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ON THE ELIMINATION OF THE SHORT PERIOD TERMS OF A FIRST ORDER GENERAL PLANETARY THEORY THROUGH VON ZEIPEL'S METHOD

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ABSTRACT

We previously eliminated, through Von Zeipel's method, the short period terms of a first order general planetary theory in which we neglect the powers of eccentricities and mutual inclination higher than the third power and in which we reduce the Fourier series of the principal part F_{1p} of the disturbing function and the Fourier series of the determining function S_{1p} to the sum of their three first terms. In the present paper, we extend our results by reducing, more generally, the Fourier series of F_{1p} and that of S_{1p} to the sum of their $2p+1$ first terms ($p > 1$). In doing so, we much enlarge the field of application of our results.

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1. In a previous paper presently in press* and of which we adopt the notations, we indicated how to eliminate, through Von Zeipel's method, the short period terms of a first order general planetary theory. In order to shorten as much as we could the calculations, we considered only one disturbing planet, we referred the disturbed planet to the orbital plane of the disturbing planet, we calculated the longitudes from the longitude of the ascending node of the disturbed planet and we neglected the powers of eccentricities and mutual inclination higher than the third. Furthermore, we calculated the principal part of the disturbing function through Newcomb operators and Laplace coefficients, the latter being reduced, when we neglect the powers of eccentricities and mutual inclination higher than the third, to the four sets of coefficients $b_{1/2}^{(j)}$, $b_{3/2}^{(j)}$, $b_{3/2}^{(j-1)}$, $b_{3/2}^{(j+1)}$, $j = 0, \pm 1, \pm 2 \cdots$. We developed our calculations considering only for j the three values 0, ±1, that is to say considering only the five coefficients $b_{1/2}^{(0)}$, $b_{1/2}^{(1)}$, $b_{3/2}^{(0)}$, $b_{3/2}^{(1)}$, $b_{3/2}^{(2)}$, the developments in a Fourier series of the principal part F_{1p} of the disturbing function, of the determining function S_{1p} which eliminates the short period terms and of the partial derivatives of S_{1p} with respect to the linear and angular variables being thus reduced to their three first terms. Our formulas are therefore practically applicable only in the particular case of a set of two planets for which it is not necessary to consider, when we neglect the powers of eccentricities and mutual inclination higher than the third, Laplace coefficients differ from the five above coefficients. In order to give more generality to our formulas and in order to apply them — in the restricted frame of our hypothesis concerning the powers of eccentricities and mutual inclination — to any set of two planets, we do not specify the values of the index j and we reduce the above mentioned developments in a Fourier series not to the sum of their three first terms as we did in our previous paper but, more generally, to the sum of their $2p + 1$ first terms ($j = 0, \pm 1, \cdots, \pm p$), the value of the positive integer p depending upon the set of the two planets that we consider.

2. We recall that k is the constant of gravitation; m_0 the mass of the Sun; σ a small parameter whose order of magnitude is that of the masses of the two planets P_1 and P_2; β_1 and β_2 two finite numerical coefficients; $\beta_1 \sigma$ the mass of the disturbed planet P_1; $\beta_2 \sigma$ the mass of the disturbing

*SAO, Special Reports, Cambridge, Mass.

planet P_2; a_1, e_1, ℓ_1, g_1 the semi major axis, the eccentricity, the mean longitude and the longitude of perihelia of P_1; a_2, e_2, ℓ_2, g_2 the semi major axis, the eccentricity, the mean longitude and the longitude of perihelia of P_2; I the inclination of the orbital plane of P_1 on the orbital plane of P_2, τ the sine of the semi angle $I/2$, α the ratio a_1/a_2, D the operator $\alpha\, d/d\alpha$.

The principal part of the disturbing function is:

$$F_{1p} = \frac{\sigma k^2 \beta_1 \beta_2}{a_2} \sum_{j=-p}^{j=+p} \Bigg[\Bigg\{\frac{1}{2} b_{1/2}^{(j)} + e_1^2 \left(-\frac{j^2}{2} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(j)} + e_2^2 \left(-\frac{j^2}{2} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(j)}$$

$$+ \tau^2 \left(-\frac{\alpha}{4} b_{3/2}^{(j-1)} - \frac{\alpha}{4} b_{3/2}^{(j+1)}\right)\Bigg\} \cos\left(-j\ell_1 + j\ell_2 - jg_1 + jg_2\right)$$

$$+ e_1 e_2 \left(j^2 + \frac{j}{2} - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(j)} \cos\left((-j-1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)$$

$$+ e_1^2 e_2 \Bigg\{-\frac{j^3}{3} + \frac{7}{8} j^2 - \frac{5}{16} j + \left(-\frac{j^2}{4} + \frac{5}{16} j - \frac{3}{16}\right) D + \left(\frac{j}{8} - \frac{1}{8}\right) D^2$$

$$+ \frac{1}{16} D^3\Bigg\} b_{1/2}^{(j)} \cos\left((-j+2)\ell_1 + (j-1)\ell_2 - jg_1 + jg_2\right)$$

$$+ \Bigg\{e_1 \left(-j - \frac{1}{2} D\right) b_{1/2}^{(j)} + e_1 \tau^2 \left(\left(\frac{j}{2}\alpha + \frac{\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(j-1)} + \left(\frac{j}{2}\alpha + \frac{\alpha}{4} + \frac{\alpha}{4} D\right) b_{3/2}^{(j+1)}\right)$$

$$+ e_1^3 \left(\frac{j^3}{2} - \frac{5}{8} j^2 + \frac{j}{8} + \left(\frac{j^2}{4} - \frac{5}{16} j + \frac{3}{16}\right) D + \left(-\frac{j}{8} + \frac{1}{16}\right) D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(j)}$$

$$+ e_1 e_2^2 \left(j^3 + \left(\frac{j^2}{2} - \frac{j}{4}\right) D + \left(-\frac{j}{4} - \frac{1}{8}\right) D^2 - \frac{1}{8} D^3\right) b_{1/2}^{(j)}\Bigg\} \times \cos\left((-j+1)\ell_1 + j\ell_2 - jg_1 + jg_2\right)$$

$$+ \Bigg\{e_2 \left(j + \frac{1}{2} + \frac{1}{2} D\right) b_{1/2}^{(j)} + e_2 \tau^2 \left(\left(-\frac{j}{2}\alpha - \frac{\alpha}{2} - \frac{\alpha}{4} D\right) b_{3/2}^{(j-1)} + \left(-\frac{j}{2}\alpha - \frac{\alpha}{2} - \frac{\alpha}{4} D\right) b_{3/2}^{(j+1)}\right)$$

$$+ e_2^3 \left(-\frac{j^3}{2} - \frac{7}{8} j^2 - \frac{5}{16} j + \left(-\frac{j^2}{4} - \frac{j}{16} + \frac{1}{8}\right) D + \left(\frac{j}{8} + \frac{1}{4}\right) D^2 + \frac{1}{16} D^3\right) b_{1/2}^{(j)}$$

$$+ e_1^2 e_2 \left(- j^3 - \frac{j^2}{2} + \left(- \frac{j^2}{2} + \frac{j}{4} + \frac{1}{8} \right) D + \left(\frac{j}{4} + \frac{1}{4} \right) D^2 + \frac{1}{8} D^3 \right) b_{1/2}^{(j)} \Big\}$$

$$\times \cos \left(- j\ell_1 + (j+1)\ell_2 - jg_1 + jg_2 \right)$$

$$+ e_1 e_2^2 \left\{ \frac{j^3}{2} + \frac{9}{8} j^2 + \frac{j}{2} + \left(\frac{j^2}{4} + \frac{j}{16} - \frac{1}{4} \right) D + \left(- \frac{j}{8} - \frac{5}{16} \right) D^2 \right.$$

$$\left. - \frac{1}{16} D^3 \right\} b_{1/2}^{(j)} \cos \left((-j-1)\ell_1 + (j+2)\ell_2 - jg_1 + jg_2 \right)$$

$$+ e_1 \tau^2 \left(\frac{j}{2} \alpha - \frac{3}{4} \alpha - \frac{\alpha}{4} D \right) b_{3/2}^{(j)} \cos \left(- j\ell_1 + (j+1)\ell_2 + (-j+1) g_1 + (j+1) g_2 \right)$$

$$+ e_2 \tau^2 \left(- \frac{j}{2} \alpha + \frac{\alpha}{4} D \right) b_{3/2}^{(j)} \cos \left((-j+1)\ell_1 + j\ell_2 + (-j+1) g_1 + (j+1) g_2 \right)$$

$$+ e_1^2 \left\{ \frac{j^2}{2} - \frac{5}{8} j + \left(- \frac{j}{2} - \frac{1}{8} \right) D - \frac{1}{8} D^2 \right\} b_{1/2}^{(j)} \cos \left((-j+2)\ell_1 + j\ell_2 - jg_1 + jg_2 \right)$$

$$+ e_1 e_2 \left\{ - j^2 - \frac{j}{2} + \left(- j - \frac{1}{4} \right) D - \frac{1}{4} D^2 \right\} b_{1/2}^{(j)} \cos \left((-j+1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2 \right)$$

$$+ e_2^2 \left\{ \frac{j^2}{2} + \frac{9}{8} j + \frac{1}{2} + \left(\frac{j}{2} + \frac{5}{8} \right) D + \frac{1}{8} D^2 \right\} b_{1/2}^{(j)} \cos \left(- j\ell_1 + (j+2)\ell_2 - jg_1 + jg_2 \right)$$

$$+ \tau^2 \frac{\alpha}{2} b_{3/2}^{(j)} \cos \left((-j+1)\ell_1 + (j+1)\ell_2 + (-j+1) g_1 + (j+1) g_2 \right)$$

$$+ e_1^3 \left\{ - \frac{j^3}{6} + \frac{5}{8} j^2 - \frac{13}{24} j + \left(- \frac{j^2}{4} + \frac{33}{48} j - \frac{17}{48} \right) D + \left(- \frac{j}{8} + \frac{9}{48} \right) D^2 \right.$$

$$\left. - \frac{1}{48} D^3 \right\} b_{1/2}^{(j)} \cos \left((-j+3)\ell_1 + j\ell_2 - jg_1 + jg_2 \right)$$

$$+ e_1^2 e_2 \left\{ \frac{j^3}{2} - \frac{3}{8} j^2 - \frac{5}{16} j + \left(\frac{3}{4} j^2 - \frac{7}{16} j - \frac{3}{16} \right) D + \left(\frac{3}{8} j - \frac{1}{8} \right) D^2 \right.$$

$$+ \frac{1}{16} D^3 \Big\} b_{1/2}^{(j)} \cos\left((-j+2)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)$$

$$+ e_1 e_2^2 \Big\{ -\frac{j^3}{2} - \frac{9}{8} j^2 - \frac{j}{2} + \left(-\frac{3}{4} j^2 - \frac{19}{16} j - \frac{1}{4}\right) D + \left(-\frac{3}{8} j - \frac{5}{16}\right) D^2$$

$$- \frac{1}{16} D^3 \Big\} b_{1/2}^{(j)} \cos\left((-j+1)\ell_1 + (j+2)\ell_2 - jg_1 + jg_2\right)$$

$$+ e_2^3 \Big\{ \frac{j^3}{6} + \frac{7}{8} j^2 + \frac{65}{48} j + \frac{9}{16} + \left(\frac{j^2}{4} + \frac{45}{48} j + \frac{19}{24}\right) D$$

$$+ \left(\frac{j}{8} + \frac{1}{4}\right) D^2 + \frac{1}{48} D^3 \Big\} b_{1/2}^{(j)} \cos\left(-j\ell_1 + (j+3)\ell_2 - jg_1 + jg_2\right)$$

$$+ e_1 \tau^2 \left(-\frac{j}{2}\alpha + \frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(j)} \cos\left((-j+2)\ell_1 + (j+1)\ell_2 + (-j+1) g_1 + (j+1) g_2\right)$$

$$+ e_2 \tau^2 \left(\frac{j}{2}\alpha + \alpha + \frac{\alpha}{4} D\right) b_{3/2}^{(j)} \cos\left((-j+1)\ell_1 + (j+2)\ell_2 + (-j+1) g_1 + (j+1) g_2\right) \Bigg] \tag{1}$$

The determining function which eliminates the short period terms is:

$$S_{1p} = \frac{\sigma k \beta_1 \beta_2}{m_0^{1/2}} a_1'^{1/2} \alpha \sum_{j=-p}^{j=+p} \Bigg[\Big\{ \frac{1}{2} b_{1/2}^{(j)} + e_1'^2 \left(-\frac{j^2}{2} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(j)} + e_2'^2 \left(-\frac{j^2}{2} + \frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(j)}$$

$$+ \tau'^2 \left(-\frac{\alpha}{4} b_{3/2}^{(j-1)} - \frac{\alpha}{4} b_{3/2}^{(j+1)}\right) \Big\} \frac{\sin\left(-j\ell_1 + j\ell_2 - jg_1 + jg_2\right)}{-j + j\alpha^{3/2}}$$

$$+ e_1' e_2' \left(j^2 + \frac{j}{2} - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(j)} \frac{\sin\left((-j-1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{-j - 1 + (j+1)\alpha^{3/2}}$$

$$+ e_1'^2 e_2' \Big\{ -\frac{j^3}{2} + \frac{7}{8} j^2 - \frac{5}{16} j + \left(-\frac{j^2}{4} + \frac{5}{16} j - \frac{3}{16}\right) D + \left(\frac{j}{8} - \frac{1}{8}\right) D^2$$

$$+ \frac{1}{16} D^3 \Big\} b_{1/2}^{(j)} \frac{\sin\left((-j+2)\ell_1 + (j-1)\ell_2 - jg_1 + jg_2\right)}{-j + 2 + (j-1)\alpha^{3/2}}$$

$$+\left\{e_1'\left(-j-\frac{1}{2}D\right)b_{1/2}^{(j)}+e_1'\tau'^2\left(\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)\right.$$

$$+e_1'^3\left(\frac{j^3}{2}-\frac{5}{8}j^2+\frac{j}{8}+\left(\frac{j^2}{4}-\frac{5}{16}j+\frac{3}{16}\right)D+\left(-\frac{j}{8}+\frac{1}{16}\right)D^2-\frac{1}{16}D^3\right)b_{1/2}^{(j)}$$

$$\left.+e_1'e_2'^2\left(j^3+\left(\frac{j^2}{2}-\frac{j}{4}\right)D+\left(-\frac{j}{4}-\frac{1}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left((-j+1)\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+1+j\alpha^{3/2}}$$

$$+\left\{e_2'\left(j+\frac{1}{2}+\frac{1}{2}D\right)b_{1/2}^{(j)}+e_2'\tau'^2\left(\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)\right.$$

$$+e_2'^3\left(-\frac{j^3}{2}-\frac{7}{8}j^2-\frac{5}{16}j-\frac{1}{16}+\left(-\frac{j^2}{4}-\frac{j}{16}+\frac{1}{8}\right)D+\left(\frac{j}{8}+\frac{1}{4}\right)D^2+\frac{1}{16}D^3\right)b_{1/2}^{(j)}$$

$$\left.+e_1'^2e_2'\left(-j^3-\frac{j^2}{2}+\left(-\frac{j^2}{2}+\frac{j}{4}+\frac{1}{8}\right)D+\left(\frac{j}{4}+\frac{1}{4}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left(-j\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{-j+(j+1)\alpha^{3/2}}$$

$$+e_1'e_2'^2\left(\frac{j^3}{2}+\frac{9}{8}j^2+\frac{j}{2}+\left(\frac{j^2}{4}+\frac{j}{16}-\frac{1}{4}\right)D+\left(-\frac{j}{8}-\frac{5}{16}\right)D^2-\frac{1}{16}D^3\right)b_{1/2}^{(j)}$$

$$\times\frac{\sin\left((-j-1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{-j-1+(j+2)\alpha^{3/2}}$$

$$+e_1'\tau'^2\left(\frac{j}{2}\alpha-\frac{3}{4}\alpha-\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\frac{\sin\left(-j\ell_1+(j+1)\ell_2+(-j+1)g_1+(j+1)g_2\right)}{-j+(j+1)\alpha^{3/2}}$$

$$+e_2'\tau'^2\left(-\frac{j}{2}\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\frac{\sin\left((-j+1)\ell_1+j\ell_2+(-j+1)g_1+(j+1)g_2\right)}{-j+1+j\alpha^{3/2}}$$

$$+e_1'^2\left\{\frac{j^2}{2}-\frac{5}{8}j+\left(\frac{j}{2}-\frac{3}{8}\right)D+\frac{1}{8}D^2\right\}b_{1/2}^{(j)}\frac{\sin\left((-j+2)\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+2+j\alpha^{3/2}}$$

$$+ e_1' e_2' \left\{- j^2 - \frac{j}{2} + \left(- j - \frac{1}{4}\right) D - \frac{1}{4} D^2\right\} b_{1/2}^{(j)} \frac{\sin\left((- j + 1)\ell_1 + (j + 1)\ell_2 - j g_1 + j g_2\right)}{- j + 1 + (j + 1)\alpha^{3/2}}$$

$$+ e_2'^2 \left\{\frac{j^2}{2} + \frac{9}{8} j + \frac{1}{2} + \left(\frac{j}{2} + \frac{5}{8}\right) D + \frac{1}{8} D^2\right\} b_{1/2}^{(j)} \frac{\sin\left(- j\ell_1 + (j + 2)\ell_2 - j g_1 + j g_2\right)}{- j + (j + 2)\alpha^{3/2}}$$

$$+ \tau'^2 \frac{\alpha}{2} b_{3/2}^{(j)} \frac{\sin\left((- j + 1)\ell_1 + j + 1)\ell_2 + (- j + 1) g_1 + (j + 1) g_2\right)}{- j + 1 + (j + 1)\alpha^{3/2}}$$

$$+ e_1'^3 \left\{- \frac{j^3}{6} + \frac{5}{8} j^2 - \frac{13}{24} j + \left(- \frac{j^2}{4} + \frac{33}{48} j - \frac{17}{48}\right) D + \left(- \frac{j}{8} + \frac{9}{48}\right) D^2\right.$$

$$\left. - \frac{1}{48} D^3\right\} b_{1/2}^{(j)} \frac{\sin\left((- j + 3)\ell_1 + j\ell_2 - j g_1 + j g_2\right)}{- j + 3 + j\alpha^{3/2}}$$

$$+ e_1'^2 e_2' \left\{\frac{j^3}{2} - \frac{3}{8} j^2 - \frac{5}{16} j + \left(\frac{3}{4} j^2 - \frac{7}{16} j - \frac{3}{16}\right) D + \left(\frac{3}{8} j - \frac{1}{8}\right) D^2\right.$$

$$\left. + \frac{1}{16} D^3\right\} b_{1/2}^{(j)} \frac{\sin\left((- j + 2)\ell_1 + (j + 1)\ell_2 - j g_1 + j g_2\right)}{- j + 2 + (j + 1)\alpha^{3/2}}$$

$$+ e_1' e_2'^2 \left\{- \frac{j^3}{2} - \frac{9}{8} j^2 - \frac{j}{2} + \left(- \frac{3}{4} j^2 - \frac{19}{16} j - \frac{1}{4}\right) D + \left(- \frac{3}{8} j - \frac{5}{16}\right) D^2\right.$$

$$\left. - \frac{1}{16} D^3\right\} b_{1/2}^{(j)} \frac{\sin\left((- j + 1)\ell_1 + (j + 2)\ell_2 - j g_1 + j g_2\right)}{- j + 1 + (j + 2)\alpha^{3/2}}$$

$$+ e_2'^3 \left\{\frac{j^3}{6} + \frac{7}{8} j^2 + \frac{65}{48} j + \frac{9}{16} + \left(\frac{j^2}{4} + \frac{45}{48} j + \frac{19}{24}\right) D + \left(\frac{j}{8} + \frac{1}{4}\right) D^2\right.$$

$$\left. + \frac{1}{48} D^3\right\} b_{1/2}^{(j)} \frac{\sin\left(- j\ell_1 + (j + 3)\ell_2 - j g_1 + j g_2\right)}{- j + (j + 3)\alpha^{3/2}}$$

$$+ e_1' \tau'^2 \left(- \frac{j}{2} \alpha + \frac{\alpha}{4} - \frac{\alpha}{4} D\right) b_{3/2}^{(j)} \frac{\sin\left((- j + 2)\ell_1 + (j + 1)\ell_2 + (- j + 1) g_1 + (j + 1) g_2\right)}{- j + 2 + (j + 1)\alpha^{3/2}}$$

$$\left. + e_2' \tau'^2 \left(\frac{j}{2}\alpha + \alpha + \frac{\alpha}{4} D\right) b_{3/2}^{(j)} \frac{\sin\left((- j + 1)\ell_1 + (j + 2)\ell_2 + (- j + 1) g_1 + (j + 1) g_2\right)}{- j + 1 + (j + 2)\alpha^{3/2}}\right] \qquad (2)$$

the prime letters designating the new linear variables that is to say the linear variables which arise from the elimination of the short period terms and α designating the ratio a_1'/a_2'.

The partial derivatives of S_{1p} with respect to the old angular variables ℓ_1, g_1, ℓ_2, g_2 may be obtained at once from the above expression of S_{1p}. We shall not write them. The partial derivatives of S_{1p} with respect to the new linear variables $L_1', G_1', H_1', L_2', G_2', H_2'$ are obtained from the equality

$$S_{1p} = \frac{\sigma k \beta_1 \beta_2}{m_0^{1/2}} a_1'^{1/2} \alpha \sum_{j=-p}^{j=+p} s_j \tag{3}$$

with

$$s_j = \frac{f_j(\alpha) g_j(e_1', e_2', \tau')}{p(j) + q(j)\alpha^{3/2}} \sin\left(p(j)\ell_1 + q(j)\ell_2 + y(j) g_1 + z(j) g_2\right)$$

$p(j), q(j), y(j), z(j)$ being relative integers which depend upon the relative integer j.

From (3) we have:

$$\frac{\partial s_j}{\partial L_1'} = \frac{1}{k m_0^{1/2} \beta_1 a_1'^{1/2}} \left[p(j)\left(g_j\left(3f_j + 2Df_j\right) + \frac{1-e_1'^2}{e_1'} f_j \frac{\partial g_j}{\partial e_1'} \right) + \alpha^{3/2} q(j) \left(g_j\, 2Df_j + \frac{1-e_1'^2}{e_1'} f_j \frac{\partial g_j}{\partial e_1'} \right) \right]$$

$$\times \frac{\sin\left(p(j)\ell_1 + q(j)\ell_2 + y(j) g_1 + z(j) g_2\right)}{\left(p(j) + q(j)\alpha^{3/2}\right)^2}, \tag{4}$$

$$\frac{\partial s_j}{\partial L_2'} = \frac{1}{k m_0^{1/2} \beta_2 a_2'^{1/2}} \left[p(j)\left(g_j\left(-2f_j - 2Df_j\right) + \frac{1-e_2'^2}{e_2'} f_j \frac{\partial g_j}{\partial e_2'} \right) + \alpha^{3/2} q(j) \left(g_j\left(f_j - 2Df_j\right) + \frac{1-e_2'^2}{e_2'} f_j \frac{\partial g_j}{\partial e_2'} \right) \right]$$

$$\times \frac{\sin\left(p(j)\ell_1 + q(j)\ell_2 + y(j) g_1 + z(j) g_2\right)}{\left(p(j) + q(j)\alpha^{3/2}\right)^2}, \tag{5}$$

$$\frac{\partial s_j}{\partial G_1'} = \frac{1}{k m_0^{1/2} \beta_1 a_1'^{1/2}} f_j \left[\frac{\partial g_j}{\partial e_1'} \left(-\frac{1}{e_1'} + \frac{e_1'}{2} \right) + \frac{1}{4} \frac{\partial g_j}{\partial \tau'} \left(\frac{1}{\tau'} + \frac{1}{2} \frac{e_1'^2}{\tau'} - 2\tau' \right) \right] \frac{\sin\left(p(j)\ell_1 + q(j)\ell_2 + y(j) g_1 + z(j) g_2\right)}{p(j) + q(j)\alpha^{3/2}} \tag{6}$$

$$\frac{\partial s_j}{\partial G_2'} = \frac{1}{k m_0^{1/2} \beta_2 a_2'^{1/2}} f_j \frac{\partial g_j}{\partial e_2'} \left(- \frac{1}{e_2'} + \frac{e_2'}{2} \right) \frac{\sin \left(p(j) \ell_1 + q(j) \ell_2 + y(j) g_1 + z(j) g_2 \right)}{p(j) + q(j) \alpha^{3/2}} , \tag{7}$$

$$\frac{\partial s_j}{\partial H_1'} = \frac{1}{k m_0^{1/2} \beta_1 a_1'^{1/2}} f_j \frac{-1}{4} \frac{\partial g_j}{\partial \tau'} \left(\frac{1}{\tau'} + \frac{1}{2} \frac{e_1'^2}{\tau'} \right) \frac{\sin \left(p(j) \ell_1 + q(j) \ell_2 + y(j) g_1 + z(j) g_2 \right)}{p(j) + q(j) \alpha^{3/2}} , \tag{8}$$

$$\frac{\partial s_j}{\partial H_2'} = \frac{\partial s_j}{\partial G_2'} \tag{9}$$

with $f_j = f_j(\alpha)$, $g_j = g_j(e_1', e_2', \tau')$.

From (2), (4), (5), (6), (7), (8), (9) we have:

$$\frac{\partial S_{1p}}{\partial L_1'} = \frac{\sigma \beta_2 \alpha}{m_0} \sum_{j=-p}^{j=+p} \Bigg[\Bigg\{ (-j) \left(\frac{3}{2} - j^2 + \frac{5}{4} D + \frac{1}{4} D^2 \right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} j \left(-j^2 + \frac{5}{4} D + \frac{1}{4} D^2 \right) b_{1/2}^{(j)} \Bigg\} \frac{\sin \left(-j\ell_1 + j\ell_2 - jg_1 + jg_2 \right)}{\left(-j + j\alpha^{3/2} \right)^2}$$

$$+ \Bigg\{ (-j-1) \frac{e_2'}{e_1'} \left(j^2 + \frac{1}{2} j - \frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} (j+1) \frac{e_2'}{e_1'} \left(j^2 + \frac{1}{2} j - \frac{1}{4} D - \frac{1}{4} D^2 \right) b_{1/2}^{(j)} \Bigg\} \frac{\sin \left((-j-1) \ell_1 + (j+1) \ell_2 - jg_1 + jg_2 \right)}{\left(-j - 1 + (j+1) \alpha^{3/2} \right)^2}$$

$$+ \Bigg\{ (-j+2) e_2' \left(-j^3 + \frac{7}{4} j^2 - \frac{5}{8} j + \left(-\frac{j^2}{2} + \frac{5}{8} j - \frac{3}{8} \right) D + \left(\frac{j}{4} - \frac{1}{4} \right) D^2 + \frac{1}{8} D^3 \right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} (j-1) e_2' \left(-j^3 + \frac{7}{4} j^2 - \frac{5}{8} j + \left(-\frac{j^2}{2} + \frac{5}{8} j - \frac{3}{8} \right) D + \left(\frac{j}{4} - \frac{1}{4} \right) D^2 + \frac{1}{8} D^3 \right) b_{1/2}^{(j)} \Bigg\}$$

$$\times \frac{\sin\left((-j+2)\ell_1+(j-1)\ell_2-jg_1+jg_2\right)}{\left(-j+2+(j-1)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+1)\left[\frac{1}{e_1'}\left(-j-\frac{1}{2}D\right)b_{1/2}^{(j)}\right.\right.$$

$$+e_1'\left(\frac{3}{2}j^3-\frac{15}{8}j^2-\frac{13}{8}j+\left(\frac{3}{4}j^2-\frac{47}{16}j-\frac{7}{16}\right)D+\left(-\frac{3}{8}j-\frac{13}{16}\right)D^2-\frac{3}{16}D^3\right)b_{1/2}^{(j)}$$

$$+\frac{\tau'^2}{e_1'}\left(\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)$$

$$\left.+\frac{e_2'^2}{e_1'}\left(j^3+\left(\frac{j^2}{2}-\frac{j}{4}\right)D+\left(-\frac{j}{4}-\frac{1}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right]$$

$$+\alpha^{3/2}j\left[\frac{1}{e_1'}\left(-j-\frac{1}{2}D\right)b_{1/2}^{(j)}\right.$$

$$+e_1'\left(\frac{3}{2}j^3-\frac{15}{8}j^2+\frac{11}{8}j+\left(\frac{3}{4}j^2-\frac{47}{16}j+\frac{17}{16}\right)D+\left(-\frac{3}{8}j-\frac{13}{16}\right)D^2-\frac{3}{16}D^3\right)b_{1/2}^{(j)}$$

$$+\frac{\tau'^2}{e_1'}\left(\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(\frac{j}{2}\alpha+\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)$$

$$\left.\left.+\frac{e_2'^2}{e_1'}\left(j^3+\left(\frac{j^2}{2}-\frac{j}{4}\right)D+\left(-\frac{j}{4}-\frac{1}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right]\right\}$$

$$\times \frac{\sin\left((-j+1)\ell_1+j\ell_2-jg_1+jg_2\right)}{\left(-j+1+j\alpha^{3/2}\right)^2}$$

$$+\left\{(-j)e_2'\left(-2j^3-j^2+3j+\frac{3}{2}+\left(-j^2+\frac{5}{2}j+\frac{11}{4}\right)D+\left(\frac{j}{2}+\frac{3}{2}\right)D^2+\frac{1}{4}D^3\right)b_{1/2}^{(j)}\right.$$

$$+ \alpha^{3/2} (j+1) e_2' \left(-2j^3 - j^2 + \left(-j^2 + \frac{5}{2} j + \frac{5}{4}\right) D + \left(\frac{j}{2} + \frac{3}{2}\right) D^2\right.$$

$$\left. + \frac{1}{4} D^3\right) b_{1/2}^{(j)} \Bigg\} \frac{\sin\left(-j\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{\left(-j + (j+1)\alpha^{3/2}\right)^2}$$

$$+ \Bigg\{ (-j-1) \frac{e_2'^2}{e_1'} \left(\frac{j^3}{2} + \frac{9}{8} j^2 + \frac{j}{2} + \left(\frac{j^2}{4} + \frac{j}{16} - \frac{1}{4}\right) D + \left(-\frac{j}{8} - \frac{5}{16}\right) D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} (j+2) \frac{e_2'^2}{e_1'} \left(\frac{j^3}{2} + \frac{9}{8} j^2 + \frac{j}{2} + \left(\frac{j^2}{4} + \frac{j}{16} - \frac{1}{4}\right) D + \left(-\frac{j}{8} - \frac{5}{16}\right) D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(j)} \Bigg\}$$

$$\times \frac{\sin\left((-j-1)\ell_1 + (j+2)\ell_2 - jg_1 + jg_2\right)}{\left(-j - 1 + (j+2)\alpha^{3/2}\right)^2}$$

$$+ \Bigg\{ (-j) \frac{\tau'^2}{e_1'} \left(\frac{j}{2} \alpha - \frac{3}{4} \alpha - \frac{\alpha}{4} D\right) b_{3/2}^{(j)}$$

$$+ \alpha^{3/2} (j+1) \frac{\tau'^2}{e_1'} \left(\frac{j}{2} \alpha - \frac{3}{4} \alpha - \frac{\alpha}{4} D\right) b_{3/2}^{(j)} \Bigg\} \frac{\sin\left(-j\ell_1 + (j+1)\ell_2 + (-j+1) g_1 + (j+1) g_2\right)}{\left(-j + (j+1)\alpha^{3/2}\right)^2}$$

$$+ \Bigg\{ (-j+2) \left(j^2 - \frac{5}{4} j + \left(j - \frac{3}{4}\right) D + \frac{1}{4} D^2\right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} j \left(j^2 - \frac{5}{4} j + \left(j - \frac{3}{4}\right) D + \frac{1}{4} D^2\right) b_{1/2}^{(j)} \Bigg\} \frac{\sin\left((-j+2)\ell_1 + j\ell_2 - jg_1 + jg_2\right)}{\left(-j + 2 + j\alpha^{3/2}\right)^2}$$

$$+ \Bigg\{ (-j+1) \frac{e_2'}{e_1'} \left(-j^2 - \frac{j}{2} + \left(-j - \frac{1}{4}\right) D - \frac{1}{4} D^2\right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} (j+1) \frac{e_2'}{e_1'} \left(-j^2 - \frac{j}{2} + \left(-j - \frac{1}{4}\right) D - \frac{1}{4} D^2\right) b_{1/2}^{(j)} \Bigg\} \frac{\sin\left((-j+1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{\left((-j + 1 + (j+1)\alpha^{3/2}\right)^2}$$

$$+ \Bigg\{ (-j+3) e_1' \left(-\frac{j^3}{2} + \frac{15}{8} j^2 - \frac{13}{8} j + \left(-\frac{3}{4} j^2 + \frac{33}{16} j - \frac{17}{16}\right) D + \left(-\frac{3}{8} j + \frac{9}{16}\right) D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(j)}$$

$$+ \alpha^{3/2} j e_1' \left(-\frac{j^3}{2} + \frac{15}{8} j^2 - \frac{13}{8} j + \left(-\frac{3}{4} j^2 + \frac{33}{16} j - \frac{17}{16}\right) D + \left(-\frac{3}{8} j + \frac{9}{16}\right) D^2 - \frac{1}{16} D^3\right) b_{1/2}^{(j)} \Bigg\}$$

$$\times \frac{\sin\left((-j+3)\ell_1 + j\ell_2 - jg_1 + jg_2\right)}{\left(-j + 3 + j\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+2)e_2'\left(j^3-\frac{3}{4}j^2-\frac{5}{8}j+\left(\frac{3}{2}j^2-\frac{7}{8}j-\frac{3}{8}\right)D+\left(\frac{3}{4}j-\frac{1}{4}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right.$$

$$\left.+\alpha^{3/2}(j+1)e_2'\left(j^3-\frac{3}{4}j^2-\frac{5}{8}j+\left(\frac{3}{2}j^2-\frac{7}{8}j-\frac{3}{8}\right)D+\left(\frac{3}{4}j-\frac{1}{4}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left((-j+2)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{\left(-j+2+(j+1)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+1)\frac{e_2'^2}{e_1'}\left(-\frac{j^3}{2}-\frac{9}{8}j^2-\frac{j}{2}+\left(-\frac{3}{4}j^2-\frac{19}{16}j-\frac{1}{4}\right)D+\left(-\frac{3}{8}j-\frac{5}{16}\right)D^2-\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right.$$

$$+\alpha^{3/2}(j+2)\frac{e_2'^2}{e_1'}\left(-\frac{j^3}{2}-\frac{9}{8}j^2-\frac{j}{2}+\left(-\frac{3}{4}j^2-\frac{19}{16}j-\frac{1}{4}\right)D+\left(-\frac{3}{8}j-\frac{5}{16}\right)D^2\right.$$

$$\left.\left.-\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right\}\frac{\sin\left((-j+1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{\left(-j+1+(j+2)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+2)\frac{\tau'^2}{e_1'}\left(-\frac{j}{2}\alpha+\frac{\alpha}{4}-\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right.$$

$$\left.\left.+\alpha^{3/2}(j+1)\frac{\tau'^2}{e_1'}\left(-\frac{j}{2}\alpha+\frac{\alpha}{4}-\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right\}\frac{\sin\left((-j+2)\ell_1+(j+1)\ell_2+(-j+1)g_1+(j+1)g_2\right)}{\left(-j+2+(j+1)\alpha^{3/2}\right)^2}\right],\qquad\textbf{(10)}$$

$$\frac{\partial S_{1p}}{\partial L_2'}=\frac{\sigma\beta_1\alpha^{3/2}}{m_0}\sum_{j=-p}^{j=+p}\left[\left\{(-j)\left(-j^2-1-\frac{3}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(j)}\right.\right.$$

$$\left.+\alpha^{3/2}j\left(-j^2+\frac{1}{2}-\frac{3}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(j)}\right\}\frac{\sin\left(-j\ell_1+j\ell_2-jg_1+jg_2\right)}{\left(-j+j\alpha^{3/2}\right)^2}$$

$$+\left\{(-j-1)\frac{e_1'}{e_2'}\left(j^2+\frac{j}{2}-\frac{1}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(j)}\right.$$

$$\left.+\alpha^{3/2}(j+1)\frac{e_1'}{e_2'}\left(j^2+\frac{j}{2}-\frac{1}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(j)}\right\}\frac{\sin\left((-j-1)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{\left(-j-1+(j+1)\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j+2)\frac{e_1'^2}{e_2'}\left(-\frac{j^3}{2}+\frac{7}{8}j^2-\frac{5}{16}j+\left(-\frac{j^2}{4}+\frac{5}{16}j-\frac{3}{16}\right)D+\left(\frac{j}{8}-\frac{1}{8}\right)D^2+\frac{1}{16}D^3\right)b_{1/2}^{(j)}$$

$$+\alpha^{3/2}(j-1)\frac{e_1'^2}{e_2'}\left(-\frac{j^3}{2}+\frac{7}{8}j^2-\frac{5}{16}j+\left(-\frac{j^2}{4}+\frac{5}{16}j-\frac{3}{16}\right)D+\left(\frac{j}{8}-\frac{1}{8}\right)D^2\right.$$

$$\left.+\frac{1}{16}D^3\right)b_{1/2}^{(j)}\Bigg\}\frac{\sin\left((-j+2)\ell_1+(j-1)\ell_2-jg_1+jg_2\right)}{\left(-j+2+(j-1)\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j+1)e_1'\left(2j^3+2j+\left(j^2+\frac{3}{2}j+1\right)D+\left(-\frac{j}{2}+\frac{3}{4}\right)D^2-\frac{1}{4}D^3\right)b_{1/2}^{(j)}$$

$$+\alpha^{3/2}je_1'\left(2j^3-j+\left(j^2+\frac{3}{2}j-\frac{1}{2}\right)D+\left(-\frac{j}{2}+\frac{3}{4}\right)D^2-\frac{1}{4}D^3\right)b_{1/2}^{(j)}\Bigg\}$$

$$\times\frac{\sin\left((-j+1)\ell_1+j\ell_2-jg_1+jg_2\right)}{\left(-j+1+j\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j)\Bigg[\frac{1}{e_2'}\left(j+\frac{1}{2}+\frac{1}{2}D\right)b_{1/2}^{(j)}$$

$$+e_2'\left(-\frac{3}{2}j^3-\frac{21}{8}j^2-\frac{63}{16}j-\frac{27}{16}+\left(-\frac{3}{4}j^2-\frac{35}{16}j-\frac{17}{8}\right)D\right.$$

$$\left.+\left(\frac{3}{8}j-\frac{1}{4}\right)D^2+\frac{3}{16}D^3\right)b_{1/2}^{(j)}$$

$$+\frac{\tau'^2}{e_2'}\left(\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)$$

$$+\frac{e_1'^2}{e_2'}\left(-j^3-\frac{j^2}{2}+\left(-\frac{j^2}{2}+\frac{j}{4}+\frac{1}{8}\right)D+\left(\frac{j}{4}+\frac{1}{4}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\Bigg]$$

$$+\alpha^{3/2}(j+1)\Bigg[\frac{1}{e_2'}\left(j+\frac{1}{2}+\frac{1}{2}D\right)b_{1/2}^{(j)}$$

$$+e_2'\left(-\frac{3}{2}j^3-\frac{21}{8}j^2-\frac{15}{16}j-\frac{3}{16}+\left(-\frac{3}{4}j^2-\frac{35}{16}j-\frac{5}{8}\right)D\right.$$

$$+\left(\frac{3}{8}j-\frac{1}{4}\right)D^2+\frac{3}{16}D^3\Big)\,b_{1/2}^{(j)}$$

$$+\frac{\tau'^2}{e_2'}\left(\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(-\frac{j}{2}\alpha-\frac{\alpha}{2}-\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)$$

$$+\frac{e_1'^2}{e_2'}\left(-j^3-\frac{j^2}{2}+\left(-\frac{j^2}{2}+\frac{j}{4}+\frac{1}{8}\right)D+\left(\frac{j}{4}+\frac{1}{4}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\Bigg]\Bigg\}$$

$$\times\frac{\sin\left(-j\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{\left(-j+(j+1)\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j-1)\,e_1'\left(j^3+\frac{9}{4}j^2+j+\left(\frac{j^2}{2}+\frac{j}{8}-\frac{1}{2}\right)D+\left(-\frac{j}{4}-\frac{5}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}$$

$$+\alpha^{3/2}(j+2)\,e_1'\left(j^3+\frac{9}{4}j^2+j+\left(\frac{j^2}{2}+\frac{j}{8}-\frac{1}{2}\right)D+\left(-\frac{j}{4}-\frac{5}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\Bigg\}$$

$$\times\frac{\sin\left((-j-1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{\left(-j-1+(j+2)\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j+1)\frac{\tau'^2}{e_2'}\left(-\frac{j}{2}\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}$$

$$+\alpha^{3/2}j\frac{\tau'^2}{e_2'}\left(-\frac{j}{2}\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\Bigg\}\frac{\sin\left((-j+1)\ell_1+j\ell_2+(-j+1)g_1+(j+1)g_2\right)}{\left(-j+1+j\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j+1)\frac{e_1'}{e_2'}\left(-j^2-\frac{j}{2}+\left(-j-\frac{1}{4}\right)D-\frac{1}{4}D^2\right)b_{1/2}^{(j)}$$

$$+\alpha^{3/2}(j+1)\frac{e_1'}{e_2'}\left(-j^2-\frac{j}{2}+\left(-j-\frac{1}{4}\right)D-\frac{1}{4}D^2\right)b_{1/2}^{(j)}\Bigg\}\frac{\sin\left((-j+1)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{\left(-j+1+(j+1)\alpha^{3/2}\right)^2}$$

$$+\Bigg\{(-j)\left(j^2+\frac{9}{4}j+1+\left(j+\frac{5}{4}\right)D+\frac{1}{4}D^2\right)b_{1/2}^{(j)}$$

$$+\alpha^{3/2}(j+2)\left(j^2+\frac{9}{4}j+1+\left(j+\frac{5}{4}\right)D+\frac{1}{4}D^2\right)b_{1/2}^{(j)}\Bigg\}\frac{\sin\left(-j\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{\left(-j+(j+2)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+2)\frac{e_1'^2}{e_2'}\left(\frac{j^3}{2}-\frac{3}{8}j^2-\frac{5}{16}j+\left(\frac{3}{4}j^2-\frac{7}{16}j-\frac{3}{16}\right)D+\left(\frac{3}{8}j-\frac{1}{8}\right)D^2+\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right.$$

$$+\alpha^{3/2}(j+1)\frac{e_1'^2}{e_2'}\left(\frac{j^3}{2}-\frac{3}{8}j^2-\frac{5}{16}j+\left(\frac{3}{4}j^2-\frac{7}{16}j-\frac{3}{16}\right)D+\left(\frac{3}{8}j-\frac{1}{8}\right)D^2\right.$$

$$\left.\left.+\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right\}\frac{\sin\left((-j+2)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{\left(-j+2+(j+1)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+1)e_1'\left(-j^3-\frac{9}{4}j^2-j+\left(-\frac{3}{2}j^2-\frac{19}{8}j-\frac{1}{2}\right)D+\left(-\frac{3}{4}j-\frac{5}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right.$$

$$\left.+\alpha^{3/2}(j+2)e_1'\left(-j^3-\frac{9}{4}j^2-j+\left(-\frac{3}{2}j^2-\frac{19}{8}j-\frac{1}{2}\right)D+\left(-\frac{3}{4}j-\frac{5}{8}\right)D^2-\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left((-j+1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{\left(-j+1+(j+2)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j)e_2'\left(\frac{j^3}{2}+\frac{21}{8}j^2+\frac{65}{16}j+\frac{27}{16}+\left(\frac{3}{4}j^2+\frac{45}{16}j+\frac{19}{8}\right)D+\left(\frac{3j}{8}+\frac{3}{4}\right)D^2+\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right.$$

$$+\alpha^{3/2}(j+3)e_2'\left(\frac{j^3}{2}+\frac{21}{8}j^2+\frac{65}{16}j+\frac{27}{16}+\left(\frac{3}{4}j^2+\frac{45}{16}j+\frac{19}{8}\right)D+\left(\frac{3j}{8}+\frac{3}{4}\right)D^2\right.$$

$$\left.\left.+\frac{1}{16}D^3\right)b_{1/2}^{(j)}\right\}\frac{\sin\left(-j\ell_1+(j+3)\ell_2-jg_1+jg_2\right)}{\left(-j+(j+3)\alpha^{3/2}\right)^2}$$

$$+\left\{(-j+1)\frac{\tau'^2}{e_2'}\left(\frac{j}{2}\alpha+\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right.$$

$$\left.\left.+\alpha^{3/2}(j+2)\frac{\tau'^2}{e_2'}\left(\frac{j}{2}\alpha+\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right\}\frac{\sin\left((-j+1)\ell_1+(j+2)\ell_2+(-j+1)g_1+(j+1)g_2\right)}{\left(-j+1+(j+2)\alpha^{3/2}\right)^2}\right], \tag{11}$$

$$\frac{\partial S_{1p}}{\partial G_1'}=\frac{\sigma\beta_2\alpha}{m_0}\sum_{j=-p}^{j=+p}\left[\left\{\left(j^2-\frac{1}{4}D-\frac{1}{4}D^2\right)b_{1/2}^{(j)}+\left(-\frac{\alpha}{8}b_{3/2}^{(j-1)}-\frac{\alpha}{8}b_{3/2}^{(j+1)}\right)\right\}\frac{\sin\left(-j\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+j\alpha^{3/2}}\right.$$

$$+\frac{e_2'}{e_1'}\left(-j^2-\frac{j}{2}+\frac{1}{4}D+\frac{1}{4}D^2\right)b_{1/2}^{(j)}\frac{\sin\left((-j-1)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{-j-1+(j+1)\alpha^{3/2}}$$

$$+e_2'\left\{j^3-\frac{7}{4}j^2+\frac{5}{8}j+\left(\frac{j^2}{2}-\frac{5}{8}j+\frac{3}{8}\right)D+\left(-\frac{j}{4}+\frac{1}{4}\right)D^2-\frac{1}{8}D^3\right\}b_{1/2}^{(j)}$$

$$\times\frac{\sin\left((-j+2)\ell_1+(j-1)\ell_2-jg_1+jg_2\right)}{-j+2+(j-1)\alpha^{3/2}}$$

$$+\left\{\frac{1}{e_1'}\left(j+\frac{1}{2}D\right)b_{1/2}^{(j)}\right.$$

$$+e_1'\left(-\frac{3}{2}j^3+\frac{15}{8}j^2-\frac{7}{8}j+\left(-\frac{3}{4}j^2+\frac{15}{16}j-\frac{13}{16}\right)D+\left(\frac{3}{8}j-\frac{3}{16}\right)D^2+\frac{3}{16}D^3\right)b_{1/2}^{(j)}$$

$$+\frac{\tau'^2}{e_1'}\left(\left(-\frac{j}{2}\alpha-\frac{\alpha}{4}-\frac{\alpha}{4}D\right)b_{3/2}^{(j-1)}+\left(-\frac{j}{2}\alpha-\frac{\alpha}{4}-\frac{\alpha}{4}D\right)b_{3/2}^{(j+1)}\right)$$

$$+e_1'\left(\left(\frac{j}{4}\alpha+\frac{\alpha}{8}+\frac{\alpha}{8}D\right)b_{3/2}^{(j-1)}+\left(\frac{j}{4}\alpha+\frac{\alpha}{8}+\frac{\alpha}{8}D\right)b_{3/2}^{(j+1)}\right)$$

$$\left.+\frac{e_2'^2}{e_1'}\left(-j^3+\left(-\frac{j^2}{2}+\frac{j}{4}\right)D+\left(\frac{j}{4}+\frac{1}{8}\right)D^2+\frac{1}{8}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left((-j+1)\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+1+j\alpha^{3/2}}$$

$$+\left\{e_2'\left(\left(-\frac{j}{4}\alpha-\frac{\alpha}{4}-\frac{\alpha}{8}D\right)b_{3/2}^{(j-1)}+\left(-\frac{j}{4}\alpha-\frac{\alpha}{4}-\frac{\alpha}{8}D\right)b_{3/2}^{(j+1)}\right)\right.$$

$$\left.+e_2'\left(2j^3+j^2+\left(j^2-\frac{j}{2}-\frac{1}{4}\right)D+\left(-\frac{j}{2}-\frac{1}{2}\right)D^2-\frac{1}{4}D^3\right)b_{1/2}^{(j)}\right\}$$

$$\times\frac{\sin\left(-j\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{-j+(j+1)\alpha^{3/2}}$$

$$+\frac{e_2'^2}{e_1'}\left(-\frac{j^3}{2}-\frac{9}{8}j^2-\frac{j}{2}+\left(-\frac{j^2}{4}-\frac{j}{16}+\frac{1}{4}\right)D+\left(\frac{j}{8}+\frac{5}{16}\right)D^2+\frac{1}{16}D^3\right)b_{1/2}^{(j)}$$

$$\times\frac{\sin\left((-j-1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{-j-1+(j+2)\alpha^{3/2}}$$

$$+\left\{\frac{\tau'^2}{e_1'}\left(-\frac{j}{2}\alpha+\frac{3}{4}\alpha+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right.$$

$$\left.+e_1'\left(\frac{j}{4}\alpha-\frac{3}{8}\alpha-\frac{\alpha}{8}D\right)b_{3/2}^{(j)}\right\}\frac{\sin\left(-j\ell_1+(j+1)\ell_2+(-j+1)g_1+(j+1)g_2\right)}{-j+(j+1)\alpha^{3/2}}$$

$$+e_2'\left(-\frac{j}{4}\alpha+\frac{\alpha}{8}D\right)b_{3/2}^{(j)}\frac{\sin\left((-j+1)\ell_1+j\ell_2+(-j+1)g_1+(j+1)g_2\right)}{-j+1+j\alpha^{3/2}}$$

$$+\left\{-j^2+\frac{5}{4}j+\left(j+\frac{1}{4}\right)D-\frac{1}{4}D^2\right\}b_{1/2}^{(j)}\frac{\sin\left((-j+2)\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+2+j\alpha^{3/2}}$$

$$+\frac{e_2'}{e_1'}\left\{j^2+\frac{j}{2}+\left(j+\frac{1}{4}\right)D+\frac{1}{4}D^2\right\}b_{1/2}^{(j)}\frac{\sin\left((-j+1)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{-j+1+(j+1)\alpha^{3/2}}$$

$$+\frac{\alpha}{4}b_{3/2}^{(j)}\frac{\sin\left((-j+1)\ell_1+(j+1)\ell_2+(-j+1)g_1+(j+1)g_2\right)}{-j+1+(j+1)\alpha^{3/2}}$$

$$+e_1'\left\{\frac{j^3}{2}-\frac{15}{8}j^2+\frac{13}{8}j+\left(\frac{3}{4}j^2-\frac{33}{16}j+\frac{17}{16}\right)D+\left(\frac{3}{8}j-\frac{9}{16}\right)D^2\right.$$

$$\left.+\frac{1}{16}D^3\right\}b_{1/2}^{(j)}\frac{\sin\left((-j+3)\ell_1+j\ell_2-jg_1+jg_2\right)}{-j+3+j\alpha^{3/2}}$$

$$+e_2'\left\{-j^3+\frac{3}{4}j^2+\frac{5}{8}j+\left(-\frac{3}{2}j^2+\frac{7}{8}j+\frac{3}{8}\right)D+\left(-\frac{3}{4}j+\frac{1}{4}\right)D^2\right.$$

$$\left.-\frac{1}{8}D^3\right\}b_{1/2}^{(j)}\frac{\sin\left((-j+2)\ell_1+(j+1)\ell_2-jg_1+jg_2\right)}{-j+2+(j+1)\alpha^{3/2}}$$

$$+\frac{e_2'^2}{e_1'}\left\{\frac{j^3}{2}+\frac{9}{8}j^2+\frac{j}{2}+\left(\frac{3}{4}j^2+\frac{19}{16}j+\frac{1}{4}\right)D+\left(\frac{3}{8}j+\frac{5}{16}\right)D^2\right.$$

$$\left.+\frac{1}{16}D^3\right\}\frac{\sin\left((-j+1)\ell_1+(j+2)\ell_2-jg_1+jg_2\right)}{-j+1+(j+2)\alpha^{3/2}}$$

$$+\left\{\frac{\tau'^2}{e_1'}\left(\frac{j}{2}\alpha-\frac{\alpha}{4}+\frac{\alpha}{4}D\right)b_{3/2}^{(j)}\right.$$

$$+ e_1' \left(-\frac{j}{4}\alpha + \frac{\alpha}{8} - \frac{\alpha}{8} D\right) b_{3/2}^{(j)} \Bigg\} \frac{\sin\left((-j+2)\ell_1 + (j+1)\ell_2 + (-j+1) g_1 + (j+1) g_2\right)}{-j+2+(j+1)\alpha^{3/2}}$$

$$+ e_2' \left(\frac{j}{4}\alpha + \frac{\alpha}{2} + \frac{\alpha}{8} D\right) b_{3/2}^{(j)} \frac{\sin\left((-j+1)\ell_1 + (j+2)\ell_2 + (-j+1) g_1 + (j+1) g_2\right)}{-j+1+(j+2)\alpha^{3/2}} \Bigg] , \tag{12}$$

$$\frac{\partial S_{1p}}{\partial G_2'} = \frac{\sigma \beta_1 \alpha^{3/2}}{m_0} \sum_{j=-p}^{j=+p} \Bigg[\left(j^2 - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(j)} \frac{\sin\left(-j\ell_1 + j\ell_2 - jg_1 + jg_2\right)}{-j + j\alpha^{3/2}}$$

$$+ \frac{e_1'}{e_2'} \left(-j^2 - \frac{j}{2} + \frac{1}{4} D + \frac{1}{4} D^2\right) b_{1/2}^{(j)} \frac{\sin\left((-j-1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{-j-1+(j+1)\alpha^{3/2}}$$

$$+ \frac{e_1'^2}{e_2'} \left\{\frac{j^3}{2} - \frac{7}{8} j^2 + \frac{15}{16} j + \left(\frac{j^2}{4} - \frac{5}{16} j + \frac{3}{16}\right) D + \left(-\frac{j}{8} + \frac{1}{8}\right) D^2 - \frac{1}{16} D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left((-j+2)\ell_1 + (j-1)\ell_2 - jg_1 + jg_2\right)}{-j+2+(j-1)\alpha^{3/2}}$$

$$+ e_1' \left\{-2j^3 + \left(-j^2 + \frac{j}{2}\right) D + \left(\frac{j}{2} + \frac{1}{4}\right) D^2 + \frac{1}{4} D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left((-j+1)\ell_1 + j\ell_2 - jg_1 + jg_2\right)}{-j+1+j\alpha^{3/2}}$$

$$+ \Bigg\{ \frac{1}{e_2'} \left(-j^2 - \frac{1}{2} - \frac{1}{2} D\right) b_{1/2}^{(j)}$$

$$+ e_2' \left(\frac{3}{2} j^3 + \frac{21}{8} j^2 + \frac{23}{16} j + \frac{3}{16} + \left(\frac{3}{4} j^2 + \frac{3}{16} j - \frac{1}{8}\right) D\right.$$

$$\left. + \left(-\frac{3}{8} j - \frac{3}{4}\right) D^2 - \frac{3}{16} D^3\right) b_{1/2}^{(j)}$$

$$+ \frac{\tau'^2}{e_2'} \left(\left(\frac{j}{2}\alpha + \frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(j-1)} + \left(\frac{j}{2}\alpha + \frac{\alpha}{2} + \frac{\alpha}{4} D\right) b_{3/2}^{(j+1)}\right)$$

$$+ \frac{e_1'^2}{e_2'}\left(j^3 + \frac{j^2}{2} + \left(\frac{j^2}{2} - \frac{j}{4} - \frac{1}{8}\right)D + \left(-\frac{j}{4} - \frac{1}{4}\right)D^2 - \frac{1}{8}D^3\right) b_{1/2}^{(j)}\Bigg\}$$

$$\times \frac{\sin\left(-j\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{-j + (j+1)\alpha^{3/2}}$$

$$+ e_1'\left\{-j^3 - \frac{9}{4}j^2 - j + \left(-\frac{j^2}{2} - \frac{j}{8} + \frac{1}{2}\right)D + \left(\frac{j}{4} + \frac{5}{8}\right)D^2 + \frac{1}{8}D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left((-j-1)\ell_1 + (j+2)\ell_2 - jg_1 + jg_2\right)}{-j - 1 + (j+2)\alpha^{3/2}}$$

$$+ \frac{\tau'^2}{e_2'}\left(\frac{j}{2}\alpha - \frac{\alpha}{4}D\right) b_{3/2}^{(j)} \frac{\sin\left((-j+1)\ell_1 + j\ell_2 + (-j+1)g_1 + (j+1)g_2\right)}{-j + 1 + j\alpha^{3/2}}$$

$$+ \frac{e_1'}{e_2'}\left\{j^2 + \frac{j}{2} + \left(j + \frac{1}{4}\right)D + \frac{1}{4}D^2\right\} b_{1/2}^{(j)} \frac{\sin\left((-j+1)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{-j + 1 + (j+1)\alpha^{3/2}}$$

$$+ \left\{-j^2 - \frac{9}{4}j - 1 + \left(-j - \frac{5}{4}\right)D - \frac{1}{4}D^2\right\} b_{1/2}^{(j)} \frac{\sin\left(-j\ell_1 + (j+2)\ell_2 - jg_1 + jg_2\right)}{-j + (j+2)\alpha^{3/2}}$$

$$+ \frac{e_1'^2}{e_2'}\left\{-\frac{j^3}{2} + \frac{3}{8}j^2 + \frac{5}{16}j + \left(-\frac{3}{4}j^2 + \frac{7}{16}j + \frac{3}{16}\right)D + \left(-\frac{3}{8}j + \frac{1}{8}\right)D^2 - \frac{1}{16}D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left((-j+2)\ell_1 + (j+1)\ell_2 - jg_1 + jg_2\right)}{-j + 2 + (j+1)\alpha^{3/2}}$$

$$+ e_1'\left\{j^3 + \frac{9}{4}j^2 + j + \left(\frac{3}{2}j^2 + \frac{19}{8}j + \frac{1}{2}\right)D + \left(\frac{3}{4}j + \frac{5}{8}\right)D^2 + \frac{1}{8}D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left((-j+1)\ell_1 + (j+2)\ell_2 - jg_1 + jg_2\right)}{-j + 1 + (j+2)\alpha^{3/2}}$$

$$+ e_2'\left\{-\frac{j^3}{2} - \frac{21}{8}j^2 - \frac{65}{16}j - \frac{27}{16} + \left(-\frac{3}{4}j^2 - \frac{45}{16}j - \frac{19}{8}\right)D + \left(-\frac{3}{8}j - \frac{3}{4}\right)D^2 - \frac{1}{16}D^3\right\} b_{1/2}^{(j)}$$

$$\times \frac{\sin\left(-j\ell_1 + (j+3)\ell_2 - jg_1 + jg_2\right)}{-j + (j+3)\alpha^{3/2}}$$

$$+ \frac{\tau'^2}{e_2'}\left(-\frac{j}{2}\alpha - \alpha - \frac{\alpha}{4}D\right) b_{3/2}^{(j)} \frac{\sin\left((-j+1)\ell_1 + (j+2)\ell_2 + (-j+1)g_1 + (j+1)g_2\right)}{-j + 1 + (j+2)\alpha^{3/2}}\Bigg] , \quad (13)$$

$$\frac{\partial S_{1p}}{\partial H_1'} = \frac{\sigma \beta_2 \alpha}{m_0} \sum_{j=-p}^{j=+p} \left[\left(\frac{\alpha}{8} b_{3/2}^{(j-1)} + \frac{\alpha}{8} b_{3/2}^{(j+1)} \right) \frac{\sin(-j\ell_1 + j\ell_2 - jg_1 + jg_2)}{-j + j\,\alpha^{3/2}} \right.$$

$$+ e_1' \left\{ \left(-\frac{j}{4}\alpha - \frac{\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(j-1)} + \left(-\frac{j}{4}\alpha - \frac{\alpha}{8} - \frac{\alpha}{8} D \right) b_{3/2}^{(j+1)} \right\}$$

$$\times \frac{\sin((-j+1)\ell_1 + j\ell_2 - jg_1 + jg_2)}{-j + 1 + j\,\alpha^{3/2}}$$

$$+ e_2' \left\{ \left(\frac{j}{4}\alpha + \frac{\alpha}{4} + \frac{\alpha}{8} D \right) b_{3/2}^{(j-1)} + \left(\frac{j}{4}\alpha + \frac{\alpha}{4} + \frac{\alpha}{8} D \right) b_{3/2}^{(j+1)} \right\}$$

$$\times \frac{\sin(-j\ell_1 + (j+1)\ell_2 - jg_1 + jg_2)}{-j + (j+1)\,\alpha^{3/2}}$$

$$+ e_1' \left(-\frac{j}{4}\alpha + \frac{3}{8}\alpha + \frac{\alpha}{8} D \right) b_{3/2}^{(j)} \frac{\sin(-j\ell_1 + (j+1)\ell_2 + (-j+1)g_1 + (j+1)g_2)}{-j + (j+1)\,\alpha^{3/2}}$$

$$+ e_2' \left(\frac{j}{4}\alpha - \frac{\alpha}{8} D \right) b_{3/2}^{(j)} \frac{\sin((-j+1)\ell_1 + j\ell_2 + (-j+1)g_1 + (j+1)g_2)}{-j + 1 + j\,\alpha^{3/2}}$$

$$- \frac{\alpha}{4} b_{3/2}^{(j)} \frac{\sin((-j+1)\ell_1 + (j+1)\ell_2 + (-j+1)g_1 + (j+1)g_2)}{-j + 1 + (j+1)\,\alpha^{3/2}}$$

$$+ e_1' \left(\frac{j}{4}\alpha - \frac{\alpha}{8} + \frac{\alpha}{8} D \right) b_{3/2}^{(j)} \frac{\sin((-j+2)\ell_1 + (j+1)\ell_2 + (-j+1)g_1 + (j+1)g_2)}{-j + 2 + (j+1)\,\alpha^{3/2}}$$

$$\left. + e_2' \left(-\frac{j}{4}\alpha - \frac{\alpha}{2} - \frac{\alpha}{8} D \right) b_{3/2}^{(j)} \frac{\sin((-j+1)\ell_1 + (j+2)\ell_2 + (-j+1)g_1 + (j+1)g_2)}{-j + 1 + (j+2)\,\alpha^{3/2}} \right], \tag{14}$$

$$\frac{\partial S_{1p}}{\partial H_2'} = \frac{\partial S_{1p}}{\partial G_2'} = (13)\,. \tag{15}$$

3. The longitudes having been calculated, as we said previously, from the longitude of the ascending node of the disturbed planet P_1, the old Delaunay variables L_1, G_1, H_1, ℓ_1, g_1, h_1 of P_1 are connected to its new Delaunay variables L_1', G_1', H_1', ℓ_1', g_1', h_1' through the equalities:

$$L_1 = L_1' + \frac{\partial S_{1p}}{\partial \ell_1}, \qquad G_1 = G_1' + \frac{\partial S_{1p}}{\partial g_1}, \qquad H_1 = H_1',$$

$$\ell_1 = \ell_1' - \frac{\partial S_{1p}}{\partial L_1'}, \qquad g_1 = g_1' - \frac{\partial S_{1p}}{\partial G_1'}, \qquad h_1 = h_1' - \frac{\partial S_{1p}}{\partial H_1'}, \tag{16}$$

and the old Delaunay variables $L_2, G_2, H_2, \ell_2, g_2, h_2$ of the disturbing planet P_2 are connected to its new Delaunay variables $L_2', G_2', H_2', \ell_2', g_2', h_2'$ through the equalities:

$$L_2 = L_2' + \frac{\partial S_{1p}}{\partial \ell_2}, \qquad G_2 = G_2' + \frac{\partial S_{1p}}{\partial g_2}, \qquad H_2 = H_2',$$

$$\ell_2 = \ell_2' - \frac{\partial S_{1p}}{\partial L_2'}, \qquad g_2 = g_2' - \frac{\partial S_{1p}}{\partial G_2'}, \qquad h_2 = h_2' - \frac{\partial S_{1p}}{\partial H_2'}. \tag{17}$$

In (16) and (17), the four partial derivatives $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial g_1$, $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_2$ are replaced by their values obtained from (2) and the six partial derivatives $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial H_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_2'$, $\partial S_{1p}/\partial H_2'$ are replaced by their values (3), (4), (5), (6), (7), (8).

The two equalities (16) in $\partial S_{1p}/\partial L_1'$ and $\partial S_{1p}/\partial G_1'$ and the two equalities (17) in $\partial S_{1p}/\partial L_2'$ and $\partial S_{1p}/\partial G_2'$ do together a system of four equations with the four unknowns ℓ_1, g_1, ℓ_2, g_2 which is solvable by the method of the "retour des suites" of Lagrange applied to functions of several variables. Since we deal only with a first order theory, ℓ_1, g_1, ℓ_2, g_2 are obtained at once by replacing everywhere, in the arguments of the sines which appear in $\partial S_{1p}/\partial L_1'$, $\partial S_{1p}/\partial G_1'$, $\partial S_{1p}/\partial L_2'$, $\partial S_{1p}/\partial G_2'$, the old variables ℓ_1, g_1, ℓ_2, g_2 by the new variables $\ell_1', g_1', \ell_2', g_2'$ and the developments of $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial g_1$, $\partial S_{1p}/\partial H_1'$, $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_2$, $\partial S_{1p}/\partial G_2'$ in Taylor series according to the powers of σ when ℓ_1, g_1, ℓ_2, g_2 are replaced by their values previously calculated show that $L_1, G_1, h_1, L_2, G_2, h_2$ are also obtained by replacing everywhere, in the arguments of the cosines which appear in $\partial S_{1p}/\partial \ell_1$, $\partial S_{1p}/\partial g_1$, $\partial S_{1p}/\partial \ell_2$, $\partial S_{1p}/\partial g_2$ and in the arguments of the sines which appear in $\partial S_{1p}/\partial H_1'$ and $\partial S_{1p}/\partial H_2'$ the old variables ℓ_1, g_1, ℓ_2, g_2 by the new variables $\ell_1', g_1', \ell_2', g_2'$.

4. The system of the twelve canonical equations

$$\frac{dL_1}{dt} = \frac{\partial F_{1p}}{\partial \ell_1}, \quad \frac{d\ell_1}{dt} = -\frac{\partial F_{1p}}{\partial L_1}, \quad \frac{dL_2}{dt} = \frac{\partial F_{1p}}{\partial \ell_2}, \quad \frac{d\ell_2}{dt} = -\frac{\partial F_{1p}}{\partial L_2},$$

$$\frac{dG_1}{dt} = \frac{\partial F_{1p}}{\partial g_1}, \quad \frac{dg_1}{dt} = -\frac{\partial F_{1p}}{\partial G_1}, \quad \frac{dG_2}{dt} = \frac{\partial F_{1p}}{\partial g_2}, \quad \frac{dg_2}{dt} = -\frac{\partial F_{1p}}{\partial G_2},$$

$$\frac{dH_1}{dt} = \frac{\partial F_{1p}}{\partial h_1}, \quad \frac{dh_1}{dt} = -\frac{\partial F_{1p}}{\partial H_1}, \quad \frac{dH_2}{dt} = \frac{\partial F_{1p}}{\partial h_2}, \quad \frac{dh_2}{dt} = -\frac{\partial F_{1p}}{\partial H_2}, \tag{18}$$

in which F_{1p} is replaced by its value (1) is transformed, after the elimination of the short period terms, into the system of the twelve canonical equations

$$\frac{dL_1'}{dt} = \frac{\partial F_{1p}'}{\partial \ell_1'} = 0, \quad \frac{d\ell_1'}{dt} = -\frac{\partial F_{1p}'}{\partial L_1'}, \quad \frac{dL_2'}{dt} = \frac{\partial F_{1p}'}{\partial \ell_2'} = 0, \quad \frac{d\ell_2'}{dt} = -\frac{\partial F_{1p}'}{\partial L_2'},$$

$$\frac{dG_1'}{dt} = \frac{\partial F_{1p}'}{\partial g_1'}, \qquad \frac{dg_1'}{dt} = -\frac{\partial F_{1p}'}{\partial G_1'}, \qquad \frac{dG_2'}{dt} = \frac{\partial F_{1p}'}{\partial g_2'}, \qquad \frac{dg_2'}{dt} = -\frac{\partial F_{1p}'}{\partial G_2'},$$

$$\frac{dH_1'}{dt} = \frac{\partial F_{1p}'}{\partial h_1'}, \qquad \frac{dh_1'}{dt} = -\frac{\partial F_{1p}'}{\partial H_1'}, \qquad \frac{dH_2'}{dt} = \frac{\partial F_{1p}'}{\partial h_2'}, \qquad \frac{dh_2'}{dt} = -\frac{\partial F_{1p}'}{\partial H_2'}, \tag{19}$$

in which F_{1p}' has for expression

$$\frac{\sigma k^2 \beta_1 \beta_2}{a_2'} \left[\frac{1}{2} b_{1/2}^{(0)} + e_1'^2 \left(\frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(0)} + e_2'^2 \left(\frac{1}{8} D + \frac{1}{8} D^2\right) b_{1/2}^{(0)} + \tau'^2 \left(-\frac{\alpha}{2}\right) b_{3/2}^{(1)}\right.$$

$$\left. + e_1' e_2' \left(\frac{1}{2} - \frac{1}{4} D - \frac{1}{4} D^2\right) b_{1/2}^{(1)} \cos\left(-g_1' + g_2'\right)\right] \cdot \tag{20}$$

Conclusions

a) In reducing the Fourier series of the disturbing function F_{1p} and that of the determining function S_{1p} to the sum of their $2p+1$ first terms ($p > 1$) instead of reducing them only to the sum of their three first terms as we did in our previous paper, we increased the accuracy of the sums (16) and (17) expressing the old Delaunay variables L_i, G_i, H_i, ℓ_i, g_i, h_i as functions of the new ones L_i', G_i', H_i', ℓ_i', g_i', h_i', ($i = 1, 2$) and we obtained formulas that can be applied to any set of two planets, the value of p depending upon the set of the two planets we consider and being so much the more larger than the ratio α of the semi major axis is itself more close to unity. An upper boundary of the error we commit by reducing the Fourier series of F_{1p}, S_{1p} and the partial derivatives of S_{1p} with respect to the linear and angular variables to the sum of their $2p+1$ first terms, may be obtained from the values of the integer p, the ratio α, the Laplace coefficients $b_{1/2}^{(j)}$, $b_{3/2}^{(j)}$, $b_{3/2}^{(j-1)}$, $b_{3/2}^{(j+1)}$ and their derivatives with respect to $D = \alpha\, d/d\alpha$.

b) On the other hand, in so far as we neglect the powers of eccentricities and mutual inclination higher than the third power, the secular terms of F_{1p}' are obtained for $j = 0$ and its long period term is obtained for $j = -1$ which means that if we truncate the Fourier series of F_{1p} to the sum of its $2p+1$ first terms instead of truncating it only to the sum of its three first terms, we do not add in the expression of F_{1p}' new secular terms and new long period terms.

c) It would be worthy to see if our conclusion of b) is still true when we consider higher powers of eccentricities and mutual inclination. Terms of order four with respect to the eccentricities and mutual inclination introduce in F_{1p}', for $j = 0$, new secular terms and a long period term in $\cos\left(g_1' + g_2'\right)$; for $j = -1$ new long period terms in $\cos\left(g_1' - g_2'\right)$ and a long period term in $\cos 2g_1'$; for $j = 1$ a long period term in $\cos 2g_2'$; for $j = -2$ a long period term in $\cos\left(2g_1' - 2g_2'\right)$. Terms of order six and eight with respect to the eccentricities and mutual inclination introduce in F_{1p}', for $j = 0$, new secular terms and new long period terms in $\cos\left(g_1' + g_2'\right)$; for $j = -1$ new long period terms in $\cos\left(g_1' - g_1'\right)$ and new long period terms in $\cos 2g_1'$; for $j = 1$ new long period terms in

$\cos 2g_2'$; for $j = -2$ new long period terms in $\cos(2g_1' - 2g_2')$ and long period terms in $\cos(3g_1' - g_2')$; for $j = 2$ long period terms in $\cos(g_1' - 3g_2')$; for $j = -3$ long period terms in $\cos(3g_1' - 3g_2')$. A truncation of the Fourier series of F_{1p} in the sum of its $2p+1$ first terms ($p > 2$) adds therefore no purely secular terms and no long period terms to its truncation in the sum of its five first terms. Its truncation in the sum of its $2p+1$ first terms ($p > 3$) adds no purely secular terms and no long period terms to its truncation in the sum of its seven first terms. Our conclusion of b) concerning the truncation of the Fourier series of F_{1p} when we neglect the powers of eccentricities and mutual inclination higher than the third power is therefore still true when we take into account the fourth, six and eight powers of eccentricities and mutual inclination, which means that it is still true when we consider a complete general planetary theory, the later requiring a calculation of F_{1p} up to the eight powers of eccentricities and mutual inclination.