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ANALYSIS AND DESIGN OF SPACE VEHICLE FLIGHT CONTROL SYSTEMS

VOLUME XV - ELASTIC BODY EQUATIONS

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FOREWORD

This report was prepared under NASA Contract NAS 8-11494 and is one of a series intended to illustrate methods used for the design and analysis of space vehicle flight control systems. Below is a complete list of the reports in the series:

Volume I	Short Period Dynamics
Volume II	Trajectory Equations
Volume III	Linear Systems
Volume IV	Nonlinear Systems
Volume V	Sensitivity Theory
Volume VI	Stochastic Effects
Volume VII	Attitude Control During Launch
Volume VIII	Rendezvous and Docking
Volume IX	Optimization Methods
Volume X	Man in the Loop
Volume XI	Component Dynamics
Volume XII	Attitude Control in Space
Volume XIII	Adaptive Control
Volume XIV	Load Relief
Volume XV	Elastic Body Equations
Volume XVI	Abort

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1. STATEMENT OF THE PROBLEM

The satisfactory performance of an autopilot-controlled flexible launch vehicle is crucially dependent on an accurate representation of the vehicle's elastic motion under prescribed forces. This is due to the fact that the forces and moments applied by the propulsion device (rocket engines) are governed both in magnitude and direction by the information derived from sensors located along the vehicle. The output from these sensors is composed of signals representing both rigid body motions and local elastic distortions. It is not difficult to visualize intuitively that, under certain adverse conditions, the autopilot will act to reinforce the amplitudes of oscillation — leading ultimately to structural failure of the vehicle.

A prerequisite for control system synthesis is therefore an adequate representation of the vehicle's elastic motion under prescribed external forces. It turns out that this representation can be conveniently expressed in terms of the vibration frequencies and mode shapes of the "free-free" structure.

Several types of vibration must be distinguished, viz., longitudinal, torsional, and lateral. Longitudinal oscillations, as the name implies, concern the elastic displacements parallel to the longitudinal axis. These displacements are induced by axial loads on the vehicle during flight. The main effect is a slight decrease in lateral bending mode frequency due to the axial load on the beam. This effect is generally very small and can be ignored in nearly all cases. Longitudinal oscillations are significant primarily in the tank pressurization control systems.

Torsional vibrations, due to elastic angular displacements around the longitudinal axis, may be studied by treating the vehicle as a system of lumped inertias connected by rotation springs. The analytical development, for purposes of mode and frequency computation, is quite analogous to that for lateral vibrations with the exception that shear stiffness replaces bending stiffness. This question is treated in detail in Ref. 16. Torsional oscillations are normally of little concern for conventional launch vehicles since these modes are uncoupled from the lateral and longitudinal motion, because of symmetry and because there is no major excitation of the modes during flight.

Of primary importance, for purposes of control system stability analysis, are the lateral vibration modes, since this motion is sensed by the autopilot instrumentation. The computation and interpretation of the lateral vibration properties of a launch vehicle are the primary aim of this monograph.

For conventional vehicles having a high degree of axial symmetry there is negligible coupling between the pitch and yaw lateral modes, as well as between lateral and longitudinal or torsional modes. Thus an analysis of the planar elastic properties of the vehicle is usually sufficient for purposes of control system analysis.

Analytical and control system aspects are stressed in the present monograph. Numerical methods are treated in detail in Ref. 15.

The picture changes considerably for the case of clustered boosters where coupling between lateral and torsional modes becomes significant.

The problems here are mainly computational rather than conceptual, since the large number of lumped mass stations required for an adequate mathematical representation strains the capabilities of even the largest computers. Furthermore, good basic data, especially for junction points, is hard to come by.

The present monograph, which treats the question of elastic motion in detail, is a supplement and extension of "Short Period Dynamics" (Ref. 22). Taken together, these documents provide a complete mathematical model of the launch vehicle attitude control system.

2. STATE OF THE ART

The importance of the dynamic (rather than static) effects of structural elasticity have long been recognized by aircraft designers. With airplanes, the problem arises because of interaction between the elastic and aerodynamic forces. It manifests itself primarily in the phenomenon of flutter. Additional problems, stemming from essentially the same source, are buffeting, divergence, loss of control effectiveness, etc.

The emergence of the space launch vehicle added a new dimension to the problem: that of interaction of elastic deformation with the control system. A launch vehicle is essentially a long slender beam. If devices are placed along the vehicle to sense angular displacement or rate (for purposes of feedback control), the devices will, in general, measure local elastic distortions as well as rigid body motion. With unfavorable phase relationships, the control device (rocket engine) will act to reinforce elastic oscillations, leading ultimately to structural failure of the vehicle.

For launch vehicles, because of their beam-like shape and the manner in which sensing instrumentation is located, the primary elastic phenomenon of interest is the lateral vibration. By obtaining the expressions for bending displacement and superimposing these as added degrees of freedom in the vehicle control system, the essential stability and performance aspects may be rationally analyzed.

In one of the earliest studies of the problem as related to the development of the Atlas autopilot, Beharrel and Friedrich[13] showed how the added elastic degrees of freedom could be expressed in the transfer function format, thereby enabling conventional analytical techniques to be applied. The elastic deflection under applied forces was expressed in terms of the natural frequencies and modes of vibration of the free system.

With the basic analytical format of the problem clearly defined, attention was next focused on remedial measures and the calculation of good bending mode data.

The first of these is a problem of control system design and is treated in other monographs in this series[23,32]. The second problem is crucial in that errors in bending mode data may degrade control system performance (sometimes catastrophically).

Conceptually, the determination of natural frequencies and modes is simple even for nonhomogeneous structures. The main problem lies in formulating an adequate lumped-mass model which on the one hand is sufficiently detailed to exhibit all phenomena of interest, and yet is not so complex as to overwhelm the capabilities of available computers. Typical numerical schemes employing classical methods such as

Myklestad, or Rayleigh-Ritz have been presented in Refs. 5, 14, 15, and 17. Theoretical motivations are discussed in standard texts[1 - 4].

For a launch vehicle exhibiting a high degree of geometric symmetry, the above methods are adequate. Various complicating factors are nevertheless present, the predominant of which are attached masses whose junction points have stiffness properties which are difficult to ascertain experimentally. These include engine and turbopump mounts, instrumentation packages, payload interfaces, etc. A detailed account of this problem is contained in Ref. 17. The result is that bending mode data seldom exhibits an accuracy better then 5 to 10% for the first mode, with progressively poorer accuracies for the higher modes.

The situation degrades measurably for the case of clustered boosters. Here, coupling between lateral and torsional vibration modes becomes significant, and a good lumped-mass model becomes more difficult to formulate. The determination of modes and frequencies for this case has been studied by Lianis and Fontenot[8], Milner[6], and Storey[7], among others. A useful technique has been developed by Hurty[9] for analyzing complex structures. This method involves the decomposition of the structure into components, each of which is analyzed individually. The properties of the entire structure are then obtained by assembling the component modes after taking due account of the restraints at the junction points. The basic method may be applied to any type of composite structure, i.e., tank and swivelled engine, clustered boosters, etc.

It should be noted that the assumptions which characterize present analytical techniques may not be valid for certain advanced configurations. Among these are finite elastic deformations (nonlinear), and the interaction of sloshing motion with elastic tanks.

3. RECOMMENDED PROCEDURES

The so-called modal method has been found to be most appropriate in expressing the oscillatory motion of an elastic body in which the driving forces are contained within a closed-loop automatic control system. In this approach, the elastic deflections are expressed in terms of the free modes of vibration of the system. The conceptual framework for this scheme is presented in Subsection 3.1, in which the general equations of motion for an unrestrained elastic body are derived from first principles. This is followed by a specialization to the case of an elastic beam in which the particular features which characterize a flexible launch vehicle are discussed.

Combining these results with the equations representing the control system, one may then formulate a complete mathematical model of an autopilot-controlled flexible launch vehicle.

Some general features of the resulting system are discussed in Subsection 3.6, and this is followed by a consideration of various special problems associated with the analysis of launch vehicles.

3.1 GENERAL EQUATIONS OF MOTION FOR AN UNRESTRAINED ELASTIC BODY

The relevant geometry for the problem is shown in Fig. 1. A body coordinate system, represented by the right-hand unit vector triad ($\bar{i}_b$, $\bar{j}_b$, $\bar{k}_b$) with its origin at the mass center of the body is assumed fixed to the body. The radius vector from O to a generic mass particle within the body is deonted by $\bar{r}$. A radius vector from an inertial reference frame ($\bar{i}_I$, $\bar{j}_I$, $\bar{k}_I$) to this same mass particle, is denoted by $\bar{r}_c$.

Via the laws for rate of change of linear and angular momentum, we have

$$\frac{d}{dt}\int_V \frac{d\bar{r}_c}{dt}\rho\, dV = \int_S \bar{F}\, dS \tag{1}$$

$$\frac{d}{dt}\int_V \bar{r}\times\frac{d\bar{r}}{dt}\rho\, dV = \int_S \bar{r}\times\bar{F}\, dS \tag{2}$$

where

$$\bar{r}_c = \bar{r}_o + \bar{r} \tag{3}$$

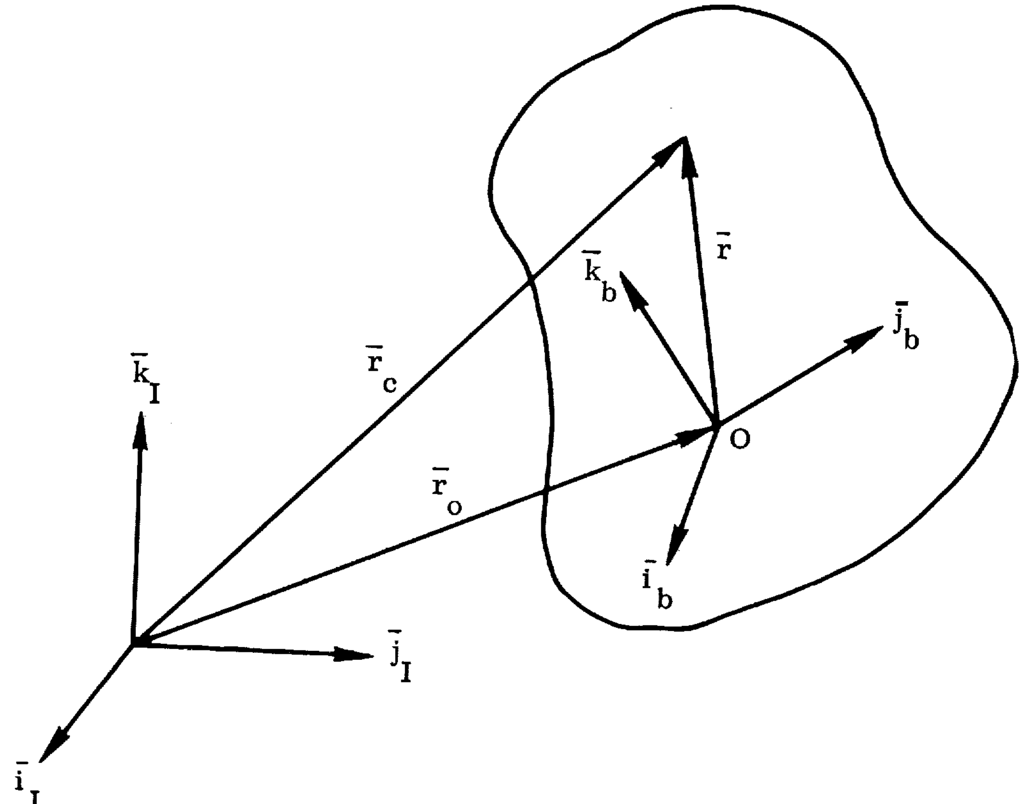

Figure 1. Coordinate Systems for Unrestrained Elastic Body

The integrals on the left-hand side of Eqs. (1) and (2) are taken over the entire volume, V, while those on the right-hand side are taken over the surface, S. The quantities $\bar{F}$ and ρ denote force and mass density respectively.

For a completely rigid body the magnitude of the vector, $\bar{r}$, is constant. However, in general, there will be relative motion between a mass particle within the body and the body-fixed reference frame. In a launch vehicle, for example, such relative motion may arise from various sources such as:

a. Vehicle flexibility.

b. Propellant sloshing.

c. Engine deflection.

etc.

In the development which follows, we take the viewpoint adopted in Ref. 22; namely, the motion of the "main" body (exclusive of sloshing and engine deflection) is derived, after which the coupling terms due to sloshing and engine deflection are introduced as inertial forces (in the sense of D'Alembert) and appropriate kinematic constraints. This approach has the virtue of retaining a physical grasp of the problem, which might otherwise be obscured. It should be emphasized, however, that by suitably interpreting $\bar{r}$ and including the necessary kinematic constraints, Equations (1) and (2) provide a complete description of the motion.

For present purposes we write

$$\bar{r} = \bar{r}_R + \bar{\xi} \tag{4}$$

Here, $\bar{r}_R$ denotes the radius vector from the origin of the body-fixed coordinate system to the undistorted position of the mass particle, and the vector, $\bar{\xi}$, denotes the elastic displacement. The coupling terms due to attached bodies having relative motion with respect to the main body will be added in the manner described in Ref. 22.

The elastic motions must satisfy a generalized form of Hooke's law which is written as follows:

$$\mu \frac{\partial^2 \bar{\xi}}{\partial t^2} + \mathcal{L}(\bar{\xi}) = \bar{F} \tag{5}$$

Here, μ and $\mathcal{L}$ are linear differential operators which involve the spacial variables only. Table 1 lists the form of μ and $\mathcal{L}$ for a number of common systems.

In many important special cases, μ does not involve partial derivatives and is merely a function of the space variables. Thus it is not an operator but merely a function. All of the cases shown in Table 1 are of this type.

In the analysis which follows, crucial simplifications are effected with this assumption and we will therefore adopt it.

The derivation of the equations of motion will proceed, first, by obtaining the solution for free vibration. Then, using the normal modes of vibration, the forced motion will be expressed in terms of these modes.

3.1.1 Free Vibrations in a Vacuum

In the absence of external forces, the system remains inert and we have

$$\bar{F} = \frac{d\bar{r}_o}{dt} = \frac{d\bar{r}_R}{dt} = 0 \tag{6}$$

Consequently, Eqs. (1), (2), and (5) become

Table 1. Linear Differential Operators for Some Common Elastic Systems

SYSTEM	DEFLECTION	μ	$\mathcal{L}$
String	lateral	$\rho\, A(x)$	$T \dfrac{\partial^2}{\partial x^2}$
Bar	longitudinal	$\rho\, A(x)$	$-\dfrac{\partial}{\partial x}\left(E A \dfrac{\partial}{\partial x}\right)$
Bar	torsional	$\rho\, J(x)$	$-\dfrac{\partial}{\partial x}\left(K G \dfrac{\partial}{\partial x}\right)$
Bar	lateral	$\rho\, A(x)$	$\dfrac{\partial^2}{\partial x^2}\left(E I \dfrac{\partial^2}{\partial x^2}\right)$
Membrane	lateral	$\rho\, h\,(x,y)$	$-S\, \nabla^2$
Plate	lateral	$\rho\, h\,(x,y)$	$\nabla^2\,(D\, \nabla^2)$

ρ = density

A = cross-sectional area

I = centroidal moment of inertia of A

J = polar moment of inertia of A

h = plate thickness

T = tensile force

S = tension per unit length

E = Young's modulus

G = shear modulus

K = torsional constant for A (=J for circular sections)

D = $E h^3/12\,(1 - \nu^2)$

ν = Poisson's ratio

$$\int_V \frac{d^2 \bar{\xi}}{dt^2} \rho \, dV = 0 \tag{7}$$

$$\int_V (\bar{r}_R + \bar{\xi}) \frac{d^2 \bar{\xi}}{dt} \rho \, dV = 0 \tag{8}$$

$$\mu \frac{\partial^2 \bar{\xi}}{\partial t^2} + \mathcal{L} (\bar{\xi}) = 0 \tag{9}$$

We now assume that $\bar{\xi}$ may be expressed as the product of a space function and time function as follows

$$\bar{\xi} = \bar{\phi} \, (x, y, z) \, T(t) \tag{10}$$

where $\bar{\phi} = \phi_x \bar{i}_b + \phi_y \bar{j}_b + \phi_z \bar{k}_b$ is an eigenfunction in vector form which represents the natural mode shape. After substituting this in Eqs. (7) - (9), we obtain

$$\int_V \bar{\phi} \rho \, dV = 0 \tag{11}$$

$$\int_V \bar{r}_R \bar{\phi} \rho \, dV = 0 \tag{12}$$

$$\mu \ddot{T} \bar{\phi} + T \mathcal{L} (\bar{\phi}) = 0 \tag{13}$$

This last equation implies that

$$\ddot{T} + \omega^2 T = 0 \tag{14}$$

and

$$\mathcal{L} (\bar{\phi}) - \omega^2 \mu \bar{\phi} = 0 \tag{15}$$

where ω^2 is a constant which represents physically the square of the frequency of the natural vibration modes.

A fundamental property of the eigenfunction vector, $\bar{\phi}$, is that of orthogonality. It is this property which permits one to express the forced motion in terms of these natural modes.

We may derive the orthogonality condition as follows.

If $\bar{\phi}_i$ and $\bar{\phi}_j$ are two distinct modes, then from Eq. (15)

$$\mathcal{L}(\bar{\phi}_i) - \mu \left[\omega^{(i)}\right]^2 \bar{\phi}_i$$

$$\mathcal{L}(\bar{\phi}_j) - \mu \left[\omega^{(j)}\right]^2 \bar{\phi}_j$$

Taking the dot product of the first of these by $\bar{\phi}_j$ and the second by $\bar{\phi}_i$, integrating over the whole volume, and then subtracting the first from the second yields

$$\left(\left[\omega^{(i)}\right]^2 - \left[\omega^{(j)}\right]^2\right)\int_V \mu\, \bar{\phi}_i \cdot \bar{\phi}_j \, dV - \int_V \left[\bar{\phi}_j \cdot \mathcal{L}(\bar{\phi}_i) - \bar{\phi}_i \cdot \mathcal{L}(\bar{\phi}_j)\right] dV = 0 \tag{16}$$

In cases where the boundary conditions are such that the second integral vanishes, we have

$$\int_V \mu\, \bar{\phi}_i \cdot \bar{\phi}_j \, dV = 0 \qquad i \neq j \tag{17}$$

This is the orthogonality condition for the free modes of vibration. It holds for all cases where it can be shown that

$$\int_V \left[\bar{\phi}_j \cdot \mathcal{L}(\bar{\phi}_i) - \bar{\phi}_i \cdot \mathcal{L}(\bar{\phi}_j)\right] dV = 0 \tag{18}$$

To examine the implications of this condition, we consider a few simple cases.

For the lateral vibrations of a bar, $\bar{\phi}$ is merely a scalar and (see Table 1)

$$\begin{aligned} \mathcal{L}(\phi) &= \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \phi}{\partial x^2}\right) \\ &= \frac{d^2}{dx^2}\left(EI \frac{d^2 \phi}{dx^2}\right) \\ &\equiv (EI\phi'')'' \end{aligned} \tag{19}$$

where the primes denote differentiation with respect to x.

Via a repeated integration by parts we have

$$\int_0^L \phi_j \mathcal{L}(\phi_i)\, dx = \int_0^L \phi_j (E I \phi_i'')''\, dx$$

$$= \Big[\phi_j (E I \phi_i'')'\Big]_0^L - \int_0^L \phi_j' (E I \phi_i'')'\, dx$$

$$= \Big[\phi_j (E I \phi_i'')' - \phi_j' (E I \phi_i'')\Big]_0^L + \int_0^L \phi_j'' E I \phi_i''\, dx$$

Similarly

$$\int_0^L \phi_i \mathcal{L}(\phi_j)\, dx = \Big[\phi_i (E I \phi_j'')' - \phi_i' (E I \phi_j'')\Big]_0^L$$

$$+ \int_0^L \phi_i'' E I \phi_j''\, dx$$

Consequently

$$\int_0^L \Big[\phi_j \mathcal{L}(\phi_i) - \phi_i \mathcal{L}(\phi_j)\Big]\, dx$$

$$= \Big[\phi_j (E I \phi_i'')' - \phi_i (E I \phi_j'')' - \phi_j' E I \phi_i'' + \phi_i' E I \phi_j''\Big]_0^L \tag{20}$$

Typical boundary conditions are

Built-in ends: $\phi = 0, \quad \phi' = 0$

Pinned ends: $\phi = 0, \quad E I \phi'' = 0$

Free ends: $E I \phi'' = 0, \quad (E I \phi'')' = 0$

For any one of these, each of the four terms on the right-hand side of Eq. (20) vanishes. Therefore, in this case the orthogonality property (17) is assured.

A somewhat more difficult situation prevails in the case of a thin plate where (see Table 1)

$$\mathcal{L} = \nabla^2 (D \nabla^2) \tag{21}$$

and the del operator is two-dimensional.

From a standard formula in vector analysis, we have ($\bar{\phi}$ is again a scalar)

$$\phi_j \nabla^2 (D\nabla^2 \phi_i) = \nabla \cdot \left[\phi_j \nabla(D \nabla^2 \phi_i)\right] - \nabla \phi_j \cdot \nabla (D \nabla^2 \phi_i) \tag{22}$$

and

$$\nabla \phi_j \cdot \nabla (D \nabla^2 \phi_i) = \nabla \cdot \left[D \nabla^2 \phi_i (\nabla \phi_j)\right] - D \nabla^2 \phi_i \nabla^2 \phi_j \tag{23}$$

Therefore

$$\int_S \left[\phi_j \nabla^2 (D \nabla^2 \phi_i) - \phi_j \nabla^2 (D \nabla^2 \phi_i)\right] dS$$

$$= \int_S \nabla \cdot \left[\phi_j \nabla(D \nabla^2 \phi_i) - D \nabla^2 \phi_i (\nabla \phi_j) - \phi_i \nabla(D \nabla^2 \phi_j)\right.$$

$$\left. + D \nabla^2 \phi_j (\nabla \phi_i)\right] dS$$

$$= \oint_C \left[\phi_j \frac{\partial}{\partial n} (D \nabla^2 \phi_i) - D \nabla^2 \phi_i \frac{\partial \phi_j}{\partial n} - \phi_i \frac{\partial}{\partial n} (D \nabla^2 \phi_j)\right.$$

$$\left. + D \nabla^2 \phi_j \frac{\partial \phi_i}{\partial n}\right] d\ell \tag{24}$$

where the $\frac{\partial}{\partial n}$ denote the directional derivative along the normal in the usual terminology of vector analysis.

If the boundary conditions for the plate are such that the integral (24) vanishes, the orthogonality condition (17) holds. It is not difficult to show that for many simple plate boundary conditions, (24) does indeed vanish.

The solution to the problem of free vibrations is embodied in Eqs. (11), (12), and (15) together with appropriate boundary and initial conditions.

Note that for a free body which has three translational and three rotational degrees of freedom, there will be six modes of zero frequency as follows:

$$
\left.\begin{aligned}
\bar{\phi}_1 &= a_1\,\bar{i}_b \\
\bar{\phi}_2 &= a_2\,\bar{j}_b \\
\bar{\phi}_3 &= a_3\,\bar{k}_b \\
\bar{\phi}_4 &= -z\,\bar{j}_b + y\,\bar{k}_b \\
\bar{\phi}_5 &= z\,\bar{i}_b - x\,\bar{k}_b \\
\bar{\phi}_6 &= -y\,\bar{i}_b + x\,\bar{j}_b
\end{aligned}\right\} \quad (25)
$$

In addition, there are an infinite number of modes of finite frequency corresponding to the elastic degrees of freedom.

3.1.2 Forced Vibrations in Terms of Natural Modes

We assume that the displacement $\bar{\xi}$ may be expressed as

$$
\bar{\xi} = \sum_{i=1}^{\infty} \bar{\phi}_i\, q^{(i)} \tag{26}
$$

Here $\bar{\phi}_i$ is the mode shape (which is a function of spacial variables only), and the $q^{(i)}$ are (as yet undetermined) functions of time only

Expanding Eq. (1) and making use of (26), we find†

$$
\int_V \left[\frac{d^2\bar{r}_o}{dt^2} + \frac{\delta}{\delta t}(\bar{\omega} \times \bar{r}_R) + \bar{\omega} \times (\bar{\omega} \times \bar{r}_R) + \sum_{i=1}^{\infty} \bar{\phi}_i\, \ddot{q}^{(i)}\right] \rho\, dV
$$

$$
= \int_S \bar{F}\, dS \tag{27}
$$

where $\bar{\omega}$ is the angular velocity of the body reference frame.

Since, by assumption, the origin of the body coordinate system coincides with the mass center, we have

$$
\int_V \bar{r}_R\, \rho\, dV = 0 \tag{28}
$$

---

† $\frac{\delta}{\delta t}(\ ) \equiv$ time derivative with respect to body coordinate system.

Using this and the mode property (11), Eq. (27) reduces to

$$m \frac{d^2 \bar{r}_o}{dt^2} = \int_S \bar{F}\, dS \tag{29}$$

where

$$m = \int_V \rho\, dV \tag{30}$$

Similarly, after expanding Eq. (2), substituting (26) and making use of the mode properties (11) and (12), we obtain

$$\frac{d}{dt} \int_V \bar{r}_R \times (\bar{\omega} \times \bar{r}_R)\, \rho\, dV = \int_S \bar{r} \times \bar{F}\, dS \tag{31}$$

Here it has also been assumed that $\bar{\omega}$ and $q^{(i)}$ are small, permitting one to discard higher order terms in these quantities.

In terms of the inertia dyadic

$$I = I_{xx} \bar{i}_b + I_{yy} \bar{j}_b + I_{zz} \bar{k}_b - 2 I_{xy} \bar{i}_b \bar{j}_b - 2 I_{xz} \bar{i}_b \bar{k}_b - 2 I_{yz} \bar{j}_b \bar{k}_b \tag{32}$$

where

$$I_{xx} = \int_V (y^2 + z^2)\, \rho\, dV$$

$$I_{xy} = \int_V xy \rho\, dV$$

etc.

We may express Eq. (31) in the form

$$\frac{d}{dt} (I \cdot \bar{\omega}) = \int_S \bar{r} \times \bar{F}\, dS \tag{33}$$

Eqs. (29) and (33) describe the six rigid body degrees of freedom for the body. These equations are identical to those derived in Ref. 22, although the latter are somewhat more general since the mass center is not coincident with the origin of the body reference frame.

Turning now to the equations which describe the elastic displacement, we find after substituting (26) in Eq. (5)

$$\mu \sum_{i=1}^{\infty} \bar{\phi}_i \ddot{q}^{(i)} + \mathcal{S}\left(\sum_{i=1}^{\infty} \bar{\phi}_i q^{(i)}\right) = \bar{F}$$

Via Eq. (15), this simplifies to

$$\mu \sum_{i=1}^{\infty} \left(\ddot{q}^{(i)} + \left[\omega^{(i)}\right]^2 q^{(i)}\right) \bar{\phi}_i = \bar{F}$$

Taking the dot product of both sides of this equation by $\bar{\phi}_j$, integrating over the whole volume, and making use of the orthogonality property (17), we obtain

$$\ddot{q}^{(j)} + \left[\omega^{(j)}\right]^2 q^{(j)} = \frac{Q^{(j)}}{\mathfrak{m}^{(j)}} \tag{34}$$

$$j = 1, 2, 3, \cdots$$

where

$$\mathfrak{m}^{(j)} = \int_V \mu \, \bar{\phi}_j \cdot \bar{\phi}_j \, dV \tag{35}$$

is the generalized mass (sometimes called "modal mass") and

$$Q^{(j)} = \int_S \bar{F} \cdot \bar{\phi}_j \, dS \tag{36}$$

is the generalized force.

Equations (29), (33), and (34) provide a complete description of the motion for an unrestrained elastic body. These equations are specialized to the case of an elastic beam in Subsection 3.2, and, for the particular case of a launch vehicle, in Subsection 3.3.

3.1.3 Energy Methods

In certain instances it is more convenient to deal with the energy expressions which characterize the force and displacement properties of an elastic system rather than the equations of motion directly. It follows from elementary considerations that the kinetic energy is given by

$$T_E = \frac{1}{2} \int_V \mu \frac{d\bar{r}_c}{dt} \cdot \frac{d\bar{r}_c}{dt} \, dV \tag{37}$$

and the potential energy

$$U_E = \frac{1}{2} \int_V \bar{\xi} \cdot \mathcal{L}(\bar{\xi}) \, dV \tag{38}$$

After expansion of terms in Eq. (37), and making use of Eqs. (11), (12), (17), (26), and (28) together with the definition of the inertia dyadic (32), the kinetic energy expression reduces to

$$T_E = \frac{1}{2} m \frac{d\bar{r}_o}{dt} \cdot \frac{d\bar{r}_o}{dt} + \frac{1}{2} \bar{\omega} \cdot (I \cdot \bar{\omega}) + \frac{1}{2} \sum_{i=1}^{\infty} \mathfrak{M}^{(i)} \dot{q}^{(i)^2} \tag{39}$$

Furthermore, after substituting (26) in (38) and making use of (17), the potential energy takes the form

$$U_E = \frac{1}{2} \sum_{i=1}^{\infty} \mathfrak{M}^{(i)} \left[\omega^{(i)}\right]^2 q^{(i)^2} \tag{40}$$

Viewing $\bar{r}_o$, $\bar{\omega}$, and $q^{(i)}$ as the generalized coordinates (degrees of freedom) of the problem, we may write the expression for virtual work as

$$\delta W = \int_S \bar{F} \cdot \left[\delta \bar{r}_o + \sum_{i=1}^{\infty} \bar{\phi}_i \, \delta q^{(i)}\right] dS + \int_S (\bar{r} \times \bar{F}) \cdot \left[\delta \int_o^t \bar{\omega} \, dt\right] dS \tag{41}$$

where $\delta(\ )$ is the variation operator.

It is now readily found that the above expressions, when substituted in Lagranges equation,

$$\frac{d}{dt}\left(\frac{\partial T_E}{\partial \dot{u}_i}\right) - \frac{\partial}{\partial u_i}(T_E - U_E) = F_i \tag{42}$$

where F_i is the generalized force corresponding to the generalized coordinate u_i, yields the equations (29), (33), and (34) obtained previously.

3.2 BEAM EQUATIONS

Depending on the form of the μ and $\mathcal{L}$ operators, the normal modes derived in the previous section may represent torsional, lateral or longitudinal deflections, or some combination thereof. In this monograph, we shall deal exclusively with the lateral vibrations of a beam, since this effect is predominant as far as interaction with the control system is concerned.

For purposes of deriving the lateral deflections of a beam due to bending and shear strains, we consider the free body diagram of a beam element shown in Fig. 2. The following nomenclature is employed.

B = mass moment of inertia (rotary inertia) per unit length

ℓ = length

m = mass per unit length

M = bending moment

Γ = shear

ϵ = angular displacement due to bending

ξ = beam deflection

F_z = force per unit length

EI = bending stiffness

KG = shearing stiffness

Summing the vertical forces on the beam element we obtain

$$m \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \Gamma}{\partial \ell} = F_z \tag{43}$$

Similarly, by summing moments, we find

$$\Gamma + B \frac{\partial^2 \epsilon}{\partial t^2} - \frac{\partial M}{\partial \ell} = 0 \tag{44}$$

From elementary beam theory we have

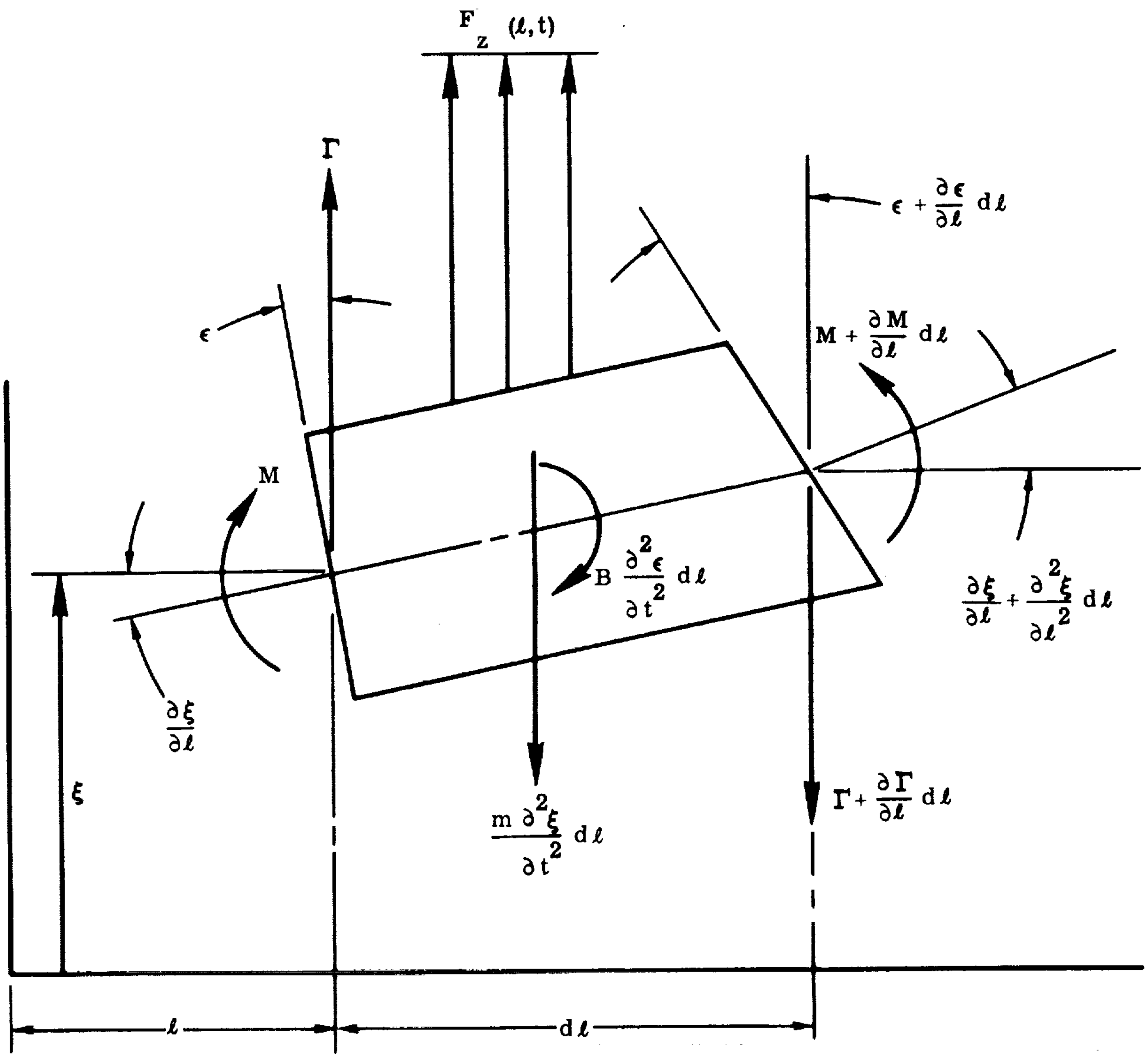

Figure 2. Free Body Diagram of Beam Segment

$$M = EI \frac{\partial \epsilon}{\partial \ell} \tag{45}$$

$$\Gamma = KG \left(\epsilon - \frac{\partial \xi}{\partial \ell}\right) \tag{46}$$

In principle, when the initial and boundary conditions are specified, the set of equations (43) - (46) can be solved for ξ. However, for nonuniform beams, where EI and KG are functions of ℓ, closed-form solutions are not available. Even in the case of free vibrations for uniform beams, the equations are quite complicated. For example, with $F_z = 0$ and B, E, I, K, and G constant, Equations (43) - (46) may be combined to give the form

$$\left[m - \left(\frac{EIm}{KG} + B\right) \frac{\partial^2}{\partial \ell^2} + \frac{Bm}{KG} \frac{\partial^2}{\partial t^2}\right] \frac{\partial^2 \xi}{\partial t^2} + EI \frac{\partial^4 \xi}{\partial \ell^4} = 0 \tag{47}$$

For a slender beam, $KG \to \infty$ and $B \to 0$. In this case, the above equation reduces to

$$m \frac{\partial^2 \xi}{\partial t^2} + EI \frac{\partial^4 \xi}{\partial \ell^4} = 0 \tag{48}$$

For a nonhomogeneous beam, Eq. (48) takes the form

$$m(\ell) \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial^2}{\partial \ell^2} \left[E(\ell)\, I(\ell) \frac{\partial^2 \xi}{\partial \ell^2}\right] = 0 \tag{49}$$

Comparing this expression with Eq. (5), we see that in this case

$$\mu = m(\ell)$$

$$\mathcal{L} = \frac{\partial^2}{\partial \ell^2} \left[E(\ell)\, I(\ell) \frac{\partial^2}{\partial \ell^2}\right]$$

The elastic displacement for most types of flexible launch vehicles is described with sufficient accuracy by Eq. (49), which represents the case of free vibrations. The corresponding modes and frequencies are used to express the motion when external forces are applied. In the general case, when the beam is not uniform, various numerical methods must be used to obtain these modes and frequencies.

It is instructive, however, to examine the simplest case, wherein the beam is uniform. The results of this elementary analysis are helpful in providing a physical grasp and a qualitative indication of the situation which prevails in the more general case. This is done in the following section.

3.2.1 Free Vibrations of Uniform Beam

The motion in this case is described by Eq. (48). Assuming that there is a solution of the form

$$\xi(\ell, t) = \phi(\ell)\, T(t) \tag{50}$$

we obtain, after substituting back in (48)†,

$$-\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{EI}{m\phi}\frac{d^4 \phi}{\partial \ell^4} \tag{51}$$

This equation is valid only if the functions on either side of the equal sign are each equal to some constant, ω^2. Thus (51) is equivalent to the two equations

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0 \tag{52}$$

$$EI\frac{d^4 \phi}{d\ell^4} - m\omega^2 \phi = 0 \tag{53}$$

Physically, ϕ represents the shape of a natural vibration mode and ω is the vibration frequency corresponding to this mode. There are an infinite number of values of ω which satisfy (52) and (53), and to each one there corresponds a particular ϕ.

Using the methods of Subsection 3.1.1, it is easy to show that

$$\int_0^L m\,\phi_i\,\phi_j\, d\ell = 0 \qquad i \neq j \tag{54}$$

where ϕ_i and ϕ_j are two distinct vibration modes. This is a special case of the more general orthogonality condition (17).

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† The analysis which follows is a scalar equivalent of the vector case considered in Subsection 3.1.1. The presentation parallels that of Bisplinghoff, Ref. 4.

The present aim is two-fold:

a. Determine the ω and ϕ values.

b. Express the motion in terms of ω and ϕ.

To do this, we begin by writing the solution to Eqs. (52) and (53), viz.

$$T = C_1 \sin \omega t + C_2 \cos \omega t \tag{55}$$

$$\phi = C_3 \sinh \left(\frac{\omega}{a}\right)^{1/2} \ell + C_4 \cosh \left(\frac{\omega}{a}\right)^{1/2} \ell + C_5 \sin\left(\frac{\omega}{a}\right)^{1/2} \ell + C_6 \cos\left(\frac{\omega}{a}\right)^{1/2} \ell \tag{56}$$

where

$$a^2 = \frac{EI}{m}$$

The constants C_1 through C_6 and ω are obtained from the initial and boundary conditions. We are primarily interested in the free-free case where the shear and bending moment at the ends of the beam are zero; i.e.

$$\phi'' (o) = 0 \tag{57a}$$

$$\phi''' (o) = 0 \tag{57b}$$

$$\phi'' (L) = 0 \tag{57c}$$

$$\phi''' (L) = 0 \tag{57d}$$

Substituting (56) into (57a) and (57b), we obtain

$$\left.\begin{aligned} C_3 - C_5 &= 0 \\ C_4 - C_6 &= 0 \end{aligned}\right\} \tag{58}$$

Furthermore, by substituting (56) into (57c) and (57d) and applying (58), we obtain

$$\left.\begin{aligned}\left[\sinh\left(\frac{\omega}{a}\right)^{1/2} L - \sin\left(\frac{\omega}{a}\right)^{1/2} L\right]C_3 + \left[\cosh\left(\frac{\omega}{a}\right)^{1/2} L - \cos\left(\frac{\omega}{a}\right)^{1/2} L\right]C_4 = 0\\ \left[\cosh\left(\frac{\omega}{a}\right)^{1/2} L - \cos\left(\frac{\omega}{a}\right)^{1/2} L\right]C_3 + \left[\sinh\left(\frac{\omega}{a}\right)^{1/2} L + \sin\left(\frac{\omega}{a}\right)^{1/2} L\right]C_4 = 0\end{aligned}\right\} \quad (59)$$

The value of ω which satisfies the relations (59) is obtained by setting the determinant of the coefficients of C_3 and C_4 equal to zero. After simplification, this operation yields

$$\left[\cos\left(\frac{\omega}{a}\right)^{1/2} L\right]\left[\cosh\left(\frac{\omega}{a}\right)^{1/2} L\right] = 1 \quad (60)$$

The roots of this equation are obtained graphically from the intersections of the curves plotted in Fig. 3. In this case, one of the intersections occurs at a zero value of the argument, indicating that one of the natural frequencies is zero. This situation arises as a result of the fact that the beam is unrestrained, and we may say that this zero frequency corresponds to a gross motion of the unrestrained beam, which we shall term a rigid body mode of motion.

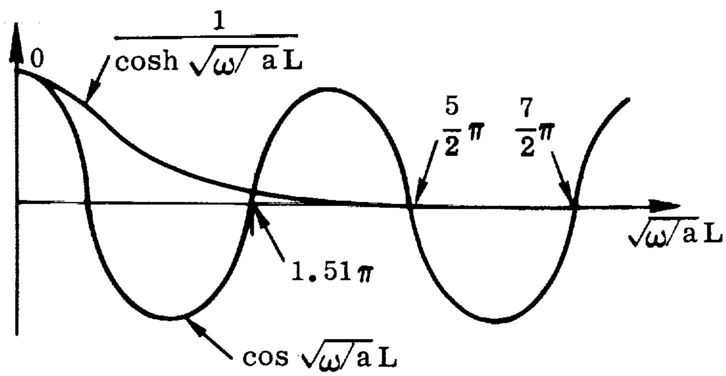


Figure 3. Graphical Solution of Transcendental Equation of Uniform Free-Free Beam

The rigid body mode shape can be obtained by putting $\omega = 0$ into the basic differential equation, (53), and applying the boundary conditions given by Eqs. (57). Putting $\omega = 0$ in Eq. (53) gives

$$\frac{d^4 \phi}{d \ell^4} = 0 \quad (61)$$

which integrates to

$$\phi(\ell) = A_1 \ell^3 + A_2 \ell^2 + A_3 \ell + A_4 \tag{62}$$

Via the boundary conditions (57) we obtain the rigid mode shape

$$\phi_0(\ell) = A_3 \ell + A_4 \tag{63}$$

where A_3 and A_4 are arbitrary constants. Thus the rigid body mode shape of an unrestrained beam can be described by a rigid translation and a rigid rotation.

The elastic mode shapes are obtained by applying (58) and (59) to determine C_3, C_5, and C_6 in terms of C_4, and substituting the result in Eq. (56). This yields

$$\phi_n(\ell) = C_4\left[\left(\frac{\cos\gamma L - \cosh\gamma L}{\sinh\gamma L - \sin\gamma L}\right)\left(\sinh\gamma\ell + \sin\gamma\ell\right) + (\cosh\gamma\ell + \cos\gamma\ell)\right] \tag{64}$$

where

$$\gamma^2 = \frac{\omega^{(n)}}{a} \tag{65}$$

Elastic mode shapes $\phi_1(\ell)$, $\phi_2(\ell)$, $\cdots$, $\phi_n(\ell)$ can be computed from Eq. (64) by substituting corresponding frequency values $\omega^{(1)}$, $\omega^{(2)}$, $\cdots$, $\omega^{(n)}$ obtained from the points of intersection in Fig. 3. The various mode shapes of the unrestrained beam and the corresponding frequencies are illustrated in Fig. 4.

As is apparent from Eq. (64), the elastic modes define <u>relative</u> displacements, since they are determined only to within an arbitrary constant. One may normalize those modes in various ways. The simplest perhaps is the following.

$$\phi^{(i)}(\ell) = \frac{\phi_i(\ell)}{\phi_i(\ell_T)} \tag{66}$$

where ℓ_T is a predetermined location on the beam (usually an end). We call the $\phi^{(i)}(\ell)$ the <u>normalized mode shapes</u>.

The complete solution for the elastic motion in the case of <u>free vibrations</u> may now be written as

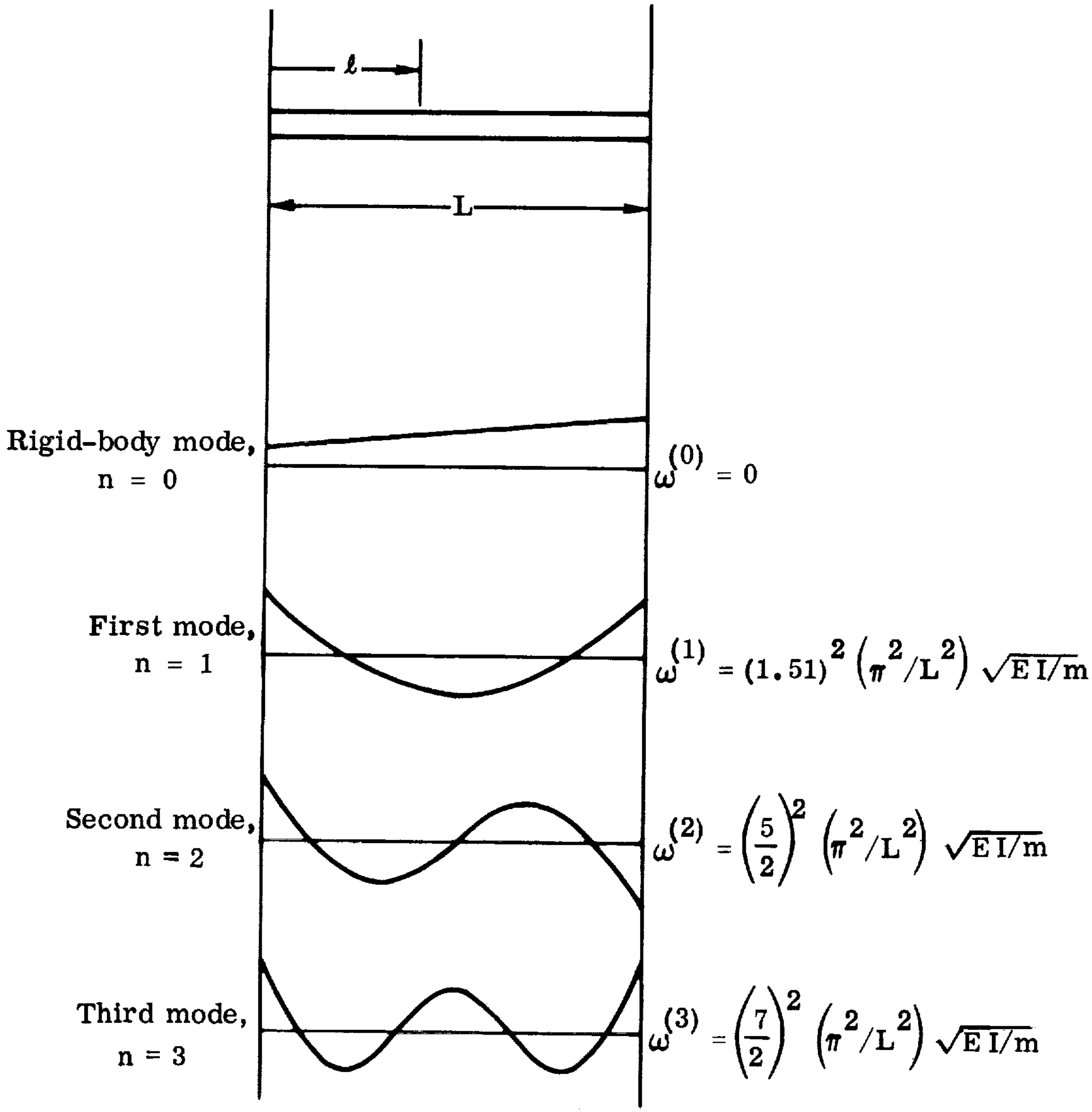


Figure 4. Mode Shapes of Uniform Free-Free Beam

$$\xi(\ell, t) = \sum_{n=1}^{\infty} \phi_n(\ell)\, T_n(t) \tag{67}$$

where $\phi_n(\ell)$ is given by (64) and

$$T_n(t) = C_1 \sin \omega^{(n)} t + C_2 \cos \omega^{(n)} t \tag{68}$$

The constants C_1 and C_2 are determined from the initial conditions.

3.2.2 Forced Vibrations of Nonuniform Beam

In the previous section, a uniform beam was considered for purposes of simplicity in order to exhibit the basic forms of the free-free mode shapes. As a rule, in realistic applications one must deal with nonuniform beams. Here, numerical methods of one kind or another must be used to calculate the natural frequencies and mode shapes. Once these are known, it is a straightforward matter to express the forced motion in these terms.

Neglecting shear and rotary inertia, the equation for forced vibration of a nonhomogeneous beam is given by†

$$m(\ell)\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial^2}{\partial \ell^2}\left[E(\ell)\, I(\ell)\frac{\partial^2 \xi}{\partial \ell^2}\right] = \sum F_z(\ell, t) \tag{69}$$

If we now assume that the modes and frequencies for prescribed boundary conditions have been calculated, the forced displacement may be represented in terms of the normalized modes as follows.

$$\xi(\ell, t) = \sum_{i=1}^{\infty} \phi^{(i)}(\ell)\, q^{(i)}(t) \tag{70}$$

The quantities, $q^{(i)}(t)$, are functions of time, called <u>normal coordinates</u>, that are to be determined.

By substituting (70) in (69), we obtain

$$m\sum_{i=1}^{\infty}\phi^{(i)}\ddot{q}^{(i)} + \sum_{i=1}^{\infty}\frac{d^2}{d\ell^2}\left(E I \frac{d^2\phi^{(i)}}{d\ell^2}\right) q^{(i)} = \sum F_z \tag{71}$$

Multiplying through by $\phi^{(j)}$ and integrating over the length, we have

$$\sum_{i=1}^{\infty}\ddot{q}^{(i)}\int_o^L \phi^{(i)}\phi^{(j)}\, m\, d\ell + \sum_{i=1}^{\infty} q^{(i)}\int_o^L\left[\frac{d^2}{d\ell^2}\left(E I\frac{d^2\phi^{(i)}}{d\ell^2}\right)\right]\phi^{(j)}\, d\ell$$

$$= \int_o^L \sum F_z\, \phi^{(j)}\, d\ell \tag{72}$$

---

†We now write ΣF_z to emphasize that there are applied forces derived from several sources.

From the scalar version of Eq. (15), where in the present case, the linear operator $\mathcal{L}$ is given by (19) and $\mu = m$, it follows that

$$\frac{d^2}{d\ell^2}\left[EI\,\frac{d^2\phi^{(i)}}{d\ell^2}\right] = m\left[\omega^{(i)}\right]^2\phi^{(i)} \tag{73}$$

Substituting this expression in (72) and using the scalar version of the orthogonality property (17), the end result is

$$\mathfrak{M}^{(j)}\ddot{q}^{(j)} + \mathfrak{M}^{(j)}\left[\omega^{(j)}\right]^2 q^{(j)} = Q^{(j)} \tag{74}$$

where

$$\mathfrak{M}^{(j)} = \int_0^L m\left[\phi^{(j)}\right]^2 d\ell \tag{75}$$

and

$$Q^{(j)} = \int_0^L \Sigma F_z\,\phi^{(j)}\,d\ell \tag{76}$$

$\mathfrak{M}^{(j)}$ and $Q^{(j)}$ are the generalized mass and force respectively for the j^{th} mode.

The final solution is obtained by putting the solution of (74) in Eq. (70).

Remark: If $\Sigma F(\ell, t)$ does not depend on the motion of the beam, we see that the differential equations defining the response of the modes are uncoupled and can be solved separately. We shall see, however, (Subsection 3.3) that in the case of a launch vehicle, coupling arises between the elastic, slosh, and rigid body modes since the applied force in each case is dependent on the motion of all of these modes.

For ease of numerical computation, elastic modes are computed with the sloshing mass removed and the engine fixed. Slosh modes are computed assuming a rigid vehicle. In this sense, the modes are often referred to as artifically uncoupled. The coupling is reintroduced via the forcing terms and inertial forces.

In order to take account of structural damping, it is convenient to write Eq. (74) as

$$\ddot{q}^{(i)} + 2\zeta^{(i)} \omega^{(i)} \dot{q}^{(i)} + \left[\omega^{(i)}\right]^2 q^{(i)} = \frac{Q^{(i)}}{\mathfrak{m}^{(i)}} \tag{77}$$

3.2.3 Orthogonality Properties with Shear and Rotary Inertia Included

In many cases the omission of shear and rotary inertia in the beam equations of motion leads to significant errors. For this case, the mode orthogonality properties developed in Subsection 3.1.1 do not apply since now the μ in Eq. (5) is a linear operator. The orthogonality property (17) was derived, however, on the assumption that μ was merely a function.

For purposes of expressing the forced motion, with shear and rotary inertia included, in terms of natural modes and frequencies, suitable orthogonality properties must be derived.

We begin by considering the relevant equations for this case, (43) - (46). After substituting (46) in (43), and putting (45) and (46) in (44), we obtain (with $F_z = 0$)

$$m \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial}{\partial \ell} \left[KG \left(\epsilon - \frac{\partial \xi}{\partial \ell} \right) \right] = 0 \tag{78}$$

$$KG \left(\epsilon - \frac{\partial \xi}{\partial \ell} \right) + B \frac{\partial^2 \epsilon}{\partial t^2} - \frac{\partial}{\partial \ell} \left(EI \frac{\partial \epsilon}{\partial \ell} \right) = 0 \tag{79}$$

The solution is assumed to have the form

$$\xi (\ell, t) = \phi (\ell) \, T (t) \tag{80}$$

$$\epsilon (\ell, t) = \psi (\ell) \, T (t) \tag{81}$$

Substituting this back in (78) and (79), the latter becomes

$$m \phi \frac{d^2 T}{d t^2} + T \frac{\partial}{\partial \ell} \left[KG \left(\psi - \frac{d\phi}{d\ell} \right) \right] = 0 \tag{82}$$

$$B \psi \frac{d^2 T}{d t^2} + T \left[KG \left(\psi - \frac{d\phi}{d\ell} \right) - \frac{\partial}{\partial \ell} \left(EI \frac{\partial \psi}{\partial \ell} \right) \right] = 0 \tag{83}$$

or equivalently†

$$-\frac{\ddot{T}}{T} = \frac{[KG(\psi - \phi')]'}{m\,\phi} \tag{84}$$

$$-\frac{\ddot{T}}{T} = \frac{KG\,(\psi - \phi') - (EI\,\psi')'}{B\psi} \tag{85}$$

This means that

$$-\frac{\ddot{T}}{T} = \frac{[KG\,(\psi - \phi')]'}{m\,\phi} = \frac{KG\,(\psi - \phi') - (EI\,\psi')'}{B\psi} = \omega^2 \tag{86}$$

where ω^2 is an as yet undetermined constant which physically represents the square of a vibration frequency. From (86) we obtain the following three equations

$$\ddot{T} + \omega^2\,T = 0 \tag{87}$$

$$m\,\omega^2\,\phi - [KG\,(\psi - \phi')]' = 0 \tag{88}$$

$$KG\,(\psi - \phi') - (EI\,\psi')' - B\,\omega^2\,\psi = 0 \tag{89}$$

For a specific mode, designated by the subscript "r", we write Eqs. (88) and (89) as

$$m\left[\omega^{(r)}\right]^2 \phi_r - [KG\,(\psi_r - \phi_r')]' = 0 \tag{90}$$

$$KG\,(\psi_r - \phi_r') - (EI\,\psi_r')' - B\left[\omega^{(r)}\right]^2 \psi_r = 0 \tag{91}$$

Multiplying Eq. (90) by ϕ_n and (91) by ψ_n, adding the results, and integrating over the length of the beam, we obtain

$$\left[\omega^{(r)}\right]^2 \int_0^L (m\,\phi_r\,\phi_n + B\,\psi_r\,\psi_n)\,d\ell = -\int_0^L \phi_n\,[KG\,(\phi_r' - \psi_r)]'\,d\ell$$

$$-\int_0^L \psi_n\,(EI\,\psi_r')'\,d\ell - \int_0^L KG\,(\psi_n\,\phi_r' - \psi_r\,\psi_n)\,d\ell \tag{92}$$

† $(\,)' \equiv \frac{\partial}{\partial \ell}(\,)$

Since the subscripts r and n are arbitrary, Eq. (92) is valid if the subscripts are reversed, i.e.

$$\left[\omega^{(n)}\right]^2 \int_0^L (m\,\phi_r\,\phi_n + B\,\psi_r\,\psi_n)\,d\ell = -\int_0^L \phi_r\,[KG\,(\phi_n' - \psi_n)]'\,d\ell$$

$$-\int_0^L \psi_r\,(EI\,\psi_n')'\,d\ell - \int_0^L KG\,(\psi_r\,\phi_n' - \psi_r\,\psi_n)\,d\ell \tag{93}$$

Applying an integration by parts to the first and second terms on the right-hand side of Eqs. (92) and (93) and then subtracting the resulting equations [(93) from (92)], we obtain

$$\left\{\left[\omega^{(r)}\right]^2 - \left[\omega^{(n)}\right]^2\right\} \int_0^L (m\,\phi_n\,\phi_r + B\,\psi_n\,\psi_r)\,d\ell$$

$$= -\phi_n\,KG\,(\phi_r' - \psi_r)\Big|_o^L + \phi_r\,KG\,(\phi_n' - \psi_n)\Big|_o^L$$

$$-\psi_n\,EI\,\psi_r'\Big|_o^L + \psi_r\,EI\,\psi_n'\Big|_o^L \tag{94}$$

For free-free boundary conditions, the shear and slope at the ends of the beam are zero, viz.†

$$M(o) = EI\,\psi'\Big]_{\ell=0} = 0$$

$$\Gamma(o) = KG\,(\psi - \phi') = 0$$

$$M(L) = EI\,\psi'\Big]_{\ell=L} = 0$$

$$\Gamma(L) = KG\,(\psi - \phi')\Big]_{\ell=L} = 0 \tag{95}$$

†See Eqs. (45) and (46).

Consequently, Eq. (94) reduces to

$$\int_0^L (m\,\phi_n\,\phi_r + B\,\psi_n\,\psi_r)\,d\ell = 0 \qquad n \neq r \tag{96}$$

This is the orthogonality condition for the modes of vibration when shear and rotary inertia are included.

Assuming now that the frequencies, $\omega^{(i)}$, and the corresponding modes, ϕ_i and ψ_i, have been calculated, we express the forced motion as follows

$$\xi\,(\ell\,,t) = \sum_{i=1}^{\infty} \phi_i\,q^{(i)} \tag{97}$$

$$\epsilon\,(\ell\,,t) = \sum_{i=1}^{\infty} \psi_i\,q^{(i)} \tag{98}$$

The motion in the case of forced vibration is given by

$$m\,\ddot{\xi} + [KG\,(\epsilon - \xi')]' = F_z \tag{99}$$

which is merely (78) with the forcing term included.

Substituting (97) and (98) in (99) yields

$$m \sum_{i=1}^{\infty} \ddot{q}^{(i)}\,\phi_i + \sum_{i=1}^{\infty} q^{(i)}\,[KG\,(\psi_i - \phi_i')]' = F_z$$

Using (88), this reduces to

$$m \sum_{i=1}^{\infty} \ddot{q}^{(i)} + \sum_{i=1}^{\infty} q^{(i)}\,m\left[\omega^{(i)}\right]^2 \phi_i = F_z \tag{100}$$

Also, after substituting (97) and (98) in Eq. (79), we find

$$B \sum_{i=1}^{\infty} \ddot{q}^{(i)}\,\psi_i + \sum_{i=1}^{\infty} q^{(i)}[\,KG\,(\psi_i - \phi_i') - (EI\,\psi_i')'\,] = 0$$

By virtue of (89), this expression simplifies to

$$B \sum_{i=1}^{\infty} \ddot{q}^{(i)} \psi_i + B \sum_{i=1}^{\infty} q^{(i)} \left[\omega^{(i)}\right]^2 \psi_i = 0 \tag{101}$$

Now, multiplying Eq. (100) by ϕ_j and Eq. (101) by ψ_j, adding the resulting equations, and integrating over the whole beam, we obtain

$$\int_o^L \sum_{i=1}^{\infty} (m\,\phi_i\,\phi_j + B\,\psi_i\,\psi_j)\,\ddot{q}^{(i)}\,d\ell + \int_o^L \sum_{i=1}^{\infty} \left[\omega^{(i)}\right]^2 (m\,\phi_i\,\phi_j$$

$$+ B\,\psi_i\,\psi_j)\,q^{(i)}\,d\ell = \int_o^L F_z\,\phi_j\,d\ell$$

Making use of the orthogonality property (96) this equation reduces to

$$\mathfrak{M}^{(j)}\,\ddot{q}^{(j)} + \mathfrak{M}^{(j)} \left[\omega^{(j)}\right]^2 q^{(j)} = Q^{(j)} \tag{102}$$

where

$$\mathfrak{M}^{(j)} = \int_o^L \left(m\,\phi_j^2 + B\,\psi_j^2\right) d\ell \tag{103}$$

is the generalized mass, and

$$Q^{(j)} = \int_o^L F_z\,\phi_j\,d\ell \tag{104}$$

is the generalized force.

From (102) we see that the form of the equation for the generalized coordinate, $q^{(i)}$, is the same as that with shear and rotary inertia neglected. However, the generalized mass includes a term due to the rotary inertia, B, and a mode, ψ_j, due to shear deflection.

3.2.4 Numerical Methods

For any but the simplest cases, the free modes of vibration must be determined by some appropriate iterative procedure which involves replacing the continuous system by a lumped-mass model.

The classical methods for doing this, due to Holzer, Myklestad, and Rayleigh, are discussed in standard texts[1,4] as well as in companion monographs in the Design Criteria series [15,17]. A recent technique, which is based on the theory of dynamic programming, is described in Ref. 10.

Alley and Guillotte[14] have developed a method for calculating the modal data when shear and rotary inertia are included.

Clustered booster type launch vehicles (e.g. Saturn, Titan III) represent more complex structures requiring extensions of the classical methods for determination of modal data. Computational methods for this case are contained in the studies by Storey[7], Kiefling[24], and Milner[6].

Since a detailed treatment of numerical procedures is beyond the scope of the present monograph, the reader is advised to consult the aforementioned references for specific methods.

3.3 LATERAL VIBRATIONS OF A LAUNCH VEHICLE

In an autopilot-controlled launch vehicle, the vehicle's angular displacement and rate are obtained from gyros located along the vehicle. These devices measure both the rigid body motion and local elastic distortion at the gyro location.

Considered as a beam, the elastic displacement of the launch vehicle is given by Eq. (70), viz.

$$\xi(\ell, t) = \sum_{i=1}^{\infty} \phi^{(i)}(\ell)\, q^{(i)}(t) \tag{105}$$

Only pitch plane motion is considered. The symmetry of most launch vehicles permits one to discard coupling between pitch, yaw, and roll modes, so that each may be analyzed separately. The relevant geometry is shown in Fig. 5, which corresponds to the coordinate system and sign convention of Ref. 22.

We have

$$\frac{\partial \xi(\ell, t)}{\partial \ell} = -\sum_{i=1}^{\infty} \sigma^{(i)}(\ell)\, q^{(i)}(t) \tag{106}$$

where

$$\sigma^{(i)}(\ell) = -\frac{\partial \phi^{(i)}(\ell)}{\partial \ell} \tag{107}$$

is called the normalized mode slope of the i^{th} bending mode.

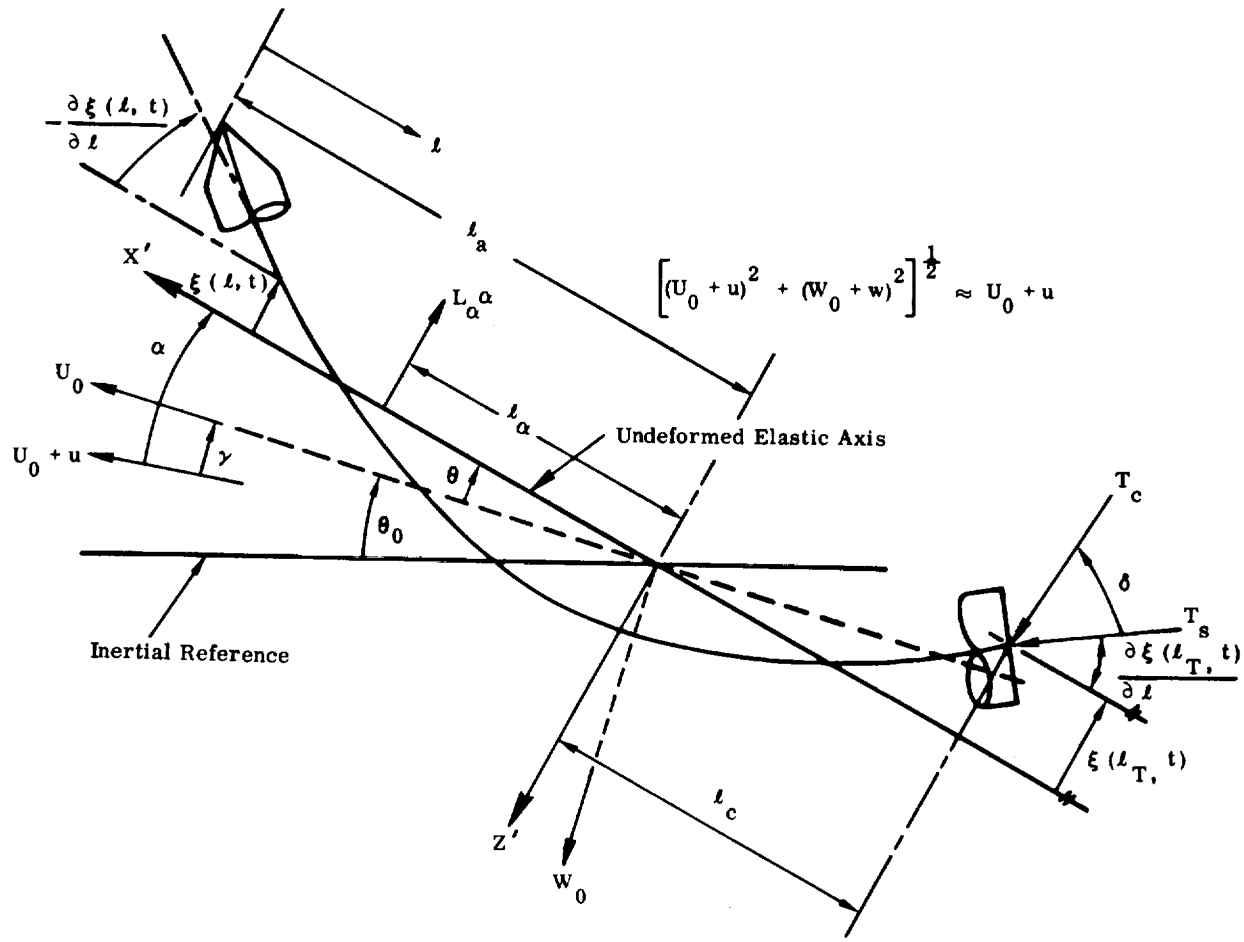


Figure 5. Elastic Vehicle in the Pitch Plane

The quantities of interest for purposes of control system analysis are the following:

Rate Gyro Output:

$$\theta_{RG} = \frac{K_R \omega_R^2 s}{\left(s^2 + 2\zeta_R \omega_R s + \omega_R^2\right)} \left[\theta + \sum_i \sigma^{(i)} (\ell_G) q^{(i)}\right] \tag{108}$$

Position Gyro Output:

$$\theta_{PG} = \frac{1}{(\tau_P s + 1)} \left[\theta + \sum_i \sigma^{(i)} (\ell_G) q^{(i)}\right] \tag{109}$$

This notation corresponds to that defined in Refs. 22 and 23.

The generalized coordinate, $q^{(i)}$, satisfies the equation

$$\ddot{q}^{(i)} + 2\zeta^{(i)}\,\omega^{(i)}\,\dot{q}^{(i)} + \left[\omega^{(i)}\right]^2 q^{(i)} = \frac{Q^{(i)}}{\mathfrak{M}^{(i)}} \tag{110}$$

$$\mathfrak{M}^{(i)} = \int_0^L m\left[\phi^{(i)}\right]^2 d\ell \tag{111}$$

$$Q^{(i)} = \int_0^L \Sigma F_z\,\phi^{(i)}\,d\ell \tag{112}$$

which are merely Eqs. (74) - (76) repeated here for convenience and where a damping term has been included in (110).

It is assumed that the modal data ($\omega^{(i)}$, $\phi^{(i)}$, $\sigma^{(i)}$, and $\mathfrak{M}^{(i)}$) has already been calculated. In order to proceed with the control system analysis, it remains to determine the generalized forces, $Q^{(i)}$. This will be done for the case of a typical launch vehicle in the following subsection.

3.3.1 Generalized Forces

We write the total normal force, ΣF_z, in Eq. (112) in the following form

$$\Sigma F_z = F_{zg} + F_{zT} + F_{za} + F_{zs} + F_{zE} \tag{113}$$

The terms on the right-hand side of this equation denote forces due to gravity, thrust, aerodynamics, propellant sloshing, and rocket engine inertia, respectively. The magnitudes of these forces are derived in Ref. 22 and are repeated below†

Gravity:

$$F_{zg} = -M_t\,g\,\theta\cos\theta_o \tag{114}$$

Thrust:

$$F_{zT} = T_c\,\delta - (T_c + T_s)\sum_i q^{(i)}\,\sigma^{(i)}\,(\ell_T) \tag{115}$$

---

†Notation corresponds to that of Ref. 22.

Aerodynamics:[†]

$$F_{za} = - q_D A \left[\alpha \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} d\ell - \frac{\dot{\theta}}{U_o} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) d\ell \right.$$

$$\left. + \sum_i q^{(i)} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} \sigma^{(i)} (\ell) d\ell - \sum_i \frac{\dot{q}^{(i)}}{U_o} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} \phi^{(i)}(\ell) d\ell \right] \tag{116}$$

Sloshing:

$$F_{zs} = \dot{U}_o \sum_i M_{pi} \Gamma_{pi} \tag{117}$$

Engine Inertia:

$$F_{zE} = M_R \left\{ \ell_R \ddot{\delta} + (\ell_c - \ell_R) \ddot{\theta} - \dot{w} + \dot{U}_o \theta \right.$$

$$\left. - \sum_i \left[\phi^{(i)} (\ell_T) + \ell_R \sigma^{(i)} (\ell_T) \right] \ddot{q}^{(i)} \right\} \tag{118}$$

The dominant terms are $T_c \delta$ and $M_R \ell_R \ddot{\delta}$, which are due to thrust deflection and engine inertia respectively. These are sufficient for purposes of a simplified analysis.

After a straightforward substitution of Eqs. (113) - (118) in Eq. (112), we obtain

† q_D ≡ dynamic pressure
A ≡ reference area

$$Q^{(i)} = \left\{ M_R \dot{U}_o \phi^{(i)} (\ell_T) - g \cos \theta_o \int_o^L \phi^{(i)} (\ell)\, m (\ell)\, d\ell \right\} \theta$$

$$+ \left\{ \frac{q_D A}{U_o} \int_o^L \phi^{(i)} (\ell) \frac{\partial C_N (\ell)}{\partial \alpha} (\ell_a - \ell)\, d\ell \right\} \dot{\theta}$$

$$+ \left\{ M_R \phi^{(i)} (\ell_T) (\ell_c - \ell_R) \right\} \ddot{\theta} + \left\{ T_c \phi^{(i)} (\ell_T) \right\} \delta$$

$$+ \left\{ M_R \ell_R \phi^{(i)} (\ell_T) \right\} \ddot{\delta} - \left\{ q_D A \int_o^L \phi^{(i)} (\ell) \frac{\partial C_N (\ell)}{\partial \alpha} d\ell \right\} \alpha$$

$$- \left\{ M_R \phi^{(i)} (\ell_T) \right\} \dot{w} + \sum_k \left\{ \dot{U}_o M_{Pk} \phi^{(i)} (\ell_{Pk}) \right\} \Gamma_{Pk}$$

$$- \sum_j \left\{ (T_c + T_s) \phi^{(i)} (\ell_T) \sigma^{(j)} (\ell_T) + q_D A \int_o^L \phi^{(i)} (\ell) \frac{\partial C_N (\ell)}{\partial \alpha} \sigma^{(j)} (\ell)\, d\ell \right\} q^{(j)}$$

$$+ \sum_j \left\{ \frac{q_D A}{U_o} \int_o^L \phi^{(i)} (\ell) \frac{\partial C_N (\ell)}{\partial \alpha} \phi^{(j)} (\ell)\, d\ell \right\} \dot{q}^{(j)}$$

$$- \sum_j \left\{ M_R \phi^{(i)} (\ell_T) \left[\phi^{(j)} (\ell_T) - \ell_R \sigma^{(j)} (\ell_T) \right] \right\} \ddot{q}^{(j)} \qquad (119)$$

3.4 FLEXIBLE BODY MOTION OF VEHICLE

The rigid body motion of the vehicle has three translatioal and three rotational degrees of freedom which are described by the general equations (29) and (33) respectively. These, together with the equations for bending and slosh are developed in detail in Ref. 22. Considering pitch plane motion only, the dynamics of the flexible vehicle is described by (all notation conforms to that of Ref. 22):

Moment Equation

$$
\begin{aligned}
I_{yy}\ddot{\theta} = {} & \ell_c\left[T_c\,\delta - (T_c + T_s)\sum_j q^{(j)}\,\sigma^{(j)}\,(\ell_T)\right] - (T_c + T_s)\sum_j q^{(j)}\,\phi^{(j)}\,(\ell_T) \\
& -\sum_k M_{Pk}\,\ell_{Pk}\,\dot{U}_o\,\Gamma_{Pk} + (I_R + M_R\,\ell_R\,\ell_c)\,\ddot{\delta} + M_R\,\ell_R\,\dot{U}_o\,\delta \\
& - (I_R - M_R\,\ell_c^{\,2})\,\ddot{\theta} - M_R\,(\ell_R + \ell_c)\,\dot{w} + M_R\,\ell_c\,\dot{U}_o\,\theta \\
& - M_R\,\ell_R\,\dot{U}_o\sum_j \sigma^{(j)}\,(\ell_T)\,q^{(j)} - \sum_j\Big[M_R\,(\ell_R + \ell_c)\,\phi^{(j)}\,(\ell_T) \\
& + (I_R + M_R\,\ell_R\,\ell_c)\,\sigma^{(j)}\,(\ell_T)\Big]\ddot{q}^{(j)} + q_D\,A\Bigg[\alpha\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a - \ell)\,d\ell \\
& - \frac{\dot{\theta}}{U_o}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a - \ell)^2\,d\ell + \sum_j q^{(i)}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a - \ell)\,\sigma^{(j)}\,(\ell)\,d\ell \\
& - \sum_j \frac{1}{U_o}\,\dot{q}^{(j)}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a - \ell)\,\phi^{(j)}\,(\ell)\,d\ell\Bigg]
\end{aligned} \tag{120}
$$

Normal Force Equation

$$
\begin{aligned}
m_o\,(\dot{w} - U_o\,\dot{\theta}) = {} & - (M_t\,g\cos\theta_o)\,\theta - (T_c + T_s)\sum_j q^{(j)}\,\sigma^{(j)}\,(\ell_T) + T_c\,\delta \\
& + \sum_k M_{Pk}\,\dot{U}_o\,\Gamma_{Pk} + M_R\Bigg\{\ell_R\,\ddot{\delta} + (\ell_c - \ell_R)\,\ddot{\theta} - \dot{w} + \dot{U}_o\,\theta \\
& - \sum_j\Big[\phi^{(j)}\,(\ell_T) + \ell_R\,\sigma^{(j)}\,(\ell_T)\Big]\ddot{q}^{(j)}\Bigg\} - q_D\,A\Bigg[\alpha\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,d\ell \\
& - \frac{\dot{\theta}}{U_o}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a - \ell)\,d\ell + \sum_j q^{(j)}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,\sigma^{(j)}\,(\ell)\,d\ell \\
& - \sum_j \frac{1}{U_o}\,\dot{q}^{(j)}\int_0^L \frac{\partial C_N(\ell)}{\partial\alpha}\,\phi^{(j)}\,(\ell)\,d\ell\Bigg]
\end{aligned} \tag{121}
$$

Sloshing

$$\ddot{\Gamma}_{Pk} + \omega_{Pk}^{2} \Gamma_{Pk} = \frac{1}{L_{Pk}} \left[U_o \dot{\theta} - \dot{w} + \ddot{\theta} \; (\ell_{Pk} - L_{Pk}) \right.$$

$$\left. + \sum_j \phi^{(j)} (\ell_{Pk}) \right] \ddot{q}^{(j)} \tag{122}$$

$k = 1, 2, \cdots$

Bending

$$\ddot{q}^{(i)} + 2\zeta^{(i)} \omega^{(i)} \dot{q}^{(i)} + [\omega^{(i)}]^2 q^{(i)} = \frac{Q^{(i)}}{\mathfrak{m}^{(i)}} \tag{123}$$

$i = 1, 2, \cdots$

Usually, the bending modes are normalized at the engine gimbal point, which means that

$$\phi^{(i)} (\ell_T) = 1$$

for all i.

Sign convention and coordinate systems are shown in Figs. 5 and 6. We note also that

$$\alpha = \frac{w}{U_o} + \alpha_w \tag{124}$$

The (linearized) engine actuator control system is described by

Engine Servo

$$\left(s^3 + 2 \zeta_e \omega_e s^2 + \omega_e^2 s + K_c \omega_e^2 \right) \delta = K_c \omega_e^2 \delta_c - \frac{(s + K_b)}{I_R} T_L \tag{125}$$

where

$$T_L = - (I_R - M_R \ell_c^2) \ddot{\theta} - M_R (\ell_R + \ell_c) \dot{w} + M_R \ell_c \dot{U}_o \theta$$

$$- \sum_i \left[M_R (\ell_R + \ell_c) \phi^{(i)} (\ell_T) + (I_R + M_R \ell_R \ell_c) \sigma^{(i)} (\ell_T) \right] \ddot{q}^{(i)}$$

$$- M_R \ell_R \dot{U}_o \sum_i \sigma^{(i)} (\ell_T) q^{(i)} \tag{126}$$

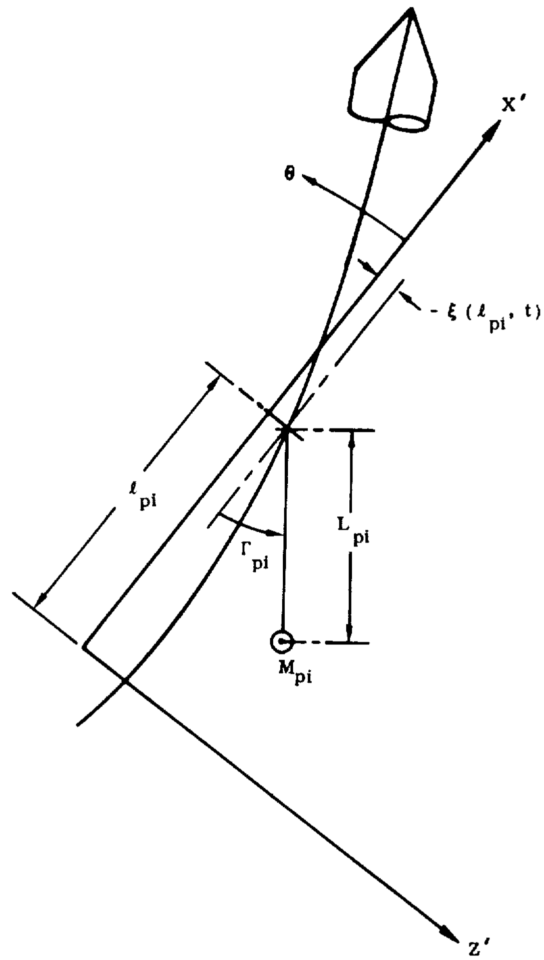


Figure 6. Schematic of Sloshing Pendulum, Pitch Plane

3.5 AUTOPILOT EQUATIONS

The attitude control system for the vehicle employs feedback loops from various sensors for purposes of stability augmentation, as well as to meet additional performance requirements such as load relief and drift minimization. These feedback signals are also given in Ref. 22 and are repeated below.

Feedback Signal

$$\theta_F = \theta_{RG} + \theta_{PG} + \theta_\alpha + \theta_a \tag{127}$$

Position Gyro

$$(\tau s + 1)\,\theta_{PG} = \theta + \sum_i \sigma^{(i)}(\ell_G)\, q^{(i)} \tag{128}$$

Rate Gyro

$$\left(s^2 + 2\zeta_R \omega_R s + \omega_R^2\right)\theta_{RG} = \omega_R^2 K_R s\left[\theta + \sum_i \sigma^{(i)}(\ell_G)\, q^{(i)}\right] \tag{129}$$

Accelerometer

$$\left(s^2 + 2\zeta_a \omega_a s + \omega_a^2\right)\theta_a = \omega_a^2 K_a \left\{\frac{\Sigma F_z}{M_t} - \ell_A \ddot{\theta} + \dot{U}_o \theta - \sum_i \left[\phi^{(i)}(\ell_A)\, \ddot{q}^{(i)} - \sigma^{(i)}(\ell_A)\, \dot{U}_o\, q^{(i)}\right]\right\} \tag{130}$$

where ΣF_z is given by Eq. (113).

Angle-of-Attack Meter (Vane Sensor)

$$\left(s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2\right)\theta_\alpha = \omega_\alpha^2 K_\alpha\left[\alpha - \sum_i \sigma^{(i)}(\ell_m)\, q^{(i)}\right] - \frac{\omega_\alpha^2\left[\ell_m \dot{\theta} + \sum_i \phi^{(i)}(\ell_m)\, \dot{q}^{(i)}\right]}{V} \tag{131}$$

Engine Command Signal

$$\delta_c = K_A\left(1 + \frac{K_I}{s}\right) G_F(s)\,(\theta_c - \theta_F) \tag{132}$$

where $G_F(s)$ represents a filter transfer function.

3.6 COMPLETE EQUATIONS OF MOTION

There are two basic blocks of equations which describe the dynamics of the vehicle autopilot. The first is that which relates the plant variables, θ, w, Γ_{Pk}, and bending variables, $q^{(i)}$, to the thrust deflection angle, δ. These are contained in

Eqs. (110) and (119): Bending

Eqs. (120) - (122): Rigid Body, slosh, bending effects

Because of the formidable complexity of these equations, some systematic procedure must be adopted in order to exhibit the salient features of the system. We begin by writing the equations in matrix form as follows.

$$[A]\{X\} = \{B\}\,\delta \tag{133}$$

Here

$[\;\,]$ ≡ rectangular matrix

$\{\;\,\}$ ≡ column matrix

$\lfloor\;\,\rfloor$ ≡ row matrix

and

$$\{X\} = \begin{bmatrix} \theta \\ w \\ \Gamma_{P1} \\ \Gamma_{P2} \\ q^{(1)} \\ q^{(2)} \end{bmatrix} \tag{134}$$

It is assumed that two bending and two sloshing modes are significant. The components of $[A]$ and $\{B\}$ are, in general, functions of the Laplace operator, s. It is a straightforward, though tedious, matter to evaluate the components of these matrices from Eqs. (110) and (119) - (122). They are written out in full in Appendix A.

We may now partition the matrix equation (133) as follows

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} \theta \\ w \\ \Gamma_{P1} \\ \Gamma_{P2} \\ \hline q^{(1)} \\ q^{(2)} \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \hline b_5 \\ b_6 \end{array}\right] \delta \tag{135}$$

where

A_{11} ≡ 4 × 4 matrix

A_{12} ≡ 4 × 2 matrix

A_{21} ≡ 2 × 4 matrix

A_{22} ≡ 2 × 2 matrix

and write it as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_R \\ X_B \end{bmatrix} = \begin{bmatrix} b_R \\ b_B \end{bmatrix} \delta \tag{134}$$

From this relationship it is easy to see that all coupling between rigid and elastic modes is contained in the matrices $[A_{12}]$ and $[A_{21}]$.

The second major block of equations describes the autopilot and feedback loops. These are (in matrix form)

$$\delta = G_c \delta_c - G_E \lfloor D \rfloor \{X\} \tag{135}$$

$$\delta_c = G_M (\theta_c - \theta_F) \tag{136}$$

$$\theta_F = \lfloor E \rfloor \{X\} \tag{137}$$

The scalar quantities G_c, G_M, and G_E as well as the components of the row matrix $\lfloor D \rfloor$ (all of which are functions of s) are obtained by direct comparison with Eqs. (125), (126), and (132). The components of the row matrix, $\lfloor E \rfloor$ (which are also functions of s) are obtained by comparison with Eqs. (127) - (131), and depend on the types of sensors employed.

Using the set of equations (134) - (137), we may represent the signal flow in the manner shown in Fig. 7. Here the single lines represent scalars and the heavy lines represent vector quantities.

The matrices

$$\left[P_R\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[P_B\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are introduced to preserve the vector formalism. This diagram is useful only in a limited sense in that it exhibits graphically the major coupling effects. Attempts to derive quantitative indications of the influence of mode coupling are speedily enmeshed in a computational quagmire.

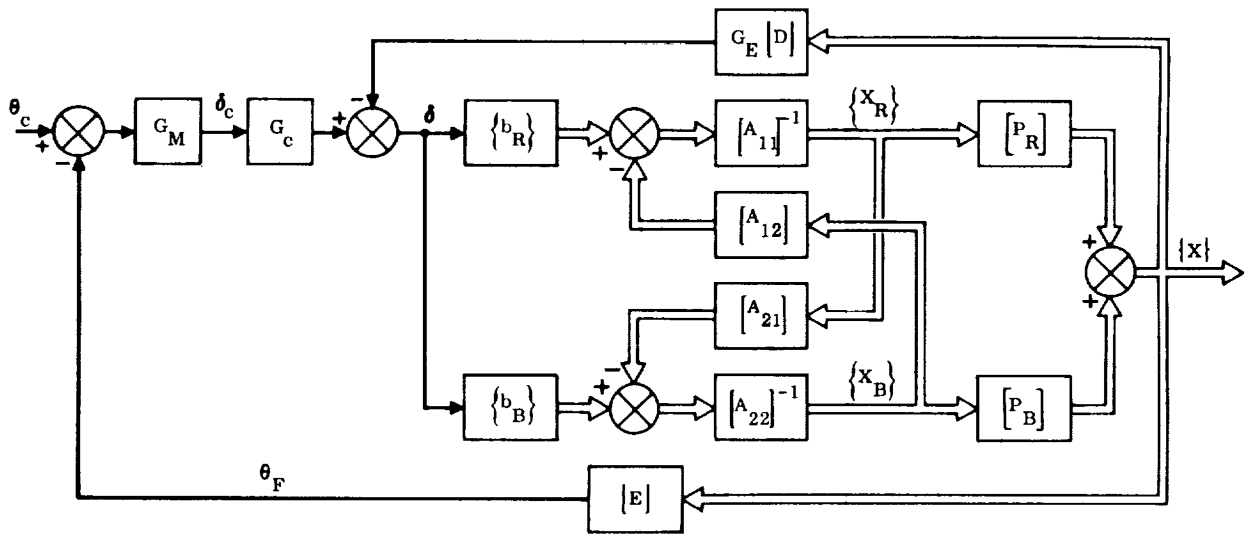


Figure 7. Matrix Schematic of Autopilot

Since a conventional autopilot controls attitude via rate and position feedback, we may represent the signal flow schematic in the alternate form shown in Fig. 8. This more closely resembles the actual configuration which is used for analysis. It is still too complex for simple evaluation, however. Nevertheless, it is in a form which permits selective simplification in a rational manner. The monograph on "Attitude Control During Launch" (Ref. 23) treats this subject in detail.

3.7 SPECIAL PROBLEMS

The type of vehicle considered thus far is a representative and somewhat simplified version of actual launch vehicles. In any particular situation, various special problems arise which require modification or extension of the methods previously described. Some of the more important problems of this type are discussed in the following subsections.

3.7.1 Clustered Boosters

For certain space missions, the requirement for increased thrust dictates the use of multiple rocket engines, which, in turn, means much larger tank structures for fuel storage. Cost and fabrication constraints limit permissible increases in tank diameters. This has led to the clustered tank configuration as typified by the Titan IIIC and certain classes of the Saturn vehicle.

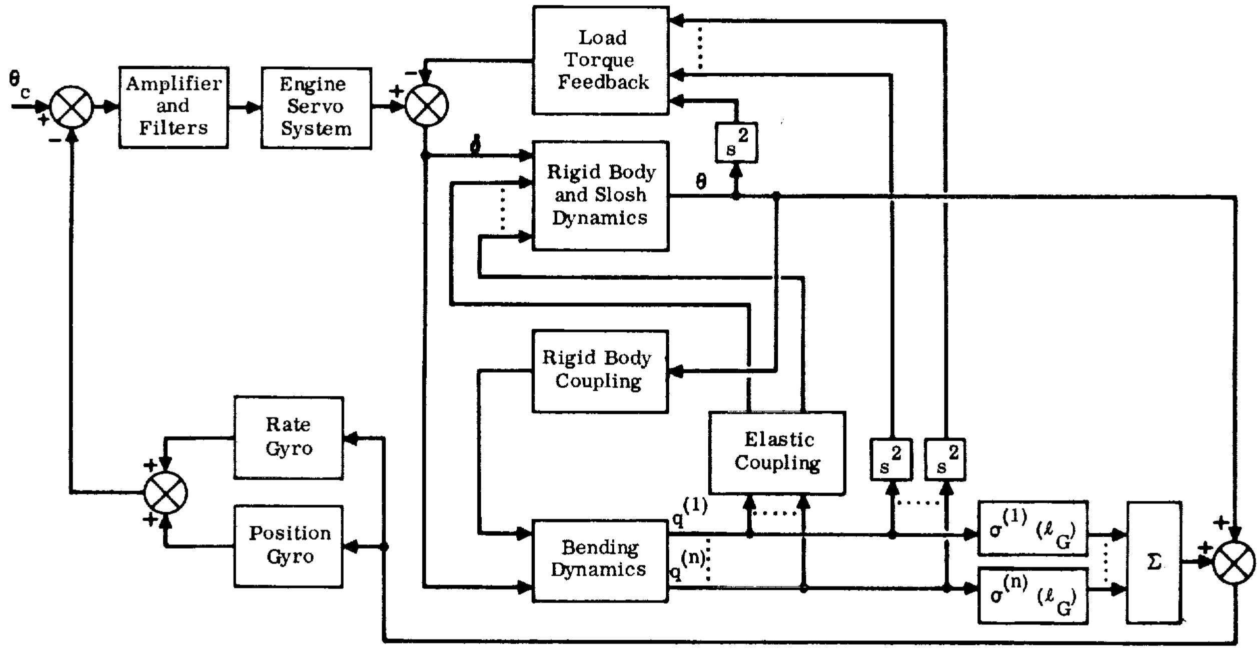


Figure 8. Schematic of Attitude Control System

With liquid-propellant boosters a peripheral ring of propellant tanks is attached to a center tank and the engines are supported on truss members connecting the tanks; with solid-propellant boosters the motors are attached to a central solid- or liquid-propellant booster. These clustered tank designs destroy axial symmetry and also, in some cases, planes of symmetry. This results in a more complicated lateral model where a number of cylindrical tanks are coupled by their elastic connections.

A vibration analysis for this case must take account of the displacement and rotation in two mutually perpendicular planes: torsion and longitudinal motion. The model of the tanks for displacement and rotation in each of the two planes would be very similar to that discussed for the cylindrical booster. Provision must be made to account for the motion of the outer tanks in these two directions due to the torsional displacement of the center tank and the elastic connections. Longitudinal motions of the outer tanks can couple with the bending motion of the center tank; it is also possible that longitudinal motion will couple with lateral and torsional displacement.

The torsional properties in the model can be represented by the torsional stiffness and roll inertia of each tank. The tanks must then be connected by the elastic properties of the truss. The complete model for the clustered booster then consists of the lateral model in two planes: the torsional model and the longitudinal model. These models are then combined through the elasticity and geometry of the connections to provide the stiffness and/or mass coupling.

The usual numerical techniques for obtaining the system eigenvalues may in principle be applied directly. However, with clustered boosters, difficulties are encountered due to inadequate basic data (especially at the elastic junctions), on the one hand, and the large number of lumped-mass points needed to adequately represent the vehicle, on the other. Thus, inversions of very large matrices are required, with the possibility of intolerable computational errors. Furthermore, there may be modes of nearly equal frequency which will be difficult to separate both analytically and experimentally.

One useful approach for calculating the modes of complex systems is the component mode synthesis method(9). Here the modes of the individual pieces are calculated individually and the combined modes are obtained from the modes of the component parts. This method is based on the assumption that significant motions of the individual tanks can be described by a small number of modes. If this is true, then the solutions for the combined system can be performed in terms of fewer coordinates.

Most of the clustered booster work to this date has involved two vehicles — the Titan IIIC and the Saturn I (and the second generation Saturn IB). Titan IIIC is comprised of a center core liquid-propellant booster with two attached solid-propellant boosters (Fig. 9). The connections at the bottom transmit axial load, shear, moment, and torque; the top connection transmits only shear. Because of the nature of the connections, it can be seen that yaw bending and longitudinal coupling can occur; pitch bending and torsion is another possible coupling mechanism. Storey in Ref. 7, develops the coupled flexibility matrices for these two conditions. This method encountered difficulty in that the number of stations required for adequate representation of the system with the required transformations exceeded computer capacity.

The final Titan IIIC analysis presented in Ref. 26 utilizes the mode synthesis approach. The longitudinal, torsional, and pitch and yaw bending modes are determined for each tank and are then coupled by the elasticity of the connecting elements. The influence coefficients for these trusses were obtained experimentally.

The Saturn I vehicle consists of a center liquid oxygen tank with eight peripheral tanks, as shown in Fig. 10.

The Saturn I vehicle consists of a center LO_2 tank with eight peripheral tanks for, alternately, LO_2 and RP-1. These tanks are connected at top and bottom by trusses providing axial, shear, and torsion restraint in both planes at the bottom plus moment restraint in the tangential planes. The top connection provides similar restraint except for the fuel tanks, which do not transmit axial load. The trusses are not symmetric with respect to planes of symmetry of the tanks, but this effect is small so that planes of symmetry as defined by the tanks do not introduce large errors.

Kiefling(24) uses a mode synthesis approach for calculation of Saturn I modes. Pitch, yaw, and torsion are considered uncoupled, and the effect of longitudinal

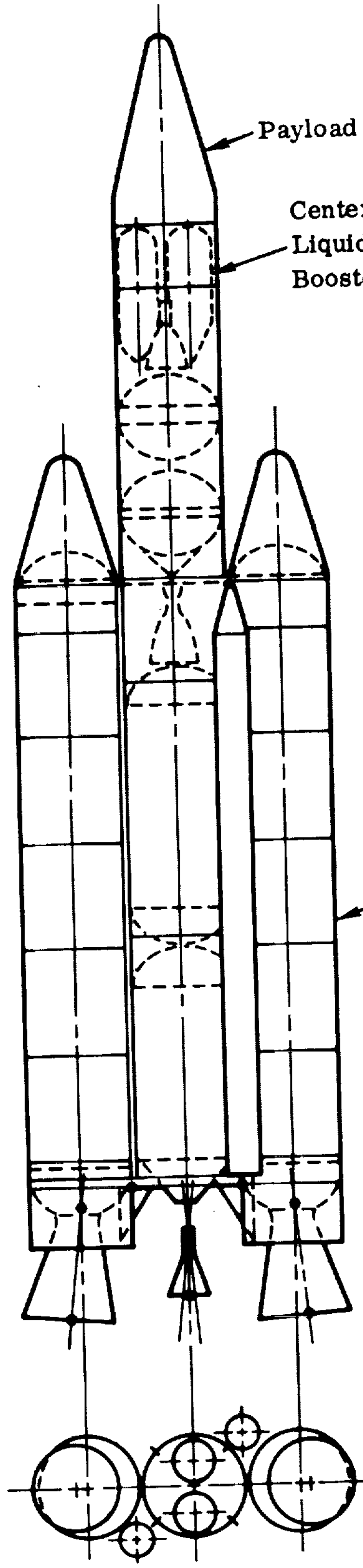


Figure 9. Titan IIIC Configuration

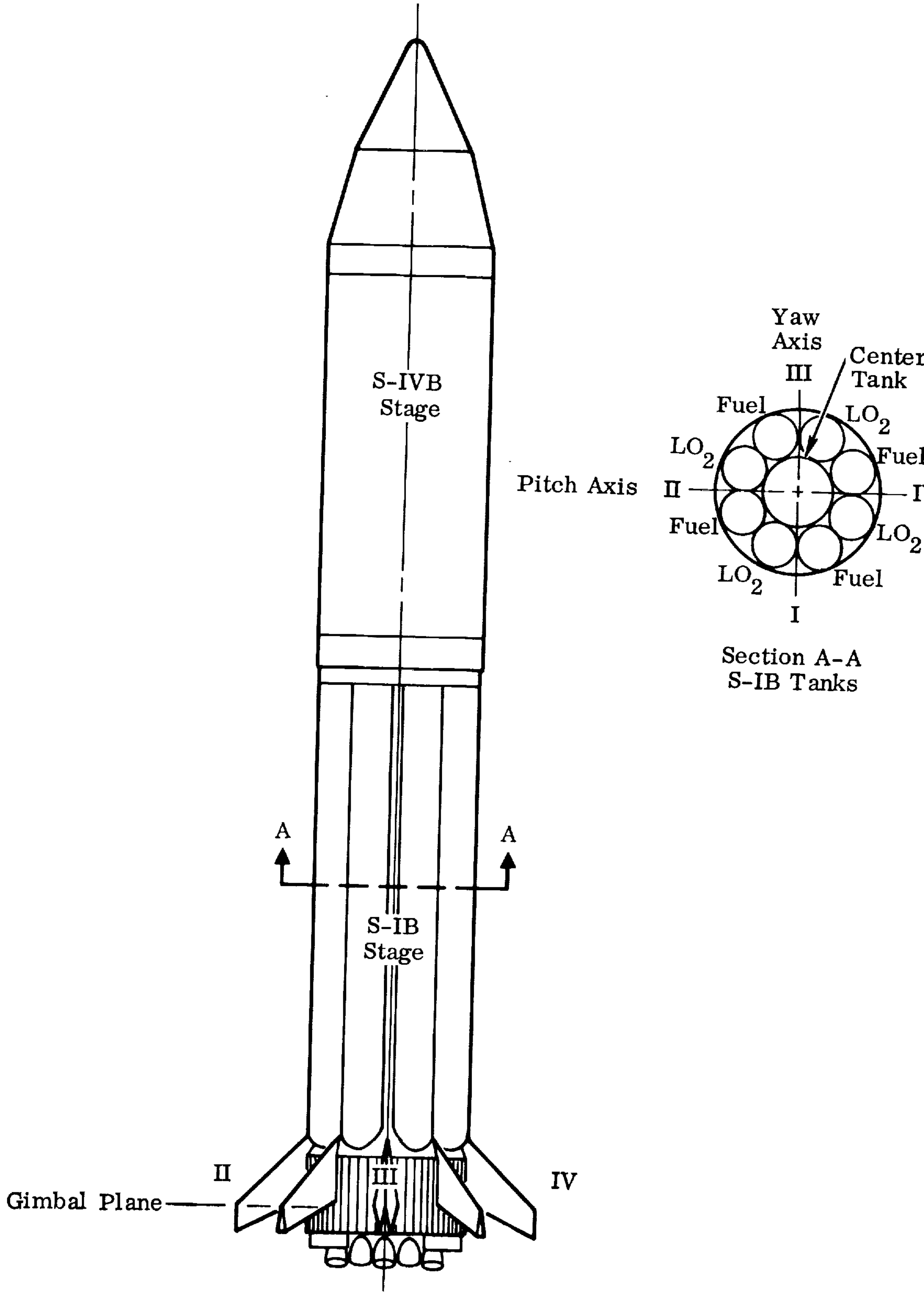


Figure 10. Saturn SA-203 Configuration

propellant vibrations in the outer tanks is coupled with bending. A comparison with test data shows very good agreement in frequency, while agreement of mode shapes is fairly good. The discrepancy, which is seen in the seventh and some higher modes, is in the displacements of the booster center tank. Since no control sensors are located in this area, this discrepancy is of limited importance for stability and control studies. The mode shape differences are due to deflections of the spider beam, the structure connecting the top of the booster tanks and the second stage.

Milner[6] establishes theoretically the uncoupling of pitch, yaw, and torsion modes for a symmetrical clustered booster and investigates the effect of minor asymmetry. Results of this study indicate that the effect of such coupling on natural frequencies is minor; mode shapes are not presented.

The basic analytical features of the clustered booster problem are described in a paper by Lianis and Fontenot[8] which considers an idealized booster with uniformly distributed flexibility and mass. Taking a four-tank configuration (Fig. 11), for example, it is assumed that each cross-section of a tank has two components (x direction and y direction) of rigid body translation and that each cross-section of the four tanks considered as a unit has two components (x direction and y direction) of rigid body translation. The tanks are assumed to be pin-joined at the points A, B, C, D, A′, B′, C′, and D′ as shown in Fig. 11. They are free to vibrate independently at all other points. It is obvious from the way the tanks are joined that the axis of rotation is the central z axis of the booster. Thus, the rigid body motion of an end section is described by three independent components, namely, translation in the x direction, translation in the y direction, and rotation around the z axis.

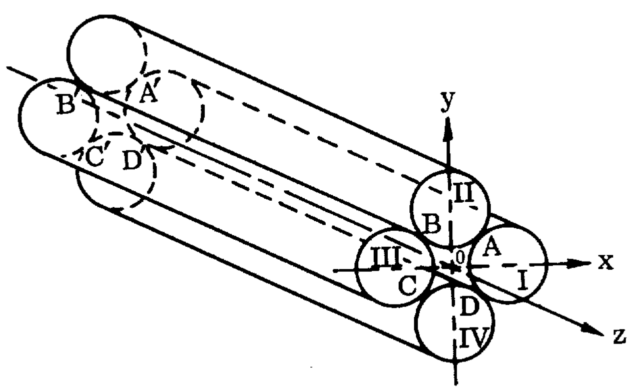


Figure 11. Geometry of Clustered Booster and Axes of Reference

The z axis was chosen as the axis of rotation for each cross-section of any tank since this choice does not cause any additional rigid body motion (for small displacements) and it furnishes simple, kinematic boundary conditions. However, the flexural rotation of the end section of each tank in the xz and yz planes need not be equal because it was assumed that the points A, B, C, D, A′, B′, C′, and D′ are pin-joints. This assumption eliminates the need for introducing additional reaction bending moments transmitted through the points of connection between the tanks, and it simplifies the static boundary conditions.

Using the conventional theory of bending and torsion of beams, we may then write for each tank[8]

$$\frac{\partial^4 x_i}{\partial z^4} + a^2\left(\frac{\partial^2 x_i}{\partial t^2} + \epsilon_i \frac{\partial^2 \phi_i}{\partial t^2}\right) = 0 \tag{138}$$

$$\frac{\partial^4 y_i}{\partial z^4} + a^2\left(\frac{\partial^2 y_i}{\partial t^2} + \nu_i \frac{\partial^2 \phi_i}{\partial t^2}\right) = 0 \tag{139}$$

$$\frac{\partial^2 \phi_i}{\partial z^2} - b\left(\frac{5+h}{2}\right) R^2 a^2 \frac{\partial^2 \phi_i}{\partial t^2}$$

$$- 2 b R^2 a^2 \left(\nu_i \frac{\partial^2 y_i}{\partial t^2} + \epsilon_i \frac{\partial^2 x_i}{\partial t^2}\right) = 0 \tag{140}$$

Here

i = I, II, III, or IV (the tank number in Fig. 11)

and the numbers ϵ_i and ν_i have the following values

i	ϵ_i	ν_i
I	0	1
II	-1	0
III	0	-1
IV	1	0

Symbols are defined as follows:

$$a = \sqrt{\frac{W}{g E I}}$$

$$b = \frac{E I}{G C}$$

h = ratio of weight of tank to total weight of tank plus liquid

EI = uniform flexural stiffness of each beam

GC = uniform torsional stiffness of each beam

W = weight of tank per unit length

R = radius of each tank

g = gravity acceleration

One may proceed in the usual way by assuming a solution of the form

$$x_i(z, t) = X_i(z)\, T(t) \tag{141}$$

$$y_i(z, t) = Y_i(z)\, T(t) \tag{142}$$

$$\phi_i(z, t) = \Phi(z)\, T(t) \tag{143}$$

$$i = \mathrm{I, II, III, IV}$$

After substituting (141) - (143) into (138) - (140) we obtain a system of 12 equations involving a separation constant, ω^2, and 40 constants of integration.† The latter are evaluated from the boundary conditions which may be summarized as follows:

a. The x displacement of both end-sections must be the same for all tanks, i.e., $X_I = X_{II} = X_{III} = X_{IV}$ for $z = 0$ and $z = \ell$, where ℓ = the length of the booster. This condition provides six equations.

b. The y displacement of both end-sections must be the same for all tanks, i.e., $Y_I = Y_{II} = Y_{III} = Y_{IV}$ for $z = 0$ and $z = \ell$. This condition furnishes six equations.

c. The twists of both end-sections must be the same for all tanks, i.e., $\Phi_I = \Phi_{II} = \Phi_{III} = \Phi_{IV}$ for $z = 0$ and $z = \ell$. This condition furnishes six equations.

d. The total shear force in the x direction at each end must be zero. One knows from the engineering theory of bending that the shear force in a beam is proportional to the third derivative of the transverse displacement with respect to the axial co-ordinate. Therefore, this condition is satisfied if $\sum_{i=I}^{IV} d^3 X_i / dz^3 = 0$ for $z = 0$ and $z = \ell$. This condition furnishes two equations. The total shear force in the y direction at each end also must be zero, i.e., $\sum_{i=I}^{IV} d^3 Y_i / dz^3 = 0$ and $z = \ell$. This

†As usual, it is found that physically ω represents the frequency of vibration.

condition furnishes two equations.

e. The total torque at both ends is zero. Since the torque of a beam is proportional to the first derivative of the angle of twist, this condition leads to $\sum_{i=I}^{IV} d\Phi_i/dz = 0$ for $z = 0$ and $z = \ell$, which furnishes two equations.

f. It is assumed here that each tank is simply supported at both ends on the two adjacent tanks. The bending moment in a certain direction is proportional to the second derivative of the transverse displacement in the corresponding direction with respect to the axial coordinates. This condition leads to the following 16 equations: $d^2 X_i/dz^2 = 0$ for $z = 0$ and $z = \ell$, $d^2 Y_i/dz^2 = 0$ for $z = 0$ and $z = \ell$, where i = I, II, III, and IV.

By counting the foregoing conditions, one observes that there are exactly 40 equations for the 40 unknowns.

One thus obtains a system of 40 linear homogeneous equations for the 40 unknowns. A nontrivial solution exists if the determinant of the coefficients is zero. This condition permits one to obtain the natural frequencies. Except for certain obvious cases, the solution must be obtained via some iterative procedure on a computer.

In general, the Lianis-Fontenot study indicates the presence of closely coupled bending-torsional modes, the closeness of certain natural frequencies, and the multiple occurrence of others. For this case, the difficulties associated with the control system design are not at all trivial.

3.7.2 Sloshing in Flexible Tanks

The importance of the liquid sloshing phenomenon in the autopilot control of launch vehicles has long been recognized.† In the usual methods of analysis, it is found that the dynamic properties of the sloshing liquid may be closely approximated by a series of pendulums (or harmonic oscillators) whose size and location along the vehicle depend on the vehicle's mass, inertial, and geometric properties as well as on the properties of the liquid and its level relative to the tank. The assumption is always made that the tank is rigid.

If, however, the tank wall is flexible, and the liquid has a free surface, the interaction between the liquid motions and elastic deformations of the tank wall could become significant. Furthermore, conventional analytic investigations consider the liquid-filled structure as a beam, assuming the tank long enough so that beam properties predominate. Actually, many tank walls have a comparatively small length-

†Cf. Refs. 20, 29, and 30.

to-diameter ratio, which means that shell action could be the overriding factor.

The analysis for the case of a coupled elastic tank and liquid propellant becomes quite complicated even in highly simplified versions. The most complete study of the problem is by Lianis and Fontenot[31]. Their preliminary results show that the low frequency oscillations (the only significant ones) are not appreciably affected by the presence of elasticity and inertia of the shell.

Thus, except possibly for radical new tank configurations, the elasticity of the launch vehicle structure does not significantly affect the slosh modes and frequencies calculated by assuming a completely rigid tank.

3.7.3 Attached Masses and Component Modes

A typical launch vehicle has many secondary structures attached to the main beam structure. Among these are rocket engines, instrument packages, turbopumps, sloshing masses, etc. In the computation of bending mode data, the usual practice is to artificially uncouple all secondary masses and determine the natural frequencies and bending modes for the basic beam structure. The degrees of freedom associated with the secondary masses are then introduced via appropriate forcing functions and kinematic contraints. The generalized coordinates obtained in this way are simply related to the actual elastic displacements of the beam, even though the bending modes are coupled to the other degrees of freedom as well as to each other. Alternately, one may include the secondary masses in the mode calculations, in which case the resulting generalized coordinates are not simply related to easily measured displacements. Either approach is valid so long as the results are properly used and interpreted.

For more complex structures such as clustered boosters, it would appear desirable to calculate the modal properties for the individual tanks and then somehow integrate the results to give the mode properties for the entire structure. A method for doing this, called "component mode synthesis," has been developed by Hurty[9]. It can be applied to any structure which is arranged basically as a series of interconnected components whose individual mode properties may be separately determined. A complete description of the method in the general case is quite lengthy and does not appear to be warranted for present purposes. However, a simplified version, which may be applied to most vehicles of interest, can be developed with relatively little effort.

We suppose that each of the (artificially disconnected) components of a structure may be described by the matrix equation

$$[m]_r \{\ddot{x}\}_r + [c]_r \{x\}_r = \{f\}_r \tag{144}$$

where

$[m]_r \equiv$ mass matrix of r^{th} component

$[c]_r \equiv$ stiffness matrix of r^{th} component

$\{f\}_r \equiv$ applied force vector for r^{th} component

$\{x\}_r \equiv$ coordinate vector of r^{th} component

The modes and frequencies may be obtained in the usual way. After defining a new coordinate vector by

$$\{x\}_r = [R]_r \{q\}_r \tag{145}$$

where $[R]_r$ is the modal matrix (i.e., the matrix whose columns are the eigenvectors of the r^{th} component), we obtain after substituting (145) in (144)

$$[\mathfrak{m}]_r \{\ddot{q}\}_r + [k]_r \{q\}_r = \{Q\}_r \tag{146}$$

where

$[\mathfrak{m}]_r = [R]_r^T [m]_r [R]_r \equiv$ generalized mass for r^{th} component

$[k]_r = [R]_r^T [c]_r [R]_r \equiv$ generalized stiffness for r^{th} component

$\{Q\}_r = [R]_r^T \{f\}_r \equiv$ generalized force for r^{th} component

The vector $\{q\}_r$ is the generalized (normal) coordinate vector.

We note that $[\mathfrak{m}]_r$ and $[k]_r$ will be diagonal matrices if the eigenvalues are distinct.

It can also be shown that

$$[k]_r = [\mathfrak{m}]_r [W]_r \tag{147}$$

where

$$[W]_r = \begin{bmatrix} \omega_{r1}^2 & & 0 \\ & \omega_{r2}^2 \cdots\cdots & \\ 0 & & \omega_{rn}^2 \end{bmatrix} \tag{148}$$

and the ω_{rj} are the natural frequencies for the r^{th} component.

For the complete system we may write

$$\{x\} = [R]\{q\} \tag{149}$$

where

$$\{x\} \begin{bmatrix} \{x\}_1 \\ \vdots \\ \{x\}_m \end{bmatrix} \qquad \{q\} \begin{bmatrix} \{q\}_1 \\ \vdots \\ \{q\}_m \end{bmatrix}$$

$$[R] = \begin{bmatrix} [R]_1 & & & 0 \\ & [R]_2 & & \\ & & \ddots & \\ 0 & & & [R]_m \end{bmatrix}$$

We will also use the notation

$$[m] = \begin{bmatrix} [m]_1 & & & 0 \\ & [m]_2 & & \\ & & \ddots & \\ 0 & & & [m]_m \end{bmatrix}$$

$$[k] = \begin{bmatrix} [k]_1 & & & 0 \\ & [k]_2 & & \\ & & \ddots & \\ 0 & & & [k]_m \end{bmatrix}$$

The total kinetic energy for the system (m components) is then found as

$$T = \frac{1}{2}\{\dot{x}\}^T [m] \{x\}$$

which by virtue of (149) becomes

$$T = \frac{1}{2}\{\dot{q}\}^T [\mathfrak{m}] \{\dot{q}\} \tag{150}$$

Furthermore, the potential energy in the m components is

$$U_m = \frac{1}{2}\{x\}^T [c] \{x\}$$

which takes the form

$$U_m = \frac{1}{2}\{q\}^T [k] \{q\} \tag{151}$$

after using (149).

Thus far, nothing has been said about the constraints introduced by the points of connection between the component members of the structure. If we let

Δ_k = deflection of the k^{th} connection

γ_k = stiffness of k^{th} connection

then the potential energy in the connection joints is given by

$$U_c = \frac{1}{2}\sum_k \gamma_k \Delta_k^2$$

This may be expressed in matrix form as

$$U_c = \frac{1}{2}\{\Delta\}^T [\gamma] \{\Delta\} \tag{152}$$

However, the deflection at a connection point may be expressed in terms of the system coordinates by

$$\{\Delta\} = [B]\{x\} = [B][R]\{q\} \tag{153}$$

where the matrix $[B]$ is obtained from the geometry of the problem.

The total potential energy is then given by the sum of (151) and (152) which takes the form

$$U = \frac{1}{2}\{q\}^T \Big([k] + [A]\Big)\{q\} \tag{154}$$

where

$$[A] = [R]^T [B]^T [\gamma] [B] [R] \tag{155}$$

The dynamical representation for the complete system is then written from the Lagrangian formulation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \tag{156}$$

$$i = 1, 2, \cdots, p$$

where Q_i is the force corresponding to the generalized coordinate q_i. The total number of equations, p, in (156) is equal to the sum of the individual degrees of freedom of all the components in the structure.

Modes and frequencies for the system (156) may then be calculated in the usual way. These frequencies will be identical to those obtained if the structure were analyzed in the conventional classical manner. However, the eigenvectors (modes) obtained from (156) must be premultiplied by the [R] matrix to obtain the eigenvectors relative to the original $\{x\}$ coordinate system.

The main advantage of the component mode method is that the complete system representation (156) may be systematically built up from mode data obtained from the individual components. This approach has significant advantages in many cases.

The following simple example will serve to clarify the general procedure. Consider the simple spring mass system shown in Fig. 12a.

For purposes of determining natural modes and frequencies by the component mode method, we "break" the system at the spring c_2 (which is therefore taken as a connection point later). The two components thus formed are shown in Fig. 12b.

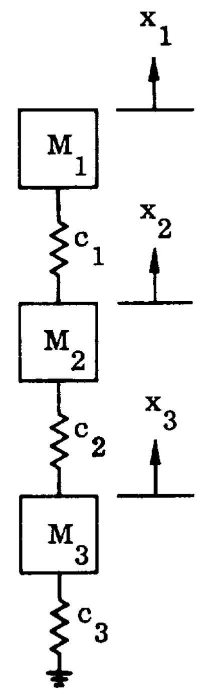
Taking the first component we have

$$m_1 \ddot{x}_1 + c_1 (x_1 - x_2) = 0$$

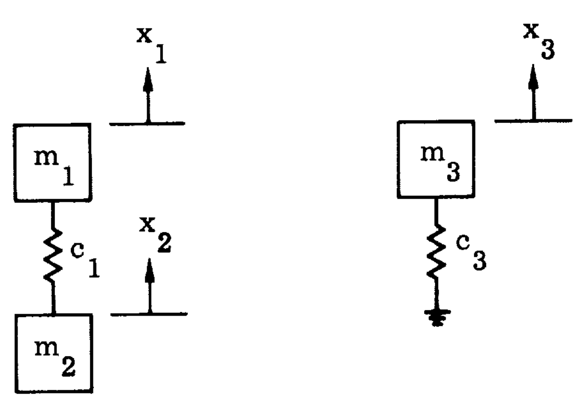
$$m_2 \ddot{x}_2 + c_1 (x_2 - x_1) = 0$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



a. Complete System



b. Components

Figure 12. Simple Spring-Mass System

or

$$[m]_1 \{\ddot{x}\}_1 + [c]_1 \{x\}_1 = \{0\}$$

The frequency equation is

$$\left| [c]_1 - \omega^2 [m]_1 \right| = 0$$

which leads to

$$\omega^2 \left[\omega^2 m_1 m_2 - c_1 (m_1 + m_2) \right] = 0$$

Consequently, the eigenvalues are

$$\omega_1^2 = \frac{c_1 (m_1 + m_2)}{m_1 m_2}$$

$$\omega_2^2 = 0$$

The modal matrix is

$$[R]_1 = \begin{bmatrix} 1 & 1 \\ -\dfrac{m_1}{m_2} & 1 \end{bmatrix}$$

and

$$[\mathfrak{m}]_1 = \begin{bmatrix} \dfrac{m_1 (m_1 + m_2)}{m_2} & 0 \\ 0 & (m_1 + m_2) \end{bmatrix}$$

For the second component, we have simply

$$m_3 \ddot{x}_3 + c_3 x_3 = 0$$

From this, we find

$$\omega_3^{\;2} = \frac{c_3}{m_3}$$

$$[\mathfrak{m}]_2 = m_3$$

$$[R]_2 = 1$$

Consequently, the composite modal matrix takes the form

$$[R] = \begin{bmatrix} 1 & 1 & 0 \\ -\dfrac{m_1}{m_2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{157}$$

Now the deflection at the connection point may be written as

$$\Delta = x_2 - x_3$$

$$= [0 \quad 1 \quad -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which means that the matrix [B] is

$$[B] = [0 \quad 1 \quad -1]$$

Taking note of the fact that at the connection point, the stiffness, $\gamma = c_2$, we obtain for the matrix [A] of Eq. (155)

$$[A] = c_2 \begin{bmatrix} \left(\dfrac{m_1}{m_2}\right)^2 & -\dfrac{m_1}{m_2} & \dfrac{m_1}{m_2} \\ -\dfrac{m_1}{m_2} & 1 & -1 \\ \dfrac{m_1}{m_2} & -1 & 1 \end{bmatrix}$$

Also

$$
[\mathfrak{m}] = \begin{bmatrix} \dfrac{m_1 (m_1 + m_2)}{m_2} & 0 & 0 \\ 0 & (m_1 + m_2) & 0 \\ 0 & 0 & m_3 \end{bmatrix}
$$

$$
[k] = [R]^T [c] [R]
$$

$$
= \begin{bmatrix} \dfrac{m_1 (m_1 + m_2)}{m_2} \omega_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_3 \omega_3^2 \end{bmatrix}
$$

Now, with the $\{q\}$ vector defined by Eq. (149), a straightforward application of Lagrange's equation (156) yields

$$
\begin{bmatrix} \mathfrak{m}_1 & 0 & 0 \\ 0 & \mathfrak{m}_2 & 0 \\ 0 & 0 & \mathfrak{m}_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \left(\mathfrak{m}_1 \omega_1^2 + \dfrac{m_1^2}{m_2^2} c_2 \right) & - \dfrac{m_1}{m_2} c_2 & \dfrac{m_1}{m_2} c_2 \\ - \dfrac{m_1}{m_2} c_2 & c_2 & -c_2 \\ \dfrac{m_1}{m_2} c_2 & -c_2 & (c_2 + c_3) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (158)
$$

The modes and frequencies for the composite system are obtained from the above matrix equation in the conventional way.

In order to compare numerical values generated by the component mode technique with those obtained via classical methods, we assume the following numerical values

$$m_1 = 10 \text{ slugs} \qquad c_1 = 5,000 \text{ lb/ft}$$

$$m_2 = 20 \text{ slugs} \qquad c_2 = 1,000 \text{ lb/ft}$$

$$m_3 = 40 \text{ slugs} \qquad c_3 = 6,000 \text{ lb/ft}$$

From this, we find

$$\omega_1^2 = 750 \qquad \mathfrak{m}_1 = 15$$

$$\omega_2^2 = 0 \qquad \mathfrak{m}_2 = 30$$

$$\omega_3^2 = 150 \qquad \mathfrak{m}_3 = 40$$

The eigenvalue determinant equation for the system (158) becomes

$$\begin{vmatrix} (11,500 - 15\,\omega^2) & -500 & 500 \\ -500 & (1,000 - 30\,\omega^2) & -1,000 \\ 500 & -1,000 & (7,000 - 40\,\omega^2) \end{vmatrix} = 0$$

or

$$\omega^6 - 975\,\omega^4 + 16.375 \times 10^4\,\omega^2 - 375 \times 10^4 = 0 \tag{159}$$

We find, therefore, that the eigenvalues together with the respective eigenvectors are:†

$$\omega^2 = 27.176\ , \quad \begin{bmatrix} 1 \\ 26.605 \\ 4.415 \end{bmatrix}$$

† Eigenvectors are normalized by setting the leading component equal to unity.

$$\omega^2 = 179.65 \, , \quad \begin{bmatrix} 1 \\ 3.175 \\ -14.433 \end{bmatrix}$$

$$\omega^2 = 768.15 \, , \quad \begin{bmatrix} 1 \\ -0.0227 \\ 0.0212 \end{bmatrix}$$

To check this value with the results obtained via classical methods, we write the equations of motion for the system of Fig. 12a, viz.

$$m_1 \ddot{x}_1 + c_1 (x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + c_1 (x_2 - x_1) + c_2 (x_2 - x_3) = 0$$

$$m_3 \ddot{x}_3 + c_3 x_3 + c_2 (x_3 - x_2) = 0$$

In matrix form

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_1 & c_1 & 0 \\ -c_1 & (c_1 + c_2) & -c_2 \\ 0 & -c_2 & (c_2 + c_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{160}$$

Using the given numerical values, the eigenvalue determinant equation is

$$\begin{vmatrix} (5,000 - 10\,\omega^2) & -5,000 & 0 \\ -5,000 & (6,000 - 20\,\omega^2) & -1,000 \\ 0 & -1,000 & (7,000 - 40\,\omega^2) \end{vmatrix} = 0$$

or

$$\omega^6 - 975\,\omega^4 + 16.375 \times 10^4\,\omega^2 - 375 \times 10^4 = 0 \tag{161}$$

which is the same as (159). Hence, the eigenvalues are the same.

The eigenvectors for the system (160) are found to be:

$$\omega^2 = 27.176 \ , \quad \begin{bmatrix} 1 \\ 0.9456 \\ 0.1599 \end{bmatrix}$$

$$\omega^2 = 179.65 \ , \quad \begin{bmatrix} 1 \\ 0.6407 \\ -3.445 \end{bmatrix}$$

$$\omega^2 = 768.15 \ , \quad \begin{bmatrix} 1 \\ -0.5363 \\ 0.0226 \end{bmatrix}$$

If now each of the eigenvectors obtained for the system (158) is premultiplied by the matrix [R] of (157), and then normalized by setting the leading element equal to unity, we find that they are identical with those obtained above.

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APPENDIX A

COMPONENTS OF THE MATRICES [A] AND {B} IN EQUATION (133)

Listed below are the components of the 6×6 matrix [A] and the 6×1 matrix {B} which appear in Eq. (133). These components are evaluated by direct comparison between the matrix representation (133) and the corresponding scalar equations (110) and (119) - (122). The proper format is provided by the definition of the state vector {X} as given in Eq. (134).

$$a_{11} = \left[I_{yy} + I_R - M_R \ell_c^2\right] s^2 + \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)^2 \, d\ell\right] s$$

$$- M_R \ell_c \dot{U}_o$$

$$a_{12} = \left[M_R (\ell_R + \ell_c)\right] s - \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) \, d\ell\right]$$

$$a_{13} = M_{P1} \ell_{P1} \dot{U}_o$$

$$a_{14} = M_{P2} \ell_{P2} \dot{U}_o$$

$$a_{15} = \left[M_R (\ell_R + \ell_c) \phi^{(1)} (\ell_T) + (I_R + M_R \ell_R \ell_c) \sigma^{(1)} (\ell_T)\right] s^2$$

$$+ \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) \phi^{(1)} (\ell) \, d\ell\right] s$$

$$+ (T_c + T_s) \left[\ell_c \sigma^{(1)} (\ell_T) + \phi^{(1)} (\ell_T)\right] + M_R \ell_R \dot{U}_o \sigma^{(1)} (\ell_T)$$

$$- q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) \sigma^{(1)} (\ell) \, d\ell$$

$$a_{16} = \left[M_R (\ell_R + \ell_c)\, \phi^{(2)}(\ell_T) + (I_R + M_R\, \ell_R\, \ell_c)\, \sigma^{(2)}(\ell_T)\right] s^2$$

$$+ \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, \phi^{(2)}(\ell)\, d\ell\right] s$$

$$+ (T_c + T_s)\left[\ell_c\, \sigma^{(2)}(\ell_T) + \phi^{(2)}(\ell_T)\right] + M_R\, \ell_R\, \dot{U}_o\, \sigma^{(2)}(\ell_T)$$

$$- q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, \sigma^{(2)}(\ell)\, d\ell$$

$$a_{21} = - M_R (\ell_c - \ell_R)\, s^2 - \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, d\ell\right.$$

$$\left. + m_o U_o\right] s - \left[M_R\, \dot{U}_o - M_t\, g \cos \theta_o\right]$$

$$a_{22} = (m_o + M_R)\; s + \frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha}\, d\ell$$

$$a_{23} = - M_{P1}\, \dot{U}_o$$

$$a_{24} = - M_{P2}\, \dot{U}_o$$

$$a_{25} = \left[\phi^{(1)}(\ell_T) + \ell_R\, \sigma^{(1)}(\ell_T)\right] s^2 - \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha}\, \phi^{(1)}(\ell)\, d\ell\right] s$$

$$+ \left[q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha}\, \sigma^{(1)}(\ell)\, d\ell + (T_c + T_s)\, \sigma^{(1)}(\ell_T)\right]$$

$$a_{26} = \phi^{(2)}(\ell_T) + \ell_R\, \sigma^{(2)}(\ell_T)\; s^2 - \left[\frac{q_D A}{U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \phi^{(2)}(\ell)\, d\ell\right] s$$

$$+ \left[q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \sigma^{(2)}(\ell)\, d\ell + (T_c + T_s)\sigma^{(2)}(\ell_T)\right]$$

$$a_{31} = - \frac{(\ell_{P1} - L_{P1})}{L_{P1}}\; s^2 - \frac{U_o}{L_{P1}}$$

$$a_{32} = \left(\frac{1}{L_{P1}}\right) s$$

$$a_{33} = s^2 + \omega_{P1}^2$$

$$a_{34} = 0$$

$$a_{35} = - \left[\frac{\phi^{(1)}(\ell_{P1})}{L_{P1}}\right] s^2$$

$$a_{36} = - \left[\frac{\phi^{(2)}(\ell_{P1})}{L_{P1}}\right] s^2$$

$$a_{41} = - \frac{(\ell_{P2} - L_{P2})}{L_{P2}}\; s^2 - \frac{U_o}{L_{P2}}$$

$$a_{42} = \left(\frac{1}{L_{P2}}\right) s$$

$$a_{43} = 0$$

$$a_{44} = s^2 + \omega_{P2}^2$$

$$a_{45} = \left[\frac{\phi^{(1)}(\ell_{P2})}{L_{P2}}\right] s^2$$

$$a_{46} = -\left[\frac{\phi^{(2)}(\ell_{P2})}{L_{P2}}\right] s^2$$

$$a_{51} = -\frac{1}{\mathfrak{m}^{(1)}}\left[M_R\, \phi^{(1)}(\ell_T)\ (\ell_c - \ell_R)\right] s^2$$

$$-\frac{1}{\mathfrak{m}^{(1)}}\left[\frac{q_D A}{U_o}\int_o^L \phi^{(1)}(\ell)\ \frac{\partial C_N(\ell)}{\partial \alpha}\ (\ell_a - \ell)\, d\ell\right] s$$

$$-\frac{1}{\mathfrak{m}^{(1)}}\left[M_R\, \dot{U}_o\, \phi^{(1)}(\ell_T) - g\cos\theta_o \int_o^L \phi^{(1)}(\ell)\ m(\ell)\, d\ell\right]$$

$$a_{52} = \frac{1}{\mathfrak{m}^{(1)}}\left[M_R\, \phi^{(1)}(\ell_T)\right] s + \frac{q_D A}{\mathfrak{m}^{(1)} U_o}\int_o^L \phi^{(1)}(\ell)\ \frac{\partial C_N(\ell)}{\partial \alpha}\ d\ell$$

$$a_{53} = -\frac{1}{\mathfrak{m}^{(1)}}\left[M_{P1}\, \dot{U}_o\, \phi^{(1)}(\ell_{P1})\right]$$

$$a_{54} = -\frac{1}{\mathfrak{m}^{(1)}}\left[M_{P2}\, \dot{U}_o\, \phi^{(1)}(\ell_{P2})\right]$$

$$a_{55} = \left\{1 + \frac{M_R\, \phi^{(1)}(\ell_T)}{\mathfrak{m}^{(1)}}\left[\phi^{(1)}(\ell_T) - \ell_R\, \sigma^{(1)}(\ell_T)\right]\right\} s^2$$

$$+\left\{2\zeta^{(1)}\omega^{(1)} - \frac{q_D A}{\mathfrak{m}^{(1)} U_o}\int_o^L \frac{\partial C_N(\ell)}{\partial \alpha}\left[\phi^{(1)}(\ell)\right]^2 d\ell\right\} s$$

$$+\left\{\left[\omega^{(1)}\right]^2+\frac{1}{\mathfrak{M}^{(1)}}\left[(T_c+T_s)\phi^{(1)}(\ell_T)\,\sigma^{(1)}(\ell_T)\right.\right.$$

$$\left.\left.+\,q_D A\int_o^L \frac{\partial C_N(\ell)}{\partial\alpha}\,\phi^{(1)}(\ell)\,\sigma^{(1)}(\ell)\,d\ell\right]\right\}$$

$$a_{56}=\frac{M_R\,\phi^{(1)}(\ell_T)}{\mathfrak{M}^{(1)}}\left[\phi^{(2)}(\ell_T)-\ell_R\,\sigma^{(2)}(\ell_T)\right]s^2$$

$$-\left[\frac{q_D A}{\mathfrak{M}^{(1)}U_o}\int_o^L \frac{\partial C_N(\ell)}{\partial\alpha}\,\phi^{(1)}(\ell)\,\phi^{(2)}(\ell)\,d\ell\right]s$$

$$+\frac{1}{\mathfrak{M}^{(1)}}\left[(T_c+T_s)\phi^{(1)}(\ell_T)\sigma^{(2)}(\ell_T)\right.$$

$$\left.+\,q_D A\int_o^L \frac{\partial C_N(\ell)}{\partial\alpha}\,\phi^{(1)}(\ell)\,\sigma^{(2)}(\ell)\,d\ell\right]$$

$$a_{61}=-\frac{1}{\mathfrak{M}^{(2)}}\left[M_R\,\phi^{(2)}(\ell_T)(\ell_c-\ell_R)\right]s^2$$

$$-\frac{1}{\mathfrak{M}^{(2)}}\left[\frac{q_D A}{U_o}\int_o^L \phi^{(2)}(\ell)\,\frac{\partial C_N(\ell)}{\partial\alpha}\,(\ell_a-\ell)\,d\ell\right]s$$

$$-\frac{1}{\mathfrak{M}^{(2)}}\left[M_R\,\dot{U}_o\,\phi^{(2)}(\ell_T)-g\cos\theta_o\int_o^L \phi^{(2)}(\ell)\;m(\ell)\;d\ell\right]$$

$$a_{62}=\frac{1}{\mathfrak{M}^{(2)}}\left[M_R\,\phi^{(2)}(\ell_T)\right]s+\frac{q_D A}{\mathfrak{M}^{(2)}U_o}\int_o^L \phi^{(2)}(\ell)\,\frac{\partial C_N(\ell)}{\partial\alpha}\,d\ell$$

$$a_{63} = -\frac{1}{\mathfrak{m}^{(2)}} \left[M_{P1}\, \dot{U}_o\, \phi^{(2)} (\ell_{P1}) \right]$$

$$a_{64} = -\frac{1}{\mathfrak{m}^{(2)}} \left[M_{P2}\, \dot{U}_o\, \phi^{(2)} (\ell_{P2}) \right]$$

$$a_{65} = \frac{M_R\, \phi^{(2)}(\ell_T)}{\mathfrak{m}^{(2)}} \left[\phi^{(1)}(\ell_T) - \ell_R\, \sigma^{(1)}(\ell_T) \right] s^2$$

$$- \left[\frac{q_D A}{\mathfrak{m}^{(2)} U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \phi^{(2)}(\ell)\, \phi^{(1)}(\ell)\, d\ell \right] s$$

$$+ \frac{1}{\mathfrak{m}^{(2)}} \left[(T_c + T_s) \phi^{(2)}(\ell_T) \sigma^{(1)}(\ell_T) \right.$$

$$\left. + q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \phi^{(2)}(\ell)\, \sigma^{(1)}(\ell)\, d\ell \right]$$

$$a_{66} = \left\{ 1 + \frac{M_R\, \phi^{(2)}(\ell_T)}{\mathfrak{m}^{(2)}} \left[\phi^{(2)}(\ell_T) - \ell_R\, \sigma^{(2)}(\ell_T) \right] \right\} s^2$$

$$+ \left\{ 2\zeta^{(2)}\, \omega^{(2)} - \frac{q_D A}{\mathfrak{m}^{(2)} U_o} \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \left[\phi^{(2)}(\ell) \right]^2 d\ell \right\} s$$

$$+ \left\{ \left[\omega^{(2)} \right]^2 + \frac{1}{\mathfrak{m}^{(2)}} \left[(T_c + T_s)\, \phi^{(2)}(\ell_T)\, \sigma^{(2)}(\ell_T) \right. \right.$$

$$\left. \left. + q_D A \int_o^L \frac{\partial C_N(\ell)}{\partial \alpha} \phi^{(2)}(\ell)\, \sigma^{(2)}(\ell)\, d\ell \right] \right\}$$

$$b_1 = \left[I_R + M_R\, \ell_R\, \ell_c \right] s^2 + (T_c\, \ell_c + M_R\, \ell_R\, \dot{U}_o)$$

$$b_2 = M_R \ell_R s^2 + T_c$$

$$b_3 = 0$$

$$b_4 = 0$$

$$b_5 = \frac{1}{\mathfrak{M}^{(1)}}\left[M_R \ell_R \phi^{(1)}(\ell_T)\right] s^2 + \frac{T_c \phi^{(1)}(\ell_T)}{\mathfrak{M}^{(1)}}$$

$$b_6 = \frac{1}{\mathfrak{M}^{(2)}}\left[M_R \ell_R \phi^{(2)}(\ell_T)\right] s^2 + \frac{T_c \phi^{(2)}(\ell_T)}{\mathfrak{M}^{(2)}}$$

APPENDIX B

EQUATIONS OF MOTION USING ONE DIMENSIONAL INFLUENCE FUNCTIONS

Instead of deriving the motion of a flexible beam from the differential equation approach, we may instead start with the associated integral equation in which the concept of an influence function plays a fundamental role. This approach is indeed more common in conventional structural theory and does provide an added degree of physical insight. It will accordingly be developed here.†

The pertinent geometry and coordinate system is shown in Fig. B1.

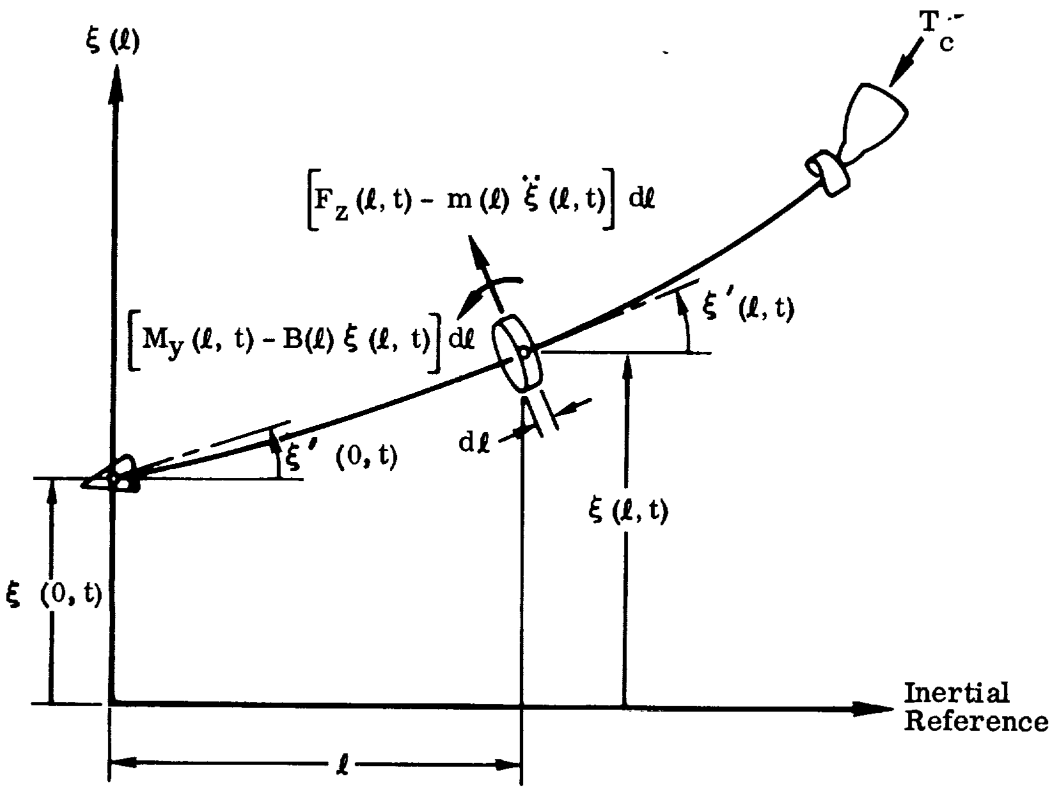


Figure B1. Elastic Vehicle Deformation

The origin for the ℓ-axis coincides with the nose of the vehicle. The displacements are described by a function $\xi(\ell,t)$ which represents the instantaneous position of the elastic axis of the vehicle with respect to the coordinate axes. The vehicle is assumed to be subjected to distributed external forces and moments, $F_z(\ell,t)$ and $r(\ell,t)$, respectively. The kinetic reactions are $m(\ell)\,\ddot{\xi}(\ell,t)$ and $B(\ell)\,\ddot{\xi}(\ell,t)$ for force and

† The material contained in this Appendix has been written by Robert Westerwick.

moment respectively. Dot notation indicates differentiation with respect to time; prime ($'$) notation indicates differentiation with respect to ℓ. Quantity $m(\ell)$ is the mass distribution and $B(\ell)$ is the rotary inertia distribution of the vehicle.

The four required equations of motion are as follows:

a. Equilibrium of lateral forces:

$$\int_0^L \left[F_Z(\ell, t) - m(\ell)\,\ddot{\xi}(\ell, t)\right] d\ell = 0 \tag{B1}$$

b. Moment equilibrium of forced vibration about a transverse axis through the origin:

$$\int_0^L \left\{\left[F_z(\ell, t) - m(\ell)\,\ddot{\xi}(\ell, t)\right]\ell + M_y(\ell, t) - B(\ell)\,\ddot{\xi}'(\ell, t)\right\} d\ell = 0 \tag{B2}$$

c. Displacement of the beam with respect to the inertial reference:

$$\xi(\ell, t) = \xi(0, t) + \xi'(0, t)\,\ell + \int_0^L C^{\delta\delta}(\ell, \nu)\left[F_Z(\nu, t) - m(\nu)\ddot{\xi}(\nu, t)\right] d\nu$$

$$-\int_0^L C^{\delta\alpha}(\ell, \nu)\left[M_y(\nu, t) - B(\nu)\,\ddot{\xi}'(\nu, t)\right] d\nu \tag{B3}$$

d. Rotation of the beam with respect to the inertial reference:

$$\xi'(\ell, t) = \xi'(0, t) + \int C^{\alpha\delta}(\ell, \nu)\left[F_z(\nu, t) - m(\nu)\,\ddot{\xi}(\nu, t)\right] d\nu$$

$$-\int C^{\alpha\alpha}(\ell, \nu)\left[M_y(\nu, t) - B(\nu)\,\ddot{\xi}'(\nu, t)\right] d\nu \tag{B4}$$

In the above, $C^{\delta\delta}(\ell, \nu)$, $C^{\delta\alpha}(\ell, \nu)$, $C^{\alpha\delta}(\ell, \nu)$, and $C^{\alpha\alpha}(\ell, \nu)$ are influence functions that relate the displacement (or rotation) at station ℓ to a unit load (or moment) at station ν, i.e.,

$C^{\delta\delta}(\ell, \nu)$ = displacement at ℓ due to a unit load at ν

$C^{\delta\alpha}(\ell, \nu)$ = displacement at ℓ due to a unit moment at ν

$C^{\alpha\delta}(\ell, \nu)$ = rotation at ℓ due to a unit load at ν

$C^{\alpha\alpha}(\ell, \nu)$ = rotation at ℓ due to a unit moment at ν

Equations (B3) and (B4) provide the relation between the elastic, inertial, and externally applied forces. The integral equations (B1) through (B4), in conjunction with the additional equations required to define the externally applied forces and moments, $F_z(\ell, t)$ and $M_y(\ell, t)$, govern the motion of the unrestrained elastic vehicle. However, practical solutions can only be obtained by resorting to numerical methods.

B.1 NUMERICAL METHOD

Consider the vehicle as being a free-free beam composed of "n" discrete masses located at points along the length of the beam (Fig. B2).

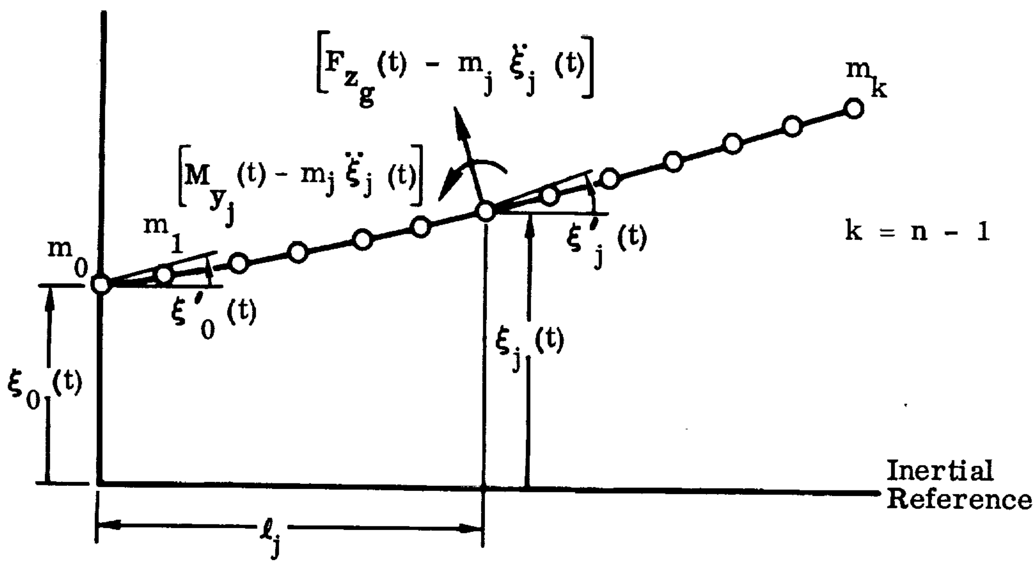


Figure B2. Point-Mass Representation of a Flexible Vehicle

For the system shown in Fig. B2, the integrals in Eqs. (B1) through (B4) may be replaced by discrete summations and the following equations obtained:

a. Lateral Forces

$$F_{z_0} - m_0\ddot{\xi}_0 + F_{z_1} - m_1\ddot{\xi}_1 + \dots + F_{z_j} - m_j\ddot{\xi}_j + \dots + F_{z_k} - m_k\ddot{\xi}_k = 0 \qquad \text{(B1a)}$$

b. Moments

$$(F_{z_0} - m_0\ddot{\xi}_0)\ell_0 - B_0\ddot{\xi}_0' + M_{y_0} + (F_{z_1} - m_1\ddot{\xi}_1)\ell_1' - B_1\ddot{\xi}_1' + M_{y_1} + \dots$$
$$+ (F_{z_j} - m_j\ddot{\xi}_j')\ell_j - B_j\ddot{\xi}_j' + M_{y_j} + \dots + (F_{z_k} - m_k\ddot{\xi}_k')\ell_k - B_k\ddot{\xi}_k'$$
$$+ M_{y_k} = 0 \qquad \text{(B2a)}$$

c. Displacements

$$\xi_i = \xi_0 + \xi_0' \ell_i + \sum_{j=0}^{k} C_{ij}^{\delta\delta} (F_{z_j} - m_j \ddot{\xi}_j) + \sum_{j=0}^{k} C_{ij}^{\delta\alpha} (M_{y_j} - B_j \ddot{\xi}_j) \tag{B3a}$$

where

$i = 0, 1, \ldots, k$

d. Slopes

$$\xi_i' = \xi_0' + \sum_{j=0}^{k} C_{ij}^{\alpha\delta} (F_{z_j} - m_j \ddot{\xi}_j) + \sum_{j=0}^{k} C_{ij}^{\alpha\alpha} (M_{y_j} - B_j \ddot{\xi}_j') \tag{B4a}$$

where

$i = 0, 1, \ldots, k$

In Eqs. (B3a) and (B4a) the influence coefficients $C_{ij}^{\delta\delta}$, $C_{ij}^{\delta\alpha}$, $C_{ij}^{\alpha\delta}$, and $C_{ij}^{\alpha\alpha}$ replace their counterparts, influence functions, which were used in the integral equations (B3) and (B4). Influence coefficients are defined as follows:

$C_{ij}^{\delta\delta}$ = displacement at i due to a unit load at j

$C_{ij}^{\delta\alpha}$ = displacement at i due to a unit moment at j

$C_{ij}^{\alpha\delta}$ = rotation at i due to a unit load at j

$C_{ij}^{\alpha\alpha}$ = rotation at i due to a unit moment at j

The influence coefficients are symmetric, i.e.,

$$C_{ij}^{\delta\delta} = C_{ji}^{\delta\delta}$$

$$C_{ij}^{\delta\alpha} = C_{ji}^{\alpha\delta}$$

$$C_{ij}^{\alpha\alpha} = C_{ji}^{\alpha\alpha}$$

(see Ref. 4, pp. 17-22, for a discussion of influence coefficients and the proof of symmetry.)

To facilitate writing equations in matrix notation, the following example is used. Let the free-free beam of Fig. B2 consist of three discrete mass stations numbered 0, 1, 2 from left to right. Eqs. (B1a) through (B4a) become:

a. Lateral Forces

$$F_{z0}+F_{z1} + F_{z2} - m_0 \ddot{\xi}_0 - m_1 \ddot{\xi}_1 - m_2 \ddot{\xi}_2 = 0 \qquad \text{(B1b)}$$

b. Moments

$$\begin{aligned} &F_{z0}\ell_0 + F_{z1}\ell_1 + F_{z2}\ell_2 - m_0\ell_0\ddot{\xi}_0 - m_1\ell_1\ddot{\xi}_1 - m_2\ell_2\ddot{\xi}_2 - B_0\ddot{\xi}_0' \\ &\quad - B_1\ddot{\xi}_1' - B_2\ddot{\xi}_2' + M_{y0} + M_{y1} + M_{y2} = 0 \end{aligned} \qquad \text{(B2b)}$$

c. Displacements

$$\left.\begin{aligned} \xi_0 = \xi_0 + \xi_0'\ell_0 &+ C_{0,0}^{\delta\delta}(F_{z0} - m_0\ddot{\xi}_0) + C_{0,1}^{\delta\delta}(F_{z1} - m_1\ddot{\xi}_1) \\ &+ C_{0,2}^{\delta\delta}(F_{z2} - m_2\ddot{\xi}_2) + C_{0,0}^{\delta\alpha}(M_{y0} - B_0\ddot{\xi}_0') \\ &+ C_{0,1}^{\delta\alpha}(M_{y1} - B_1\ddot{\xi}_1' + C_{0,2}^{\delta\alpha}(M_{y2} - B_2\ddot{\xi}_2') \\ \\ \xi_1 = \xi_0 + \xi_0'\ell_1 &+ C_{1,0}^{\delta\delta}(F_{z0} - m_0\ddot{\xi}_0) + C_{1,1}^{\delta\delta}(F_{z1} - m_1\ddot{\xi}_1) \\ &+ C_{1,2}^{\delta\delta}(F_{z2} - m_2\ddot{\xi}_2) + C_{1,0}^{\delta\alpha}(M_{y0} - B_0\ddot{\xi}_0') \\ &+ C_{1,1}^{\delta\alpha}(M_{y1} - B_1\ddot{\xi}_1') + C_{1,2}^{\delta\alpha}(M_{y2} - B_2\ddot{\xi}_2') \end{aligned}\right\} \qquad \text{(B3b)}$$

$$\left.\begin{aligned}\xi_2 = \xi_0 + \xi_0' \ell_2 &+ C_{2,0}^{\delta\delta}(F_{z_0} - m_0 \ddot{\xi}_0) + C_{2,1}^{\delta\delta}(F_{z_1} - m_1 \ddot{\xi}_1)\\ &+ C_{2,2}^{\delta\delta}(F_{z_2} - m_2 \ddot{\xi}_2) + C_{2,0}^{\delta\alpha}(M_{y_0} - B_0 \ddot{\xi}_0')\\ &+ C_{2,1}^{\delta\alpha}(M_{y_1} - B_1 \ddot{\xi}_1') + C_{2,2}^{\delta\alpha}(M_{y_2} - B_2 \ddot{\xi}_2')\end{aligned}\right\}$$

d. Slopes

$$\left.\begin{aligned}\xi_0' = \xi_0' &+ C_{0,0}^{\alpha\delta}(F_{z_0} - m_0 \ddot{\xi}_0) + C_{0,1}^{\alpha\delta}(F_{z_1} - m_1 \ddot{\xi}_1)\\ &+ C_{0,2}^{\alpha\delta}(F_{z_2} - m_2 \ddot{\xi}_2) + C_{0,0}^{\alpha\alpha}(M_{y_0} - B_0 \ddot{\xi}_0')\\ &+ C_{0,1}^{\alpha\alpha}(M_{y_1} - B_1 \ddot{\xi}_1') + C_{0,2}^{\alpha\alpha}(M_{y_2} - B_2 \ddot{\xi}_2')\\ \xi_1' = \xi_0' &+ C_{1,0}^{\alpha\delta}(F_{z_0} - m_0 \ddot{\xi}_0) + C_{1,1}^{\alpha\delta}(F_{z_1} - m_1 \ddot{\xi}_1)\\ &+ C_{1,2}^{\alpha\delta}(F_{z_2} - m_2 \ddot{\xi}_2) + C_{1,0}^{\alpha\alpha}(M_{y_0} - B_0 \ddot{\xi}_0')\\ &+ C_{1,1}^{\alpha\alpha}(M_{y_1} - B_1 \ddot{\xi}_1') + C_{1,2}^{\alpha\alpha}(M_{y_2} - B_2 \ddot{\xi}_2'\\ \xi_2' = \xi_0' &+ C_{2,0}^{\alpha\delta}(F_{z_0} - m_0 \ddot{\xi}_0) + C_{2,1}^{\alpha\delta}(F_{z_1} - m_1 \ddot{\xi}_1)\\ &+ C_{2,2}^{\alpha\delta}(F_{z_2} - m_2 \ddot{\xi}_2) + C_{2,0}^{\alpha\alpha}(M_{y_0} - B_0 \ddot{\xi}_0')\\ &+ C_{2,1}^{\alpha\alpha}(M_{y_1} - B_1 \ddot{\xi}_1') + C_{2,2}^{\alpha\alpha}(M_{y_2} - B_2 \ddot{\xi}_2')\end{aligned}\right\} \quad \text{(B4b)}$$

The equations of the forces and moments may be written as:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline \ell_0 & \ell_1 & \ell_2 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{c} F_{z_0} \\ F_{z_1} \\ F_{z_2} \\ \hline M_{y_0} \\ M_{y_1} \\ M_{y_2} \end{array}\right] - \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ \hline \ell_0 & \ell_1 & \ell_2 & 1 & 1 & 1 \end{array}\right]$$

$$\times \left[\begin{array}{ccc|ccc} m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_3 \end{array}\right] \left[\begin{array}{c} \ddot{\xi}_0 \\ \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \hline \ddot{\xi}_0' \\ \ddot{\xi}_1' \\ \ddot{\xi}_2' \end{array}\right] = 0$$

The equations of the displacements and slopes may be written as

$$\left[\begin{array}{c} \xi_0 \\ \xi_1 \\ \xi_2 \\ \hline \xi_0' \\ \xi_1' \\ \xi_2' \end{array}\right] = \left[\begin{array}{c|c} 1 & \ell_0 \\ 1 & \ell_1 \\ 1 & \ell_2 \\ \hline 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} \xi_0 \\ \xi_0' \end{array}\right]$$

$$
+\left[\begin{array}{ccc|ccc}
C^{\delta\delta}_{0,0} & C^{\delta\delta}_{0,1} & C^{\delta\delta}_{0,2} & C^{\delta\alpha}_{0,0} & C^{\delta\alpha}_{0,1} & C^{\delta\alpha}_{0,2} \\
C^{\delta\delta}_{1,0} & C^{\delta\delta}_{1,1} & C^{\delta\delta}_{1,2} & C^{\delta\alpha}_{1,0} & C^{\delta\alpha}_{1,1} & C^{\delta\alpha}_{1,2} \\
C^{\delta\delta}_{2,0} & C^{\delta\delta}_{2,1} & C^{\delta\delta}_{2,2} & C^{\delta\alpha}_{2,0} & C^{\delta\alpha}_{2,1} & C^{\delta\alpha}_{2,2} \\
\hline
C^{\alpha\delta}_{0,0} & C^{\alpha\delta}_{0,1} & C^{\alpha\delta}_{0,2} & C^{\alpha\alpha}_{0,0} & C^{\alpha\alpha}_{0,1} & C^{\alpha\alpha}_{0,2} \\
C^{\alpha\delta}_{1,0} & C^{\alpha\delta}_{1,1} & C^{\alpha\delta}_{1,2} & C^{\alpha\alpha}_{1,0} & C^{\alpha\alpha}_{1,1} & C^{\alpha\alpha}_{1,2} \\
C^{\alpha\delta}_{2,0} & C^{\alpha\delta}_{2,1} & C^{\alpha\delta}_{2,2} & C^{\alpha\alpha}_{2,0} & C^{\alpha\alpha}_{2,1} & C^{\alpha\alpha}_{2,2}
\end{array}\right]
$$

$$
\times \left\{ \left[\begin{array}{c}
F_{z_0} \\ F_{z_1} \\ F_{z_2} \\ \hline M_{y_0} \\ M_{y_1} \\ M_{y_2}
\end{array}\right]
-
\left[\begin{array}{ccc|ccc}
m_0 & 0 & 0 & 0 & 0 & 0 \\
0 & m_1 & 0 & 0 & 0 & 0 \\
0 & 0 & m_2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & B_0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_1 & 0 \\
0 & 0 & 0 & 0 & 0 & B_2
\end{array}\right]
\left[\begin{array}{c}
\ddot{\xi}_0 \\ \ddot{\xi}_1 \\ \ddot{\xi}_2 \\ \hline \ddot{\xi}'_0 \\ \ddot{\xi}'_1 \\ \ddot{\xi}'_2
\end{array}\right]
\right\}
$$

After partitioning the above matrices as shown by the broken lines and inferring an "(n+1)" mass station model from the 3-mass station example we can define the following submatrices:

$$
C^{\delta\delta} \equiv \begin{bmatrix}
C^{\delta\delta}_{0,0} & C^{\delta\delta}_{0,1} \cdots & C^{\delta\delta}_{0,n-1} \\
\vdots & & \vdots \\
C^{\delta\delta}_{n-1,0} & \cdots\cdots & C^{\delta\delta}_{n-1,n-1}
\end{bmatrix}
\quad
C^{\alpha\alpha} \equiv \begin{bmatrix}
C^{\alpha\alpha}_{0,0} & \cdots\cdots & C^{\alpha\alpha}_{0,n-1} \\
\vdots & & \vdots \\
C^{\alpha\alpha}_{n-1,0} & \cdots & C^{\alpha\alpha}_{n-1,n-1}
\end{bmatrix}
$$

$$C^{\alpha\delta} \equiv \begin{bmatrix} C^{\alpha\delta}_{0,0} & \cdots\cdots & C^{\alpha\delta}_{0,n-1} \\ \vdots & & \vdots \\ C^{\alpha\delta}_{n-1,0} & \cdots & C^{\alpha\delta}_{n-1,n-1} \end{bmatrix} \qquad C^{\delta\alpha} \equiv \begin{bmatrix} C^{\delta\alpha}_{0,0} & \cdots\cdots & C^{\delta\alpha}_{0,n-1} \\ \vdots & & \vdots \\ C^{\delta\alpha}_{n-1,0} & \cdots & C^{\delta\alpha}_{n-1,n-1} \end{bmatrix}$$

$$\xi \equiv \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{n-1} \end{bmatrix} \qquad \xi' \equiv \begin{bmatrix} \xi'_0 \\ \xi'_1 \\ \vdots \\ \xi'_{n-1} \end{bmatrix} \qquad 1 \equiv \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \ell \equiv \begin{bmatrix} \ell_0 \\ \ell_1 \\ \vdots \\ \ell_{n-1} \end{bmatrix} \qquad \begin{matrix} \xi_0 \equiv \begin{bmatrix} \xi_0 \end{bmatrix} \\ \\ \xi'_0 \equiv \begin{bmatrix} \xi'_0 \end{bmatrix} \end{matrix}$$

$$F_z \equiv \begin{bmatrix} F_{z_0} \\ F_{z_1} \\ \vdots \\ F_{z_{n-1}} \end{bmatrix} \qquad M_y \equiv \begin{bmatrix} M_{y_0} \\ M_{y_1} \\ \vdots \\ M_{y_{n-1}} \end{bmatrix} \qquad D_m \equiv \begin{bmatrix} m_0 & 0 & \cdots\cdots & 0 \\ 0 & m_1 & 0 \cdot & 0 \\ \vdots & & \ddots & \vdots \\ 0 \cdots 0 & & 0 & m_{n-1} \end{bmatrix}$$

$$D_B \equiv \begin{bmatrix} B_0 & 0 & \cdots\cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots 0 & B_{n-1} \end{bmatrix}$$

The previous matrix equations may now be expressed as:

$$\begin{bmatrix} 1^T & 0 \\ \ell^T & 1^T \end{bmatrix} \left\{ \begin{bmatrix} F_z \\ M_y \end{bmatrix} - \begin{bmatrix} D_m & 0 \\ 0 & D_B \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ \ddot{\xi}' \end{bmatrix} \right\} = 0 \tag{B5}$$

$$\begin{bmatrix} \xi \\ \xi' \end{bmatrix} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_0' \end{bmatrix} + \begin{bmatrix} C^{\delta\delta} & C^{\delta\alpha} \\ C^{\alpha\delta} & C^{\alpha\alpha} \end{bmatrix} \left\{ \begin{bmatrix} F_z \\ M_y \end{bmatrix} - \begin{bmatrix} D_m & 0 \\ 0 & D_B \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ \ddot{\xi}' \end{bmatrix} \right\} \tag{B6}$$

where the $()^T$ notation denotes transpose of a matrix.

Equations (B5) and (B6) are ordinary differential equations that represent the equations of motion for a simplified mathematical model. The adequacy of this mathematical model is primarily dependent on the number of mass stations chosen and the accuracy of the structural influence coefficients.

It may be appropriate at this time to mention that the matrix of influence coefficients is singular, i.e., there are two rows and two columns of zeros. These zeros arise because the influence coefficients are calculated with respect to the reference station, which in the previous example was station 0; therefore, any influence coefficient $C_{0,i}^{\ell\ell}$ or $C_{i,0}^{\ell\ell}$ will be equal to zero. The fact that the matrix of influence coefficients is singular is of no consequence, since no inverse operation is ever performed on this matrix.

B.2 SOLUTION OF MATRIX EQUATIONS OF FORCED VIBRATION IN NORMAL COORDINATES

We now define the matrices of Eqs. (B5) and (B6) as follows:

$$A^T = \begin{bmatrix} 1^T & 0 \\ \ell^T & 1^T \end{bmatrix} \qquad F = \begin{bmatrix} F_z \\ M_y \end{bmatrix} \qquad D_I = \begin{bmatrix} D_m & \\ & D_B \end{bmatrix} \qquad \ddot{z} = \begin{bmatrix} \ddot{\xi} \\ \ddot{\xi}' \end{bmatrix}$$

$$z = \begin{bmatrix} \xi \\ \xi' \end{bmatrix} \qquad A = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \qquad z_0 = \begin{bmatrix} \xi_0 \\ \xi_0' \end{bmatrix} \qquad C = \begin{bmatrix} C^{\delta\delta} & C^{\delta\alpha} \\ C^{\alpha\delta} & C^{\alpha\alpha} \end{bmatrix}$$

Equations (B5) and (B6) may now be written in the following abbreviated form:

$$A^T (F - D_I \ddot{z}) = 0 \tag{B7}$$

$$z = A\, z_0 + C\, (F - D_I \ddot{z}) \tag{B8}$$

Consider the homogeneous equations obtained from Eqs. (B7) and (B8) by equating the externally applied forces and moments to zero, i.e., vector $F = 0$.

From the theory of ordinary linear differential equations it is known that solutions of the form

$$z = \eta \, e^{i\omega t} \tag{B9}$$

exist. After differentiating twice with respect to time and substituting the result into Eqs. (B7) and (B8), one has:

$$\omega^2 A^T D_I \eta = 0 \tag{B10}$$

$$\eta = A \eta_0 + \omega^2 C D_I \eta \tag{B11}$$

Equations (B10) and (B11) constitute a set of 2n homogeneous linear algebraic equations. Physically, solutions to these equations represent the natural modes of vibration for an unrestrained elastic vehicle. The natural frequencies are given by the ω's and the natural mode shapes by their corresponding η's. The vector η is partitioned as

$$\eta_{(2n \times 1)} = \begin{bmatrix} \phi \\ \hline \sigma \end{bmatrix} \begin{matrix} \} \; (n \times 1) \\ \} \; (n \times 1) \end{matrix}$$

where the vector ϕ represents the modal deflections and the vector σ represents the modal slopes at stations 0, 1, 2, ..., n-1 respectively.

The ϕ, σ notation is introduced at this point to remain consistent with usual notation for modal deflections and slopes. The z-vector components ξ and ξ' which correspond to total deflections and slopes have been abandoned since the introduction of η, the modal vector.

η_0 is a vector of order two defined by:

$$\eta_0 = \begin{bmatrix} \phi_0 \\ \sigma_0 \end{bmatrix}$$

where ϕ_0 and σ_0 are the deflection and slope respectively at station 0, the reference point.

Consider Eqs. (B10) and (B11) for $\omega = 0$. $\eta = A\,\eta_0$ is a nontrivial solution that satisfies the equations. Since η_0 is arbitrary, any two linearly independent choices are admissible. The resulting vectors $\eta_i = A\,\eta_{0,i}$ $(i=1,2)$ are the so-called "rigid body" modes. It will be found convenient to choose

$$\eta_{0,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{B12}$$

$$\eta_{0,2} = \begin{bmatrix} \bar{\ell} \\ -1 \end{bmatrix} \tag{B13}$$

where $\bar{\ell}$ is the ℓ-coordinate to the center of gravity of the vehicle and is given by

$$\bar{\ell} = \frac{1^T D_m \ell}{1^T D_m 1} \tag{B14}$$

If the equation, $\eta_i = A\eta_{o,i}$, is rewritten in terms of the elements of the matrices, one has:

$$\begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{n-1} \\ \sigma_0 \\ \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \ell_0 \\ 1 & \ell_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \ell_{n-1} \\ 0 & 1 \\ 0 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \sigma_0 \end{bmatrix} \tag{B11a}$$

Substituting $\eta_{0,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\begin{bmatrix} \phi_0 \\ \sigma_0 \end{bmatrix}$ in (B11a) one obtains

$$\eta_1 = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{n-1} \\ \sigma_0 \\ \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \ell_0 \\ 1 & \ell_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \ell_{n-1} \\ 0 & 1 \\ 0 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}; \text{ or } \begin{array}{ll} \phi_0 & = 1 \\ \phi_1 & = 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \phi_{n-1} & = 1 \\ \sigma_0 & = 0 \\ \sigma_1 & = 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \sigma_{n-1} & = 0 \end{array} \qquad \text{(B15)}$$

Equivalently:

$$\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now substituting $\eta_{0,2} = \begin{bmatrix} \bar{\ell} \\ -1 \end{bmatrix}$ for $\begin{bmatrix} \phi_0 \\ \sigma_0 \end{bmatrix}$ in (B11a), one has

$$\eta_2 = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{n-1} \\ \sigma_0 \\ \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \ell_0 \\ 1 & \ell_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \ell_{n-1} \\ 0 & 1 \\ 0 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\ell} \\ -1 \end{bmatrix} = \begin{bmatrix} (\bar{\ell} - \ell_0) \\ (\bar{\ell} - \ell_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ (\bar{\ell} - \ell_{n-1}) \\ -1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{bmatrix}; \text{ or } \begin{array}{ll} \phi_0 & = \Delta\ell_0 \\ \phi_1 & = \Delta\ell_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \phi_{n-1} & = \Delta\ell_{n-1} \\ \sigma_0 & = -1 \\ \sigma_1 & = -1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \sigma_{n-1} & = -1 \end{array} \qquad \text{(B16)}$$

Equivalently:

$$\eta_2 = \begin{bmatrix} \Delta \ell \\ \\ -1 \end{bmatrix}$$

where $\Delta \ell$ is the distance from the center of gravity to the point in question.

Plunging and pitching modes are shown in Figs. B3 and B4.

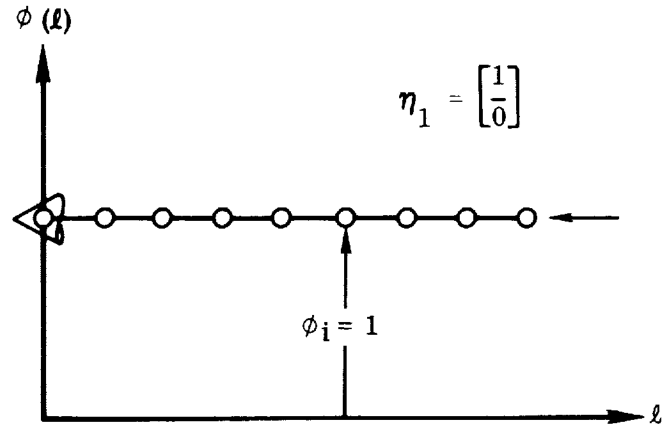


Figure B3. Plunging Mode

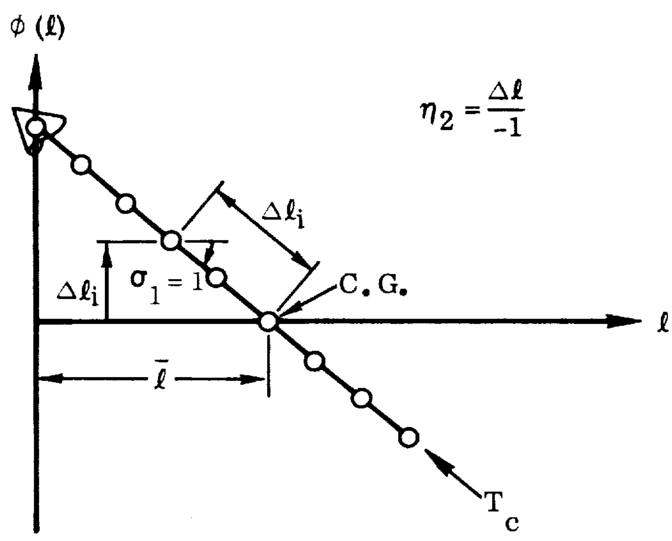


Figure B4. Pitching Mode

The remaining 2n - 2 solutions to Eqs. (B10) and (B11) represent the elastic modes of vibration. The vector η_0 can be eliminated from (B11) by its premultiplication by the matrix $A^T D_I$. This gives:

$$A^T D_I \eta = A^T D_I A \eta_0 + \omega^2 A^T D_I C D_I \eta \tag{B17}$$

From Eq. (B10):

$$\omega^2 A^T D_I \eta = 0 \text{ for } \omega \neq 0$$

Therefore, the left side of Eq. (B17) vanishes and one gets:

$$A^T D_I A \eta_0 = -\omega^2 A^T D_I C D_I \eta$$

$$\eta_0 = -\omega^2 (A^T D_I A)^{-1} A^T D_I C D_I \eta \tag{B18}$$

Substitution of η_0 from Eq. (B18) back into Eq. (B11) gives:

$$\eta = -\omega^2 A (A^T D_I A)^{-1} A^T D_I C D_I \eta + \omega^2 C D_I \eta$$

$$\left[I - A (A^T D_I A)^{-1} A^T D_I \right] C D_I \eta = \frac{1}{\omega^2} \eta \tag{B19}$$

Let $\lambda = \dfrac{1}{\omega^2}$

$$\left[I - A (A^T D_I A)^{-1} A^T D_I \right] C D_I \eta = \lambda \eta \tag{B20}$$

Equation (B20) is in the form of a standard eigenvalue problem for a conservative system. The 2n - 2 nontrivial solutions to Eq. (B20) yield the elastic bending modes and their associated frequencies. The 2n - 2 η_i's are the mode shapes (eigenvectors), and the λ_i's are the reciprocals of the squares of the modal frequencies (eigenvalues).

The η_i's and λ_i's may be efficiently obtained via digital computer using the method of matrix iteration. Reference 25 (p. 62, p. 269, p. 356) presents the method of matrix iteration currently used in the "General Missile Vibration Mk II" digital routine, which is a general program to obtain frequencies, mode shapes, and generalized masses for lumped-mass systems.

B.3 ORTHOGONALITY OF THE EIGENVECTORS (MODE SHAPES)

Let η_i and η_j be two distinct eigenvectors having λ_i and λ_j as their associated nonzero and distinct eigenvalues. From Eq. (B20) one has:

$$\left[I - A\,(A^T D_I A)^{-1} A^T D_I\right] C\, D_I\, \eta_i = \lambda_i\, \eta_i \tag{B20a}$$

$$\left[I - A\,(A^T D_I A)^{-1} A^T D_I\right] C\, D_I\, \eta_j = \lambda_j\, \eta_j \tag{B20b}$$

Premultiplying Eq. (B20a) by $\eta_j^T D_I$:

$$\eta_j^T D_I \left[I - A\,(A^T D_I A)^{-1} A^T D_I\right] C\, D_I\, \eta_i = \lambda_i\, \eta_j^T D_I\, \eta_i$$

$$\eta_j^T D_I\, C\, D_I\, \eta_i - \eta_j^T D_I\, A\,(A^T D_I A)^{-1} A^T D_I\, C\, D_I\, \eta_i = \lambda_i\, \eta_j^T D_I\, \eta_i \tag{B20a$'$}$$

From Eq. (B10), $A^T D_I\, \eta = 0$ for $\omega \neq 0$; therefore $\eta^T D_I^T A = 0$. Since D_I is a diagonal matrix, $D_I^T = D_I$; therefore $\eta_j^T D_I A = 0$.

Equation (B20a$'$) now becomes

$$\eta_j^T D_I\, C\, D_I\, \eta_i = \lambda_i\, \eta_j^T D_I\, \eta_i \tag{B21}$$

Now transposing Eq. (B20b), one has:

$$\eta_j^T D_I\, C - \eta_j^T D_I\, C\, D_I\, A \left[(A^T D_I A)^{-1}\right]^T A^T = \lambda_j\, \eta_j^T$$

Since C is symmetric, $C^T = C$ above.

Postmultiplying by $D_I\, \eta_i$:

$$\eta_j^T D_I\, C\, D_I\, \eta_i - \eta_j^T D_I\, C\, D_I\, A \left[(A^T D_I A)^{-1}\right]^T A^T D_I\, \eta_i = \lambda_j \eta_j^T D_I\, \eta_i$$

But $A^T D_I\, \eta_i = 0$ by Eq. (B10), so:

$$\eta_j^T D_I\, C\, D_I\, \eta_i = \lambda_j\, \eta_j^T D_I\, \eta_i \tag{B22}$$

Subtracting (B22) from (B21) gives:

$$(\lambda_i - \lambda_j)\, \eta_j^T D_I \eta_i = 0; \; i \neq j \tag{B23}$$

Since λ_i and λ_j were assumed to be nonzero and distinct, it follows that

$$\eta_j^T D_I \eta_i = 0; \; i \neq j \tag{B24}$$

Equation (B24) states the orthogonality properties of the eigenvectors; i.e., any eigenvector η_i is orthogonal to any other (different) eigenvector η_j with respect to the inertial matrix D_I. Although (B24) explicitly applies only to the eigenvectors corresponding to the elastic modes, it can be shown by direct substitution that it also encompasses the rigid body modes.

To illustrate, let us again consider a beam composed of 3 discrete mass stations: 0, 1, 2. We have already found $\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\eta_2 = \begin{bmatrix} \Delta \ell \\ -1 \end{bmatrix}$.

$$\eta_1^T D_I \eta_2 = [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0] \begin{bmatrix} m_0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & m_1 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & m_2 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & i_0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & i_1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & i_2 \end{bmatrix} \begin{bmatrix} \bar{\ell} - \ell_0 \\ \bar{\ell} - \ell_1 \\ \bar{\ell} - \ell_2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The m_i are the discrete lumped masses, and the i_i are the moments of inertia of the masses about their centers of gravity.

Expanding the above, one has:

$$\eta_1^T D_I \eta_2 = [m_0 \;\; m_1 \;\; m_2 \;\; 0 \;\; 0 \;\; 0] \begin{bmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= m_0 \Delta\ell_0 + m_1 \Delta\ell_1 + m_2 \Delta\ell_2 = 0$$

Now consider the case $\eta_i^T D_I \eta_j$; where $i = j$

$$\eta_i^T D_I \eta_i = [\phi_0 \;\; \phi_1 \cdots \phi_{n-1} \;\; \sigma_0 \cdots \sigma_{n-1}] \begin{bmatrix} m_0 & & & & & \\ & \ddots & & & & \\ & & m_{n-1} & & & \\ & & & B_0 & & \\ & & & & \ddots & \\ & & & & & B_{n-1} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \\ \sigma_0 \\ \vdots \\ \sigma_{n-1} \end{bmatrix}$$

$$= m_0 (\phi_0)^2 + \ldots + m_{n-1} (\phi_{n-1})^2 + B_0 (\sigma_0)^2 + \ldots + B_{n-1} (\sigma_{n-1})^2$$

$$= \sum_{i=0}^{n-1} \left(m_i (\phi_i)^2 + B_i (\sigma_i)^2 \right) \equiv \mathfrak{m}^{(i)}; \text{ generalized mass.}$$

Equation (B24) can be generalized by expressing it in the form,

$$\eta_i^T D_I \eta_j = \delta_{ij} \mathfrak{m}^{(i)}; \quad i = 0, 1, 2, \ldots, n-1 \tag{B25}$$

where

δ_{ij} is the Kronecker delta, i.e., $\delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$

$\mathfrak{m}^{(i)}$ is defined as the "generalized mass" of the i^{th} mode

It has been previously stated that $\eta_{0,i}$ was "conveniently" chosen to give $(i = 1, 2)$

$$\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \eta_2 = \begin{bmatrix} \Delta\ell \\ -1 \end{bmatrix}$$

The convenience arises if one considers the expression for the generalized mass of the rigid body modes. Again, using the 3-mass-station example as before:

$$\mathfrak{M}^{(1)} = \eta_1{}^T D_I \eta_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_0 & 0 & 0 & . & . & 0 \\ 0 & m_1 & 0 & . & . & 0 \\ 0 & 0 & m_2 & . & . & 0 \\ 0 & 0 & . & i_0 & . & 0 \\ 0 & 0 & . & . & i_1 & 0 \\ 0 & 0 & . & . & . & i_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathfrak{M}^{(1)} = m_0 + m_1 + m_2 = M_t$ total mass of the vehicle for the rigid body plunging mode

$$\mathfrak{M}^{(2)} = \eta_2{}^T D_I \eta_2 = \begin{bmatrix} \Delta\ell_0 & \Delta\ell_1 & \Delta\ell_2 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} m_0 & & & & & \\ & m_1 & & & & \\ & & m_2 & & & \\ & & & i_0 & & \\ & & & & i_1 & \\ & & & & & i_2 \end{bmatrix} \begin{bmatrix} \Delta\ell_0 \\ \Delta\ell_1 \\ \Delta\ell_2 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\mathfrak{M}^{(2)} = \left[i_0 + m_0 (\Delta\ell_0)^2\right] + \left[i_1 + m_1 (\Delta\ell_1)^2\right] + \left[i_2 + m_2 (\Delta\ell_2)^2\right]$$

$\mathfrak{M}^{(2)} = I_{yy}$; the moment of inertia of the vehicle about the center of gravity for the rigid body pitching mode

B.4 EQUATIONS OF MOTION OF FORCED VIBRATION

Equations (B7) and (B8) are rewritten below for convenience:

$$A^T (F - D_I \ddot{z}) = 0 \tag{B7}$$

$$z = A\, z_0 + C\,(F - D_I \ddot{z}) \tag{B8}$$

Consider a coordinate transformation given by

$$z = H\, q \tag{B26}$$

The transformation matrix, H, can in general be any nonsingular matrix of order $2n \times 2n$. However, the orthogonality properties of the eigenvectors can be used to great advantage by letting H be the modal matrix of the unrestrained elastic vehicle. Thus, let

$$H = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \ldots & \eta_{2n} \end{bmatrix} \tag{B27}$$

The fact that the eigenvectors $\eta_1, \eta_2, \ldots, \eta_{2n}$ are mutually orthogonal guarantees that H is nonsingular and guarantees the existence of H^{-1}.

After transforming coordinates a la (B26), Eqs. (B7) and (B8) become:

$$A^T (F - D_I H \ddot{q}) = 0 \tag{B28}$$

$$Hq = A H_0 q + C (F - D_I H\ddot{q}) \tag{B29}$$

where

$$H_0 = \begin{bmatrix} \eta_{0,1}, & \eta_{0,2}, & \ldots, & \eta_{0,2n} \end{bmatrix} \tag{B30}$$

The vectors $\eta_{0,1}$, $\eta_{0,2}$, etc., are

$$\eta_{0,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \eta_{0,2} = \begin{bmatrix} \bar{\ell} \\ -1 \end{bmatrix}$$

and from Eq. (B18):

$$\eta_{0,i} = -\omega_i^2 (A^T D_I A)^{-1} A^T D_I C D_I \eta_i; \; i = 3, 4, \ldots, 2n \tag{B31}$$

Premultiplying (B28) by

$$\begin{bmatrix} \eta_1^T \\ \left(-\eta_2^T + \bar{\ell}\, \eta_1^T\right) \end{bmatrix} (A^T)^{-1}$$

one gets, in terms of the rigid body modes,

$$\begin{bmatrix} \eta_1^T \\ \left(-\eta_2^T + \bar{\ell}\, \eta_1^T\right) \end{bmatrix} (F - D_I H \ddot{q}) = 0 \tag{B32}$$

Rearranging (B32) and expressing H and $\ddot{q}$ in terms of their elements gives:

$$\begin{bmatrix} \eta_1^T \\ \left(\eta_n^T + \bar{\ell}\,\eta_1^T\right) \end{bmatrix} \begin{bmatrix} D_I\,\eta_1\,\eta_2 \dots \eta_{2n} \end{bmatrix} \begin{bmatrix} \ddot{q}(1) \\ \ddot{q}(2) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{q}(2n) \end{bmatrix} = \begin{bmatrix} \eta_1^T \\ \left(-\eta_2^T + \bar{\ell}\,\eta_1^T\right) \end{bmatrix} F \qquad \text{(B32a)}$$

Expanding (B32a):

$$\begin{bmatrix} \eta_1^T D_I \eta_1 & \eta_1^T D_I \eta_2 & \dots \eta_1^T D_I \eta_{2n} \\ \left(-\eta_2^T + \bar{\ell}\,\eta_1^T\right) D_I \eta_1 & \left(-\eta_2^T + \bar{\ell}\,\eta_1^T\right) D_I \eta_2 & \dots \;(\;)\; D_I \eta_{2n} \end{bmatrix} \begin{bmatrix} \ddot{q}(1) \\ \ddot{q}(2) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{q}(2n) \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1^T F \\ -\eta_2^T F + \bar{\ell}\,\eta_1^T F \end{bmatrix} \qquad \text{(B32b)}$$

Since $\eta_i^T D_I \eta_j = \delta_{ij} \mathfrak{m}^{(i)}$; (B32b) becomes

$$\begin{bmatrix} \mathfrak{m}^{(1)} & 0 & & 0 \\ \bar{\ell}\,\mathfrak{m}^{(1)} & -\mathfrak{m}^{(2)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}(1) \\ \ddot{q}(2) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{q}(2n) \end{bmatrix} = \begin{bmatrix} \eta_1^T \\ -\eta_2^T F + \bar{\ell}\,\eta_1^T F \end{bmatrix} \qquad \text{(B33)}$$

Expanding the first row of (B33):

$$\mathfrak{m}^{(1)} \ddot{q}^{(1)} = \eta_1^T F \qquad \text{(B34)}$$

Expanding the second row of (B33):

$$\bar{\ell}\,\mathfrak{m}^{(1)} \ddot{q}^{(1)} - \mathfrak{m}^{(2)} \ddot{q}^{(2)} = -\eta_2^T F + \bar{\ell}\,\eta_1^T F \qquad \text{(B33a)}$$

Using Eq. (B34) in (B33a), one obtains:

$$\mathfrak{m}^{(2)}\,\ddot{q}^{(2)} = \eta_2^T F \tag{B35}$$

Eqs. (B34) and (B35) represent the rigid body pitching and plunging modes.

Returning to Eq. (B29), premultiplication by $\eta_i^T D_I$ for $i > 2$ gives:

$$\eta_i^T D_I H q = \eta_i^T D_I A H_0 q + \eta_i^T D_I C (F - D_I H \ddot{q}) \tag{B36}$$

Using Eq. (B10), $\eta_i^T D_I A = 0$, and expanding (B36),

$$\eta_i^T D_I \begin{bmatrix} \eta_1 & \eta_2 \dots \eta_i \dots \eta_{2n} \end{bmatrix} \begin{bmatrix} q^{(1)} \\ q^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ q^{(i)} \\ \cdot \\ \cdot \\ q^{(2n)} \end{bmatrix} = \eta_i^T D_I C (F - D_I H \ddot{q}) \tag{B36a}$$

Using the orthogonality property, $\eta_i^T D_I \eta_j = \delta_{ij} \mathfrak{m}^{(i)}$

$$\mathfrak{m}^{(i)}\, q^{(i)} = \eta_i^T D_I C (F - D_I H \ddot{q}) \tag{B37}$$

Now referring to Eq. (B11) which says

$$\eta_i = A\,\eta_{0,i} + \left[\omega^{(i)}\right]^2 C\, D_I\, \eta_i$$

transpose both sides:

$$\eta_i^T = \eta_{0,i}^T A^T + \left[\omega^{(i)}\right]^2 \eta_i^T D_I C$$

rearrange and divide by $\left[\omega^{(i)}\right]^2$:

$$\eta_i^T D_I C = \frac{1}{\left[\omega^{(i)}\right]^2} \left(\eta_i^T - \eta_{0,i}^T A^T\right) \tag{B11b}$$

Substituting (B11b) into (B37), one gets:

$$\mathfrak{M}^{(i)}\, q^{(i)} = \frac{1}{\left[\omega^{(i)}\right]^2} \left(\eta_i^T - \eta_{0,i}^T\, A^T\right) \left(F - D_I\, H\, \ddot{q}\right) \tag{B37a}$$

Expanding (B37a):

$$\mathfrak{M}^{(i)}\, q^{(i)} = \frac{1}{\left[\omega^{(i)}\right]^2} \left(\eta_i^T\, F - \eta_i^T\, D_I\, H\, \ddot{q}\right) - \frac{1}{\left[\omega^{(i)}\right]^2}\, \eta_{0,i}^T\, A^T\, (F - D_I H \ddot{q}) \tag{B37b}$$

From Eq. (B28), $A^T\,(F - D_I\, H\, \ddot{q}) = 0$, so

$$\mathfrak{M}^{(i)}\, q^{(i)} = \frac{1}{\left[\omega^{(i)}\right]^2} \left(\eta_i^T\, F - \eta_i^T\, D_I\, H\, \ddot{q}\right) \tag{B37c}$$

Because of orthogonality, $\eta_i^T\, D_I\, H\, \ddot{q} = \eta_i^T\, D_I\, \eta_i\, \ddot{q}^{(i)}$; and $\eta_i^T\, D_I \equiv \mathfrak{M}^{(i)}$. Therefore (B37c) becomes:

$$\mathfrak{M}^{(i)}\, q^{(i)} = \frac{1}{\left[\omega^{(i)}\right]^2} \left(\eta_i^T\, F - \mathfrak{M}^{(i)}\, \ddot{q}^{(i)}\right)$$

Multiplying through by $\left[\omega^{(i)}\right]^2$ and rearranging terms, one has:

$$\mathfrak{M}^{(i)}\, \ddot{q}^{(i)} + \mathfrak{M}^{(i)} \left[\omega^{(i)}\right]^2 q^{(i)} = Q^{(i)} \qquad (i = 3, 4, \ldots, 2n) \tag{B38}$$

where

$$Q^{(i)} = \eta_i^T F$$

is the generalized force for the i^{th} mode.

Comparing (B38) to Eqs. (B34) and (B35), it is noted that they are all of the same form. Therefore, one may write the single equation,

$$\mathfrak{M}^{(i)}\, \ddot{q}^{(i)} + \mathfrak{M}^{(i)} \left[\omega^{(i)}\right]^2 q^{(i)} = Q^{(i)} \qquad (i = 1, 2, \ldots, 2n) \tag{B39}$$

where $\omega^{(i)} = \omega^{(2)} = 0$. The $q^{(i)}$ are known as normal coordinates because they express the displacements of natural modes of motion.

The solutions to (B39) now constitute the solutions to Eqs. (B5) and (B6).

B.5 DISCUSSION OF THE COMPONENTS OF THE F-VECTOR

The components of the F-vector, as was previously mentioned, reflect the externally applied forces and moments on the vehicle. The number and type of external forces and moments which may be considered depend upon the type of lumped-mass model one chooses to represent the elastic vehicle.

Currently there are two different lumped-mass models used. The first model considers the vehicle as being a single beam and does not include the masses or moments of inertia of the engines. The second model considers the vehicle as a split beam with the engines attached (masses and moments of inertia included) to their respective gimbals by torsional springs to simulate the elasticity of the backup structure and the actuator (engines-in-the-modes). Both models are applicable to the previous analytical development. Figure B5 provides a typical illustration of the engines-in-the-modes model.

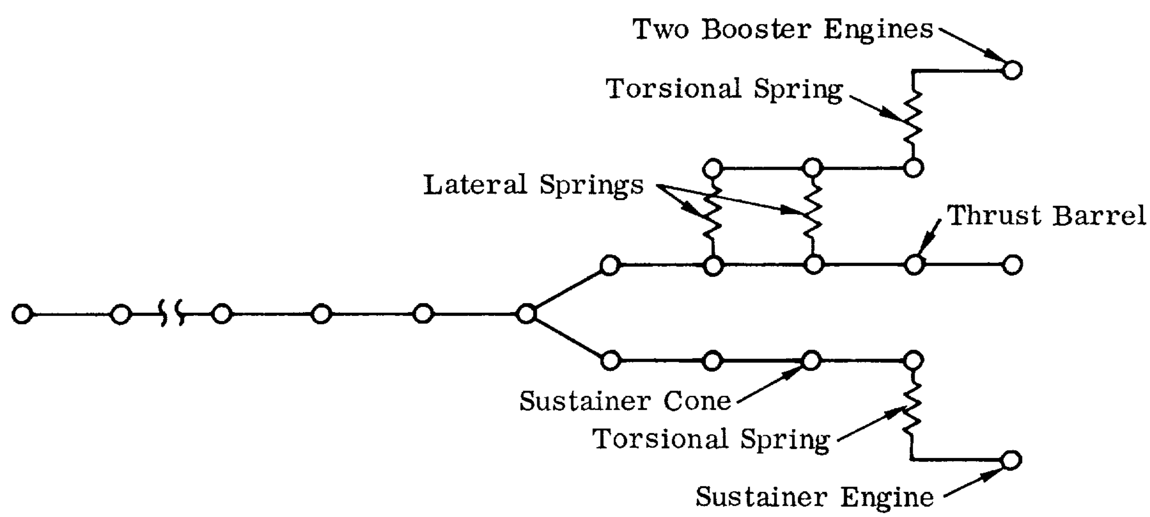


Figure B5. Typical Spring-Mass Model of the Vehicle Including Engines-in-the-Mode

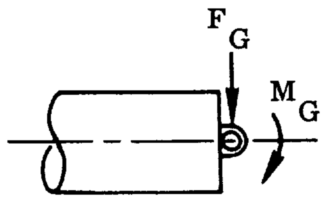
The differences between the mode shapes obtained from the two models previously discussed are:

a. The engines-in-the-modes mode shape will contain deflections and slopes at the engine center of gravity, whereas the single beam mode shape will not.

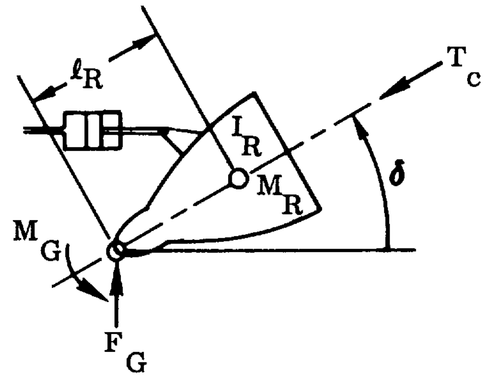
b. The engines-in-the-modes mode shape will contain deflections and slopes at the booster gimbal point and sustainer gimbal point, whereas the single beam mode shape assumes only one gimbal point, which is common for the booster and sustainer engines.

References 18 and 19 derive in detail the specific forces and moments which act on the vehicle as applied to autopilot stability analysis. Reference 18 pertains to the engines-in-the-modes model, and Ref. 19 pertains to the single beam model.

To illustrate some typical components of the F vector, consider the example of Fig. B6.



a. Vehicle



b. Gimbaled Rocket Engine

Figure B6. Vehicle and Engine Geometry

The external moment at the gimbal, M_G, is

$$M_G = I_R \ddot{\delta}$$

where

I_R = moment of inertia of engine with respect to the gimbal

The external force at the gimbal, F_G, is

$$F_G = - T_c \delta - M_R (\ell_R \ddot{\delta})$$

where

T_c = engine thrust

ℓ_R = length from gimbal to engine center of gravity

M_R = mass of engine

δ = engine angular displacement

We may now define:

$$
F \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -(T_c\,\delta + M_R\,\ell_R\,\ddot{\delta}) \\ 0 \\ \vdots \\ I_R\,\ddot{\delta} \\ 0 \end{bmatrix}
\begin{array}{l} \\ \\ \\ \longleftarrow \text{gimbal point deflection position} \\ \\ \\ \longleftarrow \text{gimbal point slope position} \\ \\ \end{array}
$$

Now, referring back to Eq. (B39):

$$
\mathfrak{m}^{(i)}\,\ddot{q}^{(i)} + \mathfrak{m}^{(i)}\left[\omega(i)\right]^2 q^{(i)} = Q^{(i)} \tag{B39}
$$

one may expand $Q^{(i)}$ in terms of its elements:

$$
Q^{(i)} = \eta_i{}^T F = \left[\phi_0^{(i)}\phi_1^{(i)} \ldots \phi_T^{(i)} \ldots \sigma_0^{(i)}\sigma_1^{(i)} \ldots \sigma_T^{(i)} \ldots\right] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -(T_c\,\delta + M_R\,\ell_R\,\ddot{\delta}) \\ 0 \\ 0 \\ \vdots \\ I_R\,\ddot{\delta} \\ 0 \end{bmatrix}
$$

$$
= -(T\,\delta + M_R\,\ell_R\,\ddot{\delta})\,\phi_T^{(i)} + I_R\,\ddot{\delta}\,\sigma_T^{(i)} \quad \text{(subscript T indicates the gimbal point)}
$$

Therefore, rearranging (B39) and substituting for $\eta_i^T F$,

$$\ddot{q}^{(i)} + \left[\omega^{(i)}\right]^2 q^{(i)} = \frac{-1}{\mathfrak{M}^{(i)}} \left[\left(T_c \delta + M_R \ell_R \ddot{\delta}\right) \phi_T^{(i)} - I_R \ddot{\delta}\, \sigma_T^{(i)}\right] \tag{B40}$$

If one compares Eq. (B40) with the equations derived in Refs. 18 and 19, one finds that the terms on the right side of the equations are of similar form, i.e., the product of an external force or moment and a modal deflection or slope. Thus, the preceding example shows that the product $\eta_i^T F$ transforms the externally applied forces and moments to "generalized forces." Further inspection of (B39) shows that it is identical to the elastic equations of motion as derived via LaGrange's equations, the method used in Refs. 18 and 19.