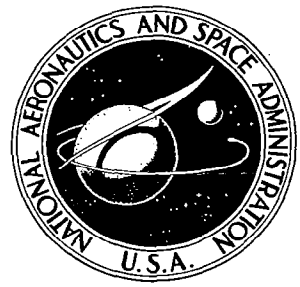
NASA CONTRACTOR REPORT



NASA CI

NASA CR-825

ANALYSIS AND DESIGN OF SPACE VEHICLE FLIGHT CONTROL SYSTEMS

VOLUME VI - STOCHASTIC EFFECTS

by Arthur L. Greensite

Prepared by
GENERAL DYNAMICS CORPORATION
San Diego, Calif.
for George C. Marshall Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1967

NASA CR-825

ANALYSIS AND DESIGN OF SPACE VEHICLE FLIGHT CONTROL SYSTEMS

VOLUME VI - STOCHASTIC EFFECTS

By Arthur L. Greensite

Issued by Originator as GDC DDE66-006

Prepared under Contract No. NAS 8-11494 by
GENERAL DYNAMICS CONVAIR
A DIVISION OF GENERAL DYNAMICS CORPORATION
San Diego, Calif.

for George C. Marshall Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

FOREWORD

This report was prepared under NASA Contract NAS 8-11494 and is one of a series intended to illustrate methods used for the design and analysis of space vehicle flight control systems. Below is a complete list of the reports in the series:

Volume I	Short Period Dynamics
Volume II	Trajectory Equations
Volume III	Linear Systems
Volume IV	Nonlinear Systems
Volume V	Sensitivity Theory
Volume VI	Stochastic Effects
Volume VII	Attitude Control During Launch
Volume VIII	Rendezvous and Docking
Volume IX	Optimization Methods
Volume X	Man in the Loop
Volume XI	Component Dynamics
Volume XII	Attitude Control in Space
Volume XIII	Adaptive Control
Volume XIV	Load Relief
Volume XV	Elastic Body Equations
Volume XVI	Abort

The work was conducted under the direction of Clyde D. Baker, Billy G. Davis and Fred W. Swift, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The General Dynamics Convair program was conducted under the direction of Arthur L. Greensite.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. STATEMENT OF THE PROBLEM

A comprehensive treatment of the control problem for space vehicles requires a consideration of the various random effects that influence overall performance. In this category, we include instrumentation noise, parameter uncertainty, extraneous disturbances, etc. Underlying all these cases is the fact that one is dealing with phenomena that are not predictable in a deterministic sense. Statistical methods must therefore be employed in some rational manner that yields useful results.

In this monograph, we are mainly concerned with random (hereafter called "stochastic") effects insofar as they influence the design of space vehicle control systems. Attention will therefore be focused on three areas where stochastic control principles have been employed to enhance system performance.

The first of these deals with instrumentation noise, the word noise being used in a generic sense to denote extraneous effects that contaminate a given signal. The classical solution to this problem involved the design of filters to suppress unwanted signals. The design of these filters presupposed that the desired and parasitic signals could be characterized by well defined frequency bands. But if both the signal and noise are described only in a statistical sense, then a more sophisticated approach is required. The modern approaches to this problem all stem from the classic work of Wiener,[6] who laid the foundation for the optimal design of filters that minimize the influence of random noise. The features and use of some of these techniques are described in this monograph, especially as they pertain to instrumentation noise commonly encountered in space vehicle control systems.

A second and related problem is that of separating signal from noise in a generic sense; in other words, when instrumentation noise is not the sole source of extraneous signals. It may be desired, for example, to control a system optimally when there are extraneous disturbances whose general features can be described only in a statistical manner. This would include the design of a launch vehicle autopilot subjected to winds whose description is given by suitable probability distributions.

Finally, it may be desired to determine various control system parameters to minimize some designated error function when parasitic signals are present.

All these methods are presently being used or are potentially useful in the design of space vehicle control systems. Various applications are described in this monograph.

So that this document will be reasonably self-contained, the exposition proceeds from first principles. Some derivations have been included, primarily as an aid to

understanding the basic principles and limitations of the method. These derivations are characterized by extreme brevity and generally phrased in the engineering vernacular, a procedure that has a high degree of plausibility and intuitive appeal. The mathematical purist may therefore find distress in such steps as unhesitatingly interchanging the order of integration and assuming that various limiting processes are valid. In all instances, however, suitable reference is made to rigorous demonstrations in the literature. Because the development is in many instances simpler (albeit less rigorous) than that available in standard texts, it was deemed justified to include it here, especially as it contributes to the understanding of what is generally a difficult subject.

A summary of typical applications in aerospace control problems is included, and a guide to detailed analysis of specialized cases is contained in an extensive list of references.

2. STATE OF THE ART

The period immediately following publication of Wiener's classic work[6] on linear filters and stationary time series witnessed the appearance of a broad range of studies dealing with various extensions and generalizations. The newer developments enlarged the scope of the theory to include nonstationary time series and finite memory (time-varying) linear filters as well as some special results for nonlinear filters[7, 12]. These results were, for the most part, highly theoretical, and the sophisticated mathematics made them inaccessible to most engineers. This condition was remedied somewhat by the appearance of several texts[5, 47] aimed specifically for control engineers.

Until only a few years ago, engineering application of the theory was very limited. Wiener's original problem related to the design of a realizable linear filter that minimized the mean-square error when the filter input was composed of a signal corrupted by additive noise and both the signal and noise had well-defined statistical properties. A related study by Phillips[47] showed how to vary some parameters of a fixed filter in order to achieve mean-square-error minimization.

The wide scope of the Wiener theory, however, stimulated fresh approaches to related problems. Press and Houbolt[49] showed how some of Wiener's equations could be applied to gust loads on airplanes in the sense of determining output power spectra from random inputs expressed in power spectral form.

Perhaps the most significant new result since Wiener was obtained by Kalman[13], who expressed Wiener's formulation in state transition concepts in the time domain (rather than frequency) and thereby simplified the mathematical structure immensely; more important, he greatly enlarged the area of application. Most of the engineering literature in the last three or four years in stochastic control has been expressed in Kalman's format. The method enables one to treat problems of noise and disturbance minimization in a highly systematic manner; it has been used to design optimal filters and control functions as well as to identify unknown system parameters. Bryson and Johansen[29] extended Kalman's technique to include the case of "colored noise."

The theory of Wiener-Kalman filtering (as it is now called) has become virtually classical in a very short time. The basic results are firmly and rigorously established, and the theory has assumed a prominent role in modern control system design. Recent studies have demonstrated that an intimate relationship exists between the Wiener-Kalman theory and related statistical optimization methods such as linear regression[53], maximum likelihood[54], and dynamic programming[26]. A summary of these ideas is contained in a stimulating paper by Smith[55]. A judicious blend of advances in related fields will no doubt further enlarge the scope and application of stochastic control concepts.

3. RECOMMENDED PROCEDURES

3.1 MATHEMATICAL PRELIMINARIES

This section presents some of the essential aspects of the theory of probability and random processes. The treatment is extremely brief, highlighting definitions and results pertinent to the subject matter of the monograph. In keeping with the theme of the presentation, we emphasize physical understanding rather than mathematical abstraction, with the hope that the underlying ideas can be made intuitively plausible. We do not seek, however, to disparage the merits of a rigorous treatment, which is necessary for some of the more sophisticated applications of the theory. For this, we must, of necessity, refer the reader to standard references[1,3].

3.1.1 Random Variables

The concept of a random variable is fundamental to all that follows. A random variable is defined as a function whose value depends on the outcome of a chance event. Generally, in any chance event, certain outcomes are "more likely" than others. This idea is made precise by defining a distribution function, $F(x)$, as follows:

$$F(x) = \text{Prob}\,(R \leq x)$$

In words, $F(x)$ is the probability that the random variable, R, takes on a value equal to or less than x.

Since the probability of an event must lie between zero (impossibility) and one (certainty), it follows that $F(x)$ must satisfy the following:

$$F(a) \leq F(b) \quad \text{if } a \leq b \tag{1}$$

$$F(-\infty) = 0 \tag{2}$$

$$F(\infty) = 1 \tag{3}$$

If $F(x)$ is differentiable, as will be assumed for the most part hereafter, then we define the probability density function by

$$f(x) = \frac{d\,F(x)}{dx}$$

Since $F(x)$ is nondecreasing, it follows that

$$f(x) \geq 0 \quad \text{for all } x \tag{4}$$

Furthermore, if $a < b$, then

$$\int_a^b f(x)\,dx = \int_a^b dF(x) = F(b) - F(a)$$

$$= \text{Prob}\,(R \le b) - \text{Prob}\,(R \le a)$$

$$= \text{Prob}\,(a \le R \le b) \tag{5}$$

Also,

$$\int_{-\infty}^{x} f(x)\,dx = F(x) - F(-\infty) = F(x) \tag{6}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = F(\infty) - F(-\infty) = 1 \tag{7}$$

Associated with each probability density function, $f(x)$, of a random variable, R, are the following:

Mean:

$$E(R) \equiv m = \int_{-\infty}^{\infty} x f(x)\,dx \tag{8}$$

Mean Square:

$$E(R^2) \equiv \nu = \int_{-\infty}^{\infty} x^2 f(x)\,dx \tag{9}$$

Variance:

$$E\left[(R - m)^2\right] \equiv \sigma^2 = \nu - m^2$$

$$= \int_{-\infty}^{\infty} (x - m)^2 f(x)\,dx \tag{10}$$

Quantity σ is called the standard deviation.

The moments of the density function are defined by

$$\alpha_n = E(R^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx \tag{11}$$

while the central moments are defined by

$$\mu_n = E\left[(R-m)^n\right] = \int_{-\infty}^{\infty} (x-m)^n f(x)\, dx \tag{12}$$

Note that

$$\alpha_0 = 1 \qquad \mu_0 = 1$$
$$\alpha_1 = m \qquad \mu_1 = 0$$
$$\alpha_2 = \nu \qquad \mu_2 = \sigma^2$$
$$\sigma^2 = \alpha_2 - \alpha_1^2$$

A knowledge of the density function serves in defining the properties of the random variable completely. If the density function is not known, then the statistical properties of the random variable are described in terms of its first and higher moments. The more of these that are known, the more complete the description of the random variable. In most cases of interest, a knowledge of the first two moments is sufficient to describe the properties of the random variable. Some of the more common probability density functions are listed below.

Binomial:

$$f(x) = \frac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x} \tag{13}$$

f(x) is the probability that an event will occur x times in n trials, where p is the probability that the event will occur in a given trial.

Here n and x are positive integers such that $0 \le x \le n$, and p is a real number between zero and one. It follows readily that

$$m = np \tag{14}$$

$$\sigma^2 = np(1-p) \tag{15}$$

Poisson:

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad (16)$$

$$x = 0, 1, 2, \ldots\ldots$$

$$\lambda > 0$$

In this case,

$$m = \sigma^2 = \lambda \qquad (17)$$

Gaussian:

$$f(x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right) \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \qquad (18)$$

The above results are readily generalized to the case of n random variables, $R_1, R_2, \ldots\ldots, R_n$. For example, the joint distribution function, $F(x_1, x_2, \ldots\ldots, x_n)$ is defined by

$$F(x_1, x_2, \ldots\ldots, x_n) = \text{Prob}\,[R_1 \leq x_1, R_2 \leq x_2, \ldots\ldots$$

$$\ldots\ldots, R_n \leq x_n] \qquad (19)$$

In words, $F(x_1, x_2, \ldots\ldots, x_n)$ is the probability that R_i takes on a value equal to or less than x_i for $i = 1, 2, \ldots\ldots, n$. All the inequalities must be satisfied simultaneously.

The joint probability density function is defined by

$$f(x_1, x_2, \ldots\ldots, x_n) = \frac{\partial^n F(x_1, x_2, \ldots\ldots x_n)}{\partial x_1 \partial x_2 \ldots\ldots\ldots \partial x_n} \qquad (20)$$

The quantities of interest are the following:

Mean:

$$m_i = E(R_i) \qquad (21)$$

Variance and Covariance:

$$\sigma_{ij} = E\left[(R_i - m_i)(R_j - m_j)\right] \qquad \begin{cases} i = j; & \text{variance} \\ i \neq j; & \text{covariance} \end{cases} \tag{22}$$

Correlation:

$$\rho_{ij} = \sigma_{ij}\left(\sigma_{ii}\,\sigma_{jj}\right)^{-1/2} \tag{23}$$

Of special interest in the subsequent analyses is the multivariate Gaussian (normal) density function given by

$$f(x_1, x_2, \ldots\ldots, x_n) = \left[(2\pi)^n \det V\right]^{-1/2} \exp\left[-\frac{1}{2}(\bar{x} - \bar{m})^T V^{-1} (\bar{x} - \bar{m})\right] \tag{24}$$

Here $\bar{x}$ and $\bar{m}$ are n vectors and V is an n × n matrix whose typical element is σ_{ij}; viz.,

$$V = E\left[(\bar{R} - \bar{m})(\bar{R} - \bar{m})^T\right] \tag{25}$$

In the sections that follow, we shall deal extensively with probability distributions of two variables. The following definitions define the parameters of interest in this case.

Moments:

$$\alpha_{jk} = E(R_1^j R_2^k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^j x_2^k f(x_1, x_2)\, dx_1\, dx_2 \tag{26}$$

Central Moments:

$$\mu_{jk} = E\left[(R_1 - m_1)^j (R_2 - m_2)^k\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - m_1)^j (x_2 - m_2)^k f(x_1, x_2)\, dx_1\, dx_2 \tag{27}$$

Note that, in particular,

$$\alpha_{10} = E(R_1) = m_1$$

$$\alpha_{01} = E(R_2) = m_2$$

$$\mu_{20} = E[(R_1 - m_1)^2]$$

$$= \alpha_{20} - \alpha_{10}^2 \equiv \text{variance of } R_1$$

$$\mu_{02} = E[(R_2 - m_2)^2]$$

$$= \alpha_{02} - \alpha_{01}^2 \equiv \text{variance of } R_2$$

$$\mu_{11} = E[(R_1 - m_1)(R_2 - m_2)]$$

$$= \alpha_{11} - \alpha_{10}\alpha_{01} \equiv \text{covariance of } R_1 \text{ and } R_2$$

Note also the following equivalence:

$$\sigma_{12} = \mu_{11}$$

$$\sigma_{11} = \mu_{20}$$

$$\sigma_{22} = \mu_{02}$$

The quantity

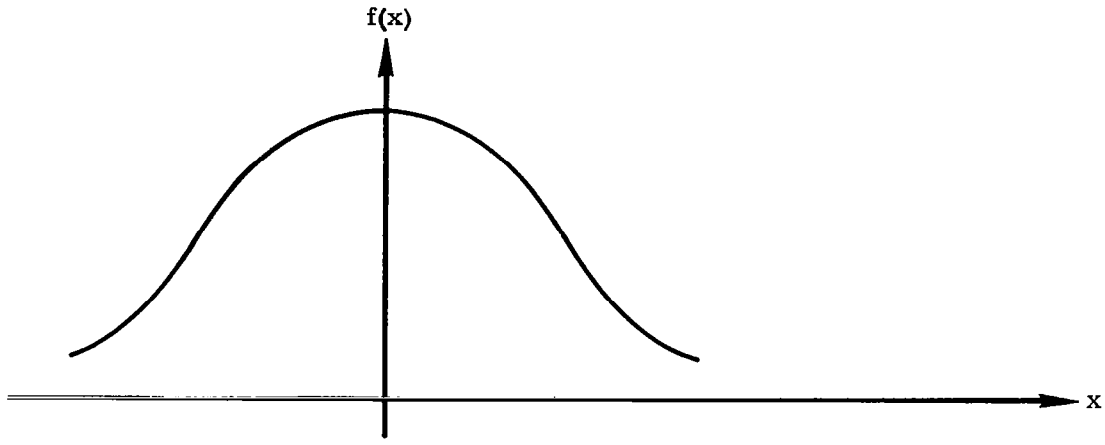
$$\rho_{12} = \sigma_{12}(\sigma_{11}\sigma_{22})^{-1/2}$$

$$= \mu_{11}(\mu_{20}\mu_{02})^{-1/2}$$

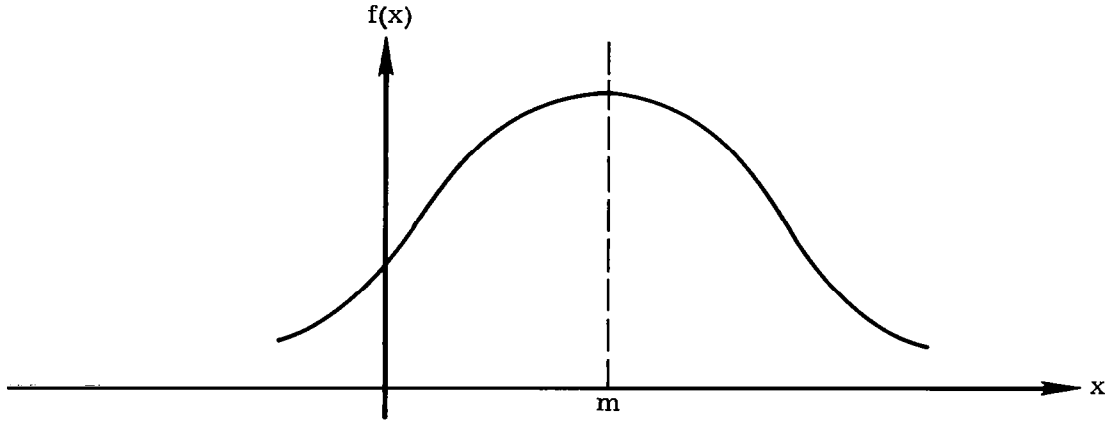
is called the correlation coefficient of R_1 and R_2.

The fundamental quantities of interest in relation to a random variable are the mean and the variance of the probability density function. To obtain some physical insight into the significance of these quantities, consider the Gaussian (or normal) density function shown in Fig. 1. When the mean, m, equals zero, the curve f(x) vs x is symmetrical about the x = 0 axis as shown in Fig. 1a. For any nonzero value of m, the curve is merely shifted parallel to the f(x) axis in the manner shown in Fig. 1b. Parameter σ gives a measure of the rapidity with which the curve drops off. Fig. 1c shows two Gaussian density functions where $\sigma_2 > \sigma_1$. It is apparent that the smaller the value of σ, the steeper the curve.

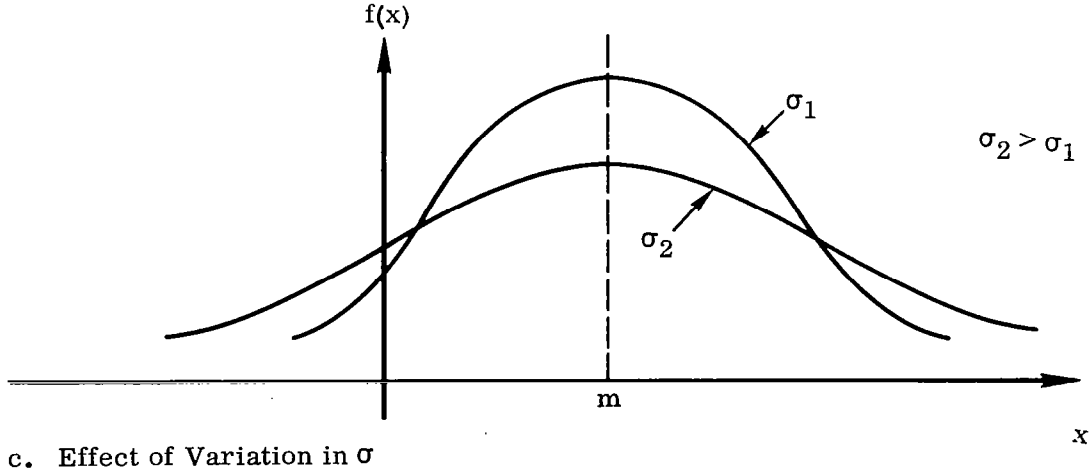
As an illustration of the use of this curve and the order of magnitude of numerical quantities involved, let it be required to determine the probability that a ran-



a. Zero Mean



b. Nonzero Mean



c. Effect of Variation in σ

Figure 1. The Gaussian Density Function

dom variable, R (whose density function is Gaussian), assumes a value in the range (m - b) to (m + b) for a given value of b. From Eq. (5),

$$\text{Prob}\,[m-b \le R \le m+b] = \int_{m-b}^{m+b} f(x)\,dx$$

where

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

In order to use available tables for the Gaussian density function, we convert to "standard" form via the substitution $x = t\sigma + m$. This leads to

$$\text{Prob}\,[m-b \le R \le m+b] = F(m+b) - F(m-b)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b/\sigma} e^{-1/2\,t^2}\,dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-b/\sigma} e^{-1/2\,t^2}\,dt$$

Using standard tables[2], we find

$$\text{Prob}\,[m-b \le R \le m+b] = 0.6827 \quad \text{if } b = \sigma$$

$$= 0.9545 \quad \text{if } b = 2\sigma$$

$$= 0.9973 \quad \text{if } b = 3\sigma$$

In other words, the smaller the value of σ, the greater the tendency of the random variable to assume a value close to the mean.

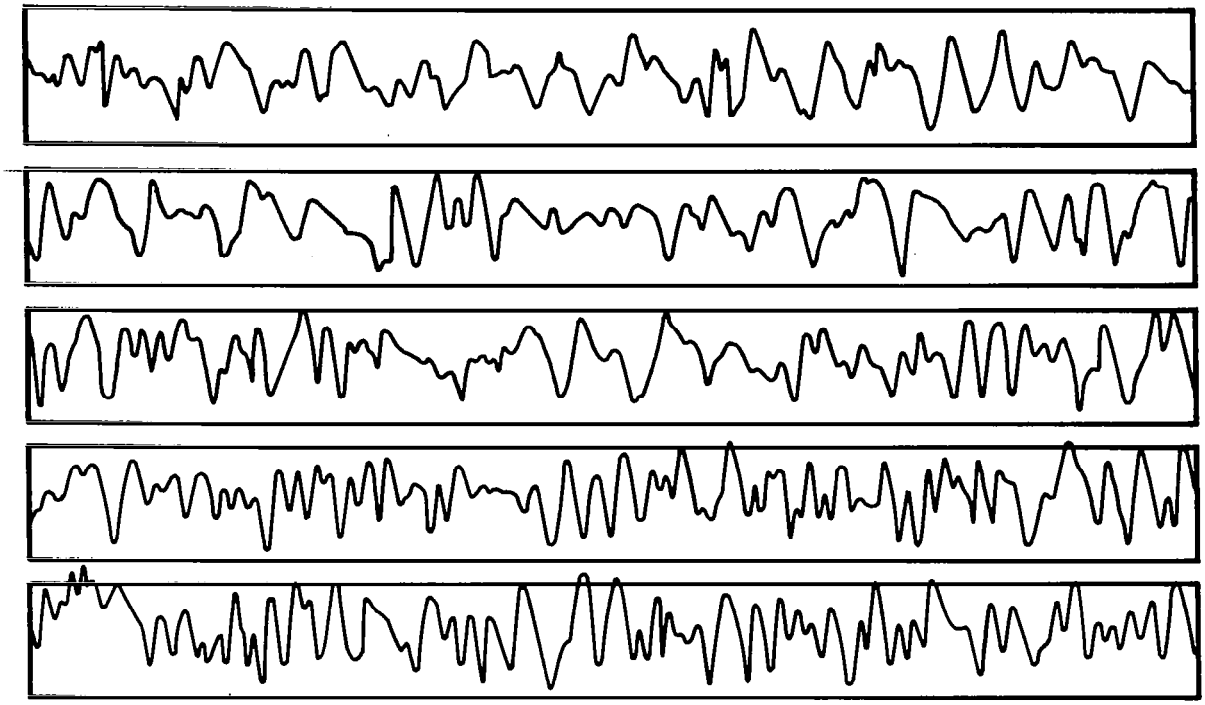
3.1.2 Random Processes

By a random process we shall mean a collection or ensemble of functions of time having certain statistical properties. A typical case is shown in Fig. 2. The random process is characterized by probability distribution functions defined as follows. Let x(t) denote a representative member of the ensemble $\{x(t)\}$. Then $x(t_1)$ is a random variable in the sense defined in Sec. 3.1.1. A probability distribution function, $F_1(x_1, t_1)$ is defined by

$$F_1(x_1, t_1) = \text{Prob}\,[x(t_1) \le x_1] \tag{28}$$

where x_1 is a prescribed number. The probability density function is

$$f_1(x_1, t_1) = \frac{\partial F(x_1, t_1)}{\partial x_1} \tag{29}$$



Time

Figure 2. An Ensemble of Random Functions of Time

It follows that

$$\text{Prob}\,[x_1 \leq x(t_1) \leq x_1 + dx_1] = f_1(x_1, t_1)\, dx_1 \tag{30}$$

Similarly, the probability that simultaneously $x(t_1) \leq x_1$ and $x(t_2) \leq x_2$ is given by the second probability distribution function

$$F_2(x_1, t_1 ; x_2, t_2) = \text{Prob}\,[x(t_1) \leq x_1 ; x(t_2) \leq x_2] \tag{31}$$

The corresponding probability density function is given by

$$f_2(x_1, t_1 ; x_2, t_2) = \frac{\partial^2 F_2(x_1, t_1 ; x_2, t_2)}{\partial x_1 \partial x_2} \tag{32}$$

Similarly

$$F_3(x_1, t_1 ; x_2, t_2 ; x_3, t_3) = \text{Prob}\,[x(t_1) \leq x_1 ; x(t_2) \leq x_2 ; x(t_3) \leq x_3]$$

etc. Each f_n implies all previous f_k for $k < n$ by the relation

$$f_{n-1}(x_1, t_1; \ldots\ldots; x_{n-1}, t_{n-1}) = \int_{-\infty}^{\infty} f_n(x_1, t_1, \ldots\ldots, x_n, t_n)\, dx_n \qquad (33)$$

As in Sec. 3.1.1, we define the mean by

$$m_i = E[x(t_i)] \qquad (34)$$

and the covariance by

$$\sigma_{ij} = E\{[x(t_i) - m_i][x(t_j) - m_j]\} \qquad (35)$$

It is important to note the distinction between the definitions for a random variable and for a random process.

The covariance may be written in expanded form as follows.

$$\sigma_{ij} = E[x(t_i)\, x(t_j)] - m_i\, m_j$$

The first term on the right-hand side of this expression plays a fundamental role in stochastic control theory, and it is accorded a separate name and symbol.

$$\varphi_{xx}(t_i, t_j) = E[x(t_i)\, x(t_j)] \qquad (36)$$

We call this the autocorrelation function of x, the subscript xx indicating correlation of x(t) with itself as distinguished from cross-correlations to be discussed later.

The set of density functions $f_1, f_2, \ldots\ldots, f_n$ describes the random process in ever-increasing detail. In addition to the mean and covariance already defined, higher-order moments may also be defined in a manner analogous to that of Sec. 3.1.1. However, these will not be required for the topics to be dealt with in this monograph, and they are therefore not pursued further.

It is often required to consider the statistical relationships between two random processes, $\{x(t)\}$ and $\{y(t)\}$. For this case, we define a general joint distribution function as follows.

$$F^{(c)}_{mn}(x_1, t_1, \ldots\ldots, x_m, t_m; y_1, t'_1, \ldots\ldots\ldots, y_n, t'_n)$$

$$= \text{Prob}\,[x(t_1) \le x_1, \ldots\ldots\ldots\ldots, x(t_m) \le x_m; y(t'_1) \le y_1,$$

$$\ldots\ldots\ldots, y(t'_n) \le y_n]$$

The associated joint density function is

$$f^{(c)}_{mn}(x_1, t_1, \ldots\ldots, x_m, t_m ; y_1, t'_1, \ldots\ldots, y_n, t'_n)$$

$$= \frac{\partial^{m+n} F^{(c)}_{mn}}{\partial x_1 \ldots\ldots \partial x_m \partial y_1 \ldots\ldots \partial y_n} \tag{37}$$

In practical applications, only the joint density function, $f^{(c)}_{11}(x_1, t_1; y_1, t'_1)$, is used extensively. Of the moments associated with these joint density functions, the only one that will be used subsequently is

$$E[x(t_1)\, y(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\, f^{(c)}_{11}(x, t_1 ; y, t_2)\, dx\, dy \tag{38}$$

This quantity is called the cross-correlation function and is denoted by the symbol

$$\varphi_{xy}(t_1, t_2) = E[x(t_1)\, y(t_2)] \tag{39}$$

If $\{x(t)\}$ and $\{y(t)\}$ are statistically independent -- that is, if the value of $x(t)$ does not depend in any way on the value of $y(t)$ -- then

$$E[x(t_1)\, y(t_2)] = E[x(t_1)]\, E[y(t_2)] \tag{40}$$

We sometimes deal with a signal of the form

$$z(t) = x(t) + y(t)$$

where $\{x(t)\}$ and $\{y(t)\}$ are two given random processes. In this case, we find, directly from the definitions,

$$\varphi_{zz}(t_1, t_2) = \varphi_{xx}(t_1, t_2) + \varphi_{xy}(t_1, t_2) + \varphi_{yx}(t_1, t_2) + \varphi_{yy}(t_1, t_2) \tag{41}$$

From (39), it is obvious that in general, $\varphi_{xy}(t_1, t_2) \neq \varphi_{yx}(t_1, t_2)$. Note that if $\{x(t)\}$ and $\{y(t)\}$ are statistically independent, then

$$\varphi_{xy}(t_1, t_2) = \phi_{yx}(t_1, t_2) = 0 \tag{42}$$

if one or both of the random processes has a zero mean value. In the particular case when (40) and 42) hold, then Eq. (41) reduces to

$$\varphi_{zz}(t_1, t_2) = \varphi_{xx}(t_1, t_2) + \varphi_{yy}(t_1, t_2) \tag{43}$$

A random process is said to be stationary if its statistical properties do not vary with time. An important consequence of this property is that

$$E[x(t+\tau)] = E[x(t)]$$

The particular choice $\tau = -t$ gives

$$E[x(0)] = E[x(t)]$$

The autocorrelation function for a stationary random process satisfies

$$\varphi_{xx}(t_1+\tau, t_2+\tau) = \varphi_{xx}(t_1, t_2)$$

for all τ. In particular, if $\tau = -t_1$, then

$$\varphi_{xx}(t_1, t_2) = \varphi_{xx}(0, t_2 - t_1)$$

In other words, the autocorrelation function for a stationary random process depends only on the time interval, not the specific values of time. It is appropriate to denote the autocorrelation function for a stationary random process by a special symbol; thus

$$\Gamma_{xx}(\tau) = \varphi_{xx}(0, t_2 - t_1) \tag{44}$$

where $\tau = t_2 - t_1$. In similar manner, for the cross-correlation function,

$$\Gamma_{xy}(\tau) = \varphi_{xy}(0, t_2 - t_1) \tag{45}$$

Note that

$$\Gamma_{xy}(\tau) = E[x(t)\, y(t+\tau)] \tag{46}$$

The autocorrelation function satisfies the following.

$$\Gamma_{xx}(\tau) = \Gamma_{xx}(-\tau) \tag{47}$$

$$|\Gamma_{xx}(\tau)| \leq \Gamma_{xx}(0) \tag{48}$$

If x(t) *does not contain any periodic component, then*

$$\lim_{\tau \to \infty} \Gamma_{xx}(\tau) \to 0 \tag{49}$$

A stationary random process is often assumed to be ergodic. Generally speaking, this property permits one to equate ensemble averages and averages with respect to time performed on a single representative member of the ensemble. More precisely, a stationary random process is said to be ergodic if every member, $x(t)$, of the ensemble $\{x(t)\}$ satisfies

$$E\{V[x(t)]\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V[x(t+\tau)]\,dt \tag{50}$$

where $V[x(t)]$ is any random variable associated with $x(t)$; e.g., $V = x(t)$, $x^2(t)$, $x(t_1)\,x(t_2)$.

When the ergodic property holds, we have

$$E[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt \tag{51}$$

$$E[x^2(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt \tag{52}$$

$$\Gamma_{xx}(\tau) = E[x(t)\,x(t+\tau)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x(t+\tau)\,dt \tag{53}$$

The ergodic property is important because it eliminates the need for dealing with a large ensemble in order to calculate probability density functions. For example, if the random process is stationary, then

$$E[x(t)] = \int_{-\infty}^{\infty} x_1 f_1(x_1)\,dx_1$$

$$\Gamma_{xx}(\tau) = \alpha_{11} = \int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, x_2, \tau)\,dx_1\,dx_2$$

However, if the process is also ergodic, then the two foregoing quantities are given in much simpler form by Eqs. (51) and (53) which do not involve probability density functions. This results in a crucial simplification for computational purposes.

3.1.3 Random Signals and Linear Filters

Given a random stationary signal characterized by appropriate statistical parameters, we may pose the question, "If this signal is applied at the input of a linear filter, what is the output?" A linear filter is best described by its transfer function, which requires that the input signal be decomposed into its harmonic components if the signal is periodic. The action on each harmonic component is then determined and the results superimposed.

When the signal is not periodic, it cannot be decomposed into discrete harmonic components; but if it has a Fourier (or Laplace) transform, then it has a continuous frequency spectrum that can be treated mathematically as a spectrum of harmonics. It is natural, therefore, to investigate the properties of the Fourier transform of x(t); viz.,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

An immediate difficulty is encountered. If x(t) is stationary, then its statistical properties do not vary with time; however, x(t) can wander randomly ad infinitum with respect to time. In short, the integral in the above expression may not converge, in which case its Fourier transform does not exist. However, while an expression such as

$$\int_{-\infty}^{\infty} x(t)\, dt \qquad \text{or} \qquad \int_{-\infty}^{\infty} x^2(t)\, dt$$

may not converge, it happens that the quantity

$$E\left[x^2(t)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt \tag{54}$$

which represents the mean square* of x(t), is finite for nearly all random phenomena of interest. It is natural to think of $\int_{-T}^{T} x^2(t)\, dt$ as being a measure of power, in which case, Eq. (54) represents average power in the interval (-T, T).

We now have a foothold on a method of converting a quantity of interest, expressed in the time domain, to an equivalent formulation in terms of frequency spectrum. The procedure is as follows. Define

$$\begin{aligned} x_T(t) &= x(t) && -T \le t \le T \\ &= 0 && \text{otherwise} \end{aligned} \tag{55}$$

*See Eq. (52).

The Fourier transform of $x_T(t)$ exists and is given by

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-T}^{T} x(t)\, e^{-i\omega t}\, dt \tag{56}$$

$$x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_T(\omega)\, e^{i\omega t}\, d\omega \tag{57}$$

The quantity defined by

$$G_{xx}(\omega) \equiv \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T} \tag{58}$$

is called the power spectral density*. We will show that this quantity exists and is finite if the mean-square value of $x(t)$ is finite.

We have

$$\int_{-\infty}^{\infty} |X_T(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} X_T(\omega)\, X_T(-\omega)\, d\omega$$

$$= \int_{-\infty}^{\infty} X_T(\omega)\, d\omega \int_{-\infty}^{\infty} x_T(t)\, e^{i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x_T(t)\, dt \int_{-\infty}^{\infty} X_T(\omega)\, e^{i\omega t}\, d\omega$$

$$= 2\pi \int_{-\infty}^{\infty} x_T^2(t)\, dt$$

*Some authors call this the power density spectrum, spectral density, power spectrum, etc.

Therefore,

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} x_T^2(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

This last quantity is the mean-square value of $x(t)$, which is finite by assumption. Consequently, the power spectral density defined by Eq. (58) exists, since its integral converges.

It may be shown that the autocorrelation function is the Fourier transform of the power spectral density. To do this, define

$$C_T(\tau) = \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

which is merely Eq. (53) with T finite. Note that

$$\lim_{T\to\infty} C_T(\tau) = \Gamma_{xx}(\tau)$$

The Fourier transform of $C_T(\tau)$ is

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} C_T(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau \int_{-\infty}^{\infty} x_T(t)\, x_T(t+\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} [x_T(t)\, e^{i\omega t}]\, [x_T(t+\tau)\, e^{-i\omega(t+\tau)}]\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} x_T(t)\, e^{i\omega t}\, dt \int_{-\infty}^{\infty} x_T(t_1)\, e^{-i\omega t_1}\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} X_T(-\omega)\, X_T(\omega)$$

$$= \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}$$

$$= G_{xx}(\omega)$$

by (58).

This means that

$$G_{xx}(\omega) = \int_{-\infty}^{\infty} \Gamma_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau \tag{59}$$

and

$$\Gamma_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{xx}(\omega)\, e^{i\omega\tau}\, d\omega \tag{60}$$

Since both $\Gamma_{xx}(\tau)$ and $G_{xx}(\omega)$ are real-valued even functions of their respective arguments, we may write

$$G_{xx}(\omega) = 2 \int_{0}^{\infty} \Gamma_{xx}(\tau) \cos\, \omega\tau\, d\tau \tag{61}$$

$$\Gamma_{xx}(\tau) = \frac{1}{\pi} \int_{0}^{\infty} G_{xx}(\omega) \cos\, \omega\tau\, d\omega \tag{62}$$

These are known as the Wiener-Khinchin equations.

Remark: There is a distressing lack of uniformity in the literature regarding the definitions of $G_{xx}(\omega)$ and the Fourier transform. The former is variously defined as

$$\frac{|X_T(\omega)|^2}{T}, \quad \frac{|X_T(\omega)|^2}{2\pi T}, \text{ etc.}$$

while the Fourier transform pairs are defined by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad \text{(Ref. 5)}$$

or

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad \text{(Ref. 48)}$$

The Fourier transform pair, Eqs. (56) and (57), is apparently the most widely adopted*. This form will be used consistently in this monograph.

There even seems to be some inconsistency by the same author. Laning and Battin (Ref. 5, p. 123) use

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

to define the Fourier transform, while in deriving the Wiener-Khinchin equations (Ref. 5, p. 132), they write

$$F(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Needless to say, great care must be exercised in comparing results derived by different authors.

---

*e.g., Refs. 12, 46, 47.

We now return to the question posed at the beginning of this section, namely: how does a linear filter affect a random signal? To answer this we proceed as follows.

If x(t) is any signal applied at the input of a linear filter whose weighting function is h(t), then the output is given by*

$$y(t) = \int_0^\infty h(u)\, x(t-u)\, du$$

The autocorrelation function of y(t) is

$$\Gamma_{yy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, y(t+\tau)\, dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \left[\int_0^\infty h(u)\, x(t-u)\, du\right] \left[\int_0^\infty h(v)\, x(t+\tau-v)\, dv\right] dt$$

By interchanging the order of integration, we have

$$\Gamma_{yy}(\tau) = \int_0^\infty \int_0^\infty h(u)\, h(v) \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-u)\, x(t+\tau-v)\, dt\right] du\, dv$$

If we make the change of variable, $t_1 = t - u$, inside the bracketed quantity, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-u)\, x(t+\tau-v)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t_1)\, x(t_1+u+\tau-v)\, dt_1 = \Gamma_{xx}(u-v+\tau)$$

via Eq. (53).

Therefore, $\Gamma_{yy}(\tau)$ may be written as

$$\Gamma_{yy}(\tau) = \int_0^\infty \int_0^\infty h(u)\, h(v)\, \Gamma_{xx}(u-v+\tau)\, du\, dv$$

---

*See Appendix B.

Taking the Fourier transform of both sides and making use of Eqs. (56) and (59), we find

$$G_{yy}(\omega) = \int_{-\infty}^{\infty} \left[\int_0^{\infty} \int_0^{\infty} h(u)\, h(v)\, \Gamma_{xx}(u - v + \tau)\, du\, dv \right] e^{-i\omega\tau} d\tau$$

After a change in the order of integration, this becomes

$$G_{yy}(\omega) = \int_0^{\infty} \int_0^{\infty} h(u)\, h(v) \left[\int_{-\infty}^{\infty} \Gamma_{xx}(u - v + \tau)\, e^{-i\omega\tau} d\tau \right] du\, dv$$

Now by making the change of variable, $\zeta = u - v + \tau$, this simplifies to

$$G_{yy}(\omega) = \int_0^{\infty} h(u)\, e^{i\omega u}\, du \int_0^{\infty} h(v)\, e^{-i\omega v}\, dv \int_{-\infty}^{\infty} \Gamma_{xx}(\zeta)\, e^{-i\omega\zeta} d\zeta$$

However, by the definition of the Laplace transform,

$$H(s) \equiv \mathcal{L}\,[h(t)] = \int_0^{\infty} h(t)\, e^{-st} dt$$

$$s = \sigma + i\omega$$

while

$$\int_{-\infty}^{\infty} \Gamma_{xx}(\zeta)\, e^{-i\omega\zeta} d\zeta = G_{xx}(\omega)$$

by Eq. (59).

We have, finally,

$$G_{yy}(\omega) = H(-i\omega)\, H(i\omega)\, G_{xx}(\omega)$$

$$= |H(i\omega)|^2\, G_{xx}(\omega) \qquad (63)$$

In the usual terminology, $H(i\omega)$ is the transfer function of the linear filter.

Eq. (63) is of fundamental importance. It shows how the power spectral density of a random signal is altered when the signal passes through a linear filter. This relation will be used often in the following sections.

The discussion thus far has not been limited to any specific type of random process. It turns out that much random phenomena of interest can be described in terms of normal or Gaussian distributions. More specifically, a random process is termed Gaussian (normal) if it is characterized by the n dimensional probability density function (17)

$$f_n(x_1, t_1; \ldots\ldots; x_n, t_n) = [(2\pi)^n \det M]^{-1/2} \exp[-\frac{1}{2}(\bar{x}-\bar{m})^T M^{-1}(\bar{x}-\bar{m})] \quad (64)$$

Here $\bar{x}$ is an n vector whose typical element is x_i, while $\bar{m}$ is the mean vector, a typical element of which is

$$m_i = E[x(t_i)] \quad (65)$$

M is the covariance matrix whose ij^{th} component is given by

$$\sigma_{ij} = E\{[x(t_i) - m_i][x(t_j) - m_j]\}$$

$$= \varphi_{xx}(t_i, t_j) - m_i m_j \quad (66)$$

If the process is stationary, then $m_i = m_j \equiv m$ and

$$\sigma_{ij} = \Gamma_{xx}(\tau_{ij}) - m^2 \quad (67)$$

where $\tau_{ij} = t_i - t_j$. The Gaussian random process is completely determined from its joint density function $f_2(x_1, t_i; x_2 t_j)$ because the autocorrelation function is expressed in terms of this density function by

$$\varphi_{xx}(t_i, t_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, t_i; x_2, t_j)\, dx_1\, dx_2 \quad (68)$$

The Gaussian random process has the important property that it remains Gaussian after passing through a linear filter.

3.1.4 Practical Considerations

A fundamental problem in the analysis of random (stochastic) processes is the determination of the statistical properties that characterize the process. In the general case, the problem is very formidable. For purposes of computational and theoretical expediency, one is compelled to make various assumptions that are valid in greater or lesser degree in practical situations. In the first place, most random phenomena of interest are stationary; that is, in general, the statistical properties are invariant with respect to translation along the time axis. This property affords crucial simplifications

in both theoretical analysis and computational methods. One often assumes also that the process under consideration is ergodic. This assumption enables one to compute the statistical properties from a single long record rather than an ensemble of the type shown in Fig. 2. Hereafter, unless otherwise noted, we shall treat only those random processes that are both stationary and ergodic.

Consider now a single noise record of the type shown in Fig. 3. This is assumed to be a typical member of an ensemble representing a stationary, ergodic, random process. We are interested in calculating the mean and autocorrelation function. One may proceed as follows. Divide the trace of Fig. 3 into n equally spaced points a distance, d, apart. With the k^{th} point is associated an ordinate, a_k, that may be positive or negative. The mean is simply determined by using the discrete version of Eq. (51); vis.,

$$E[x(t)] = \frac{1}{n} \sum_{i=1}^{n} a_i \tag{69}$$

The autocorrelation function is obtained from the discrete form of Eq. (53; viz.,

$$\Gamma_{xx}(\tau) = \frac{1}{n} \sum_{i=1}^{n} a_i a_{i+k} \tag{70}$$

where $\tau = kd$. The curve of $\Gamma_{xx}(\tau)$ vs τ is plotted in Fig. 4. The general shape of this curve is typical of a wide variety of random phenomena of interest, such as vacuum tube noise, radar fading records, and atmospheric turbulence. It may be very closely approximated by

$$\Gamma_{xx}(\tau) = A\, e^{-k|\tau|} \cos c\,\tau \tag{71}$$

where A, k, and c are positive constants. The power spectral density may be obtained from Eq. (61); viz.,

$$G_{xx}(\omega) = 2Ak \left[\frac{\omega^2 + (k^2 + c^2)}{\omega^4 + 2(k^2 - c^2)\,\omega^2 + (k^2 + c^2)^2} \right] \tag{72}$$

The shape of this curve depends on the quantity $(3c^2 - k^2)$ and is depicted in Fig. 5. Consider now a special case of Eq. (71) in which $A = k/2$ and $c = 0$. It is not difficult to show that for these parameters,

$$\lim_{k \to \infty} \left[\frac{1}{2} k\, e^{-k|\tau|} \right] = 0 \qquad \tau \neq 0$$

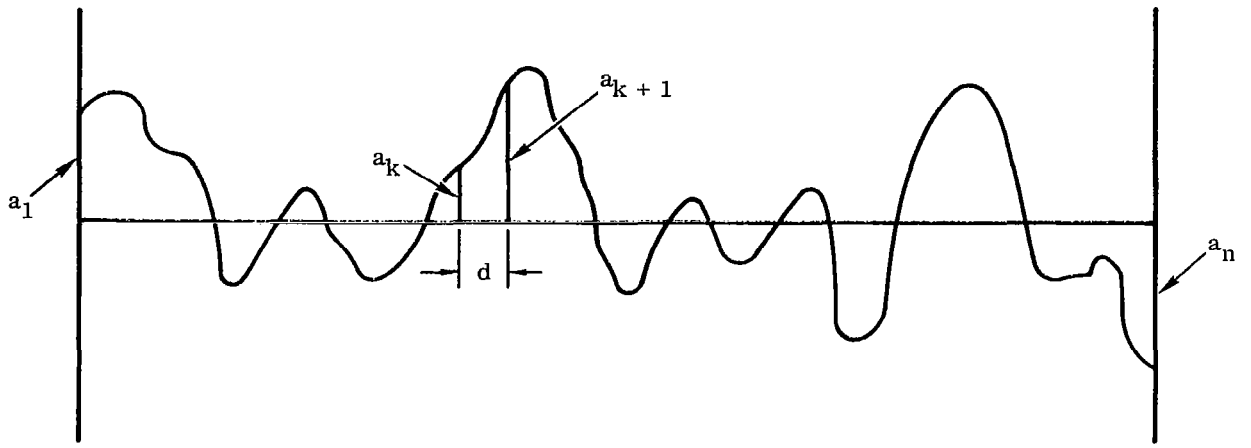


Figure 3. Single Noise Record

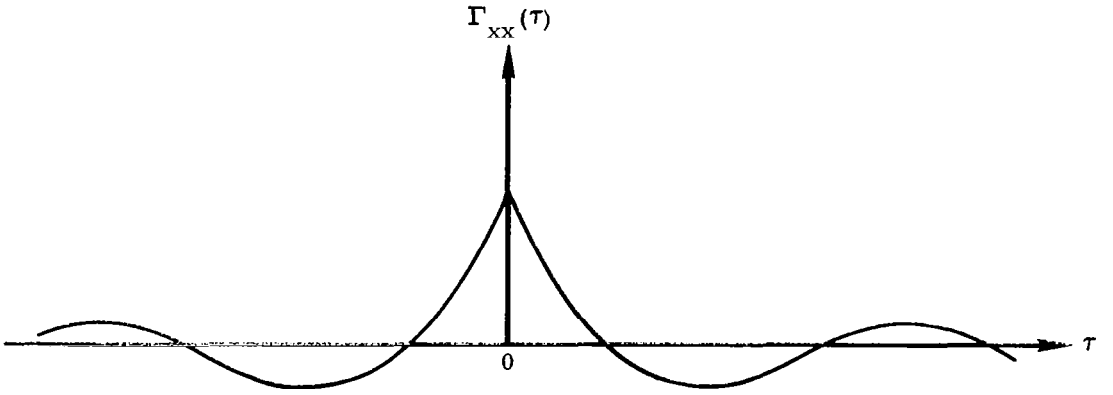
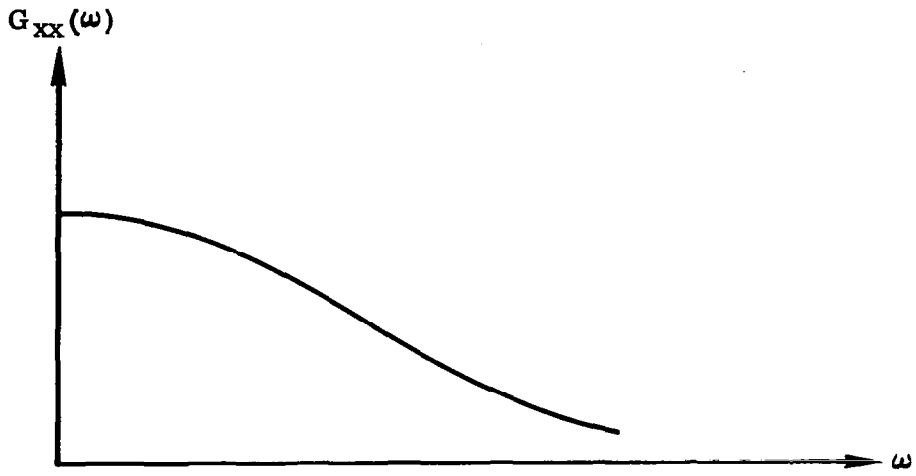
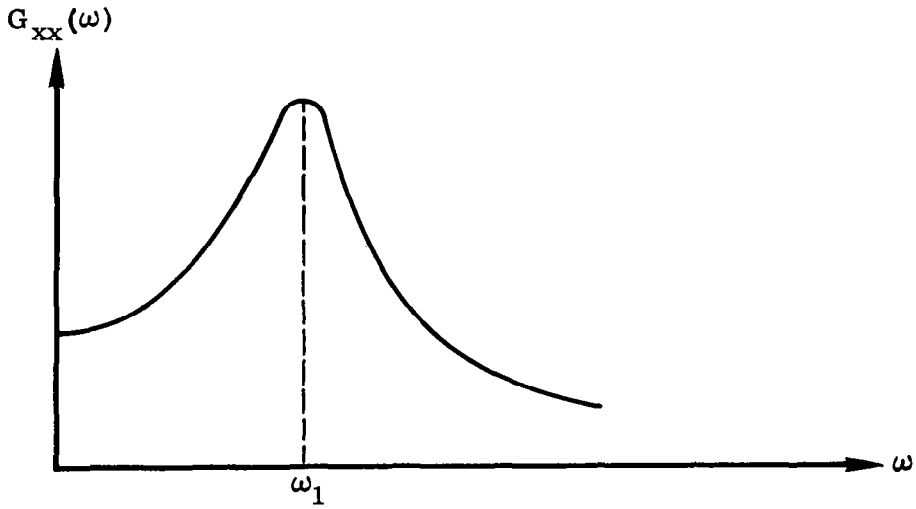


Figure 4. Plot of Correlation Function



a. $3C^2 - K^2 < 0$



b. $3C^2 - K^2 > 0$

Figure 5. Curves of Power Spectral Density

and

$$\lim_{k\to\infty}\left[\frac{k}{2}\int_{-\infty}^{\infty} e^{-k|\tau|}\,d\tau\right] = 0$$

Consequently, with $\Gamma_{xx}(\tau)$ defined by

$$\Gamma_{xx}(\tau) = \frac{k}{2}\,e^{-k|\tau|}$$

it follows that

$$\lim_{k\to\infty}\left[\Gamma_{xx}(\tau)\right] = \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function*. This last relation simply states that successive values of the random function are completely uncorrelated, no matter how small the interval between successive samples. This is, of course, a mathematical abstraction that is never realized completely in practice. Nevertheless, this concept is a highly useful theoretical tool and closely approximates many important random phenomena (e.g., Brownian motion). The idea of a purely random process, often called white noise, may be arrived at in many ways. Some of these are discussed in Appendix A. However, in general, we will say that a random process is white if its autocorrelation function is given by

$$\Gamma_{xx}(\tau) = B\,\delta(\tau) \tag{73}$$

where B is a positive constant. The power spectral density for Eq. (73) is obtained by direct application of Eq. (61); viz.,

$$G_{xx}(\omega) = B \tag{74}$$

Thus the power spectral density is a constant, which, by analogy with the spectrum for visible light, accounts for the name "white noise."

We now proceed to a discussion of the Wiener theory and some of its ramifications.

3.2 THE WIENER THEORY

The transmission of information, whether by electrical, mechanical, social, or, indeed, biological channels, often has the effect of introducing extraneous signals

*See Appendix A.

that materially corrupt the basic signal. To fix ideas, we consider a signal, $f_i(t)$, made up of two parts as follows.

$$f_i(t) = f_s(t) + f_n(t) \tag{75}$$

Quantity $f_s(t)$ represents the uncorrupted signal, while $f_n(t)$ is the extraneous information. In the real world, we have available only $f_i(t)$, and it is natural to investigate means of extracting the true signal, $f_s(t)$, from its "noisy" environment. The term noise will be used in a generic sense to describe any extraneous and undesirable signals that corrupt useful information.

Simple means of achieving this are well known and elementary. Thus, for example, if $f_s(t)$ is known to be concentrated in one frequency band while the noise is generally contained in another (often much higher) frequency band, simple passive filtering will serve to eliminate virtually all of $f_n(t)$ without disturbing $f_s(t)$. Again, if the noise is known to be restricted to one frequency or a very narrow range of frequencies (e.g., the 60-cycle hum in radio receivers), a narrow-band attenuator or notch filter will effectively "clean up" the signal.

The problem becomes substantially more difficult when one considers not single signals, but whole classes of signals; both the basic signal and the noise can be described only in some statistical sense. Furthermore, $f_s(t)$ can be separated from $f_n(t)$ only if their statistical descriptions contain some distinguishing features. In addition, there exists the problem of adopting some criterion of how well this separation is accomplished.

Posed in this fashion, the problem is very formidable. By making three basic assumptions, Wiener[6] obtained an elegant solution and laid the ground work for all subsequent research in stochastic control theory. These assumptions are:

a. The signal, $f_s(t)$, and the noise, $f_n(t)$, are each members of a stationary random process.

b. The device used to operate on $f_i(t)$ is a physically realizable linear filter.*

c. The criterion to be used in selecting the "best possible" linear filter is the rms difference between the actual and desired outputs.

By restricting the permissible operations to linear filters, we immediately have for the filter output

$$f_0(t) = \int_0^\infty h(\tau)\, f_i(t-\tau)\, d\tau \tag{76}$$

*A linear filter is said to be physically realizable if its transfer function has no poles in the right-half plane or on the imaginary axis.

where $h(\tau)$ is the weighting function* of the filter. Since this filter is to be physically realizable, we must have

$$h(\tau) = 0 \qquad \text{for } \tau < 0 \tag{77}$$

since a real system cannot respond before an input is applied. In this case, nothing is changed if instead of Eq. (76) we write

$$f_0(t) = \int_{-\infty}^{\infty} h(\tau)\, f_i(t-\tau)\, d\tau \tag{78}$$

In what follows, it will be more convenient to use (78) rather than (76). If we denote the desired signal by $f_d(t)$, then the rms error is given by

$$\overline{\epsilon^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f_0(t) - f_d(t)]^2\, dt \tag{79}$$

The desired signal, $f_d(t)$, may be any one of a multitude of functions of $f_s(t)$. Usually, we take

$$f_d(t) = f_s(t)$$

which constitutes the filter problem. If we let

$$f_d(t) = f_s(t+a)$$

we have the filtering and prediction problem. In the absence of noise, the latter is called simply pure prediction.

By substituting Eq. (78) in Eq. (79) and making use of the fact** that

$$\Gamma_{ab}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_a(t)\, f_b(t+\tau)\, dt \tag{80}$$

we obtain***

---

* See Appendix B.

** See Sec. 3.1.2.

***It has been assumed here that the signal and noise are uncorrelated; i.e., $\Gamma_{sn}(\tau) = 0$.

$$\overline{\epsilon^2} = \int_{-\infty}^{\infty} h(\tau)\, d\tau \int_{-\infty}^{\infty} h(\sigma)\, \Gamma_{ii}(\tau - \sigma)\, d\sigma$$

$$- 2 \int_{-\infty}^{\infty} h(\tau)\, \Gamma_{id}(\tau)\, d\tau + \Gamma_{dd}(0) \tag{81}$$

Via conventional application of the variational calculus, it is found that the value of $h(\tau)$ that minimizes Eq. (81) is given by the solution of the integral equation

$$\Gamma_{id}(\tau) - \int_{-\infty}^{\infty} h(\sigma)\, \Gamma_{ii}(\tau - \sigma)\, d\sigma = 0\,, \quad \tau \geq 0 \tag{82}$$

This is the well-known Wiener-Hopf equation. This equation could be solved in routine fashion were it not for the constraint (77). As a matter of fact, if this latter constraint were neglected, one would obtain

$$H(s) = \frac{G_{ss}(\omega)}{G_{ss}(\omega) + G_{nn}(\omega)}\, H_d(s) \tag{83}$$

where

$$H(s) = \mathcal{L}\,[h(t)]$$

$$H_d(s) = 1 \qquad \text{if } f_d(t) = f_i(t)$$

$$= e^{as} \qquad \text{if } f_d(t) = f_i(t+a) \tag{84}$$

and $G_{ss}(\omega)$ and $G_{nn}(\omega)$ are the power spectral densities for $f_s(t)$ and $f_n(t)$ respectively.

The solution of Eq. (82), which takes account of condition (77), is summarized here as follows*.

a. Given the power spectral densities, $G_{ss}(\omega)$ and $G_{nn}(\omega)$, for the signal and noise respectively, replace ω by s/j and form the sum

$$G_{ii}(s) = G_{ss}(s) + G_{nn}(s) \tag{85}$$

*Details of the analysis leading to the result given here may be found in Refs. 5 and 6. The method as outlined in this monograph appears to be substantially simpler than that given in most references.

Factor this quantity as follows.

$$G_{ii}(s) = G_1(s)\, G_1(-s) = |G_1(s)|^2 \tag{86}$$

b. Form the function

$$H_1^{(0)}(s) = \frac{G_{ss}(s)}{G_1(-s)} H_d(s) \tag{87}$$

Here, $H_d(s)$ is the transfer function relating $f_d(t)$ to $f_s(t)$. For example, if $f_d(t) = f_s(t)$, then $H_d(s) = 1$. This is the case of simple filtering. If $f_d(t) = f_s(t+a)$ (i.e., filtering and prediction or pure prediction), then $H_d(s) = e^{as}$. The right-hand side of Eq. (87) is expanded in partial fractions, and the inverse Laplace transformation of the physically realizable* terms is calculated. As a result of this operation, we have

$$\begin{aligned} h_1(t) &= 0 && t < 0 \\ &= h_1^{(0)}(t) && t > 0 \end{aligned} \tag{88}$$

c. By taking the Laplace transform of (88), we obtain

$$H_1(s) = \mathcal{L}\,[h_1(t)] \tag{89}$$

d. The transfer function of the required physically realizable linear filter is then given by

$$H(s) = \frac{H_1(s)}{G_1(s)} \tag{90}$$

The method is illustrated in the following examples.

Example 1: Given

$$G_{ss}(\omega) = \frac{1}{1+\omega^2}$$

$$G_{nn}(\omega) = k^2$$

We seek to determine the optimum realizable filter for a prediction time interval, a.

---

*i.e., those terms having poles in the left half plane only.

The following steps are a direct application of the procedure outlined above and require no further explanation.

$$G_{ii}(s) = \frac{1}{1-s^2} + k^2 = \frac{1+k^2-k^2s^2}{1-s^2}$$

$$= G_1(s)\, G_1(-s)$$

where

$$G_1(s) = \frac{\sqrt{1+k^2} + ks}{1+s}$$

Now

$$H_1^{(0)}(s) = \left[\frac{1}{1-s^2}\right]\left[\frac{1-s}{\sqrt{1+k^2} - ks}\right] e^{as}$$

$$= \left[\frac{A_1}{s+1} + \frac{A_2}{\sqrt{1+k^2} - ks}\right] e^{as}$$

where

$$A_1 = \frac{1}{k + \sqrt{1+k^2}}$$

$$A_2 = \frac{k}{k + \sqrt{1+k^2}}$$

Only the first term inside the bracket represents a physically realizable filter. Taking the Laplace transform of this yields

$$h_1(t) = 0 \qquad t < 0$$

$$= A_1 e^{-(t+a)} \qquad t > 0$$

Consequently,

$$H_1(s) = \mathcal{L}\left[A_1 e^{-(t+a)}\right]$$

$$= \frac{A_1 e^{-a}}{s+1}$$

We have, finally,

$$H(s) = \frac{H_1(s)}{G_1(s)} = \frac{A_1 e^{-a}}{ks + \sqrt{1+k^2}}.$$

This is the transfer function of the required optimal linear filter.

Example 2: The given power spectral densities are

$$G_{ss}(\omega) = \frac{6}{\pi} \frac{1}{(\omega^2 + \frac{9}{4})}$$

$$G_{nn}(\omega) = \frac{2}{\pi} \frac{(\omega^2 + 2)}{(\omega^4 + 4)}$$

As in the previous example, it is required to determine the optimum filter for a prediction time interval, a.

Proceeding as before,

$$G_{ii}(s) = \frac{6}{\pi} \frac{1}{(\frac{9}{4} - s^2)} + \frac{2}{\pi} \frac{(2 - s^2)}{(4 + s^4)}$$

$$= \frac{8}{\pi} \frac{(s^2 + 2.264s + 2.031)(s^2 - 2.264s + 2.031)}{(\frac{3}{2} + s)(\frac{3}{2} - s)(s^2 + 2s + 2)(s^2 - 2s + 2)}$$

We have therefore,

$$G_1(s) = \left(\frac{8}{\pi}\right)^{1/2} \frac{(s^2 + 2.264s + 2.031)}{(s + \frac{3}{2})(s^2 + 2s + 2)}$$

$$H_1^{(0)}(s) = \frac{G_{ss}(s)}{G_1(-s)} H_d(s)$$

where $H_d(s) = e^{as}$. This leads to

$$H_1^{(0)}(s) = \frac{9}{2\pi} \frac{(s^2 - 2s + 2)\, e^{as}}{(s + \frac{3}{2})(s^2 - 2.264s + 2.031)}$$

Expanding this in partial fractions and taking the Laplace transform of the physically realizable portion results in

$$h_1(t) = 0 \qquad t < 0$$

$$= \left(\frac{9}{2\pi}\right)^{1/2} \times 0.9444\, e^{-1.5(t+a)} \qquad t > 0$$

Therefore,

$$H_1(s) = \frac{2.004}{\sqrt{\pi}} \frac{e^{-1.5a}}{(s+\frac{3}{2})}$$

and finally,

$$H(s) = \frac{H_1(s)}{G_1(s)}$$

which, in expanded form, becomes

$$H(s) = \frac{0.7085\, e^{-1.5a}\, (s^2 + 2s + 2)}{(s^2 + 2.264s + 2.031)}$$

Example 3: The given spectral densities are

$$G_{ss}(\omega) = \frac{1}{4+\omega^2}$$

$$G_{nn}(\omega) = \frac{25}{25+\omega^2}$$

We seek to determine an optimal realizable filter (no prediction in the present case).

The following steps are again self-explanatory.

$$G_{ss}(s) = \frac{1}{4-s^2}$$

$$G_{nn}(s) = \frac{25}{25-s^2}$$

and

$$G_{ii}(s) = \frac{1}{4 - s^2} + \frac{25}{25 - s^2}$$

$$= \left[\frac{5.0990\,(s + 2.1926)}{(s+2)\,(s+5)}\right] \left[\frac{5.0990\,(2.1926 - s)}{(2 - s)\,(5 - s)}\right]$$

$$G_1(s) = \frac{5.0990\,(s + 2.1926)}{(s+2)\,(s+5)}$$

$$H_1^{(0)}(s) = \frac{1}{(4 - s^2)} \times \frac{(2 - s)\,(5 - s)}{5.0990\,(2.1926 - s)}$$

$$= 0.1961 \left[\frac{1.6696}{(s+2)} + \frac{0.6696}{(2.1926 - s)}\right]$$

$$h_1(t) = 0 \qquad t < 0$$

$$= 0.3274\, e^{-2t}$$

$$H_1(s) = \frac{0.3274}{s+2}$$

and finally,

$$H(s) = \frac{0.3274}{(s+2)} \times \frac{(s+2)\,(s+5)}{5.0990\,(s+2.1926)}$$

$$= \frac{0.0642\,(s+5)}{(s+2.1926)}$$

Remark: It is pertinent at this point to review the significance and interpretation of the results thus far obtained. What has been done essentially is to consider a class or ensemble of signals about which only limited statistical data is available, namely the autocorrelation function*. The signals are corrupted by additive noise for which the autocorrelation function is also known. What has been done is to derive the form of a physically realizable linear filter that is optimum in the sense that the rms error between the actual and desired output is minimized.

*The power spectral density gives the same information since these are related by Eqs. (61) and (62).

Let us suppose that in Example 1, for instance, there is no noise present. We find in this case that the optimum filter is simply an attenuator of the form $H(s) = e^{-a}$, where "a" is the prediction interval. This result is somewhat surprising, since it states that the "best" prediction of the signal is merely a constant (less than unity) times the present value of the signal. The usefulness of this result may be seriously questioned. It is, however, the solution to the problem under the conditions stated. It is certainly the most that can be expected if the available statistical information of the signal is limited to the correlation function (or power spectral density), and the optimum operator is to be a linear filter, optimal in the least-square sense. In short, limited data can yield only limited information. We note in passing that in the case of pure filtering (and no noise), the optimum filter reduces to $H(s) = 1$, which of course is in accord with physical intuition.

The methods discussed here have been extended in various ways. In the cases thus far considered, it has been assumed that an infinitely long record of the signal has been available. This assumption simplified the mathematics leading to Eq. (82), the Wiener Hopf equation. If the integration limits are replaced by finite values, the mathematical complexities are increased considerably. Furthermore, if the signal is not of the stationary type (that is, if its autocorrelation function is not invariant under a time translation), then the previous results are not applicable. These extensions are treated in the literature[7, 12]. The analyses are extremely complex, and it is difficult to retain a physical grasp of the situation, which is so necessary for practical application. In this respect, a significant breakthrough has been made by Kalman[13, 14], who has interpreted the Wiener problem in terms of conditional expectations and state transition concepts. He has thus not only simplified the mathematical aspects, but has also displayed the results in a manner permitting application to a wider class of problems and reducing the problems of finite data and nonstationary effects to manageable proportions.

This approach will be considered in the following sections.

3.3 THE KALMAN THEORY

The Wiener problem is concerned with finding a linear filter that minimizes the mean-square error between actual and desired output when information on the signal and noise is specified in terms of power spectral densities (or correlation functions). As noted in the preceding section, the analysis leads to an integral equation whose solution yields the impulse response of the required filter. Serious complications are introduced in the case of finite memory or nonstationary processes.

The approach adopted by Kalman[13, 14] yields a much simplified mathematical structure that includes all the situations mentioned above as special cases. Further-

more, the form of the solution permits a wider range of application; the result is not merely an optimal filter, but an algorithm for obtaining optimal measurements of (deterministic) dynamic systems corrupted by additive noise. This noise may act on the dynamic system, the measurement, or both. The fundamental premises in Kalman's analysis are:

a. Arbitrary random signals are represented as the output of a linear dynamic system excited by white noise.

b. The concepts of state and state transition are used throughout.

c. If the minimum mean-square error is adopted as the optimality criterion, then the best estimate of a signal is given by the conditional expectation

$$E[x_1(t_k) \mid y(t_0),\ y(t_1),\ \ldots\ldots,\ y(t_k)]$$

We shall discuss these in turn. The first premise involves the concept of a "shaping filter," which has been considered in the previous section in a somewhat disguised form. By definition, a shaping filter for an arbitrary random process with a power spectral density, $G_{yy}(\omega)$, has the property that a white noise input generates a random process having the same power spectral density. Referring to Eq. (63), and letting $G_{xx}(\omega) = k^2$ (i.e., white noise), we see that the shaping filter, $Y(j\omega)$, is defined by

$$G_{yy}(\omega) = k^2 \left| Y(j\omega) \right|^2 \tag{91}$$

When the power density spectrum, $G_{yy}(\omega)$, is a real-valued, even, nonnegative function of ω for all real values of ω, then Eq. (91) may be solved for $Y(j\omega)$, which is in fact physically realizable (i.e., there are no poles or zeros in the right half plane).

The simplest means of doing this has already been indicated in Eq. (86); namely: replace ω by s/j in $G_{yy}(\omega)$ and form the expression*

$$G_{yy}(s) = Y_1(s)\ Y_1(-s) \tag{92}$$

The term $Y_1(s)$ is the shaping filter for $G_{yy}(\omega)$.

The following examples illustrate the calculation of the shaping filter for given power spectral densities.

Example 4:

$$G_{ss}(\omega) = \frac{1}{1+\omega^2}$$

*We have assumed that the white noise, $G_{xx}(\omega) = 1$.

$$G_{ss}(s) = \frac{1}{1-s^2} = \left(\frac{1}{1+s}\right)\left(\frac{1}{1-s}\right)$$

$$Y_1(s) = \frac{1}{1+s}$$

Example 5:

$$G_{ss}(\omega) = \frac{25+\omega^2}{4+\omega^4}$$

$$G_{ss}(s) = \frac{25-s^2}{4+s^4} = \left[\frac{5+s}{s^2+2s+2}\right]\left[\frac{5-s}{s^2-2s+2}\right]$$

and

$$Y_1(s) = \frac{5+s}{s^2+2s+2}$$

Example 6:

$$G_{ss}(\omega) = \frac{169+\omega^2}{\omega^4+238\omega^2+169^2}$$

$$G_{ss}(s) = \left[\frac{13+s}{s^2+24s+169}\right]\left[\frac{13-s}{s^2-24s+169}\right]$$

$$Y_1(s) = \frac{s+13}{s^2+24s+169}$$

Kalman's second premise involves the use of the concepts of state and state transition. Thus, instead of expressing the transfer function in terms of the complex variable, s, it is used merely to relate the output-input function in state transition terms in the time domain. In general, the shaping filter, $Y_1(s)$, may be written as

$$Y_1(s) = \frac{\theta(s)}{\varphi(s)} = \frac{\sum_{i=0}^{n} \beta_{n-i} s^i}{\sum_{j=0}^{m} \alpha_{m-j} s^i}, \quad m \geq n \tag{93}$$

It can be shown* that the input-output relation expressed by this transfer function may be written in terms of a vector matrix differential equation as follows:

$$\dot{x} = Ax + b\varphi \tag{94}$$

Here A is a constant $n \times n$ matrix and x and b are n vectors. The components of the state vector x are linear functions of θ and φ, while the components of the b vector are constants. In what follows, the form (94) will be used in preference to (93).

The third premise adopted by Kalman is that conditional expectation rather than variational calculus may be used to solve the problem of optimal filtering and prediction. This observation requires closer scrutiny, involving the concept of a conditional probability distribution, which is defined as follows [5].

$$\text{Prob}[x(t_n) \leq x_n \mid x(t_1) = x_1; x(t_2) = x_2; \ldots\ldots; x(t_{n-1}) = x_{n-1}]$$

$$\equiv \Psi(x_n) = \frac{\int_{-\infty}^{x_n} f_n(x_1, t_1; x_2, t_2; \ldots\ldots; x_n, t_n)\, dx_n}{f_{n-1}(x_1, t_1; x_2, t_2; \ldots\ldots; x_{n-1}, t_{n-1})} \tag{95}$$

where $f_n(\ )$ has the meaning defined by Eqs. (32) and (33). In words, $\Psi(x_n)$ is the probability that $x(t_n)$ takes on a value less than or equal to x_n, given that $x(t_1)$, $x(t_2)$, ..., $x(t_{n-1})$ have taken on the values, $x_1, x_2; \ldots x_{n-1}$, respectively. The conditional mean or expectation is defined by

$$E[x(t_n) \mid x(t_1) = x_1; x(t_2) = x_2; \ldots\ldots; x(t_{n-1}) = x_{n-1}]$$

$$= \int_{-\infty}^{\infty} x(t_n)\, d\Psi_n(x_n) \tag{96}$$

Suppose now that we are given a signal x(t) and a noise n(t). Only the sum $y(t) = x(t) + n(t)$ can be observed. Assume that we have observed and know the values of $y(t_1)$, $y(t_2)$, ... $y(t_{n-1})$ exactly. What can be inferred about the value of x(t)? If we know the conditional probability distribution

$$\Psi(x_n) = \text{Prob}[x(t_n) \leq x_n \mid y(t_1) = y_1; \ldots\ldots; y(t_{n-1}) = y_{n-1}] \tag{97}$$

then this conveys all the information derived by the measurements $y(t_1)$, ... $y(t_{n-1})$. Any statistical estimate of $x(t_1)$, denoted by $X(t_1)$, will then be some function of $\Psi(x_n)$. In general, the value of $X(t_n)$ will be different from the actual (but unknown) value of

*Details may be found in Ref. 15, p. 108-110.

$x(t_n)$. A rational method of determining some optimal estimate $X(t_n)$ is to define a function of the estimation error, ϵ, and try to minimize this. Let a loss function be defined by

$$L(\epsilon) = L(-\epsilon)$$

$$L(0) = 0 \tag{98}$$

$$L(\epsilon_2) \geq L(\epsilon_1) \text{ when } \epsilon_2 > \epsilon_1 \geq 0$$

where

$$\epsilon = x(t_n) - X(t_n)$$

An optimal estimate may then be determined in straightforward fashion, using the following result due to Sherman[16].

Theorem I: Assume that $L(\epsilon)$ satisfies (98) and that the conditional distribution function, $\Psi(x_n)$, defined by Eq. (97) is such that

(A) $$\Psi(x_n - \bar{x}_n) = 1 - \Psi(\bar{x}_n - x_n)$$

(B) $$\Psi[\lambda x_n^{(1)} + (1-\lambda) x_n^{(2)}] \leq \lambda \Psi(x_n^{(1)}) + (1-\lambda) \Psi(x_n^{(2)})$$

for all $x_n^{(1)}$, $x_n^{(2)} \leq \bar{x}_n$ and $0 \leq \lambda \leq 1$, where $\bar{x}_n$ is the mean of $\Psi(x_n)$.

Then the estimate $X(t_n)$ of $x(t_n)$ which minimizes $E[L(\epsilon)]$ is the conditional expectation

$$E[x(t_n) \mid y(t_1) = y_1; \ldots\ldots; y(t_{n-1}) = y_{n-1}]$$

In short, if the conditional distribution function is known to satisfy certain conditions, then the optimal estimate is merely the conditional expectation. In certain cases, even these restrictions may be removed. According to a theorem by Doob[1], if $L(\epsilon) = \epsilon^2$, then Theorem I holds without restrictions (A) and (B). Furthermore, if random processes x(t) and n(t) are Gaussian, then Theorem I holds, since quantity y(t) must also be Gaussian.

We have shown that under certain conditions, an optimal estimate of a signal corrupted by additive noise may be determined via conditional expectation. But, in general, the conditional distribution function is not known, nor is it known what general class of such functions satisfies (A) and (B) of Theorem I. Some progress can be made only if one can justifiably assume that the random processes considered are Gaussian.

Now if the random processes are Gaussian, then it is known[17] that the mean-square-error criterion yields a linear optimal estimate. Also, since the calculation of this estimate involves only the means and covariances of the Gaussian process, then this estimate is also optimal for any random process with the same means and variances if the optimal estimate is required to be linear.

This, then, forms the basis for formulating the problem in the following manner.

Statement of the Problem

I. Continuous Case

Given the dynamic system

$$\frac{dx}{dt} = A(t)\,x + B(t)\,w(t) \tag{99}$$

The observed signal is

$$z(t) = M(t)\,x + v(t) \tag{100}$$

where $A(t)$ and $B(t)$ are $n \times n$ matrices, x is an n vector, $z(t)$ is an m vector, and $M(t)$ is an $m \times n$ matrix; $w(t)$ and $v(t)$ are n and m vectors respectively, representing independent Gaussian random processes with zero means and covariance matrices*

$$E[w(t)\,w^T(\tau)] = Q(t)\,\delta(t-\tau) \tag{101}$$

$$E[v(t)\,v^T(\tau)] = R(t)\,\delta(t-\tau) \tag{102}$$

$$E[v(t)\,w^T(\tau)] = 0 \tag{103}$$

where δ is the Dirac delta function and $Q(t)$ and $R(t)$ are symmetric nonnegative definite matrices continuously differentiable in t.

It is assumed that the measurement of $z(t)$ starts at some fixed instant, t_0, (which may be $-\infty$), at which time $P(t_0) = E[x(t_0)\,x^T(t_0)]$ is known.

The optimal estimation problem is then formulated as follows. Given the known values of $z(\tau)$ in the interval $t_0 \le \tau \le t_1$; find an estimate $\hat{x}(t_1|t)$ of $x(t_1)$ that minimizes the function, $E(\epsilon^2)$, where

$$\epsilon = x(t_1) - \hat{x}(t_1|t)$$

*The superscript T denotes matrix transpose.

If $t_1 < t$, this is called the smoothing problem. If $t_1 = t$, this is called the filtering problem, while for $t_1 > t$, we have the prediction problem.

II. Discrete Case

Given the dynamic system

$$x(t+1) = \Phi(t+1, t)\, x(t) + \Gamma(t+1, t)\, w(t) \tag{104}$$

The observed signal is

$$z(t) = M(t)\, x(t) + v(t) \tag{105}$$

where $\Phi(t+1, t)$ and $\Gamma(t+1, t)$ are transition matrices* and the other symbols have the same meaning as in the continuous case except that in Eqs. (101) - (103) the numbers t and τ are integers. In other words, w(t) and v(t) are constant during the sampling interval (which in the present case has been normalized to unity).

The optimal estimation problem is as stated in the continuous case except that instead of being given $z(\tau)$ in the interval $t_0 \leq \tau \leq t$, we are given the sequence of measurements, z(0), z(1), ..., z(t). As before, we are also given the initial covariance of x(0); namely, $P(0) = E[x(0)\, x^T(0)]$.

Remark: In the problem statement, Eqs. (99) and (104) constitute models of the message process; that is, the statistical properties of the message are represented as the output of a linear system excited by white noise. If the white noise is Gaussian, then the system output is Gaussian. Furthermore, given any random process with known first- and second-order statistical moments (averages), one can find a unique Gaussian process with the same moments. In line with the earlier discussion, if one seeks an optimal linear estimate that minimizes an rms error, the same result will be obtained if the process is assumed Gaussian; for then, under the same optimality criteria, the result will be an optimal estimate that is linear. It should be emphasized that these observations are valid only when one considers statistical moments no greater than second order.

Given now the first and second statistical moments, how does one obtain (99) and (104)? This is largely an unsolved problem, and we are compelled to start with these models and consider the question of how to obtain them as a separate problem.

*Transition matrices are discussed by Tou[18] and also in part 1 of Vol. II of the present series of monographs.

There is one exception. If the statistics of the message are known in the form of a power spectral density or an autocorrelation function (of the stationary type), then one may obtain Eq. (99) from the shaping filter for the message in the manner described by Eqs. (92) - (94).

Having stated the basic problem, how does one obtain the solution? As noted earlier, Kalman showed that the optimal estimate may be obtained by calculating conditional probabilities; the details of the procedure may be found in References (13) and (14). The analysis relies heavily on the methods of abstract probability theory, which are not readily accessible to control engineers. It has been shown by Greensite[26] that Kalman's solution may be derived from first principles in dynamic programming theory. Since most control engineers are now well acquainted with the basic ideas of dynamic programming, we will adopt this approach in obtaining the solution to the problem.*

We consider first the discrete case as given by Eqs. (104) and (105). For ease of writing, we will use the following abbreviations.

$$x(t) \equiv x_t$$

$$z(t) \equiv z_t$$

etc. Also

$$\Phi(t+1,\ t) \equiv \Phi_t$$

$$\Gamma(t+1,\ t) \equiv \Gamma_t$$

$$M(t) \equiv M_t$$

By virtue of Bayes law, we have

$$\text{Prob}[x_0, \ldots\ldots, x_{t+1} \mid z_0, \ldots\ldots, z_t] = \frac{\text{Prob}[z_0, \ldots\ldots, z_t \mid x_0, \ldots\ldots, x_{t+1}]\ \text{Prob}[x_0, \ldots\ldots, x_{t+1}]}{\text{Prob}[z_0, \ldots\ldots, z_t]} \tag{106}$$

We note that

$$\text{Prob}[z_0, \ldots\ldots, z_t \mid x_0, \ldots\ldots, x_{t+1}] = \text{Prob}[z_0, \ldots\ldots, z_t \mid x_0, \ldots\ldots, x_t] \tag{107}$$

since the probability of the sequence $z_0, \ldots\ldots, z_t$ is independent of x_{t+1}.

*An outline of Kalman's approach is contained in Appendix E.

Furthermore,

$$\text{Prob}[z_0, \ldots\ldots, z_t | x_0, \ldots\ldots, x_t] = \prod_{i=0}^{t} \text{Prob}[z_i - M_i x_i] \tag{108}$$

via (105) and the independence of v_t.

We have also[19]

$$\text{Prob}[x_0, \ldots\ldots, x_{t+1}] = \text{Prob}[x_0]\, \text{Prob}[x_1 | x_0]\, \text{Prob}[x_2 | x_1, x_0] \ldots$$

$$\ldots\ldots\ldots \text{Prob}[x_{t+1} | x_t, \ldots\ldots, x_0]$$

But since the random disturbance is independent, it follows that

$$\text{Prob}[x_i | x_{i-1}, \ldots\ldots\ldots, x_0] = \text{Prob}[x_i | x_{i-1}]$$

Consequently, Eq. (106) may be written as

$$\text{Prob}[x_0, \ldots\ldots, x_{t+1} | z_0, \ldots\ldots, z_t]$$

$$= \frac{\prod_{i=0}^{t} \text{Prob}[z_i - M_i x_i]\, \text{Prob}[x_0] \prod_{i=1}^{t+1} \text{Prob}[x_i | x_{i-1}]}{\text{Prob}[z_0, \ldots\ldots\ldots, z_t]} \tag{109}$$

In accordance with the previous discussions, we assume that x_0 is Gaussian with mean

$$E(x_0) = \mu \tag{110}$$

and covariance

$$E[(x_0 - \mu)(x_0 - \mu)^T] = P_0 \tag{111}$$

But since

$$\text{Prob}[x_t | x_{t-1}] = \zeta_t \exp \left\{ -\frac{1}{2} \left[\Gamma_{t-1}^{-1} (x_t - \Phi_{t-1} x_{t-1}) \right]^T Q_{t-1}^{-1} \left[\Gamma_{t-1}^{-1} (x_t - \Phi_{t-1} x_{t-1}) \right] \right\} = \zeta_t \exp \left[-\frac{1}{2} (x_t - \Phi_{t-1} x_{t-1})^T U_{t-1}^{-1} (x_t - \Phi_{t-1} x_{t-1}) \right]$$

where

$$U_t = \Gamma_t Q_t \Gamma_t^T \tag{112}$$

and

$$\text{Prob}[z_t - M_t x_t] = \gamma_t \exp\left[-\frac{1}{2}(z_t - M_t x_t)^T R_t^{-1}(z_t - M_t x_t)\right]$$

equation (109) may be expressed in the form

$$\text{Prob}[x_0, \ldots\ldots, x_{t+1} | z_0, \ldots\ldots, z_t] = \theta_t \exp\left\{-\frac{1}{2}(x_0 - \mu)^T P_0^{-1}(x_0 - \mu)\right.$$
$$-\frac{1}{2}\sum_{i=0}^{t}\left[(z_i - M_i x_i)^T R_i^{-1}(z_i - M_i x_i)\right.$$
$$\left.\left. + (x_{i+1} - \Phi_i x_i)^T U_i^{-1}(x_{i+1} - \Phi_i x_i)\right]\right\} \tag{113}$$

where θ_t depends only on the sequence, $z_0, \ldots\ldots, z_t$, and known constants.

Since the random effects are white and Gaussian, the least-square estimate and the maximum likelihood[19] estimate are identical[20]. The problem is then reduced to one of choosing the sequence $x_0, \ldots\ldots, x_t$ that maximizes the quantity in the braces of Eq. (113) or minimizing the quantity

$$J_{t+1} = (x_0 - \mu)^T P_0^{-1}(x_0 - \mu) + \sum_{i=0}^{t}\left[(z_i - M_i x_i)^T R_i^{-1}(z_i - M_i x_i)\right.$$
$$\left. + (x_{i+1} - \Phi_i x_i)^T U_i^{-1}(x_{i+1} - \Phi_i x_i)\right] \tag{114}$$

This is precisely a multistage decision process to which the methods of dynamic programming are directly applicable.

Let us define

$$f_{t+1}(x_{t+1}) = \underset{x_0, \ldots\ldots, x_t}{\text{Min}}\left\{(x_0 - \mu)^T P_0^{-1}(x_0 - \mu)\right.$$
$$\left. + \sum_{i=1}^{t}\left[(z_i - M_i x_i)^T R_i^{-1}(z_i - M_i x_i) + (x_{i+1} - \Phi_i x_i)^T U_i^{-1}(x_{i+1} - \Phi_i x_i)\right]\right\} \tag{115}$$

Via the Principle of Optimality[21], we have

$$f_{t+1}(x_{t+1}) = \min_{x_t} [(z_t - M_t x_t)^T R_t^{-1} (z_t - M_t x_t)$$

$$+ (x_{t+1} - \Phi_t x_t)^T U_t^{-1} (x_{t+1} - \Phi_t x_t) + f_t(x_t)]$$

which, after some rearrangement, may be written as

$$f_{t+1}(x_{t+1}) = \min_{x_t} [x_t^T (M_t^T R_t^{-1} M_t + \Phi_t^T U_t^{-1} \Phi_t) x_t$$

$$- 2 (z_t^T R_t^{-1} M_t + x_{t+1}^T U_t^{-1} \Phi_t) x_t + z_t R_t^{-1} z_t$$

$$+ x_{t+1}^T U_t^{-1} x_{t+1} + f_t(x_t)] \tag{116}$$

We now assume a solution of the form*

$$f_t(x_t) = x_t^T \Lambda_t x_t - 2 b_t^T x_t + c_t \tag{117}$$

where Λ_t is a symmetric $n \times n$ matrix, b_t is an n vector, and c_t is a scalar, all of which are for the moment unknown. Substituting this back in (116), we obtain, after some reduction,

$$f_{t+1}(x_{t+1}) = \min_{x_t} [x_t^T L_t x_t - 2 \sigma_t^T x_t + z_t^T R_t^{-1} z_t$$

$$+ x_{t+1}^T U_t^{-1} x_{t+1} + c_t] \tag{118}$$

where

$$L_t = M_t^T R_t^{-1} M_t + \Phi_t^T U_t^{-1} \Phi_t + \Lambda_t \tag{119}$$

$$\sigma_t = b_t + \Phi_t^T U_t^{-1} x_{t+1} + M_t^T R_t^{-1} z_t \tag{120}$$

The value of x_t that minimizes the quantity in the brackets in Eq. (118) is readily found to be

$$x_t = L_t^{-1} \sigma_t \tag{121}$$

*This particular form of $f_t(x_t)$ is suggested by the fact that the criterion function (104) is a quadratic form, and the state equations are linear.

Substituting this back in Eq. (118) and simplifying, we obtain

$$f_{t+1}(x_{t+1}) = x_{t+1}^T \Lambda_{t+1} x_{t+1} - 2 b_{t+1}^T x_{t+1} + c_{t+1} \tag{122}$$

where

$$\Lambda_{t+1} = U_t^{-1}(I - \Phi_t L_t^{-1} \Phi_t^T U_t^{-1}) \tag{123}$$

$$b_{t+1} = U_t^{-1} \Phi_t L_t^{-1} (b_t + M_t^T R_t^{-1} z_t) \tag{124}$$

$$c_{t+1} = z_t^T R_t^{-1} (I - M_t L_t^{-1} M_t^T R_t^{-1}) z_t$$

$$- 2 b_t^T L_t^{-1} M_t^T R_t^{-1} z_t - b_t^T L_t^{-1} b_t - c_t \tag{125}$$

The function $f_{t+1}(x_{t+1})$ may be interpreted as follows. Let "a" be any value of x_{t+1}. Then $-f_{t+1}(a)$ is a measure of the likelihood of the most probable sequence of states, $x_0, \ldots\ldots, x_{t+1}$, in which x_{t+1} takes on the particular value, "a", given the observed sequence, $z_0, \ldots\ldots, z_t$ and the a priori distribution on x_0. An optimal choice of "a" is that which maximizes the likelihood function and which, for white Gaussian statistics, is also the one which minimizes the mean-square error. From Eq. (122), the optimal value of x_{t+1} is found to be

$$x_{t+1}^* = \Lambda_{t+1}^{-1} b_{t+1} \tag{126}$$

Substituting Eqs. (119), (123), and (124), we find, after some lengthy reduction,

$$x_{t+1}^* = \Phi_t (I + H_t^{-1} \Phi_t^T U_t^{-1} \Phi_t)(H_t + \Phi_t^T U_t^{-1} \Phi_t)^{-1} (b_t + M_t^T R_t^{-1} z_t) \tag{127}$$

where

$$H_t = \Lambda_t + M_t^T R_t^{-1} H_t \tag{128}$$

A crucial simplification of Eq. (127) may be obtained by making use of the following relation[22].

$$(C_1 - C_2 C_3^{-1} C_4)^{-1} = C_1^{-1} + C_1^{-1} C_2 (C_3 - C_4 C_1^{-1} C_2)^{-1} C_4 C_1^{-1} \tag{129}$$

$C_1 \equiv$ m × m matrix (nonsingular)

$C_2 \equiv$ m × n matrix

$C_3 \equiv$ n × n matrix (nonsingular)

$C_4 \equiv$ n × m matrix

Using the above relation, Eq. (127) reduces to

$$x_{t+1}^{*} = \Phi_t \Lambda_t^{-1} b_t - \Phi_t \Lambda_t^{-1} M_t^T (M_t \Lambda_t^{-1} M_t^T + R_t)^{-1} M_t \Lambda_t^{-1} b_t$$

$$+ \Phi_t \Lambda_t^{-1} M_t^T (M_t \Lambda_t^{-1} M_t^T + R_t)^{-1} z_t \tag{130}$$

We now seek to obtain the physical significance of the matrix Λ_t. For $t=-1$, we find, from Eq. (115),

$$f_0(x_0) = (x_0 - \mu)^T P_0^{-1} (x_0 - \mu)$$

$$= x_0^T P_0^{-1} x_0 - 2\mu^T P_0^{-1} x_0 + \mu^T P_0^{-1} \mu$$

while from Eq. (122),

$$f_0(x_0) = x_0^T \Lambda_0 x_0 - 2 b_0^T x_0 + c_0$$

This permits us to equate terms as follows.

$$\Lambda_0 = P_0^{-1}$$

$$b_0 = P_0^{-1} \mu$$

$$c_0 = \mu^T P_0^{-1} \mu$$

Consequently, we may interpret Λ_t^{-1} as the covariance matrix, P_t.

Noting further that Eq. (126) enables us to write x_t^* for $P_t b_t$, we find that Eq. (130) reduces to

$$\hat{x}(t+1|t) = \Phi(t+1, t)\, \hat{x}(t|t-1) + K(t)\, [z(t) - M(t)\, \hat{x}(t|t-1)] \tag{131}$$

$$K(t) = \Phi(t+1, t)\, P(t|t-1)\, M^T(t)\, [M(t)\, P(t|t-1)\, M^T(t) + R(t)]^{-1} \tag{132}$$

Here we have reverted to the original notation of the problem, and have written $\hat{x}(t+1|t)$ for x_{t+1}^* to conform with the common notation in the literature.

It remains to determine the recurrence relation for the covariance matrix, $P(t+1|t)$. This may be obtained from Eq. (123) by taking the inverse, applying Eq. (129), and then substituting (112) and (119). The end result is

$$P(t+1|t) = \Phi(t+1, t)\{P(t|t-1) - P(t|t-1)M^T(t)[M(t)P(t|t-1)M^T(t) + R(t)]^{-1}M(t)P(t|t-1)\}\Phi^T(t+1, t) + \Gamma(t+1, t)Q(t)\Gamma^T(t+1, t) \quad (133)$$

Here, obviously, $P(t_0|t_0-1) = P(t_0) \equiv P_0$, which is assumed given. This represents a type of initial condition. Eqs. (131) - (133) represent the solution to the problem in the discrete case. It can be shown[23] that

$$P(t|t-1) = E[\tilde{x}(t|t-1)\,\tilde{x}^T(t|t-1)] \quad (134)$$

where

$$\tilde{x}(t|t-1) = x(t) - \hat{x}(t|t-1) \quad (135)$$

Fig. 6 depicts both the dynamic system of Eqs. (104) and (105) (i.e., the message) and the optimal filter described by Eqs. (131) - (133).

If the unit interval for the discrete case solution is allowed to approach zero, we obtain the solution for the continuous case, Eqs. (99) and (100); viz.,

$$\frac{d\hat{x}(t|t)}{dt} = A(t)\hat{x}(t|t) + K(t)[z(t) - M(t)\hat{x}(t|t)] \quad , \text{ optimal estimate} \quad (136)$$

$$K(t) = P(t)M^T(t)R^{-1}(t) \quad , \text{ optimal gain} \quad (137)$$

$$\frac{dP(t)}{dt} = A(t)P(t) + P(t)A^T(t) - P(t)M^T(t)R^{-1}(t)M(t)P(t) + B(t)Q(t)B^T(t) \quad , \text{ variance equation} \quad (138)$$

The limiting process is a matter of some delicacy because of the presence of Dirac delta functions. However, the procedure can be made mathematically legitimate by a sufficiently sophisticated analysis. A typically "engineering type" of proof (which therefore involves some compromise with rigor) is contained in Ref. 23.

Eq. (136) is the optimal estimate for the filtering case. For optimal filtering and prediction, we add the relation

$$\hat{x}(t_1|t) = \Phi(t_1, t)\hat{x}(t|t) \quad (139)$$

$$t_1 \geq t$$

Fig. 7 is a block diagram for the message and optimal filter for the continuous case.

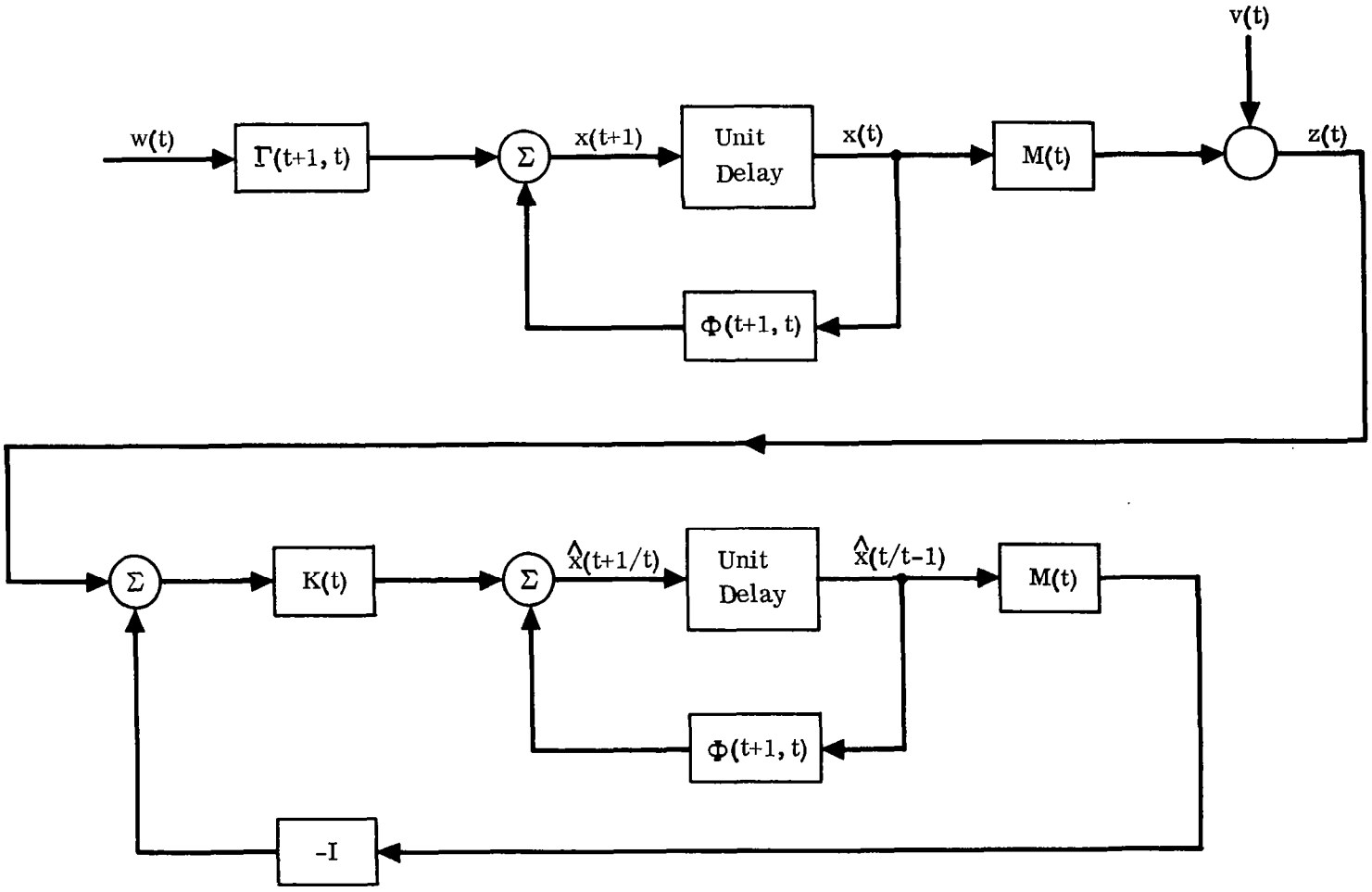


Figure 6. Model of the Message and Optimal Filter Discrete Case

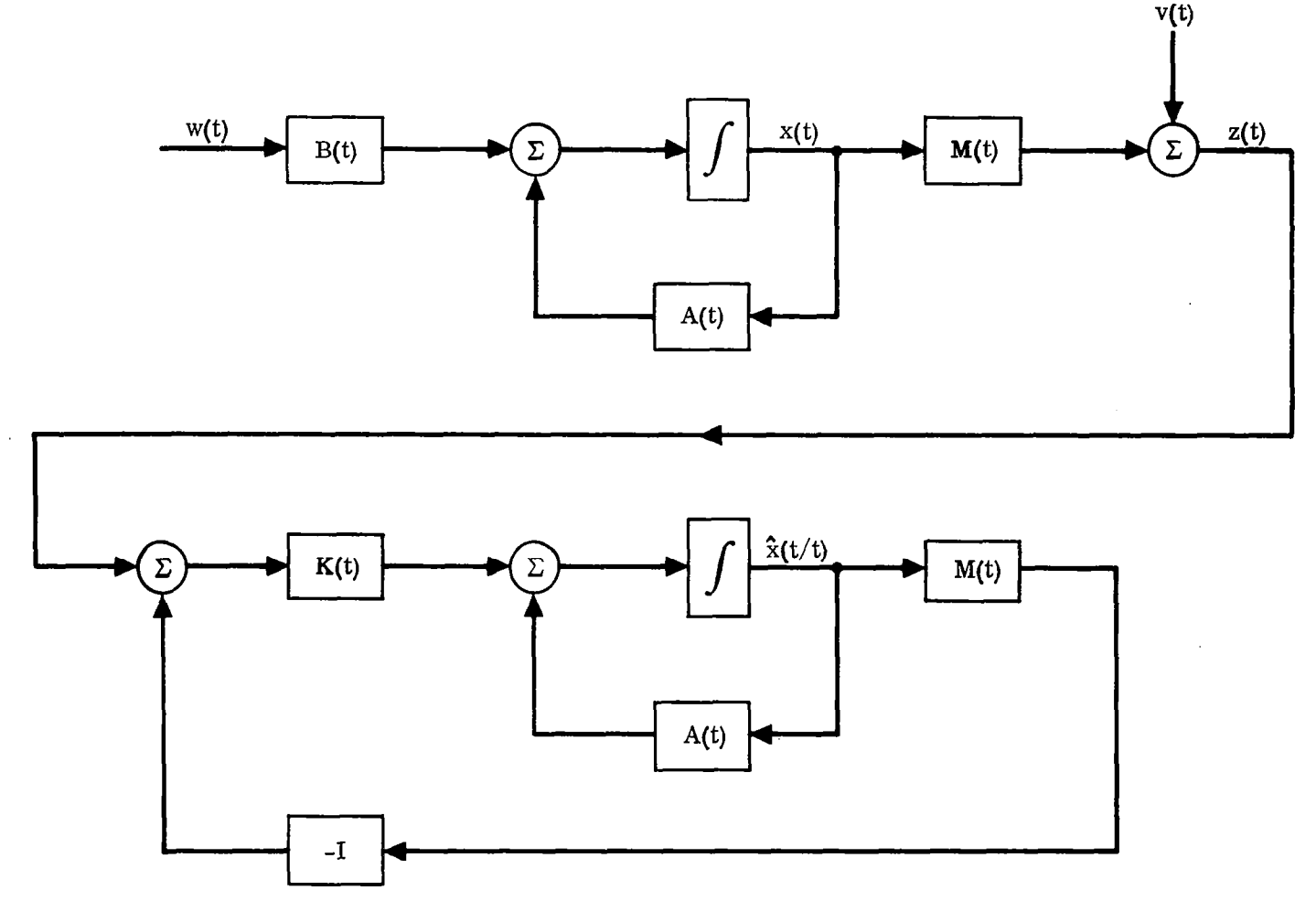


Figure 7. Model of the Message and Optimal Filter Continuous Case

The optimal filter is in general nonstationary, because A(t), $\Phi(t+1, t)$, and M(t) are time dependent. If, in fact, these quantities are constant, then as $t \to \infty$, the system (136) - (138) reduces to the classical Wiener solution.

It is worthy of note that the derivations given here are not affected materially by the fact that the system is nonstationary or that the available data is finite. The analysis leads directly to the form of the optimal filter without the extraordinary mathematical complications that characterize the classical approach.

3.3.1 Interpretation of the Kalman Filter

Before considering some direct applications of the Kalman theory, it is instructive to examine the physical significance of the operations described by the estimation equations. In order to simplify the situation without eliminating the essential features, consider the problem of estimating the components of a constant n vector, x. The only available measurement is of a linear combination of the components of x which is contaminated by white noise. Mathematically, this corresponds to the system (104) and (105) with $\Phi(t+1, t) = I$, $w(t) \equiv 0$, and $M(t) = M$ = constant. In the present case, M is a known $m \times n$ matrix and we are given the initial covariance, P_0. For simplicity, it is assumed that $R(t) = I$.

The optimal estimation equations (131) - (133) may therefore be written as

$$\hat{x}_{t+1} = \hat{x}_t + K_t(z_t - M\hat{x}_t) \tag{140}$$

$$K_t = P_t M^T (M P_t M^T + I)^{-1} \tag{141}$$

$$P_{t+1} = P_t - P_t M^T (M P_t M^T + I)^{-1} M P_t \tag{142}$$

where we are again using the simplified notation.

Making use of the relation (129), Eq. (142) simplifies to

$$P_{t+1}^{-1} = P_t^{-1} + M^T M \tag{143}$$

But Eq. (142) can also be written as

$$P_{t+1}(P_{t+1}^{-1} - P_t^{-1}) = P_t M^T (M P_t M^T + I)^{-1} M$$

which, by virtue of (143), becomes

$$P_{t+1} M^T = P_t M^T (M P_t M^T + I)^{-1} \tag{144}$$

Repeated application of this equation yields

$$P_t M^T = P_{t-1} M^T (M P_{t-1} M^T + I)^{-1}$$
$$= P_{t-2} M^T (2 M P_{t-2} M^T + I)^{-1}$$
$$= P_0 M^T (t M P_0 M^T + I)^{-1}$$

For t sufficiently large, this becomes

$$P_t M^T = \frac{1}{t} P_0 M^T (M P_0 M^T)^{-1} \tag{145}$$

Substituting (141) and (145) in Eq. (140), we find

$$\hat{x}_{t+1} = \hat{x}_t + \left(\frac{1}{t+1}\right) P_0 M^T (M P_0 M^T)^{-1} (z_t - M \hat{x}_t) \tag{146}$$

This equation is in the form of a multidimensional stochastic approximation[24,25]. The correspondence between the two concepts is intriguing and suggestive of the underlying unity between statistical estimation procedures.

Let us now assume that M is an $n \times n$ matrix which is nonsingular. This corresponds to the usual observability condition[27]. Eq. (146) then simplifies to

$$\hat{x}_{t+1} = \frac{1}{t+1} \hat{x}_t + \frac{1}{t+1} (x + M^{-1} v_t)$$

By repeated application of this equation we have

$$\hat{x}_{t+1} = \frac{t-1}{t+1} \hat{x}_{t-1} + \frac{1}{t+1} [2x + M^{-1}(v_t + v_{t-1})]$$
$$= \frac{t-2}{t+1} \hat{x}_{t-2} + \frac{1}{t+1} [3x + M^{-1}(v_t + v_{t-1} + v_{t-2})]$$
$$= \frac{t-j}{t+1} \hat{x}_{t-j} + \frac{1}{t+1} \left[(j+1) x + M^{-1} \sum_{i=t-j}^{t} v_i \right]$$

For j=t, this becomes

$$\hat{x}_{t+1} = \frac{1}{t+1} \left[(t+1) x + \sum_{i=0}^{t} M^{-1} v_i \right]$$
$$= \frac{1}{t+1} \sum_{i=1}^{t+1} M^{-1} z_i \tag{147}$$

which is precisely the weak law of large numbers[2]. Another way of writing Eq. (147) is

$$\hat{x}_{t+1} = x + \frac{1}{t+1} \sum_{i=0}^{t} M^{-1} v_i$$

In other words, for sufficiently large t, there is virtual certainty that $\hat{x}_{t+1}$ is the true value of x. Note that the variance matrix (145) becomes (for M an n × n matrix)

$$P_t = \frac{1}{t} (M^T M)^{-1}$$

For sufficiently large t, this approaches zero as expected.

3.3.2 Calculation of the Kalman Filter

In this section, we will illustrate the determination of the optimal filter, via the Kalman equations, in two simple cases. More realistic applications of practical importance will be considered in Sec. 3.3.4.

Example 7: The given data is the same as that of Example 1 in Sec. 3.2. In order to use the Kalman theory, it is necessary to express the power spectral density of the message as the output of a linear dynamic system excited by white noise. Thus the shaping filter for

$$G_{ss}(\omega) = \frac{1}{1+\omega^2}$$

is readily found to be

$$G_1(s) = \frac{1}{s+1}$$

Expressed in the time domain, in the form analogous to Eq. (99), this becomes

$$\dot{x} = -x + w$$

where w is white noise having a power spectral density of one, which means that

$$E[w(t)\, w^T(\tau)] = \delta(t-\tau)$$

The observed signal is

$$z = x + v$$

where the power spectral density of v has been given as $G_{nn}(\omega) = k^2$; consequently*

*It is assumed that $E[w(t)] = E[v(t)] = 0$.

$$E[v(t)\, v^T(\tau)] = k^2 \delta(t-\tau)$$

Fig. 8 is the schematic of this system. The solution for the optimal filter is given by Eqs. (136) - (138), where, in the present case,

$$A(t) = 1 \qquad Q(t) = 1$$

$$B(t) = 1 \qquad R(t) = k^2$$

$$M(t) = 1$$

Each of these quantities is scalar. We have, therefore,

$$\frac{d\hat{x}(t|t)}{dt} = -\hat{x}(t|t) + K[z(t) - \hat{x}(t|t)]$$

$$K = \frac{P}{R}$$

$$0 = -2P - \frac{P^2}{R} + Q$$

The quantity dP/dt has been set equal to zero in Eq. (138), since we are seeking the steady-state (time-invariant) form of the optimal filter. From the above

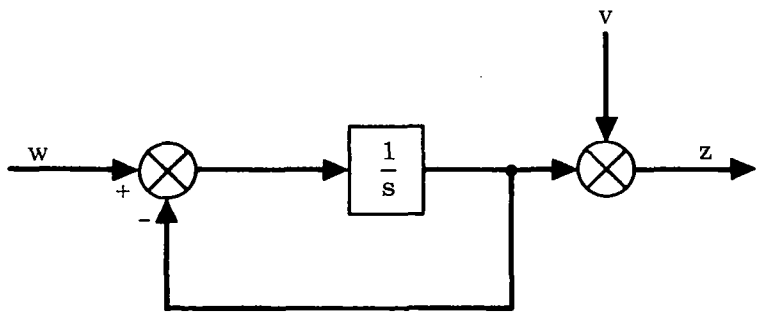


Figure 8. Schematic of Message Model for Example 7

equations, we find that this filter has the form shown in Fig. 9. The corresponding transfer function is

$$\frac{\hat{x}(t|t)}{z} = \frac{K}{s+1+K}$$

Solving for P in the quadratic equation given previously, and substituting in K, we find that this solution corresponds with that given in Example 1 of Sec. 3.2. The inclusion of the prediction interval, a, requires that we use Eq. (139), where, in the present case,

$$\Phi(t_1, t) = e^{-(t_1-t)} = e^{-a}$$

where $t_1 - t = a$ is the prediction interval.

Example 8: It is sometimes required to design an optimal realizable filter for two related messages, the measurement of each of which is corrupted by additive noise. For example, suppose that the power spectral density of a velocity signal is h_{22}^2/ω^2 while for the position signal it is h_{11}^2/ω^4. Measurements of velocity and position are corrupted by additive white noise. Some problems of this type have been treated by Bendat[28] using the classical Wiener theory. The analysis is characterized by some intricate mathematics in which it is difficult to obtain physical insight. Using the

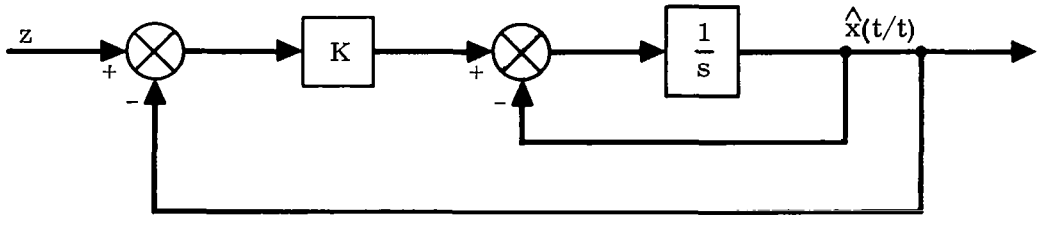


Figure 9. Schematic of Optimal Filter for Example 7

Kalman approach, however, a simple and physically plausible solution is obtained in a straightforward manner. From the power spectral densities given above, it is easy to obtain the shaping filters which, in turn, lead to the message model shown in Fig. 10. Here x_2 and x_1 represent the velocity and position respectively, while z_2 and z_1 are the respective measurements. The equations of the message model are given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = w_1$$

and for the measurements,

$$z_1 = h_{11} x_1 + v_1$$

$$z_2 = h_{22} x_2 + v_2$$

These equations may be written in the matrix form corresponding to (99) and (100), noting the following equivalence.

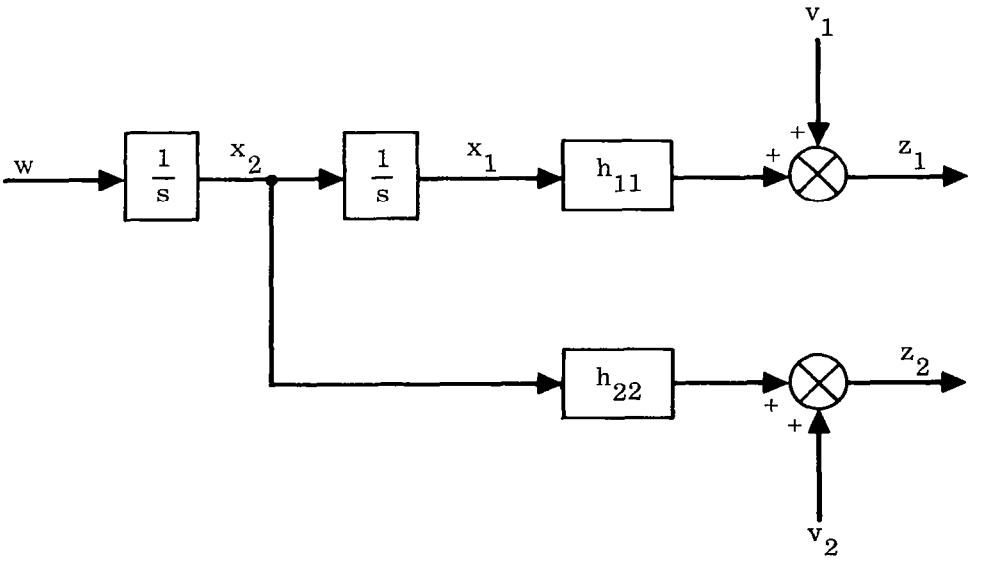


Figure 10. Model of Messages for Example 8

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} h_{11} & 0 \\ 0 & h_{22} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$$

The optimal filter is given by Eq. (136); viz.,

$$\frac{d\hat{x}(t|t)}{dt} = [A - k(t)M]\,\hat{x}(t|t) + k(t)\,z(t)$$

Fig. 11 is the schematic for this filter. Note that this is time varying, since, in the general case, only finite data is available. A time-invariant form results when $t \to \infty$, since, for this case, P becomes a constant (corresponding to the classical assumption that an infinitely long past record of the signal is available).

To solve the problem in the case of finite data one must be given the quantity*, P(0) which is written in the form

$$P(0) = \begin{bmatrix} \sigma_{11}(0) & \sigma_{12}(0) \\ \sigma_{21}(0) & \sigma_{22}(0) \end{bmatrix}$$

where $\sigma_{12}(t) = \sigma_{21}(t)$ since P(t) is a symmetric matrix.

From Eq. (138), we find

$$\dot{\sigma}_{11} = 2\sigma_{12} - \frac{h_{11}^2 \sigma_{11}^2}{r_{11}} - \frac{h_{22}^2 \sigma_{12}^2}{r_{22}}$$

$$\dot{\sigma}_{12} = \sigma_{22} - \frac{h_{11}^2 \sigma_{11} \sigma_{12}}{r_{11}} - \frac{h_{22}^2 \sigma_{12} \sigma_{22}}{r_{22}}$$

$$\dot{\sigma}_{22} = - \frac{h_{11}^2 \sigma_{12}^2}{r_{11}} - \frac{h_{22}^2 \sigma_{22}^2}{r_{22}} + q_{11}$$

*See Eq. (111) and the discussion following Eq. (133).

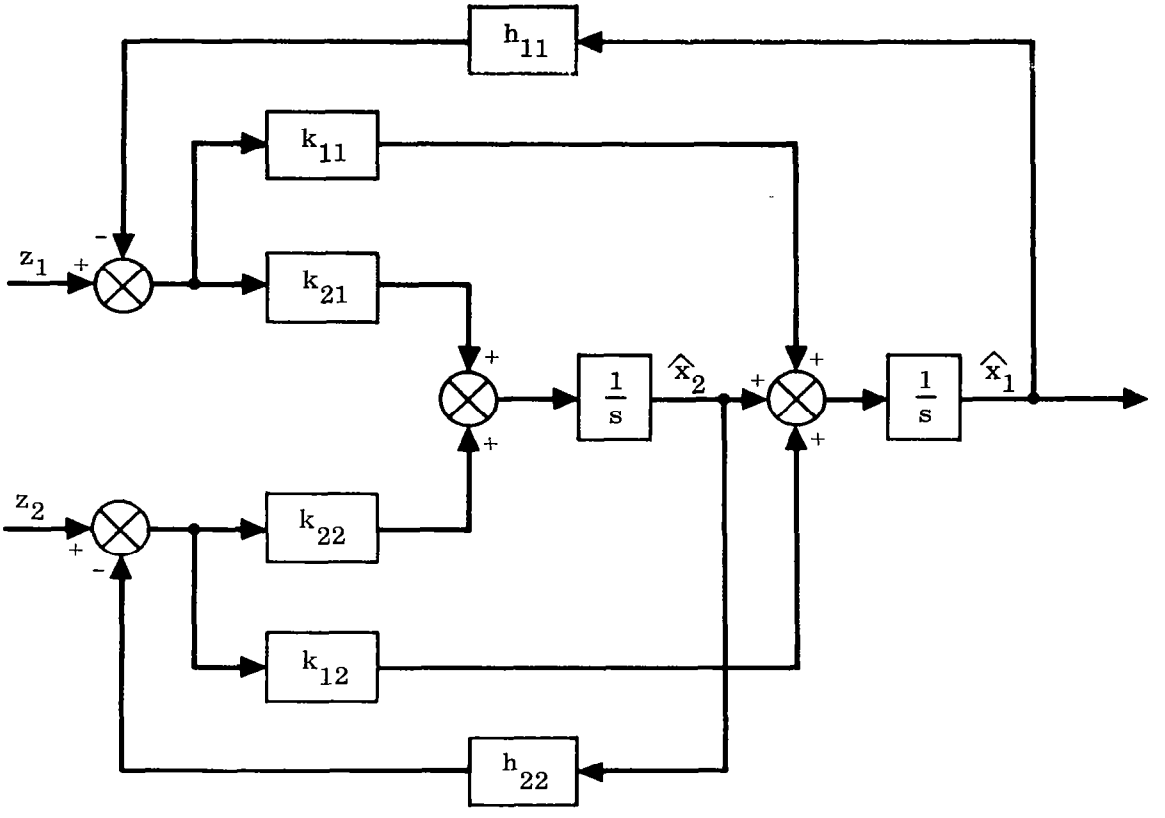


Figure 11. Model of Optimal Filter for Example 8

The optimal gains are given by Eq. (137); viz.,

$$k_{11} = \frac{h_{11}\sigma_{11}}{r_{11}}$$

$$k_{21} = \frac{h_{11}\sigma_{12}}{r_{11}}$$

$$k_{12} = \frac{h_{22}\sigma_{12}}{r_{22}}$$

$$k_{22} = \frac{h_{22}\sigma_{22}}{r_{22}}$$

In order to obtain some numerical results, let us assume that

$h_{11} = 1$ $\quad$ $q_{11} = 1$ $\quad$ $\sigma_{11}(0) = 1$

$h_{22} = 2$ $\quad$ $r_{11} = 16$ $\quad$ $\sigma_{12}(0) = 0$

$r_{22} = 1$ $\quad$ $\sigma_{22}(0) = 0$

The solution for the variance and optimal gains is shown in Figs. 12 and 13 respectively. Kalman[14] shows that under mild restrictions, a steady state is reached which is equivalent to the time-invariant optimal filter. This steady state is apparent in the present case. Note that the initial values of the time-varying parameters of the filter are strongly dependent on the assumed values for σ_{ij}. However, the final steady state is independent of the assumed values, which is plausible, since with increased information available, the initial assumptions play a vanishingly decreasing role in determining the form of the filter. Note that the steady-state values for k_{ij} and σ_{ij} could have been obtained directly from the σ equations by assuming $\dot{\sigma}_{ij} = 0$ and solving the

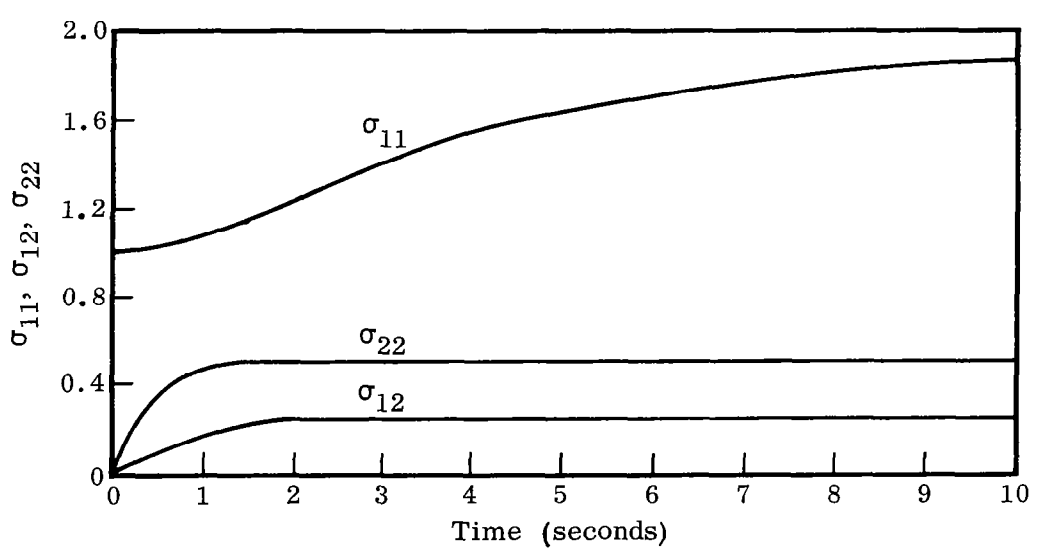


Figure 12. Solution of the Variance Equations of Example 8

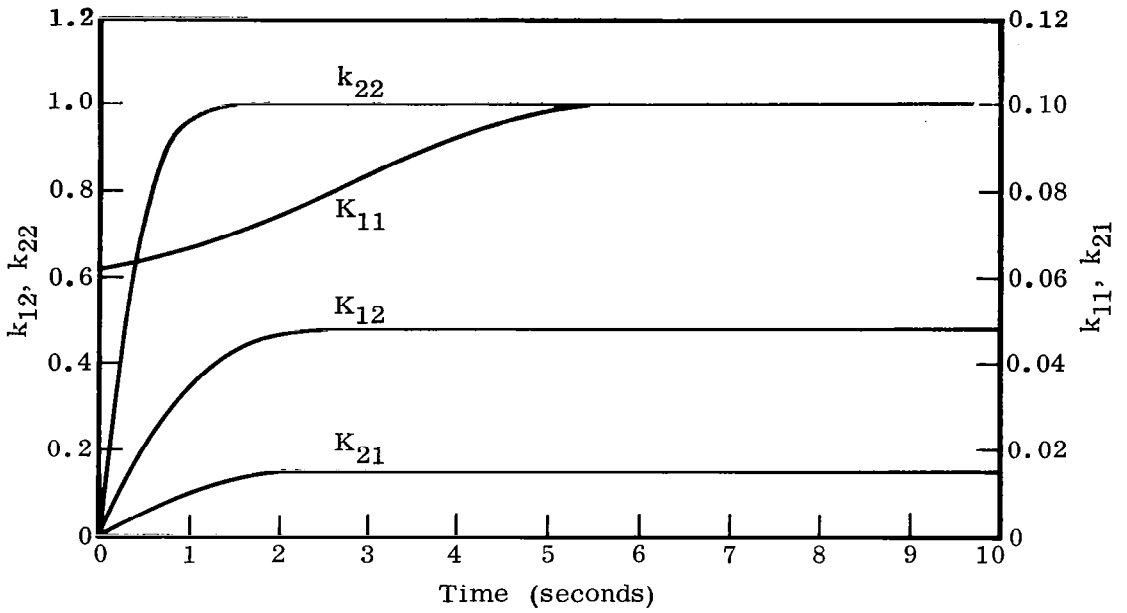


Figure 13. Optimal Gains in Example 8

resulting set of algebraic equations. For high-order systems, however, this is a difficult process, and it is much simpler to integrate the variance equations until a steady state is reached. The speed with which one converges to this steady state is, of course, dependent on how good is the initial estimate.

3.3.3 The Problem of "Colored" Noise

The Kalman filter for the continuous case, as described by Eqs. (136) - (138), is based on the fact that the measurement noise, v(t) in Eq. (100), is white and nonzero. If indeed v(t) were identically zero (i.e., the measurement is perfect), then the variance equation (138) would be singular, since R(t) would be a null matrix. Furthermore, the analysis leading to Eqs. (136) - (138) is based on the assumption that v(t) is white; i.e., successive values for small intervals are essentially uncorrelated. The case where v(t) has a power spectral density that is not a constant cannot be treated within the framework of the theory thus far presented. This gap has been filled in a recent paper by Bryson and Johansen[29], which treats the general case of measurements corrupted by colored noise, white noise, no noise, or any combination thereof. The details of the general procedure are quite lengthy, and we will accordingly describe only one special, though important, case. The reader is referred to the aforementioned reference for a complete treatment.

It is assumed that the process (message) is described by

$$\dot{x}_1 = F_{11} x_1 + u_1 \tag{148}$$

where x_1 and u_1 are r vectors, and F_{11} is an $r \times r$ matrix, which, in general, is time varying. Also

$$E[u_1(t)] = E[x_1(t)] = 0 \tag{149}$$

$$E[u_1(t)\, u_1^T(\tau)] = Q_1(t)\, \delta(t-\tau) \tag{150}$$

$$E[x_1(0)\, x_1^T(0)] = P_1(0) \tag{151}$$

The measurements are given by

$$y = C_1 x_1 + m \tag{152}$$

where y is a p vector, C_1 a $p \times r$ matrix, and m a p vector representing colored noise.

As might be expected, the procedure involves the representation of the statistical properties of m as the output of a linear system excited by white noise. Specifically, in the present case, we assume that

$$\dot{m} = Am + Bu_2 \tag{153}$$

where A and B are $p \times p$ matrices, u_2 is a p vector, and

$$E[u_2(t)] = E[u_1(t)\, u_2^T(\tau)] = 0 \tag{154}$$

$$E[u_2(t)\, u_2^T(\tau)] = Q_2(t)\, \delta(t-\tau) \tag{155}$$

$$E[m(0)] = E[m(t)\, x_1^T(t)] = 0 \tag{156}$$

$$E[m(0)\, m^T(0)] = N(0) \tag{157}$$

The optimal filter is then given by*

$$\hat{x}_1(t|t) = x_1^*(t|t) + Ky \tag{158}$$

$$\dot{x}_1^*(t|t) = F_{11} \hat{x}_1(t|t) - K[Ay + H\hat{x}_1(t|t)] - \dot{K}y \tag{159}$$

*See Ref. 29 for the derivation. The result given here is a slightly generalized version of Example 1 of that paper.

$$K = (P_1 H^T + Q_1 C_1^T) R^{-1} \tag{160}$$

$$\dot{P}_1 = F_{11} P_1 + P_1 F_{11}^T + Q_1 - (P_1 H^T + Q_1 C_1^T) R^{-1} (C_1 Q_1 + H P_1) \tag{161}$$

and

$$H = \dot{C}_1 + C_1 F_{11} - A C_1 \tag{162}$$

$$R = C_1 Q_1 C_1^T + B Q_2 B^T \tag{163}$$

The initial conditions are

$$P_1(0+) = \{P_1 - P_1 C_1^T (C_1 P_1 C_1^T + N)^{-1} C_1 P_1\}_{t=0} \tag{164}$$

$$\hat{x}_1(0+) = \{P_1 C_1^T (C_1 P_1 C_1^T + N)^{-1}\}_{t=0} y(0) \tag{165}$$

The discontinuities at the initial time are due essentially to the fact that an initial estimate and an exact measurement are simultaneously available at the start.

We illustrate the application of these ideas in the following.

Example 9: Consider the problem stated in Example 3 and solved there by the Wiener method. The same problem will be solved here by the techniques presented in this section.

The shaping filters for the message and noise are found to be

$$G_1^{(s)}(s) = \frac{1}{s+2}$$

$$G_1^{(n)}(s) = \frac{5}{s+5}$$

Consequently, the message can be represented as

$$\dot{x}_1 = -2x_1 + u_1$$

with

$$E[u_1(t) u_1^T(\tau)] = \delta(t-\tau)$$

The measurement is

$$y = x_1 + m$$

where

$$\dot{m} = -5m + 5u_2$$

$$E[u_2(t)\, u_2^T(\tau)] = \delta(t-\tau)$$

Comparison of these equations with the set (148 - 155) indicates the following equivalence.

$$F_{11} = -2 \qquad A = -5$$

$$Q_1 = 1 \qquad B = 5$$

$$C_1 = 1 \qquad Q_2 = 1$$

Since we are seeking the steady-state solution (time-invariant filter), K and P_1 are constants, with the latter obtained from Eq. (161) with $\dot{P}_1 = 0$.

Substituting known values, we find from Eqs. (160 - 163),

$$R = 26.0$$

$$H = 3.0$$

$$P_1 = 0.2232$$

$$K = 0.0642$$

The equations of the optimal filter are then given by (158) and (159); viz.,

$$\hat{x}_1(t|t) = x_1^*(t|t) + Ky$$

$$\dot{x}_1^*(t|t) = -2\,\hat{x}_1(t|t) - K[-5y + 3\hat{x}_1(t|t)]$$

Eliminating x_1^* between these equations results in

$$\frac{\hat{x}_1(t|t)}{y} = \frac{0.0642\,(s+5)}{(s+2.1926)}$$

which is identical to the result obtained in Example 3 by the Wiener method.

3.3.4 Aerospace Applications

Direct applications of the Wiener theory in aerospace problems have been rather limited. For any but the simplest type of systems, the mathematical structure becomes too unwieldy. On the other hand, Kalman's approach has the virtue of ex-

hibiting immediately the form of the optimal filter and makes use of either finite or infinite data with equal facility. This makes it very attractive for use in systems where optimal estimates based on noisy measurements are to be improved as additional information becomes available. The computational algorithms are easily adapted for computer processing in real time for operational systems. It is required only that the system be linear and that initial variance estimates be available. The latter restriction is not too serious, since the effect of poor initial estimates diminishes as more and more measurements become available. The linearity stipulation is more fundamental; usually this is satisfied by treating the estimation error (which may be taken as the state variable) as a linear expansion about some reference condition. The prospects are therefore very favorable for applying the theory in order to obtain significant improvements in system performance for a wide variety of realistic situations in aerospace guidance and control. A sampling of the literature in this area is contained in Refs. 30-36.

In what follows, we will discuss three problems which are fairly typical of present applications of the theory.

3.3.4.1 Optimal Estimation of Position and Velocity[30]

The problem to be considered involves the in-flight determination of the position and velocity of a space vehicle for purposes of midcourse guidance. It is presumed that a reference trajectory is known, but because of various random effects, the vehicle will never be precisely on this reference trajectory. Consequently, it is proposed to make a series of measurements that will, in fact, give the actual position and velocity of the vehicle and that will thereby permit the necessary guidance corrections to be applied. However, the measurements obtained are contaminated by noise, so that again, it is not possible to know the position and velocity precisely. The problem reduces to one of making optimum use of the measurements for purposes of guidance correction.

In order to apply the Kalman theory, certain conditions must be satisfied. First of all, the system must be linear. The equations describing the vehicle motion are, however, highly nonlinear.

Consider the situation depicted in Fig. 14. The motion of the vehicle is derived on the basis of including the gravitational effects of the earth, moon, and sun. For a vehicle on a lunar mission, this accounts for the predominant effects influencing the motion. The sun and moon are assumed spherical and homogeneous, while the gravitational field of the earth is modified to take account of surface oblateness.

An inertial geocentric coodinate frame* is adopted where the Z axis lies along the earth's polar axis and is positive to the north; X and Y are in the equatorial plane

*See Ref. 57.

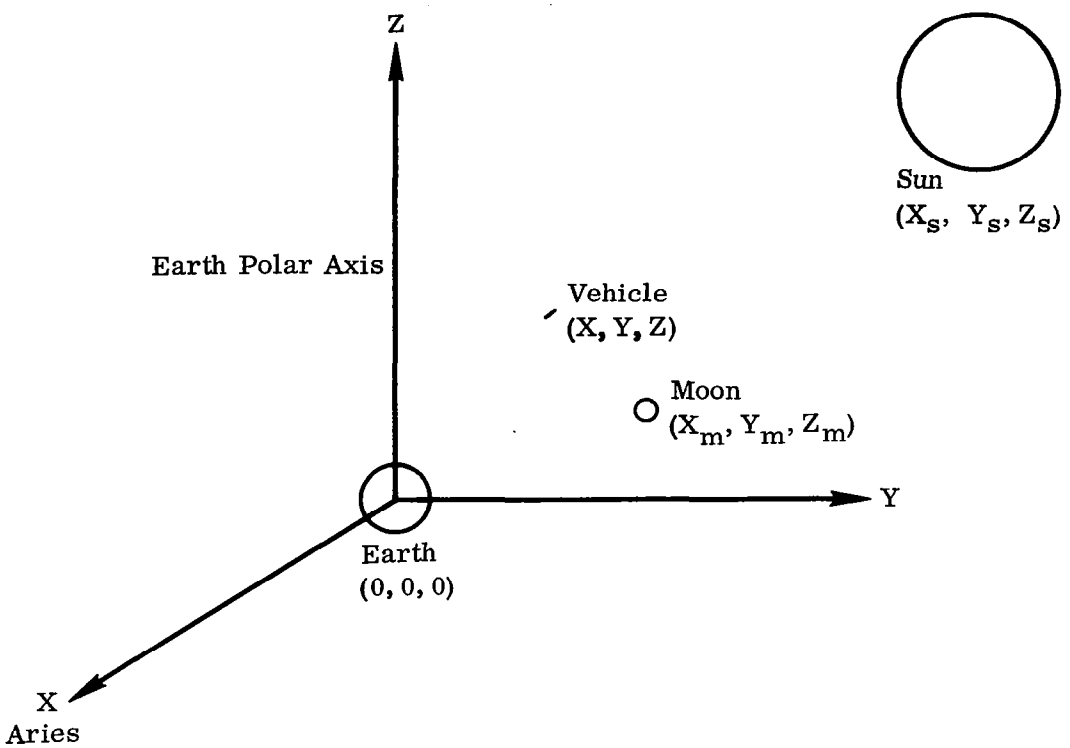


Figure 14. Geocentric Coordinate System

with the positive X axis pointed in the direction of the first point of Aries. The Y axis is oriented to complete a right-handed orthogonal system.

The vehicle equations of motion then take the following form[37].

$$\ddot{X} = -\frac{\mu_e X}{r^3}\left[1+J\left(\frac{a}{r}\right)^2\left(1-\frac{5Z^2}{r^2}\right)\right] - \frac{\mu_m (X-X_m)}{\Delta_m^3}$$

$$-\frac{\mu_m X_m}{r_m^3} - \frac{\mu_s (X-X_s)}{\Delta_s^3} - \frac{\mu_s X_s}{r_s^3} \equiv f_1(X, Y, Z) \qquad (166)$$

$$\ddot{Y} = -\frac{\mu_e Y}{r^3}\left[1+J\left(\frac{a}{r}\right)^2\left(1-\frac{5Z^2}{r^2}\right)\right] - \frac{\mu_m (Y-Y_m)}{\Delta_m^3}$$

$$-\frac{\mu_m Y_m}{r_m^3} - \frac{\mu_s (Y-Y_s)}{\Delta_s^3} - \frac{\mu_s Y_s}{r_s^3} \equiv f_2(X, Y, Z) \qquad (167)$$

$$\ddot{Z} = -\frac{\mu_e Z}{r^3}\left[1+J\left(\frac{a}{r}\right)^2\left(3-\frac{5Z^2}{r^2}\right)\right] - \frac{\mu_m (Z-Z_m)}{\Delta_m^3}$$

$$-\frac{\mu_m Z_m}{r_m^3} - \frac{\mu_s (Z-Z_s)}{\Delta_s^3} - \frac{\mu_s Z_s}{r_s^3} \equiv f_3(X, Y, Z) \quad (168)$$

where

$$r = (X^2+Y^2+Z^2)^{1/2}$$

$$r_m = (X_m^2+Y_m^2+Z_m^2)^{1/2}$$

$$r_s = (X_s^2+Y_s^2+Z_s^2)^{1/2}$$

$$\Delta_m = [(X-X_m)^2 + (Y-Y_m)^2 + (Z-Z_m)^2]^{1/2}$$

$$\Delta_s = [(X-X_s)^2 + (Y-Y_s)^2 + (Z-Z_m)^2]^{1/2}$$

$$\mu_e = 1.407683 \times 10^{16} \text{ ft}^3/\text{sec}^2$$

$$\mu_m = 1.729774 \times 10^{14} \text{ ft}^3/\text{sec}^2$$

$$\mu_s = 4.68023 \times 10^{21} \text{ ft}^3/\text{sec}^2$$

$$a = 20.9258 \times 10^6 \text{ ft} = \text{radius of earth at equator}$$

$$J = 1.6246 \times 10^{-3}$$

In order to linearize the set of equations (166) - (168), we expand each in a Taylor series about the reference trajectory; viz.,

$$\ddot{X} = f_1(X_R, Y_R, Z_R) + \frac{\partial f_1}{\partial X}(X-X_R) + \frac{\partial f_1}{\partial Y}(Y-Y_R)$$

$$+ \frac{\partial f_1}{\partial Z}(Z-Z_R) + \text{higher-order terms}$$

with similar equations for $\ddot{Y}$ and $\ddot{Z}$.

If we now define a vector, x, whose components are

$$
\begin{aligned}
x_1 &= X - X_R \\
x_2 &= Y - Y_R \\
x_3 &= Z - Z_R \\
x_4 &= \dot{X} - \dot{X}_R = \dot{x}_1 \\
x_5 &= \dot{Y} - \dot{Y}_R = \dot{x}_2 \\
x_6 &= \dot{Z} - \dot{Z}_R = \dot{x}_3
\end{aligned}
\tag{169}
$$

then, in view of the preceding results, we have

$$\dot{x} = A(t)\, x \tag{170}$$

$$A(t) = \left[\begin{array}{c:c} 0 & I \\ \hdashline A_1(t) & 0 \end{array}\right]$$

where

$$A_1(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial X} & \dfrac{\partial f_1}{\partial Y} & \dfrac{\partial f_1}{\partial Z} \\ \dfrac{\partial f_2}{\partial X} & \dfrac{\partial f_2}{\partial Y} & \dfrac{\partial f_2}{\partial Z} \\ \dfrac{\partial f_3}{\partial X} & \dfrac{\partial f_3}{\partial Y} & \dfrac{\partial f_3}{\partial Z} \end{bmatrix}$$

$I \equiv 3 \times 3$ unit matrix

$0 \equiv 3 \times 3$ null matrix

All the partial derivatives are evaluated along the reference trajectory.

Since measurements are to be made at discrete intervals, it is convenient to express Eq. (170) in the form

$$x(t+\Delta t) = \Phi(t+\Delta t,\, t)\, x(t) \tag{171}$$

where $\Phi(t+\Delta t, t)$ is the transition matrix[18] for the system (170). It is assumed that

$$E[x(t_0)] = 0$$

and that the covariance matrix

$$P_0 = E[x(t_0)\, x^T(t_0)]$$

is given, where t_0 is the time at the start of the estimation scheme. In other words, at the start of the process, it is "expected" that the deviation vector, x, is zero (i.e., the vehicle is on the reference trajectory) and our confidence in this assertion is expressed quantitatively by the initial value of the covariance matrix, P_0. As noted earlier, the estimation scheme is not too sensitive to P_0 in the long run. A large P_0 merely means that more measurements must be taken to ensure that P(t) ultimately is reduced to below a preselected level. Some care must be taken, however, that P_0 not be too large, since, in some instances, the numerical processing may produce a negative definite P(t) which will invalidate the computation. The question of uncertainty in the a priori estimates of covariance has been clarified in recent papers by Soong[38] and Nishimura[39].

It now remains to consider the instrumentation to be employed and the means of incorporating this in the estimation scheme. For present purposes, it will be assumed that one can measure the angles, α, β, and γ, as shown in Fig. 15. From the geometry depicted here, one can readily derive the equations which relate these angles to the vehicle position; viz.,

$$\alpha = \sin^{-1}\frac{Z}{r} \tag{172}$$

$$\beta = \sin^{-1}\frac{Y}{(X^2+Y^2)} \tag{173}$$

$$\gamma = \sin^{-1}\frac{r_0}{r} \tag{174}$$

The instrumentation output is of the form

$$\alpha_m = \alpha_{act} + v_1$$

$$\beta_m = \beta_{act} + v_2$$

$$\gamma_m = \gamma_{act} + v_3$$

where the subscript "act" denotes actual value, and the v_i represents noise. It will be assumed that the noise has zero mean and is uncorrelated between successive measuring instants (i.e., white).

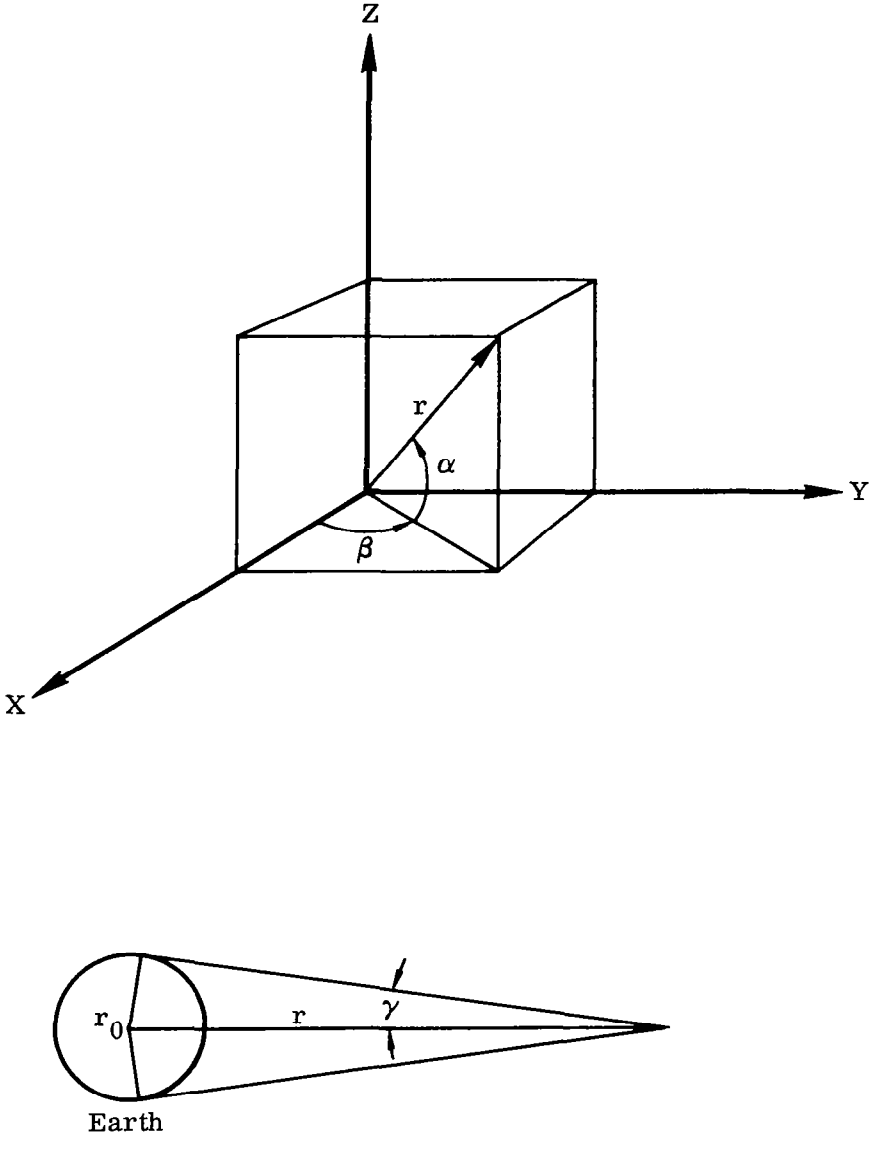


Figure 15. Measurement Scheme

We now define

$$\Delta\alpha_m = \alpha_m - \alpha_{nom}$$

$$\Delta\beta_m = \beta_m - \beta_{nom}$$

$$\Delta\gamma_m = \gamma_m - \gamma_{nom}$$

where the subscript "nom" means nominal value -- a known quantity if the reference trajectory is known.

If we now let

$$\Delta\alpha = \alpha_{act} - \alpha_{nom}$$

and similarly for $\Delta\beta$ and $\Delta\gamma$, we may write

$$\Delta\alpha_m = \Delta\alpha + v_1 \tag{175}$$

$$\Delta\beta_m = \Delta\beta + v_2 \tag{176}$$

$$\Delta\gamma_m = \Delta\gamma + v_3 \tag{177}$$

The quantities $\Delta\alpha$, $\Delta\beta$, and $\Delta\gamma$ represent deviations from a nominal or reference value, and if these are assumed small, then a Taylor series expansion about this nominal results in

$$\begin{bmatrix} \Delta\alpha \\ \Delta\beta \\ \Delta\gamma \end{bmatrix} = \begin{bmatrix} \frac{\partial\alpha}{\partial X} & \frac{\partial\alpha}{\partial Y} & \frac{\partial\alpha}{\partial Z} \\ \frac{\partial\beta}{\partial X} & \frac{\partial\beta}{\partial Y} & \frac{\partial\beta}{\partial Z} \\ \frac{\partial\gamma}{\partial X} & \frac{\partial\gamma}{\partial Y} & \frac{\partial\gamma}{\partial Z} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By combining this with the previous results, we may write

$$z(t) = M(t)\,x + v(t) \tag{178}$$

where v(t) is the noise vector whose components are v_1, v_2, v_3; x is the state vector defined by Eq. (169); and

$$z(t) \equiv \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} \equiv \begin{bmatrix} \Delta\alpha_m \\ \Delta\beta_m \\ \Delta\gamma_m \end{bmatrix}$$

$$M(t) \equiv \left[M_1(t) \vdots \; 0 \right]$$

where

$$M_1(t) = \begin{bmatrix} \frac{\partial\alpha}{\partial X} & \frac{\partial\alpha}{\partial Y} & \frac{\partial\alpha}{\partial Z} \\ \frac{\partial\beta}{\partial X} & \frac{\partial\beta}{\partial Y} & \frac{\partial\beta}{\partial Z} \\ \frac{\partial\gamma}{\partial X} & \frac{\partial\gamma}{\partial Y} & \frac{\partial\gamma}{\partial Z} \end{bmatrix}$$

and 0 is a 3×3 null matrix.

The estimation scheme is now in the form of Eqs. (104) and (105), where, in the present case, $w(t) \equiv 0$. Consequently, the optimal estimates are given by Eqs. (131) - (133) as follows.

$$\hat{x}(t+\Delta t|t) = \Phi(t+\Delta t, t)\, \hat{x}(t|t-\Delta t) + K(t)\, [z(t) - M(t)\, \hat{x}(t|t-\Delta t)] \tag{179}$$

$$K(t) = \Phi(t+\Delta t, t)\, P(t|t-\Delta t)\, M^T(t)\, [M(t)\, P(t|t-\Delta t)\, M^T(t) + R(t)]^{-1} \tag{180}$$

$$P(t+\Delta t|t) = \Phi(t+\Delta t, t)\, \{P(t|t-\Delta t) - P(t|t-\Delta t)\, M^T(t)\, [M(t)\, P(t|t-\Delta t)\, M^T(t) + R(t)]^{-1} M(t)\, P(t|t-\Delta t)\}\, \Phi^T(t+\Delta t, t) \tag{181}$$

Recall that $x(t)$ is a vector representing deviations from the reference trajectory. Therefore $\hat{x}(t+\Delta t|t)$ is an optimal estimate of deviations from nominal. This, in turn, permits one to estimate the actual trajectory.

Regarding the actual implementation of this procedure in an on-board digital computer, the following points may be noted. First of all, as far as calculating the actual trajectory, the set of equations (166) - (168) could be used, given initial values of position and velocity rather than the linearized version (171).

In other words, the estimated values of position and velocity can be used to calculate future position and velocity based on the precise equations of motion, rather than linearized versions, which are inherently less accurate, being used.

However, the computations involving P and K still require the linearized approach because of the manner in which Eqs. (132) and (133) were derived. The matrix, R, which represents the covariance matrix of the noise vector, v(t), is constant and can be initially stored in the computer. Theoretically, one can also store $\Phi(t+\Delta t, t)$ and M(t), since these are known for all values of t, given the reference trajectory. However, this may lead to computer storage capacity problems, and also to less flexibility in choosing measurement times and intervals. There is the additional disadvantage that the linearizations are about a reference trajectory that is now less accurate than the new estimated trajectory. Consequently, it appears preferable to linearize about the estimated rather than a reference trajectory, from the point of view of minimizing error buildup. This is clearly the correct procedure, since P has to do with the difference between the estimate and the true state; the estimate is, on the average, closer to the true state than is the reference. Since $\Phi(t+\Delta t, t)$ and M(t) are to be recomputed between successive measurements, it is of course necessary that the computation time on the computer be less than the time interval, Δt, between successive observations.

Some results of a digital computer study of this problem are given by Smith et al[30].

3.3.4.2 Optimum Stellar Inertial Navigation System[34]

A general discussion of the dynamic properties of an inertial navigation system is presented in Chap. 4 of Ref. 40. As discussed there, the system dynamics depend upon the mode and duration of operation. The problem to be analyzed in this section is concerned with an inertial system operating in conjunction with a star tracker (stellar-inertial mode) and represents a specialized case of the more general dynamics.

The basic problem derives from the fact that over a period of time, the drift rates of the gyros in the inertial platform introduce significant error in the navigation system. There are various ways of alleviating this problem*. The approach to be considered here makes use of the Kalman theory with the result that significant improvement over previous methods can be demonstrated[34].

To begin the discussion, we define three coordinate systems in the nomenclature of Ref. 40.

(I). φ is the vector angle relating the platform coordinate system to a true coordinate system (attitude error).

*Cf. Ref. 40, p. 146.

(II). $\delta\theta$ is the vector angle relating the computer coordinate system to a true coordinate system (position error).

(III). ψ is the vector angle relating the platform coordinate system to the computer coordinate system.

The vector angles are related by

$$\varphi = \psi + \delta\theta \tag{182}$$

We shall be especially concerned with the vector angle, ψ, that can be shown[40] to satisfy the equation

$$\dot{\psi} = \epsilon \tag{183}$$

in the stellar-inertial mode of operation. Here, ϵ represents the gyro drift rate vector whose effects on the system must somehow be compensated for. The information available on ϵ is generally of a statistical nature, and the problem is one of making an optimal estimate of ψ based on some reasonable stochastic description of ϵ.

In what follows, we shall assume that ϵ is composed of two parts,

$$\epsilon = \epsilon_c + \epsilon_r \tag{184}$$

the first of which is an unknown constant vector (gyro bias) and the second a zero mean random vector whose power spectral density is known. Specifically, we assume that each component ϵ_{ri} of ϵ_r, corresponding to each of the three platform gyros, has a power spectral density given by

$$G_i(\omega) = 2\sigma_i^2 \beta_i \left(\frac{1}{\omega^2 + \beta_i^2} \right) \tag{185}$$

The corresponding autocorrelation function is

$$\Gamma_i(\tau) = \sigma_i^2 \, e^{-\beta_i |\tau|} \tag{186}$$

It is assumed that the gyros are uncorrelated; i.e.,

$$E[\epsilon_{ri}(t)\, \epsilon_{rj}(\tau)] = 0 \qquad i \neq j$$

Now according to the definition of a shaping filter*, if white noise having a power spectral density given by

$$G_w^{(i)}(\omega) = 2\sigma_i^2 \beta_i \tag{187}$$

*See Eq. 91.

is passed through the linear (shaping) filter having the transfer function

$$\frac{1}{s+\beta}$$

then the output power spectral density is equal to that of Eq. (185). Accordingly, we may represent the statistical properties of ϵ_{ri} by the dynamic system

$$\dot{\epsilon}_{ri} = -\beta_i \epsilon_{ri} + w_i \tag{188}$$

where

$$E[w_i(t)\, w_i(\tau)] = 2\sigma_i^2 \beta_i \delta(t-\tau) \tag{189}$$

This may be written in matrix form as

$$\dot{\epsilon}_r = H\epsilon_r + w \tag{190}$$

The equation for the unknown bias constant is

$$\dot{\epsilon}_c = 0 \tag{191}$$

Combining Eqs. (183), (184), (190), and (191), we have

$$\dot{x} = Ax + Bw \tag{192}$$

where

$$x = \begin{bmatrix} \psi \\ \cdots \\ \epsilon_r \\ \cdots \\ \epsilon_c \end{bmatrix}$$

$$A = \left[\begin{array}{c:c:c} 0 & I & I \\ \hdashline 0 & H & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right]$$

$$B = \begin{bmatrix} 0 \\ \cdots \\ I \\ \cdots \\ 0 \end{bmatrix}$$

Notice that Eq. (192) is expressed in terms of partitioned matrices -- not scalars.

We seek to estimate the state vector, x, making optimal use of appropriate measurements. The present situation is distinguished from that of Sec. 3.3.4.1 in that both correlated noise (the ϵ_r vector) and an unknown constant (the bias vector, ϵ_c) have been incorporated in an augmented state vector. In other words, these additional complications may be treated in straightforward fashion at the cost of dealing with a higher-order system. Since there is no associated difficulty of a theoretical nature, the limitations are primarily computer time and storage requirements.

We now turn to a consideration of the measurement procedure. A star tracker that is physically mounted on the stable platform can be driven in both azimuth and elevation. Tracking is accomplished by selecting a star from the catalog stored in the system computer. The computer then automatically computes the telescope pointing angles; i.e., the azimuth angle, α, and the elevation angle, γ, through which the telescope must turn to point at the selected star. Neglecting errors in the telescope drives, it would be possible to point the telescope directly at the star if the computer coordinate system and the platform coordinate system were coincident. The platform system, however, is rotated from the computer system by the vector angle, ψ. That is to say, the vector angle, ψ, is the telescope pointing error, or the error in pointing a platform-mounted telescope at a star.

We therefore seek to determine the relationship between the errors in the pointing angles, α and γ, and the vector angle ψ.

The position vector of the telescope relative to the platform coordinate system is given by (see Fig. 16)

$$S_P = \begin{bmatrix} \cos\gamma \cos\alpha \\ \cos\gamma \sin\alpha \\ \sin\gamma \end{bmatrix}$$

This vector has components in the computer coordinate system as follows

$$S_c = \Lambda S_P$$

$$\Lambda = \begin{bmatrix} 1 & \psi_z & -\psi_y \\ -\psi_z & 1 & \psi_x \\ \psi_y & -\psi_x & 1 \end{bmatrix}$$

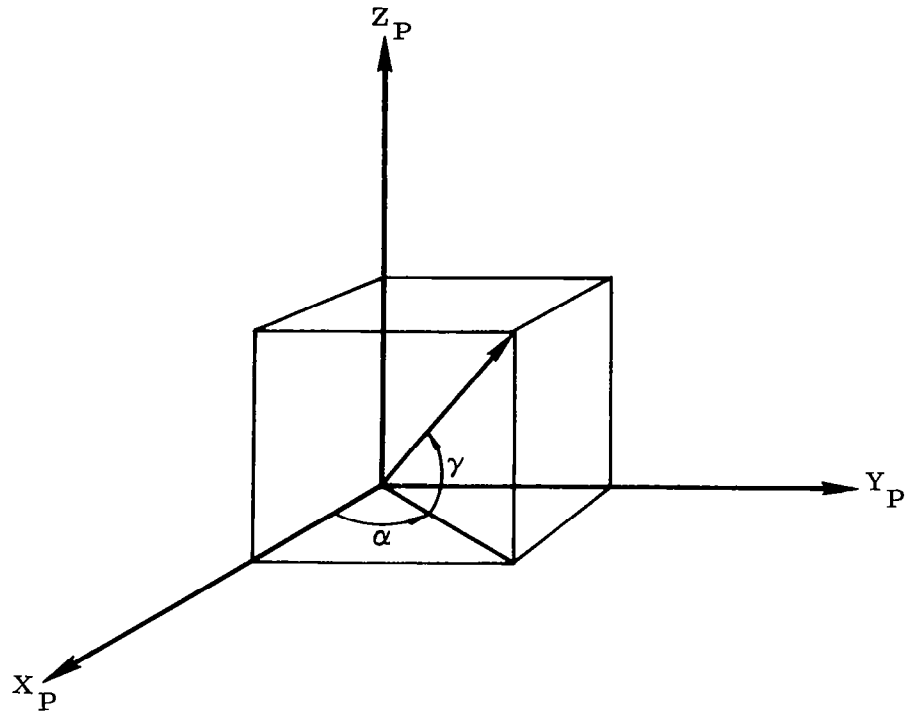


Figure 16. Orientation of Telescope in Computer Coordinate System

or*

$$S_c = \begin{bmatrix} c\gamma c\alpha & + \psi_z c\gamma s\alpha & - \psi_y s\gamma \\ - \psi_z c\gamma c\alpha & + c\gamma s\alpha & + \psi_x s\gamma \\ \psi_y c\gamma c\alpha & - \psi_x c\gamma s\alpha & + s\gamma \end{bmatrix}$$

This may be written as

$$S_c = S_P + \Delta S_P$$

where

$$\Delta S_P = \begin{bmatrix} \psi_z s\alpha c\gamma & - \psi_y s\gamma \\ - \psi_z c\alpha c\gamma & + \psi_x s\gamma \\ \psi_y c\alpha c\gamma & - \psi_x s\alpha c\gamma \end{bmatrix} \qquad (193)$$

*We have used the abbreviations, $s\alpha$, $c\alpha$, for $\sin \alpha$, $\cos \alpha$, etc.

The increment in S_P due to increments $\Delta\alpha$ and $\Delta\gamma$ is

$$\Delta S_P = \begin{bmatrix} c(\gamma+\Delta\gamma) & c(\alpha+\Delta\alpha) \\ c(\gamma+\Delta\gamma) & s(\alpha+\Delta\alpha) \\ & s(\gamma+\Delta\gamma) \end{bmatrix} - \begin{bmatrix} c\gamma\, c\alpha \\ c\gamma\, s\alpha \\ s\gamma \end{bmatrix}$$

which reduces to

$$\Delta S_P = \begin{bmatrix} -\Delta\alpha\, c\gamma\, s\alpha & -\Delta\gamma\, s\gamma\, c\alpha \\ \Delta\alpha\, c\alpha\, c\gamma & -\Delta\gamma\, s\alpha\, s\gamma \\ & \Delta\gamma\, c\gamma \end{bmatrix} \tag{194}$$

By equating the expressions (193) and (194) we obtain the relations between the ψ vector and the errors in pointing angles; viz.,

$$\Delta\alpha = \psi_x \cos\alpha \tan\gamma + \psi_y \sin\alpha \tan\gamma - \psi_z$$

$$\Delta\gamma = -\psi_x \sin\alpha + \psi_y \cos\alpha$$

We now define the measurement vector as

$$z \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \equiv \begin{bmatrix} \Delta\alpha \\ \Delta\gamma \end{bmatrix} = M_s \psi + v \tag{195}$$

where

$$M_s = \begin{bmatrix} \tan\gamma \cos\alpha & \tan\gamma \sin\alpha & -1 \\ -\sin\alpha & \cos\alpha & 0 \end{bmatrix}$$

and v is a two-dimensional white-noise vector. In order to be compatible with Eq. (192), we write Eq. (195) as

$$z = \begin{bmatrix} M_s & \vdots & 0 & \vdots & 0 \end{bmatrix} x + v \tag{196}$$

where 0 is a 2×3 null matrix.

We will consider the discrete version of Eq. (192), which may be written as

$$x(t+1) = \Phi(t+1, t)\, x(t) + \Gamma(t+1, t)\, w(t) \tag{197}$$

where $\Phi(t+1, t)$ is the transition matrix for the system, and

$$\Gamma(t+1, t) = \int_{t}^{t+1} \Phi(t+1, \tau)\, B\, d\tau$$

Eqs. (197) and (196) are in the form of (104) and (105), so that the optimal estimation equations are given by (131) - (133). The optimal estimates thus obtained are used to correct the system in some appropriate fashion. In the paper by Bona and Hutchinson[34] this information is used to "reset" the inertial system to compensate for gyro drift rates. Typical results obtained are shown in Figs. 17 and 18. The upper curve in each figure indicates the accuracy obtainable by conventional methods*. The runs were made with the system operated as described above, employing optimal estimation followed by corrections for a period of six hours, after which a conventional correction procedure was used. The improvement in accuracy achieved by employing optimal estimation procedures is quite dramatic.

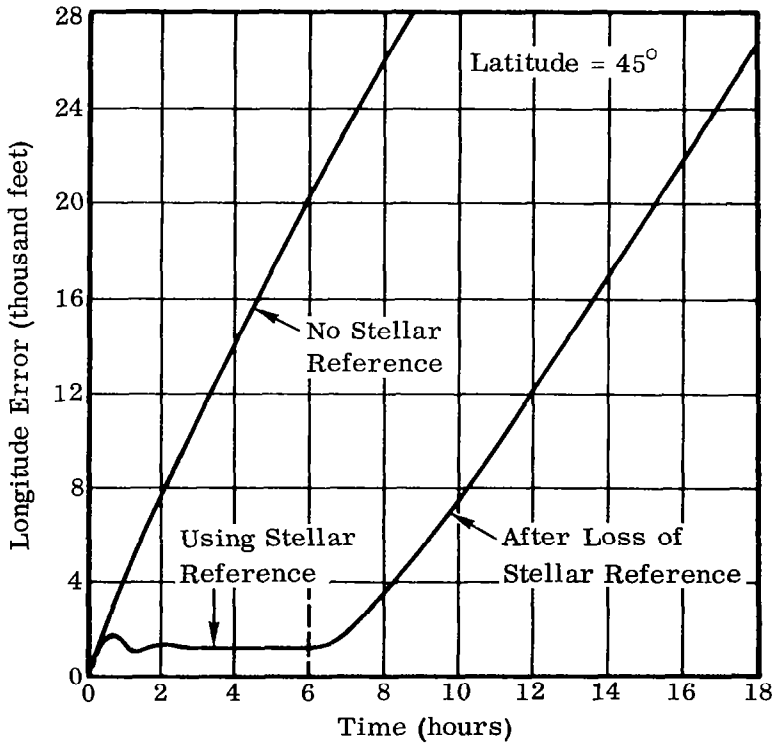


Figure 17. Longitude Error Comparison

*i.e., the so called "damped inertial mode." Cf. Ref. 40 - p. 146.

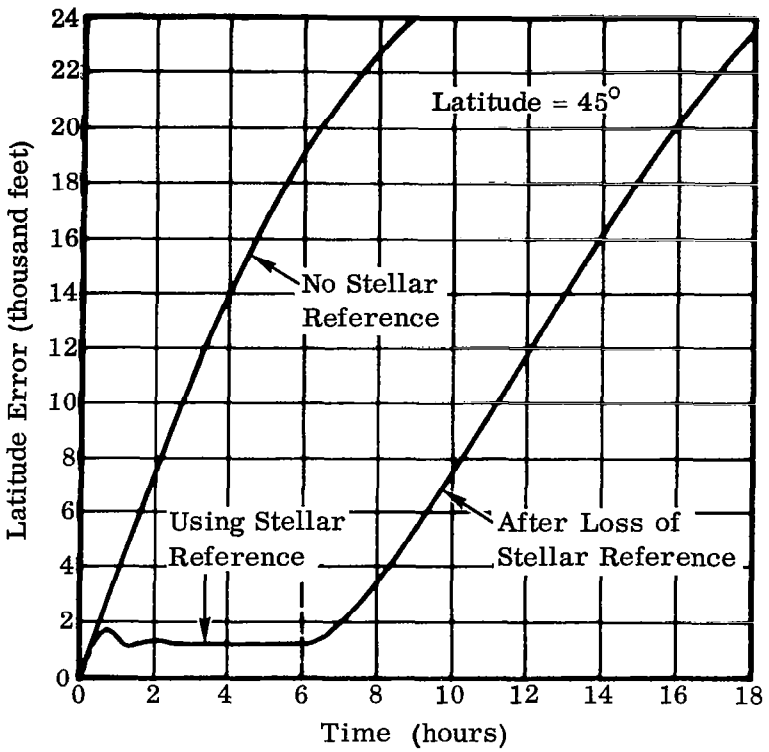


Figure 18. Latitude Error Comparison

3.3.4.3 Optimal Estimation of Local Vertical and Orbital Parameters[41]

In this section, we discuss the application of the Kalman theory to the problem of determining the direction of the normal to the earth's surface from aboard an earth satellite making use of horizon sensor measurements that are contaminated by noise. In addition to providing an optimal estimate of the local vertical, the estimation scheme corrects for errors in the assumed values of the elliptical orbit parameters that determine the motion in the orbital plane.

Much of the simplicity of the scheme results from the assumption that all measurements are taken in the orbital plane and that the vehicle rotation is about an axis normal to the plane. After injection into orbit, the vehicle attitude control system removes rotations about all vehicle axes except one. The attitude control system is also used to align this axis with the normal to the orbital plane. A good estimate of the orbital parameters (which are constant if orbital perturbations are neglected) is available. The estimation scheme will refine these values as measurements are processed.

If a vehicle fixed reference direction is chosen in the plane of rotation, each horizon sensor measurement may be interpreted as an angle between local vertical and this reference direction. The simplest situation to consider is when the reference is in the direction of local vertical at perigee and the satellite is rotating at the mean

motion of the orbit, ($\omega = \omega_s = \frac{2\pi}{\tau}$, where τ is the period). At any time, t, after perigee passage, the angle, α, between local vertical and reference is simply the difference between the true anomaly, φ, and the mean anomaly, γ, of the elliptic orbit. (See Fig. 19.)

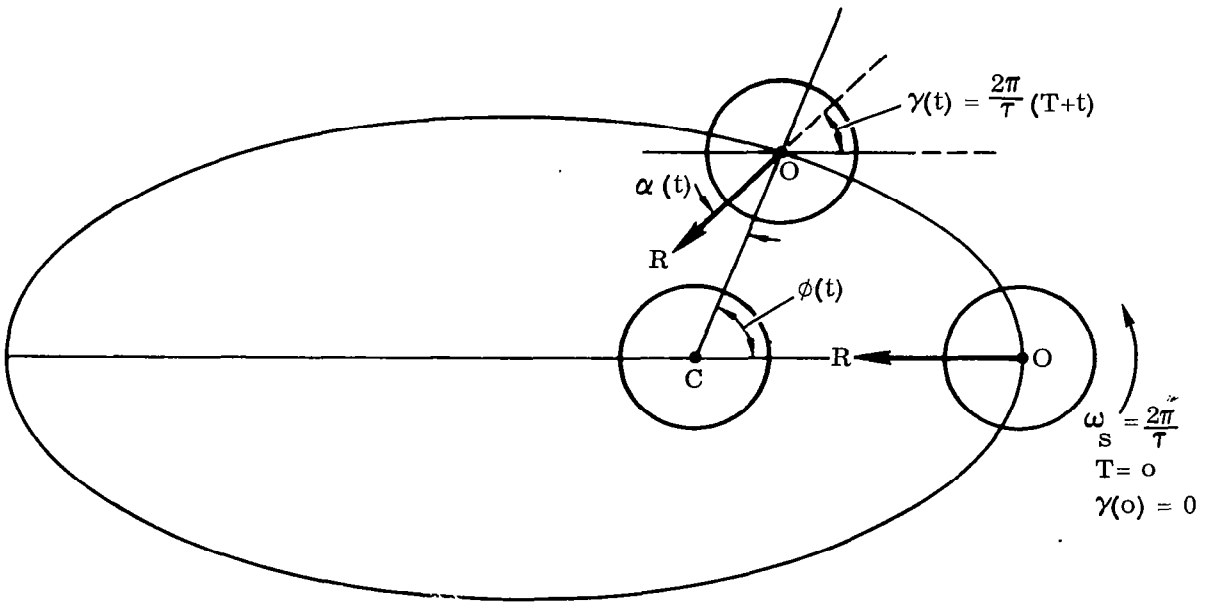
$$\alpha(t) = \varphi(t) - \gamma(t) \equiv \zeta(t) \tag{198}$$

More generally, for any constant rate of rotation and an arbitrary injection point, (t=0), $\alpha(t)$ is given by

$$\alpha(t) = \beta + \zeta(t) - \zeta(0) - \left[\omega - \frac{2\pi}{\tau}\right] t \tag{199}$$

The quantity, β, is simply a bias angle depending on the arbitrary choice of the reference direction. Following injection, $\alpha(t)$ depends upon the behavior of $\zeta(t)$ as compared to $\zeta(0)$, and the difference between the actual rotational rate and the mean motion of the orbit.

For orbits of small eccentricity, e, $\zeta(t)$ may be expressed[42] as a series in e and $\gamma(t)$.



$\overrightarrow{OC}$ = local vertical

$\overrightarrow{OR}$ = reference direction

Figure 19. Satellite and Orbital Parameters

$$\zeta(t) = \left(2e - \frac{1}{4}e^3 + \frac{5}{96}e^5 + \frac{107}{4608}e^7\right)\sin\gamma(t)$$

$$+ \ldots\ldots\ldots + \frac{47273}{32256}e^7 \sin 7\gamma(t) \tag{200}$$

Also,

$$\gamma(t) = \frac{2\pi}{\tau}(T+t) \tag{201}$$

where T is the (virtual) time from perigee passage to the injection point, and

$$\tau = C\,a^{3/2} \tag{202}$$

which is simply Kepler's third law. Here, "a" is the semimajor axis of the ellipse, and C is a known constant.

Consequently, $\alpha(t)$ may be expressed as a function of five independent parameters and time, t; viz.,

$$\alpha(t) = f(e, a, T, \omega, \beta, t) \tag{203}$$

The orbital parameters are e, a, and T, while ω and β relate to the orientation of the vehicle.

The measured angle, $\alpha_m(t_i)$, determined from a noisy horizon sensor measurement of the local vertical at time, t_i, is

$$\alpha_m(t_i) = \alpha(t_i) + v(t_i) \tag{204}$$

where $v(t_i)$ is random white Gaussian noise whose statistical properties are given by

$$E[v(t_i)] = 0 \tag{205}$$

$$E[v^2(t_i)] = \sigma^2 \tag{206}$$

$$E[v(t_i)\,v(t_j)] = 0 \qquad i \neq j \tag{207}$$

If a nominal or calculated value of $\alpha(t_i)$ is denoted by $\alpha_{nom}(t_i)$, then

$$\Delta\alpha_m(t_i) = \alpha_m(t_i) - \alpha_{nom}(t_i) \tag{208}$$

In calculating $\alpha_{nom}(t_i)$, the best current estimates of e, a, T, ω, and β are used. At time t=0, an initial estimate of these quantities is available.

From Eqs. (204) and (208),

$$\Delta\alpha_m(t_i) = \Delta\alpha(t_i) + v(t_i) \tag{209}$$

where

$$\Delta\alpha(t_i) = \alpha(t_i) - \alpha_{nom}(t_i) \tag{210}$$

If the error parameters

$$\begin{aligned} \Delta e_i &= e - e_i \\ \Delta a_i &= a - a_i \\ \Delta T_i &= T - T_i \\ \Delta\omega_i &= \omega - \omega_i \\ \Delta\beta_i &= \beta - \beta_i \end{aligned} \tag{211}$$

are small, then $\Delta\alpha(t_i)$ may be approximated by*

$$\Delta\alpha(t_i) = \left[\frac{\partial\alpha(t_i)}{\partial e}\right]\Delta e + \left[\frac{\partial\alpha(t_i)}{\partial a}\right]\Delta a + \left[\frac{\partial\alpha(t_i)}{\partial T}\right]\Delta T + \left[\frac{\partial\alpha(t_i)}{\partial\omega}\right]\Delta\omega + \left[\frac{\partial\alpha(t_i)}{\partial\beta}\right]\Delta\beta \tag{212}$$

which is obtained via a Taylor series expansion of (203) and dropping higher-order terms. In (211), e_i denotes the updated nominal value of e calculated after measurement at time t_i, and similarly for a_i, T_i, etc.

From (209) and (212),

$$z_i = M_i x_i + v_i \tag{213}$$

where

$$z_i \equiv z(t_i) \equiv \Delta\alpha_m(t_i)$$

$$v_i \equiv v(t_i)$$

$$M_i = \left[\frac{\partial\alpha(t_i)}{\partial e} \quad \frac{\partial\alpha(t_i)}{\partial a} \quad \cdots\cdots\cdots \quad \frac{\partial\alpha(t_i)}{\partial\beta}\right]$$

*The partial derivatives are evaluated using nominal parameter values.

a 1×5 matrix

$$x_i = \begin{bmatrix} \Delta e_i \\ \Delta a_i \\ \Delta T_i \\ \Delta \omega_i \\ \Delta \beta_i \end{bmatrix}$$

Eq. (213) is in the form of (105). In the companion equation (104), $w(t) \equiv 0$ in the present case, while $\Phi(t+1, t)$ is the unit matrix, since the state vector, x_i, is constant. Consequently, the associated estimation scheme, (131) - (133), reduces to

$$\hat{x}_{i+1} = \hat{x}_i + K_i [z_i - M_i \hat{x}_i] \tag{214}$$

$$K_i = P_i M_i^T [M_i P_i M_i^T + R_i]^{-1} \tag{215}$$

$$P_{i+1} = P_i - P_i M_i^T [M_i P_i M_i^T + R_i]^{-1} M_i P_i \tag{216}$$

where we have used the abbreviated notation. Here

$$R_i = \sigma^2$$

and we are given

$$P_0 = E[x_0 x_0^T]$$

and

$$E[x_0] = 0$$

The computational procedure is as follows. At t=0, we are given a set of nominal values for the parameters e, a, T, ω, and β. Using Eq. (203), we calculate $\alpha_{nom}(0)$. Furthermore, we have $\hat{x}_0 = 0$, and the given value of P_0. This permits us to calculate P_1 and $\hat{x}_1$ from Eqs. (214) - (216), which in turn yields a revised set of values for e, a, T, ω, and β. With these, we calculate $\alpha_{nom}(t_1)$, an improved estimate of $\alpha(t)$. The estimation scheme then proceeds recursively, obtaining more refined values of $\alpha(t)$ and the parameters e, a, T, ω, and β.

A more complete discussion of the problem and the results of some computer studies are contained in the paper by Knoll and Edelstein[41].

3.3.5 Use of Quasilinearization

In applying the Kalman estimation procedure, it is essential that the system under consideration be linear. When this is not the case, a linearizing procedure may be employed by the usual perturbation methods. Thus by considering only small deviations from a nominal trajectory, a linear system is obtained. This procedure was illustrated in the previous examples.

A somewhat less restrictive procedure is that of quasilinearization[43,44], which essentially replaces a nonlinear system by a sequence of linear equations, whose solution approaches that of the original nonlinear system. The main advantage of quasilinearization over perturbation methods is that a nominal or reference trajectory need not be specified a priori.

3.3.5.1 The Quasilinearization Method

Consider the vector differential equation

$$\dot{x} = f(x, \tau) \tag{217}$$

where x is an n dimensional state vector. The boundary conditions are

$$x_j(\tau_i) = a_{ij} \qquad i=1,2,\ldots\ldots,n \tag{218}$$

where the τ_j are, in general, not all equal; i.e., we are considering a multipoint boundary value problem.

Let $x^{(0)}(\tau)$ be an initial approximation to the solution of (217). The initial approximation may be any function that satisfies (218). Then the $(k+1)^{th}$ approximation is determined from the k^{th} approximation by the linear differential equation

$$\dot{x}^{(k+1)}(\tau) = F^{(k)}(\tau)\, x^{(k+1)}(\tau) + u^{(k)}(\tau) \tag{219}$$

where $F^{(k)}(\tau)$ is an $n \times n$ matrix whose ij^{th} component is

$$F_{ij}^{(k)}(\tau) = \frac{\partial f_i(x^{(k)}, \tau)}{\partial x_j} \qquad i,j=1,2,\ldots\ldots,n \tag{220}$$

and

$$u^{(k)}(\tau) = f(x^{(k)}, \tau) - F^{(k)}(\tau)\, x^{(k)}(\tau) \tag{221}$$

The maximum value of k should satisfy some error criterion, for example

$$|x^{(k+1)} - x^{(k)}| \leq \epsilon \tag{222}$$

where ϵ is a predetermined error vector.

If $x^{(k+1)}(\tau)$ is sufficiently close to $x^{(k)}(\tau)$, then the differential equation (219) is a sufficiently close approximation to the nonlinear equation (217).

Kalaba[43] showed that the sequence of functions, $x^{(0)}(\tau)$, $x^{(1)}(\tau)$, ..., $x^{(k+1)}(\tau)$, converges quadratically to $x(\tau)$, the solution of (217), if $f(x, \tau)$ is a strictly convex function, and the off-diagonal terms of the Jacobian of $f(x, \tau)$ are all positive.

The general solution of Eq. (219) is

$$x^{(k+1)}(\tau) = H^{(k+1)}(\tau)\, c^{(k+1)} + p^{(k+1)}(\tau) \tag{223}$$

where

$$\dot{H}^{(k+1)}(\tau) = F^{(k)}(\tau)\, H^{(k+1)}(\tau) \tag{224}$$

$$H^{(k+1)}(\tau_0) = I \text{ the unit matrix}$$

$$\dot{p}^{(k+1)}(\tau) = F^{(k)}(\tau)\, p^{(k+1)}(\tau) + u^{(k+1)}(\tau) \tag{225}$$

$$p^{(k+1)}(\tau_0) = 0$$

$$c^{(k+1)} \equiv \text{initial condition vector}$$

$$\tau \geq \tau_0$$

The n components of $c^{(k+1)}$ are found by writing the expression for the j^{th} component of $x^{(k+1)}(\tau)$; viz.,

$$x_j^{(k+1)}(\tau) = \sum_{q=1}^{n} H_{jq}^{(k+1)}(\tau)\, c_q^{(k+1)} + p_j^{(k+1)}(\tau)$$

Using the n boundary values given by (218), this may be written as

$$a_{ij} = \sum_{q=1}^{n} H_{jq}^{(k+1)}(\tau_i)\, c_q^{(k+1)} + p_j^{(k+1)}(\tau) \tag{226}$$

An application of this method to the solution of a simple nonlinear boundary value problem is contained in Appendix C.

The main disadvantage in the use of quasilinearization, aside from the fact that it is more complicated then perturbation methods, is the requirement that $f(x, \tau)$ be strictly convex and that the off-diagonal terms of the Jacobian of $f(x, \tau)$ be positive. However, these conditions are sufficient; they may not be necessary. Ohap and Stubberud[45] analyze a specific problem in orbit estimation where the quasilinearization method converges even though $f(x, \tau)$ do not satisfy the aforementioned conditions.

A detailed procedure for combining Kalman filtering and quasilinearization as a general estimation technique is outlined in the following section.

3.3.5.2 Combination with Kalman Filtering

The nonlinear system to be considered is given by Eq. (217), and its quasilinear equivalent is described by Eq. (219). We will first transform the latter into a linear difference equation as follows. For simplicity, we will drop the superscripts in the ensuing discussion. Letting $\tau = t$ and $t+1$ in Eq. (223), we have

$$\begin{aligned} x(t) &= H(t)\, c + p(t) \\ x(t+1) &= H(t+1)\, c + p(t+1) \end{aligned} \tag{227}$$

Eliminating c from these equations yields

$$x(t+1) = \Phi(t+1, t)\, x(t) + w(t) \tag{228}$$

where

$$\Phi(t+1, t) = H(t+1)\, H^{-1}(t) \tag{229}$$

$$w(t) = p(t+1) - \Phi(t+1, t)\, p(t) \tag{230}$$

Eq. (228) is identical to Eq. (104) except that $\Gamma(t+1, t) = I$ and $w(t)$ is a nonrandom vector in the present case. Consequently,

$$E[w(t)] = w(t)$$

and*

$$Q(t) = 0$$

In this special case, the optimal estimation equations (131) - (133) become

*See Eq. (101).

$$\hat{x}(t+1|t) = \Phi(t+1, t)\,\hat{x}(t|t-1) + K(t)\,[z(t) - M(t)\,\hat{x}(t|t-1)] + w(t) \tag{231}$$

$$K(t) = \Phi(t+1, t)\,P(t|t-1)\,M^{T}(t)\,[M(t)\,P(t|t-1)\,M^{T}(t) + R(t)]^{-1} \tag{232}$$

$$P(t+1|t) = \Phi(t+1, t)\,\{P(t|t-1) - P(t|t-1)\,M^{T}(t)\,[M(t)\,P(t|t-1)\,M^{T}(t) + R(t)]^{-1} M(t)\,P(t|t-1)\}\,\Phi^{T}(t+1, t) \tag{233}$$

The general procedure requires that a sufficient number of measurements be taken such that the n boundary values necessary for the iterative solution of (219) can be determined. The measurements are of the form analogous to Eq. (105); viz.,

$$z(t) = M(t)\,x(t) + v(t) \tag{234}$$

where $z(t)$ is an m vector.

In order to start the process, we take h measurements, $z(0), z(1), \ldots\ldots, z(h-1)$, where h is the first positive integer that is greater than the ratio n/m. This then yields hm ($\geq$n) scalar equations*, the first n of which are a set of simultaneous algebraic equations used to determine the initial vector c. The random vector $v(t)$ in (234) is assumed zero for the moment. In other words, the initial vector, c, is only approximate, since the noise effect has been discarded. Specifically, we have

$$z(0) = M(0)\,x(0)$$

$$z(1) = M(1)\,x(1)$$

$$\vdots$$

$$z(h-1) = M(h-1)\,x(h-1)$$

Replacing $x(t)$ by (223),

$$z(0) = M(0)\,H(0)\,c + M(0)\,p(0)$$

$$z(1) = M(1)\,H(1)\,c + M(1)\,p(1)$$

$$\vdots$$

$$z(h-1) = M(h-1)\,H(h-1)\,c + M(h-1)\,p(1)$$

*The (hm-n) excess equations are merely discarded, since they are redundant for present purposes.

These equations are solved for c. We now determine the (2h-1) optimal estimates $\hat{x}(1|0)$, $\hat{x}(2|1)$, ..., $\hat{x}(2h-1|2h-2)$, and the corresponding covariance matrices, $P(1|0)$, $P(2|1)$, ..., $P(2h-1|2h-2)$ from Eqs. (231) - (233), where P(0) is assumed given and the initial value of the state vector is taken as

$$\hat{x}(0) = x^{(k+1)}(0) = H^{(k+1)}(0)\, c + p^{(k+1)}(0) \tag{235}$$

We have included the superscripts to emphasize that the final iterated value of the quasilinearization is used; i.e., when the condition (222) has been satisfied. The quantities $\Phi(t+1, t)$ and w(t) in Eqs. (231) - (233) are calculated from Eqs. (229) and (230).

The procedure then continues as follows:

1. Compute the optimal estimates of the next set of observations, $\hat{z}(h|h-1)$, $\hat{z}(h+1|h)$, ..., $\hat{z}(2h-1|2h-2)$, where

$$\hat{z}(t+1|t) = M(t+1)\,\hat{x}(t+1|t) \tag{236}$$

2. Make the next set of (h-1) measurements, z(h), z(h+1), ..., z(2h-2).

3. Determine the prediction errors, $\tilde{z}(h|h-1)$, $\tilde{z}(h+1|h)$, ..., $\tilde{z}(2h-2|2h-3)$, where

$$\tilde{z}(t+1|t) = z(t+1) - \hat{z}(t+1|t) \tag{237}$$

4. Compute the (h-1) weighted observed quantities z*(h), z*(h+1), ..., z*(2h-2), where

$$z^*(t) = z(t) + \Lambda\tilde{z}(t|t-1) \tag{238}$$

The quantity Λ is an m × m weighting matrix to be discussed subsequently. These (h-1) values of z* together with $\hat{z}(h-1|h-2)$ are used to determine a new initial condition vector, c, for the quasilinearization of Eq. (217) over the interval $(h-1) \leq \tau \leq (3h-3)$.

5. Compute the (2h-1) optimal estimates $\hat{x}(2h|2h-1)$, $\hat{x}(2h+1|2h)$, ..., $\hat{x}(3h-2|3h-3)$, and the corresponding covariance matrices, $P(2h|2h-1)$, $P(2h+1|2h)$, ..., $P(3h-2|3h-3)$ using the new initial condition vector and the previously determined optimal estimates, $\hat{x}(2h-1|2h)$ and $P(2h-1|2h)$.

6. Continue the process by returning to step 1 and incrementing all of the arguments by (h-1).

The presence of the weighting matrix in Eq. (238) is necessary in order to ensure some degree of stability in the computations. For example, if $\Lambda = 0$, then the new

information provided by the optimal estimates is not fed back to help improve the accuracy of the computed transition matrices. If the transition matrices are computed using only estimates of the observations, (i.e., with $\Lambda = -I$) then the computations may diverge with time, since the estimate $\hat{z}(t+1|t)$ is really composed of two quantities; one is the true optimal estimate of $M(t+1)\,x(t+1)$, and the other is an error due to the fact that the computed transition matrix is not the true transition matrix. Hence the computational errors would be cumulative. It is sufficient to take $\Lambda = -\frac{1}{2}I$ to ensure computational stability.

The paper by Ohap and Stubberud[45] gives some numerical results for an orbit estimation problem.

3.4 STOCHASTIC EFFECTS IN SYSTEM DESIGN

The discussion of the Wiener and Kalman methods contained in Sections 3.2 and 3.3 does not exhaust the stochastic techniques available for control system application. Of the many design tools that have evolved in recent years for the rational design of control systems subject to stochastic effects, we have chosen two examples that are especially useful in aerospace applications. These are discussed in the following sections.

3.4.1 Minimization of Error

Whereas the Wiener method is normally concerned with the design of a physically realizable filter to minimize a mean-square-error criterion, one often has a fixed transfer function where only a few parameters may be varied to achieve this minimization. In one of the earliest practical applications of the Wiener approach, Phillips[47] considered a radar tracking device where certain "equalizer" parameters could be varied to achieve a minimum mean-square error. In order to describe the general approach, wherein the basic ideas are not drowned in a tidal wave of mathematics, we will consider a highly simplified version.

A message consisting of a signal, x(t), and additive noise, n(t), is applied at the input of a device having a transfer function, L(s). If the output signal is denoted by, y(t), then we have, in Laplace transform notation,

$$Y(s) = L(s)\,[X(s) + N(s)] \tag{239}$$

If we define the error by

$$e(t) = y(t) - x(t)$$

or, equivalently,

$$E(s) = Y(s) - X(s) \tag{240}$$

then it follows that

$$E(s) = [L(s) - 1]\, X(s) + L(s)\, N(s) \tag{241}$$

We are given the power spectral densities for the signal and noise as follows.

$$G_{xx}(\omega) = \frac{2\sigma^2 \beta}{\beta^2 \omega^2} \tag{242}$$

$$G_{nn}(\omega) = k^2 \tag{243}$$

where k, σ, and β are positive constants.

The transfer function, L(s) is of the form

$$L(s) = \frac{1}{Ts+1} \tag{244}$$

It is required to determine the value of T such that the mean-square error is minimized.

Using Eqs. (52), (53), and (60), we find that

$$E[e^2(t)] \equiv \overline{e^2} = \Gamma_{ee}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{ee}(\omega)\, d\omega \tag{245}$$

Also, via Eqs. (63) and (241),

$$G_{ee}(\omega) = |L(i\omega) - 1|^2 G_{xx}(\omega) + |L(i\omega)|^2 G_n(\omega) \tag{246}$$

Combining the last two equations, we have

$$\overline{e^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega) - 1|^2 G_{xx}(\omega)\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega)|^2 G_{nn}(\omega)\, d\omega \tag{247}$$

Taking the first integral on the right-hand side of this equation, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega) - 1|^2 G_{xx}(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} [L(i\omega) - 1][L(-i\omega) - 1]\, G_{xx}(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[-\frac{i\omega T}{1+i\omega T} \right] \left[\frac{i\omega T}{1-i\omega T} \right] \frac{2\sigma^2 \beta}{(\beta^2 + \omega^2)} \, d\omega$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2i\sigma^2 \beta T^2 \omega^2 \, d\omega}{(1+i\omega T)(\beta+i\omega)(1-i\omega T)(\beta-i\omega)}$$

This may be evaluated by the method given in Appendix D. The final result is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega) - 1|^2 G_{xx}(\omega) \, d\omega = \frac{\sigma^2 \beta T}{1+T\beta}$$

Similarly, the second integral in Eq. (247) is found to be

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega)|^2 G_{nn}(\omega) \, d\omega = \frac{k^2}{2T}$$

Therefore,

$$\overline{e^2} = \frac{\sigma^2 \beta T}{1+\beta T} + \frac{k^2}{2T} \tag{248}$$

In order that this expression be a minimum, we must have

$$T = \frac{k}{\sigma\sqrt{2\beta} - k\beta} \tag{249}$$

When more complex transfer functions are considered, it is often found that the value of a parameter that minimizes the mean-square error results in a system with unsatisfactory performance from a conventional point of view. This is generally in the form of a highly oscillatory response, since the mean-square-error criterion places a heavy weight on large errors. Consequently, results obtained by this analysis must be interpreted with due regard for other factors.

3.4.2 Wind Loads on a Launch Vehicle

The short-period dynamics of an autopilot-controlled launch vehicle in the pitch plane may be described by*

*These equations are derived in the monograph, "Short Period Dynamics," which constitutes part 1 of Vol. I in the present series.

$$mU(\dot{\alpha} - \dot{\theta}) = T_c \delta - L_\alpha (\alpha + \alpha_w) \tag{250}$$

$$I\ddot{\theta} = T_c \ell_c \delta + L_\alpha \ell_\alpha (\alpha + \alpha_w) \tag{251}$$

$$\delta = -K_A(\theta + k_R \dot{\theta}) \tag{252}$$

$$\alpha_w = -\frac{W_w}{U} \tag{253}$$

The vehicle geometry is shown in Fig. 20; the symbols have the following meaning.

I = vehicle moment of inertia

K_A = servoamplifier gain

K_R = rate gyro gain

ℓ_α = aerodynamic moment arm

ℓ_c = control thrust moment arm

L_α = aerodynamic load coefficient

m = vehicle mass

T_c = control thrust

U = forward velocity of vehicle

W_w = wind velocity

α = angle of attack

α_w = defined by Eq. (253)

δ = control thrust deflection angle

θ = pitch angle

For simplicity, all higher-order effects, such as bending, sloshing, engine inertia, and instrumentation dynamics, have been neglected. This is valid if one is interested mainly in crude approximations of vehicle response to wind loads. More specifically, we shall be concerned with the bending moments induced in the vehicle due to wind loads.

The bending moment at some station, j, (see Fig. 20) is given by

$$M_j = -L_\alpha \ell_j (\alpha + \alpha_w) - \sum_i m_i \ell_i U(\dot{\alpha} - \dot{\theta}) \tag{254}$$

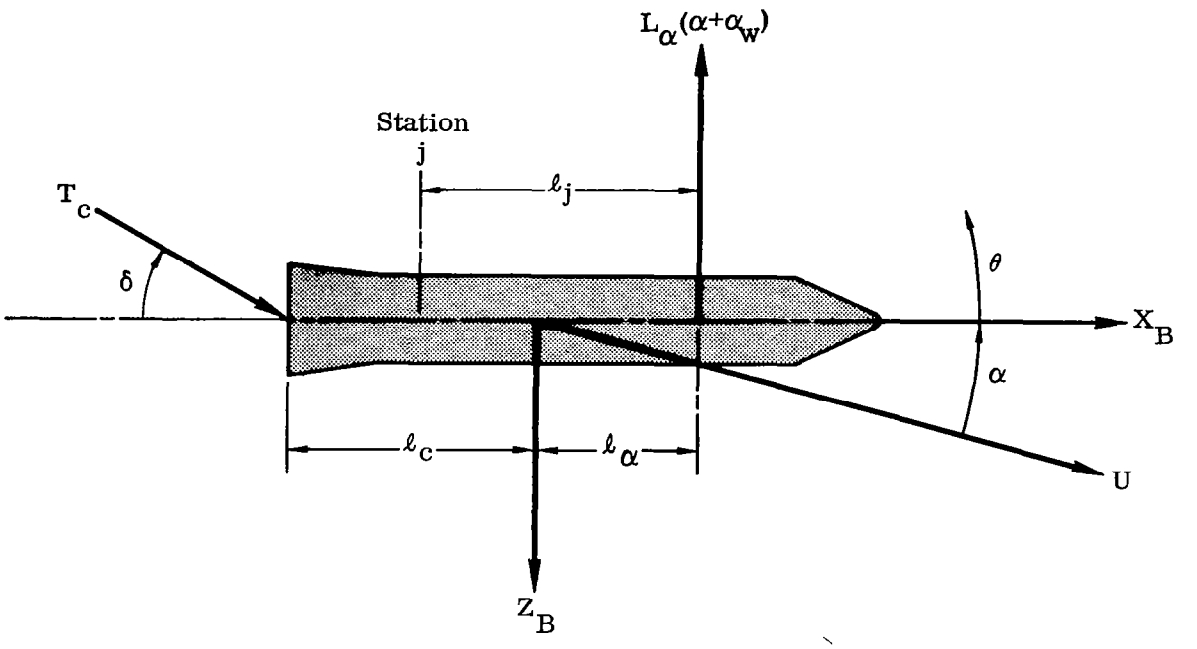


Figure 20. Vehicle Geometry

where ℓ_i is the distance from the i^{th} mass to station j, and the summation is taken over all discrete masses forward of station j.

By combining Eqs. (250) - (254), we obtain the transfer function relating the output $M_j(t)$ to wind input $W_w(t)$ as follows.

$$\mathcal{L}[M_j(t)] = G(s)\,\mathcal{L}[W_w(t)] \tag{255}$$

where

$$G(s) = \left(\frac{L_\alpha \ell_j}{U}\right)\frac{(s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)}{(s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0)} \tag{256}$$

and the a's and b's are known constants.

If the power spectral density of the wind is given, one may calculate the power spectral density of the bending moment response via Eq. (63). Such calculations have been made by Press and Houbolt[49] for aircraft. However, the implied assumption (other than system linearity) is that the wind statistics may be represented by a stationary time series. For a vertically rising launch vehicle, it is known that the wind statistics are decidedly nonstationary. Consequently, a somewhat expanded effort is required in order to obtain meaningful results.

To begin with, the basic quantity influencing the bending moment response is $W_w(t)$. In the present case this is assumed to be a random variable. Normally, W_w is taken to be a function of altitude (the wind profile). However, for a launch vehicle, the mission profile is known; we can therefore relate altitude to time (based on some initially prescribed time reference).

We now assume that the wind velocity satisfies a Gaussian probability density function; viz.,

$$f(x_1, x_2, \ldots\ldots, x_n) = [(2\pi)^n \det \Lambda(W_w)]^{-1/2} \exp \left\{ -\frac{1}{2} [\bar{x} - \bar{\mu}(W_w)]^T \Lambda^{-1}(W_w) \cdot [\bar{x} - \bar{\mu}(W_w)] \right\} \tag{257}$$

where

$$\Lambda(W_w) = E\{[\bar{W}_w - \bar{\mu}(W_w)][\bar{W}_w - \bar{\mu}(W_w)]^T\} \tag{258}$$

$$\bar{\mu}(W_w) = \begin{bmatrix} \mu_1(W_w) \\ \mu_2(W_w) \\ \vdots \\ \mu_n(W_w) \end{bmatrix} \qquad \bar{W}_w = \begin{bmatrix} W_w(t_1) \\ W_w(t_2) \\ \vdots \\ W_w(t_n) \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mu_i(W_w) = E[W_w(t_i)] \tag{259}$$

and the ij^{th} component of $\Lambda(W_w)$ is given by

$$\sigma_{ij}(W_w) = E\{[W_w(t_i) - \mu_i(W_w)][W_w(t_j) - \mu_j(W_w)]\} \tag{260}$$

This assumption is less restrictive than that of stationary wind statistics. It is, in fact, supported by certain empirical evidence. If the wind statistics are Gaussian, then so is the bending moment response. Consequently, one need only compute the mean and covariance in order to define the probability distribution of the bending moment response.

In order to compute the means, $\mu_i(W_w)$, and variances, $\sigma_{ij}(W_w)$, assume that N samples* of the wind profile are taken. As noted earlier, the wind profile is usually expressed as wind velocity vs altitude. However, with each value of altitude we will associate a corresponding value of time that is related to the specific mission profile.

We then calculate

$$\mu_i(W_w) = \frac{1}{N} \sum_{p=1}^{N} W_w^{(p)}(t_i) \tag{261}$$

and

$$\sigma_{ij}(W_w) = \frac{1}{N} \sum_{p=1}^{N} [W_w^{(p)}(t_i) - \mu_i(W_w)][W_w^{(p)}(t_j) - \mu_j(W_w)] \tag{262}$$

where $W_w^{(p)}(t_i)$ denotes a measurement of the p^{th} sample.

Strictly speaking, this procedure is not correct, since the true means and variances are given by

$$\mu_i(W_w) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1 f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n \tag{263}$$

and

$$\sigma_{ij}(W_w) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [x_i - \mu_i(W_w)][x_j - \mu_j(W_w)]\, f(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n \tag{264}$$

However, for large values of N, the sample means and variances calculated from Eqs. (261) and (262) will be a close approximation to the true values.

*N should be a fairly large number, perhaps a few hundred, for the results to be meaningful.

From Eq. (255), we find

$$\mu_r(M_j) \equiv E[M_j(t_r)] = \int_0^{t_r} g(t_r - \tau)\, \mu_\tau(W_w)\, d\tau \tag{265}$$

where

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

Eq. (265) gives the mean value of the bending moment at station j at time, t_r.

The bending moment variance is calculated from

$$\sigma_{rs}(M_j) = E\{[M_j(t_r) - \mu_r(M_j)][M_j(t_s) - \mu_s(M_j)]\} \tag{266}$$

In order to obtain this result in terms of wind variance, we proceed as follows.

$$M_j(t_r) - \mu_r(M_j) = \int_0^{t_r} g(t_r - \tau)\, W_w(\tau)\, d\tau$$

$$- \int_0^{t_r} g(t_r - \tau)\, \mu_\tau(W_w)\, d\tau$$

$$= \int_0^{t_r} g(t_r - \tau)\, [W_w(\tau) - \mu_\tau(W_w)]\, d\tau$$

Similarly,

$$M_j(t_s) - \mu_s(M_j) = \int_0^{t_s} g(t_s - \zeta)\, [W_w(\zeta) - \mu_\zeta(W_w)]\, d\zeta$$

Therefore

$$[M_j(t_r) - \mu_r(M_j)][M_j(t_s) - \mu_s(M_j)]$$

$$= \int_0^{t_r} g(t_r - \tau)\, [W_w(\tau) - \mu_\tau(W_w)]\, d\tau \int_0^{t_s} g(t_s - \zeta)\, [W_w(\zeta) - \mu_\zeta(W_w)]\, d\zeta$$

$$= \int_0^{t_r} \int_0^{t_s} g(t_r - \tau)\, g(t_s - \zeta)\, [W_w(\tau) - \mu_\tau(W_w)]\, [W_w(\zeta)$$

$$- \mu_\zeta(W_w)]\, d\tau\, d\zeta$$

Taking the expectation of both sides of this equation results in

$$\sigma_{rs}(M_j) = \int_0^{t_r} \int_0^{t_s} g(t_r - \tau)\, g(t_s - \zeta)\, \sigma_{\tau\zeta}(W_w)\, d\tau\, d\zeta \tag{267}$$

Thus the mean and variance of the bending moment response is given in terms of the mean and variance of the wind by Eqs. (265) and (267). Furthermore, since the wind statistics are Gaussian, so is the bending moment response; viz.,

$$f(y_1, y_2, \ldots\ldots, y_n) = [(2\pi)^n \det \Lambda(M_j)]^{-1/2} \exp\left\{-\frac{1}{2} [\bar{y} - \bar{\mu}(M_j)]^T \right.$$

$$\left. \cdot\, \Lambda^{-1}(M_j)\, [\bar{y} - \bar{\mu}(M_j)]\right\} \tag{268}$$

where

$$\Lambda(M_j) = E\left\{[\bar{M}_j - \bar{\mu}(M_j)]\, [\bar{M}_j - \bar{\mu}(M_j)]^T\right\}$$

$$\bar{M}_j = \begin{bmatrix} M_j(t_1) \\ M_j(t_2) \\ \vdots \\ M_j(t_n) \end{bmatrix} \qquad \bar{\mu}(M_j) = \begin{bmatrix} \mu_1(M_j) \\ \mu_2(M_j) \\ \vdots \\ \mu_n(M_j) \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Since there are structural design limits for any launch vehicle, it is essential that one calculate the probability of exceeding a given value of bending moment, M_j^*, at various values of station j. Using the previous results, we find that the probability of not exceeding the value M_j^* at any time of flight, $t_1, t_2, \ldots\ldots, t_n$, is given by

$$\int_{-\infty}^{M_j^*} \ldots\ldots \int_{-\infty}^{M_j^*} f(y_1, y_2, \ldots\ldots, y_n)\, dy_1\, dy_2 \ldots\ldots dy_n$$

If we are interested only in the probability that M_j^* is not exceeded at a particular time, t_r, then

$$\text{Prob}\left[M_j(t_r) \le M_j^*\right] = \int_{-\infty}^{M_j^*} \frac{1}{\sigma_{rr}(M_j)\sqrt{2\pi}} \exp\left\{-\frac{\left[y_r - \mu_r(M_j)\right]^2}{2\sigma_{rr}^2(M_j)}\right\} dy_r \qquad (269)$$

which is a substantially simpler computation. Quantities $\mu_r(M_j)$ and $\sigma_{rr}(M_j)$ are given by (265) and (267) respectively.

Let us write M_j^* in the form

$$M_j^* = \mu_r(M_j) + a\,\sigma_{rr}(M_j) \qquad (270)$$

where a is a positive constant.

Then if we make the change of variable

$$y_r = \sigma_{rr}(M_j)\, z_r + \mu_r(M_j) \qquad (271)$$

Eq. (269) becomes

$$\text{Prob}\left[M_j(t_r) \le M_j^*\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left(-\frac{z_r^2}{2}\right) dz_r \qquad (272)$$

This integral may be evaluated from standard tables (e.g., Ref. 2, p. 132). We find

$$\text{Prob}\left[M_j(t_r) \le M_j^*\right] = 0.841 \quad \text{if} \quad a = 1$$
$$= 0.999 \quad \text{if} \quad a = 3$$

For the case under consideration, $\mu_r(M_j)$ and $\sigma_{rr}(M_j)$ are available from Eqs. (265) and (267) respectively; therefore, for a given M_j^*, the appropriate value of a is obtained from (270), thus permitting the evaluation of the required probability from Eq. (272).

The scheme herein presented, while having a certain rational and intuitive appeal, has several drawbacks in practice. First of all, the available evidence that the wind statistics satisfy a Gaussian distribution is far from conclusive. Also, the validity of Eq. (265) is based on the assumption that the system under consideration is linear and stationary, which of course is not strictly true except for very short time intervals. Finally, the computations become extremely cumbersome for systems of only moderate order. The method is therefore useful only for obtaining crude estimates, which must be interpreted with caution.[58]

In actual practice, the complete nonlinear time-varying system dynamics are used to simulate the vehicle on a computer, and a large number of runs are made for different wind profiles. Time histories of critical parameters, such as bending moment, angle of attack, and engine deflection, then become available. Various criteria, such as not exceeding a limiting value of a parameter 95 or 99 percent of the time, are used to evaluate the system performance. A detailed discussion of these methods is contained in the monograph, "Response Studies," which is part 10 of Vol. III in the present series.

APPENDIX A

THE DELTA FUNCTION

The delta function is defined by the properties

$$\delta(x) = 0 \qquad x \neq 0 \tag{A1}$$

$$\int_a^b \delta(x)\, dx = 1 \qquad a < 0 < b \tag{A2}$$

This is more precise than the common engineering definition

$$\delta(x) = 0 \qquad x \neq 0$$

$$\delta(x) = \infty \qquad x = 0$$

The delta function is not a mathematical function in the strict sense. In all legitimate applications, this function is visualized as the result of a limiting process involving a function, $\delta(x, \epsilon)$, which satisfies the following conditions.

$$\delta(x, \epsilon) \geq 0 \qquad -\infty < x < \infty \qquad 0 < \epsilon < \infty \tag{A3}$$

$$\int_{-\infty}^{\infty} \delta(x, \epsilon)\, dx = 1 \qquad 0 < \epsilon < \infty \tag{A4}$$

$$\lim_{\epsilon \to 0} \delta(x, \epsilon) = 0 \qquad x \neq 0 \tag{A5}$$

Examples of such functions are

I. $\delta(x, \epsilon) = \dfrac{1}{2\epsilon} \qquad -\epsilon \leq x \leq \epsilon$

$= 0 \qquad$ otherwise

II. $\delta(x, \epsilon) = \dfrac{1}{\epsilon} \epsilon^{-x/\epsilon} \qquad x \geq 0$

$= 0 \qquad x < 0$

III. $$\delta(x, \epsilon) = \frac{1}{\epsilon\sqrt{\pi}} e^{-(x/\epsilon)^2}$$

IV. $$\delta(x, \epsilon) = \frac{\epsilon}{\pi} \frac{\sin^2(x/\epsilon)}{x^2}$$

The most useful relation involving delta functions is

$$\int_{-\infty}^{\infty} g(x)\, \delta(x - x_0)\, dx = g(x_0) \tag{A6}$$

where

$g(x)$ is continuous at $x = x_0$.

A rigorous treatment of the delta function is contained in Ref. 52.

APPENDIX B

THE WEIGHTING FUNCTION

The response of a linear filter to a unit impulse (delta function) applied at time t = 0 is called the weighting function. It follows that h(t) = 0 for t < 0 in the case of physically realizable filters. The response to an impulse input at time t_1, $\delta (t - t_1)$, is given by $h (t - t_1)$.

In order to develop an expression for the output, $f_0 (t)$, in terms of the weighting function and an arbitrary input, $f_i (t)$, one may proceed as follows.

Let the input signal, $f_i (t)$, have the form shown in Fig. B1. The response to a differential input, $f_i (t_1) \delta (t - t_1) \Delta t_1$, is given by $f_i (t_1) h (t - t_1) \Delta t_1$. By virtue of linearity, the response to an arbitrary input, $f_i (t)$, may be approximated by

$$f_0 (t) = \sum_n f_i (t_n) h (t - t_n) \Delta t_n \tag{B1}$$

where the input signal is approximated by the sequence of impulses depicted in Fig. B1.

In the limit, as $\Delta t_n \to 0$, we obtain

$$f_0 (t) = \int_0^t h (t - \tau_1) f_i (\tau_1) d\tau_1 \tag{B2}$$

An alternate derivation of this expression may be obtained via Laplace transform theory. If $F_i (s)$ and $F_0 (s)$ denote the Laplace transforms of $f_i (t)$ and $f_0 (t)$ respectively, then

$$F_0(s) = H(s) F_i(s) \tag{B3}$$

where H (s) is the transfer function relating the output to the input.

Applying the complex multiplication theorem* to Eq. (B3) yields

$$f_0 (t) \equiv \mathfrak{L}^{-1} \left[F_0 (s) \right] = \int_0^t h (t - \tau_1) f_i (\tau_1) d\tau_1 \tag{B4}$$

*Ref. 46 p. 228.

Via the change of variables, $\tau = t - \tau_1$, we find

$$f_0(t) = \int_0^t h(\tau)\, f_i(t - \tau)\, d\tau \tag{B5}$$

It is sometimes convenient to write this as

$$f_0(t) = \int_0^\infty h(\tau)\, f_i(t - \tau)\, d\tau \tag{B6}$$

which has the same value as (B5), since

$$f_i(\tau_1) = 0 \quad \text{for} \quad \tau_1 < 0.$$

The weighting function may therefore be interpreted as the inverse Laplace transform of the transfer function.

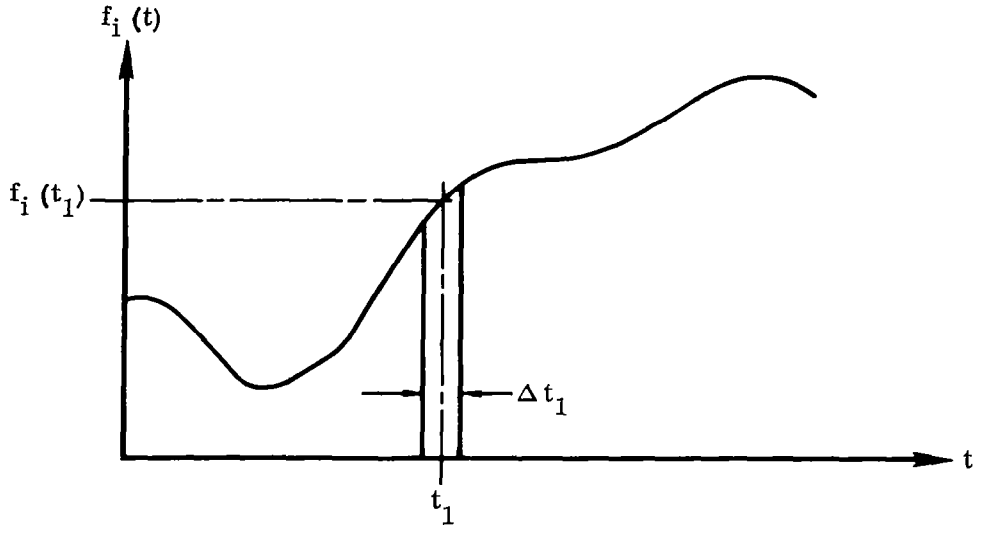


Figure B1. Form of Arbitrary Input Signal

APPENDIX C

A SIMPLE APPLICATION OF QUASILINEARIZATION

In order to clarify the basic ideas of the quasilinearization method discussed in Sec. 3.3.5.1, we consider the following nonlinear equation.

$$\ddot{x}_1 - e^{x_1} = 0 \tag{C1}$$

with the boundary conditions

$$x_1(0) = x_1(1) = 0 \tag{C2}$$

According to the discussion of Sec. 3.3.5.1, we may write this as

$$\begin{aligned} \dot{x}_1 &= x_2 \equiv f_1 \\ \dot{x}_2 &= e^{x_1} \equiv f_2 \end{aligned} \tag{C3}$$

with

$$\begin{aligned} x_1(0) &= 0 \equiv a_{01} \\ x_1(1) &= 0 \equiv a_{11} \end{aligned} \tag{C4}$$

From Eq. (220).

$$\begin{aligned} F_{11}^{(k)}(\tau) &= 0 & F_{12}^{(k)}(\tau) &= 1 \\ F_{21}^{(k)}(\tau) &= e^{x_1^{(k)}} & F_{22}^{(k)}(\tau) &= 0 \end{aligned} \tag{C5}$$

Therefore, from (221),

$$u_1^{(k)}(\tau) = 0$$

$$u_2^{(k)}(\tau) = (1 - x_1^{(k)})\, e^{x_1^{(k)}} \tag{C5}$$

As an initial approximation, we assume

$$x^{(0)}(\tau) = \begin{bmatrix} x_1^{(0)}(\tau) \\ x_2^{(0)}(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C6}$$

Then

$$F_{11}^{(0)}(\tau) = 0 \qquad F_{12}^{(0)}(\tau) = 1$$

$$F_{21}^{(0)}(\tau) = 1 \qquad F_{22}^{(0)}(\tau) = 0$$

From Eq. (224)

$$\dot{H}_{11}^{(1)}(\tau) = H_{21}^{(1)}(\tau)$$

$$\dot{H}_{12}^{(1)}(\tau) = H_{22}^{(1)}(\tau)$$

$$\dot{H}_{21}^{(1)}(\tau) = H_{11}^{(1)}(\tau)$$

$$\dot{H}_{22}^{(1)}(\tau) = H_{12}^{(1)}(\tau)$$

$$H_{11}^{(1)}(0) = H_{22}^{(1)}(0) = 1$$

$$H_{12}^{(1)}(0) = H_{21}^{(1)}(0) = 0$$

The solution of this system is given by

$$H_{11}^{(1)}(\tau) = H_{22}^{(1)}(\tau) = \frac{1}{2}\left(e^{\tau} + e^{-\tau}\right)$$
$$H_{12}^{(1)}(\tau) = H_{21}^{(1)}(\tau) = \frac{1}{2}\left(e^{\tau} - e^{-\tau}\right) \qquad \text{(C7)}$$

Also, from Eqs. (225), (C5), and (C6),

$$\dot{p}_1^{(1)}(\tau) = p_2^{(1)}$$

$$\dot{p}_2^{(1)}(\tau) = p_1^{(1)}(\tau) + 1$$

$$p_1^{(1)}(0) = p_2^{(1)}(0) = 0$$

The solution to this set of equations is given by

$$p_1^{(1)}(\tau) = \frac{1}{2}\left(e^{\tau} + e^{-\tau}\right) - 1$$
$$p_2^{(1)}(\tau) = \frac{1}{2}\left(e^{\tau} - e^{-\tau}\right) \qquad \text{(C8)}$$

The first component of the vector equation (223) is

$$x_1^{(1)}(\tau) = H_{11}^{(1)}(\tau)\, c_1^{(1)} + H_{12}^{(1)}(\tau)\, c_2^{(1)} + p_1^{(1)}(\tau)$$

Substituting known values for $\tau = 0$ and $\tau = 1$ yields two equations for the two unknowns $c_1^{(1)}$ and $c_2^{(1)}$. These turn out to be $c_1^{(1)} = 0$ and $c_2^{(1)} = -0.46212$. Therefore the first quasilinear approximation to Eq. (C1) is

$$x_1^{(1)}(\tau) = -0.46212\, H_{12}^{(1)}(\tau) + p_1^{(1)}(\tau) \qquad \text{(C9)}$$

where $H_{12}^{(1)}(\tau)$ and $p_1^{(1)}(\tau)$ are given by Eqs. (C7) and (C8) respectively.

The calculations of successive iterations is straightforward. The following table summarizes the results obtained after two iterations.

τ	$x_1^{(0)}$	$x_1^{(1)}$	$x_1^{(2)}$	x_1 actual
0	0	0	0	0
0.1	0	-0.04128	-0.04144	-0.04144
0.2	0	-0.07297	-0.07327	-0.07327
0.3	0	-0.09539	-0.09580	-0.09580
0.4	0	-0.10874	-0.10924	-0.10924
0.5	0	-0.11318	-0.11370	-0.11370
0.6	0	-0.10874	-0.10924	-0.10924
0.7	0	-0.09539	-0.09580	-0.09580
0.8	0	-0.07297	-0.07327	-0.07327
0.9	0	-0.04128	-0.04144	-0.04144
1.0	0	0	0	0

The last column is calculated from the closed-form solution of Eq. (C1), which is known to be

$$x(\tau) = -\ln 2 + 2 \ln \left\{ \alpha \sec \left[0.5\, \alpha\, (\tau - 0.5) \right] \right\}$$

where α is the root of

$$\alpha \sec (0.25\alpha) - \sqrt{2} = 0$$

or $\alpha = 1.33606$

APPENDIX D

THE PHILLIPS INTEGRAL

In calculating mean-square error in the manner of that in Sec. 3.4.1, one is required to evaluate integrals of the form

$$I_n = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g_n(x)}{h_n(x)\, h_n(-x)}\, dx \tag{D1}$$

where

$$h_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots\cdots\cdots + a_n \tag{D2}$$

$$g_n(x) = b_0 x^{2n-2} + b_1 x^{2n-4} + \cdots\cdots\cdots + b_{n-1} \tag{D3}$$

$$i \equiv \sqrt{-1}$$

Note that the order of the numerator is at least two less than the order of the denominator. Also, it is required that the roots of h (x) be in the upper half plane. Any function, f (x), which contains only even powers of x can be factored in the required form, since if x_0 is a root of f (x), so is $(-x_0)$.

The value of the integral of Eq. (D1) is tabulated in Ref. 47 for n = 1, 2, ... , 7.

The values of I_n for n = 1, 2, 3, 4 are given below.

$$I_1 = \frac{b_0}{2\, a_0\, a_1}$$

$$I_2 = \frac{-b_0 + \left(\frac{a_0\, b_1}{a_2}\right)}{2\, a_0\, a_1}$$

$$I_3 = \frac{-a_2 b_0 + a_0 b_1 - \left(\frac{a_0 a_1 b_2}{a_3}\right)}{2 a_0 (a_0 a_3 - a_1 a_2)}$$

$$I_4 = \frac{b_0 (a_2 a_3 - a_1 a_4) - a_0 a_3 b_1 + a_0 a_1 b_2 + (a_0 a_3 - a_1 a_2) \left(\frac{a_0 b_3}{a_4}\right)}{2 a_0 (a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}$$

The evaluation of the Phillips integral is based on Parseval's theorem. See Ref. 56 for a more complete treatment.

APPENDIX E

KALMAN'S FILTER AND PREDICTION EQUATIONS

As noted in Sec. 3.3, Kalman's derivation of the optimal filter equations makes use of some sophisticated mathematical concepts. In order to exhibit the basic ideas in Kalman's development of the main results, we shall outline the essential steps of his derivation in simplified form. In doing this, we will use the following properties of Gaussian random vectors.

Let x_1 and x_2 be Gaussian random vectors with

Mean: $\mu_1 = E(x_1)$

$\mu_2 = E(x_2)$

Covariance:

$$\sigma_{11} = E\left[(x_1 - \mu_1)(x_1 - \mu_1)^T\right] \equiv \text{cov}(x_1)$$

$$\sigma_{22} = E\left[(x_2 - \mu_2)(x_2 - \mu_2)^T\right] \equiv \text{cov}(x_2)$$

$$\sigma_{12} = E\left[(x_1 - \mu_1)(x_2 - \mu_2)^T\right] \equiv \text{cov}(x_1, x_2)$$

Then

$$E(x_1 \mid x_2) = \mu_1 + \sigma_{12}\sigma_{22}^{-1}(x_2 - \mu_2) \tag{E1}$$

If we let

$$\hat{x}_1 = E(x_1 \mid x_2)$$

$$\tilde{x} = x_1 - \hat{x}_1$$

Then

$$E(\tilde{x}\,\hat{x}^T) = 0 \tag{E2}$$

Finally, if x_1, x_2, and x_3 are Gaussian random vectors, and the pair x_2, x_3 are independent, then

$$E(x_1 | x_2, x_3) = E(x_1 | x_2) + E(x_1 | x_2) \tag{E3}$$

Properties (E1), (E2), and (E3) are derived in Ref. 59.

We now consider the discrete form of the optimal filter problem as stated on p. 44 and repeated here for convenience. Given the dynamic system

$$x(t+1) = \Phi(t+1, t)\, x(t) + \Gamma(t+1, t)\, w(t) \tag{E4}$$

$$z(t) = M(t)\, x(t) + v(t) \tag{E5}$$

The random vectors w (t) and v (t) are Gaussian with means and covariance given by

$$E\left[w(t)\right] = E\left[v(t)\right] = 0 \tag{E6}$$

$$E\left[w(t)\, w^T(\tau)\right] \equiv \operatorname{cov}\left[w(t)\right] = Q(t) \qquad t < \tau \leq t+1$$
$$= 0 \qquad \text{otherwise} \tag{E7}$$

$$E\left[v(t)\, v^T(\tau)\right] \equiv \operatorname{cov}\left[v(t)\right] = R(t) \qquad t < \tau \leq t+1$$
$$= 0 \qquad \text{otherwise} \tag{E8}$$

$$E\left[v(t)\, w^T(\tau)\right] = 0$$

We are given the observed values, z (0), z (1), ... , z (t). Kalman shows that the optimal estimate of x (t + 1), given that z (0), z (1), ... , z (t) have occurred, is merely the conditional mean

$$\hat{x}(t+1|t) \equiv E\left[x(t+1) | z(0), z(1), \ldots\ldots, z(t)\right]$$
$$\equiv E\left[x(t+1) | \mathcal{Z}(t)\right] \tag{E9}$$

The problem is then reduced to that of calculating (E9). For this purpose, it is assumed that x (0) is a Gaussian random vector of zero mean with a known convariance. It then follows that x (0), x (1), ... , x (t) is a sequence of Gaussian random vectors with zero mean. Furthermore, it follows from Eq. (E5) that z (0), z (1), ... , z (t) is also a sequence of Gaussian random variables with zero mean.

It is convenient to adopt the following notation, in a manner similar to (E9).

$$\hat{z}(t+1|t) \equiv E\left[z(t+1)|\mathcal{Z}(t)\right] \tag{E10}$$

$$\tilde{x}(t+1|t) = x(t+1) - \hat{x}(t+1|t) \tag{E11}$$

and similarly for $\tilde{z}(t+1|t)$, etc.

The fundamental problem,as previously noted, involves the computation of the conditional expectation (E9), which,because of the Gaussian statistical properties of the system, may be expressed in terms of the given means and variances. To begin with, we note that (E9) may be expressed as

$$\hat{x}(t+1|t) = E\left[x(t+1) \mid z(t), \mathcal{Z}(t-1)\right] \tag{E12}$$

But

$$z(t) = \tilde{z}(t \mid t-1 + \hat{z}(t \mid t-1) \tag{E13}$$

by definition, while by (E1), $\hat{z}(t \mid t-1)$ is a linear function of $\mathcal{Z}(t-1)$. It follows therefore that Eq. (E12) may be written as

$$\hat{x}(t+1 \mid t) = E\left[x(t+1) \mid \tilde{z}(t \mid t-1), \mathcal{Z}(t-1)\right] \tag{E14}$$

Furthermore $\tilde{z}(t \mid t-1)$ and $\mathcal{Z}(t-1)$ are independent random vectors by (E2). Consequently, via (E3), we obtain

$$\begin{aligned}\hat{x}(t+1 \mid t) &= E\left[x(t+1) \mid \tilde{z}(t \mid t-1)\right] + E\left[x(t+1) \mid \mathcal{Z}(t-1)\right] \\ &= E\left[x(t+1) \mid \tilde{z}(t \mid t-1)\right] + \hat{x}(t+1 \mid t-1)\end{aligned} \tag{E15}$$

From (E5), we have

$$\begin{aligned}\hat{z}(t \mid t-1) &= M(t)\,\hat{x}(t \mid t-1) + \hat{v}(t \mid t-1) \\ &= M(t)\,\hat{x}(t \mid t-1)\end{aligned} \tag{E16}$$

since v (t) is uncorrelated for successive time instants. Substituting this in (E13) yields

$$\tilde{z}(t \mid t-1) = z(t) - M(t)\,\hat{x}(t \mid t-1)$$

$$= M(t)\,x(t) + v(t) - M(t)\,\hat{x}(t \mid t-1)$$

$$= M(t)\,\tilde{x}(t \mid t-1) + v(t) \qquad \text{(E17)}$$

Also from (E4),

$$\hat{x}(t+1 \mid t-1) = \phi(t+1,t)\,\hat{x}(t \mid t-1) + \Gamma(t+1,t)\,\hat{w}(t \mid t-1)$$

$$= \phi(t+1,t)\,\hat{x}(t \mid t-1)$$

which becomes

$$\hat{x}(t+1 \mid t) = \phi(t+1,t)\,\hat{x}(t \mid t-1) + E\left[x(t+1) \mid \tilde{z}(t \mid t-1)\right]$$

via (E15). Applying (E1), this may be written as

$$\hat{x}(t+1 \mid t) = \phi(t+1,t)\,\hat{x}(t \mid t-1) + \left\{\operatorname{cov}\left[x(t+1),\ \tilde{z}(t \mid t-1)\right]\right\}$$
$$\times \left\{\operatorname{cov}\left[\tilde{z}(t \mid t-1)\right]\right\}^{-1} \tilde{z}(t \mid t-1) \qquad \text{(E18)}$$

Also,

$$\operatorname{cov}\left[\tilde{z}(t \mid t-1)\right] = E\left\{\left[\tilde{z}(t \mid t-1)\right]\left[\tilde{z}(t \mid t-1)\right]^T\right\}$$
$$= E\left\{\left[M(t)\,\tilde{x}(t \mid t-1) + v(t)\right]\left[\tilde{x}^T(t \mid t-1)\,M^T(t) + v^T(t)\right]\right\}$$
$$= E\left\{M(t)\,\tilde{x}(t \mid t-1)\,\tilde{x}^T(t \mid t-1)\,M^T(t) + v(t)\,v^T(t)\right\}$$

since x (t) and v (t) are independent.

Defining

$$P(t \mid t-1) = E\left[\tilde{x}(t \mid t-1)\,\tilde{x}^T(t \mid t-1)\right] \qquad \text{(E19)}$$

we have

$$\operatorname{cov}\left[\tilde{z}(t \mid t-1)\right] = M(t)\,P(t \mid t-1)\,M^T(t) + R(t) \qquad \text{(E20)}$$

Finally

$$\begin{aligned}
\text{cov}\left[x(t+1), \tilde{z}(t \mid t-1)\right] &= E\Big\{\left[\phi(t+1,t)\,x(t) + \Gamma(t+1,t)\,w(t)\right] \\
&\quad x\left[M(t)\,\tilde{x}(t \mid t-1) + v(t)\right]^T\Big\} \\
&= E\left\{\left[\phi(t+1,t)\,x(t)\right]\left[M(t)\,\tilde{x}(t \mid t-1)\right]^T\right\} \\
&= E\left\{\left[\phi(t+1,t)\,\tilde{x}(t \mid t-1) + \phi(t+1,t)\,\hat{x}(t \mid t-1)\right]\right. \\
&\quad \left. x\left[M(t)\,\tilde{x}(t \mid t-1)\right]^T\right\} \\
&= E\left[\phi(t+1,t)\,\tilde{x}(t \mid t-1)\,\tilde{x}^T(t \mid t-1)\,M^T(t)\right] \\
&= \phi(t+1,t)\,P(t \mid t-1)\,M^T(t) \qquad \text{(E21)}
\end{aligned}$$

Here we have used the fact that

a) $x(t)$, $w(t)$, and $v(t)$ are all independent of each other.

b) $\tilde{x}(t \mid t-1)$ and $\hat{x}(t \mid t-1)$ are independent of each other by (E2).

Substituting (E20) and (E21) in (E18), and making use of (E17) yields

$$\hat{x}(t+1 \mid t) = \phi(t+1,\,t)\,\hat{x}(t \mid t-1) + K(t)\left[z(t) - M(t)\,\hat{x}(t \mid t-1)\right] \qquad \text{(E22)}$$

$$K(t) = \phi(t+1,\,t)\,P(t \mid t-1)\,M^T(t)\left[M(t)\,P(t \mid t-1)\,M^T(t) + R(t)\right]^{-1} \qquad \text{(E23)}$$

while

$$\begin{aligned}
P(t+1 \mid t) &= \text{cov}\left[\tilde{x}(t+1 \mid t)\right] = \text{cov}\left[x(t+1) - \hat{x}(t+1 \mid t)\right] \\
&= \text{cov}\left[x(t+1) - \phi(t+1,t)\,\hat{x}(t \mid t-1) - K(t)\,\tilde{z}(t \mid t-1)\right]
\end{aligned}$$

using (E22) and (E17).

Expanding this:

$$\begin{aligned}
P(t+1 \mid t) &= \text{cov}\left[\phi(t+1,\,t)\,x(t) + \Gamma(t+1,\,t)\,w(t) - \phi(t+1,\,t)\,\hat{x}(t \mid t-1)\right. \\
&\qquad \left. - K(t)\,\tilde{z}(t \mid t-1)\right] \\
&= \text{cov}\left[\phi(t+1,\,t)\,\tilde{x}(t \mid t-1) + \phi(t+1,\,t)\,\hat{x}(t \mid t-1)\right. \\
&\qquad \left. + \Gamma(t+1,\,t)\,w(t) - \phi(t+1,\,t)\,\hat{x}(t \mid t-1) - K(t)\,\tilde{z}(t \mid t-1)\right]
\end{aligned}$$

$$= \text{cov} \left\{ \left[\phi (t+1, t) - K(t) M(t) \right] \tilde{x} (t \mid t-1) + \Gamma (t+1, t) w(t) - K(t) v(t) \right\}$$

via (E17). Performing the expectation operation, we have

$$P(t+1 \mid t) = \left[\phi (t+1, t) - K(t) M(t) \right] P(t \mid t-1) \left[\phi (t+1, t) - K(t) M(t) \right]^T + \Gamma (t+1, t) Q(t) \Gamma^T (t+1, t) + K(t) R(t) K^T (t)$$

using the fact that x (t), w (t), and v (t) are all independent of each other.

Rearranging, simplifying, and eliminating K (t) via (E23), we obtain

$$P(t+1 \mid t) = \phi (t+1, t) \left\{ P(t \mid t-1) - P(t \mid t-1) M^T (t) \left[M(t) \times P(t \mid t-1) M^T (t) + R(t) \right]^{-1} M(t) P(t \mid t-1) \right\} \phi^T (t+1, t) + \Gamma (t+1, t) Q(t) \Gamma^T (t+1, t) \qquad \text{(E24)}$$

The solution to the problem of obtaining the optimal filter is embodied in Equations (E22) - (E24), which requires that the initial value of the covariance matrix $P(t_0 \mid t_0 - 1)$ be given. Here obviously, $P(t_0 \mid t_0 - 1) \equiv P(t_0)$.

REFERENCES

1. Doob, J. L. — Stochastic Processes, John Wiley & Sons, Inc., New York, 1953.

2. Feller, W. — An Introduction to Probability Theory and Its Applications, John Wiley & Sons, Inc., New York, 1950.

3. Cramer, H. — Mathematical Methods of Statistics, Princeton Univ. Press, Princeton, N. J., 1946.

4. Titchmarch, E. C. — Introduction to the Theory of Fourier Integrals, Oxford Univ. Press, 1937.

5. Laning, J. H. and Battin, R. H. — Random Processes in Automatic Control, McGraw Hill Book Co., Inc., New York, 1956.

6. Wiener, N. — The Extrapolation, Interpolation, and Smoothing of Stationary Time Series, John Wiley & Sons, Inc., New York, 1949.

7. Zadeh, L. and Ragazzini, J. — "An Extension of Wiener's Theory of Prediction," J. Appl. Phys., Vol. 21, 1950, p. 645.

8. Booton, R. C. — "An Optimization Theory for Time Varying Linear Systems with Nonstationary Statistical Inputs," Proc. IRE, Vol. 40, 1952, p. 977.

9. Zadeh, L. — "Optimum Nonlinear Filters," J. Appl. Phys., Vol. 24, 1953, p. 396.

10. Davis, R. C. — "On the Theory of Prediction of Nonstationary Stochastic Processes," J. Appl. Phys., Vol. 23, 1952, p. 1047.

11. Dolph, C. L. and Woodbury, M. A. — "On the Relation Between Green's Functions and Covariances of Certain Stochastic Processes and Its Application to Unbiased Linear Prediction," Trans. Am. Math. Soc., Vol. 72, 1952, p. 519.

12. Bendat, J. S. — "A General Theory for Linear Prediction and Filtering," J. Soc. Ind. & Appl. Math., Vol. 4, 1956, p. 131.

13. Kalman, R. — "A New Approach to Linear Filtering and Prediction Problems," J. Basic Eng., Vol. 82, part D, 1960, p. 35.

14. Kalman, R. and Bucy, R. — "New Results in Linear Filtering and Prediction Theory," J. Basic Eng., Vol. 83, part D, 1961, p. 95.

15. Greensite, A. — Design Criteria for Control of Space Vehicles; Vol. II, part 2, Nonlinear Systems, General Dynamics Convair Report GDC-DDE65-056, 1 Sept. 1965.

16. Sherman, S. — "Non Mean Square Error Criteria," Trans. IRE, IT-4, 1958, p. 125.

17. Davenport, W. B. and Root, W. L. — An Introduction to the Theory of Random Signals and Noise, McGraw Hill Book Co., Inc., New York, 1958.

18. Tou, J. T. — Modern Control Theory, McGraw Hill Book Co., Inc., New York, 1964.

19. Mood, A. M. — Introduction to the Theory of Statistics, McGraw Hill Book Co., Inc., New York, 1950.

20. Levin, M. J. — "Optimum Estimation of Impulse Response in the Presence of Noise," IRE Trans. on Circuit Theory, 1960, p. 50.

21. Bellman, R. — Dynamic Programming, Princeton Univ. Press, Princeton, N. J., 1957.

22. Bodewig, E. — Matrix Calculus, Interscience Pub. Co., New York, N. Y., 1959, 2nd Ed.

23. Greensite, A. — Optimization Theory for Linear Stochastic Control Systems, General Dynamics Convair Report ERR-AN-427, 15 Nov. 1963.

24. Robbins, H. and Monro, S. — "A Stochastic Approximation Model," Ann. Math. Stat., Vol. 22, 1951, p. 400.

25. Blum, J. — "Multidimensional Stochastic Approximation Method," Ann. Math. Stat., Vol. 25, 1954, p. 737.

26. Greensite, A. — "Dynamic Programming and the Kalman Filter Theory," Proc. Nat'l. Elec. Conf., Vol. 20, Chicago, Ill., 1964, p. 601.

27. Kalman, R. — "On the General Theory of Control Systems," Proc. 1st Int. Cong. on Automatic Contr., Butterworth Scientific Publications, London, Vol. 1, 1961, p. 481.

28. Bendat, J. S. — "Optimum Filters for Independent Measurements of Two Related Perturbed Messages," Trans. IRE, Vol. CT-4, March 1957, p. 14.

29. Bryson, A. E. and Johansen, D. E. — "Linear Filtering for Time Varying Systems Using Measurements Containing Colored Noise," IEEE Trans. on Automatic Control, Jan. 1965, p. 4.

30. Smith, G. L., Schmidt, S. F. and McGee, L. A. — Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity on Board a Circumlunar Vehicle, NASA TR R-135, 1962.

31. Schmidt, S. F. — "State Space Techniques Applied to the Design of a Space Navigation System," Proc. Joint Automatic Control Conf., New York, 1962, paper No. 11-3.

32. Farrell, J. L. — "Simulation of a Minimum Variance Orbital Navigation System," Proc. AIAA/ION Third Manned Space Flight Meeting, Houston, Texas, 1964, p. 46.

33. Wilcox, J. C. — "Self Contained Orbital Navigation Systems with Correlated Measurement Errors," Proc. AIAA/ION Guidance & Contr. Conf., Minneapolis, Minn., 1965, p. 231.

34. Bona, B. E., and Hutchinson, C. E. — "An Optimum Stellar-Inertial Navigation System," J. of Inst. of Navigation, Vol. 12, No. 2, 1965, p. 171.

35. Gunkel, T. L. "Orbit Determination Using Kalman's Method," J. of Inst. of Navigation, Vol. 10, No. 3, 1963, p. 273.

36. Knoll, A. L., and Edelstein, M. M. "Estimation of Local Vertical and Orbital Parameters for an Earth Satellite Using Horizon Sensor Measurements," AIAA Journ. Vol. 3, No. 2, 1965, p. 338.

37. Baker, R. M., and Makemson, M. W. An Introduction to Astrodynamics, Academic Press, Inc., New York, 1960.

38. Soong, T. T. "On A Priori Statistics in Minimum Variance Estimation Problems," Proc. Joint Automatic Control Conf., Stanford, Calif., 1964, p. 338.

39. Nishimura, T. "On the A Priori Information in Sequential Estimation Problems," Proc. Nat'l. Elec. Conf., Chicago, Ill., 1965, p. 511.

40. Leondes, C. T. Guidance and Control of Aerospace Vehicles, McGraw Hill Book Co., Inc., New York, 1963.

41. Knoll, A. L., and Edelstein, M. M. "Estimation of Local Vertical and Orbital Parameters for An Earth Satellite Using Horizon Sensor Measurements," AIAA Journ. , Vol. 3, No. 2, 1965, p. 338.

42. Brouwer, D., and Clemence, B. Methods of Celestial Mechanics, Academic Press, Inc., New York, N. Y., 1961.

43. Kalaba, R. "On Nonlinear Differential Equations; The Maximum Operation and Monotone Convergence," J. Math., Mech., Vol. 8, 1959, p. 519.

44. Bellman, R., Kagiwada, H., and Kalaba, R. "Orbit Determination as a Multipoint Boundary Value Problem and Quasilinearization," Proc. Nat. Acad. Sci., Vol. 48, 1962, p. 1327.

45. Ohap, R. F., and Stubberud, A. R. "A Technique for Estimating the State of a Nonlinear System," IEEE Trans. on Automatic Control, 1965, p. 150.

46. Gardner, M. F., and Barnes, J. L. — Transients in Linear Systems, John Wiley & Sons, Inc., New York, 1942.

47. James, H. M., Nichols, N. B., and Phillips, R. S. — Theory of Servomechanisms, McGraw Hill Book Co., Inc., New York, 1947.

48. Lee, Y. W. — Statistical Theory of Communication, John Wiley & Sons, Inc., New York, 1960.

49. Press, H., and Houbolt, J. C. — "Some Applications of Generalized Harmonic Analysis to Gust Loads on Airplanes," J. Aero. Sci., Vol. 22, 1955, p. 17.

50. Freeman, J. J. — Principles of Noise, John Wiley & Sons, Inc., New York, 1958.

51. Rice, S. O. — "Mathematical Analysis of Random Noise," Bell System Tech. J., Vol. 23, p. 282, and Vol. 24, p. 46.

52. Friedman, B. — Principles and Techniques of Applied Mathematics, John Wiley & Sons, Inc., New York, 1956.

53. Fagin, S. L. — "Recursive Linear Regression Theory, Optimal Filter Theory, and Error Analysis of Optimal Systems," IEEE Int. Conv., New York City, March 1964.

54. Rauch, H. E., et al. — "Maximum Likelihood Estimates of Linear Dynamic Systems," AIAA Journ., Vol. 3, No. 8, 1965, p. 1445.

55. Smith, G. L. — "The Scientific Inferential Relationships Between Statistical Estimation, Decision Theory, and Modern Filter Theory," Proc. Joint Automatic Contr. Conf., Troy, N. Y., 1965, p. 350.

56. Newton, G. C., Gould, L. A., and Kaiser, J. F. — Analytical Design of Linear Feedback Controls, John Wiley & Sons, Inc., New York, 1957.