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THE POTENTIAL FUNCTION AND FIELD OF FORCE OF A SPHEROID

BY

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SUMMARY

General formulas are presented for the potential function of a spheroid in rectangular coordinates. The components of the force, the partial derivatives of the potential with respect to the coordinates, are also derived.

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LIST OF SYMBOLS

b	mean radius of the spheroid
$C_n^{(m)}$, $S_n^{(m)}$	spherical harmonic coefficients
$\overline{C}_n^{(m)}$, $\overline{S}_n^{(m)}$	normalized spherical harmonic coefficients
$\vec{F}$	field of force for a spheroid
$\vec{F}'$	$\vec{F}$ in an inertial system for a rotating spheroid
i, j, k	indices of summation
$\vec{i}, \vec{j}, \vec{k}$	orthogonal unit vectors
$\vec{i}', \vec{j}', \vec{k}'$	inertial orthogonal unit vectors
J_n	zonal harmonics for axially symmetric spheroid
$N_n^{(m)}$	normalizing factor
P_n	Legendre polynomial
$P_n^{(m)}$	Legendre associated function
R	disturbing function
$R_n^{(m)}$	disturbing function due to (m, n) harmonic
$R_3(\alpha)$	rotation matrix—by angle α about the z axis
(r, ϕ, λ)	spherical coordinates
U	potential of a spheroid
x, y, z	rectangular coordinates
α	the angle between $\vec{i}$ and $\vec{i}'$
$\alpha(m)$, $\beta(m)$, $\gamma(m, n)$, $\tau(m,n)$	defined by Equations 7 and 14

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CONTENTS

THE POTENTIAL FUNCTION AND FIELD OF FORCE OF A SPHEROID

INTRODUCTION

With progress in tracking systems and the availability of more and better tracking data from the many artificial satellites launched and about to be launched, the need arises to include smaller forces in analyzing the motion of the satellites. Among these smaller forces are the so called "higher harmonics" in the earth's gravitational potential. The purpose of this report is to present general expressions depending only on m, the order, and n, the degree, for the components of the force due to a particular harmonic. With such a formulation in a system for analyzing data, generating tracking predictions, deriving harmonic coefficients etc., there would be no need to revise the programs to include any additional harmonic, but simply to imput the various (m, n) that one wished to consider.

THE POTENTIAL FUNCTION

The potential of a spheroid at a point, p, with spherical coordinates (r, ϕ, λ) is

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{b}{r}\right)^n P_n^{(m)}(\sin\phi)\left(C_n^{(m)}\cos m\lambda + S_n^{(m)}\sin m\lambda\right)\right\} \tag{1}$$

where μ is the product of the gravitational constant, G, times the mass of the spheroid, M, and b is the mean radius of the spheroid. The Legendre associated function, $P_n^{(m)}(\sin\phi)$, is defined by

$$P_n^{(m)}(t) = \frac{1}{2^n n!}\left(1 - t^2\right)^{m/2}\frac{d^{n+m}\left(t^2 - 1\right)^n}{dt^{n+m}} . \tag{2}$$

The quantity $R = U - \mu/r$, is called the disturbing function and $R_n^{(m)}$ is that part of the disturbing function due to the harmonic of order m and degree n. Therefore,

$$R_n^{(m)} = \frac{\mu b^n}{r^{n+1}} P_n^{(m)}(\sin\phi)\left(C_n^{(m)}\cos m\lambda + S_n^{(m)}\sin m\lambda\right) . \tag{3}$$

One of the factors of $R_n^{(m)}$, $P_n^{(m)}(\sin\phi)$, will now be simplified:

$$P_n^{(m)}(\sin\phi) = \frac{1}{2^n\, n!}\left(1-\sin^2\phi\right)^{m/2} \frac{d^{n+m}\left(\sin^2\phi - 1\right)^n}{d(\sin\phi)^{n+m}}$$

$$= \frac{1}{2^n\, n!}\cos^m\phi \, \frac{d^{n+m}\left\{\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sin^{2(n-1)}\phi\right\}}{d(\sin\phi)^{n+m}}$$

$$= \frac{1}{2^n\, n!}\cos^m\phi \sum_{i=0}^{\substack{(n-m)/2\ [(n+m)\ \text{even}]\\ (n-m-1)/2\ [(n+m)\ \text{odd}]}} (-1)^i\binom{n}{i}\frac{\left[2(n-i)\right]!}{(n-m-2i)!}\sin^{n-m-2i}\phi\ . \tag{4}$$

After substituting Equation 4 into Equation 3, and after making use of the following relationships between spherical and rectangular coordinates:

$$\sin\phi = \frac{z}{r}$$

$$\cos^m\phi\cos m\lambda = r^{-m}\sum_{j=0}^{\substack{m/2\ (m\ \text{even})\\ (m-1)/2\ (m\ \text{odd})}}(-1)^j\binom{m}{2j}x^{m-2j}\,y^{2j}$$

$$\cos^m\phi\sin m\lambda = r^{-m}\sum_{k=0}^{\substack{(m-2)/2\ (m\ \text{even})\\ (m-1)/2\ (m\ \text{odd})}}(-1)^k\binom{m}{2k+1}x^{m-2k-1}\,y^{2k+1} \tag{5}$$

where

$$r = \sqrt{x^2+y^2+z^2}\ ,$$

Equation 3 becomes

$$R_n^{(m)} = \frac{\mu b^n}{2^n \, n!} \, r^{-(n+m+1)} \, \gamma(m, n) \left[C_n^{(m)} \, \alpha(m) + S_n^{(m)} \, \beta(m) \right] , \tag{6}$$

where

$$\alpha(m) = \sum_{j=0}^{\substack{m/2 \ (m \text{ even}) \\ (m-1)/2 \ (m \text{ odd})}} (-1)^j \binom{m}{2j} x^{m-2j} \, y^{2j} \qquad \text{if} \qquad m \neq 0$$

$$\beta(m) = \sum_{k=0}^{\substack{(m-2)/2 \ (m \text{ even}) \\ (m-1)/2 \ (m \text{ odd})}} (-1)^k \binom{m}{2k+1} x^{m-2k-1} \, y^{2k+1} \qquad \text{if} \qquad m \neq 0$$

$$\gamma(m, n) = \sum_{i=0}^{\substack{(n-m)/2 \ (m+n \text{ even}) \\ (n-m-1)/2 \ (m+n \text{ odd})}} (-1)^i \binom{n}{i} \frac{[2(n-i)]!}{(n-m-2i)!} \left(\frac{z}{r}\right)^{n-m-2i} . \tag{7}$$

and

$$\alpha(0) = 1 \quad \text{if} \quad m = 0$$

$$\beta(0) = 0 \quad \text{if} \quad m = 0 .$$

THE FIELD OF FORCE

The field of force is given by the gradient of the potential function. Thus,

$$\vec{F} = \nabla U = \frac{\partial U}{\partial x} \vec{i} + \frac{\partial U}{\partial y} \vec{j} + \frac{\partial U}{\partial z} \vec{k} , \tag{8}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the orthogonal unit vectors in the spheroid centered coordinate system. If we denote any one of the coordinates x, y, or z by x_p, then

$$\frac{\partial U}{\partial x_p} = -\frac{\mu x_p}{r^3} + \sum_{m,n} \frac{\partial R_n^{(m)}}{\partial x_p}, \tag{9}$$

where

$$\begin{aligned}\frac{\partial R_n^{(m)}}{\partial x_p} = \frac{\mu b^n}{2^n\, n!}\, r^{-(n+m+1)} \Bigg\{ & \frac{-x_p}{r^2}\,(n+m+1)\,\gamma(m,n)\left[C_n^{(m)}\,\alpha(m) + S_n^{(m)}\,\beta(m)\right] \\ & + \frac{\partial \gamma(m,n)}{\partial x_p}\left[C_n^{(m)}\,\alpha(m) + S_n^{(m)}\,\beta(m)\right] \\ & + \gamma(m,n)\left[C_n^{(m)}\,\frac{\partial \alpha(m)}{\partial x_p} + S_n^{(m)}\,\frac{\partial \beta(m)}{\partial x_p}\right]\Bigg\}\,.\end{aligned} \tag{10}$$

The various partial derivatives of α, β, and γ for x_p equal to x, y, and z are now given. If $m = 0$, $\partial\alpha(0)/\partial x_p = \partial\beta(0)/\partial x_p = 0$. If $m \neq 0$

$$\begin{aligned}\frac{\partial \alpha(m)}{\partial x} &= \sum_{j=0}^{\substack{m/2 \ (m\ \mathrm{even}) \\ (m-1)/2\ (m\ \mathrm{odd})}} (-1)^j \binom{m}{2j} (m-2j)\, x^{m-2j-1}\, y^{2j} \\ \frac{\partial \alpha(m)}{\partial y} &= 2 \sum_{j=0}^{\substack{m/2 \ (m\ \mathrm{even}) \\ (m-1)/2\ (m\ \mathrm{odd})}} j\,(-1)^j \binom{m}{2j} x^{m-2j}\, y^{2j-1} \\ \frac{\partial \alpha(m)}{\partial z} &= 0\end{aligned} \tag{11}$$

$$\frac{\partial\beta(m)}{\partial x} = \sum_{k=0}^{\substack{(m-2)/2\ (m\ \text{even})\\(m-1)/2\ (m\ \text{odd})}} (-1)^k \binom{m}{2k+1} (m - 2k - 1)\, x^{m-2k-2}\, y^{2k+1}$$

$$\frac{\partial\beta(m)}{\partial y} = \sum_{k=0}^{\substack{(m-2)/2\ (m\ \text{even})\\(m-1)/2\ (m\ \text{odd})}} (-1)^k \binom{m}{2k+1} (2k + 1)\, x^{m-2k-1}\, y^{2k}$$

$$\frac{\partial\beta(m)}{\partial z} = 0 \ . \tag{12}$$

If $m = n$, $\partial\gamma(m, n)/\partial x_p = 0$. If $m \neq n$

$$\frac{\partial\gamma(m, n)}{\partial x} = -\frac{x}{r^2}\,\tau(m, n)$$

$$\frac{\partial\gamma(m, n)}{\partial y} = -\frac{y}{r^2}\,\tau(m, n)$$

$$\frac{\partial\gamma(m, n)}{\partial z} = \frac{x^2 + y^2}{r^3} \sum_{i=0}^{\substack{(n-m-2)/2\ (n+m\ \text{even})\\(n-m-1)/2\ (n+m\ \text{odd})}} (-1)^i \binom{n}{i} \frac{[2(n - i)]!}{(n - m - 2i - 1)!} \left(\frac{z}{r}\right)^{n-m-2i-1} \tag{13}$$

where

$$\tau(m, n) = \sum_{i=0}^{\substack{(n-m-2)/2\ (n+m\ \text{even})\\(n-m-1)/2\ (n+m\ \text{odd})}} (-1)^i \binom{n}{i} \frac{[2(n - i)]!}{(n - m - 2i - 1)!} \left(\frac{z}{r}\right)^{n-m-2i} \ . \tag{14}$$

Therefore, by employing the relations given by Equations 8 through 14, the potential and field of force for any spheroid comprised of any combination of zonal ($m = 0$), tesseral ($m \neq 0$, $n \neq m$) or sectorial harmonics ($m \neq 0, m = n$) may be derived.

If the spheroid is rotating about the z-axis, then the influence of the tesseral and sectorial harmonics vary at a fixed point, p, in space due to the rotation. The force at p in a fixed, or inertial, coordinate system is given by

$$\vec{F}' = R_3(-\alpha)F = \left(\frac{\partial U}{\partial x}\cos\alpha - \frac{\partial U}{\partial y}\sin\alpha\right)\vec{i}\,' + \left(\frac{\partial U}{\partial x}\sin\alpha + \frac{\partial U}{\partial y}\cos\alpha\right)\vec{j}\,' + \frac{\partial U}{\partial z}\vec{k}\,' , \quad (15)$$

where $\vec{i}\,'$, $\vec{j}\,'$, and $\vec{k}\,'$ are the orthogonal unit vectors in the inertial system and α is the angular separation between $\vec{i}$ and $\vec{i}\,'$.

APPLICATION TO AN EARTH SATELLITE

In the analysis of the motion of an earth satellite, many find it more convenient to use the normalized harmonic coefficients $\overline{C}_n^{(m)}$ and $\overline{S}_n^{(m)}$. They may be obtained from $C_n^{(m)}$ and $S_n^{(m)}$ by

$$\overline{C}_n^{(m)} = N_n^{(m)} C_n^{(m)}$$

$$\overline{S}_n^{(m)} = N_n^{(m)} S_n^{(m)} , \quad (16)$$

where

$$N_n^{(m)} = \left[\frac{N(2n+1)(n-m)!}{(n+m)!}\right]^{-1/2} \quad \begin{cases} N = 2 & m \neq 0 \\ N = 1 & m = 0 \end{cases} . \quad (17)$$

Values for many of the $\overline{C}_n^{(m)}$ and $\overline{S}_n^{(m)}$ for the earth may be found in Reference 1.

For any earth satellite, the previously referred to quantity α becomes the Greenwich Mean Sidereal Time. Note that the rotation given by Equation 15 may be bypassed if only zonal harmonics are considered since the z coordinate is the same in both the rotating and inertial systems and r is invariant with respect to the rotation.

Finally, for an axially symmetric potential (zonal harmonics only), the recommended potential function Reference 2 is

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{b}{r}\right)^n P_n(\sin\phi)\right] . \quad (18)$$

Therefore, one must note that, if certain J_n zonal harmonics are known but the tesseral and sectorial harmonic are to be considered also, in the equations above we have

$$C_{n,0} = - J_n$$

$$S_{n,0} = 0 .$$

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2. Hagihara, Y., Astron. J., Vol. 67, p. 108, 1962