{1/2}^{(1)}$$

$$+ e_2'^3 \left(-\frac{1}{8} - \frac{1}{16} D + \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \Bigg] \sin(\ell_1 + g_1 - g_2)$$

$$+ e_1'^2 \left(-\frac{1}{8} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \frac{\sin(\ell_1 + \ell_2 - g_1 + g_2)}{1 + \alpha^{3/2}}$$

$$+ e_1'^2 \left(-\frac{3}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(0)} \frac{\sin 2\ell_1}{2}$$

$$+ e_1'^2 \left(\frac{9}{8} - \frac{7}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \frac{\sin(3\ell_1 - \ell_2 + g_1 - g_2)}{3 - \alpha^{3/2}}$$

$$+ e_1' e_2' \left(-\frac{3}{2} - \frac{5}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin(2\ell_2 - g_1 + g_2)}{2\alpha^{3/2}}$$

$$+ e_1' e_2' \left(-\frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)} \frac{\sin(\ell_1 + \ell_2)}{1 + \alpha^{3/2}}$$

$$+ e_1' e_2' \left(-\frac{1}{2} + \frac{3}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 + g_1 - g_2)}{2}$$

$$+ e_1' e_2' \left(-\frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)} \frac{\sin(\ell_1 - \ell_2)}{1 - \alpha^{3/2}}$$

$$+ e_1' e_2' \left(\frac{3}{2} - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{2 - 2\alpha^{3/2}}$$

$$+ e_2'^2 \left(\frac{17}{8} + \frac{9}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \frac{\sin(-\ell_1 + 3\ell_2 - g_1 + g_2)}{-1 + 3\alpha^{3/2}}$$

$$+ e_2'^2 \left(\frac{1}{2} + \frac{5}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(0)} \frac{\sin 2\ell_2}{2\alpha^{3/2}}$$

$$+ e_2'^2 \left(- \frac{1}{8} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(1)} \frac{\sin (\ell_1 + \ell_2 + g_1 - g_2)}{1 + \alpha^{3/2}}$$

$$+ \tau'^2 \frac{\alpha}{2} b_{3/2}^{(1)} \frac{\sin (-2\ell_1 - 2g_1)}{-2}$$

$$+ \tau'^2 \frac{\alpha}{2} b_{3/2}^{(0)} \frac{\sin (-\ell_1 - \ell_2 - g_1 - g_2)}{-1 - \alpha^{3/2}}$$

$$+ \tau'^2 \frac{\alpha}{2} b_{3/2}^{(1)} \frac{\sin (-2\ell_2 - 2g_2)}{-2\alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(- \frac{5\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (-\ell_1 - 2g_1)}{-1}$$

$$+ e_1' \tau'^2 \left(- \frac{3\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} \frac{\sin (-\ell_2 - g_1 - g_2)}{-\alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(- \frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (\ell_1 - 2\ell_2 - 2g_2)}{1 - 2\alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(- \frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (\ell_1 + 2\ell_2 + 2g_2)}{1 + 2\alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(\frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} \frac{\sin (2\ell_1 + \ell_2 + g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(\frac{3\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (3\ell_1 + 2g_1)}{3}$$

$$+ e_2' \tau'^2 \left(\frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (-2\ell_1 + \ell_2 - 2g_1)}{-2 + \alpha^{3/2}}$$

$$+ e_2' \tau'^2 \; \frac{\alpha}{4} D \, b_{3/2}^{(0)} \; \frac{\sin(-\ell_1 - g_1 - g_2)}{-1}$$

$$+ e_2' \tau'^2 \left(-\frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \; \frac{\sin(-\ell_2 - 2g_2)}{-\alpha^{3/2}}$$

$$+ e_2' \tau'^2 \left(\frac{3\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \; \frac{\sin(3\ell_2 + 2g_2)}{3\alpha^{3/2}}$$

$$+ e_2' \tau'^2 \left(\alpha + \frac{\alpha}{4} D\right) b_{3/2}^{(0)} \; \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$+ e_2' \tau'^2 \left(\frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \; \frac{\sin(2\ell_1 + \ell_2 + 2g_1)}{2 + \alpha^{3/2}}$$

$$+ e_1'^3 \left(-\frac{1}{12} + \frac{1}{12} D + \frac{1}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(1)} \; \frac{\sin(2\ell_1 + \ell_2 - g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$+ e_1'^3 \left(-\frac{17}{48} D + \frac{3}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(0)} \; \frac{\sin 3\ell_1}{3}$$

$$+ e_1'^3 \left(\frac{4}{3} - \frac{31}{24} D + \frac{5}{16} D^2 - \frac{1}{48} D^3\right) b_{1/2}^{(1)} \; \frac{\sin(4\ell_1 - \ell_2 + g_1 - g_2)}{4 - \alpha^{3/2}}$$

$$+ e_1'^2 e_2' \left(-\frac{3}{16} + \frac{1}{8} D + \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \; \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$+ e_1'^2 e_2' \left(-\frac{3}{16} D - \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \; \frac{\sin(2\ell_1 + \ell_2)}{2 + \alpha^{3/2}}$$

$$+ e_1'^2 e_2' \left(-\frac{9}{16} + D - \frac{1}{2} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \; \frac{\sin(3\ell_1 + g_1 - g_2)}{3}$$

$$+ e_1'^2 e_2' \left(\frac{1}{16} - \frac{1}{8} D \, b_{1/2}^{(1)} + \frac{1}{16} D^3 \, b_{1/2}^{(1)}\right) \sin(\ell_1 - g_1 + g_2)$$

$$+ e_1'^2 e_2' \left(-\frac{3}{16} D - \frac{1}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin(2\ell_1 - \ell_2)}{2 - a^{3/2}}$$

$$+ e_1'^2 e_2' \left(\frac{27}{16} - \frac{3}{4} D - \frac{1}{4} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2a^{3/2}}$$

$$+ e_1' e_2'^2 \left(-\frac{17}{8} - \frac{35}{16} D - \frac{11}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(3\ell_2 - g_1 + g_2)}{3a^{3/2}}$$

$$+ e_1' e_2'^2 \left(-\frac{1}{4} D - \frac{5}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin(\ell_1 + 2\ell_2)}{1 + 2a^{3/2}}$$

$$+ e_1' e_2'^2 \left(-\frac{1}{8} + \frac{3}{16} D + \frac{1}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 + \ell_2 + g_1 - g_2)}{2 + a^{3/2}}$$

$$+ e_1' e_2'^2 \left(\frac{1}{8} - \frac{1}{16} D - \frac{3}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(-\ell_2 - g_1 + g_2)}{-a^{3/2}}$$

$$+ e_1' e_2'^2 \left(-\frac{1}{4} D - \frac{5}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin(\ell_1 - 2\ell_2)}{1 - 2a^{3/2}}$$

$$+ e_1' e_2'^2 \left(\frac{17}{8} + \frac{1}{16} D - \frac{7}{16} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3a^{3/2}}$$

$$+ e_2'^3 \left(\frac{71}{24} + \frac{95}{48} D + \frac{3}{8} D^2 + \frac{1}{48} D^3\right) b_{1/2}^{(1)} \frac{\sin(-\ell_1 + 4\ell_2 - g_1 + g_2)}{-1 + 4a^{3/2}}$$

$$+ e_2'^3 \left(\frac{27}{48} + \frac{19}{24} D + \frac{1}{4} D^2 + \frac{1}{48} D^3\right) b_{1/2}^{(0)} \frac{\sin 3\ell_2}{3a^{3/2}}$$

$$\left. + e_2'^3 \left(-\frac{1}{12} + \frac{5}{48} D + \frac{1}{8} D^2 + \frac{1}{48} D^3\right) b_{1/2}^{(1)} \frac{\sin(\ell_1 + 2\ell_2 + g_1 - g_2)}{1 + 2a^{3/2}}\right) . \tag{23}$$

The values of $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_1$, $\partial S_{1p}/\partial g_2$ are easily obtained from (23), and will not be given here. The values of $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial G_2'$, $\partial S_{1p}/\partial H_1'$, $\partial S_{1p}/\partial H_2'$ require further calculations. We obtain from (22):

$$\frac{\partial S_{1p}}{\partial L_1'} = \frac{\sigma \beta_2 \alpha}{m_0} \left\{ p \left[g(3f + 2\,Df) + \frac{1 - e_1'^2}{e_1'} f \frac{\partial g}{\partial e_1'} \right] \right.$$

$$\left. + \alpha^{3/2} q \left(g\, 2\,Df + \frac{1 - e_1'^2}{e_1'} f \frac{\partial g}{\partial e_1'} \right) \right\} \frac{\sin X}{\left(p + q\,\alpha^{3/2}\right)^2} , \qquad (24)$$

$$\frac{\partial S_{1p}}{\partial L_2'} = \frac{\sigma \beta_1 \alpha^{3/2}}{m_0} \left\{ p \left[g(-\,2f - 2\,Df) + \frac{1 - e_2'^2}{e_2'} f \frac{\partial g}{\partial e_2'} \right] \right.$$

$$\left. + \alpha^{3/2} q \left[g(f - 2\,Df) + \frac{1 - e_2'^2}{e_2'} f \frac{\partial g}{\partial e_2'} \right] \right\} \frac{\sin X}{\left(p + q\,\alpha^{3/2}\right)^2} , \qquad (25)$$

$$\frac{\partial S_{1p}}{\partial G_1'} = \frac{\sigma \beta_2 \alpha}{m_0} f \left[\frac{\partial g}{\partial e_1'} \left(-\frac{1}{e_1'} + \frac{1}{2} e_1' + \frac{1}{8} e_1'^3 \right) \right.$$

$$\left. + \frac{1}{4} \frac{\partial g}{\partial \tau'} \left(\frac{1}{\tau'} + \frac{1}{2} \frac{e_1'^2}{\tau'} - 2\,\tau' \right) \right] \frac{\sin X}{p + q\,\alpha^{3/2}} , \qquad (26)$$

$$\frac{\partial S_{1p}}{\partial G_2'} = \frac{\sigma \beta_1 \alpha^{3/2}}{m_0} f \frac{\partial g}{\partial e_2'} \left(-\frac{1}{e_2'} + \frac{1}{2} e_2' + \frac{1}{8} e_2'^3 \right) \frac{\sin X}{p + q\,\alpha^{3/2}} , \qquad (27)$$

$$\frac{\partial S_{1p}}{\partial H_1'} = \frac{\sigma \beta_2 \alpha}{m_0} f \left(-\frac{1}{4} \right) \frac{\partial g}{\partial \tau'} \left(\frac{1}{\tau'} + \frac{1}{2} \frac{e_1'^2}{\tau'} \right) \frac{\sin X}{p + q\,\alpha^{3/2}} , \qquad (28)$$

$$\frac{\partial S_{1p}}{\partial H_2'} = \frac{\partial S_{1p}}{\partial G_2'} = \text{equation (27)} ,$$

with

$$X = p\ell_1 + q\ell_2 + yg_1 + zg_2 \quad .$$

4.2 Terms of F_{1p} of orders 4 and 5 in e'_1 introduce in (24) and (26), through the quantitity $(1/e'_1)(\partial g/\partial e'_1)$, terms of orders 2 and 3 in e'_1, respectively. Terms of F_{1p} of orders 4 and 5 in e'_2 introduce in (25) and (27), through the quantity $(1/e'_2)(\partial g/\partial e'_2)$, terms of orders 2 and 3 in e'_2, respectively. Terms of F_{1p} of order 4 in τ' introduce in (26) and (28), through the quantity $(1/\tau')(\partial g/\partial \tau')$, terms of order 2 in τ'. Since we neglect in F_{1p} powers of e'_1, e'_2, τ' higher than the third, we shall neglect in $\partial S_{1p}/\partial L'_1$, $\partial S_{1p}/\partial L'_2$, $\partial S_{1p}/\partial G'_1$, $\partial S_{1p}/\partial G'_2$, $\partial S_{1p}/\partial H'_1$ the powers of e'_1, e'_2, τ' higher than the first.

4.3 For each S_{1p} defined by (22), we have to calculate equations (24), (25), (26), (27), and (28). Let us consider, for instance, the fourth term of the expression (23) of S_{1p}. We have

$$f(\alpha) = \left(1 - \frac{1}{2}D\right) b_{1/2}^{(1)}, \quad g(e'_1, e'_2, \tau') = e'_1,$$

$$p = 2, \quad q = -1, \quad y = 1, \quad z = -1 \quad ,$$

whence

$$3f + 2\,Df = \left(3 + \frac{1}{2}D - D^2\right) b_{1/2}^{(1)},$$

$$2\,Df = (2\,D - D^2)\, b_{1/2}^{(1)},$$

$$-\,2f - 2\,Df = (-\,2 - D + D^2)\, b_{1/2}^{(1)},$$

$$f - 2\,Df = \left(1 - \frac{5}{2}D + D^2\right) b_{1/2}^{(1)},$$

$$\frac{\partial g}{\partial e'_1} = 1, \quad \frac{\partial g}{\partial e'_2} = 0, \quad \frac{\partial g}{\partial \tau'} = 0,$$

and

$$g(3f + 2\,Df) + \frac{1 - e_1'^2}{e_1'}\, f\, \frac{\partial g}{\partial e_1'} = \frac{1}{e_1'}\left(1 - \frac{1}{2}D\right) b_{1/2}^{(1)} + e_1'\left(2 + D - D^2\right) b_{1/2}^{(1)} ,$$

$$g\; 2\,Df + \frac{1 - e_1'^2}{e_1'}\, f\, \frac{\partial g}{\partial e_1'} = \frac{1}{e_1'}\left(1 - \frac{1}{2}D\right) b_{1/2}^{(1)} + e_1'\left(-1 + \frac{5}{2}D - D^2\right) b_{1/2}^{(1)} .$$

Thus

$$\frac{\partial S_{1p}}{\partial L_1'} = \frac{\sigma\beta_2\alpha}{m_0}\left\{\frac{1}{e_1'}(2 - D)\, b_{1/2}^{(1)} + e_1'\left(4 + 2\,D - 2\,D^2\right) b_{1/2}^{(1)}\right.$$

$$+ \alpha^{3/2}\left[\frac{1}{e_1'}\left(-1 + \frac{1}{2}D\right) b_{1/2}^{(1)}\right.$$

$$\left.\left. + e_1'\left(1 - \frac{5}{2}D + D^2\right) b_{1/2}^{(1)}\right]\right\} \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{(2 - \alpha^{3/2})^2} \quad ,$$

$$\frac{\partial S_{1p}}{\partial L_2'} = \frac{\sigma\beta_1\alpha^{3/2}}{m_0}\left[e_1'(-4 - 2\,D + 2\,D^2)\, b_{1/2}^{(1)}\right.$$

$$\left. + \alpha^{3/2} e_1'\left(-1 + \frac{5}{2}D - D^2\right) b_{1/2}^{(1)}\right] \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{(2 - \alpha^{3/2})^2} ,$$

$$\frac{\partial S_{1p}}{\partial G_1'} = \frac{\sigma\beta_2\alpha}{m_0}\left[\frac{-1}{e_1'}\left(1 - \frac{1}{2}D\right) b_{1/2}^{(1)}\right.$$

$$\left. + \frac{1}{2} e_1'\left(1 - \frac{1}{2}D\right) b_{1/2}^{(1)}\right] \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{2 - \alpha^{3/2}} ,$$

$$\frac{\partial S_{1p}}{\partial G_2'} = \frac{\partial S_{1p}}{\partial H_1'} = \frac{\partial S_{1p}}{\partial H_2'} = 0 \quad .$$

Let us also consider the 31st term of equation (23). We have

$$f(\alpha) = \left(\frac{3\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)}, \quad g = e_2' \tau'^2, \quad p = 0, \quad q = 3, \quad y = 0, \quad z = 2,$$

whence

$$\frac{\partial g}{\partial e_1'} = 0, \quad \frac{\partial g}{\partial e_2'} = \tau'^2, \quad \frac{\partial g}{\partial \tau'} = 2\, e_2' \tau',$$

and

$$\frac{1}{e_2'} \frac{\partial g}{\partial e_2'} = \frac{\tau'^2}{e_2'}, \qquad \frac{1}{\tau'} \frac{\partial g}{\partial \tau'} = 2\, e_2' \quad .$$

We have, therefore, since we neglect the powers of e_1', e_2', and τ' higher than the third,

$$\frac{\partial S_{1p}}{\partial L_1'} = 0 \quad ,$$

$$\frac{\partial S_{1p}}{\partial L_2'} = \frac{\sigma \beta_1 \alpha^{3/2}}{m_0} \alpha^{3/2} \frac{\tau'^2}{e_2'} \left(\frac{9\alpha}{2} + \frac{3\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (3\ell_2 + 2g_2)}{(3\, \alpha^{3/2})^2} \quad ,$$

$$\frac{\partial S_{1p}}{\partial G_1'} = \frac{\sigma \beta_2 \alpha}{m_0} e_2' \left(\frac{3\alpha}{4} + \frac{\alpha}{8} D\right) b_{3/2}^{(1)} \frac{\sin (3\ell_2 + 2g_2)}{3\, \alpha^{3/2}} \quad ,$$

$$\frac{\partial S_{1p}}{\partial G_2'} = \frac{\partial S_{1p}}{\partial H_2'} = \frac{\sigma \beta_1 \alpha^{3/2}}{m_0} \frac{\tau'^2}{e_2'} \left(- \frac{3\alpha}{2} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin (3\ell_2 + 2g_2)}{3\alpha^{3/2}} \quad ,$$

$$\frac{\partial S_{1p}}{\partial H_1'} = \frac{\sigma \beta_2 \alpha}{m_0} e_2' \left(- \frac{3\alpha}{4} - \frac{\alpha}{8} D\right) b_{3/2}^{(1)} \frac{\sin (3\ell_2 + 2g_2)}{3\alpha^{3/2}} \quad .$$

4.4 This calculation has to be repeated for each of the remaining 49 terms of S_{1p} defined by (23). Each $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial G_2'$, $\partial S_{1p}/\partial H_1'$ thus obtained is reduced to its terms of order -1, 0, 1 in e_1', e_2', and to its terms of order 0, 1 in τ', and we sum up all the $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial G_2'$, $\partial S_{1p}/\partial H_1'$. We have finally:

$$\frac{\partial S_{1p}}{\partial L_1'} = \frac{\sigma\beta_2 a}{m_0}\Bigg[\Bigg[\left(-1 - \frac{5}{2}D - \frac{1}{2}D^2\right)b_{1/2}^{(1)}$$

$$+ a^{3/2}\left(-2 + \frac{5}{2}D + \frac{1}{2}D^2\right)b_{1/2}^{(1)}\Bigg]\frac{\sin(-\ell_1 + \ell_2 - g_1 + g_2)}{(-1 + a^{3/2})^2}$$

$$+ a^{3/2}\Bigg\{\frac{1}{e_1'}\left(-1 - \frac{1}{2}D\right)b_{1/2}^{(1)}$$

$$+ e_1'\left(1 - \frac{9}{8}D - \frac{19}{16}D^2 - \frac{3}{16}D^3\right)b_{1/2}^{(1)}$$

$$+ \frac{\tau'^2}{e_1'}\left[\left(\frac{3a}{4} + \frac{a}{4}D\right)b_{3/2}^{(0)} + \left(\frac{3a}{4} + \frac{a}{4}D\right)b_{3/2}^{(2)}\right]$$

$$+ \frac{e_2'^2}{e_1'}\left(1 + \frac{1}{4}D - \frac{3}{8}D^2 - \frac{1}{8}D^3\right)b_{1/2}^{(1)}\Bigg\}\frac{\sin(\ell_2 - g_1 + g_2)}{(a^{3/2})^2}$$

$$+\Bigg[\frac{1}{e_1'}\left(-\frac{1}{2}D\right)b_{1/2}^{(0)}$$

$$+ e_1'\left(-\frac{7}{16}D - \frac{13}{16}D^2 - \frac{3}{16}D^3\right)b_{1/2}^{(0)}$$

$$+ \frac{\tau'^2}{e_1'}\left(\frac{a}{2} + \frac{a}{2}D\right)b_{3/2}^{(1)}$$

$$+ \frac{e_2'^2}{e_1'}\left(-\frac{1}{8}D^2 - \frac{1}{8}D^3\right)b_{1/2}^{(0)}\Bigg]\sin\ell_1$$

$$+\Bigg(\frac{1}{e_1'}(2 - D)\,b_{1/2}^{(1)}$$

$$+ e_1' \left(- \frac{7}{2} + \frac{13}{2} D - \frac{7}{8} D^2 - \frac{3}{8} D^3\right) b_{1/2}^{(1)}$$

$$+ \frac{\tau'^2}{e_1'} \left[\left(- \frac{\alpha}{2} + \frac{\alpha}{2} D\right) b_{3/2}^{(0)} + \left(- \frac{\alpha}{2} + \frac{\alpha}{2} D\right) b_{3/2}^{(2)}\right]$$

$$+ \frac{e_2'^2}{e_1'} \left(- 2 + \frac{3}{2} D + \frac{1}{4} D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(1)}$$

$$+ \alpha^{3/2} \left\{ \frac{1}{e_1'} \left(- 1 + \frac{1}{2} D\right) b_{1/2}^{(1)}\right.$$

$$+ e_1' \left(\frac{19}{4} - \frac{19}{4} D + \frac{7}{16} D^2 + \frac{3}{16} D^3\right) b_{1/2}^{(1)}$$

$$+ \frac{\tau'^2}{e_1'} \left[\left(\frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(\frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]$$

$$\left.+ \frac{e_2'^2}{e_1'} \left(1 - \frac{3}{4} D - \frac{1}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \right\} \Bigg) \frac{\sin (2\ell_1 - \ell_2 + g_1 - g_2)}{(2 - \alpha^{3/2})^2}$$

$$+ e_2' \left[\left(- \frac{3}{2} - \frac{17}{4} D - 2 D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(1)}\right.$$

$$\left.+ \alpha^{3/2} e_2' \left(- 6 + \frac{11}{2} D + 4 D^2 + \frac{1}{2} D^3\right) b_{1/2}^{(1)}\right] \frac{\sin (- \ell_1 + 2\ell_2 - g_1 + g_2)}{(-1 + 2 \alpha^{3/2})^2}$$

$$+ \alpha^{3/2} e_2' \left(\frac{5}{4} D + \frac{3}{2} D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(0)} \frac{\sin \ell_2}{(\alpha^{3/2})^2}$$

$$+ e_2' \left(- \frac{1}{2} - \frac{3}{4} D + D^2\right) b_{1/2}^{(1)} \sin (\ell_1 + g_1 - g_2)$$

$$+ \left[\left(- \frac{1}{4} + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)}\right.$$

$$\left.+ \alpha^{3/2} \left(- \frac{1}{4} + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)}\right] \frac{\sin (\ell_1 + \ell_2 - g_1 + g_2)}{(1 + \alpha^{3/2})^2}$$

$$+ \left(- \frac{3}{2} D + \frac{1}{2} D^2\right) b_{1/2}^{(0)} \frac{\sin 2\ell_1}{4}$$

$$+ \left[\left(\frac{27}{4} - \frac{21}{4} D + \frac{3}{4} D^2\right) b_{1/2}^{(1)}\right.$$

$$\left. + \alpha^{3/2} \left(- \frac{9}{4} + \frac{7}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)}\right] \frac{\sin (3\ell_1 - 2\ell_2 + g_1 - g_2)}{(3 - 2\alpha^{3/2})^2}$$

$$+ \alpha^{3/2} \frac{e'_2}{e'_1} \left(- 3 - \frac{5}{2} D - \frac{1}{2} D^2\right) b_{1/2}^{(1)} \frac{\sin (2\ell_2 - g_1 + g_2)}{(2 \alpha^{3/2})^2}$$

$$+ \left[\frac{e'_2}{e'_1} \left(- \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)}\right.$$

$$\left. + \alpha^{3/2} \frac{e'_2}{e'_1} \left(- \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)}\right] \frac{\sin (\ell_1 + \ell_2)}{(1 + \alpha^{3/2})^2}$$

$$+ \frac{e'_2}{e'_1} \left(- \frac{1}{4} + \frac{3}{2} D - \frac{1}{2} D^2\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 + g_1 - g_2)}{4}$$

$$+ \left[\frac{e'_2}{e'_1} \left(- \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)}\right.$$

$$\left. + \alpha^{3/2} \frac{e'_2}{e'_1} \left(\frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(0)}\right] \frac{\sin (\ell_1 - \ell_2)}{(1 - \alpha^{3/2})^2}$$

$$+ \left[\frac{e'_2}{e'_1} \left(3 - \frac{1}{2} D - \frac{1}{2} D^2\right) b_{1/2}^{(1)}\right.$$

$$\left. + \alpha^{3/2} \frac{e'_2}{e'_1} \left(- 3 + \frac{1}{2} D + \frac{1}{2} D^2\right) b_{1/2}^{(1)}\right] \frac{\sin (2\ell_1 - 2\ell_2 + g_1 - g_2)}{(2 - 2 \alpha^{3/2})^2}$$

$$+ \alpha^{3/2} \left(2 D + \frac{5}{2} D^2 + \frac{1}{2} D^3\right) b_{1/2}^{(0)} \frac{\sin 2\ell_2}{(2 \alpha^{3/2})^2}$$

$$+ \frac{\tau'^2}{e_1'} \left(\frac{5\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \quad \sin(-\ell_1 - 2g_1)$$

$$+ \alpha^{3/2} \frac{\tau'^2}{e_1'} \left(\frac{3\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \quad \frac{\sin(-\ell_2 - g_1 - g_2)}{(-\alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e_1'} \left(-\frac{\alpha}{4} - \frac{\alpha}{4} D \right) b_{3/2}^{(1)} + \alpha^{3/2} \frac{\tau'^2}{e_1'} \left(\frac{\alpha}{2} + \frac{\alpha}{2} D \right) b_{3/2}^{(1)} \right] \frac{\sin(\ell_1 - 2\ell_2 - 2g_2)}{(1 - 2\alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e_1'} \left(-\frac{\alpha}{4} - \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + \alpha^{3/2} \frac{\tau'^2}{e_1'} \left(-\frac{\alpha}{2} - \frac{\alpha}{2} D \right) b_{3/2}^{(1)} \right] \frac{\sin(\ell_1 + 2\ell_2 + 2g_2)}{(1 + 2\alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e_1'} \left(\frac{\alpha}{2} - \frac{\alpha}{2} D \right) b_{3/2}^{(0)} + \alpha^{3/2} \frac{\tau'^2}{e_1'} \left(\frac{\alpha}{4} - \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \right] \frac{\sin(2\ell_1 + \ell_2 + g_1 + g_2)}{(2 + \alpha^{3/2})^2}$$

$$+ \frac{\tau'^2}{e_1'} \left(\frac{9\alpha}{4} - \frac{3\alpha}{4} D \right) b_{3/2}^{(1)} \quad \frac{\sin(3\ell_1 + 2g_1)}{9}$$

$$+ \left[e_1' \left(-\frac{1}{2} + \frac{1}{2} D + \frac{3}{8} D^2 - \frac{1}{8} D^3 \right) b_{1/2}^{(1)} \right.$$

$$\left. + \alpha^{3/2} e_1' \left(-\frac{1}{4} + \frac{1}{4} D + \frac{3}{16} D^2 - \frac{1}{16} D^3 \right) b_{1/2}^{(1)} \right] \frac{\sin(2\ell_1 + \ell_2 - g_1 + g_2)}{(2 + \alpha^{3/2})^2}$$

$$+ e_1' \left(-\frac{51}{16} D + \frac{27}{16} D^2 - \frac{3}{16} D^3 \right) b_{1/2}^{(0)} \quad \frac{\sin 3\ell_1}{9}$$

$$+ \left[e_1' \left(16 - \frac{31}{2} D + \frac{15}{4} D^2 - \frac{1}{4} D^3 \right) b_{1/2}^{(1)} \right.$$

$$\left. + \alpha^{3/2} e_1' \left(-4 + \frac{31}{8} D - \frac{15}{16} D^2 + \frac{1}{16} D^3 \right) b_{1/2}^{(1)} \right] \frac{\sin(4\ell_1 - \ell_2 + g_1 - g_2)}{(4 - \alpha^{3/2})^2}$$

$$+\left[e_2'\left(-\frac{3}{8}+\frac{1}{4}D+\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}e_2'\left(-\frac{3}{4}+\frac{1}{2}D+\frac{1}{2}D^2+\frac{1}{4}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(\ell_1+2\ell_2-g_1+g_2)}{(1+2\alpha^{3/2})^2}$$

$$+\left[e_2'\left(-\frac{3}{4}D-\frac{1}{2}D^2+\frac{1}{4}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}e_2'\left(-\frac{3}{8}D-\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(2\ell_1+\ell_2)}{(2+\alpha^{3/2})^2}$$

$$+e_2'\left(-\frac{27}{8}+6D-3D^2+\frac{3}{8}D^3\right)b_{1/2}^{(1)}\frac{\sin(3\ell_1+g_1-g_2)}{9}$$

$$+e_2'\left(\frac{1}{8}-\frac{1}{4}D+\frac{1}{8}D^3\right)b_{1/2}^{(1)}\sin(\ell_1-g_1+g_2)$$

$$+\left[e_2'\left(-\frac{3}{4}D-\frac{1}{2}D^2+\frac{1}{4}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}e_2'\left(\frac{3}{8}D+\frac{1}{4}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(2\ell_1-\ell_2)}{(2-\alpha^{3/2})^2}$$

$$+\left[e_2'\left(\frac{81}{8}-\frac{9}{2}D-\frac{3}{2}D^2+\frac{3}{8}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}e_2'\left(-\frac{27}{4}+3D+D^2-\frac{1}{4}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(3\ell_1-2\ell_2+g_1-g_2)}{(3-2\alpha^{3/2})^2}$$

$$+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(-\frac{51}{8}-\frac{105}{16}D-\frac{33}{16}D^2-\frac{3}{16}D^3\right)b_{1/2}^{(1)}\frac{\sin(3\ell_2-g_1+g_2)}{(3\alpha^{3/2})^2}$$

$$+\left[\frac{e_2'^2}{e_1'}\left(-\frac{1}{4}D-\frac{5}{16}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(-\frac{1}{2}D-\frac{5}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(\ell_1+2\ell_2)}{(1+2\alpha^{3/2})^2}$$

$$+\left[\frac{e_2'^2}{e_1'}\left(-\frac{1}{4}+\frac{3}{8}D+\frac{1}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(-\frac{1}{8}+\frac{3}{16}D+\frac{1}{16}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(2\ell_1+\ell_2+g_1-g_2)}{(2+\alpha^{3/2})^2}$$

$$+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(-\frac{1}{8}+\frac{1}{16}D+\frac{3}{16}D^2+\frac{1}{16}D^3\right)b_{1/2}^{(1)}\frac{\sin(-\ell_2-g_1+g_2)}{(-\alpha^{3/2})^2}$$

$$+\left[\frac{e_2'^2}{e_1'}\left(-\frac{1}{4}D-\frac{5}{16}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(\frac{1}{2}D+\frac{5}{8}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(\ell_1-2\ell_2)}{(1-2\alpha^{3/2})^2}$$

$$+\left[\frac{e_2'^2}{e_1'}\left(\frac{17}{4}+\frac{1}{8}D-\frac{7}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.\left.+\alpha^{3/2}\frac{e_2'^2}{e_1'}\left(-\frac{51}{8}-\frac{3}{16}D+\frac{21}{16}D^2+\frac{3}{16}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(2\ell_1-3\ell_2+g_1-g_2)}{(2-3\alpha^{3/2})^2}\right],$$

(29)

$$\frac{\partial S_{1p}}{\partial L_2'}=\frac{\sigma\beta_1\alpha^{3/2}}{m_0}\left[\left[\left(4+\frac{3}{2}D-\frac{1}{2}D^2\right)b_{1/2}^{(1)}\right.\right.$$

$$\left.+\alpha^{3/2}\left(-1+\frac{3}{2}D+\frac{1}{2}D^2\right)b_{1/2}^{(1)}\right]\frac{\sin(-\ell_1+\ell_2-g_1+g_2)}{(-1+\alpha^{3/2})^2}$$

$$+ \alpha^{3/2} e_1' \left(1 + 2 D + \frac{1}{4} D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(1)} \frac{\sin(\ell_2 - g_1 + g_2)}{(\alpha^{3/2})^2}$$

$$+ e_1' \left(D + \frac{3}{4} D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(0)} \sin \ell_1$$

$$+ \left[e_1' \left(- 8 + D + \frac{5}{2} D^2 - \frac{1}{2} D^3\right) b_{1/2}^{(1)}\right.$$

$$\left. + \alpha^{3/2} e_1' \left(1 + D - \frac{5}{4} D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(1)}\right] \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{(2 - \alpha^{3/2})^2}$$

$$+ \left(\frac{1}{e_2'} \left(- \frac{3}{2} - \frac{1}{2} D\right) b_{1/2}^{(1)}\right.$$

$$+ e_2' \left(\frac{39}{4} + \frac{81}{16} D - \frac{17}{8} D^2 - \frac{3}{16} D^3\right) b_{1/2}^{(1)}$$

$$+ \frac{e_1'^2}{e_2'} \left(\frac{3}{2} + \frac{1}{8} D - \frac{1}{2} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)}$$

$$+ \frac{\tau'^2}{e_2'} \left[\left(\alpha + \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(\alpha + \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]$$

$$+ \alpha^{3/2} \left\{ \frac{1}{e_2'} (3 + D) b_{1/2}^{(1)}\right.$$

$$+ e_2' \left(- \frac{21}{2} - \frac{57}{8} D + \frac{17}{4} D^2 + \frac{3}{8} D^3\right) b_{1/2}^{(1)}$$

$$+ \frac{e_1'^2}{e_2'} \left(- 3 - \frac{1}{4} D + D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(1)}$$

$$\left.\left. + \frac{\tau'^2}{e_2'} \left[\left(- 2\alpha - \frac{\alpha}{2} D\right) b_{3/2}^{(0)} + \left(- 2\alpha - \frac{\alpha}{2} D\right) b_{3/2}^{(2)}\right]\right\}\right) \frac{\sin(- \ell_1 + 2\ell_2 - g_1 + g_2)}{(- 1 + 2\alpha^{3/2})^2}$$

$$+ \alpha^{3/2} \left[\frac{1}{e'_2} \left(\frac{1}{2} + \frac{1}{2} D \right) b^{(0)}_{1/2} \right.$$

$$+ e'_2 \left(- \frac{3}{16} - \frac{5}{8} D - \frac{1}{4} D^2 + \frac{3}{16} D^3 \right) b^{(0)}_{1/2}$$

$$+ \frac{e'^2_1}{e'_2} \left(\frac{1}{8} D + \frac{1}{4} D^2 + \frac{1}{8} D^3 \right) b^{(0)}_{1/2}$$

$$\left. + \frac{\tau'^2}{e'_2} \left(- \alpha - \frac{\alpha}{2} D \right) b^{(1)}_{3/2} \right] \frac{\sin \ell_2}{(\alpha^{3/2})^2}$$

$$+ \left[\frac{1}{e'_2} \left(- \frac{1}{2} + \frac{1}{2} D \right) b^{(1)}_{1/2} \right.$$

$$+ e'_2 \left(\frac{9}{8} - \frac{11}{16} D - \frac{5}{8} D^2 + 3 D^3 \right) b^{(1)}_{1/2}$$

$$+ \frac{\tau'^2}{e'_2} \left(- \frac{\alpha}{4} D b^{(0)}_{3/2} - \frac{\alpha}{4} D b^{(2)}_{3/2} \right)$$

$$\left. + \frac{e'^2_1}{e'_2} \left(\frac{1}{2} - \frac{5}{8} D + \frac{1}{8} D^3 \right) b^{(1)}_{1/2} \right] \sin (\ell_1 + g_1 - g_2)$$

$$+ \alpha^{3/2} \frac{e'_1}{e'_2} \left(- 3 - \frac{5}{2} D - \frac{1}{2} D^2 \right) b^{(1)}_{1/2} \frac{\sin (2\ell_2 - g_1 + g_2)}{(2 \alpha^{3/2})^2}$$

$$+ \left[\frac{e'_1}{e'_2} \left(- \frac{1}{4} D - \frac{1}{4} D^2 \right) b^{(0)}_{1/2} \right.$$

$$\left. + \alpha^{3/2} \frac{e'_1}{e'_2} \left(- \frac{1}{4} D - \frac{1}{4} D^2 \right) b^{(0)}_{1/2} \right] \frac{\sin (\ell_1 + \ell_2)}{(1 + \alpha^{3/2})^2}$$

$$+ \frac{e'_1}{e'_2} (- 2 + 3 D - D^2) b^{(1)}_{1/2} \frac{\sin (2\ell_1 + g_1 - g_2)}{4}$$

$$+\left[\frac{e'_1}{e'_2}\left(-\frac{1}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}\frac{e'_1}{e'_2}\left(\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(0)}\right]\frac{\sin(\ell_1-\ell_2)}{(1-\alpha^{3/2})^2}$$

$$+\left[\frac{e'_1}{e'_2}\left(3-\frac{1}{2}D-\frac{1}{2}D^2\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\frac{e'_1}{e'_2}\left(-3+\frac{1}{2}D+\frac{1}{2}D^2\right)b_{1/2}^{(1)}\right]\frac{\sin(2\ell_1-2\ell_2+g_1-g_2)}{(2-2\alpha^{3/2})^2}$$

$$+\left[\left(-\frac{17}{4}-\frac{9}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\left(\frac{51}{4}+\frac{27}{4}D+\frac{3}{4}D^2\right)b_{1/2}^{(1)}\right]\frac{\sin(-\ell_1+3\ell_2-g_1+g_2)}{(-1+3\alpha^{3/2})^2}$$

$$+\alpha^{3/2}\left(2+\frac{5}{2}D+\frac{1}{2}D^2\right)b_{1/2}^{(0)}\frac{\sin 2\ell_2}{(2\alpha^{3/2})^2}$$

$$+\left[\left(-\frac{1}{4}+\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\left(-\frac{1}{4}+\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(1)}\right]\frac{\sin(\ell_1+\ell_2+g_1-g_2)}{(1+\alpha^{3/2})^2}$$

$$+\alpha^{3/2}(\alpha+2\alpha D)\,b_{3/2}^{(1)}\frac{\sin(-2\ell_2-2g_2)}{(-2\alpha^{3/2})^2}$$

$$+\alpha^{3/2}\left(-\frac{3}{2}\alpha-2\alpha D-\frac{\alpha}{2}D^2\right)b_{3/2}^{(0)}\frac{\sin(-\ell_2-g_1-g_2)}{(-\alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e'_2} \left(- \alpha - \frac{\alpha}{2} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + \alpha^{3/2} \frac{\tau'^2}{e'_2} \left(\frac{\alpha}{2} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right] \frac{\sin (- 2\ell_1 + \ell_2 - 2g_1)}{(- 2 + \alpha^{3/2})^2}$$

$$+ \frac{\tau'^2}{e'_2} \left(- \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \quad \sin (- \ell_1 - g_1 - g_2)$$

$$+ \alpha^{3/2} \frac{\tau'^2}{e'_2} \left(\frac{\alpha}{2} - \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \frac{\sin (- \ell_2 - 2g_2)}{(- \alpha^{3/2})^2}$$

$$+ \alpha^{3/2} \frac{\tau'^2}{e'_2} \left(\frac{9\alpha}{2} + \frac{3\alpha}{4} D \right) b_{3/2}^{(1)} \frac{\sin (3\ell_2 + 2g_2)}{(3 \alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e'_2} \left(\alpha + \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \right.$$

$$\left. + \alpha^{3/2} \frac{\tau'^2}{e'_2} \left(2 \alpha + \frac{\alpha}{2} D \right) b_{3/2}^{(0)} \right] \frac{\sin (\ell_1 + 2\ell_2 + g_1 + g_2)}{(1 + 2 \alpha^{3/2})^2}$$

$$+ \left[\frac{\tau'^2}{e'_2} \left(\alpha + \frac{\alpha}{2} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + \alpha^{3/2} \frac{\tau'^2}{e'_2} \left(\frac{\alpha}{2} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right] \frac{\sin (2\ell_1 + \ell_2 + 2g_1)}{(2 + \alpha^{3/2})^2}$$

$$+\left[\frac{e_1'^2}{e_2'}\left(-\frac{3}{16}+\frac{1}{8}D+\frac{1}{8}D^2+\frac{1}{16}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_1'^2}{e_2'}\left(-\frac{3}{8}+\frac{1}{4}D+\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(\ell_1+2\ell_2-g_1+g_2)}{(1+2\alpha^{3/2})^2}$$

$$+\left[\frac{e_1'^2}{e_2'}\left(-\frac{3}{8}D-\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_1'^2}{e_2'}\left(-\frac{3}{16}D-\frac{1}{8}D^2+\frac{1}{16}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(2\ell_1+\ell_2)}{(2+\alpha^{3/2})^2}$$

$$+\frac{e_1'^2}{e_2'}\left(-\frac{27}{16}+3D-\frac{3}{2}D^2+\frac{3}{16}D^3\right)b_{1/2}^{(1)}\frac{\sin(3\ell_1+g_1-g_2)}{9}$$

$$+\frac{e_1'^2}{e_2'}\left(\frac{1}{16}-\frac{1}{8}D+\frac{1}{16}D^3\right)b_{1/2}^{(1)}\sin(\ell_1-g_1+g_2)$$

$$+\left[\frac{e_1'^2}{e_2'}\left(-\frac{3}{8}D-\frac{1}{4}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_1'^2}{e_2'}\left(\frac{3}{16}D+\frac{1}{8}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(2\ell_1-\ell_2)}{(2-\alpha^{3/2})^2}$$

$$+\left[\frac{e_1'^2}{e_2'}\left(\frac{81}{16}-\frac{9}{4}D-\frac{3}{4}D^2+\frac{3}{16}D^3\right)b_{1/2}^{(1)}\right.$$

$$\left.+\alpha^{3/2}\frac{e_1'^2}{e_2'}\left(-\frac{27}{8}+\frac{3}{8}D+\frac{1}{2}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(3\ell_1-2\ell_2+g_1-g_2)}{(3-2\alpha^{3/2})^2}$$

$$+\alpha^{3/2}e_1'\left(-\frac{51}{4}-\frac{105}{8}D-\frac{33}{8}D^2-\frac{3}{8}D^3\right)b_{1/2}^{(1)}\frac{\sin(3\ell_2-g_1+g_2)}{(3\alpha^{3/2})^2}$$

$$+\left[e_1'\left(-\frac{1}{2}D-\frac{5}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right.$$
$$\left.+\alpha^{3/2}e_1'\left(-D-\frac{5}{4}D^2-\frac{1}{4}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(\ell_1+2\ell_2)}{(1+2\alpha^{3/2})^2}$$

$$+\left[e_1'\left(-\frac{1}{2}+\frac{3}{4}D+\frac{1}{4}D^2-\frac{1}{4}D^3\right)b_{1/2}^{(1)}\right.$$
$$\left.+\alpha^{3/2}e_1'\left(-\frac{1}{4}+\frac{3}{8}D+\frac{1}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(2\ell_1+\ell_2+g_1-g_2)}{(2+\alpha^{3/2})^2}$$

$$+\alpha^{3/2}e_1'\left(-\frac{1}{4}+\frac{1}{8}D+\frac{3}{8}D^2+\frac{1}{8}D^3\right)b_{1/2}^{(1)}\frac{\sin(-\ell_2-g_1+g_2)}{(-\alpha^{3/2})^2}$$

$$+\left[e_1'\left(-\frac{1}{2}D-\frac{5}{8}D^2-\frac{1}{8}D^3\right)b_{1/2}^{(0)}\right.$$
$$\left.+\alpha^{3/2}e_1'\left(D+\frac{5}{4}D^2+\frac{1}{4}D^3\right)b_{1/2}^{(0)}\right]\frac{\sin(\ell_1-2\ell_2)}{(1-2\alpha^{3/2})^2}$$

$$+\left[e_1'\left(\frac{17}{2}+\frac{1}{4}D-\frac{7}{4}D^2-\frac{1}{4}D^3\right)b_{1/2}^{(1)}\right.$$
$$\left.+\alpha^{3/2}e_1'\left(-\frac{51}{4}-\frac{3}{8}D+\frac{21}{8}D^2+\frac{3}{8}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(2\ell_1-3\ell_2+g_1-g_2)}{(2-3\alpha^{3/2})^2}$$

$$+\left[e_2'\left(-\frac{71}{8}-\frac{95}{16}D-\frac{9}{8}D^2-\frac{1}{16}D^3\right)b_{1/2}^{(1)}\right.$$
$$\left.+\alpha^{3/2}e_2'\left(\frac{71}{2}+\frac{95}{4}D+\frac{9}{2}D^2+\frac{1}{4}D^3\right)b_{1/2}^{(1)}\right]\frac{\sin(-\ell_1+4\ell_2-g_1+g_2)}{(-1+4\alpha^{3/2})^2}$$

$$+ \alpha^{3/2}\left(\frac{27}{16} + \frac{19}{8} D + \frac{3}{4} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin 3\ell_2}{(3\alpha^{3/2})^2}$$

$$+\left[e_2' \left(- \frac{1}{4} + \frac{5}{16} D + \frac{3}{8} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)}\right.$$

$$\left. + \alpha^{3/2} e_2' \left(- \frac{1}{2} + \frac{5}{8} D + \frac{3}{4} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)}\right] \left.\frac{\sin(\ell_1 + 2\ell_2 + g_1 - g_2)}{(1 + 2\alpha^{3/2})^2}\right] ,$$

(30)

$$\frac{\partial S_{1p}}{\partial G_1'} = \frac{\sigma\beta_2\alpha}{m_0}\left(\left[\left(2 - \frac{1}{2} D - \frac{1}{2} D^2\right) b_{1/2}^{(1)} - \frac{\alpha}{4} b_{3/2}^{(0)} - \frac{\alpha}{4} b_{3/2}^{(2)}\right] \frac{\sin(-\ell_1 + \ell_2 - g_1 + g_2)}{1 + \alpha^{3/2}}\right.$$

$$+\left\{ \frac{1}{e_1'} \left(1 + \frac{1}{2} D\right) b_{1/2}^{(1)}\right.$$

$$+ \frac{\tau'^2}{e_1'} \left[\left(- \frac{3\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(- \frac{3\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]$$

$$+ e_1' \left[\left(- \frac{1}{2} - \frac{5}{8} D + \frac{3}{16} D^2 + \frac{3}{16} D^3\right) b_{1/2}^{(1)} + \left(\frac{3\alpha}{8} + \frac{\alpha}{8} D\right) b_{3/2}^{(0)}\right.$$

$$\left. + \left(\frac{3\alpha}{8} + \frac{\alpha}{8} D\right) b_{3/2}^{(2)}\right]$$

$$\left. + \frac{e_2'^2}{e_1'} \left(- 1 - \frac{1}{4} D + \frac{3}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \right\} \frac{\sin(\ell_2 - g_1 + g_2)}{\alpha^{3/2}}$$

$$+ \left[\frac{1}{e_1'} \left(\frac{1}{2} D\right) b_{1/2}^{(0)}\right.$$

$$+ e_1' \left(- \frac{13}{16} D - \frac{3}{16} D^2 + \frac{3}{16} D^3\right) b_{1/2}^{(0)} + \left(\frac{\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)}$$

$$+ \frac{e_2'^2}{e_1'} \left(\frac{1}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(0)}$$

$$\left. + \frac{\tau'^2}{e_1'} \left(- \frac{\alpha}{2} - \frac{\alpha}{2} D\right) b_{3/2}^{(1)}\right] \sin \ell_1$$

$$
+ \left\{ \frac{1}{e'_1} \left(- 1 + \frac{1}{2} D\right) b_{1/2}^{(1)} \right.
$$

$$
+ e'_1 \left[\left(\frac{17}{4} - \frac{5}{2} D - \frac{9}{16} D^2 + \frac{3}{16} D^3\right) b_{1/2}^{(1)} + \left(- \frac{\alpha}{8} + \frac{\alpha}{8} D\right) b_{3/2}^{(0)} \right.
$$

$$
\left. + \left(- \frac{\alpha}{8} + \frac{\alpha}{8} D\right) b_{3/2}^{(2)}\right]
$$

$$
+ \frac{\tau'^2}{e'_1} \left[\left(\frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(0)} + \left(\frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(2)}\right]
$$

$$
\left. + \frac{e'^2_2}{e'_1} \left(1 - \frac{3}{4} D - \frac{1}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \right\} \frac{\sin (2\ell_1 - \ell_2 + g_1 - g_2)}{2 - \alpha^{3/2}}
$$

$$
+ \left[e'_2 \left(- \frac{\alpha}{2} - \frac{\alpha}{8} D\right) b_{3/2}^{(0)} + \left(- \frac{\alpha}{2} - \frac{\alpha}{8} D\right) b_{3/2}^{(2)} \right.
$$

$$
\left. + \left(3 + \frac{1}{4} D - D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(1)}\right] \frac{\sin (- \ell_1 + 2\ell_2 - g_1 + g_2)}{- 1 + 2 \alpha^{3/2}}
$$

$$
+ \left[e'_2 \left(- \frac{\alpha}{2} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} + \left(- \frac{1}{4} D - \frac{1}{2} D^2 - \frac{1}{4} D^3\right) b_{1/2}^{(0)}\right] \frac{\sin \ell_2}{\alpha^{3/2}}
$$

$$
+ e'_2 \left[- \frac{\alpha}{8} D\, b_{3/2}^{(0)} - \frac{\alpha}{8} D\, b_{3/2}^{(2)} + \left(- 1 + \frac{5}{4} D - \frac{1}{4} D^3\right) b_{1/2}^{(1)}\right] \sin (\ell_1 + g_1 - g_2)
$$

$$
+ \left(\frac{1}{4} - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin (\ell_1 + \ell_2 - g_1 + g_2)}{1 + \alpha^{3/2}}
$$

$$
+ \left(\frac{3}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(0)} \frac{\sin 2\ell_1}{2}
$$

$$
+ \left(- \frac{9}{4} + \frac{7}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin (3\ell_1 - \ell_2 + g_1 - g_2)}{3 - \alpha^{3/2}}
$$

$$+ \frac{e_2'}{e_1'} \left(\frac{3}{2} + \frac{5}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin (2\ell_2 - g_1 + g_2)}{2 a^{3/2}}$$

$$+ \frac{e_2'}{e_1'} \left(\frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(0)} \frac{\sin (\ell_1 + \ell_2)}{1 + a^{3/2}}$$

$$+ \frac{e_2'}{e_1'} \left(\frac{1}{2} - \frac{3}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 + g_1 - g_2)}{2}$$

$$+ \frac{e_2'}{e_1'} \left(\frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(0)} \frac{\sin (\ell_1 - \ell_2)}{1 - a^{3/2}}$$

$$+ \frac{e_2'}{e_1'} \left(- \frac{3}{2} + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 - 2\ell_2 + g_1 - g_2)}{2 - 2 a^{3/2}}$$

$$+ \frac{a}{4} b_{3/2}^{(1)} \frac{\sin (- 2\ell_1 - 2g_1)}{-2}$$

$$+ \frac{a}{4} b_{3/2}^{(0)} \frac{\sin (- \ell_1 - \ell_2 - g_1 - g_2)}{- 1 - a^{3/2}}$$

$$+ \frac{a}{4} b_{3/2}^{(1)} \frac{\sin (- 2\ell_2 - 2g_2)}{- 2 a^{3/2}}$$

$$+ \left[\frac{\tau'^2}{e_1'} \left(\frac{5a}{4} + \frac{a}{4} D\right) b_{3/2}^{(1)} \right.$$
$$\left. + e_1' \left(- \frac{5a}{8} - \frac{a}{8} D\right) b_{3/2}^{(1)}\right] \frac{\sin (- \ell_1 - 2g_1)}{-1}$$

$$+ \left[\frac{\tau'^2}{e'_1} \left(\frac{3\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \right.$$

$$\left. + e'_1 \left(- \frac{3\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(0)} \right] \frac{\sin(-\ell_2 - g_1 - g_2)}{-\alpha^{3/2}}$$

$$+ \left[\frac{\tau'^2}{e'_1} \left(\frac{\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + e'_1 \left(- \frac{\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(1)} \right] \frac{\sin(\ell_1 - 2\ell_2 - 2g_2)}{1 - 2\alpha^{3/2}}$$

$$+ \left[\frac{\tau'^2}{e'_1} \left(\frac{\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + e'_1 \left(- \frac{\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(1)} \right] \frac{\sin(\ell_1 + 2\ell_2 + 2g_2)}{1 + 2\alpha^{3/2}}$$

$$+ \left[\frac{\tau'^2}{e'_1} \left(- \frac{\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(0)} \right.$$

$$\left. + e'_1 \left(\frac{\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(0)} \right] \frac{\sin(2\ell_1 + \ell_2 + g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$+ \left[\frac{\tau'^2}{e'_1} \left(- \frac{3\alpha}{4} + \frac{\alpha}{4} D \right) b_{3/2}^{(1)} \right.$$

$$\left. + e'_1 \left(\frac{3\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(1)} \right] \frac{\sin(3\ell_1 + 2g_1)}{3}$$

$$+ e'_2 \left(\frac{\alpha}{4} + \frac{\alpha}{8} D \right) b_{3/2}^{(1)} \frac{\sin(-2\ell_1 + \ell_2 - 2g_1)}{-2 + \alpha^{3/2}}$$

$$+ e'_2 \frac{\alpha}{8} D \, b_{3/2}^{(0)} \frac{\sin(-\ell_1 - g_1 - g_2)}{-1}$$

$$+ e_2' \left(-\frac{a}{4} + \frac{a}{8} D\right) b_{3/2}^{(1)} \frac{\sin(-\ell_2 - 2g_2)}{-a^{3/2}}$$

$$+ e_2' \left(\frac{3a}{4} + \frac{a}{8} D\right) b_{3/2}^{(1)} \frac{\sin(3\ell_2 + 2g_2)}{3\ a^{3/2}}$$

$$+ e_2' \left(\frac{a}{2} + \frac{a}{8} D\right) b_{3/2}^{(0)} \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2\ a^{3/2}}$$

$$+ e_2' \left(\frac{a}{4} + \frac{a}{8} D\right) b_{3/2}^{(1)} \frac{\sin(2\ell_1 + \ell_2 + 2g_1)}{2 + a^{3/2}}$$

$$+ e_1' \left(\frac{1}{4} - \frac{1}{4} D - \frac{3}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 + \ell_2 - g_1 + g_2)}{2 + a^{3/2}}$$

$$+ e_1' \left(\frac{17}{16} D - \frac{9}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin 3\ell_1}{3}$$

$$+ e_1' \left(-4 + \frac{31}{8} D - \frac{15}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(4\ell_1 - \ell_2 + g_1 - g_2)}{4 - a^{3/2}}$$

$$+ e_2' \left(\frac{3}{8} - \frac{1}{4} D - \frac{1}{4} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2\ a^{3/2}}$$

$$+ e_2' \left(\frac{3}{8} D + \frac{1}{4} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(0)} \frac{\sin(2\ell_1 + \ell_2)}{2 + a^{3/2}}$$

$$+ e_2' \left(\frac{9}{8} - 2\ D + D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin(3\ell_1 + g_1 - g_2)}{3}$$

$$+ e_2' \left(-\frac{1}{8} + \frac{1}{4} D - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \sin(\ell_1 - g_1 + g_2)$$

$$+ e_2'\left(\frac{3}{8} D + \frac{1}{4} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(0)} \frac{\sin (2\ell_1 - \ell_2)}{2 - a^{3/2}}$$

$$+ e_2' \left(- \frac{27}{8} + \frac{3}{2} D + \frac{1}{2} D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin (3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2 a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(\frac{17}{8} + \frac{35}{16} D + \frac{11}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (3\ell_2 - g_1 + g_2)}{3 a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(\frac{1}{4} D + \frac{5}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin (\ell_1 + 2\ell_2)}{1 + 2 a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(\frac{1}{8} - \frac{3}{16} D - \frac{1}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 + \ell_2 + g_1 - g_2)}{2 + a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(- \frac{1}{8} + \frac{1}{16} D + \frac{3}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (- \ell_2 - g_1 + g_2)}{- a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(\frac{1}{4} D + \frac{5}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin (\ell_1 - 2\ell_2)}{1 - 2 a^{3/2}}$$

$$+ \frac{e_2'^2}{e_1'} \left(- \frac{17}{8} - \frac{1}{16} D + \frac{7}{16} D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(1)} \left.\frac{\sin (2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3 a^{3/2}}\right) , \tag{31}$$

$$\frac{\partial S_{1p}}{\partial G_2'} = \frac{\sigma \beta_1 a^{3/2}}{m_0}\left(\left(2 - \frac{1}{2} D - \frac{1}{2} D^2\right) b_{1/2}^{(1)} \frac{\sin (- \ell_1 + \ell_2 - g_1 + g_2)}{- 1 + a^{3/2}}\right.$$

$$+ e_1' \left(- 2 - \frac{1}{2} D + \frac{3}{4} D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(1)} \frac{\sin (\ell_2 - g_1 + g_2)}{a^{3/2}}$$

$$+ e_1' \left(\frac{1}{4} D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(0)} \sin \ell_1$$

$$+ e_1' \left(2 - \frac{3}{2} D - \frac{1}{4} D^2 + \frac{1}{4} D^3\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 - \ell_2 + g_1 - g_2)}{2 - a^{3/2}}$$

$$
\begin{aligned}
&+\left\{\frac{1}{e'_2}\left(-\frac{3}{2}-\frac{1}{2}D\right)b^{(1)}_{1/2}\right.\\
&+e'_2\left(6+\frac{13}{16}D-\frac{9}{8}D^2-\frac{3}{16}D^3\right)b^{(1)}_{1/2}\\
&+\frac{\tau'^2}{e'_2}\left[\left(\alpha+\frac{\alpha}{4}D\right)b^{(0)}_{3/2}+\left(\alpha+\frac{\alpha}{4}D\right)b^{(2)}_{3/2}\right]\\
&\left.+\frac{e_1'^2}{e'_2}\left(\frac{3}{2}+\frac{1}{8}D-\frac{1}{2}D^2-\frac{1}{8}D^3\right)b^{(1)}_{1/2}\right\}\frac{\sin(-\ell_1+2\ell_2-g_1+g_2)}{-1+2\alpha^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1}{e'_2}\left(-\frac{1}{2}-\frac{1}{2}D\right)b^{(0)}_{1/2}\right.\\
&+e'_2\left(\frac{7}{16}-\frac{1}{8}D-\frac{3}{4}D^2-\frac{3}{16}D^3\right)b^{(0)}_{1/2}\\
&+\frac{\tau'^2}{e'_2}\left(\alpha+\frac{\alpha}{2}D\right)b^{(1)}_{3/2}\\
&\left.+\frac{e_1'^2}{e'_2}\left(-\frac{1}{8}D-\frac{1}{4}D^2-\frac{1}{8}D^3\right)b^{(0)}_{1/2}\right]\frac{\sin\ell_2}{\alpha^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1}{e'_2}\left(\frac{1}{2}-\frac{1}{2}D\right)b^{(1)}_{1/2}\right.\\
&+e'_2\left(\frac{1}{8}+\frac{7}{16}D-\frac{3}{8}D^2-\frac{3}{16}D^3\right)b^{(1)}_{1/2}\\
&+\frac{\tau'^2}{e'_2}\left(\frac{\alpha}{4}D\,b^{(0)}_{3/2}+\frac{\alpha}{4}D\,b^{(2)}_{3/2}\right)\\
&\left.+\frac{e_1'^2}{e'_2}\left(-\frac{1}{2}+\frac{5}{8}D-\frac{1}{8}D^3\right)b^{(1)}_{1/2}\right]\sin(\ell_1+g_1-g_2)
\end{aligned}
$$

$$+\frac{e_1'}{e_2'}\left(\frac{3}{2}+\frac{5}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(1)}\ \frac{\sin(2\ell_2-g_1+g_2)}{2\,\alpha^{3/2}}$$

$$+\frac{e_1'}{e_2'}\left(\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(0)}\ \frac{\sin(\ell_1+\ell_2)}{1+\alpha^{3/2}}$$

$$+\frac{e_1'}{e_2'}\left(\frac{1}{2}-\frac{3}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(1)}\ \frac{\sin(2\ell_1+g_1-g_2)}{2}$$

$$+\frac{e_1'}{e_2'}\left(\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(0)}\ \frac{\sin(\ell_1-\ell_2)}{1-\alpha^{3/2}}$$

$$+\frac{e_1'}{e_2'}\left(-\frac{3}{2}+\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(1)}\ \frac{\sin(2\ell_1-2\ell_2+g_1-g_2)}{2-2\,\alpha^{3/2}}$$

$$+\left(-\frac{17}{4}-\frac{9}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(1)}\ \frac{\sin(-\ell_1+3\ell_2-g_1+g_2)}{-1+3\,\alpha^{3/2}}$$

$$+\left(-1-\frac{5}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(0)}\ \frac{\sin 2\ell_2}{2\,\alpha^{3/2}}$$

$$+\left(\frac{1}{4}-\frac{1}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(1)}\ \frac{\sin(\ell_1+\ell_2+g_1-g_2)}{1+\alpha^{3/2}}$$

$$+\frac{\tau'^2}{e_2'}\left(-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(1)}\ \frac{\sin(-2\ell_1+\ell_2-2g_1)}{-2+\alpha^{3/2}}$$

$$+\frac{\tau'^2}{e_2'}\left(-\frac{\alpha}{4}D\,b_{3/2}^{(0)}\right)\frac{\sin(-\ell_1-g_1-g_2)}{-1}$$

$$+\frac{\tau'^2}{e_2'}\left(\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(1)}\ \frac{\sin(-\ell_2-2g_2)}{-\alpha^{3/2}}$$

$$+\frac{\tau'^2}{e_2'}\left(-\frac{3\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(1)}\ \frac{\sin(3\ell_2+2g_2)}{3\,\alpha^{3/2}}$$

$$+ \frac{\tau'^2}{e'_2} \left(- a - \frac{a}{4} D\right) b_{3/2}^{(0)} \frac{\sin (\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2 a^{3/2}}$$

$$+ \frac{\tau'^2}{e'_2} \left(- \frac{a}{2} - \frac{a}{4} D\right) b_{3/2}^{(1)} \frac{\sin (2\ell_1 + \ell_2 + 2g_1)}{2 + a^{3/2}}$$

$$+ \frac{e_1'^2}{e'_2} \left(\frac{3}{16} - \frac{1}{8} D - \frac{1}{8} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2 a^{3/2}}$$

$$+ \frac{e_1'^2}{e'_2} \left(\frac{3}{16} D + \frac{1}{8} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin (2\ell_1 + \ell_2)}{2 + a^{3/2}}$$

$$+ \frac{e_1'^2}{e'_2} \left(\frac{9}{16} - D + \frac{1}{2} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (3\ell_1 + g_1 - g_2)}{3}$$

$$+ \frac{e_1'^2}{e'_2} \left(- \frac{1}{16} + \frac{1}{8} D - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \sin (\ell_1 - g_1 + g_2)$$

$$+ \frac{e_1'^2}{e'_2} \left(\frac{3}{16} D + \frac{1}{8} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin (2\ell_1 - \ell_2)}{2 - a^{3/2}}$$

$$+ \frac{e_1'^2}{e'_2} \left(- \frac{27}{16} + \frac{3}{4} D + \frac{1}{4} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin (3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2 a^{3/2}}$$

$$+ e'_1 \left(\frac{17}{4} + \frac{35}{8} D + \frac{11}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin (3\ell_2 - g_1 + g_2)}{3 a^{3/2}}$$

$$+ e'_1 \left(\frac{1}{2} D + \frac{5}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(0)} \frac{\sin (\ell_1 + 2\ell_2)}{1 + 2 a^{3/2}}$$

$$+ e'_1 \left(\frac{1}{4} - \frac{3}{8} D - \frac{1}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin (2\ell_1 + \ell_2 + g_1 - g_2)}{2 + a^{3/2}}$$

$$+ e'_1 \left(- \frac{1}{4} + \frac{1}{8} D + \frac{3}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin (- \ell_2 - g_1 + g_2)}{- a^{3/2}}$$

$$+ e_1' \left(\frac{1}{2} D + \frac{5}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(0)} \frac{\sin(\ell_1 - 2\ell_2)}{1 - 2\alpha^{3/2}}$$

$$+ e_1' \left(- \frac{17}{4} - \frac{1}{8} D + \frac{7}{8} D^2 + \frac{1}{8} D^3\right) b_{1/2}^{(1)} \frac{\sin(2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3\alpha^{3/2}}$$

$$+ e_2' \left(- \frac{71}{8} - \frac{95}{16} D - \frac{9}{8} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(-\ell_1 + 4\ell_2 - g_1 + g_2)}{-1 + 4\alpha^{3/2}}$$

$$+ e_2' \left(- \frac{27}{16} - \frac{19}{8} D - \frac{3}{4} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(0)} \frac{\sin 3\ell_2}{3\alpha^{3/2}}$$

$$\left. + e_2' \left(\frac{1}{4} - \frac{5}{16} D - \frac{3}{8} D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(1)} \frac{\sin(\ell_1 + 2\ell_2 + g_1 - g_2)}{1 + 2\alpha^{3/2}}\right) .$$

(32)

$$\frac{\partial S_{1p}}{\partial H_1'} = \frac{\sigma \beta_2 \alpha}{m_0} \left\{ \left(\frac{\alpha}{4} b_{3/2}^{(0)} + \frac{\alpha}{4} b_{3/2}^{(2)}\right) \frac{\sin(-\ell_1 + \ell_2 - g_1 + g_2)}{-1 + \alpha^{3/2}} \right.$$

$$+ e_1' \left[\left(- \frac{3\alpha}{8} - \frac{\alpha}{8} D\right) b_{3/2}^{(0)} + \left(- \frac{3\alpha}{8} - \frac{\alpha}{8} D\right) b_{3/2}^{(2)}\right] \frac{\sin(\ell_2 - g_1 + g_2)}{\alpha^{3/2}}$$

$$+ e_1' \left(- \frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \sin \ell_1$$

$$+ e_1' \left[\left(\frac{\alpha}{8} - \frac{\alpha}{8} D\right) b_{3/2}^{(0)} + \left(\frac{\alpha}{8} - \frac{\alpha}{8} D\right) b_{3/2}^{(2)}\right] \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{2 - \alpha^{3/2}}$$

$$+ e_2' \left[\left(\frac{\alpha}{2} + \frac{\alpha}{8} D\right) b_{3/2}^{(0)} + \left(\frac{\alpha}{2} + \frac{\alpha}{8} D\right) b_{3/2}^{(2)}\right] \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{-1 + 2\alpha^{3/2}}$$

$$+ e_2' \left(\frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(1)} \frac{\sin \ell_2}{\alpha^{3/2}}$$

$$+ e_2' \left(\frac{\alpha}{8} D\, b_{3/2}^{(0)} + \frac{\alpha}{8} D\, b_{3/2}^{(2)}\right) \sin(\ell_1 + g_1 - g_2)$$

$$+\left(-\frac{a}{4}\right)b_{3/2}^{(1)}\ \frac{\sin(-2\ell_1-2g_1)}{-2}$$

$$+\left(-\frac{a}{4}\right)b_{3/2}^{(0)}\ \frac{\sin(-\ell_1-\ell_2-g_1-g_2)}{-1-a^{3/2}}$$

$$+\left(-\frac{a}{4}\right)b_{3/2}^{(1)}\ \frac{\sin(-2\ell_2-2g_2)}{-2\,a^{3/2}}$$

$$+e_1'\left(\frac{5a}{8}+\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(-\ell_1-2g_1)}{-1}$$

$$+e_1'\left(\frac{3a}{8}+\frac{a}{8}D\right)b_{3/2}^{(0)}\ \frac{\sin(-\ell_2-g_1-g_2)}{-a^{3/2}}$$

$$+e_1'\left(\frac{a}{8}+\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(\ell_1-2\ell_2-2g_2)}{1-2\,a^{3/2}}$$

$$+e_1'\left(\frac{a}{8}+\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(\ell_1+2\ell_2+2g_2)}{1+2\,a^{3/2}}$$

$$+e_1'\left(-\frac{a}{8}+\frac{a}{8}D\right)b_{3/2}^{(0)}\ \frac{\sin(2\ell_1+\ell_2+g_1+g_2)}{2+a^{3/2}}$$

$$+e_1'\left(-\frac{3a}{8}+\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(3\ell_1+2g_1)}{3}$$

$$+e_2'\left(-\frac{a}{4}-\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(-2\ell_1+\ell_2-2g_1)}{-2+a^{3/2}}$$

$$+e_2'\left(-\frac{a}{8}D\,b_{3/2}^{(0)}\right)\frac{\sin(-\ell_1-g_1-g_2)}{-1}$$

$$+e_2'\left(\frac{a}{4}-\frac{a}{8}D\ \ b_{3/2}^{(1)}\right)\frac{\sin(-\ell_2-2g_2)}{-a^{3/2}}$$

$$+e_2'\left(-\frac{3a}{4}-\frac{a}{8}D\right)b_{3/2}^{(1)}\ \frac{\sin(3\ell_2+2g_2)}{3\,a^{3/2}}$$

$$+ e'_2 \left(- \frac{\alpha}{2} - \frac{\alpha}{8} D\right) b_{3/2}^{(0)} \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$+ e'_2 \left(- \frac{\alpha}{4} - \frac{\alpha}{8} D\right) b_{3/2}^{(1)} \frac{\sin(2\ell_1 + \ell_2 + 2g_1)}{2 + \alpha^{3/2}} \Bigg\} , \qquad (33)$$

$$\frac{\partial S_{1p}}{\partial H'_2} = \frac{\partial S_{1p}}{\partial G'_2} = (32) \quad . \qquad (34)$$

4.5 The expressions (29), (30), (31), (32), (33) show that $\partial S_{1p}/\partial L'_1$, $\partial S_{1p}/\partial L'_2$, $\partial S_{1p}/\partial G'_1$, $\partial S_{1p}/\partial G'_2$, $\partial S_{1p}/\partial H'_1$ are sums of, respectively, 37, 38, 45, 36, 22 terms. In each of these five partial derivatives, let us call terms of class zero those that arise from the terms of F_{1p} of class zero in the Newcomb sense and terms of class one those that arise from the terms of F_{1p} of class one in the Newcomb sense. The first 16 and the last 15 terms of $\partial S_{1p}/\partial L'_1$ are of class zero, the 6 others are of class one. The first 15 and the last 15 terms of $\partial S_{1p}/\partial L'_2$ are of class zero, the remaining 8 are of class one. The first 15 and the last 15 terms of $\partial S_{1p}/\partial G'_1$ are of class zero, the remaining 15 are of class one. The first 15 and the last 15 terms of $\partial S_{1p}/\partial G'_2$ are of class zero, the remaining 6 are of class one. The first 7 terms of $\partial S_{1p}/\partial H'_1$ are of class zero, the remaining 15 are of class one.

Small divisors in e'_1 appear in 20 terms of $\partial S_{1p}/\partial L'_1$ and in the 20 corresponding terms of $\partial S_{1p}/\partial G'_1$. They arise from the terms of S_{1p} in which e'_1 has the power one. In the same manner, small divisors in e'_2 appear in 20 terms of $\partial S_{1p}/\partial L'_2$ and in the 20 corresponding terms of $\partial S_{1p}/\partial G'_2$. They arise from the terms of S_{1p} in which e'_2 has the power one. Fourteen of the 20 small divisors in e'_1 of $\partial S_{1p}/\partial L'_1$ and $\partial S_{1p}/\partial G'_1$ appear in terms of class zero, the other 6 appear in terms of class one. Those that appear in terms of class one arise from the terms in $e_1' \tau'^2$ of F_{1p}. Fourteen of the 20 small divisors in e_2' of $\partial S_{1p}/\partial L_2'$ and $\partial S_{1p}/\partial G_2'$ appear in terms of class zero, the other 6 appear in terms of class one. Those that appear in terms of class one arise from the terms in $e_2' \tau'^2$ of F_{1p}.

The 20 small divisors in e_1' of $\partial S_{1p}/\partial L_1'$ and $\partial S_{1p}/\partial G_1'$ cancel out in the sum $\ell_1 + g_1$. In the same manner, the 20 small divisors in e_2' of $\partial S_{1p}/\partial L_2'$ and $\partial S_{1p}/\partial G_2'$ cancel out in the sum $\ell_2 + g_2$. The former do not appear in the Cartesian rectangular coordinates x_{1p}, y_{1p}, z_{1p} of the disturbed planet P_1 referred to the Sun S and dealing with the Hamiltonian F_{1p}. The latter do not appear in the Cartesian rectangular coordinates x_{2p}, y_{2p}, z_{2p} of the disturbing planet P_2 referred to the center of mass of S and P_1 and dealing with F_{1p} because x_{1p}, y_{1p}, z_{1p} depend upon ℓ_1 and g_1 through the sum $\ell_1 + g_1$, and x_{2p}, y_{2p}, z_{2p} depend upon ℓ_2 and g_2 through the sum $\ell_2 + g_2$.

<u>4.6</u> The old Delaunay variables L_1, G_1, H_1, ℓ_1, g_1, h_1 of P_1 are connected to its new Delaunay variables L_1', G_1', H_1', ℓ_1', g_1', h_1' that result from the elimination of the short-period terms of F_{1p} through the equations

$$
\begin{aligned}
L_1 &= L_1' + \frac{\partial S_{1p}}{\partial \ell_1}, & \ell_1 &= \ell_1' - \frac{\partial S_{1p}}{\partial L_1'}, \\
G_1 &= G_1' + \frac{\partial S_{1p}}{\partial g_1}, & g_1 &= g_1' - \frac{\partial S_{1p}}{\partial G_1'}, \\
H_1 &= H_1', & h_1 &= h_1' - \frac{\partial S_{1p}}{\partial H_1'},
\end{aligned}
\qquad (35)
$$

in which $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial H_1'$ are replaced by their values (29), (31), (33), and in which $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial g_1$ are replaced by their values obtained from (23).

In the same manner, the old Delaunay variables L_2, G_2, H_2, ℓ_2, g_2, h_2 of P_2 are connected to its new Delaunay variables L_2', G_2', H_2', ℓ_2', g_2' h_2' that result from the elimination of the short-period terms of F_{1p} through the equations

$$
\begin{aligned}
L_2 &= L_2' + \frac{\partial S_{1p}}{\partial \ell_2} , & \ell_2 &= \ell_2' - \frac{\partial S_{1p}}{\partial L_2'} , \\
G_2 &= G_2' + \frac{\partial S_{1p}}{\partial g_2} , & g_2 &= g_2' - \frac{\partial S_{1p}}{\partial G_2'} , \\
H_2 &= H_2' , & h_2 &= h_2' - \frac{\partial S_{1p}}{\partial G_2'} ,
\end{aligned}
\qquad (36)
$$

in which $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_2'$ are replaced by their values (30) (32), and in which $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_2$ are replaced by their values obtained from (23).

4.7 The two equations (35) containing $\partial S_{1p}/\partial L_1'$ and $\partial S_{1p}/\partial G_1'$ and the two equations (36) containing $\partial S_{1p}/\partial L_2'$ and $\partial S_{1p}/\partial G_2'$ together give a system of four equations with the four unknowns ℓ_1, ℓ_2, g_1, g_2. This system may be written

$$
\begin{aligned}
\ell_1 &= \ell_1' + \sigma\, M(\ell_1, \ell_2, g_1, g_2) , \\
g_1 &= g_1' + \sigma\, N(\ell_1, \ell_2, g_1, g_2) , \\
\ell_2 &= \ell_2' + \sigma\, P(\ell_1, \ell_2, g_1, g_2) , \\
g_2 &= g_2' + \sigma\, Q(\ell_1, \ell_2, g_1, g_2) ,
\end{aligned}
\qquad (37)
$$

with

$$
M = -\frac{1}{\sigma}\frac{\partial S_{1p}}{\partial L_1'} , \quad N = -\frac{1}{\sigma}\frac{\partial S_{1p}}{\partial G_1'} , \quad P = -\frac{1}{\sigma}\frac{\partial S_{1p}}{\partial L_2'} , \quad Q = -\frac{1}{\sigma}\frac{\partial S_{1p}}{\partial G_2'} .
$$

We obtain a solution of (37) by expanding ℓ_1, ℓ_2, g_1, g_2 in powers of σ according to Lagrange's formula extended to functions of several variables. Since we neglect the powers of σ higher than the first, we reduce these expansions to their first two terms and the solution of (37) is

$$\ell_1 = \ell_1' + \sigma M (\ell_1', \ell_2', g_1', g_2') \quad ,$$

$$g_1 = g_1' + \sigma N (\ell_1', \ell_2', g_1', g_2') \quad ,$$

$$\ell_2 = \ell_2' + \sigma P (\ell_1', \ell_2', g_1', g_2') \quad ,$$

$$g_2 = g_2' + \sigma Q (\ell_1', \ell_2', g_1', g_2') \quad .$$

These values of ℓ_1, ℓ_2, g_1, g_2 are introduced into the equations (35) containing $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial g_1$, $\partial S_{1p}/\partial H_1'$ and into the equations (36) containing $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_2$, h_2', $\partial S_{1p}/\partial G_2'$. Each expression

$$\sigma \begin{matrix} \sin \\ \cos \end{matrix} (p\ell_1 + q\ell_2 + y\, g_1 + z\, g_2)$$

becomes

$$\sigma \begin{matrix} \sin \\ \cos \end{matrix} \left[p\ell_1' + q\ell_2' + y g_1' + z\, g_2' + \sigma(pM + qP + rN + sQ) \right];$$

or, since we neglect the powers of σ higher than the first,

$$\sigma \begin{matrix} \sin \\ \cos \end{matrix} \left(p\ell_1' + q\ell_2' + yg_1' + zg_2' \right) \quad .$$

Thus, the old Delaunay variables are expressed in terms of the new Delaunay variables that result from the elimination of the short-period terms by replacing everywhere, under the signs sin and cos of the second members of equations (35) and (36), the old angular variables ℓ_1, ℓ_2, g_1, g_2 by the new ones ℓ_1', ℓ_2', g_1', g_2'.

5. ELIMINATION OF THE SHORT-PERIOD TERMS ARISING FROM F_{1i}

5.1 Equation (5) shows that $F_{1i} = F_{1i*}$, which means that F_{1i*} is the sum of 24 terms. To each of these terms there corresponds an equation (16), which may be written

$$A\frac{\partial S_{1i}}{\partial \ell_1} + B\frac{\partial S_{1i}}{\partial \ell_2} = C\cos(p\ell_1 + q\ell_2 + yg_1 + zg_2) \quad , \qquad (38)$$

with

$$C = \frac{-\sigma\alpha \; g(e'_1, e'_2, \tau')}{a'_2} \quad ,$$

and where A, B, p, q, y, z are the quantities that also appeared in equation (20). We consider the particular solution of (38):

$$S_{1i} = \frac{\sigma}{km_0^{1/2}} a_1'^{\,1/2} \frac{\alpha^2 g(e'_1, e'_2, \tau')}{p + q\alpha^{3/2}} \sin(p\ell_1 + q\ell_2 + yg_1 + zg_2) \quad . \, (39)$$

Each solution (39) is characterized by the five quantities $g(e'_1, e'_2, \tau')$, p, q, y, z. The sum of the 24 solutions (39) we thus obtain is S_{1i}. We have

$$S_{1i} = \frac{\sigma}{km_0^{1/2}} a_1'^{\,1/2} \alpha^2 \left[\tau'^2 \frac{\sin(\ell_1 + \ell_2 + g_1 + g_2)}{1 + \alpha^{3/2}} \right.$$

$$+ \left(1 - \tau'^2 - \frac{1}{2} e_1'^2 - \frac{1}{2} e_2'^2\right) \frac{\sin(\ell_1 - \ell_2 + g_1 - g_2)}{1 - \alpha^{3/2}}$$

$$+ \frac{1}{2} e'_1 \tau'^2 \frac{\sin(2\ell_1 + \ell_2 + g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$+\left(\frac{1}{2}e_1' - \frac{1}{2}e_1'\tau'^2 - \frac{3}{8}e_1'^3 - \frac{1}{4}e_1' e_2'^2\right)\frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{2 - a^{3/2}}$$

$$+\left(-\frac{3}{2}e_1' + \frac{3}{2}e_1'\tau'^2 + \frac{3}{4}e_1' e_2'^2\right)\frac{\sin(\ell_2 - g_1 + g_2)}{a^{3/2}}$$

$$+\left(-\frac{3}{2}e_1'\tau'^2\right)\frac{\sin(\ell_2 + g_1 + g_2)}{a^{3/2}}$$

$$+ 2e_2'\tau'^2\frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2a^{3/2}}$$

$$+\left(2e_2' - 2e_2'\tau'^2 - e_1'^2 e_2'\right)\frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{-1 + 2a^{3/2}}$$

$$+\frac{3}{8}e_1'^2\frac{\sin(3\ell_1 - \ell_2 + g_1 - g_2)}{3 - a^{3/2}}$$

$$+\frac{1}{8}e_1'^2\frac{\sin(\ell_1 + \ell_2 - g_1 + g_2)}{1 + a^{3/2}}$$

$$+(-3e_1' e_2')\frac{\sin(2\ell_2 - g_1 + g_2)}{2a^{3/2}}$$

$$+ e_1' e_2'\frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{2 - 2a^{3/2}}$$

$$+\frac{1}{8}e_2'^2\frac{\sin(\ell_1 + \ell_2 + g_1 - g_2)}{1 + a^{3/2}}$$

$$+ \frac{27}{8} e_2'^2 \frac{\sin(\ell_1 - 3\ell_2 + g_1 - g_2)}{1 - 3a^{3/2}}$$

$$+ \frac{1}{3} e_1'^3 \frac{\sin(4\ell_1 - \ell_2 + g_1 - g_2)}{4 - a^{3/2}}$$

$$+ \frac{1}{24} e_1'^3 \frac{\sin(2\ell_1 + \ell_2 - g_1 + g_2)}{2 + a^{3/2}}$$

$$+ \frac{1}{4} e_1'^2 e_2' \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2a^{3/2}}$$

$$+ \frac{3}{4} e_1'^2 e_2' \frac{\sin(3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2a^{3/2}}$$

$$+ \frac{1}{16} e_1' e_2'^2 \frac{\sin(2\ell_1 + \ell_2 + g_1 - g_2)}{2 + a^{3/2}}$$

$$+ \left(- \frac{81}{16} e_1' e_2'^2\right) \frac{\sin(3\ell_2 - g_1 + g_2)}{3a^{3/2}}$$

$$+ \frac{27}{16} e_1' e_2'^2 \frac{\sin(2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3a^{3/2}}$$

$$+ \left(- \frac{3}{16} e_1' e_2'^2\right) \frac{\sin(-\ell_2 - g_1 + g_2)}{-a^{3/2}}$$

$$+ \frac{1}{6} e_2'^3 \frac{\sin(\ell_1 + 2\ell_2 + g - g_2)}{1 + 2a^{3/2}}$$

$$\left. + \frac{16}{3} e_2'^3 \frac{\sin(-\ell_1 + 4\ell_2 - g_1 + g_2)}{-1 + 4a^{3/2}} \right] . \qquad (40)$$

We shall not write the values of $\partial S_{1i}/\partial \ell_1$, $\partial S_{1i}/\partial \ell_2$, $\partial S_{1i}/\partial g_1$, $\partial S_{1i}/\partial g_2$ immediately obtained from (40). The values of $\partial S_{1i}/\partial L_1'$, $\partial S_{1i}/\partial L_2'$, $\partial S_{1i}/\partial G_1'$, $\partial S_{1i}/\partial G_2'$, $\partial S_{1i}/\partial H_1'$ obtained from (40) may be written at once from the values of $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial G_2'$, $\partial S_{1p}/\partial H_1'$ obtained from (23) and expressed by equations (24), (25), (26), (27), (28) by replacing $k\beta_1\beta_2$ by $1/k$, and $f(\alpha)$ by α. Thus,

$$\frac{\partial S_{1i}}{\partial L_1'} = \frac{\sigma\alpha}{k^2 m_0 \beta_1}\left[p\left(5g\alpha + \frac{1-e_1'^2}{e_1'}\alpha\frac{\partial g}{\partial e_1'}\right)\right.$$

$$\left. + \alpha^{3/2} q\left(2g\alpha + \frac{1-e_1'^2}{e_1'}\alpha\frac{\partial g}{\partial e_1'}\right)\right]\frac{\sin(p\ell_1 + q\ell_2 + yg_1 + zg_2)}{\left(p + q\alpha^{3/2}\right)^2},$$

$$\frac{\partial S_{1i}}{\partial L_2'} = \frac{\sigma\alpha^{3/2}}{k^2 m_0 \beta_2}\left[p\left(-4g\alpha + \frac{1-e_2'^2}{e_2'}\alpha\frac{\partial g}{\partial e_2'}\right)\right.$$

$$\left. + \alpha^{3/2} q\left(-g\alpha + \frac{1-e_2'^2}{e_2'}\alpha\frac{\partial g}{\partial e_2'}\right)\right]\frac{\sin(p\ell_1 + q\ell_2 + yg_1 + zg_2)}{\left(p + q\alpha^{3/2}\right)^2},$$

$$\frac{\partial S_{1i}}{\partial G_1'} = \frac{\sigma\alpha^2}{k^2 m_0 \beta_1}\left[\frac{\partial g}{\partial e_1'}\left(-\frac{1}{e_1'} + \frac{1}{2}e_1' + \frac{1}{8}e_1'^3\right)\right.$$

$$\left. + \frac{1}{4}\frac{\partial g}{\partial \tau'}\left(\frac{1}{\tau'} + \frac{1}{2}\frac{e_1'^2}{\tau'} - 2\tau'\right)\right]\frac{\sin(p\ell_1 + q\ell_2 + yg_1 + zg_2)}{p + q\alpha^{3/2}},$$

$$\frac{\partial S_{1i}}{\partial G_2'} = \frac{\sigma a^{5/2}}{k^2 m_0 \beta_2} \frac{\partial g}{\partial e_2'} \left(- \frac{1}{e_2'} + \frac{1}{2} e_2' + \frac{1}{8} e_2'^{\,3} \right) \frac{\sin (p\ell_1 + q\ell_2 + yg_1 + zg_2)}{p + qa^{3/2}} ,$$

$$\frac{\partial S_{1i}}{\partial H_1'} = \frac{\sigma a^2}{k^2 m_0 \beta_1} \frac{\partial g}{\partial \tau'} \left(\frac{-1}{4\tau'} - \frac{1}{8} \frac{e_1'^{\,2}}{\tau'} \right) \frac{\sin (p\ell_1 + q\ell_2 + yg_1 + zg_2)}{p + qa^{3/2}} .$$

Each $\partial S_{1i}/\partial L_1'$, $\partial S_{1i}/\partial L_2'$, $\partial S_{1i}/\partial G_1'$, $\partial S_{1i}/\partial G_2'$, $\partial S_{1i}/\partial H_1'$ is reduced to its terms of order -1, 0, 1 and e_1', e_2' and to its terms of order 0, 1 in τ', and we sum up all these five partial derivatives. We have finally:

$$\begin{aligned}
\frac{\partial S_{1i}}{\partial L_1'} = \frac{\sigma a}{k^2 m_0 \beta_1} \Bigg\{ & (4a - a^{5/2}) \frac{\sin (\ell_1 - \ell_2 + g_1 - g_2)}{\left(1 - a^{3/2}\right)^2} \\
& + \left(a \frac{\tau'^{\,2}}{e_1'} + a^{5/2} \frac{1}{2} \frac{\tau'^{\,2}}{e_1'} \right) \frac{\sin (2\ell_1 + \ell_2 + g_1 + g_2)}{\left(2 + a^{3/2}\right)^2} \\
& + \Bigg[a \left(\frac{1}{e_1'} + \frac{7}{4} e_1' - \frac{\tau'^{\,2}}{e_1'} - \frac{1}{2} \frac{e_2'^{\,2}}{e_1'} \right) \\
& + a^{5/2} \left(- \frac{1}{2e_1'} + \frac{5}{8} e_1' + \frac{1}{2} \frac{\tau'^{\,2}}{e_1'} + \frac{1}{4} \frac{e_2'^{\,2}}{e_1'} \right) \Bigg] \frac{\sin (2\ell_1 - \ell_2 + g_1 - g_2)}{\left(2 - a^{3/2}\right)^2} \\
& + a^{5/2} \left(- \frac{3}{2e_1'} - \frac{3}{2} e_1' + \frac{3}{2} \frac{\tau'^{\,2}}{e_1'} + \frac{3}{4} \frac{e_2'^{\,2}}{e_1'} \right) \frac{\sin (\ell_2 - g_1 + g_2)}{\left(a^{3/2}\right)^2}
\end{aligned}$$

$$+ \alpha^{5/2} \left(-\frac{3}{2} \frac{\tau'^{2}}{e'_1}\right) \frac{\sin(\ell_2 + g_1 + g_2)}{\left(\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha(-8e'_2) + \alpha^{5/2}\, 4e'_2\right] \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{\left(-1 + 2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{9}{4} + \alpha^{5/2} \left(-\frac{3}{4}\right)\right] \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{\left(3 - \alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{4} + \alpha^{5/2} \frac{1}{4}\right) \frac{\sin(\ell_1 + \ell_2 - g_1 + g_2)}{\left(1 + \alpha^{3/2}\right)^2}$$

$$+ \alpha^{5/2} \left(-\frac{6e'_2}{e'_1}\right) \frac{\sin(2\ell_2 - g_1 + g_2)}{\left(2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{2e'_2}{e'_1} + \alpha^{5/2} \left(-\frac{2e'_2}{e'_1}\right)\right] \frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{\left(2 - 2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha\, 4e'_1 + \alpha^{5/2} (-e'_1)\right] \frac{\sin(4\ell_1 - \ell_2 + g_1 - g_2)}{\left(4 - \alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{1}{4} e'_1 + \alpha^{5/2} \frac{1}{8} e'_1\right] \frac{\sin(2\ell_1 + \ell_2 - g_1 + g_2)}{\left(2 + \alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{2} e'_2 + \alpha^{5/2} e'_2\right) \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{\left(1 + 2\alpha^{3/2}\right)^2}$$

$$+\left[\alpha\,\frac{9}{2}\,e_2' + \alpha^{5/2}\,(-\,3e_2')\right] \frac{\sin\,(3\ell_1 - 2\ell_2 + g_1 - g_2)}{\left(3 - 2\alpha^{3/2}\right)^2}$$

$$+\left[\alpha\left(\frac{1}{8}\,\frac{e_2'^2}{e_1'}\right) + \alpha^{5/2}\left(\frac{1}{16}\,\frac{e_2'^2}{e_1'}\right)\right] \frac{\sin\,(2\ell_1 + \ell_2 + g_1 - g_2)}{\left(2 + \alpha^{3/2}\right)^2}$$

$$+\,\alpha^{5/2}\left(-\,\frac{243}{16}\,\frac{e_2'^2}{e_1'}\right) \frac{\sin\,(3\ell_2 - g_1 + g_2)}{\left(3\alpha^{3/2}\right)^2}$$

$$+\left[\alpha\,\frac{27}{8}\,\frac{e_2'^2}{e_1'} + \alpha^{5/2}\left(-\,\frac{81}{16}\,\frac{e_2'^2}{e_1'}\right)\right] \frac{\sin\,(2\ell_1 - 3\ell_2 + g_1 - g_2)}{\left(2 - 3\alpha^{3/2}\right)^2}$$

$$\left. +\,\alpha^{5/2}\,\frac{3}{16}\,\frac{e_2'^2}{e_1'}\,\frac{\sin\,(-\ell_2 - g_1 + g_2)}{\left(-\alpha^{3/2}\right)^2}\right\}\,, \qquad (41)$$

$$\frac{\partial S_{1i}}{\partial L_2'} = \frac{\sigma\alpha^{3/2}}{k^2 m_0 \beta_2}\left\{\left[\alpha(-5) + \alpha^{5/2}\,2\right] \frac{\sin\,(\ell_1 - \ell_2 + g_1 - g_2)}{\left(1 - \alpha^{3/2}\right)^2}\right.$$

$$+\left[\alpha(-5e_1') + \alpha^{5/2}\,e_1'\right] \frac{\sin\,(2\ell_1 - \ell_2 + g_1 - g_2)}{\left(2 - \alpha^{3/2}\right)^2}$$

$$+\,\alpha^{5/2}\,3e_1'\,\frac{\sin\,(\ell_2 - g_1 + g_2)}{\left(\alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{2\tau'^2}{e_2'} + \alpha^{5/2} \frac{4\tau'^2}{e_2'}\right) \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{\left(1 + 2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha\left(-\frac{2}{e_2'} + \frac{2\tau'^2}{e_2'} + \frac{e_1'^2}{e_2'} + 10e_2'\right)\right.$$

$$\left. + \alpha^{5/2}\left(\frac{4}{e_2'} - \frac{4\tau'^2}{e_2'} - \frac{2e_1'^2}{e_2'} - 8e_2'\right)\right] \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{\left(-1 + 2\alpha^{3/2}\right)^2}$$

$$+ \alpha^{5/2}\left(-\frac{6e_1'}{e_2'}\right) \frac{\sin(2\ell_2 - g_1 + g_2)}{\left(2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{2e_1'}{e_2'} + \alpha^{5/2}\left(-\frac{2e_1'}{e_2'}\right)\right] \frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{\left(2 - 2\alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{4} + \alpha^{5/2} \frac{1}{4}\right) \frac{\sin(\ell_1 + \ell_2 + g_1 - g_2)}{\left(1 + \alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{27}{4} + \alpha^{5/2}\left(-\frac{81}{4}\right)\right] \frac{\sin(\ell_1 - 3\ell_2 + g_1 - g_2)}{\left(1 - 3\alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{4} \frac{e_1'^2}{e_2'} + \alpha^{5/2} \frac{1}{2} \frac{e_1'^2}{e_2'}\right) \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{\left(1 - 2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{9}{4} \frac{e_1'^2}{e_2'} + \alpha^{5/2} \left(-\frac{3}{2} \frac{e_1'^2}{e_2'} \right) \right] \frac{\sin(3\ell_1 - 2\ell_2 + g_1 - g_2)}{\left(3 - 2\alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{4} e_1' + \alpha^{5/2} \frac{1}{8} e_1' \right) \frac{\sin(2\ell_1 + \ell_2 + g_1 - g_2)}{\left(2 + \alpha^{3/2}\right)^2}$$

$$+ \alpha^{5/2} \left(-\frac{243}{8} e_1' \right) \frac{\sin(3\ell_2 - g_1 + g_2)}{\left(3\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha \frac{27}{4} e_1' + \alpha^{5/2} \left(-\frac{81}{8} e_1' \right) \right] \frac{\sin(2\ell_1 - 3\ell_2 + g_1 - g_2)}{\left(2 - 3\alpha^{3/2}\right)^2}$$

$$+ \alpha^{5/2} \frac{3}{8} e_1' \frac{\sin(-\ell_2 - g_1 + g_2)}{\left(-\alpha^{3/2}\right)^2}$$

$$+ \left(\alpha \frac{1}{2} e_1' + \alpha^{5/2} e_2' \right) \frac{\sin(\ell_1 + 2\ell_2 + g_1 - g_2)}{\left(1 + 2\alpha^{3/2}\right)^2}$$

$$+ \left[\alpha(-16 e_2') + \alpha^{5/2} 64 e_2' \right] \frac{\sin(-\ell_1 + 4\ell_2 - g_1 + g_2)}{\left(-1 + 4\alpha^{3/2}\right)^2} \Bigg\} , \qquad (42)$$

$$\frac{\partial S_{1i}}{\partial G_1'} = \frac{\sigma \alpha^2}{k^2 m_0 \beta_1} \left[\frac{1}{2} \frac{\sin(\ell_1 + \ell_2 + g_1 + g_2)}{1 + \alpha^{3/2}} \right.$$

$$+ \frac{1}{2} \frac{\sin(\ell_1 - \ell_2 + g_1 - g_2)}{1 - \alpha^{3/2}}$$

$$+ \left(- \frac{1}{2} \frac{\tau'^2}{e_1'} + \frac{1}{4} e_1'\right) \frac{\sin(2\ell_1 + \ell_2 + g_1 + g_2)}{2 + a^{3/2}}$$

$$+ \left(- \frac{1}{2e_1'} + \frac{9}{8} e_1' + \frac{1}{2} \frac{\tau'^2}{e_1'} + \frac{1}{4} \frac{e_2'^2}{e_1'}\right) \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{2 - a^{3/2}}$$

$$+ \left(\frac{3}{2e_1'} - \frac{3\tau'^2}{2e_1'} - \frac{3}{4} \frac{e_2'^2}{e_1'}\right) \frac{\sin(\ell_2 - g_1 + g_2)}{a^{3/2}}$$

$$+ \left(\frac{3}{2} \frac{\tau'^2}{e_1'} - \frac{3}{4} e_1'\right) \frac{\sin(\ell_2 + g_1 + g_2)}{a^{3/2}}$$

$$+ e_2' \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2a^{3/2}}$$

$$+ e_2' \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{-1 + 2a^{3/2}}$$

$$- \frac{3}{4} \frac{\sin(3\ell_1 - \ell_2 + g_1 - g_2)}{3 - a^{3/2}}$$

$$- \frac{1}{4} \frac{\sin(\ell_1 + \ell_2 - g_1 + g_2)}{1 + a^{3/2}}$$

$$+ \frac{3e_2'}{e_1'} \frac{\sin(2\ell_2 - g_1 + g_2)}{2a^{3/2}}$$

$$- \frac{e_2'}{e_1'} \frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{2 - 2a^{3/2}}$$

$$- e_1' \frac{\sin (4\ell_1 - \ell_2 + g_1 - g_2)}{4 - \alpha^{3/2}}$$

$$- \frac{1}{8} e_1' \frac{\sin (2\ell_1 + \ell_2 - g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$- \frac{1}{2} e_2' \frac{\sin (\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$- \frac{3}{2} e_2' \frac{\sin (3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2\alpha^{3/2}}$$

$$- \frac{1}{16} \frac{e_2'^2}{e_1'} \frac{\sin (2\ell_1 + \ell_2 + g_1 - g_2)}{2 + \alpha^{3/2}}$$

$$+ \frac{81}{16} \frac{e_2'^2}{e_1'} \frac{\sin (3\ell_2 - g_1 + g_2)}{3\alpha^{3/2}}$$

$$- \frac{27}{16} \frac{e_2'^2}{e_1'} \frac{\sin (2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3\alpha^{3/2}}$$

$$\left. + \frac{3}{16} \frac{e_2'^2}{e_1'} \frac{\sin (-\ell_2 - g_1 + g_2)}{-\alpha^{3/2}} \right] , \tag{43}$$

$$\frac{\partial S_{1i}}{\partial G_2'} = \frac{\sigma \alpha^{5/2}}{k^2 m_0 \beta_2} \left[\frac{\sin (\ell_1 - \ell_2 + g_1 - g_2)}{1 - \alpha^{3/2}} \right.$$

$$+ \frac{1}{2} e_1' \frac{\sin (2\ell_1 - \ell_2 + g_1 - g_2)}{2 - \alpha^{3/2}}$$

$$- \frac{3}{2} e'_1 \frac{\sin(\ell_2 - g_1 + g_2)}{\alpha^{3/2}}$$

$$- \frac{2\tau'^2}{e'_2} \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$+ \left(- \frac{2}{e'_2} + \frac{2\tau'^2}{e'_2} + \frac{e'^2_1}{e'_2} + e'_2 \right) \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{-1 + 2\alpha^{3/2}}$$

$$+ \frac{3e'_1}{e'_2} \frac{\sin(2\ell_2 - g_1 + g_2)}{2\alpha^{3/2}}$$

$$- \frac{e'_1}{e'_2} \frac{\sin(2\ell_1 - 2\ell_2 + g_1 - g_2)}{2 - 2\alpha^{3/2}}$$

$$- \frac{1}{4} \frac{\sin(\ell_1 + \ell_2 + g_1 - g_2)}{1 + \alpha^{3/2}}$$

$$- \frac{27}{4} \frac{\sin(\ell_1 - 3\ell_2 + g_1 - g_2)}{1 - 3\alpha^{3/2}}$$

$$- \frac{1}{4} \frac{e'^2_1}{e'_2} \frac{\sin(\ell_1 + 2\ell_2 - g_1 + g_2)}{1 + 2^{3/2}}$$

$$- \frac{3}{4} \frac{e'^2_1}{e'_2} \frac{\sin(3\ell_1 - 2\ell_2 + g_1 - g_2)}{3 - 2\alpha^{3/2}}$$

$$- \frac{1}{8} e'_1 \frac{\sin(2\ell_1 + \ell_2 + g_1 - g_2)}{2 + \alpha^{3/2}}$$

$$+\frac{81}{8}\, e'_1 \, \frac{\sin(3\ell_2 - g_1 + g_2)}{3\alpha^{3/2}}$$

$$-\frac{27}{8}\, e'_1 \, \frac{\sin(2\ell_1 - 3\ell_2 + g_1 - g_2)}{2 - 3\alpha^{3/2}}$$

$$+\frac{3}{8}\, e'_1 \, \frac{\sin(-\ell_2 - g_1 + g_2)}{-\alpha^{3/2}}$$

$$-\frac{1}{2}\, e'_2 \, \frac{\sin(\ell_1 + 2\ell_2 + g_1 - g_2)}{1 + 2\alpha^{3/2}}$$

$$\left. -\, 16 e'_2 \, \frac{\sin(-\ell_1 + 4\ell_2 - g_1 + g_2)}{-1 + 4\alpha^{3/2}} \right] , \qquad (44)$$

$$\frac{\partial S_{1i}}{\partial H'_1} = \frac{\sigma\alpha^2}{k^2 m_0 \beta_1} \left[-\frac{1}{2} \, \frac{\sin(\ell_1 + \ell_2 + g_1 + g_2)}{1 + \alpha^{3/2}} \right.$$

$$+\frac{1}{2} \, \frac{\sin(\ell_1 - \ell_2 + g_1 - g_2)}{1 - \alpha^{3/2}}$$

$$-\frac{1}{4}\, e'_1 \, \frac{\sin(2\ell_1 + \ell_2 + g_1 + g_2)}{2 + \alpha^{3/2}}$$

$$+\frac{1}{4}\, e'_1 \, \frac{\sin(2\ell_1 - \ell_2 + g_1 - g_2)}{2 - \alpha^{3/2}}$$

$$- \frac{3}{4} e'_1 \frac{\sin(\ell_2 - g_1 + g_2)}{\alpha^{3/2}}$$

$$+ \frac{3}{4} e'_1 \frac{\sin(\ell_2 + g_1 + g_2)}{\alpha^{3/2}}$$

$$- e'_2 \frac{\sin(\ell_1 + 2\ell_2 + g_1 + g_2)}{1 + 2\alpha^{3/2}}$$

$$\left. + e'_2 \frac{\sin(-\ell_1 + 2\ell_2 - g_1 + g_2)}{-1 + 2\alpha^{3/2}} \right] , \tag{45}$$

$$\frac{\partial S_{1i}}{\partial H'_2} = \frac{\partial S_{1i}}{\partial G'_2} = \text{equation (44)} \quad . \tag{46}$$

5.2 Small divisors in e'_1 appear in 10 terms of $\partial S_{1i}/\partial L'_1$ and in the 10 corresponding terms of $\partial S_{1i}/\partial G'_1$. Small divisors in e'_2 appear in six terms of $\partial S_{1i}/\partial L'_2$ and in the six corresponding terms of $\partial S_{1i}/\partial G'_2$. The former arise from the terms of S_{1i} in which e'_1 has the power one, they cancel out in the sum $\ell_1 + g_1$, and they do not appear in the Cartesian rectangular coordinates x_{1i}, y_{1i}, z_{1i} of P_1 dealing with the Hamiltonian F_{1i}. The latter arise from the terms of S_{1i} in which e'_2 has the power one, they cancel out in the sum $\ell_2 + g_2$, and they do not appear in the Cartesian rectangular coordinates x_{2i}, y_{2i}, z_{2i}, of P_2 dealing with F_{1i}. No small divisors appear in $\partial S_{1i}/\partial H'_1$.

5.3 The old Delaunay variables L_1, G_1, H_1, ℓ_1, g_1, h_1 of P_1 are connected to its new Delaunay variables L'_1, G'_1, H'_1, ℓ'_1, g'_1, h'_1, which result from the elimination of the short-period terms of F_{1i} through the equations

$$L_1 = L_1' + \frac{\partial S_{1i}}{\partial \ell_1}, \qquad \ell_1 = \ell_1' - \frac{\partial S_{1i}}{\partial L_1'},$$

$$G_1 = G_1' + \frac{\partial S_{1i}}{\partial g_1}, \qquad g_1 = g_1' - \frac{\partial S_{1i}}{\partial G_1'},$$

$$H_1 = H_1', \qquad h_1 = h_1' - \frac{\partial S_{1i}}{\partial H_1'}, \tag{47}$$

in which $\partial S_{1i}/\partial L_1'$, $\partial S_{1i}/\partial G_1'$, $\partial S_{1i}/\partial H_1'$ are replaced by their values (41), (43), (45), and in which $\partial S_{1i}/\partial \ell_1$, $\partial S_{1i}/\partial g_1$ are replaced by their values obtained from (40). In the same manner, the old Delaunay variables L_2, G_2, H_2, ℓ_2, g_2, h_2 of P_2 are connected to its new Delaunay variables L_2', G_2', H_2', ℓ_2', g_2', h_2', which result from the elimination of the short-period terms of F_{1i} through the equations

$$L_2 = L_2' + \frac{\partial S_{1i}}{\partial \ell_2}, \qquad \ell_2 = \ell_2' - \frac{\partial S_{1i}}{\partial L_2'},$$

$$G_2 = G_2' + \frac{\partial S_{1i}}{\partial g_2}, \qquad g_2 = g_2' - \frac{\partial S_{1i}}{\partial G_2'},$$

$$H_2 = H_2', \qquad h_2 = h_2' - \frac{\partial S_{1i}}{\partial G_2'}, \tag{48}$$

in which $\partial S_{1i}/\partial L_2'$ and $\partial S_{1i}/\partial G_2'$ are replaced by their values (42) and (44), and in which $\partial S_{1i}/\partial \ell_2$ and $\partial S_{1i}/\partial g_2$ are replaced by their values obtained from (40).

What we said about equations (35) and (36) is true also for equations (47) and (48). In the second members of (47) and (48), the old angular variables ℓ_1, ℓ_2, g_1, g_2 have to be replaced by the new ones ℓ_1', ℓ_2', g_1', g_2'.

6. THE CANONICAL EQUATIONS OF THE NEW HAMILTONIANS F'_{1p} AND F'_{1i}

The elimination of the short-period terms arising from F_{1p} transforms the system of canonical equations

$$
\begin{array}{llll}
\dfrac{dL_1}{dt} = \dfrac{\partial F_{1p}}{\partial \ell_1}, & \dfrac{d\ell_1}{dt} = -\dfrac{\partial F_{1p}}{\partial L_1}, & \dfrac{dL_2}{dt} = \dfrac{\partial F_{1p}}{\partial \ell_2}, & \dfrac{d\ell_2}{dt} = -\dfrac{\partial F_{1p}}{\partial L_2}, \\[2ex]
\dfrac{dG_1}{dt} = \dfrac{\partial F_{1p}}{\partial g_1}, & \dfrac{dg_1}{dt} = -\dfrac{\partial F_{1p}}{\partial G_1}, & \dfrac{dG_2}{dt} = \dfrac{\partial F_{1p}}{\partial g_2}, & \dfrac{dg_2}{dt} = -\dfrac{\partial F_{1p}}{\partial G_2}, \\[2ex]
\dfrac{dH_1}{dt} = 0, & \dfrac{dh_1}{dt} = -\dfrac{\partial F_{1p}}{\partial H_1}, & \dfrac{dH_2}{dt} = 0, & \dfrac{dh_2}{dt} = -\dfrac{\partial F_{1p}}{\partial G_2},
\end{array}
\tag{49}
$$

in which F_{1p} is expressed by equation (6), into the system

$$
\begin{array}{llll}
\dfrac{dL'_1}{dt} = 0, & \dfrac{d\ell'_1}{dt} = -\dfrac{\partial F'_{1p}}{\partial L'_1}, & \dfrac{dL'_2}{dt} = 0, & \dfrac{d\ell'_2}{dt} = -\dfrac{\partial F'_{1p}}{\partial L'_2}, \\[2ex]
\dfrac{dG'_1}{dt} = \dfrac{\partial F'_{1p}}{\partial g'_1}, & \dfrac{dg'_1}{dt} = -\dfrac{\partial F'_{1p}}{\partial G'_1}, & \dfrac{dG'_2}{dt} = \dfrac{\partial F'_{1p}}{\partial g'_2}, & \dfrac{dg'_2}{dt} = -\dfrac{\partial F'_{1p}}{\partial G'_2}, \\[2ex]
\dfrac{dH'_1}{dt} = 0, & \dfrac{dh'_1}{dt} = -\dfrac{\partial F'_{1p}}{\partial H'_1}, & \dfrac{dH'_2}{dt} = 0, & \dfrac{dh'_2}{dt} = -\dfrac{\partial F'_{1p}}{\partial G'_2},
\end{array}
\tag{50}
$$

in which F_{1p} is expressed by equation (19).

In the same manner, the elimination of the short-period terms that arise from F_{1i} transforms the system of canonical equations

$$\frac{dL_1}{dt} = \frac{\partial F_{1i}}{\partial \ell_1}, \quad \frac{d\ell_1}{dt} = -\frac{\partial F_{1i}}{\partial L_1}, \quad \frac{dL_2}{dt} = \frac{\partial F_{1i}}{\partial \ell_2}, \quad \frac{d\ell_2}{dt} = -\frac{\partial F_{1i}}{\partial L_2},$$

$$\frac{dG_1}{dt} = \frac{\partial F_{1i}}{\partial g_1}, \quad \frac{dg_1}{dt} = -\frac{\partial F_{1i}}{\partial G_1}, \quad \frac{dG_2}{dt} = \frac{\partial F_{1i}}{\partial g_2}, \quad \frac{dg_2}{dt} = -\frac{\partial F_{1i}}{\partial G_2},$$

$$\frac{dH_1}{dt} = 0, \quad \frac{dh_1}{dt} = -\frac{\partial F_{1i}}{\partial H_1}, \quad \frac{dH_2}{dt} = 0, \quad \frac{dh_2}{dt} = -\frac{\partial F_{1i}}{\partial G_2}, \tag{51}$$

in which F_{1i} is expressed by equation (5), into the system

$$\frac{dL'_1}{dt} = 0, \quad \frac{d\ell'_1}{dt} = -\frac{\partial F'_{1i}}{\partial L'_1} = 0, \quad \frac{dL'_2}{dt} = 0, \quad \frac{d\ell'_2}{dt} = -\frac{\partial F'_{1i}}{\partial L'_2} = 0,$$

$$\frac{dG'_1}{dt} = \frac{\partial F'_{1i}}{\partial g'_1} = 0, \quad \frac{dg'_1}{dt} = -\frac{\partial F'_{1i}}{\partial G'_1} = 0, \quad \frac{dG'_2}{dt} = \frac{\partial F'_{1i}}{\partial g'_2} = 0, \quad \frac{dg'_2}{dt} = -\frac{\partial F'_{1i}}{\partial G'_2} = 0,$$

$$\frac{dH'_1}{dt} = 0, \quad \frac{dh'_1}{dt} = -\frac{\partial F'_{1i}}{\partial H'_1} = 0, \quad \frac{dH'_2}{dt} = 0, \quad \frac{dh'_2}{dt} = -\frac{\partial F'_{1i}}{\partial G'_2} = 0, \tag{52}$$

whose Hamiltonian F'_{1i} is identically equal to zero. Equation (52) is solved: its linear variables L'_i, G'_i, H'_i and its angular variables ℓ'_i, g'_i, h'_i $(i = 1, 2)$ are constants.

The elimination of the long-period terms therefore concerns only the system of canonical equations whose Hamiltonian is F_{1p}, and hence is performed in equations (50).

7. CONCLUSIONS

7.1 We showed that in a first-order general planetary theory in which we neglect the powers of eccentricities and mutual inclination higher than the third, the elimination of the short-period terms that arise from the indirect part F_{1i} of the disturbing function solves the system of canonical equations whose Hamiltonian is F_{1i}, and the elimination of the short-period terms that arise from the principal part F_{1p} of the disturbing function transforms the system of canonical equations whose Hamiltonian is F_{1p} into a system of canonical equations whose Hamiltonian F'_{1p} is the sum of four secular terms and a long-period term. The elimination of the long-period terms deals therefore only with the system of canonical equations whose Hamiltonian is F'_{1p}.

7.2 The results we obtain are a consequence of our hypothesis. It would be necessary to see if they hold true when we consider higher powers of the eccentricities and mutual inclination. We know F_{1i} contains only short-period terms irrespective of the powers of the eccentricities and mutual inclination (Brown and Shook, 1933; Brouwer and Clemence, 1961). This means that the elimination of the short-period terms always solves the system of canonical equations whose Hamiltonian is F_{1i}, and that the elimination of the long-period terms deals therefore only with the system of canonical equations whose Hamiltonian is F_{1p}. But this latter elimination becomes more and more intricate as the powers of the eccentricities and mutual inclination increase. Beyond the third powers of the eccentricities and mutual inclination, F'_{1p} contains more than four secular terms and more than one long-period term.

7.3 On the other hand, we considered only one disturbing planet. It would be useful to investigate the case in which we consider two and, more generally, n disturbing planets instead of one. Let us call

$P_2, \ldots, P_{n+1}$ those n planets respectively referred to the center of mass of S and $P_1, \ldots,$ to the center of mass of P_n and an imaginary planet referred to the center of mass of $P_1, \ldots, P_{n-1}$. We then have to consider n mutual inclinations instead of one: the inclination τ_1 of the orbital plane of P_1 on the orbital plane of $P_2, \ldots,$ the inclination τ_n of the orbital plane of P_n on the orbital plane of P_{n+1}. The formulas become much more intricate.

7.4 Moreover, in a second-order theory, the old Delaunay variables $L_i, G_i, H_i, \ell_i, g_i, h_i$ cannot be expressed in terms of the new Delaunay variables $L_i', G_i', H_i', \ell_i', g_i', h_i'$ $(i = 1, 2)$ that arise from the elimination of the short-period terms only by replacing inside the partial derivatives of S_{1p} and S_{1i} with respect to $L_i', G_i', H_i', \ell_i, g_i, h_i$ the old angular variables ℓ_1, ℓ_2, g_1, g_2 by the new ones $\ell_1', \ell_2', g_1', g_2'$ as we do in a first-order theory. The Lagrange formula extended to functions of several variables shows indeed that the coefficients of the sines and cosines are not the same, from the second powers of the masses, in the equations containing the partial derivatives of S_{1p} and S_{1i} with respect to L_i', G_i' $(i = 1, 2)$ and in their solutions. Besides, according to our previous notation, the formula

$$\sigma \begin{matrix}\sin\\ \cos\end{matrix} (p\ell_1 + q\ell_2 + yg_1 + zg_2) \sim \sigma \begin{matrix}\sin\\ \cos\end{matrix} (p\ell_1' + q\ell_2' + yg_1' + zg_2') \quad ,$$

which holds true in a first-order theory, has to be replaced, in a second-order theory, by the formulas

$$\sigma \sin (p\ell_1 + q\ell_2 + yg_1 + zg_2) \sim \sigma \sin (p\ell_1' + q\ell_2' + yg_1' + zg_2')$$

$$+ \sigma^2 (pM + qP + rN + sQ) \cos (p\ell_1' + q\ell_2' + yg_1' + zg_2') \quad ,$$

$$\sigma \cos (p\ell_1 + q\ell_2 + yg_1 + zg_2) \sim \sigma \cos (p\ell'_1 + q\ell'_2 + yg'_1 + zg'_2)$$

$$- \sigma^2 (pM + qP + rN + sQ) \sin (p\ell'_1 + q\ell'_2 + yg'_1 + zg'_2) \quad ,$$

in which M, N, P, Q are the quantities that have been defined with respect to equations (37).

7.5 In order to build a first-order general planetary theory by means of von Zeipel's method and compare it with the previous theories, it is necessary to check the disturbing function up to the eighth power of the eccentricities and mutual inclination. This calculation may be carried out by the procedure developed in the present paper. An extension of the principal part of the disturbing function up to the eighth power of the eccentricities and mutual inclination was made by Newcomb (1895), but erroneous coefficients in the eighth powers were pointed out and corrected by Sharaf (1955) and emphasized by Izsak et al. (1964). It would be interesting to compare the development of Newcomb corrected by Sharaf to the development of F_{1p} carried out up to the eighth power of the eccentricities and mutual inclination according to our procedure. Such a development must also be carried out for F_{1i}. The elimination of the short-period terms from these two developments would be the first step in the building, through von Zeipel's method, of a first-order general planetary theory.

7.6 The analytical formula we obtain in the frame of our hypothesis concerning the eccentricities and mutual inclination shows us that the calculation of the angular variables that deal with the principal part of the disturbing function is reduced to that of the three expressions

$$\frac{1}{\left(p + q\alpha^{3/2}\right)^2}(a + bD + cD^2 + dD^3)b_{1/2}^{(j)} \quad ,$$

$$\frac{\alpha^{3/2}}{\left(p + q\alpha^{3/2}\right)^2}(a + bD + cD^2 + dD^3)b_{1/2}^{(j)} \quad ,$$

$$\frac{1}{p + q\alpha^{3/2}}(a + bD + cD^2 + dD^3)b_{1/2}^{(j)} \quad ,$$

with $j = 0, 1$, and to that of the three expressions

$$\frac{1}{\left(p + q\alpha^{3/2}\right)^2}(a\alpha + b\alpha D)b_{3/2}^{(j)} \quad ,$$

$$\frac{\alpha^{3/2}}{\left(p + q\alpha^{3/2}\right)^2}(a\alpha + b\alpha D)b_{3/2}^{(j)} \quad ,$$

$$\frac{1}{p + q\alpha^{3/2}}(a\alpha + b\alpha D)b_{3/2}^{(j)} \quad ,$$

with $j = 0, 1, 2$, where a, b, c, d are positive or negative rational numbers, and p and q are positive or negative integers. This calculation, which can be easily computed, has been applied to the two particular cases of Jupiter disturbed by Saturn and of Mars disturbed by the Earth. It has been programmed in such a way as to include, later on, terms of higher order in the eccentricities and mutual inclination, especially the eighth-order terms of a first-order general planetary theory. A detailed report of this calculation will be given in a later paper. Our coefficients will then be compared to those obtained by Le Verrier (1876) in his theory of Jupiter and to those of Clemence (1949) in his theory of Mars.

7.7 In the above mentioned expressions, we assumed essentialy that $p + q\alpha^{3/2} \neq 0$. If $p + q\alpha^{3/2}$ is close to zero — we let aside the case in which $p + q\alpha^{3/2}$ is strictly equal to zero — and if p and q are small and not greatly different integers, the above mentioned expressions are called critical terms and their calculation is a very important problem in general planetary theory. Phase terms must be excluded from the determining functions S_{1p} and S_{1i} and carried along instead with the secular and long period terms. We plan to study them, later on, in a systematic manner.

8. ACKNOWLEDGMENTS

I wish to express my deep gratitude to the late Professor Dirk Brouwer, Director of the Yale Observatory, who suggested this research to me. I worked with him as Research Associate during the academic year 1962-63, and benefited greatly from his unceasing encouragement. His sudden death on 31 January 1966 was a great personal loss to me.

I am grateful to Dr. Fred Whipple, Director of the Smithsonian Astrophysical Observatory, for allowing me to pursue this work in the Research and Analysis Division of SAO.

I also wish to thank my friends and colleagues at the Yale Observatory and SAO with whom I had fruitful discussions, especially with Dr. Hamid, Dr. Hori, Dr. Lundquist, and Dr. Marsden.

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