X-547-67-201

79

NASA TM X-55864

THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICAL SATELLITE DUE TO THE FOURTH DEGREE LEGENDRE POLYNOMIAL

DAVID FISHER

APRIL 1967

NASA

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

PRECEDING PAGE BLANK NOT FILMED.

X-547-67-201

THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICIAL SATELLITE DUE TO THE FOURTH DEGREE LEGENDRE POLYNOMIAL

David Fisher

April 1967

Goddard Space Flight Center
Greenbelt, Maryland

CONTENTS

THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICIAL SATELLITE DUE TO THE FOURTH DEGREE LEGENDRE POLYNOMIAL

by

David Fisher

SUMMARY

The development of the fourth degree Legendre polynomial is given in separable functions in both coordinates and elements of the satellite and the perturbing body. The result in terms of rectangular coordinates is given as a series containing 13 pairs of functions of the artificial earth satellite and the disturbing body. The results in terms of Keplerian elements are given in 13 tables which contain functions of the perturbing body as well as functions of the elements of the artificial satellite, and derivatives of these functions. For small eccentricity satellites, the tables may be used to obtain perturbations by either direct integration of the equations of motion or by the method of canonical variables. For satellites of high eccentricity, the tables are used to form Jacobi's determining function and then the perturbations are obtained by the principles of canonical variables.

LIST OF SYMBOLS
(primed quantities refer to elements of the disturbing body)

a	semimajor axis of satellite
B	function depending on quantity β of satellite
C	cosine of angle between radius vector of satellite and radius vector of disturbing body
c	$\cos \frac{I}{2}$
e	eccentricity of satellite
E	eccentric anomaly of satellite
f	true anomaly of satellite
G	Delaunay variable, $G = L\sqrt{1 - e^2}$
g	argument of perigee of satellite
h	longitude of ascending node of satellite
H	Delaunay variable, $H = G \cos I$
I	angle of inclination of orbit plane of satellite to the equatorial plane
i	summation index
ℓ	mean anomaly of satellite
M	function depending on eccentricity of satellite
m	mass of satellite
m_0	ratio of mean motion of disturbing body to that of satellite
N	inclination function of satellite
n	mean motion of satellite
$P_2(C), P_3(C), P_4(C)$	second, third, and fourth degree Legendre polynomial, respectively
Q	determining function
Q_0', Q_1', Q_2'	functions of order zero, one, and two, respectively, in $m_0 e$ which define Q
q	summation index
R	disturbing function
R_4	that part of R containing $P_4(C)$

R_{4_S}	secular part of R_4
R_{4_P}	short period part of R_4
$R_{4_{LP}}$	long period part of R_4
r	radius vector from center of earth to satellite
s	$\sin \frac{I}{2}$
t	summation index
u	$f + g - f' - g'$
v	$f + g + f' + g'$
x, y, z	rectangular coordinates of the satellite
α	argument of disturbing function
ν	index of summation
θ	$h - h'$
τ	phase angle $= q\,g + q'\,g'$
ϵ	preassigned infinitesimal defining accuracy of computation

THE LUNAR AND SOLAR PERTURBATIONS IN THE MOTION OF AN ARTIFICIAL SATELLITE DUE TO THE FOURTH DEGREE LEGENDRE POLYNOMIAL

INTRODUCTION

The launching of distant communication satellites, as well as satellites exploring the far regions between earth and moon, has created a need to investigate more thoroughly the influence of the moon and the sun on artificial satellites. In earlier researches (Reference 1, for example), the influence of the second and third Legendre polynomials of the luni-solar disturbing function has been demonstrated. In this investigation, the work is continued to show the effect of the fourth Legendre polynomial of the disturbing function on earth satellites.

For many purposes, including numerical integration techniques, expansions in terms of rectangular coordinates are useful. Consequently, the disturbing function is first developed in terms of rectangular coordinates. Each term of the resulting series is a product of two functions, one function depending solely on the coordinates of the perturbing body, the other exclusively on the coordinates of the artificial satellite.

For analytic work, the expansion of the disturbing function in terms of the Keplerian elements of the artificial satellite is useful. When the eccentricity of the artificial satellite is small, the formation of the perturbations is expedited if the disturbing function is expanded in terms of mean anomaly. In this case, the coefficients of the expansion form rapidly converging series. However, when the eccentricity is large the expansion proceeds in terms of eccentric anomaly. In either case, the perturbations may be found by the method of canonical variables, after introducing the Jacobi determining function.

THE EXPANSION IN TERMS OF RECTANGULAR COORDINATES

We start with the expression of the disturbing function R in the form

$$R = k^2 m' \frac{a^2}{a'^3}\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3 P_2(C) + k^2 m' \frac{a^3}{a'^4}\left(\frac{r}{a}\right)^3\left(\frac{a'}{r'}\right)^4 P_3(C)$$

$$+ k^2 m' \frac{a^4}{a'^5}\left(\frac{r}{a}\right)^4\left(\frac{a'}{r'}\right)^5 P_4(C) + \cdots\cdots, \tag{1}$$

where

k = Gaussian constant $\left[k^2 = 6.670 \times 10^{-5} \ (\mathrm{cm}^3/\mathrm{kg\ sec}^2)\right]$

m' = mass of the disturbing body

= 7.35×10^{22} kg. for the moon

= 1.99×10^{33} kg. for the sun

a, a' = semimajor axis of the satellite and the disturbing body, respectively

a' = 60.266011 earth radii for the moon

= 23438.524 earth radii for the sun

r, r' = absolute value of the radius vector from the center of the earth to the satellite and to the disturbing body, respectively

C = cosine of the angle between r and r'

$$C = \frac{x x' + y y' + z z'}{r r'} \tag{2}$$

The equatorial plane is taken as the basic reference plane. We have

$$r^2 = x^2 + y^2 + z^2, \qquad r'^2 = x'^2 + y'^2 + z'^2 .$$

Expressions for P_2 (C) and P_3 (C) have already been given by earlier investigators, Reference 1. The corresponding development for P_4 (C) is now outlined. The expression for P_4 (C) is

$$P_4(C) = \frac{35}{8} C^4 - \frac{15}{4} C^2 + \frac{3}{8}. \tag{3}$$

If we now substitute for C its value given by Equation (2), we find

$$\begin{aligned} P_4(C) = \frac{3}{8} + \frac{35}{8 r^4 r'^4} \Big[& x^2 x'^2 \left(2 x^2 x'^2 + 3 y^2 y'^2 + 3 z^2 z'^2 - \frac{6}{7} r^2 r'^2\right) \\ & + y^2 y'^2 \left(3 x^2 x'^2 + 2 y^2 y'^2 + 3 z^2 z'^2 - \frac{6}{7} r^2 r'^2\right) + z^2 z'^2 \left(3 x^2 x'^2 + 3 y^2 y'^2 + 2 z^2 z'^2 - \frac{6}{7} r^2 r'^2\right) \\ & + 4 x x' y y' \left(x^2 x'^2 + y^2 y'^2 + 3 z^2 z'^2 - \frac{3}{7} r^2 r'^2\right) + 4 x x' y y' \left(x^2 x'^2 + 3 y^2 y'^2 + z^2 z'^2 - \frac{3}{7} r^2 r'^2\right) \\ & + 4 y y' z z' \left(3 x^2 x'^2 + y^2 y'^2 + z^2 z'^2 - \frac{3}{7} r^2 r'^2\right) - (x^4 x'^4 + y^4 y'^4 + z^4 z'^4) \Big]. \end{aligned}$$

We illustrate the next step in the procedure of separating the variables by developing the term

$$x^2 x'^2 \left(2 x^2 x'^2 + 3 y^2 y'^2 + 3 z^2 z'^2 - \frac{6}{7} r^2 r'^2\right).$$

By considerations of symmetry, we write

$$2 x^2 x'^2 + 3 y^2 y'^2 + 3 z^2 z'^2 - \frac{6}{7} r^2 r'^2 \equiv (a r^2 + b z^2)(a r'^2 + b z'^2) + (c x^2 + d y^2)(c x'^2 + d y'^2)$$

and determine the constants a, b, c, and d so that an identity results. In this case, we find

$$2x^2x'^2 + 3y^2y'^2 + 3z^2z'^2 - \frac{6}{7}r^2r'^2 =$$

$$\frac{1}{35}[3(2r^2 - 7z^2)(2r'^2 - 7z'^2) + 7(3x^2 - 2y^2)(3x'^2 - 2y'^2)].$$

We proceed in a similar way with the remaining terms to arrive at the required expansion of $P_4(C)$:

$$\begin{aligned} P_4(C) = \frac{3}{8} + \frac{1}{8r^4r'^4}\Big[& 3(2r^2x^2 - 7z^2x^2)(2r'^2x'^2 - 7z'^2x'^2) + 3(2r^2y^2 - 7z^2y^2)(2r'^2y'^2 - 7z'^2y'^2) \\ & + 7(2x^4 - 3y^2x^2)(2x'^4 - 3y'^2x'^2) + 7(3x^2y^2 - 2y^4)(3x'^2y'^2 - 2y'^4) \\ & + \frac{5}{2}(3r^2z^2 - 7z^4)(3r'^2z'^2 - 7z'^4) + \frac{105}{2}(x^2z^2 - y^2z^2)(x'^2z'^2 - y'^2z'^2) \\ & + 10(r^2xy - 7z^2xy)(r'^2x'y' - 7z'^2x'y') + 70(x^3y - xy^3)(x'^3y' - x'y'^3) \\ & + 5(3r^2xz - 7xz^3)(3r'^2x'z' - 7x'z'^3) + 35(x^3z - 3y^2xz)(x'^3z' - 3y'^2x'z') \\ & + 5(3r^2yz - 7yz^3)(3r'^2y'z' - 7y'z'^3) + 35(3x^2yz - y^3z)(3x'^2y'z' - y'^3z') \\ & - (x^4x'^4 + y^4y'^4 + z^4z'^4)\Big] \end{aligned} \tag{4}$$

THE DISTURBING FUNCTION IN TERMS OF THE ELEMENTS

We must first express the quantity C, Equation 2, in terms of the elements. This is done by substituting into Equation 2 the following expressions:

$$\begin{aligned} \frac{x}{r} &= c^2\cos(f+g+\theta) + s^2\cos(f+g-\theta) \\ \frac{y}{r} &= c^2\sin(f+g+\theta) - s^2\sin(f+g-\theta) \\ \frac{z}{r} &= 2sc\sin(f+g) \\ \frac{x'}{r'} &= \cos(f'+g') \\ \frac{y'}{r'} &= (1-2s'^2)\sin(f'+g') \\ \frac{z'}{r'} &= 2s'c'\sin(f'+g'), \end{aligned} \tag{5}$$

where

$$c = \cos \frac{I}{2},$$

I being the angle of inclination of the orbit plane of the satellite, and

$$s = \sin \frac{I}{2}$$

f = true anomaly of the satellite

f' = true anomaly of the disturbing body

g = argument of perigee of the satellite

g' = argument of perigee of the disturbing body

h = longitude of the ascending node of the satellite

h' = longitude of the ascending node of the disturbing body

θ = $h - h'$

Upon performing the substitution of Equation 5 into Equation 2, we find

$$\begin{aligned} C = {} & c^2 c'^2 \cos(u+\theta) + 2 s c s' c' \cos u + s^2 s'^2 \cos(u-\theta) \\ & + c^2 s'^2 \cos(v+\theta) - 2 s c s' c' \cos v + s^2 c'^2 \cos(v-\theta), \end{aligned} \tag{6}$$

where

$$u = f + g - f' - g'$$
$$v = f + g + f' + g'.$$

Substituting the expression for C given by Equation (6) into the formula for the fourth degree Legendre polynomial given by Equation (3), we obtain the following series:

$$\begin{aligned} P_4(C) = {} & \Sigma N_{00\nu} N'_{00\nu} \cos \nu\theta + \Sigma N_{02\nu} N'_{02\nu} \cos(2u + \nu\theta) \\ & + \Sigma N_{20\nu} N'_{20\nu} \cos(2v + \nu\theta) + \Sigma N_{11\nu} N'_{11\nu} \cos(v + u + \nu\theta) + \Sigma N_{1-1\nu} N'_{1-1\nu} \cos(v - u + \nu\theta) \\ & + \Sigma N_{2-2\nu} N'_{2-2\nu} \cos(2v - 2u + \nu\theta) + \Sigma N_{22\nu} N'_{22\nu} \cos(2v + 2u + \nu\theta) + \Sigma N_{04\nu} N'_{04\nu} \cos(4u + \nu\theta) \\ & + \Sigma N_{40\nu} N'_{40\nu} \cos(4v + \nu\theta) + \Sigma N_{13\nu} N'_{13\nu} \cos(v + 3u + \nu\theta) + \Sigma N_{31\nu} N'_{31\nu} \cos(3v + u + \nu\theta) \\ & + \Sigma N_{-13\nu} N'_{-13\nu} \cos(3u - v + \nu\theta) + \Sigma N_{3-1\nu} N'_{3-1\nu} \cos(3v - u + \nu\theta). \end{aligned} \tag{7}$$

The quantities appearing in the summations of Equation (7) are listed in Tables 1-13. The first summation appearing in Equation (7) is called Type 14, the second summation Type 15, and so on; the last summation is Type 26. This numbering sequence is followed to show that the expansion is a continuation of the work of earlier investigators (Reference 1). The quantities $N_{ij\nu}$ are functions of the elements of the satellite, while $N'_{ij\nu}$ are functions of the elements of the disturbing body. To illustrate how the series in Equation (7) is formed, we write the first summation in Equation (7) as obtained from Table 1.

$$\Sigma N_{00\nu} N'_{00\nu} \cos \nu\theta = \frac{9}{64} (1-20\, s^2 c^2 + 70\, s^4 c^4)\, (1-20\, s'^2 c'^2 + 70\, s'^4 c'^4)$$

$$+ \frac{45}{8}\, sc\,(1-2s^2)\, (1-7s^2c^2)\, s'c'\, (1-2s'^2)\, (1-7s'^2c'^2) \cos\theta$$

$$+ \frac{45}{16}\, s^2c^2\,(3-14\,s^2c^2)\, s'^2c'^2\,(3-14\,s'^2c'^2)\cos 2\theta + \frac{315}{8}\, s^3c^3\,(1-2s^2)\, s'^3c'^3\,(1-2s'^2) \cos 3\theta$$

$$+ \frac{315}{16}\, s^4\, c^4\, s'^4\, c'^4 \cos 4\theta .$$

We denote the term of the disturbing function R, Equation (1), containing the fourth degree polynomial by R_4. Then

$$R_4 = k^2 m' \frac{a^4}{a'^5} \left(\frac{r}{a}\right)^4 \left(\frac{a'}{r'}\right)^5 P_4(C). \tag{8}$$

Using the expression for $P_4(C)$ given by Equation (7), we may write

$$R_4 = k^2 m' \frac{a^4}{a'^5} \left(\frac{r}{a}\right)^4 \sum_t N_t N'_t \left(\frac{a'}{r'}\right)^5 \cos(\alpha + q' f'), \tag{9}$$

where

$$\alpha = q f + q g + \nu\theta + q' g' .$$

The argument $\alpha + q' f'$ appearing in Equation (9) is a general form of the trigonometric arguments of Equation (7). The letters t, q, q', and ν are indices which are summed. The quantity $(a'/r')^5 \cos(\alpha + q' f')$ appearing in Equation 9 is now expanded as a Fourier series whose arguments are multiples of the mean anomaly ℓ' and whose coefficients are functions of the eccentricity e', Reference 2. Equation (9) can then be written in the form

$$R_4 = k^2 m' \frac{a^4}{a'^5} \left(\frac{r}{a}\right)^4 \sum_t \sum_{i'} N_t N'_t B'_{ti'} \cos(\alpha + i' \ell'). \tag{10}$$

The quantities $B'_{ti'}$ are functions of the eccentricity of the disturbing body, e', only. These functions are also listed in Tables 1-13.

We now form the quantity

$$S = \frac{1}{\pi} \int_0^{\pi} R_4 \, d\ell . \tag{11}$$

Those terms of S which contain no trigonometric terms are called the secular part of R_4 and denoted by the symbol R_{4S}. The remaining terms which form a trigonometric series are called the long period part of R_4 and are denoted by the symbol $R_{4_{LP}}$. Finally, the trigonometric terms of R_4 which remain when the secular and long period terms are removed are called the short period terms of R_4 and denoted by R_{4_P}.

The methods for finding the perturbations in the elements of an artificial satellite due to R_4 are now discussed. We consider first the case when the eccentricity of the satellite is small, and then when the eccentricity is large.

THE PERTURBATIONS IN THE ELEMENTS OF SMALL ECCENTRICITY SATELLITES

For satellites with small or moderate eccentricity, the product $(r/a)^4 \cos(\alpha + i' \ell')$ appearing in Equation (10) can be expanded in a Fourier series by means of Cayley's Tables, Reference 2.

Upon performing this expansion, we find that

$$R_4 = k^2 m' \frac{a^4}{a'^5} \sum M N B' N' \cos(i \ell + i' \ell' + \nu \theta + \tau). \tag{12}$$

The subscripts appearing in Equation (10) have been omitted. The functions M appearing in Equation (11) are functions of the eccentricity of the artificial earth satellite and are given in Tables 1-13.

R_4 may be used directly in the planetary equations to obtain the time derivatives of the elements of a satellite influenced by the sun or the moon.

Another, possibly more attractive, way of obtaining the perturbations in the elements is by the method of canonical variables.

The differential equation for the Jacobi determining function Q, Reference 3, is given by

$$n \frac{\partial Q}{\partial \ell} + n' \frac{\partial Q}{\partial \ell'} = R_{4_p}, \tag{13}$$

where

n = mean motion of the mean anomaly of the satellite

n' = mean motion of the disturbing body

= approximately 13.2°/day for the moon

= approximately .986°/day for the sun.

R_{4_p} is the periodic part of R_4 defined above.

The solution of Equation (13), using Equation (12), is

$$Q = k^2 m' \frac{a^4}{a'^5} \sum \frac{M N B' N' \sin(i \ell + i' \ell' + \nu \theta + \tau)}{i n + i' n'}. \tag{14}$$

Then we have

$$\delta L = \frac{\partial Q}{\partial \ell} = k^2 m' \frac{a^4}{a'^5} \Sigma \frac{i}{i n + i' n'} M N B' N' \cos (i \ell + i' \ell' + \nu \theta + \tau)$$

$$\delta G = \frac{\partial Q}{\partial g} = k^2 m' \frac{a^4}{a'^5} \Sigma \frac{\frac{\partial \tau}{\partial g}}{i n + i' n'} M N B' N' \cos (i \ell + i' \ell' + \nu \theta + \tau) \tag{15}$$

$$\delta H = \frac{\partial Q}{\partial h} = k^2 m' \frac{a^4}{a'^5} \Sigma \frac{\nu}{i n + i' n'} M N B' N' \cos (i \ell + i' \ell' + \nu \theta + \tau) .$$

The perturbations in the angular variables are somewhat more complicated. We have

$$\delta \ell = - \frac{\partial Q}{\partial L} = - \frac{2L}{\mu} \frac{\partial Q}{\delta a} + \frac{G^2}{e L^3} \frac{\partial Q}{\partial e}$$

$$\delta g = - \frac{\partial Q}{\partial G} = \frac{G}{e L^2} \frac{\partial Q}{\partial e} - \frac{(1 - 2 s^2)}{4 G s} \frac{\partial Q}{\partial s} \tag{16}$$

$$\delta h = - \frac{\partial Q}{\partial H} = \frac{1}{4 G s} \frac{\partial Q}{\partial s} .$$

We then find that

$$\delta \ell = \frac{k^2 m'}{a'^5} \frac{L}{e n^2} \Sigma N \left\{ - e M (5 i n + 8 i' n') + (1 - e^2) \frac{dM}{de} \right\} \frac{\sin (i \ell + i' \ell' + \nu \theta + \tau)}{(i n + i' n')^2}$$

$$\delta g = \frac{k^2 m'}{e a'^5} \frac{a^4}{G} \Sigma \left\{ (1 - e^2) N \frac{dM}{de} - \frac{e (1 - 2 s^2)}{s} M \frac{dN}{ds} \right\} \frac{\sin (i \ell + i' \ell' + \nu \theta + \tau)}{i n + i' n'} \tag{17}$$

$$\delta h = \frac{k^2 m' a^4}{4 G a'^5 s} \Sigma \left\{ M \frac{dN}{ds} \right\} \frac{\sin (i \ell + i' \ell' + \nu \theta + \tau)}{i n + i' n'} .$$

THE PERTURBATIONS IN THE ELEMENTS OF LARGE ECCENTRICITY SATELLITES

When the eccentricity of an artificial earth satellite is large, expanding Equation (10) by means of Cayley's Tables, Reference 2, is no longer adequate. Instead, Hansen's coefficients, Reference 4, may be employed to reduce Equation (10) to the same form as Equation (12). Another alternative is to expand Equation (10) in terms of the eccentric anomaly E of the satellite, as in Reference 1. This form has the advantage that the coefficients of the expansion are finite series in β,

where

$$\beta = \frac{e}{1 + \sqrt{1 - e^2}} .$$

The perturbations may then be found by the methods of canonical variables. As shown in Reference 1, the partial differential equation for the determining function Q is given by

$$\frac{\partial Q}{\partial E} + m_0 \frac{r}{a} \frac{\partial Q}{\partial \ell'} = \frac{r}{n a} R_{4_P}, \tag{18}$$

where $m_0 = n'/n$.

In order to solve Equation (18), the quantity $(r/a) R_{4_P}$ must be found first. Using the results obtained above, $(r/a) R_{4_P}$ can be written in the form

$$\frac{r}{a} R_{4_P} = k^2 m' \frac{a^4}{a'^5} \Sigma\, B N B' N' \cos (i E + i' \ell' + \nu \theta + \tau). \tag{19}$$

The functions N, B', and N' are given in Tables 1-13. The polynomials B and $dB/d\beta$ are tabulated in Tables 14, 15 and 16.

It has been shown in Reference 1 that the solution of Equation (18) to order $m_0^2 e^2$ is given by

$$Q = \frac{k^2 m' a^4}{a'^5} \Sigma \frac{P N B' N'}{i n + i' n'} \left[Q'_0 + m_0 e Q'_1 + m_0^2 e^2 Q'_2\right]$$

$$Q'_0 = \sin (i E + i' \ell' + \tau)$$

$$Q'_1 = \frac{i' \sin [(i+1) E + i' \ell' + \tau]}{2 [(i+1) + i' m]} + \frac{i'}{2} \frac{\sin [(i-1) E + i' \ell' + \tau]}{[(i-1) + i' m]} \tag{20}$$

$$Q'_2 = \frac{i'^2 \sin [(i-2) E + i' \ell' + \tau]}{4 [(i-1) + i' m] [(i+1) + i' m]} + \frac{i'^2}{2} \frac{\sin (i E + i' \ell' + \tau)}{[(i-1) + i' m] [(i+1) + i' m]}$$

$$+ \frac{i'^2}{4} \frac{\sin [(i+2) E + i' \ell' + \tau]}{[(i+1) + i' m] [(i+2) + i' m]}.$$

The determining function Q given above is now used to find the perturbations by the method of canonical variables. Thus,

$$\delta L = \frac{\partial Q}{\partial \ell} \qquad \delta \ell = -\frac{\partial Q}{\partial L}$$

$$\delta G = \frac{\partial Q}{\partial g} \qquad \delta g = -\frac{\partial Q}{\partial G} \tag{21}$$

$$\delta H = \frac{\partial Q}{\partial h} \qquad \delta h = -\frac{\partial Q}{\partial H}.$$

ACKNOWLEDGMENT

The author gratefully thanks Mr. T. L. Felsentreger for his careful review of the computations in this investigation. The responsibility for the accuracy of this report rests, of course, with the author.

REFERENCES

1. Fisher, D., and Murphy, J. P., "Short Period Lunar and Solar Perturbations for Artificial Satellites," Goddard Space Flight Center Report X-547-66-493.

2. Cayley, A., "Tables of the Development of Functions in the Theory of Elliptic Motion," Memoirs Royal Astronomical Society, 29:1861, pp 191-306.

3. Brouwer, D., and Clemence, G. M., "Methods of Celestial Mechanics," Academic Press, 1961, p. 549.

4. Tisserand, F., Traite de Mecanique Celeste, Vol. 1, Chap. 15, Gauthier-Villers, Paris, 1960.

Table 1
Type 14, $\tau = 0$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{9}{64}$	$1 - 20s'^2c'^2 + 70s'^4c'^4$	$1 - 20s^2c^2 + 70s^4c^4$	$-40s(1-2s^2)(1-7s^2c^2)$	0
$\frac{45}{8}$	$s'c'(1-2s'^2)(1-7s'^2c'^2)$	$sc(1-2s^2)(1-7s^2c^2)$	$c^{-1}(1-29s^2c^2+112s^4c^4)$	1
$\frac{45}{16}$	$s'^2c'^2(3-14s'^2c'^2)$	$s^2c^2(3-14s^2c^2)$	$2s(1-2s^2)(3-28s^2c^2)$	2
$\frac{315}{8}$	$s'^3c'^3(1-2s'^2)$	$s^3c^3(1-2s^2)$	$s^2c(3-16s^2c^2)$	3
$\frac{315}{16}$	$s'^4c'^4$	s^4c^4	$4s^3c^2(1-2s^2)$	4
		B'		i'
		$5e'^2$		-2
		$\frac{5}{2}e'$		-1
		$(1-e'^2)^{-7/2}\left(1+\frac{3}{2}e'^2\right)$		0
		$\frac{5}{2}e'$		1
		$5e'^2$		2
		M	$\frac{dM}{de}$	i
		$\frac{7}{48}e^4$	$\frac{7}{12}e^3$	-4
		$\frac{e^3}{4}$	$\frac{3}{4}e^2$	-3
		$\frac{e^2}{2} - \frac{7}{12}e^4$	$e - \frac{7}{3}e^3$	-2
		$-2e - \frac{9}{4}e^3$	$-2 - \frac{27}{4}e^2$	-1
		$1 + 5e^2 + \frac{15}{8}e^4$	$10e + \frac{15}{2}e^3$	0
		$-2e - \frac{9}{4}e^3$	$-2 - \frac{27}{4}e^2$	1
		$\frac{e^2}{2} - \frac{7}{12}e^4$	$e - \frac{7}{3}e^3$	2
		$\frac{e^3}{4}$	$\frac{3}{4}e^2$	3
		$\frac{7}{48}e^4$	$\frac{7}{12}e^3$	4

Table 2
Type 15, $\tau = 2g - 2g'$

	N'	N	$\frac{dN}{ds}$	[illegible]
$\frac{35}{4}$	$s'^6 c'^2$	$s^6 c^2$	$2 s^5 (3 - 4 s^2)$	- 4
$\frac{35}{8}$	$s'^5 c' (1 - 4 c'^2)$	$s^5 c (1 - 4 c^2)$	$c^{-1} s^4 (1 - 2 s^2)(1 - 16 c^2)$	- 3
$\frac{5}{16}$	$s'^4 (1 - 14 c'^2 + 28 c'^4)$	$s^4 (1 - 14 c^2 + 28 c^4)$	$4 s^3 (15 - 63 s^2 + 56 s^4)$	- 2
$\frac{5}{8}$	$s'^3 c' (3 - 21 c'^2 + 28 c'^4)$	$s^3 c (3 - 21 c^2 + 28 c^4)$	$c^{-1} s^2 (30 - 215 s^2 + 406 s^4 - 224 s^6)$	- 1
$\frac{25}{8}$	$s'^2 c'^2 (3 - 14 s'^2 c'^2)$	$s^2 c^2 (3 - 14 s^2 c^2)$	$2 s (1 - 2 s^2)(3 - 28 s^2 c^2)$	0
$\frac{5}{8}$	$s' c'^3 (3 - 21 s'^2 + 28 s'^4)$	$s c^3 (3 - 21 s^2 + 28 s^4)$	$c (3 - 75 s^2 + 266 s^4 - 224 s^6)$	1
$\frac{5}{16}$	$c'^4 (1 - 14 s'^2 + 28 s'^4)$	$c^4 (1 - 14 s^2 + 28 s^4)$	$- 4 s c^2 (8 - 49 s^2 + 56 s^4)$	2
$\frac{35}{8}$	$s' c'^5 (1 - 4 s'^2)$	$s c^5 (1 - 4 s^2)$	$c^3 (1 - 2 s^2)(1 - 16 s^2)$	3
$\frac{35}{4}$	$s'^2 c'^6$	$s^2 c^6$	$2 s c^4 (1 - 4 s^2)$	4
		B'		i'
		$\frac{53}{4} e'^2$		- 4
		$\frac{9}{2} e'$		- 3
		$1 + e'^2$		- 2
		$\frac{e'}{2}$		- 1
		$\frac{3}{4} e'^2$		0
		M	$\frac{dM}{de}$	i
		$- \frac{5}{48} e^4$	$- \frac{5}{12} e^3$	- 2
		$- \frac{13}{6} e^3$	$- \frac{13}{2} e^2$	- 1
		$\frac{21}{4} e^2 + \frac{21}{8} e^4$	$\frac{21}{2} e + \frac{21}{2} e^3$	0
		$- 4 e - 3 e^3$	$- 4 - 9 e^2$	1
		$1 + e^2 - \frac{43}{16} e^4$	$2 e - \frac{43}{4} e^3$	2
		$\frac{3}{2} e^3$	$\frac{9}{2} e^2$	3
		$- \frac{e^2}{4} + \frac{37}{24} e^4$	$- \frac{e}{2} + \frac{37}{6} e^3$	4
		$- \frac{1}{3} e^3$	$- e^2$	5
		$- \frac{3}{8} e^4$	$- \frac{3}{2} e^3$	6

Table 3
Type 16, $\tau = 2g + 2g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{4}$	$s'^2 c'^6$	$s^6 c^2$	$2s^5(3-4s^2)$	-4
$-\frac{35}{8}$	$s' c'^5 (1-4s'^2)$	$s^5 c (1-4c^2)$	$c^{-1} s^4 (1-2s^2)(1-16c^2)$	-3
$\frac{5}{16}$	$c'^4 (1-14s'^2+28s'^4)$	$s^4 (1-14c^2+28c^4)$	$4s^3(15-63s^2+56s^4)$	-2
$-\frac{5}{8}$	$s' c'^3 (3-21s'^2+28s'^4)$	$s^3 c (3-21c^2+28c^4)$	$c^{-1} s^2 (30-215s^2+406s^4-224s^6)$	-1
$\frac{25}{8}$	$s'^2 c'^2 (3-14s'^2 c'^2)$	$s^2 c^2 (3-14s^2 c^2)$	$2s(1-2s^2)(3-28s^2 c^2)$	0
$-\frac{5}{8}$	$s'^3 c' (3-21c'^2+28c'^4)$	$sc^3(3-21s^2+28s^4)$	$c(3-75s^2+266s^4-224s^6)$	1
$\frac{5}{16}$	$s'^4(1-14c'^2+28c'^4)$	$c^4(1-14s^2+28s^4)$	$-4sc^2(8-49s^2+56s^4)$	2
$-\frac{35}{8}$	$s'^5 c' (1-4c'^2)$	$sc^5(1-4s^2)$	$c^3(1-2s^2)(1-16s^2)$	3
$\frac{35}{4}$	$s'^6 c'^2$	$s^2 c^6$	$2sc^4(1-4s^2)$	4
		B'		i'
		$\frac{3}{4} e'^2$		0
		$\frac{e'}{2}$		1
		$1 + e'^2$		2
		$\frac{9}{2} e'$		3
		$\frac{53}{4} e'^2$		4
		M	$\frac{dM}{de}$	i
		$-\frac{5}{48} e^4$	$-\frac{5}{12} e^3$	-2
		$-\frac{13}{6} e^3$	$-\frac{13}{2} e^2$	-1
		$\frac{21}{4} e^2 + \frac{21}{8} e^4$	$\frac{21}{2} e + \frac{21}{2} e^3$	0
		$-4e - 3e^3$	$-4 - 9e^2$	1
		$1 + e^2 - \frac{43}{16} e^4$	$2e - \frac{43}{4} e^3$	2
		$\frac{3}{2} e^3$	$\frac{9}{2} e^2$	3
		$-\frac{e^2}{4} + \frac{37}{24} e^4$	$-\frac{e}{2} + \frac{37}{6} e^3$	4
		$-\frac{e^3}{3}$	$-e^2$	5
		$-\frac{3}{8} e^4$	$-\frac{3}{2} e^3$	6

Table 4
Type 17, $\tau = 2g$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{105}{8}$	$s'^4 c'^4$	$s^6 c^2$	$2 s^5 (3 - 4 s^2)$	-4
$\frac{105}{8}$	$s'^3 c'^3 (1 - 2 c'^2)$	$s^5 c (1 - 4 c^2)$	$c^{-1} s^4 (1 - 2 s^2)(1 - 16 c^2)$	-3
$\frac{15}{16}$	$s'^2 c'^2 (3 - 14 s'^2 c'^2)$	$s^4 (1 - 14 c^2 + 28 c^4)$	$4 s^3 (15 - 63 s^2 + 56 s^4)$	-2
$\frac{15}{8}$	$s' c' (1 - 2 s'^2)(1 - 7 s'^2 c'^2)$	$s^3 c (3 - 21 c^2 + 28 c^4)$	$c^{-1} s^2 (30 - 215 s^2 + 406 s^4 - 224 s^6)$	-1
$\frac{15}{16}$	$1 - 20 s'^2 c'^2 + 70 s'^4 c'^4$	$s^2 c^2 (3 - 14 s^2 c^2)$	$2 s (1 - 2 s^2)(3 - 28 s^2 c^2)$	0
$\frac{15}{8}$	$s' c' (1 - 2 c'^2)(1 - 7 s'^2 c'^2)$	$s c^3 (3 - 21 s^2 + 28 s^4)$	$c (3 - 75 s^2 + 266 s^4 - 224 s^6)$	1
$\frac{15}{16}$	$s'^2 c'^2 (3 - 14 s'^2 c'^2)$	$c^4 (1 - 14 s^2 + 28 s^4)$	$-4 s c^2 (8 - 49 s^2 + 56 s^4)$	2
$\frac{105}{8}$	$s'^3 c'^3 (1 - 2 s'^2)$	$s c^5 (1 - 4 s^2)$	$c^3 (1 - 2 s^2)(1 - 16 s^2)$	3
$\frac{105}{8}$	$s'^4 c'^4$	$s^2 c^6$	$2 s c^4 (1 - 4 s^2)$	4
		B'		i'
		$5 e'^2$		-2
		$\frac{5}{2} e'$		-1
		$(1 - e'^2)^{-7/2} \left(1 + \frac{3}{2} e'^2\right)$		0
		$\frac{5}{2} e'$		1
		$5 e'^2$		2
		M	$\frac{dM}{de}$	i
		$-\frac{5}{48} e^4$	$-\frac{5}{12} e^3$	-2
		$-\frac{13}{6} e^3$	$-\frac{13}{2} e^2$	-1
		$\frac{21}{4} e^2 + \frac{21}{8} e^4$	$\frac{21}{2} e + \frac{21}{2} e^3$	0
		$-4 e - 3 e^3$	$-4 - 9 e^2$	1
		$1 + e^2 - \frac{43}{16} e^4$	$2 e - \frac{43}{4} e^3$	2
		$\frac{3}{2} e^3$	$\frac{9}{2} e^2$	3
		$-\frac{e^2}{4} + \frac{37}{24} e^4$	$-\frac{e}{2} + \frac{37}{6} e^3$	4
		$-\frac{e^3}{3}$	$-e^2$	5
		$-\frac{3}{8} e^4$	$-\frac{3}{2} e^3$	6

Table 5
Type 18, $\tau = 2g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{105}{8}$	$s'^2 c'^6$	$s^4 c^4$	$4s^3 c^2 (1 - 2s^2)$	-4
$\frac{105}{8}$	$s' c'^5 (1 - 4s'^2)$	$s^3 c^3 (1 - 2s^2)$	$cs^2 (3 - 16s^2 c^2)$	-3
$\frac{15}{16}$	$c'^4 (1 - 14s'^2 + 28s'^4)$	$s^2 c^2 (3 - 14s^2 c^2)$	$2s (1 - 2s^2)(3 - 28s^2 c^2)$	-2
$\frac{15}{8}$	$s' c'^3 (3 - 21s'^2 + 28s'^4)$	$sc (1 - 2c^2)(1 - 7s^2 c^2)$	$-c^{-1} (1 - 29s^2 c^2 + 112s^4 c^4)$	-1
$\frac{15}{16}$	$s'^2 c'^2 (3 - 14s'^2 c'^2)$	$1 - 20s^2 c^2 + 70s^4 c^4$	$40s (1 - 2c^2)(1 - 7s^2 c^2)$	0
$\frac{15}{8}$	$s'^3 c' (3 - 21c'^2 + 28c'^4)$	$sc (1 - 2s^2)(1 - 7s^2 c^2)$	$c^{-1} (1 - 29s^2 c^2 + 112s^4 c^4)$	1
$\frac{15}{16}$	$s'^4 (1 - 14c'^2 + 28c'^4)$	$s^2 c^2 (3 - 14s^2 c^2)$	$2s (1 - 2s^2)(3 - 28s^2 c^2)$	2
$\frac{105}{8}$	$s'^5 c' (1 - 4c'^2)$	$s^3 c^3 (1 - 2c^2)$	$-cs^2 (3 - 16s^2 c^2)$	3
$\frac{105}{8}$	$s'^6 c'^2$	$s^4 c^4$	$4s^3 c^2 (1 - 2s^2)$	4
		B'		i'
		$\frac{3}{4} e'^2$		0
		$\frac{e'}{2}$		1
		$1 + e'^2$		2
		$\frac{9}{2} e'$		3
		$\frac{53}{4} e'^2$		4
		M	$\frac{dM}{de}$	i
		$\frac{7}{48} e^4$	$\frac{7}{12} e^3$	-4
		$\frac{e^3}{4}$	$\frac{3e^2}{4}$	-3
		$\frac{e^2}{2} - \frac{7}{12} e^4$	$e - \frac{7}{3} e^3$	-2
		$-2e - \frac{9}{4} e^3$	$-2 - \frac{27}{4} e^2$	-1
		$1 + 5e^2 + \frac{15}{8} e^4$	$10e + \frac{15}{2} e^2$	0
		$-2e - \frac{9}{4} e^3$	$-2 - \frac{27}{4} e^2$	1
		$\frac{e^2}{2} - \frac{7}{12} e^4$	$e - \frac{7}{3} e^3$	2
		$\frac{e^3}{4}$	$\frac{3e^2}{4}$	3
		$\frac{7}{48} e^4$	$\frac{7}{12} e^3$	4

Table 6
Type 19, $\tau = 4g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{105}{32}$	c'^8	$s^4 c^4$	$4 s^3 c^2 (c^2 - s^2)$	-4
$\frac{105}{8}$	$s' c'^7$	$s^3 c^3 (1 - 2c^2)$	$- s^2 c (3 - 16 s^2 c^2)$	-3
$\frac{105}{16}$	$s'^2 c'^6$	$s^2 c^2 (3 - 14 s^2 c^2)$	$2s (c^2 - s^2)(3 - 28 s^2 c^2)$	-2
$\frac{105}{8}$	$s'^3 c'^5$	$sc (1 - 2c^2)(1 - 7 s^2 c^2)$	$- c^{-1} (1 - 29 s^2 c^2 + 112 s^4 c^4)$	-1
$\frac{105}{32}$	$s'^4 c'^4$	$1 - 20 s^2 c^2 + 70 s^4 c^4$	$- 40 s (c^2 - s^2)(1 - 7 s^2 c^2)$	0
$\frac{105}{8}$	$s'^5 c'^3$	$sc (1 - 2s^2)(1 - 7 s^2 c^2)$	$c^{-1} (1 - 29 s^2 c^2 + 112 s^4 c^4)$	1
$\frac{105}{16}$	$s'^6 c'^2$	$s^2 c^2 (3 - 14 s^2 c^2)$	$2s (c^2 - s^2)(3 - 28 s^2 c^2)$	2
$\frac{105}{8}$	$s'^7 c'$	$s^3 c^3 (1 - 2s^2)$	$s^2 c (3 - 16 s^2 c^2)$	3
$\frac{105}{32}$	s'^8	$s^4 c^4$	$4 s^3 c^2 (c^2 - s^2)$	4
		B'		i'
		$\frac{e'^2}{2}$		2
		$-\frac{3e'}{2}$		3
		$1 - 11 e'^2$		4
		$\frac{13 e'}{2}$		5
		$\frac{51}{2} e'^2$		6
		M	$\frac{dM}{de}$	i
		$\frac{7}{48} e^4$	$\frac{7}{12} e^3$	-4
		$\frac{e^3}{4}$	$\frac{3}{4} e^2$	-3
		$\frac{e^2}{2} - \frac{7}{12} e^4$	$e - \frac{7}{3} e^3$	-2
		$- 2e - \frac{9}{4} e^3$	$- 2 - \frac{27}{4} e^2$	-1
		$1 + 5 e^2 + \frac{15}{8} e^4$	$10 e + \frac{15}{2} e^3$	0
		$- 2e - \frac{9}{4} e^3$	$- 2 - \frac{27}{4} e^2$	1
		$\frac{e^2}{2} - \frac{7}{12} e^4$	$e - \frac{7}{3} e^3$	2
		$\frac{e^3}{4}$	$\frac{3}{4} e^2$	3
		$\frac{7}{48} e^4$	$\frac{7}{12} e^3$	4

Table 7
Type 20, $\tau = 4g$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{105}{32}$	$s'^4 c'^4$	s^8	$8s^7$	-4
$\frac{105}{8}$	$s'^3 c'^3 (1 - 2s'^2)$	$s^7 c$	$c^{-1} s^6 (7 - 8s^2)$	-3
$\frac{105}{16}$	$s'^2 c'^2 (3 - 14s'^2 c'^2)$	$s^6 c^2$	$2s^5 (3 - 4s^2)$	-2
$\frac{105}{8}$	$s' c' (1 - 2s'^2)(1 - 7s'^2 c'^2)$	$s^5 c^3$	$cs^4 (5 - 8s^2)$	-1
$\frac{105}{32}$	$1 - 20s'^2 c'^2 + 70s'^4 c'^4$	$s^4 c^4$	$4c^2 s^3 (1 - 2s^2)$	0
$\frac{105}{8}$	$s' c' (1 - 2c'^2)(1 - 7s'^2 c'^2)$	$s^3 c^5$	$c^3 s^2 (3 - 8s^2)$	1
$\frac{105}{16}$	$s'^2 c'^2 (3 - 14s'^2 c'^2)$	$s^2 c^6$	$2c^4 s (1 - 4s^2)$	2
$\frac{105}{8}$	$s'^3 c'^3 (1 - 2c'^2)$	sc^7	$c^5 (1 - 8s^2)$	3
$\frac{105}{32}$	$s'^4 c'^4$	c^8	$-8c^6 s$	4
		B'		i'
		$5e'^2$		-2
		$\frac{5}{2} e'$		-1
		$(1 - e'^2)^{-7/2} \left(1 + \frac{3}{2} e'^2\right)$		0
		$\frac{5}{2} e'$		1
		$5e'^2$		2
		M	$\frac{dM}{de}$	i
		$\frac{63}{8} e^4$	$\frac{63}{2} e^3$	0
		$-\frac{187}{12} e^3$	$-\frac{187}{4} e^2$	1
		$14e^2 - \frac{137}{6} e^4$	$28e - \frac{274}{3} e^3$	2
		$-6e + \frac{93}{4} e^3$	$-6 + \frac{279}{4} e^2$	3
		$1 - 11e^2 + \frac{253}{8} e^4$	$-22e + \frac{253}{2} e^3$	4
		$2e - \frac{63}{4} e^3$	$2 - \frac{189}{4} e^2$	5
		$3e^2 - 21e^4$	$6e - 84e^3$	6
		$\frac{49}{12} e^3$	$\frac{49}{4} e^2$	7
		$\frac{16}{3} e^4$	$\frac{64}{3} e^3$	8

Table 8
Type 21, $\tau = 4g - 4g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{64}$	s'^8	s^8	$8s^7$	-4
$\frac{35}{8}$	$s'^7 c'$	$s^7 c$	$c^{-1} s^6 (7 - 8s^2)$	-3
$\frac{245}{16}$	$s'^6 c'^2$	$s^6 c^2$	$2s^5 (3 - 4s^2)$	-2
$\frac{245}{8}$	$s'^5 c'^3$	$s^5 c^3$	$c s^4 (5 - 8s^2)$	-1
$\frac{1225}{32}$	$s'^4 c'^4$	$s^4 c^4$	$4c^2 s^3 (1 - 2s^2)$	0
$\frac{248}{8}$	$s'^3 c'^5$	$s^3 c^5$	$c^3 s^2 (3 - 8s^2)$	1
$\frac{245}{8}$	$s'^2 c'^6$	$s^2 c^6$	$2c^4 s (1 - 4s^2)$	2
$\frac{35}{8}$	$s' c'^7$	$s c^7$	$c^5 (1 - 8s^2)$	3
$\frac{35}{64}$	c'^8	c^8	$-8c^6 s$	4
		B'		i'
		$\frac{e'^2}{2}$		-2
		$-\frac{3e'}{2}$		-3
		$1 - 11e'^2$		-4
		$\frac{13e'}{2}$		-5
		$\frac{51}{2}e'^2$		-6
		M	$\frac{dM}{de}$	i
		$\frac{63}{8}e^4$	$\frac{63}{2}e^3$	0
		$-\frac{187}{12}e^3$	$-\frac{187}{4}e^2$	1
		$14e^2 - \frac{137}{6}e^4$	$28e - \frac{274}{3}e^3$	2
		$-6e + \frac{93}{4}e^3$	$-6 + \frac{279}{4}e^2$	3
		$1 - 11e^2 + \frac{253}{8}e^4$	$-22e + \frac{253}{2}e^3$	4
		$2e - \frac{63}{4}e^3$	$2 - \frac{189}{4}e^2$	5
		$3e^2 - 21e^4$	$6e - 84e^3$	6
		$\frac{49}{12}e^3$	$\frac{49}{4}e^2$	7
		$\frac{16}{3}e^4$	$\frac{64}{3}e^3$	8

Table 9
Type 22, $\tau = 4g + 4g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{64}$	c'^8	s^8	$8s^7$	-4
$-\frac{35}{8}$	$s'c'^7$	$s^7 c$	$c^{-1} s^6 (7 - 8s^2)$	-3
$\frac{245}{16}$	$s'^2 c'^6$	$s^6 c^2$	$2s^5 (3 - 4s^2)$	-2
$-\frac{245}{8}$	$s'^3 c'^5$	$s^5 c^3$	$cs^4 (5 - 8s^2)$	-1
$\frac{1225}{32}$	$s'^4 c'^4$	$s^4 c^4$	$4c^2 s^3 (1 - 2s^2)$	0
$-\frac{245}{8}$	$s'^5 c'^3$	$s^3 c^5$	$c^3 s^2 (3 - 8s^2)$	1
$\frac{245}{16}$	$s'^6 c'^2$	$s^2 c^6$	$2c^4 s (1 - 4s^2)$	2
$-\frac{35}{8}$	$s'^7 c'$	sc^7	$c^5 (1 - 8s^2)$	3
$\frac{35}{16}$	s'^8	c^8	$-8c^6 s$	4
		B'		i'
		$\frac{e'^2}{2}$		2
		$-\frac{3e'}{2}$		3
		$1 - 11e'^2$		4
		$\frac{13e'}{2}$		5
		$\frac{51e'^2}{2}$		6
		M	$\frac{dM}{de}$	i
		$\frac{63}{8} e^4$	$\frac{63}{2} e^3$	0
		$-\frac{187}{12} e^3$	$-\frac{187}{4} e^2$	1
		$14e^2 - \frac{137}{6} e^4$	$28e - \frac{274}{3} e^3$	2
		$-6e + \frac{93}{4} e^3$	$-6 + \frac{279}{4} e^2$	3
		$1 - 11e^2 + \frac{253}{8} e^4$	$-22e + \frac{253}{2} e^3$	4
		$2e - \frac{63}{4} e^3$	$2 - \frac{189}{4} e^2$	5
		$3e^2 - 21e^4$	$6e - 84e^3$	6
		$\frac{49}{12} e^3$	$\frac{49}{4} e^2$	7
		$\frac{16}{3} e^4$	$\frac{64}{3} e^3$	8

Table 10
Type 23, $\tau = 4g - 2g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{16}$	$s'^6 c'^2$	s^8	$8 s^7$	-4
$\frac{35}{8}$	$s'^5 c' (3 - 4 s'^2)$	$s^7 c$	$c^{-1} s^6 (7 - 8 s^2)$	-3
$\frac{35}{16}$	$s'^4 (1 - 14 c'^2 + 28 c'^4)$	$s^6 c^2$	$2 s^5 (3 - 4 s^2)$	-2
$\frac{35}{8}$	$s'^3 c' (3 - 21 c'^2 + 28 c'^4)$	$s^5 c^3$	$c s^4 (5 - 8 s^2)$	-1
$\frac{175}{16}$	$s'^2 c'^2 (3 - 14 s'^2 c'^2)$	$s^4 c^4$	$4 c^2 s^3 (1 - 2 s^2)$	0
$\frac{35}{8}$	$s' c'^3 (3 - 21 s'^2 + 28 s'^4)$	$s^3 c^5$	$c^3 s^2 (3 - 8 s^2)$	1
$\frac{35}{16}$	$c'^4 (1 - 14 s'^2 + 28 s'^4)$	$s^2 c^6$	$2 c^4 s (1 - 4 s^2)$	2
$\frac{35}{8}$	$s' c'^5 (3 - 4 c'^2)$	$s c^7$	$c^5 (1 - 8 s^2)$	3
$\frac{35}{16}$	$s'^2 c'^6$	c^8	$-8 c^6 s$	4
		B'		**i'**
		$\frac{3e'^2}{4}$		0
		$\frac{e'}{2}$		-1
		$1 + e'^2$		-2
		$\frac{9}{2} e'$		-3
		$\frac{53}{4} e'^2$		-4
		M	**$\frac{dM}{de}$**	**i**
		$\frac{63}{8} e^4$	$\frac{63}{2} e^3$	0
		$-\frac{187}{12} e^3$	$-\frac{187}{4} e^2$	1
		$14 e^2 - \frac{137}{6} e^4$	$28 e - \frac{274}{3} e^3$	2
		$-6 e + \frac{93}{4} e^3$	$-6 + \frac{279}{4} e^2$	3
		$1 - 11 e^2 + \frac{253}{8} e^4$	$-22 e + \frac{253}{2} e^3$	4
		$2 e - \frac{63}{4} e^3$	$2 - \frac{189}{4} e^2$	5
		$3 e^2 - 21 e^4$	$6 e - 84 e^3$	6
		$\frac{49}{12} e^3$	$\frac{49}{4} e^2$	7
		$\frac{16}{3} e^4$	$\frac{64}{3} e^3$	8

Table 11
Type 24, $\tau = 4g + 2g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{16}$	$s'^2 c'^6$	s^8	$8s^7$	-4
$-\frac{35}{8}$	$s' c'^5 (3 - 4c'^2)$	$s^7 c$	$c^{-1} s^6 (7 - 8s^2)$	-3
$\frac{35}{16}$	$c'^4 (1 - 14s'^2 + 28s'^4)$	$s^6 c^2$	$2s^5 (3 - 4s^2)$	-2
$-\frac{35}{8}$	$s' c'^3 (3 - 21s'^2 + 28s'^4)$	$s^5 c^3$	$c s^4 (5 - 8s^2)$	-1
$\frac{175}{16}$	$s'^2 c'^2 (3 - 14s'^2 c'^2)$	$s^4 c^4$	$4c^2 s^3 (1 - 2s^2)$	0
$-\frac{35}{8}$	$s'^3 c' (3 - 21c'^2 + 28c'^4)$	$s^3 c^5$	$c^3 s^2 (3 - 8s^2)$	1
$\frac{35}{16}$	$s'^4 (1 - 14c'^2 + 28c'^4)$	$s^2 c^6$	$2c^4 s (1 - 4s^2)$	2
$-\frac{35}{8}$	$s'^5 c' (3 - 4s'^2)$	$s c^7$	$c^5 (1 - 8s^2)$	3
$\frac{35}{16}$	$s'^6 c'^2$	c^8	$-8c^6 s$	4
		B'		i'
		$\frac{3}{4} e'^2$		0
		$\frac{e'}{2}$		1
		$1 + e'^2$		2
		$\frac{9}{2} e'$		3
		$\frac{53}{4} e'^2$		4
		M	$\frac{dM}{de}$	i
		$\frac{63}{8} e^4$	$\frac{63}{2} e^3$	0
		$-\frac{187}{12} e^3$	$-\frac{187}{4} e^2$	1
		$14e^2 - \frac{137}{6} e^4$	$28e - \frac{274}{3} e^3$	2
		$-6e + \frac{93}{4} e^3$	$-6 + \frac{279}{4} e^2$	3
		$1 - 11e^2 + \frac{253}{8} e^4$	$-22e + \frac{253}{2} e^3$	4
		$2e - \frac{63}{4} e^3$	$2 - \frac{189}{4} e^2$	5
		$3e^2 - 21e^4$	$6e - 84e^3$	6
		$\frac{49}{12} e^3$	$\frac{49}{4} e^2$	7
		$\frac{16}{3} e^4$	$\frac{64}{3} e^3$	8

Table 12
Type 25, $\tau = 2g - 4g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{16}$	s'^8	$s^6 c^2$	$2s^5(3-4s^2)$	-4
$-\frac{35}{8}$	$s'^7 c'$	$s^5 c(1-4c^2)$	$c^{-1} s^4(1-2s^2)(1-16c^2)$	-3
$\frac{35}{16}$	$s'^6 c'^2$	$s^4(1-14c^2+28c^4)$	$4s^3(15-63s^2+56s^4)$	-2
$\frac{35}{8}$	$s'^5 c'^3$	$s^3 c(3-21c^2+28c^4)$	$c^{-1} s^2(30-215s^2+406s^4-224s^6)$	-1
$\frac{175}{16}$	$s'^4 c'^4$	$s^2 c^2(3-14s^2c^2)$	$2s(1-2s^2)(3-28s^2c^2)$	0
$\frac{35}{8}$	$s'^3 c'^5$	$sc^3(3-21s^2+28s^4)$	$c(3-75s^2+266s^4-224s^6)$	1
$\frac{35}{16}$	$s'^2 c'^6$	$c^4(1-14s^2+28s^4)$	$-4sc^2(8-49s^2+56s^4)$	2
$-\frac{35}{8}$	$s' c'^7$	$sc^5(1-4s^2)$	$c^3(1-2s^2)(1-16s^2)$	3
$\frac{35}{16}$	c'^8	$s^2 c^6$	$2sc^4(1-4s^2)$	4
		B'		i'
		$\frac{e'^2}{2}$		-2
		$-\frac{3e'}{2}$		-3
		$1-11e'^2$		-4
		$\frac{13e'}{2}$		-5
		$\frac{51e'^2}{2}$		-6
		M	$\frac{dM}{de}$	i
		$-\frac{5}{48}e^4$	$-\frac{5}{12}e^3$	-2
		$-\frac{13}{6}e^3$	$-\frac{13}{2}e^2$	-1
		$\frac{21}{4}e^2+\frac{21}{8}e^4$	$\frac{21}{2}e+\frac{21}{2}e^3$	0
		$-4e-3e^3$	$-4-9e^2$	1
		$1+e^2-\frac{43}{16}e^4$	$2e-\frac{43}{4}e^3$	2
		$\frac{3}{2}e^3$	$\frac{9}{2}e^2$	3
		$-\frac{e^2}{4}+\frac{37}{24}e^4$	$-\frac{e}{2}+\frac{37}{6}e^3$	4
		$-\frac{e^3}{3}$	$-e^2$	5
		$-\frac{3}{8}e^4$	$-\frac{3}{2}e^3$	6

Table 13
Type 26, $\tau = 2g + 4g'$

	N'	N	$\frac{dN}{ds}$	ν
$\frac{35}{16}$	c'^8	$s^6 c^2$	$2s^5(3-4s^2)$	-4
$\frac{35}{8}$	$s' c'^7$	$s^5 c(1-4c^2)$	$c^{-1} s^4(1-2s^2)(1-16c^2)$	-3
$\frac{35}{16}$	$s'^2 c'^6$	$s^4(1-14c^2+28c^4)$	$4s^3(15-63s^2+56s^4)$	-2
$-\frac{35}{8}$	$s'^3 c'^5$	$s^3 c(3-21c^2+28c^4)$	$c^{-1}s^2(30-215s^2+406s^4-224s^6)$	-1
$\frac{175}{16}$	$s'^4 c'^4$	$s^2c^2(3-14s^2c^2)$	$2s(1-2s^2)(3-28s^2c^2)$	0
$-\frac{35}{8}$	$s'^5 c'^3$	$sc^3(3-21s^2+28s^4)$	$c(3-75s^2+266s^4-224s^6$	1
$\frac{35}{16}$	$s'^6 c'^2$	$c^4(1-14s^2+28s^4)$	$-4sc^2(8-49s^2+56s^4)$	2
$\frac{35}{8}$	$s'^7 c'$	$sc^5(1-4s^2)$	$c^3(1-2s^2)(1-16c^2)$	3
$\frac{35}{16}$	s'^8	s^2c^6	$2sc^4(1-4s^2)$	4
		B'		i'
		$\frac{e'^2}{2}$		2
		$-\frac{3e'}{2}$		3
		$1-11e'^2$		4
		$-\frac{13e'}{2}$		5
		$\frac{51}{2}e'^2$		6
		M	$\frac{dM}{de}$	i
		$-\frac{5}{48}e^4$	$-\frac{5}{12}e^3$	-2
		$-\frac{13}{6}e^3$	$-\frac{13}{2}e^2$	-1
		$\frac{21}{4}e^2+\frac{21}{8}e^4$	$\frac{21}{2}e+\frac{21}{2}e^3$	0
		$-4e-3e^3$	$-4-9e^2$	1
		$1+e^2-\frac{43}{16}e^4$	$2e-\frac{43}{4}e^3$	2
		$\frac{3}{2}e^3$	$\frac{9}{2}e^2$	3
		$-\frac{e^2}{4}+\frac{37}{24}e^4$	$-\frac{e}{2}+\frac{37}{6}e^3$	4
		$-\frac{e^3}{3}$	$-e^2$	5
		$-\frac{3}{8}e^4$	$-\frac{3}{2}e^3$	6

Table 14
Types - 14, 18, 19

B $(1+\beta^2)^{-5}$ x	$\frac{dB}{d\beta}$	i
$-\beta^5$	$-5\beta^4(1-\beta^2)$	-5
$5\beta^4(1+\beta^2)$	$20\beta^3(1-\beta^4)$	-4
$-10\beta^3\left(1+\frac{5}{2}\beta^2+\beta^4\right)$	$-5\beta^2(1-\beta^2)(6+17\beta^2+6\beta^4)$	-3
$10\beta^2(1+5\beta^2+5\beta^4+\beta^6)$	$20\beta(1-\beta^4)(1+6\beta^2+\beta^4)$	-2
$-5\beta(1+10\beta^2+20\beta^4+10\beta^6+\beta^8)$	$-5(1-\beta^2)(1+22\beta^2+52\beta^4+22\beta^6+\beta^8)$	-1
$1+25\beta^2+100\beta^4+100\beta^6+25\beta^8+\beta^{10}$	$40\beta(1-\beta^4)(1+5\beta^2+\beta^4)$	0
$-5\beta(1+10\beta^2+20\beta^4+10\beta^6+\beta^8)$	$-5(1-\beta^2)(1+22\beta^2+52\beta^4+22\beta^6+\beta^8)$	1
$10\beta^2(1+5\beta^2+5\beta^4+\beta^6)$	$20\beta(1-\beta^4)(1+6\beta^2+\beta^4)$	2
$-10\beta^3\left(1+\frac{5}{2}\beta^2+\beta^4\right)$	$-5\beta^2(1-\beta^2)(6+17\beta^2+6\beta^4)$	3
$5\beta^4(1+\beta^2)$	$20\beta^3(1-\beta^4)$	4
$-\beta^5$	$-5\beta^4(1-\beta^2)$	5

Table 15
Types 15, 16, 17, 25, 26

B $(1+\beta^2)^{-5}$ x	$\frac{dB}{d\beta}$ $(1+\beta^2)^{-6}$ x	i
$-\beta^7$	$-\beta^6(7-3\beta^2)$	-5
$\beta^6(7+3\beta^2)$	$2\beta^5(21-2\beta^2-3\beta^4)$	-4
$-3\beta^5(7+7\beta^2+\beta^4)$	$-3\beta^4(35+14\beta^2-12\beta^4-\beta^6)$	-3
$\beta^4(35+63\beta^2+21\beta^4+\beta^6)$	$4\beta^3(35+42\beta^2-21\beta^4-8\beta^6)$	-2
$-7\beta^3(5+15\beta^2+9\beta^4+\beta^6)$	$-7\beta^2(15+40\beta^2-12\beta^4-18\beta^6-\beta^8)$	-1
$21\beta^2(1+\beta^2)(1+4\beta^2+\beta^4)$	$42\beta(1-\beta^4)(1+6\beta^2+\beta^4)$	0
$-7\beta(1+9\beta^2+15\beta^4+5\beta^6)$	$-7(1+18\beta^2+12\beta^4-40\beta^6-15\beta^8)$	1
$1+21\beta^2+63\beta^4+35\beta^6$	$4\beta(8+21\beta^2-42\beta^4-35\beta^6)$	2
$-3\beta(1+7\beta^2+7\beta^4)$	$-3(1+12\beta^2-14\beta^4-35\beta^6)$	3
$\beta^2(3+7\beta^2)$	$2\beta(3+2\beta^2-21\beta^4)$	4
$-\beta^3$	$-\beta^2(3-7\beta^2)$	5

Table 16
Types - 20, 21, 22, 23, 24

B $(1+\beta^2)^{-5}$ x	$\frac{dB}{d\beta}$ $(1+\beta^2)^{-6}$ x	i
$-\beta^9$	$-\beta^8(9-\beta^2)$	-5
$\beta^8(9+\beta^2)$	$8\beta^7(9-\beta^2)$	-4
$-9\beta^7(4+\beta^2)$	$-9\beta^6(28-3\beta^2-\beta^4)$	-3
$12\beta^6(7+3\beta^2)$	$24\beta^5(21-2\beta^2-3\beta^4)$	-2
$-42\beta^5(3+2\beta^2)$	$-42\beta^4(15-\beta^2-6\beta^4)$	-1
$126\beta^4(1+\beta^2)$	$504\beta^3(1-\beta^4)$	0
$-42\beta^3(2+3\beta^2)$	$-42\beta^2(6+\beta^2-15\beta^4)$	1
$12\beta^2(3+7\beta^2)$	$24\beta(3+2\beta^2-21\beta^4)$	2
$-9\beta(1+4\beta^2)$	$-9(1+3\beta^2-28\beta^4)$	3
$1+9\beta^2$	$8\beta(1-9\beta^2)$	4
$-\beta$	$-(1-9\beta^2)$	5