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THREE-DIMENSIONAL NONLINEAR STABILITY ANALYSIS OF THE SUN-PERTURBED EARTH-MOON EQUILATERAL POINTS+

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Abstract

This paper presents the results of a nonlinear analytic study of the long period features of the motion of a particle in the Earth-Moon system, near the L_4 libration point. Such long term effects stem from the excitations close to the particle's natural frequencies, which are introduced by the presence of resonance terms in the internal (Earth and Moon) and external (direct and indirect solar) force fields. Nonlinearities up to the fourth order in the displacements from L_4, and solar terms of comparable magnitude, were retained in the analysis. The importance of the nonlinear coupling of the out-of-plane motion with the in-plane motion was investigated. The existence of some equilibrium solutions was established, and some insight regarding the geometry of motion along the corresponding periodic elliptic particle orbits was obtained. Qualitative and quantitative information concerning the stability of these orbits was obtained by studying the slow variations around these periodic equilibrium motions.

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I. Introduction

The problem of the stability of motion of a small particle in the vicinity of the L_4 and L_5 points of the Earth-Moon system in the presence of the Sun has been studied in a number of recent papers.[1-5]

Attempts to study the problem by means of formal perturbation techniques applied to the complete nonlinear system of equations of motion near L_4, or to assess the effect of the solar force field on the motion,[3] have proven unsuccessful because of the occurrence of small divisors in many terms of the assumed series solution. These small divisors result from the presence of small combination or "detuning" frequencies in the nonlinear forcing terms of the differential equations, when the latter are evaluated along the nominal solution to the linearized set of equations. Such near-resonance conditions lead to poor, or at best questionable, convergence of the terms in the attempted series solution, and prevent its truncation after a reasonable number of terms. Analytic approaches using standard perturbation methods based on series expansions are thus unable to resolve in a satisfactory manner the question of boundedness of the motion near the equilateral points, with or even without the perturbative effects of the Sun.

Another more straightforward approach pursued by a number of researchers has consisted in the direct machine integration of the complete set of Langrangian equations of motion. The resultant numerical solutions available to date are rather limited in that they were generated only for restricted sets of initial conditions and initial Earth-Moon-Sun configurations. Consequently, they do not shed much new light on the question of the possible existence of domains of initial conditions and configurations for which bounded motions may exist and they are also unable to provide deeper insight into the dynamics of this nonlinear motion from which more general conclusions could be drawn.

The necessity and usefulness of further theoretical analysis prompted Breakwell and Pringle[5] to propose a theoretical approach to the two-dimensional case, which was based on Hamiltonian mechanics and took into account the dominant nonlinear near resonances by examining only the slowly varying terms.

The present paper extends the analysis of Ref. 5 to the three-dimensional case and analyzes in greater detail the stability of slow variations around the two periodic equilibrium solutions which were found to exist in the Sun-perturbed model. One of the equilibrium solutions was found to be stable, and the second unstable. Short-period terms are removed from the Hamiltonian via von Zeipel's method which results in an expression for a slowly varying Hamiltonian that describes the long-period features of the motion. The long-period behavior of the system is determined by its response to excitations which occur at or close to the particle's natural frequencies. These excitations are introduced by the presence of resonance terms in the nonlinear parts of the "internal" force field due to Earth and Moon, and by similar terms in the "external" force field due to the Sun. Some errors in the expression for the long-period Hamiltonian of Ref. 5 have been corrected in the present paper.

II. The Hamiltonian H Near L_4

The geometry of the present physical model is shown in Figure 1.

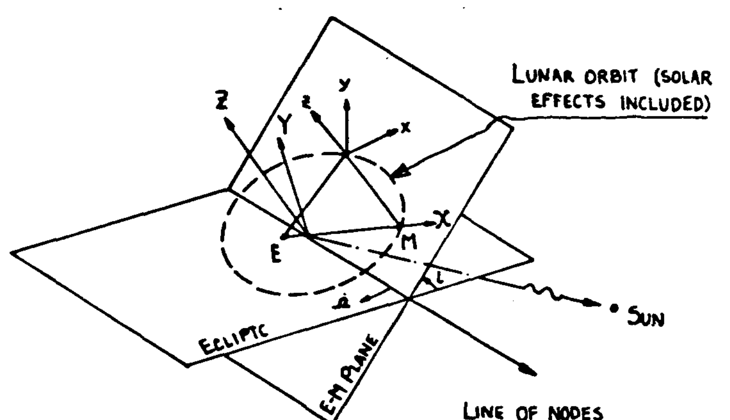

Figure 1. Geometry of the Four-Body Problem

A suitable nondimensionalization is introduced by defining the E-M distance $r_{EM}(t)$ and angular velocity n(t) by the relations $r_{EM} = 1 + \rho(t)$, $\bar{n}(t) = \bar{1}_z + \bar{\nu}(t)$; $\rho(t)$ and $\bar{\nu}(t)$ are available from classical lunar theory[6,7] in terms of lunar eccentricity e, the Earth's angular velocity around the Sun m, and the angular variables ξ, ϕ, η, Ω, i.

$$\rho(t) = -0.0079 \cos 2\xi - 0.00093 - e \cos \phi + \frac{1}{2} e^2(1 - \cos 2\phi) - \frac{15}{8} em \cos(2\xi - \phi) + \cdots$$

$$\bar{\nu}(t) = [\dot{\Omega} \sin i \sin \eta + \dot{i} \cos \eta]\bar{i}_x + [\dot{\Omega} \sin i \cos \eta - \dot{i} \sin \eta]\bar{i}_y + [0.0202 \cos 2\xi + 2e \cos \phi + \frac{15}{4} em \cos (2\xi - \phi) + \frac{5}{2} e^2 \cos 2\phi + \ldots]\bar{i}_z = \nu_x\bar{i}_x + \nu_y\bar{i}_y + \nu_z\bar{i}_z \tag{1}$$

$$\xi = (1 - m)t + \epsilon - \epsilon' \qquad m \cong 0.074801$$

$$\phi = ct + \epsilon - \varpi \qquad c \cong 0.99155$$

$$\eta = gt + \epsilon - \Omega \qquad g \cong 1.0040212$$

$$\dot{\Omega} = \frac{\sin \eta}{\sin i} (1 + \rho)W \qquad \dot{i} = (1 + \rho)W \cos \eta$$

W denotes the solar acceleration component normal to the E-M plane at the Moon's position (see page 404 of Ref. 8); $\bar{i}_x$, $\bar{i}_y$, $\bar{i}_z$ denote unit vectors in the x,y,z directions, respectively. We shall consider here only the case of a lunar orbit with a mean eccentricity $e = 0$.

Because of the near resonance $\omega_1 \cong 3\omega_2$ of the coplanar natural frequencies $\omega_1 = \pm 0.95459$, $\omega_2 = \pm 0.29791$, we must retain fourth-order terms in the Taylor expansion of the Hamiltonian H around L_4. In order to be consistent in our retention of comparable order of magnitude terms in the solar perturbation, we consider the quantities m,x,y,z, $P_x,P_y,P_z,\sqrt{\rho},\sqrt{\nu}$, and i as being of the first order of smallness; P_x,P_y,P_z denote, respectively, the momenta conjugate to x,y,z, and are introduced by the relations

$$P_x = \frac{\partial L}{\partial \dot{x}} = p_x - \frac{\sqrt{3}}{2}$$
$$P_y = \frac{\partial L}{\partial \dot{y}} = p_y + \frac{1}{2} \tag{2}$$
$$P_z = \frac{\partial L}{\partial \dot{z}} = p_z$$

where L denotes the Langrangian of the system. In matrix notation $H = p_r^T \dot{r} - L$, where p_r^T is the row matrix obtained from Eq. (2) and $\dot{r}$ is the column matrix of velocities $\dot{x},\dot{y},\dot{z}$. It is convenient to split H into the parts

$$H = H^{(o)} + H' \tag{3}$$

where

$$H' = H_3 + H_4$$

and where $H^{(o)}$ contains the second- and third-order linear and quadratic terms in the displacements and momenta, and the perturbation Hamiltonian H' contains the third-order (H_3) and the fourth-order (H_4) perturbing terms shown below:

$$\begin{aligned}
H^{(o)} = \Big\{ & \tfrac{1}{2}\left(P_x^2 + P_y^2 + P_z^2\right) + \left(yP_x - xP_y\right) \\
& + \tfrac{1}{8}\left(x^2 - 5y^2 + 4z^2\right) - \tfrac{3\sqrt{3}}{4}(1 - 2\mu)xy \\
& - \tfrac{1}{2}\dot{\rho}\left(P_x + \sqrt{3}\,P_y\right) + \\
& + \tfrac{1}{2}\left(\rho + \upsilon_z\right)\left(\sqrt{3}\,P_x - P_y\right) - \left(\rho + \tfrac{1}{2}\upsilon_z\right)\left(x + \sqrt{3y}\right) \\
& - m^2\left[\frac{3}{2r_{ES}^2}\left(x_s + \sqrt{3y_s}\right)\left(x_s x + y_s y\right) - \tfrac{1}{2}\left(x + \sqrt{3y}\right)\right]\Big\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
H_3 = \Big\{ & \frac{3\sqrt{3}}{16}\left(x^2 y + y^3\right) \\
& + \frac{1 - 2\mu}{16}\left(33xy^2 - 7x^3 - 12xz^2\right) - \frac{3\sqrt{3}}{4}yz^2\Big\}
\end{aligned} \tag{5}$$

$$\begin{aligned}
H_4 = \Big\{ & \frac{5\sqrt{3}\,(1 - 2\mu)}{32}\left(5x^3 y - 9xy^3 + 12xyz^2\right) + \frac{37}{128}x^4 \\
& + \frac{3}{16}x^2 z^2 + \frac{33}{16}y^2 z^2 - \frac{123}{64}x^2 y^2 - \frac{3}{128}y^4 \\
& - \frac{3}{8}z^4\Big\}_4 + \Big\{ - \frac{3}{8}\rho\Big(x^2 - 5y^2 + 4z^2 \\
& - 6\sqrt{3}\,(1 - 2\mu)xy\Big) + \upsilon_z\left(yP_x - xP_y\right) \\
& + \upsilon_y\left(xP_z - zP_x\right) + \upsilon_x\left(zP_y - yP_z\right) \\
& - m^2\left[\frac{3}{2r_{ES}^2}\left(x_s x + y_s y\right)^2 - \frac{1}{2}\left(x^2 + y^2 + z^2\right)\right]\Big\} \\
& - m^2\frac{3}{2r_{ES}^2}\left(x_s + \sqrt{3}\,y_s\right)z_s z + \frac{1}{2}\left(\sqrt{3}\,\upsilon_y + \upsilon_x\right)z \\
& + \frac{1}{2}\left(\upsilon_y - \sqrt{3}\,\upsilon_x\right)P_z + o(m^5, m^6) \text{ etc...}
\end{aligned} \tag{6}$$

The terms x_s, y_s, z_s denote the solar coordinates:

$$x_s = r_{ES} \cos \xi \qquad y_s = -r_{ES} \sin \xi$$
$$z_s = r_{ES} \sin i \sin (\Omega - v') \tag{7}$$

and v' is the solar longitude. The solution to the linear nonhomogeneous differential equations governed by $H^{(o)}$ consists of a complementary part, $\bar{x}$, $\bar{y}, \bar{z}$, and a forced response, $\tilde{x}, \tilde{y}$. No $\tilde{z}$ arises because the terms linear in z were included in H_4.

$$\bar{x} = 2.902\sqrt{\alpha_1} \cos \omega_1\beta_1^{\neq} + 8.003\sqrt{\alpha_2} \cos \omega_2\beta_2^{\neq}$$
$$\bar{y} = 2.103\sqrt{\alpha_1} \cos (\omega_1\beta_1^{\neq} + 123.57^\circ)$$
$$+ 4.793\sqrt{\alpha_2} \cos (\omega_2\beta_2^{\neq} + 154.82^\circ)$$
$$\bar{z} = \sqrt{2\alpha_3} \cos \beta_3^{\neq}$$
$$\tilde{x} = 0.01016 \cos (2\xi - 67.2^\circ) \tag{8}$$
$$+ 1.697m^2 \cos (2\xi - 127.7^\circ)$$
$$\tilde{y} = 0.00867 \cos (2\xi + 38.3^\circ)$$
$$+ 1.43m^2 \cos (2\xi - 20.83^\circ)$$
$$\tilde{z} = 0$$

where

$$\beta_1^{\neq} = t + \beta_1; \qquad \beta_2^{\neq} = t - \beta_2; \qquad \beta_3^{\neq} = t + \beta_3$$

Except for $\bar{z}$, Eqs. (8) are the same ones found in Ref. 5; α_i, β_i $(i = 1,2,3)$ form a polar set of integration constants, canonical with respect to a Hamiltonian $H = 0$. Further details regarding the derivation of the above or any other equations of this paper can be found in Refs. 9 or 10.

III. The Slowly Varying Hamiltonian

The inclusion of the additional terms H_3 and H_4 in the Hamiltonian can be handled by a method equivalent to the customary technique of variation of constants.

We shall require α, β to become functions of time which then satisfy Hamilton's equations with a Hamiltonian H'. Inasmuch as we are not concerned in the present investigation with an exact or detailed determination of the particle's trajectory, but rather in the overall broad features of the motion, we shall desire to obtain only the slowly varying components of α and β which will arise from the secular terms in H', and those terms containing low combination frequencies which arise from the near resonances.

This can be accomplished by means of a suitable canonical transformation of coordinates from the polar canonical set α,β to a new slowly varying canonical set α',β' associated with a new slowly varying Hamiltonian K'. K' will contain only the lowest frequency terms which arise in H' as a result of the above transformation, all other faster terms having been suitably eliminated. It is reasonable to assume that for relatively small displacements x,y,z of the particle, the effect of H' would be in the nature of a slow change of the linearized solution found earlier. With this assumption in mind we may now consider a stationary contact transformation

$$\begin{aligned} \alpha_i' &= \alpha_i + \delta\alpha_i \\ \beta_i' &= \beta_i + \delta\beta_i \end{aligned} \tag{9}$$

that may be introduced with the aid of a generating function $G(\beta,\alpha')$

$$G(\beta,\alpha') = \beta\alpha' + S(\beta,\alpha') \tag{10}$$

which satisfies the relations(11)

$$\begin{aligned} \beta' &= \frac{\partial G}{\partial \alpha'} = \beta + S_{\alpha'} \\ \alpha &= \frac{\partial G}{\partial \beta} = \alpha' + S_\beta \end{aligned} \tag{11}$$

The first term $\beta\alpha'$ in G generates the identity transformation, while the function $S(\beta,\alpha') = S_1 + S_2$ denotes an additional suitably selected generating function which is introduced for the specific purpose of eliminating all the short-period terms which occur in H'; S_1 is selected to eliminate the terms of $o(m^3)$, and S_2 those of $o(m^4)$.

Since S does not depend explicitly on time t, we can write

$$K'(\beta',\alpha',t) = \overset{(0)}{H}(\beta',\alpha',t) + H'(\beta',\alpha',t) \tag{12}$$

where H above is evaluated in terms of the new coordinates and new momenta.

When all the required steps of the transformation are carried out, one arrives at the following relation for K' (see Note at end of References)

$$K' = \bar{\bar{H}}_3 + \bar{\bar{H}}_4 + \bar{H}_4 - \frac{1}{2}\,\overline{[H_3, S_1]} \tag{13}$$

where $\bar{\bar{H}}_3$ and $\bar{\bar{H}}_4$ are the secular and long-period terms resulting from the Taylor series expansions

$$\tilde{H}_i = \sum_{n=1}^{\infty} \frac{1}{n!}\left[\tilde{x}\frac{\partial}{\partial x} + \tilde{y}\frac{\partial}{\partial y} + \tilde{z}\frac{\partial}{\partial z}\right]^n H_i \tag{14}$$

evaluated at $\bar{x}, \bar{y}, \bar{z}$. For the present case $e = 0$ and $\bar{\bar{H}}_4 = 0$.

The term $\overline{[H_3, S_1]}$ denotes the long-period part of the Poisson bracket of H_3 with S_1; $\bar{H}_4$ results from the substitution of the homogeneous solutions $\bar{x}, \bar{y}, \bar{z}$ into H_4 and consists of an internal part $\bar{H}_{4int}$ and an external part $\bar{H}_{4ext}$ which contains both the direct and indirect solar effects. The nearly resonant terms arising from z and P_z in $\bar{H}_{4ext}$ cancel out, i.e. there is no long period out-of-plane forced motion.

The above K' differs from the expression on page 63 of Ref. 5, where only one part of the Poisson bracket $[H_3, S_1]$ appeared in the function $\mathfrak{F}$. Now the transformation of a Hamiltonian from one canonical set (α, β) to another set (α', β') should give rise only to polynomial terms[12,13] (i.e., terms which arise from a binomial expansion such as $(x + y)^n$, with n being some finite integer). This clearly is the case when K' contains a Poisson bracket, but not for the function $\mathfrak{F}$ which introduced an incorrect nonpolynomial long-period term such as $\alpha_1'^{(3/2)}\alpha_2'^{(1/2)}$ in Eq. (10). Dr. Breakwell attributed the presence of this term to his use of mixed variables in the determination of the generating function S_2, which would yield a correct result only for linear Hamiltonians, and suggested here that one transform immediately to the set (α', β') before one chooses a relation for S_2.

The algebraic work needed to express K' in terms of α', β', and t is rather formidable and is one of the major stumbling blocks in what would otherwise be a relatively straightforward solution. If all the manipulations have been successfully carried out, one does eventually come up with the expression for K'

$$
\begin{aligned}
K' = \Big\{&0.1266\alpha_1'^2 - 6.000\alpha_1'\alpha_2' + 3.829\alpha_2'^2 \\
&- 29.04\alpha_1'^{(1/2)}\alpha_2'^{(3/2)} \cos[0.06086t + \omega_1\beta_1' \\
&+ 3\omega_2\beta_2' + 14.2^\circ] + \alpha'\alpha_3'[0.09316 \\
&+ 0.08608 \cos 2\Delta_{13} - 0.03934 \sin 2\Delta_{13}] \\
&+ 0.7554\alpha_2'\alpha_3' - 0.002231\alpha_3'^2\Big\}_{int} - \Big\{0.005394\alpha_1' \\
&+ 0.008208\alpha_2' + 0.02685\alpha_1' \cos[0.05878t + 2\omega_1\beta_1' \\
&+ 29.4^\circ + 2\epsilon' - 2\epsilon] + 0.004193\alpha_3'\Big\}_{ext} \qquad (15)
\end{aligned}
$$

where

$$\Delta_{13} = \omega_1(t + \beta_1') - (t + \beta_3') = -0.04541t + \omega_1\beta_1 - \beta_3'$$

The first bracket contains all the internal terms, while the second bracket includes all the external (solar) terms. The long-period contributions to the coplanar (α_1', α_2') terms resulting from the periodic parts of the indirect $\rho(t)$ and $\upsilon(t)$ terms in H' were found to cancel exactly the indirect periodic terms generated by the first term in the linear forced response $\tilde{x}$ and $\tilde{y}$ of Eq. (8). The external terms displayed in Eq. (15) which are left after the above cancellations, stem from the contribution of the indirect constant component -0.00093 in ρ, from the direct m^2 terms in H, and from the m^2 terms in the forced response $\tilde{x}$ and $\tilde{y}$ of the linear system.

The explicit presence of time t in K can be eliminated via a coordinate transformation to a new canonical set α^*, β^*

$$
\begin{aligned}
\beta_1^* &= 0.02939t + \omega_1\beta_1' + 14.7^\circ - \epsilon + \epsilon' \\
\beta_2^* &= 0.03146t + 3\omega_2\beta_2' - 0.50^\circ + \epsilon - \epsilon' \\
\beta_3^* &= 0.074801t + \beta_3' + 2.42^\circ - \epsilon + \epsilon' \qquad (16) \\
\alpha_1^* &= \frac{\alpha_1'}{\omega_1} \qquad \alpha_2^* = \frac{\alpha_2'}{\omega_2} \qquad \alpha_3^* = \alpha_3'
\end{aligned}
$$

which results in the time-independent Hamiltonian K^*

$$K^* = \Big\{0.1154\alpha_1^{*2} - 5.1\alpha_1^*\alpha_2^* + 3.059\alpha_2^{*2} - 23.97\alpha_1^{*(1/2)}\alpha_2^{*(3/2)}C_{\beta_1^*+\beta_2^*} + 0.09035\alpha_1^*\alpha_3^*C_{2(\beta_1^*-\beta_3^*)} + 0.08893\alpha_1^*\alpha_3^* + 0.6751\alpha_2^*\alpha_3^* - 0.002231\alpha_3^* + 0.02939\alpha_1^* + 0.03146\alpha_2^* + 0.074801\alpha_3^*\Big\}_{int} + \Big\{0.004193\alpha_3^* - 0.007336\alpha_2^* - \alpha_1^*\Big[0.005149 + 0.02563C_{2\beta_1^*}\Big]\Big\}_{ext} \tag{17}$$

where

$$C_{2\beta_1^*} \equiv \cos 2\beta_1^*$$

IV. Equilibrium Points and Their Stability

It can be shown, on the basis of an analysis of only the internal terms in K^* that an appreciable coplanar internal coupling exists between α_1^* and α_2^*. This analysis disclosed on the other hand that the internal coupling between α_1^* and α_3^* did not lead to any measurable transfer of energy from the out-of-plane mode to the coplanar mode of motion. The major long term solar effect causes mainly an excitation of the α_1^* mode. The α_3^* mode does not experience any external excitation to the order of magnitude of the terms retained.

Determination of Equilibrium Points

If a stable motion in the presence of the Sun is possible in which α_1^*, α_2^* and α_3^* remain small, it would suffice to retain only linear terms in K^* in order to determine long term effects. To linear terms we have the simpler Hamiltonian

$$0.02425\alpha_1^* + 0.02412\alpha_2^* + 0.07899\alpha_3^* - 0.02563\alpha_1^*C_{2\beta_1^*} \tag{18}$$

which is of the Mathieu type, and leads to parametric resonance in the α_1^* motion. Since 0.02563 > 0.02425 the motion falls into the unstable region of the Mathieu plane, and therefore no motion can exist for which α_1^* remains very small.

From a physical point of view this means that the libration point L_4 is not stable with respect to small perturbations, when the solar force field is included, and that the higher order terms in K_2^* must be retained in any analysis.

The lack of stability exhibited by the linearized Hamiltonian does not preclude the existence of equilibrium points in the α^* space for the complete Hamiltonian. In view of the negligible effect of α_3^* on the coplanar motion, it is of interest to look for equilibrium points for $\alpha_3^* = 0$. Such points in the (α_1^*,α_2^*) plane are determined by looking for solutions to Hamilton's equations of the form $\dot{\alpha}_1^* = \dot{\alpha}_2^* = 0$.

Once such points are located, it is then necessary to investigate the type of equilibrium which exists there, and to identify the stable ones.

This search is more easily carried out if one switches over to a set of normal coordinates (Q^*,P^*) defined by

$$Q_i^* = \sqrt{2\alpha_i^*}\,\sin\beta_i^* \qquad P_i^* = \sqrt{2\alpha_i^*}\,\cos\beta_i^* \qquad i = 1,2 \tag{19}$$

After setting $\alpha_3^* = 0$, the two dimensional part of K^*, which we denote here by K_2^*, becomes

$$\begin{aligned}
K_2^* = {} & \frac{0.1154}{4}\left(P_1^{*2} + Q_1^{*2}\right)^2 - \frac{5.1}{4}\left(P_1^{*2} + Q_1^{*2}\right)\left(P_2^{*2} + Q_2^{*2}\right) \\
& + \frac{3.059}{4}\left(P_2^{*2} + Q_2^{*2}\right)^2 \\
& - \frac{23.97}{4}\left(P_1^*P_2^* - Q_1^*Q_2^*\right)\left(P_2^{*2} + Q_2^{*2}\right) \\
& + \frac{0.02425}{2}\left(P_1^{*2} + Q_1^{*2}\right) + \frac{0.02412}{2}\left(P_2^{*2} + Q_2^{*2}\right) \\
& - \frac{0.02563}{2}\left(P_1^{*2} - Q_1^{*2}\right)
\end{aligned} \tag{20}$$

The equilibrium points (Q_e^*,P_e^*) are obtained from the solution of the equations

$$\begin{pmatrix} \dot{Q}^* \\ \dot{P}^* \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} K_{2P^*}^{*T} \\ K_{2Q^*}^{*T} \end{pmatrix} = 0 \tag{21}$$

From Eqs. (21) we have

$$K^*_{2P_1^*} = 0 = 0.1154P_1^*\left(P_1^{*2} + Q_1^{*2}\right) - \frac{5.1}{2} P_1^*\left(P_2^{*2} + Q_2^{*2}\right) - \frac{23.97}{4} P_2^*\left(Q_2^{*2} + Q_2^{*2}\right) - 0.001379P_1^* \qquad (22a)$$

$$K^*_{2P_2^*} = 0 = -\frac{5.1}{2} P_2^*\left(P_1^{*2} + Q_1^{*2}\right) + 3.059P_2^*\left(P_2^{*2} + Q_2^{*2}\right) - \frac{23.97}{4}\left(3P_1^*P_2^{*2} + P_1^*Q_2^{*2} - 2P_2^*Q_1^*Q_2^*\right) + 0.02412P_2^* \qquad (22b)$$

$$K^*_{2Q_1^*} = 0 = 0.1154Q_1^*\left(P_1^{*2} + Q_1^{*2}\right) - \frac{5.1}{2} Q_1^*\left(P_2^{*2} + Q_2^{*2}\right) + \frac{23.97}{4} Q_2^*\left(P_2^* + Q_2^*\right) + 0.04988Q_1^* \qquad (22c)$$

$$K^*_{2Q_2^*} = 0 = -\frac{5.1}{2} Q_2^*\left(P_1^{*2} + Q_1^{*2}\right) + 3.059Q_2^*\left(P_2^{*2} + Q_2^{*2}\right) - \frac{23.97}{4}\left(2Q_2^*P_1^*P_2^* - Q_1^*P_2^{*2} - 3Q_1^*Q_2^{*2}\right) + 0.02412Q_2^* \qquad (22d)$$

Equations (22c) and (22d) are identically satisfied if we choose $Q^*_{1e} = Q^*_{2e} = 0$. For simplicity we shall therefore restrict our search to those equilibrium points for which

$$Q^*_{1e} = Q^*_{2e} = 0 \qquad (23)$$

For the above Q^*'s, Eqs. (22a) and (22b) give

$$0.1154P_1^{*3} - 2.55P_1^*P_2^{*2} - 5.810P_2^{*3} - 0.001379P_1^* = 0 \qquad (24a)$$

$$-2.55P_1^{*2}P_2^* + 3.059P_2^{*3} - 17.43P_1^*P_2^{*2} + 0.02412P_2^* = 0 \qquad (24b)$$

One equilibrium point can be obtained by setting $P^*_{2e} = 0$ (which automatically satisfies Eq. (24b)) and then solving for P^*_{1e} from the relation

$$0.1154P_1^{*2} - 0.001379 = 0 \qquad (25)$$

or

$$P_1^* = 0.1093 \qquad (26)$$

which corresponds to

$$\alpha_{1e}^* = 0.005975$$

The first equilibrium point, which we denote by E_I, is thus specified by the coordinates

$$\begin{aligned} Q_1^* = Q_2^* = Q_3^* = P_2^* = P_3^* = 0 \qquad P_1^* = 0.1093 \\ \alpha_1^* = 0.005975 \qquad \alpha_2^* = 0 \qquad \alpha_3^* = 0 \end{aligned} \tag{27}$$

Another equilibrium point can be found for which P_1^* and $P_2^* \neq 0$, all other coordinates remaining zero. The values of P_1^* and P_2^* result from the solution of Eqs. (24a) and (24b) after P_2^* is factored out from the latter. The coordinates of the second equilibrium point E_{II} were found to be

$$\begin{aligned} Q_1^* = Q_2^* = Q_3^* = P_3^* = 0 \\ P_1^* = 0.1106 \qquad \alpha_1^* = 0.006116 \\ P_2^* = -0.003675 \qquad \alpha_2^* = 6.753 \times 10^{-6} \end{aligned} \tag{28}$$

The two points E_I and E_{II} were the only ones readily found for the present simplified conditions. A machine search might reveal the existence of additional roots of the complete set of Eqs. (21). The periodic elliptic particle trajectory of mode close to ω_1 corresponding to conditions at E_I is shown in Fig. 2. It has a semimajor axis of about 60,000 mi and a semiminor axis of half this value. These values were determined by computing $r_{max} = [\bar{x}^2 + \bar{y}^2]^{1/2}_{max}$, where the ω_1 modes of $\bar{x}$ and $\bar{y}$ of Eq. (8) were used, and the maximum was determined with respect to $\omega_1\beta_1^{\neq}$. It can be shown that this requires that

$$8.422 S_{2\omega_1\beta_1^{\neq}} + 4.423 S_{2\omega_1\beta_1^{\neq}+247.14^o} = 0$$

and results in a value $\omega_1\beta_1^{\neq} \cong 15.62$ deg. The dimensionless expression for r_{max} then becomes $r_{max} \cong 3.2\alpha_1^{*1/2}$, and at $\alpha_1^* \cong 0.006$ amounts to roughly $3.2\sqrt{0.955 \cdot .006} \times 2.4 \times 10^5 = 58{,}128 \cong 60{,}000$.

It is of interest to observe that this result indicates the particles mean motion is synchronized with that of the Sun such that their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse. To see this we recall that at equilibrium $Q_1^* = 0$, and hence $\beta_1^* = n\pi$ with $n = 0,1...$ For $n = 0$ Eq. (16) gives

$$\beta_1^* = 0 = 0.02939t + \omega_1\beta_1' + 14.7 - \epsilon + \epsilon'$$

and from here

$$\omega_1\beta_1^{\neq} = \omega_1 t - 0.02939t - 14.7 + \epsilon - \epsilon'$$

When the particle crossed the major axis, we had $\omega_1\beta_1^{\neq} = 15.62$, and from the commensurability of angular velocities at E_I, $(\omega_1 - 0.02939) = 1 - m$. Substitution above gives

$$15.62 + 14.7 = 30.32^\circ = 1 - m)t + \epsilon - \epsilon' \equiv \xi$$

as defined below Eq. (1). From the definition of x_s and y_s preceding Eq. (8), this then shows the Sun to be located 30.32 deg below the x-axis, and therefore closely aligned with the major axis of the particle's orbit.

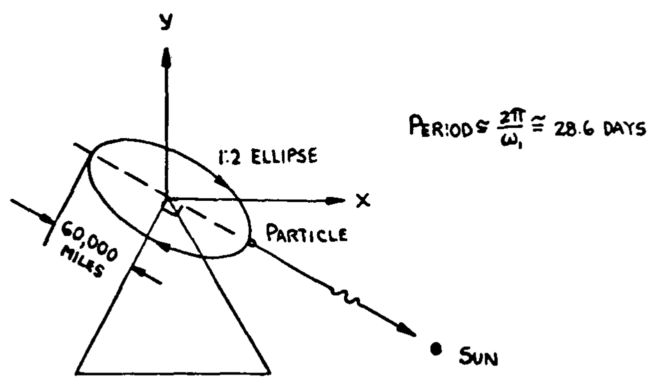

Figure 2. Periodic Orbit Corresponding to Equilibrium Point E_I

Stability of the Equilibrium Points

The stability of the slow variations around the above periodic equilibrium motions in the x-y plane can be determined by setting up the expression for the variation δK^*, which results from taking small displacements δQ^* and δP^* around the equilibrium values $Q^*_{ie} = 0$ and P^*_{ie}. Clearly, since E_I and E_{II} are equilibrium points, the coefficients of the linear terms in δK^* must vanish. The expression for δK^* at point E_I becomes

$$\delta K^* = 0.001380\delta P_1^{*2} + 0.02563\delta Q_1^{*2} - 0.003174\delta P_2^{*2} - 0.003174\delta Q_2^{*2} + 0.04008\delta P_3^{*2} + 0.03949\delta Q_3^{*2} \tag{29}$$

Since for every value of i = 1,2,3 the coefficients of δP_i^{*2} have the same sign as the coefficients of δQ^{*2}(i.e., δK^* is either positive or negative definite irrespective of the signs of δP^* or δQ^*), we can conclude that point E_I is stable for small disturbances in all principal directions. The period of the slow variations in δP_1^*, δQ_1^* is approximately 83 months.

At point E_{II} we have for δK^*

$$\delta K^* = 0.001411\delta P_1^{*2} + 0.02563\delta Q_1^{*2} + 0.001838\delta P_1^*\delta P_2^* + 0.003652\delta P_2^{*2} + 7.847 \times 10^{-5}\delta Q_1^*\delta Q_2^* - 0.001150\delta Q_2^{*2} + 0.04008\delta P_3^{*2} + 0.03949\delta Q_3^{*2} \tag{30}$$

Equation (30) indicates that the in-plane variational equations are decoupled from the out-of-plane equations, and that the latter lead to a stable solution. The nature of stability at E_{II} can thus be determined by writing down the complete system of first-order linear differential equations for δQ_i^* and δP_i^* (i = 1,2) obtained from δK^* of Eq. (30), and by examining the roots of the appropriate characteristic equations. We find that

$$\begin{pmatrix} \delta\dot{Q}_1^* \\ \delta\dot{Q}_2^* \\ \delta\dot{P}_1^* \\ \delta\dot{P}_2^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.002822 & 0.001838 \\ 0 & 0 & 0.001838 & 0.007304 \\ -0.05126 & -7.85\cdot 10^{-5} & 0 & 0 \\ -7.85\cdot 10^{-5} & -0.0023 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta Q_1^* \\ \delta Q_2^* \\ \delta P_1^* \\ \delta P_2^* \end{pmatrix} \tag{31}$$

A trial solution of the form e^{st} leads to the characteristic equation

$$s^4 + 1.282\cdot10^{-4}s^2 - 2.031\cdot10^{-8} = 0 \qquad (32)$$

which has one positive root because of the negative constant term. E_{II} is therefore unstable.

A simple geometrical description of the stable and unstable regions in the six-dimensional P^*,Q^* space is of course not feasible. On the other hand, it is possible to take advantage of the fact that the stable point E_I happens to lie very close to the unstable point E_{II}, and to determine the extent of the stable region around E_I by expanding K^* up to cubic powers in δP^* and δQ^* around E_I. (The cubic terms arise from the internal resonant terms in (15).)

The intersection of surfaces of constant K^* with the (P_2^*,Q_2^*) plane, for a value of $P_1^* = 0.11$, is shown in Fig. 3. The dashed curve shows the separatrix which passes through E_{II} and separates the stable from the unstable regions.

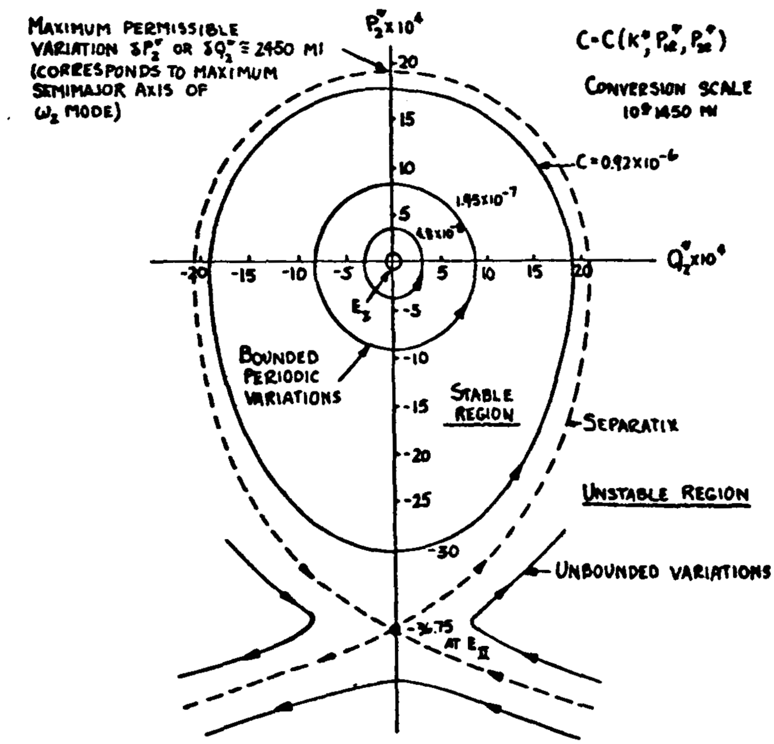

Figure 3. Stability Regions in the $(P_2^*\text{-}Q_2^*)$ Plane Near the Coplanar Equilibrium Points

In the physical x-y plane, a point in the stable region gives rise to slow variations of the elements of the periodic particle orbit corresponding to E_I. A point in the unstable region of the (P_2^*, Q_2^*) plane would lead to large particle departures from the equilibrium orbit, and thus would indicate a possible divergence. In other words, the slower free mode, unless almost absent, is unstably excited by the faster mode which is present due to the Sun.

The maximum allowable participation of the ω_2 mode in the particle's motion, before instability sets in, is limited by the least distance of the separatrix of Fig. 3 from the origin. This occurs at its intersection with the positive P_2^* axis, and results in an elliptic orbit with a maximum semimajor axis.

$$r_{max} = [(9.1 \sqrt{0.8937}/\sqrt{2}) P^*_{2max} \cdot 2.4 \times 10^5] = 2450 \text{ mi}$$

V. Summary and Conclusions

In the present paper, the three-dimensional stability of the motion of a particle near the equilateral libration points of the Earth-Moon system, in the presence of the Sun, has been investigated.

Because the inclusion of lunar eccentricity would have introduced into the problem a larger number of internal and external resonances than could have been handled by the present method of approach, it was found necessary to restrict the stability analysis to a lunar orbit perturbed by the Sun but without eccentricity.

Four major conclusions emerge from the present study. First, small coplanar motions near L_4 or L_5 will grow large because of parametric excitation by the Sun, as a result of nonlinear resonance. In fact, the growth of the energy in the faster normal mode of the linearized theory is found to be governed by a Mathieu equation.

Second, the out-of-plane motion is not seriously excited by the Sun, and has a negligible effect on the coplanar motion, which is the dominant factor as far as stability is concerned.

Third, a stable periodic coplanar orbit can exist in the presence of the Sun. It consists of a westward (clockwise) motion along the 1:2 ellipse around L_4 corresponding to the first (or faster) normal mode and has a semimajor axis of approximately 60,000 mi. The external nonlinear excitation causes the mean angular motion of the particle to become synchronized with that of the Sun. Thus to an observer located at L_4 and looking continuously in the direction of the Sun, the particle would appear to move back and forth across his line of sight in the manner of a simple harmonic oscillator. The times of crossing of the line of sight coincide closely with the times at which the line of sight is aligned with the major or minor axis of the ellipse.

Fourth, the presence of the internal resonant excitation, resulting from the near commensurability (3:1) of the two coplanar normal modes, makes the stability somewhat delicate. As a consequence, the semimajor axis of the second mode is limited to magnitudes less than approximately 2450 mi. For larger values the motion becomes unstable and may result in very large displacements, which would exceed the range of applicability of the present theory.

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Note for Eq. (13): The following key steps lead to the expression presented. Introduce a generalized canonical set q,p and transform to a new slowly varying set q',p' by using a scleronomic generating function $S = S(q,p') = S_1 + S_2$. At the end we will identify q and p with β and α respectively. Using Taylor series expansions around the new variables q',p' we can write

$$q = q' - S_{p'T} - S_{p'T_{q'}}\,\Delta q + \ldots$$

$$p = p' + S_{q'T} + S_{q'T_{q'}}\,\Delta q + \ldots$$

$$\Delta q = -S_{p'T} + S_{p'T_{q'}}S_{p'T} + \ldots$$

$$\Delta p = S_{q'T} - S_{q'T_{q'}}S_{p'T} + \ldots$$

$$K = H(q,p)\Big|_{q',p'} = H + H_{q'}\,\Delta q + H_{p'}\,\Delta p +$$

$$+ \frac{1}{2!}\left[\Delta q^T \frac{\partial}{\partial q'^T} + \Delta p^T \frac{\partial}{\partial p'^T}\right]^2 H(q',p') + \ldots$$

All functions and partial derivatives are evaluated at q', p'. In the present notation $S_q{}'T$ denotes the column matrix of partials of S, which is obtained by transposition of the row matrix of partials $S_q{}'$. When all the algebraic manipulations in K are carried out and the terms suitably combined, one can obtain the expression

$$K = H^{(o)} + H_4 + \left\{ \left[H^{(o)}, S_1 \right] - H_3 \right\}$$

$$- \left[H^{(o)}, S_2 - \frac{1}{2} S_{1q}{}' S_{1p}{}'T \right] - \frac{1}{2} [H_3, S_1]$$

Short period terms in H_4 and the two terms in the brackets can be eliminated by proper choice of S_1 and S_2. Introduction of the forced linear response then leads to Eq. (13). Note that $H^{(o)}$ will drop out if the set α, β is used since they are canonical with respect to $H = 0$.