RADIAL TIME FUNCTIONS AND OTHER SOLUTIONS OF THE RADIAL HEAT EQUATION

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ABSTRACT

For $\omega > 2$, define

$$(*) \quad V_\omega \equiv V_\omega(r,t,f) = 2^{2-\omega}(4\pi)^{\omega/2}(\Gamma(\omega/2-1))^{-1}[\int_0^t S_\omega(r,t-y)f(y)dy],$$

where $S_\omega(r,t) = (4\pi t)^{-\omega/2}\exp(-r^2/4t)$. For $\Theta < 2$, define

$$(**) \quad U_\Theta = U_\Theta(r,t,f) = r^{2-\Theta}V_{4-\Theta}(r,t,f).$$

The functions V_ω and U_Θ are solutions of the radial heat equation $[D_r^2 + \frac{\mu - 1}{r}D_r]u(r,t) = D_t u(r,t)$ for $\mu = \omega$ and $\mu = \Theta$ respectively. Moreover, when $\omega = 3, 4, 5, \ldots$, the bracketed quantity in (*) (denote it by $H(r,t)$) represents the temperature in a ω-dimensional medium with $H(r,0+) = 0$ for $r > 0$ and into which heat is liberated at the rate $f(y)$ per unit time from $y = 0$ to $y = t$. Thus, for arbitrary $\omega > 2$, V_ω can be related to a generalized diffusion process with source $f(t)$.

D.V. Widder has developed an expansion theory for solutions of the equation

$$D_r^2 u(r,t) = D_t u(r,t),$$

where $\quad u(r,0+) = 0, \quad u(0,t) = f(t).$

The function U_Θ has the property that $U_\Theta(r,0+,f) = 0$, $U_\Theta(0,t,f) = f(t)$. We develop expansion theorems, analogous to those of Widder, for the function U_Θ.

The solution U_Θ is represented in terms of the set of radial time functions $\{u(\mu,h,r,t)\}$, $\mu < 2$, $h > -1$, where $u(\mu,h,r,t) = U_\mu(r,t,t^h/\Gamma(h+1))$.

Theorems are developed which relate the asymptotic behavior of U_Θ and the time average of U_Θ to the behavior of the function $f(t)$ and its Laplace transform. By means of (**), similar theorems are obtained for the function V_ω.

We examine several sets of special solutions which have been used in expansion theories for the radial heat equation. It is proved that the radial time functions and the elements of these several sets have a common form, involving confluent hypergeometric functions.

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INTRODUCTION

D. V. Widder [12] has developed necessary and sufficient conditions for the validity of expansions of solutions of the equation

(1.1a) $\quad D_r^2 u(r,t) = D_t u(r,t), \quad r > 0, \; t > 0,$

where

(1.1b) $\quad \lim_{t\to 0+} u(r,t) = 0, \quad r \geq 0; \quad \lim_{r\to 0} u(r,t) = f(t), \quad t > 0.$

This theory uses two basic sets of special solutions of (1.1a): $\{T_n(r,t)\}_{n=0}^{\infty}$ and $\{U_n(r,t)\}_{n=0}^{\infty}$. The first set was defined earlier by H. Poritsky and R. A. Powell [11]. Explicitly,

(1.2) $\quad T_n(r,t) = 2\int_0^t k(r,t-y)y^n/n!\, dy, \quad r \geq 0, \quad t > 0,$

where $k(r,t) = (4\pi t)^{-1/2} \exp(-r^2/4t)$ is the fundamental source solution of (1.1a). These authors show that

(1.3) $\quad T_n(0,t) = t^{n+1/2}/\Gamma(n+3/2),$

therefore

(1.4) $\quad T_n(r,t) = \int_0^t r(t-y)^{-1}k(r,t-y)T_n(0,y)dy.$

D. V. Widder defined the element $U_n(r,t)$ of the second set by

(1.5) $\quad U_n(r,t) = T_{n-1/2}(r,t).$

It then follows that

(1.6) $$U_n(0,t) = t^n/n!,$$

and

(1.7) $$U_n(r,t) = \int_0^t r(t-y)^{-1}k(r,t-y)U_n(0,y)dy.$$

Thus the function $T_n(r,t)(U_n(r,t))$ may be interpreted as the temperature of the semi-infinite bar $0 \leq r < \infty$, in which $T_n(r,0) = 0$ $(U_n(r,0) = 0)$ for $r > 0$, and the point $r = 0$ is maintained at the temperature given by (1.3) ((1.6)) for $t > 0$.

The major results relating to the expansions of solutions of the problem (1.1a)-(1.1b) are the following:

(I) The expansion

(1.8) $$u(r,t) = \sum_{j=0}^{\infty} [a_j T_j(r,t) + b_j U_j(r,t)]$$

converges to a solution of (1.1a) for $0 < t < T$, if and only if

(1.9) $$u(r,t) = \int_0^{\infty} k(r+y,t)g(y)dy,$$

where $g(y)$ is an entire function of growth $(2, 1/4T)$.

(II) The expansion (1.8) converges to a solution of (1.1a) for $0 < t < \infty$ if and only if (1.9) is valid for $g(y)$ an entire function of growth $(1, q)$.

(III) If $a_j = 0$, $j = 0, 1, 2, \ldots$, in (1.8), then the function $g(y)$ in (1.9) is an even function.

(IV) If $b_j = 0$, $j = 0, 1, 2, \ldots$, in (1.8), then the function $g(y)$ in (1.9) is an odd function.

Of fundamental importance is deriving these results is the replacement of the integrals in (1.4) and (1.7) by equivalent integrals which are analytic for $|r| < \infty$.

In this thesis, we are concerned with the radial heat equation

$$(1.10) \qquad \Delta_\mu u(r,t) = D_t u(r,t),$$

where $\Delta_\mu = D_r^2 + \frac{\mu - 1}{r} D_r$. When $\mu = n$, a positive integer, the operator Δ_n is the n-dimensional Laplacian in radial coordinates. In particular, we develop results analogous to (I)-(IV) for solution of (1.10) of the form

$$(1.11) \qquad V_\mu(r,t,f) = \frac{2^{2-\mu}(4\pi)^{\mu/2}}{\Gamma(\mu/2-1)} [\int_0^t S_\mu(r,t-y)f(y)dy], \text{ for } \mu>2,$$

and

$$(1.12) \qquad U_\mu(r,t,f) = r^{2-\mu} V_{4-\mu}(r,t,f), \quad \text{for } \mu < 2,$$

where $S_\mu(r,t) = (4\pi t)^{-\mu/2} \exp(-r^2/4t)$ is the fundamental source solution of (1.10).

When $\mu = 3,4,5,\ldots,$ the bracketed quantity in (1.11) (denote it by $H(r,t)$) represents the temperature (or in the case of a general diffusion process - the concentration) in a μ-dimensional medium with $H(r,0+) = 0$ for $r > 0$, and into which heat is liberated at the rate $f(y)$ per unit time from $y = 0$ to $y = t$. Thus for arbitrary $\mu > 2$,

$V_\mu(r,t,f)$ may be interpreted in terms of a generalized diffusion process with source $f(t)$.

It is shown that $\lim_{t\to 0+} U_\mu(r,t,f) = 0$, and that, if $f(t)$ is absolutely integrable and continuous from the left on $(0,M)$, then $\lim_{r\to 0} U_\mu(r,t,f) = f(t)$ for $0 < t < M$. Defining the set of radial time functions $\{u(\mu,h,r,t)\}$, $\mu < 2$, $h > -1$, by

$$(1.13) \qquad u(\mu,h,r,t) = U_\mu(r,t,t^h/\Gamma(h+1)),$$

results analogous to (I)-(IV) are obtained by expanding solutions of the form (1.12) in terms of the set $\{u(\mu,m/2,r,t)\}_{m=0}^{\infty}$. It is shown that the function $u(\mu,h,r,t)$ can be defined in terms of confluent hypergeometric functions. This allows us to develop integral representations for the radial time functions which are analytic for $|r| < \infty$. However, certain integral representations are valid only for special combinations of h and μ, and we are then limited to examining expansions of the form

$$(1.14) \qquad u(r,t) = \sum_{j=h_0}^{\infty} a_j u(\mu,j+1/2,r,t) + \sum_{j=h_1}^{\infty} b_j\, u(\mu,j,r,t).$$

In (1.14), unlike the development in [14], h_0 and h_1 are non-negative integers depending on μ. Nonetheless, if $k(r+y,t)$, in (1.9), is replaced by $K(r,t,y)$, a more complicated kernel involving Bessel functions, it is shown that results (I)-(IV) are valid when (1.8) is replaced by (1.14). Moreover, when $\mu = 1$, then $h_0 = h_1 = 0$ in (1.14) and our results reduce to those given in [14].

We examine the asymptotic behavior of $U_\mu(r,t,f)$. Theorems are developed which relate the behavior of $U_\mu(r,t,f)$ and time averages of $U_\mu(r,t,f)$ to the asymptotic behavior of the function $f(t)$ and of its Laplace transform $L[f]$. A typical theorem is the following: Let $f(t)$ be a non-negative function and let $L[f] = \int_0^\infty e^{-st} f(t)dt$ converge for $s > 0$. If there exist constants $c > 0$ and C such that $L[f] \sim C/s^c$ as $s \to 0^+$, then

$$ct^{-1} \int_0^t U_\mu(r,y,f)dy \sim Ct^{c-1}/\Gamma(c), \quad \text{as} \quad t \to \infty.$$

By means of formula (1.12), theorems pertaining to integral representations, expansion theorems, and asymptotic behavior of the function $V_{4-\mu}(r,t)$ are obtained.

When $\mu = 2$, solutions of (1.10) which have logarithmic singularities in a neighborhood of $r = 0$ are obtained by modifying the definition of $U_\mu(r,t,f)$. The methods used for examining series representations for $\mu < 2$ are also modified to obtain similar results when $\mu = 2$. Some of the asymptotic results are also extended to this case.

P.C. Rosenbloom and D.V. Widder [12] made a detailed study of the validity of expansions for solutions of (1.1a), for $-\infty < r < \infty$, in terms of two sets of special solutions:

(a) the set of heat polynomials $\{v_n(r,t)\}_{n=0}^{\infty}$ and (b) the set of associated functions $\{w_n(r,t)\}_{n=0}^{\infty}$. These authors develop expansion theorems which show that

(A) expansions in terms of heat polynomials are valid in a time strip $|t| < \sigma$ in which the solution satisfies a Huygen's principle while

(B) expansions in terms of associated functions are valid in a half-plane $t > \sigma \geq 0$ in which the solution has certain entireness properties.

Several methods are given for determining the coefficients in these expansions, and the L^2 theory of such expansions is also examined.

More recently L.R. Bragg [2] has developed results analogous to (A) and (B) for solutions of equation (1.10) when $\mu > 1$. This theory also uses two sets of special solutions: (a) the set of radial heat polynomials $\{R_K^{\mu}(r,t)\}_{K=0}^{\infty}$ and (b) the set of associated functions $\{\tilde{R}_K^{\mu}(r,t)\}_{K=0}^{\infty}$

The expansion theorems given in [2] were developed independently by D.T. Haimo [8], using the sets $\{P_{n,(\mu-1)/2}(r,t)\}_{n=0}^{\infty}$ and $\{W_{n,(\mu-1)/2}(r,t)\}_{n=0}^{\infty}$. Analogous results for the L^2 theory of such expansions were also developed [9].

It is shown that the radial time functions and the elements of these several sets of special solutions have the common form

$$(1.15) \qquad G(r,t) = ct^h F(h+\mu/2,\ \mu/2,\ r^2/4t)\ \exp\ (-r^2/4t).$$

Here, c, h, and μ are parameters independent of r and t, and $F(a,b,z)$ is a confluent hypergeometric function with parameters a and b.

In section 2, we develop integral representations for the functions $U_\mu(r,t,f)$ and $V_\mu(r,t,f)$ and obtain some of their basic properties. In section 3, a detailed study is made of the properties and integral representations of the radial time functions. The asymptotic behavior of $U_\mu(r,t,f)$ and $V_\mu(r,t,f)$ is developed in section 4. In sections 5 and 6, expansion theorems for the finite and infinite time intervals are developed, and in section 7, these results are extended to solutions of (1.10) when $\mu = 2$. In section 8, after examining the theories associated with several sets of special solutions of (1.10), it is shown that the elements of those sets all have the form (1.15). The Appendix contains all of the results about confluent hypergeometric functions which are used in sections 1-8. The notation (A-n) is used to refer to the nth equation in the Appendix.

INTEGRAL REPRESENTATIONS

H. Poritsky and R.A. Powell [11], in examining the one-dimensional heat equation, $u_{rr}(r,t) = u_t(r,t)$, studied special solutions of the form

$$(2.1) \qquad T_n(r,t) = 2S_1(r,t) * (t^n/n!),$$
$$r > 0, \quad t > 0, \quad n = 0, 1, 2, \ldots .$$

Here $S_1(r,t) = (4\pi t)^{-1/2} \exp(-r^2/4t)$ and $*$ denotes the convolution operation defined by

$$(2.2) \qquad f(t) * g(t) = \int_0^t f(t-y)g(y)dy.$$

The authors show that

$$(2.3) \qquad T_n(0,t) = t^{n+1/2}/\Gamma(n+3/2),$$

therefore

$$(2.4) \qquad T_n(r,t) = [rt^{-1}S_1(r,t)] * T_n(0,t).$$

In this form $T_n(r,t)$ may be interpreted as the temperature in the semi-infinite bar $0 \leq r < \infty$. When $t = 0$ the temperature in the bar is zero at all points $r > 0$. For $t > 0$, the end of the bar at $r = 0$ is maintained at the temperature given by the function (2.3).

Formulas (2.1) and (2.4) each have the form

$$(2.5) \qquad u(r,t) = c[t^a r^b \exp(-r^2/4t) * f(t)].$$

The following theorem shows that the radial heat equation

$$(2.6) \qquad \left[\frac{\partial^2}{\partial r^2} + \frac{\mu-1}{r}\frac{\partial}{\partial r}\right] u(r,t) = \frac{\partial u(r,t)}{\partial t}$$

also has solutions of the form (2.5).

<u>Theorem 2.1.</u> Let the function $f(t)$ be absolutely integrable on $(0,T)$ with $T \leq \infty$. Let Ω be the set of points $t \in (0,T)$ at which $f(t)$ is continuous. If $\int_0^T |f(t)|dt > 0$, then equation (2.6) has a solution of the form (2.5) in the region $R(\Omega) = \{(r,t)|r > 0,\ t \in \Omega\}$ if and only if

$$(2.7a) \qquad a = (\mu/2) - 2, \qquad b = 2 - \mu$$

or

$$(2.7b) \qquad a = -\mu/2, \qquad b = 0.$$

<u>Proof</u>. Writing (2.5) in integral form, it follows that

$$(2.8) \qquad u(r,t) = c\int_0^t (t-y)^a r^b \exp\,(-r^2/4(t-y))f(y)dy.$$

Since $f(t)$ is absolutely integrable on $(0,T)$, the integral in (2.8) exists when $t \in (0,T)$. Also differentiation under the integral sign with respect to either r or t is permissible in $R(\Omega)$ since $f(t)$ is continuous on Ω. Substituting the form (2.8) into equation (2.6) and simplifying, we have

$$(2.9) \qquad G(r,t;a,b) * f(t) = 0,$$

where

$$(2.10) \qquad G(r,t;a,b) = \exp\,(-r^2/4t)[(b+\mu/2+a)t^{a-1}r^b - b(b+\mu-2)t^a r^{b-2}].$$

If condition (2.7a) or (2.7b) holds then $G(r,t;a,b) \equiv 0$ and (2.9) is satisfied. This completes the proof of sufficiency. From [10], page 15, footnote (2), it follows that if $f(t)$ and $g(t)$ are integrable functions and if $f * g = 0$, then at least one of the functions f and g is equal to zero almost everywhere. By hypothesis $\int_0^T |f(t)|dt > 0$, thus equation (2.9) is satisfied only if $G(r,t;a,b) = 0$ almost everywhere. In view of formula (2.10), this is only possible when a and b satisfy the following pair of equations:

$$b(b + \mu - 2) = 0, \tag{2.11}$$

$$b + (\mu/2) + a = 0. \tag{2.12}$$

The only solutions of equation (2.11) are $b = 2 - \mu$ or $b = 0$. From equation (2.12), the corresponding values of a are $a = \mu/2 - 2$ or $a = -\mu/2$. These are the alternatives stated in the theorem. This completes the proof.

When $\mu < 2$, choose $c = 2^{\mu-2}/\Gamma(1-\mu/2)$ in formula (2.5). Let a and b be given by condition (2.7a). Then (2.5) defines a solution of equation (2.6) which is denoted by

$$U_\mu(r,t) \equiv U_\mu(r,t,f) = \frac{2^{\mu-2}(4\pi)^{\mu/2}}{\Gamma(1-\mu/2)}\left[\left(\frac{r}{t}\right)^{2-\mu} S_\mu(r,t)*f(t)\right], \tag{2.13}$$

where $S_\mu(r,t) = (4\pi t)^{-\mu/2}\exp(-r^2/4t)$ is the source solution of equation (2.6).

When $\mu > 2$, choose $c = 2^{2-\mu}/\Gamma(\mu/2 - 1)$ in formula (2.5). Let a and b be given by condition (2.7b). These choices define a solution of equation (2.6) which is denoted by

$$(2.14) \qquad V_\mu(r,t) \equiv V_\mu(r,t,f) = \frac{2^{2-\mu}(4\pi)^{\mu/2}}{\Gamma(\mu/2 - 1)} \left[S_\mu(r,t) * f(t) \right].$$

Unless stated otherwise it is assumed that $\mu < 2$ in any expression for $U_\mu(r,t)$ and that $\mu > 2$ in any expression for $V_\mu(r,t)$.

Theorem 2.2. If $a > 0$, then $U_{2-a}(r,t) = r^a V_{2+a}(r,t)$.

The proof of this theorem follows immediately from formulas (2.13) and (2.14).

In integral form, formula (2.13) becomes

$$(2.15) \qquad U_\mu(r,t) = \frac{2^{\mu-2}}{\Gamma(1-\mu/2)} \int_0^t \left(\frac{r}{z}\right)^{2-\mu} \frac{\exp(-r^2/4z)}{z^{\mu/2}} f(t-z)dz.$$

Introduce the change of variable $y = r^2/4z$ to obtain

$$(2.16) \qquad U_\mu(r,t) = [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2} f(t-r^2/4y)dy.$$

We now use formula (2.16) to examine, in detail, the properties of $U_\mu(r,t)$. Then, using Theorem 2.2, similar results are obtained for $V_\mu(r,t)$.

Theorem 2.3. Let the function $f(t)$ be absolutely integrable on $(0,T)$ with $T \leq \infty$. Then

$$(2.17)\qquad \lim_{t\to 0+} U_\mu(r,t) = 0, \quad \text{pointwise for} \quad r > 0.$$

Proof. Introducing absolute values on both sides of (2.16), it follows that

$$(2.18)\qquad |U_\mu(r,t)| \le [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{2-\mu/2} y^{-2} |f(t-r^2/4y)|\,dy.$$

Since $\lim_{y\to\infty} e^{-y}y^{2-\mu/2} = 0$, the function $e^{-y}y^{2-\mu/2}$ is bounded by some positive constant M, for sufficiently large values of y. In (2.18), y varies inversely with t, hence

$$(2.19)\qquad \lim_{t\to 0+} |U_\mu(r,t)| \le M \lim_{t\to 0+} \int_{r^2/4t}^{\infty} y^{-2} |f(t-r^2/4y)|\,dy.$$

With the change of variable $z = t - r^2/4y$, the inequality (2.19) becomes

$$\lim_{t\to 0+} |U_\mu(r,t)| \le M \lim_{t\to 0+} \int_0^t |f(z)|\,dz = 0.$$

In the following theorems, we examine the behavior of $U_\mu(r,t)$ as $r \to 0$.

Theorem 2.4. Let the function $f(t)$ be Lipschitz continuous with exponent a, $0 < a \le 1$, on the interval $[0,T]$ with $T < \infty$. Then

$$(2.20)\qquad \lim_{r\to 0} U_\mu(r,t) = f(t), \quad \text{if} \quad 0 < t \le T.$$

<u>Proof.</u> Let $U_\mu(r,t)$ be given by (2.16). Define

$$\bar{U}_\mu(r,t) = [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2} f(t)\,dy. \tag{2.21}$$

Consider the function $I(r,t)$ defined by

$$\begin{aligned} I(r,t) &= \bar{U}_\mu(r,t) - U_\mu(r,t) \\ &= [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2}\left[f(t)-f(t-r^2/4y)\right]dy. \end{aligned} \tag{2.22}$$

Introducing absolute values, we find

$$|I(r,t)| \leq [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2} |f(t)-f(t-r^2/4y)|\,dy. \tag{2.23}$$

Since $f(t)$ satisfies a Lipschitz condition on $[0,T]$, $|f(t) - f(t - r^2/4y)| \leq M(r^2/4y)^a$. Substitute into (2.23) to obtain

$$|I(r,t)| \leq [\Gamma(1-\mu/2)]^{-1} M(r^2/4)^a \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2-a}\,dy. \tag{2.24}$$

Integration by parts yields the inequality

$$\begin{aligned} |I(r,t)| \leq{} & \frac{Mr^{2-\mu} \exp(-r^2/4t)}{4^a (4t)^{1-\mu/2-a}(\mu/2+a-1)\Gamma(1-\mu/2)} \\ & + \frac{Mr^{2a}}{(1-\mu/2-a)\Gamma(1-\mu/2)} \int_{r^2/4t}^{\infty} e^{-y} y^{1-\mu/2-a}\,dy, \end{aligned} \tag{2.25}$$

provided $-\mu/2 - a \neq -1$.

Hence

$$(2.26)\qquad \lim_{r\to 0} |I(r,t)| = 0, \quad \text{if} \quad t > 0,\ -\mu/2 - 1 \neq -1.$$

If $-\mu/2 - a = -1$, the integral in (2.24) reduces to

$$(2.27)\qquad E_1(r^2/4t) = \int_{r^2/4t}^{\infty} e^{-y}y^{-1}dy.$$

From [1], page 229, (5.1.11.), it follows that

$$(2.28)\qquad |E_1(z)| \leq \gamma + |\ln z| + e^{z},$$

where γ is Euler's constant. Replace z by $r^2/4t$ in (2.28) and substitute for the integral in (2.24). Then,

$$|I(r,t)| \leq \frac{M}{4^a\Gamma(1-\mu/2)} [\gamma r^{2a}+r^{2a}|\ln(r^2/4t)| + r^{2a} \exp (r^2/4t)],$$

and

$$(2.29)\qquad \lim_{r\to 0} |I(r,t)| = 0, \quad \text{if} \quad t > 0,\ -\mu/2 - a = -1.$$

From (2.22), the definition of $I(r,t)$, we conclude that,

$$(2.30)\qquad \lim_{r\to 0} U_\mu(r,t) = \lim_{r\to 0} \overline{U}_\mu(r,t), \quad \text{if} \quad t > 0.$$

By definition (2.21), the right hand limit in (2.30) is $f(t)$. This completes the proof.

<u>Definition 2.1.</u> A function $f(t)$ is continuous from the left on the interval $[a,b]$ if and only if $\lim_{\tau\to t^-} f(\tau) = f(t)$ whenever $t \in (a,b]$.

Theorem 2.5. Let the function $f(t)$ be continuous from the left and bounded on $[0,T]$ with $T < \infty$. Then

$$\lim_{r \to 0} U_\mu(r,t) = f(t), \quad \text{if} \quad 0 < t \leq T. \tag{2.31}$$

Proof. Let the point $t_0 \in (0,T]$ be fixed. Since $f(t)$ is bounded on $[0,T]$, there exist constants m, M such that $m < f(t) < M$ for $t \in [0,t_0]$. Since $f(t)$ is continuous from the left at t_0, given $\epsilon > 0$, there exists a $\delta > 0$, such that $|f(t_0) - f(t)| < \epsilon$ when $0 < t_0 - t < \delta$. Define a pair of functions $f_1(t)$, $f_2(t)$ (see Figure 1) by

$$f_1(t) = \begin{cases} M & , \ 0 < t \leq t_0 - 3\delta/4 \\ (t-t_0-3\delta/4)\left[\dfrac{f(t_0)+\epsilon-M}{\delta/2}\right] + M, & t_0-3\delta/4 < t \leq t_0-\delta/4 \\ f(t_0) + \epsilon & , \ t_0 - \delta/4 < t \leq t_0, \end{cases} \tag{2.32}$$

and

$$f_2(t) = \begin{cases} m & , \ 0 < t \leq t_0 - 3\delta/4 \\ (t-t_0+3\delta/4)\left[\dfrac{f(t_0)-\epsilon-m}{\delta/2}\right] + m, & t_0-3\delta/4 < t \leq t_0-\delta/4 \\ f(t_0) - \epsilon & , \ t_0 - \delta/4 < t \leq t_0. \end{cases} \tag{2.33}$$

By construction $f_2(t) < f(t) < f_1(t)$, and

$$U_\mu(r,t,f_2) < U_\mu(r,t,f) < U_\mu(r,t,f_1). \tag{2.34}$$

Set $t = t_0$ in (2.34), then

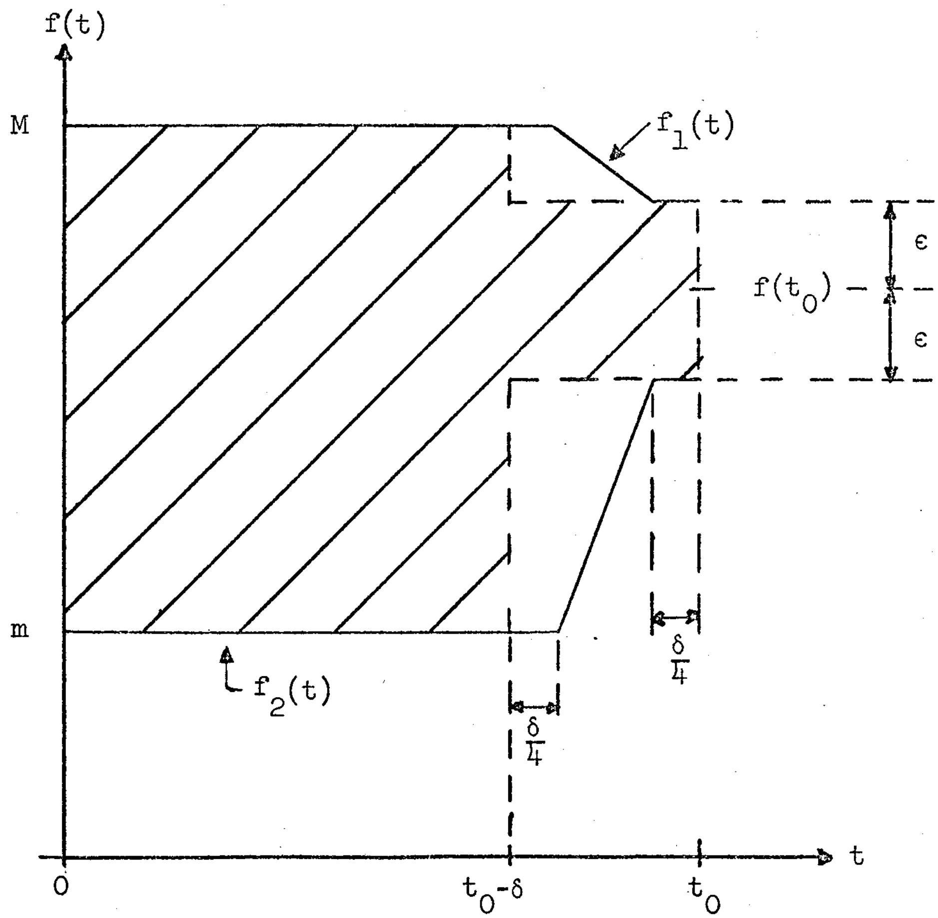

FIGURE 1

Graphs of the functions $f_1(t)$ and $f_2(t)$. The graph of the function $f(t)$ lies in the shaded region.

$$(2.35) \qquad \lim_{r\to 0} U_\mu(v,t_0,f_2) \leq \lim_{r\to 0} U_\mu(r,t_0,f) \leq \lim_{r\to 0} U_\mu(r,t_0,f_1).$$

By construction, the functions $f_1(t)$ and $f_2(t)$ satisfy the hypothesis of Theorem 2.4 on the interval $[0,t_0]$. Applying the results of that theorem to (2.35), it follows that

$$(2.36) \qquad f(t_0) - \epsilon \leq \lim_{r\to 0} U_\mu(r,t_0,f) \leq f(t_0) + \epsilon.$$

Since ϵ is arbitrary and t_0 is any point in $(0,T]$, the proof is complete.

Theorem 2.6. Let the function $f(t)$ be continuous from the left and absolutely integrable on $[0,T]$ with $T < \infty$. Then,

$$(2.37) \qquad \lim_{r\to 0} U_\mu(r,t) = f(t), \qquad \text{if} \quad 0 < t \leq T.$$

Proof. There are two cases to consider depending upon whether or not $f(t)$ is finite at the point in question.

Case (a) - f finite. Let $t_0 \in (0,T]$ be fixed and let $f(t_0) = M < \infty$. Since $f(t)$ is continuous from the left at t_0, there exists a $\delta_0 > 0$ and a positive integer N_0 such that $|f(t)| < N_0$ when $0 \leq t_0 - t \leq \delta_0$. Let $\{F_N(t)\}_{N=N_0}^{\infty}$ be a sequence of functions with

$$(2.38) \qquad F_N(t) = \begin{cases} N & , \text{ if } f(t) > N \\ f(t) & , \text{ if } |f(t)| \leq N \\ -N & , \text{ if } f(t) < -N \end{cases} .$$

Define $\Theta_N = \int_0^T |f(t) - F_N(t)|dt$. Then the sequence $\{\Theta_N\}_{N=N_0}^{\infty}$ is monotone decreasing with limit zero. Let $\epsilon > 0$ be given. Choose a positive integer K such that

(2.39a) $\qquad K > N_0,$

and

(2.39b) $\qquad \dfrac{\Theta_K C(\mu)}{\delta_0} < \epsilon,$

where

(2.40) $$C(\mu) = \frac{(1-\mu/2)^{1-\mu/2}\exp(\mu/2-1)}{\Gamma(1-\mu/2)} .$$

Consider the function

(2.41) $$I(r,t) = U_\mu(r,t,f) - U_\mu(r,t,F_K).$$

From definition (2.13), it follows that

(2.42) $$|I(r,t)| \le U_\mu(r,t\ |f-F_K|)$$

$$= \frac{2^{\mu-2}(4\pi)^{\mu/2}}{\Gamma(1-\mu/2)}\left[\left(\frac{r}{t}\right)^{2-\mu} S_\mu(r,t)*|f(t)-F_K(t)|\right] .$$

Consider $(r/t)^{2-\mu}S_\mu(r,t)$ as a function of r for fixed t. It has a maximum when $r^2 = (2-\mu)2t$ and this maximum is $\dfrac{[2(2-\mu)]^{1-\mu/2}\exp(\mu/2-1)}{t\,(4\pi)^{\mu/2}}$. Substitute into (2.42) to obtain

(2.43) $$|I(r,t)| \le C(\mu)\int_0^t \frac{1}{t-y}\,|f(y) - F_K(y)|dy,$$

where $C(\mu)$ is given by (2.40). From condition (2.39a) and the definition (2.38) of $F_K(t)$, we find that $|f(y) - F_K(y)| \equiv 0$ when $y \in [t_0 - \delta_0, t_0]$. Thus, with $t = t_0$, the inequality (2.43) becomes

$$(2.44) \qquad |I(r,t_0)| \leq C(\mu) \int_0^{t_0-\delta_0} \frac{|f(y) - F_K(y)|}{t_0-y} dy.$$

Replace $1/(t_0-y)$ by $1/\delta_0$, its maximum when $y \in [0, t_0 - \delta_0]$. Then

$$(2.45) \qquad |I(r,t_0)| \leq \frac{C(\mu)}{\delta_0} \int_0^{t_0-\delta_0} |f(y) - F_K(y)| dy.$$

The integral in (2.45) is bounded by Θ_K. From condition (2.39b), it follows that

$$(2.46) \qquad |I(r,t_0)| \leq \epsilon.$$

Since ϵ is arbitrary, we conclude from formula (2.41) that

$$(2.47) \qquad \lim_{r\to 0} U_\mu(r,t_0,f) = \lim_{r\to 0} U_\mu(r,t_0,F_K).$$

By construction, $F_K(t)$ satisfies the hypothesis of Theorem 2.5. Apply the conclusion of that theorem to the right member of (2.47) to obtain

$$(2.48) \qquad \lim_{r\to 0} U_\mu(r,t_0,f) = F_K(t_0) = f(t_0).$$

Case (b) - f infinite. Let $t_0 \in (0,T]$ be fixed and let $f(t_0) = \infty$. Since $f(t)$ is continuous from the left, for each integer $N > 0$ there exists a $\delta_N > 0$ such that $f(t) \geq N$ when $0 \leq t_0 - t \leq \delta_N$. Let $\{G_N(t)\}_{N=1}^{\infty}$ be a sequence of functions with

$$(2.49) \qquad G_N(t) = \begin{cases} f(t) & , \quad 0 \leq t \leq t_0 - \delta_N \\ N & , \quad t_0 - \delta_N < t \leq t_0 . \end{cases}$$

Then $G_N(t) \leq f(t)$ on $[0,t_0]$, and $U_\mu(r,t_0,G_N) \leq U_\mu(r,t_0,f)$ so that

$$(2.50) \qquad \lim_{r\to 0} U_\mu(r,t_0,G_N) \leq \lim_{r\to 0} U_\mu(r,t_0,f), \quad N = 1, 2, 3, \ldots .$$

By construction $G_N(t)$ satisfies the hypothesis for case (a) of this theorem. Apply the conclusion of case (a) to the limit on the left of (2.50) to obtain

$$(2.51) \qquad G_N(t_0) = N \leq \lim_{r\to 0} U_\mu(r,t_0,f), \quad N = 1, 2, 3, \ldots .$$

Thus, the limit in (2.51) is $+\infty$.

If $f(t_0) = -\infty$, the inequality (2.51) is valid when $f(t)$ is replaced by $-f(t)$. Then, multiplying (2.51) by -1, we find

$$(2.52) \qquad -\lim_{r\to 0} U_\mu(r,t_0,-f) \leq -N, \quad N = 1, 2, 3, \ldots .$$

But the expression on the left of (2.52) is just $\lim_{r\to 0} U_\mu(r,t_0,f)$. Thus $\lim_{r\to 0} U_\mu(r,t_0,f) = -\infty$. This completes the proof.

In examining the behavior of $U_\mu(r,t,f)$ as $r \to 0$, t was restricted to the interval $[0,T]$ with T finite. If $T = \infty$, we have

<u>Corollary 2.1.</u> Let the function $f(t)$ be continuous from the left and absolutely integrable on $[0,\infty)$. Then

$$\lim_{r\to 0} U_\mu(r,t,f) = f(t), \quad \text{if} \quad 0 < t < \infty. \tag{2.53}$$

<u>Proof.</u> Let $t_0 \in (0,\infty)$ be fixed. Then condition (2.53) follows for the interval $0 < t \leq t_0$ by setting $T = t_0$ in Theorem 2.6. Since t_0 is arbitrary, this completes the corollary.

Removing the restriction that $f(t)$ be continuous from the left, we have

<u>Corollary 2.2.</u> Let the function $g(t)$ be absolutely integrable on $(0,\infty)$. If the function $f(t) = \lim\limits_{\tau \to t-} g(\tau)$ is defined for $0 < t < \infty$, then

$$\lim_{r\to 0} U_\mu(r,t,g) = f(t), \quad \text{if} \quad 0 < t < \infty. \tag{2.54}$$

<u>Proof.</u> By construction $f(t) = g(t)$ almost everywhere on $(0,\infty)$. From (2.13), it follows that $U_\mu(r,t,g-f) = 0$, if $0 < t < \infty$. Hence,

$$\lim_{r\to 0} U_\mu(r,t,g) = \lim_{r\to 0} U_\mu(r,t,f). \tag{2.55}$$

But $f(t)$ satisfies the condition of Corollary 2.1, and the limit on the right of (2.55) is $f(t)$.

Remark. It is not necessary that $\lim_{t \to \infty} f(t)$ exist in either Corollary 2.1 or Corollary 2.2. For example, if

$$f(t) = \begin{cases} 2^j & , \quad j < t < j+2^{-2j}, \; j = 0,1,2,\ldots \\ 0 & , \quad \text{otherwise,} \end{cases}$$

see Figure 2, the conclusions of Corollaries 2.1 and 2.2 are valid. The same situation holds if $g(\alpha)$ is any function absolutely integrable on $[0,1]$ and $f(t)$ is defined on $[0,\infty)$ by $f(n + \alpha) = 2^{-n}g(\alpha)$, $n = 0, 1, 2, \ldots$, $0 \leq \alpha < 1$.

The following theorems develop similar properties of the solution function $V_\mu(r,t)$.

Theorem 2.7. Let the function $f(t)$ be absolutely integrable and continuous from the left on $[0,\infty)$. Then, if $V_\mu(r,t)$ is given by (2.14),

$$\lim_{t \to 0+} V_\mu(r,t) = 0, \quad \text{pointwise if} \quad r > 0, \tag{2.56}$$

and

$$\lim_{r \to 0} r^{\mu-2} V_\mu(r,t) = f(t), \quad \text{if} \quad 0 < t < \infty. \tag{2.57}$$

Proof. Set $\mu = 2 + a$ in Theorem 2.2 to obtain

$$U_{4-\mu}(r,t) = r^{\mu-2} V_\mu(r,t). \tag{2.58}$$

From (2.17), it follows that

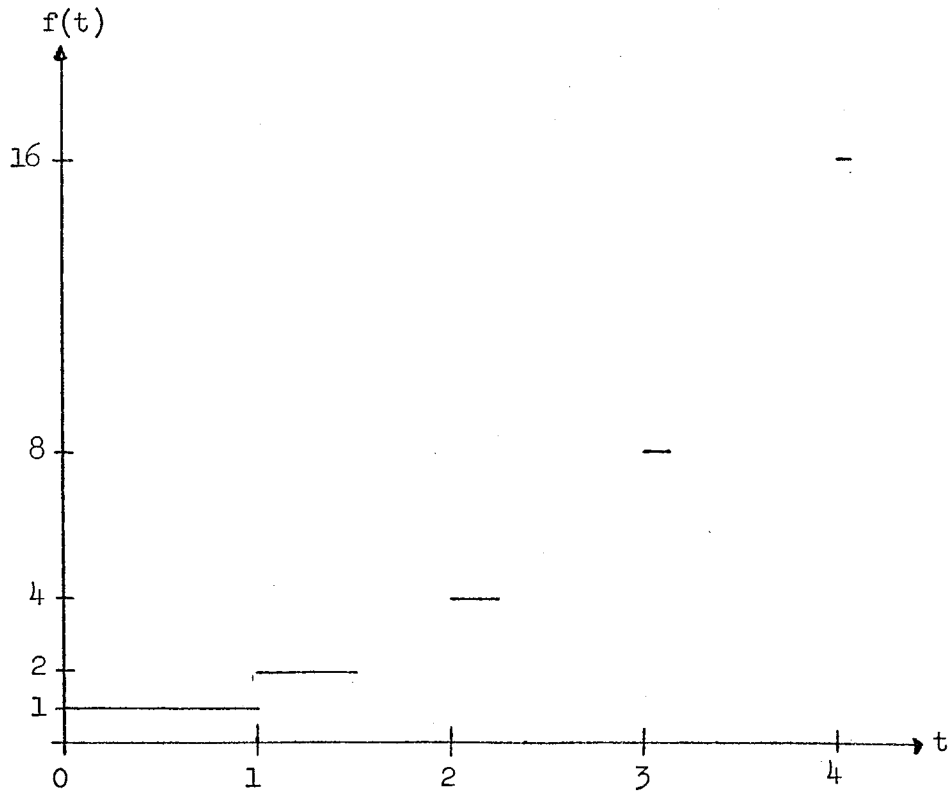

FIGURE 2

Graph of the function $f(t)$.

$$(2.59)\qquad \lim_{t\to 0+} U_{4-\mu}(r,t) = 0 = r^{\mu-2}\lim_{t\to 0+} V_\mu(r,t),$$

pointwise if $r > 0$.

This shows that (2.56) is valid. Combining formula (2.58) and the result (2.53), it follows that

$$(2.60)\qquad \lim_{r\to 0} U_{4-\mu}(r,t) = f(t) = \lim_{r\to 0} r^{\mu-2}V_\mu(r,t), \quad \text{if} \quad 0 < t < \infty.$$

This proves (2.57).

For $\mu > 2$, the following result corresponds to Corollary 2.2.

Corollary 2.3. Let the function $g(t)$ be absolutely integrable on $(0,\infty)$. If the function $f(t) = \lim_{\tau\to t-} g(\tau)$ is defined for $0 < t < \infty$, then

$$(2.61)\qquad \lim_{r\to 0} r^{\mu-2}V_\mu(r,t,g) = f(t), \quad \text{if} \quad 0 < t < \infty.$$

The following examples illustrate the theorems developed in this section.

Example 1. With $\mu < 2$ define

$$f(t) = \begin{cases} H_0, & 0 < t \le T \\ H_1, & T < t \end{cases}$$

From definition (2.13) a solution of equation (2.6) has the following integral representation:

$$U_\mu(r,t) = \begin{cases} [\Gamma(1-\mu/2)]^{-1}2^{\mu-2}H_0 \int_0^t e^{-r^2/4y} y^{-\mu/2}(r/y)^{2-\mu} dy, & 0<t\leq T \\ [\Gamma(1-\mu/2)]^{-1}2^{\mu-2}[H_0\int_0^t e^{-r^2/4y} y^{-\mu/2}(r/y)^{2-\mu} dy & \\ \qquad + H_1\int_T^t e^{-r^2/4y} y^{-\mu/2}(r/y)^{2-\mu} dy], & T<t. \end{cases}$$

Simplifying these expressions, we find

$$U_\mu(r,t) = \begin{cases} [\Gamma(1-\mu/2)]^{-1} H_0 \Gamma(1-\mu/2, r^2/4t), & 0 < t \leq T \\ [\Gamma(1-\mu/2)]^{-1}[H_1\Gamma(1-\mu/2,r^2/4t)+(H_0-H_1)\Gamma(1-\mu/2,r^2/4T)], & \\ & T < t. \end{cases} \tag{2.62}$$

Here $\Gamma(a,z)$ is the complimentary incomplete gamma function given by

$$\Gamma(a,z) = \int_z^\infty e^{-t} t^{a-1} dt, \quad a > 0. \tag{2.63}$$

From formula (2.62) we can easily verify the conclusions of Theorem 2.3 and Theorem 2.5. That is

$$\lim_{t\to 0+} U_\mu(r,t) = 0, \quad r > 0$$

and

$$\lim_{r\to 0} U_\mu(r,t) = \begin{cases} H_0, & 0 < t \leq T \\ H_1, & T < t \end{cases} .$$

<u>Example 2.</u> With $f(t) = t^K$, $K = 0, 1, 2, \ldots$, formula (2.16) becomes

$$U_\mu(r,t,t^K) = [\Gamma(1-\mu/2)]^{-1} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2} (t-r^2/4y)^K dy.$$

Using the binomial theorem and interchanging summation and integration signs, we find

$$(2.64) \quad U_\mu(r,t,t^K) = \frac{t^K}{\Gamma(1-\mu/2)} \sum_{j=0}^{K} \binom{K}{j} (-r^2/4t)^j \cdot$$

$$\int_{r^2/4t}^{\infty} e^{-y} y^{(1-\mu/2-j)-1} dy.$$

If $(1-\mu/2-j) > 0$, for $j = 0, 1, \ldots, K,$ it follows from (2.63) that

$$U_\mu(r,t,t^K) = \frac{t^K}{\Gamma(1-\mu/2)} \sum_{j=0}^{K} \binom{K}{j} (-r^2/4t)^j \Gamma(1-\mu/2-j,\ r^2/4t).$$

If $1-\mu/2-j \leq 0$ for some values of j between 0 and K, the corresponding integrals in the sum (2.64) cannot be interpreted directly in terms of (2.63). In these cases, successive integration by parts will produce an integral which can be evaluated in terms of (2.63). Hence, $U_\mu(r,t,t^K)$ can always be written as a finite sum of tabulated functions.

3

TWO BASIC SETS OF SOLUTION FUNCTIONS

Let us now examine two sets of solutions of equation (2.6). The first is the set of radial time functions $\{u(\mu,h,r,t)\}$. The elements of this set are defined by

$$(3.1) \qquad u(\mu,h,r,t) = U_\mu(r,t,t^h/\Gamma(h+1)), \ \mu < 2 \quad \text{and} \quad h > -1.$$

The elements of the second set $\{v(\mu,h,r,t)\}$ are defined by

$$(3.2) \qquad v(\mu,h,r,t) = V_\mu(r,t,t^h/\Gamma(h+1)), \ \mu > 2 \quad \text{and} \quad h > -1.$$

At this point, the function $u(\mu,h,r,t)$ is defined only for $r > 0$. However, if we substitute (2.16) into (3.1), it follows that

$$(3.3) \qquad u(\mu,h,r,t) = \frac{1}{\Gamma(1-\mu/2)} \int_{r^2/4t}^{\infty} e^{-y} y^{-\mu/2} \frac{(t-r^2/4y)^h}{\Gamma(h+1)} \, dy.$$

Thus (3.3) provides a representation for $u(\mu,h,r,t)$ which is analytic for $|r| < \infty$.

Example 2 in section 2 shows that the right side of (3.3) can be expressed as a finite sum of tabulated functions when h is a non-negative integer. The following theorem proves that (3.3) can be expressed in terms of a single tabulated function whenever $h > -1$.

Theorem 3.1. Let $u(\mu,h,r,t)$ be given by (3.3) for

$\mu < 2$ and $h > -1$. Then,

$$(3.4) \qquad u(\mu,h,r,t) = \frac{t^h e^{-r^2/4t}}{\Gamma(1-\mu/2)} \psi(h+\mu/2,\ \mu/2,\ r^2/4t),$$

where $\psi(a,b,z)$ is the confluent hypergeometric function defined in the Appendix, formula (A-3).

<u>Proof.</u> In (3.3), introduce the change of variable $y = r^2\omega/4t$. Then

$$(3.5) \qquad u(\mu,h,r,t) = \frac{(r^2/4t)^{1-\mu/2} t^h}{\Gamma(1-\mu/2)\Gamma(h+1)} \int_1^\infty e^{-\omega(r^2/4t)}(\omega-1)^h \omega^{-\mu/2-h} d\omega.$$

A comparison of (3.5) with (A-11) shows that by making the choices $a = h + 1$, $b = 2 - \mu/2$, and $z = r^2/4t$, it follows that

$$(3.6) \qquad u(\mu,h,r,t) = \frac{t^h e^{-r^2/4t}(r^2/4t)^{1-\mu/2}}{\Gamma(1-\mu/2)} \psi(h+1,2-\mu/2,r^2/4t).$$

A similar comparison of (3.6) with (A-9) shows that by making the choices $a = h + \mu/2$, $b = \mu/2$, and $z = r^2/4t$ we obtain

$$(3.7) \qquad u(\mu,h,r,t) = \frac{t^h e^{-r^2/4t}}{\Gamma(1-\mu/2)} \psi(h + \mu/2,\ \mu/2,\ r^2/4t).$$

This is precisely formula (3.4).

Although (3.4) is valid when $h > -1$, the right hand member of (3.4) is defined for all real h. That is, even though the integrals in (3.3) and (3.5) do not converge if

$h \leq -1$, formula (3.4) provides an extension to real h of the definition for $u(\mu,h,r,t)$ given in (3.1). In the following theorem, we examine the properties of the function given in (3.4) when h is real.

Theorem 3.2. Let $u(\mu,h,r,t)$ be given by (3.4) for h real. Then

(A) $u(\mu,h,r,t)$ satisfies equation (2.6),

(B) $\lim_{t \to 0+} u(\mu,h,r,t) = 0$, if $r \neq 0$,

(C) if $h \neq -1, -2, -3, \ldots$, $\lim_{r \to 0} u(\mu,h,r,t) = t^h/\Gamma(h+1)$, for $t > 0$, and

(D) if $h = -1, -2, -3, \ldots$, $\lim_{r \to 0} u(\mu,h,r,t) = 0$, for $t > 0$.

Proof of (A). Let $z = r^2/4t$ and denote differentiation with respect to z by $'$. Set

(3.8) $$F(z) = \psi(h + \mu/2, \mu/2, z).$$

Then, (3.4) becomes

(3.9) $$u(\mu,h,r,t) = [\Gamma(1-\mu/2)]^{-1}t^h e^{-z}F(z).$$

From formula (3.9), it follows that

$$D_r u = [\Gamma(1-\mu/2)]^{-1}t^{h-1}e^{-z}(r/2)[F'(z)-F(z)],$$

$$D_r^2 u = [\Gamma(1-\mu/2)]^{-1}t^{h-1}e^{-z}(1/2)[(r^2/2t)F''(z)+(1-r^2/t)F'(z) + (r^2/2t-1)F(z)],$$

and

$$D_t u = [\Gamma(1-\mu/2)]^{-1}t^{h-1}e^{-z}[-zF'(z)+(z+h)F(z)].$$

Substitute these expressions into equation (2.6) to obtain

$$(3.10) \qquad zF''(z) + (\mu/2-z)F'(z) - (\mu/2+h)F(z) = 0.$$

This is precisely the confluent hypergeometric equation (A-1) with parameters $h + \mu/2$ and $\mu/2$. Since $F(z)$ is given by (3.8), it satisfies equation (3.10) identically for all real values of h and μ. This completes the proof of (A).

<u>Remark.</u> From the proof of (A) it is clear that if $\mathcal{F}(z)$ is any confluent hypergeometric function with parameters $h+\mu/2$ and $\mu/2$, then a solution of equation (2.6) is given by $u(r,t) = ct^h e^{-r^2/4t}\mathcal{F}(r^2/4t)$. We will make use of this fact in section 8.

<u>Proof of (B).</u> From formulas (3.4) and (A-14), it follows that

$$\lim_{t\to 0+} u(\mu,h,r,t) = \lim_{t\to 0+} \frac{t^h e^{-r^2/4t}}{\Gamma(1-\mu/2)} (r^2/4t)^{-h-\mu/2} = 0, \quad \text{if } r\neq 0.$$

<u>Proof of (C).</u> From formula (3.4), we have

$$(3.11) \qquad \lim_{r\to 0} u(\mu,h,r,t) = \frac{t^h}{\Gamma(1-\mu/2)} \lim_{r\to 0} e^{-r^2/4t}\psi(h+\mu/2,\mu/2,r^2/4t).$$

When $\mu < 2$ and $h \neq -1, -2, -3, \ldots$, choose $b = \mu/2$, $a = h + \mu/2$, and $z = r^2/4t$ in Table 2, (e), (f), or (g) (depending on the actual value of μ). In each case, by applying the appropriate formula for $\psi(h+\mu/2,\mu/2,r^2/4t)$, (3.11) reduces to

$$\lim_{r\to 0} u(\mu,h,r,t) = \frac{t^h}{\Gamma(1-\mu/2)} \cdot \frac{\Gamma(1-\mu/2)}{\Gamma(h+1)} = t^h/\Gamma(h+1).$$

Proof of (D). When $\mu < 2$ and $h = -1, -2, \ldots, -K, \ldots,$ (3.4) becomes

$$(3.12) \qquad u(\mu,-K,r,t) = \frac{t^{-K}e^{-r^2/4t}}{\Gamma(1-\mu/2)} \psi(-K+\mu/2,\mu/2,r^2/4t).$$

A comparison of (3.12) with (A-13) shows that the choices $a = -K + \mu/2$, $n = K-1$, and $z = r^2/4t$ lead to

$$(3.13) \qquad u(\mu,-K,r,t) = \frac{(-1)^{K-1}(K-1)!t^{-K}}{\Gamma(1-\mu/2)(4t)^{1-\mu/2}} \left[r^2 e^{-r^2/4t} L_{K-1}^{(1-\mu/2)}(r^2/4t)\right].$$

Since $\lim_{r\to 0} L_{K-1}^{(1-\mu/2)}(r^2/4t)$ is a constant if $t > 0$, it follows that
$$\lim_{r\to 0} u(\mu,-K,r,t) = 0, \quad t > 0.$$

Corollary 3.1. When $\mu < 2$, there exist solutions of (2.6) with the property that $u(0,t) = \lim_{t\to 0+} u(r,t) = 0.$

Proof. By properties (B) and (D) of Theorem 5.2., the set $\{u(\mu,-K,r,t)\}_{K=1}^{\infty}$ provides a countable collection of such solutions for each value of $\mu < 2$.

In the following theorem an integral representation is developed for radial time functions of the form $u(\mu,m/2,r,t)$, $m = 0, 1, 2, \ldots$. This representation is used to determine what solutions of equation (2.6) have valid series representations of the form

$$(3.14) \quad u(r,t) = \sum_{m=0}^{\infty} a_m \, u(\mu,m/2,r,t).$$

<u>Theorem 3.3.</u> Let $\mu < 2$ and let m be a non-negative integer such that $m + \mu > 0$. Then,

$$(3.15) \quad u(\mu,m/2,r,t) = \frac{r^{1-\mu/2} e^{-r^2/4t}}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} e^{-z^2/4t} z^{\mu/2} K_{\mu/2-1}(rz/2t) \cdot \left[\frac{\Gamma(m/2+1/2)}{\Gamma(m/2+\mu/2)} \, \frac{z^m}{m!} \right] dz \, .$$

<u>Proof.</u> A comparison of (3.4) with (A-10) shows that by making the choices $a = h + \mu/2$, $b = \mu/2$, and $z = r^2/4t$, it follows that

$$(3.16) \quad u(\mu,h,r,t) = \frac{e^{-r^2/4t} 2^{\mu/2} r^{1-\mu/2}}{\Gamma(1-\mu/2)\Gamma(h+1)\Gamma(h+\mu/2)} \cdot \int_0^{\infty} e^{-y} (ty)^{h+\mu/4-1/2} K_{\mu/2-1}(r\, y^{1/2} t^{-1/2}) dy,$$

if $h + \mu/2 > 0$. If $h = k$, a non-negative integer, introduce the change of variable $\omega^2/4 = ty$. From the identity $\Gamma(2k+1) = 4^k \pi^{-1/2} \Gamma(k+1/2)\Gamma(k+1)$, it follows that

$$(3.17) \quad u(\mu,k,r,t) = \frac{r^{1-\mu/2} e^{-r^2/4t}}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} e^{-\omega^2/4t} \omega^{\mu/2} K_{\mu/2-1}(r\omega/2t) \cdot \left[\frac{\Gamma(k+1/2)}{\Gamma(k+\mu/2)} \, \frac{\omega^{2k}}{(2k)!} \right] d\omega, \text{ if } k + \mu/2 > 0.$$

When $m = 0, 2, 4, \ldots,$ and $k = m/2$, (3.17) reduces to

(3.15) and is valid for $m + \mu > 0$.

If k is a non-negative integer and $h = k + 1/2$, introduce again the change of variable $\omega^2/4 = ty$ in (3.16). Using the identity $\Gamma(2k+2) = 2^{2k+1}\pi^{-1/2}\Gamma(k+1)\Gamma(k+3/2)$, it follows that

$$(3.18) \qquad u(\mu,k+1/2,r,t) = \frac{r^{1-\mu/2}e^{-r^2/4t}}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^\infty e^{-\omega^2/4t}\omega^{\mu/2}K_{\mu/2-1}(r\omega/2t)\cdot$$

$$\left[\frac{\Gamma(k+1/2+1/2)}{\Gamma(k+1/2+\mu/2)} \; \frac{\omega^{2(k+1/2)}}{(2(k+1/2))!}\right] d\omega, \text{ if } k + 1/2 + \mu/2 > 0.$$

If $m = 1, 3, 5, \ldots$ and $k + 1/2 = m/2$, then (3.18) reduces to (3.15) and is valid when $m + \mu > 0$. This completes the proof.

The following theorem develops corresponding properties for the function $v(\mu,h,r,t)$ when $\mu > 2$.

<u>Theorem 3.4.</u> With $\mu > 2$ and h real, let $v(\mu,h,r,t)$ be given by

$$(3.19) \qquad v(\mu,h,r,t) = \frac{r^{2-\mu}t^h e^{-r^2/4t}}{\Gamma(\mu/2-1)} \psi(h+2-\mu/2, 2-\mu/2, r^2/4t).$$

Then the following statements are valid.

(A) Formulas (3.19) and (3.2) are identical when $h > -1$.

(B) If $a > 0$, then $u(2-a,h,r,t) = r^a v(2+a,h,r,t)$.

(C) The function $v(\mu,h,r,t)$ satisfies equation (2.6).

(D) If $r \neq 0$, $\lim_{t \to 0+} v(\mu,h,r,t) = 0$.

(E) If $h \neq -1, -2, -3, \ldots$, $\lim_{r \to 0} r^{\mu-2}v(\mu,h,r,t) = t^h/\Gamma(h+1)$, for $t > 0$.

(F) If $h = -1, -2, -3, \ldots$, $\lim_{r \to 0} r^{\mu-2} v(\mu,h,r,t) = 0$, for $t > 0$.

Proof of (A). Choose $\mu = 2 + a$ in Theorem 2.2. Then (3.2) becomes

$$v(\mu,h,r,t) = r^{2-\mu} U_{4-\mu}(r,t,t^h/\Gamma(h+1)), \quad \text{for } h > -1. \tag{3.20}$$

Applying definition (3.1) and Theorem 3.1 to the right member of (3.20) we obtain formula (3.19).

Proof of (B). Since Theorem 3.2 extended the definition (3.4) of $u(\mu,h,r,t)$ to include all real h, the validity of statement (B) follows from a comparison of formula (3.19) with (3.4).

Statements (C), (D), (E), and (F) are immediate consequences of statement (B) and Theorem 3.2, results (A), (B), (C), and (D) respectively.

4

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR LARGE t

We now examine the asymptotic behavior of the functions $U_\mu(r,t,f)$ and $V_\mu(r,t,f)$ in terms of the behavior of $f(t)$ as $t \to \infty$. Results are obtained by two methods. The first method uses results from the theory of Laplace transforms which relate the asymptotic behavior of a function to that of its Laplace transform and vice versa. The second method uses definitions (2.13) and (2.14) and relates the asymptotic behavior of the functions $U_\mu(r,t,f)$ and $V_\mu(r,t,f)$ to the special solutions defined in section 3.

The following notation is used to describe the asymptotic behavior of a function. By the expression $f(x) = O(g(x))$ as $x \to x_0$, we understand that the quotient $|f(x)/g(x)|$ is bounded in some neighborhood of x_0, and by the expression $f(x) = o(g(x))$ as $x \to x_0$, we understand that $\lim_{x \to x_0} f(x)/g(x) = 0$. A function $f(x)$ is asymptotic to the function $g(x)$ as $x \to x_0$, written $f(x) \sim g(x)$ as $x \to x_0$, if and only if $f(x) = g(x) + o(g(x))$ as $x \to x_0$. In using the 'big O' or 'little o' notation, the phrase 'as $x \to x_0$' is omitted when there is no ambiguity about the point x_0.

The following theorem is a restatement of [13], p. 182, Corollary 1a. It relates the asymptotic behavior of a function to that of its Laplace transform.

Theorem 4.1. If there exist constants $c > 0$ and C such that

(4.1) $\quad f(t) \sim Ct^{c-1}/\Gamma(c), \quad$ as $\quad t \to \infty,$

and if $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$ converges for $s > 0$, then

(4.2) $\quad \bar{f}(s) \sim C/s^c, \quad$ as $\quad s \to 0+.$

Remark. Formula (4.2) does not imply (4.1) even when $f(t)$ is restricted to non-negative values. For example, choose $f(t) = t + t \cos t$. Then, $f(t)$ is not asymptotic to t. However $\bar{f}(s) = 1/s^2 + (s^2-1)/(s^2+1)^2$, and $\bar{f}(s) \sim 1/s^2$ as $s \to 0+$.

The following theorem is a restatement of [13], p. 192, Theorem 4.3. It provides a partial converse to Theorem 4.1.

Theorem 4.2. If the function $a(t)$ is non-decreasing and the integral $g(s) = \int_0^\infty e^{-st} da(t)$ converges for $s > 0$, and if there exist constants $c > 0$ and C such that $g(s) \sim C/s^c$ as $s \to 0+$, then $a(t) \sim Ct^c/\Gamma(c+1)$ as $t \to \infty$.

We now apply these results to the functions $U_\mu(r,t,f)$ and $f(t)$. Definition (2.13) expresses $U_\mu(r,t,f)$ as a convolution integral. From [7], p. 146, (29), it follows that

(4.3) $$L[U_\mu(r,t,f)] = \frac{2}{\Gamma(1-\mu/2)} \left(\frac{r\sqrt{s}}{2}\right)^{1-\mu/2} K_{1-\mu/2}(r\sqrt{s}\,)\bar{f}(s).$$

In this expression, $\bar{f}(s)$ denotes the Laplace transform of $f(t)$ and $K_a(z)$ is the modified Bessel function of the second kind with index a. We now prove

Theorem 4.3. Let $f(t)$ be a non-negative function and let $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$ converge for $s > 0$. If there exist constants $c > 0$ and C such that $\bar{f}(s) \sim C/s^c$ as $s \to 0+$, then

$$(4.4) \qquad \frac{c \int_0^t U_\mu(r,y,f)dy}{t} \sim \frac{Ct^{c-1}}{\Gamma(c)},$$

and

$$(4.5) \qquad \frac{\int_0^t [U_\mu(r,y,f)-f(y)]dy}{t} = o(t^{c-1}), \quad \text{as} \quad t \to \infty.$$

If, in addition to the above hypotheses, $f(t) \sim Ct^{c-1}/\Gamma(c)$ as $t \to \infty$, then

$$(4.6) \qquad \frac{c}{t} \int_0^t U_\mu(r,y,f)dy - f(t) = o(t^{c-1}), \quad \text{as} \quad t \to \infty.$$

Proof of (4.4). Since $f(t) \geq 0$, it follows from (2.13) that $U_\mu(r,t,f) \geq 0$. Choose $a(t) = \int_0^t U_\mu(r,y,f)dy$ in Theorem 4.2. Then,

$$(4.7) \qquad g(s) = \int_0^\infty e^{-st} da(t) = L[U_\mu(r,t,f)].$$

By hypothesis $\bar{f}(s) \sim C/s^c$ as $s \to 0+$. Substituting (4.3) into last member of (4.7), we find

$$(4.8) \qquad g(s) \sim C/s^c, \quad \text{as} \quad s \to 0+.$$

Since $a(t)$ satisfies the hypotheses of Theorem 4.2., it follows that

(4.9) $\quad a(t) \sim Ct^c/\Gamma(c+1)$, as $t \to \infty$.

Formula 4.4 is obtained by multiplying both sides of (4.9) by c/t.

<u>Proof of (4.5).</u> Define $b(t) = \int_0^t f(y)dy$. Then $b(t)$ satisfies the hypotheses of Theorem 4.2 since (i) $b(t)$ is non-decreasing, (ii) $g(s) = \int_0^\infty e^{-st}db(t) = \int_0^\infty e^{-st}f(t)dt$ converges for $s > 0$, and (iii) $g(s) \sim C/s^c$ as $s \to 0+$. As a consequence of Theorem 4.2,

(4.10) $\quad b(t) \sim Ct^c/\Gamma(c+1)$, as $t \to \infty$.

Formula (4.5) is obtained by subtracting (4.10) from (4.9).

<u>Proof of (4.6).</u> In this case,

(4.11) $\quad f(t) = Ct^{c-1}/\Gamma(c) + \epsilon(t)t^{c-1}$,

where $\epsilon(t) \to 0$ as $t \to \infty$. Subtract (4.11) from (4.4) to obtain formula (4.6).

In the following theorem we describe the asymptotic behavior of the function $U_\mu(r,t,f)$ in terms of the radial time functions discussed in section 3.

<u>Theorem 4.4.</u> Let the function $f(t)$ be absolutely integrable on $(0,T)$ for each $T \in (0,\infty)$. If there exists a constant $c > 0$ such that $f(t) \sim t^c/\Gamma(c+1)$ as $t \to \infty$, then

(4.12) $\quad U_\mu(r,t,f) \sim u(\mu,c,r,t)$, as $t \to \infty$, $r > 0$.

<u>Proof.</u> By hypothesis $f(t) = t^c/\Gamma(c+1) + \epsilon(t)t^c$, where $\epsilon(t) \to 0$ as $t \to \infty$. Let $\epsilon_0 > 0$ be given. Choose $M > 0$

such that $|\epsilon(t)| < \epsilon_0/\Gamma(c+1)$ when $t \geq M$. Define

$G(r,t) = \frac{2^{\mu-2}}{\Gamma(1-\mu/2)} (\frac{r}{t})^{2-\mu} \frac{\exp(-r^2/4t)}{t^{\mu/2}}$. When $t > M$, it follows from definition (2.13) that

$$(4.13) \quad U_\mu(r,t,f) = \int_0^M f(y)G(r,t-y)dy$$

$$+ \int_M^t [y^c/\Gamma(c+1)+\epsilon(y)y^c]G(r,t-y)dy$$

$$= \int_0^M [f(y)-y^c/\Gamma(c+1)]G(r,t-y)dy$$

$$+ \int_0^t [y^c/\Gamma(c+1)]G(r,t-y)dy + \int_M^t \epsilon(y)y^c G(r,t-y)dy$$

$$= u(\mu,c,r,t) + \int_0^M [f(y)-y^c/\Gamma(c+1)]G(r,t-y)dy$$

$$+ \int_M^t \epsilon(y)y^c G(r,t-y)dy.$$

Consider $G(r,t)$ as a function of t for fixed r. It has a maximum when $t = r^2/(2-\mu)$ and this maximum is $\frac{2^{\mu-2}[2(4-\mu)]^{2-\mu/2}\exp\,(\mu/2-2)}{r^2\Gamma(1-\mu/2)}$. For $r > 0$, it follows that

$$|\int_0^M [f(y)-y^c/\Gamma(c+1)]G(r,t-y)dy| < C_1 \int_0^M [|f(y)|+y^c/\Gamma(c+1)]dy < C_2,$$

where C_1 and C_2 are positive constants depending on r. Moreover, $|\epsilon(t)| < \epsilon_0/\Gamma(c+1)$ when $t \geq M$, so that

$$|\int_M^t \epsilon(y)y^c G(r,t-y)dy| < \epsilon_0 \int_0^t [y^c/\Gamma(c+1)]G(r,t-y)dy = \epsilon_0\, u(\mu,c,r,t).$$

Substituting these expressions into the last member of (4.13), it follows that

$$|U_\mu(r,t,f) - u(\mu,c,r,t)| < C_2 + \epsilon_0\, u(\mu,c,r,t).$$

Since $u(\mu,c,r,t) > 0$, we have

$$(4.14) \qquad \left|\frac{U_\mu(r,t,f)-u(\mu,c,r,t)}{u(\mu,c,r,t)}\right| < \frac{C_2}{u(\mu,c,r,t)} + \epsilon_0.$$

From Theorem 3.1 and Table 2, (e), (f), or (g) (depending on the size of μ), it follows that

$$\lim_{t\to\infty} u(\mu,c,r,t) = \frac{1}{\Gamma(c+1)} \lim_{t\to\infty} t^c, \quad \text{if} \quad r > 0.$$

Hence, if $c > 0$, it follows from (4.14) that

$$(4.15) \qquad \lim_{t\to\infty} \left|\frac{U_\mu(r,t,f)-u(\mu,c,r,t)}{u(\mu,c,r,t)}\right| < \epsilon_0.$$

Since ϵ_0 is arbitrary, the limit in (4.15) is zero. This completes the proof.

For the function $V_\mu(r,t,f)$ we have

<u>Theorem 4.5.</u> Let $f(t)$ be a non-negative function and let $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$ converge for $s > 0$. If there exist constants $c > 0$ and C such that $\bar{f}(s) \sim C/s^c$ as $s \to 0+$, then

$$(4.16) \qquad \frac{c\int_0^t V_\mu(r,y,f)dy}{t} \sim \frac{r^{2-\mu} C t^{c-1}}{\Gamma(c)},$$

and

$$(4.17)\qquad \frac{\int_0^t [V_\mu(r,y,f)-r^{2-\mu}f(y)]dy}{t} = o(t^{c-1}), \quad \text{as} \quad t\to\infty.$$

If, in addition to the above hypotheses, $f(t) \sim Ct^{c-1}/\Gamma(c)$ as $t\to\infty$, then

$$(4.18)\qquad \frac{c}{t}\int_0^t V_\mu(r,y,f)dy - r^{2-\mu}f(t) = o(t^{c-1}), \quad \text{as} \quad t\to\infty.$$

Proof. Replace μ by $4-\mu$ in formulas (4.12), (4.13), and (4.14). Multiply these formulas through by $r^{2-\mu}$ and use Theorem 2.2 to obtain (4.16), (4.17), and (4.18) respectively.

By a similar argument we can prove

Theorem 4.6. Let the function $f(t)$ satisfy the hypothesis of Theorem 4.4. Then

$$(4.19)\qquad V_\mu(r,t,f) \sim v(\mu,c,r,t), \quad \text{as} \quad t\to\infty, \quad r > 0.$$

The following examples illustrate the theorems developed in this section.

Example 1. Choose $f(t) = t + t\cos t$ in Theorem 4.3. From the Remark following Theorem 4.1, it follows that $f(t)$ satisfies the hypotheses of only the first part of Theorem 4.3 with $c = 2$ and $C = 1$. Formulas (4.4) and (4.5) become

$$\frac{2\int_0^t U_\mu(r,y,t+t\cos t)dy}{t} \sim t,$$

and

$$\frac{\int_0^t [U_\mu(r,y,t+t\cos t)-(y+y\cos y)]dy}{t} = o(t), \quad \text{as} \quad t\to\infty.$$

From this example, it is clear that the condition $f(t) \sim Ct^c/\Gamma(c+1)$ is sufficient but not necessary for the validity of formulas (4.4) and (4.5).

Example 2. Define a unit step function $H(t)$ by

$$H(t) = \begin{cases} 0 & ,t < 0 \\ 1/2 & ,t = 0 \\ 1 & ,t > 0 \end{cases}$$

In Theorem 4.3, choose $f(t) = \sum_{n=1}^{\infty} a^{n-1}H(t-nk)$, with $|a| < 1$ (see Figure 3). Then $f(t) \sim \frac{1}{1-a}$ and, using [1], p. 1025, (29.3.66), $\overline{f}(s) = 1/s(e^{ks}-a)$. Thus the hypotheses of both parts of the theorem are satisfied with $c = 1$ and $C = 1/(1-a)$. Formulas (4.4)-(4.6) become

$$\frac{\int_0^t U_\mu(r,y,f)dy}{t} \sim \frac{1}{1-a},$$

and

$$\frac{\int_0^t U_\mu(r,y,f)dy}{t} - f(t) = \frac{\int_0^t [U_\mu(r,y,f)-f(y)]dy}{t} = o(1),$$

$$\text{as} \quad t\to\infty.$$

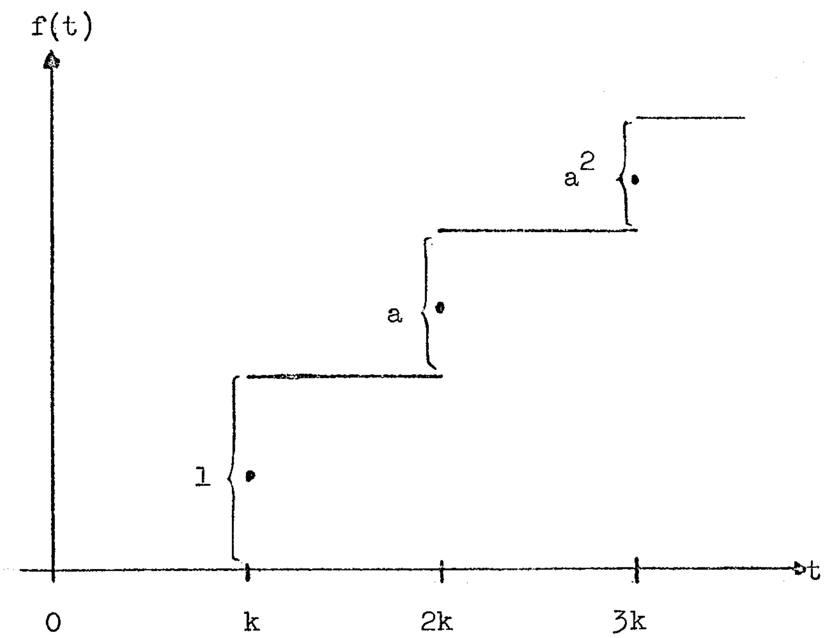

FIGURE 3

Graph of the function $f(t) = \sum_{n=1}^{\infty} a^{n-1} H(t-nk)$.

5

EXPANSION THEOREMS FOR THE FINITE INTERVAL $0 < t < T$

We now determine which solutions of equation (2.6) can be represented in a series of the form

$$(5.1) \qquad u(r,t) = \sum_{k=0}^{\infty} [a_k u(\mu,k,r,t) + b_k u(\mu,k+1/2,r,t)].$$

In this and the following section, necessary and sufficient conditions are developed in order that a solution of (2.6) have a series representation of the form

$$(5.2) \qquad \omega(r,t) = \sum_{k=h_0}^{\infty} a_k u(\mu,k,r,t) + \sum_{k=h_1}^{\infty} b_k u(\mu,k+1/2,r,t).$$

Here h_0 and h_1 are non-negative integers depending on μ. We then have the result that a solution of (2.6) has a representation of the form (5.1) if and only if

$$u(r,t) = \sum_{k=0}^{h_0-1} a_k u(\mu,k,r,t) + \sum_{k=0}^{h_1-1} b_k u(\mu,k+1/2,r,t) + \omega(r,t), \quad \text{where}$$

$\omega(r,t)$ is given by (5.2). In order to develop these expansion theorems, we need some results pertaining to entire functions.

<u>Definition 5.1.</u> A function $f(z)$ is of growth (ρ,σ) if and only if

$$(5.3) \qquad f(z) = o(\exp(|z|^{\rho}\sigma/\theta)), \quad \text{as} \quad z \to \infty,$$

for any Θ such that $0 < \Theta < 1$. For example, the functions $\sinh 3z^2$, $\exp(z^{1+1/z})$ and z^5 all have growth $(2,3)$.

Let $f(z)$ and $F(z)$ be entire functions defined by

$$(5.4) \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n/n!; \quad F(z) = \sum_{n=0}^{\infty} |c_n| z^n/n!.$$

Then the growth of the functions f and F is determined by the sequence $\{|c_n|\}_{n=0}^{\infty}$. The specific results we need are that the functions defined in (5.4) have growth $(2,\sigma)$ if and only if

$$(5.5) \qquad \limsup_{n\to\infty} \frac{|c_n|^{2/n}}{n} \leq \frac{2\sigma}{e};$$

they have growth $(1,\sigma)$ if and only if

$$(5.6) \qquad \limsup_{n\to\infty} |c_n|^{1/n} \leq \sigma.$$

We examine expansions in terms of the sets of radial time functions $\{u(\mu,k,r,t)\}_{k=0}^{\infty}$ and $\{u(\mu,k+1/2,r,t)\}_{k=0}^{\infty}$ separately. The main theorem is obtained by combining the two results.

Theorem 5.1. Let $\mu < 2$ and let h_0 be the least integer such that $h_0 + \mu/2 > 0$. A function $u(r,t)$ has an expansion of the form

$$(5.7) \qquad u(r,t) = \sum_{k=h_0}^{\infty} a_k u(\mu,k,r,t),$$

which converges for $0 < t < T$, if and only if

$$(5.8) \qquad u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^\infty T_\mu(r,t,y)g(y)dy.$$

Here, $g(y)$ is an even entire function of growth $(2,1/4T)$ defined by

$$(5.9) \qquad g(y) = \sum_{k=h_0}^{\infty} a_k \frac{\Gamma(k+1/2)}{\Gamma(k+\mu/2)} \cdot \frac{y^{2k}}{(2k)!},$$

and

$$(5.10) \qquad T_\mu(r,t,y) = e^{-y^2/4t} y^{\mu/2} r^{1-\mu/2} K_{\mu/2-1}(ry/2t).$$

Proof. Sufficiency. Substitute (5.9) into (5.8) and apply Theorem 3.3 with $m = 2k$. We formally obtain the series (5.7). Since $T_\mu(r,t,y)$ is positive for r, t, and $y > 0$, the term-by-term integration needed to derive the series (5.7) is valid if

$$(5.11) \qquad I = \int_0^\infty T_\mu(r,t,y)G(y)dy < \infty.$$

Here, $G(y)$ is obtained from $g(y)$ by replacing a_k by $|a_k|$ in (5.9). Write (5.11) in the form

$$(5.12) \qquad I = \int_0^m T_\mu(r,t,y)G(y)dy + \int_m^M T_\mu(r,t,y)G(y)dy$$
$$+ \int_M^\infty T_\mu(r,t,y)G(y)dy.$$

Denote the integrals in (5.12) by I_1, I_2, and I_3 respectively. For sufficiently small y, (5.9) shows that

$G(y) \sim |a_{h_0}| \dfrac{\Gamma(h_0+1/2)}{\Gamma(h_0+\mu/2)} \dfrac{y^{2h_0}}{(2h_0)!}$. It follows from [1], p. 375, (9.6.9) that

$$y^{\mu/2}K_{\mu/2-1}(ry/2t) \sim \frac{\Gamma(1-\mu/2)}{2}\left(\frac{r}{2t}\right)^{\mu/2-1} y^{\mu-1}, \quad \text{as} \quad y \to 0+.$$

Thus when $r \neq 0$ and $t > 0$ are fixed, and m is sufficiently small

$$(5.13) \qquad I_1 \sim C \int_0^m y^{2(h_0+\mu/2)-1}\,dy.$$

We conclude that the integral I_1 is finite. The function $g(y)$ and, hence, $G(y)$ has growth $(2,1/4T)$. When $M > 0$ is sufficiently large, it follows from (5.3) that $G(y) \leq e^{y^2/4T\theta}$ when $y \geq M$ and θ is any number between 0 and 1. Also, for sufficiently large y, if $r \neq 0$ and $t > 0$ are fixed, then $y^{\mu/2}r^{1-\mu/2}K_{\mu/2-1}(ry/2t)$ is bounded by some positive constant C_2. Then, for M sufficiently large

$$(5.14) \qquad I_3 < C_2 \int_M^\infty e^{-y^2(1/4t\ -\ 1/4T\theta)}\,dy.$$

It is clear that this integral exists and is finite if $0 < t < \theta T$ for arbitrary θ such that $0 < \theta < 1$. Finally, since $T_\mu(r,t,y)G(y)$ is a bounded continuous function of y on the interval $[m,M]$, I_2 is finite. This fact, combined with (5.13) and (5.14) prove the validity of (5.11) when $0 < t < T$.

Necessity. Assume the validity of the representation (5.7). The function $u(\mu,k,r,t)$, as given by (3.4) is an even function of r. Since

$\lim_{r \to 0} u(\mu,k,r,t)$ exists for $t > 0$,

$$(5.15) \qquad u(0,t) = \sum_{k=h_0}^{\infty} a_k t^k/k!.$$

By hypothesis, this series converges for $0 < t < T$. Applying Stirling's formula to the series (5.15), we obtain

$$(5.16) \qquad \limsup_{n \to \infty} \frac{|a_n|^{1/n}}{n} \leq \frac{1}{eT}.$$

Use formula (5.5) to examine the growth of $g(y)$, as given by (5.9). Since only even powers of y are involved, replace n by $2n$, and c_{2n} by $a_n\Gamma(n+1/2)/\Gamma(n+\mu/2)$ in formula (5.5). Simplifying, we have

$$(5.17) \qquad \limsup_{n \to \infty} \frac{|a_n|^{1/n}}{n} \leq \frac{4\sigma}{e}.$$

A comparison of (5.16) and (5.17) shows that $g(y)$ has growth $(2,1/4T)$. It follows that the integral in (5.8) converges absolutely if $0 < t < T$. Therefore, the interchange of summation and integration signs needed to obtain (5.8) from (5.7) is valid. This completes the proof.

For expressions in terms of the set $\{u(\mu,k+1/2,r,t)\}_{k=0}^{\infty}$, we have

<u>Theorem 5.2.</u> Let $\mu < 2$ and let $h_1 \geq 0$ be the least integer such that $h_1 + \frac{\mu+1}{2} > 0$. A function $u(r,t)$ has an expansion of the form

$$(5.18) \qquad u(r,t) = \sum_{k=h_1}^{\infty} b_k u(\mu, k+1/2, r, t),$$

which converges for $0 < t < T$, if and only if

$$(5.19) \qquad u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^\infty T_\mu(r,t,y)h(y)dy.$$

Here, $T_\mu(r,t,y)$ is given by (5.10) and $h(y)$ is an odd entire function of growth $(2,1/4T)$, defined by

$$(5.20) \qquad h(y) = \sum_{k=h_1}^{\infty} b_k \frac{\Gamma((k+1/2)+1/2)}{\Gamma(k+1/2+\mu/2)} \quad \frac{y^{2k+1}}{(2k+1)!}\,.$$

Proof. Sufficiency. Substitute the series (5.20) into the integral in (5.19) and apply Theorem 3.3 with $m = 2k + 1$. We formally obtain the series (5.18). The term-by-term integration needed to derive the series (5.18) is valid if

$$(5.21) \qquad J = \int_0^\infty T_\mu(r,t,y)H(y)dy < \infty,$$

where $H(y)$ is obtained from $h(y)$ by replacing b_k by $|b_k|$ in (5.20). The same argument that was used to prove inequality (5.11) can now be used to show that the inequality (5.21) is valid for $0 < t < T$.

Necessity. Evaluating the series (5.18) at $r = 0$, we obtain

$$(5.22) \qquad u(0,t) = \sum_{k=h_1}^{\infty} b_k t^{k+1/2}/\Gamma(k+3/2).$$

From the identity $\Gamma(k+1/2) = \pi^{1/2}\Gamma(2k+1)/4\,\Gamma(k+1)$, it follows that

$$(5.23)\qquad u(0,t) = 4t^{1/2}\pi^{-1/2}\sum_{k=h_1}^{\infty} 4^k(k+1)!b_k t^k/(2k+2)!.$$

By hypothesis, the series (5.23) converges for $0 < t < T$. Apply Stirling's formula to the series in (5.23) to obtain

$$(5.24)\qquad \limsup_{n\to\infty} \frac{|b_n|^{1/n}}{n} \le \frac{1}{eT}\ .$$

Use formula (5.5) to examine the growth of $h(y)$, as given by (5.20). Since only odd powers of y are involved, replace n by $2n + 1$ and c_{2n+1} by $b_n \cdot \frac{\Gamma((n+1/2)+1/2)}{\Gamma(n+1/2+\mu/2)}$ in formula (5.5). Simplifying, we have

$$(5.25)\qquad \limsup_{n\to\infty} \frac{|b_n|^{1/n}}{n} \le \frac{4\sigma}{e}.$$

A comparison of (5.25) with (5.24) shows that $h(y)$ has growth $(2,1/4T)$. It follows that the integral in (5.19) converges absolutely if $0 < t < T$. Therefore the interchange of summation and integration signs needed to obtain (5.19) from (5.18) is valid.

The main result of this section is obtained by combining Theorems 5.1 and 5.2.

Theorem 5.3. Let $\mu < 2$ and let h_0 and $h_1 \ge 0$ be the least integers such that $h_0 + \mu/2 > 0$ and $h_1 + (\mu+1)/2 > 0$. A function $u(r,t)$ has an expansion of the form

$$(5.26) \quad u(r,t) = \sum_{k=h_0}^{\infty} a_k u(\mu,k,r,t) + \sum_{k=h_1}^{\infty} b_k u(\mu,k+1/2,r,t),$$

which converges for $0 < t < T$, if and only if

$$(5.27) \quad u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} T_\mu(r,t,y) f(y)dy.$$

Here $T_\mu(r,t,y)$ is given by (5.10) and $f(y)$ is an entire function of growth $(2,1/4T)$ defined by

$$(5.28) \quad f(y) = \sum_{k=h_0}^{\infty} a_k \frac{\Gamma(k+1/2)}{\Gamma(k+\mu/2)} \frac{y^{2k}}{(2k)!} + \sum_{k=h_1}^{\infty} b_k \frac{\Gamma(k+1)}{\Gamma(k+1/2+\mu/2)} \frac{y^{2k+1}}{(2k+1)!}.$$

It is clear that function defined by the series (5.26) satisfies equation (2.6) for $0 < t < T$. Moreover, $\lim_{t\to 0+} u(r,t) = 0$ if $r \neq 0$, and $u(0,t) = \sum_{k=h_0}^{\infty} a_k t^k/k! + \sum_{k=h_1}^{\infty} b_k t^{k+1/2}/\Gamma(k+3/2)$.

When $\mu > 2$, we have

Theorem 5.4. Let $\omega > 2$ and define $\mu = 4 - \omega$. Let h_0 and $h_1 \geq 0$ be the best integers such that $h_0 + \mu/2 > 0$ and $h_1 + (\mu+1)/2 > 0$. A function $v(r,t)$ has an expansion of the form

$$(5.29) \quad v(r,t) = \sum_{k=h_0}^{\infty} a_k v(\omega,k,r,t) + \sum_{k=h_1}^{\infty} b_k v(\omega,k+1/2,r,t),$$

which converges for $0 < t < T$, if and only if the function $u(r,t) = r^{\omega-2}v(r,t)$ has an expansion of the form (5.26) which converges for $0 < t < T$.

Proof. Necessity. Since $u(r,t) = 2^{\omega-2}v(r,t)$ and $u(r,t)$ also has an expansion of the form (5.26), it follows that

$$(5.30) \quad v(r,t) = r^{2-\omega}u(r,t) = \sum_{k=h_0}^{\infty} a_k r^{2-\omega} u(4-\omega,k,r,t) + \sum_{k=h_1}^{\infty} b_k r^{2-\omega} u(4-\omega,k+1/2,r,t).$$

With $a = 2 - \omega$, apply Theorem 3.4, statement (B) to the terms in the series (5.30). We then obtain formula (5.29). A proof of sufficiency is obtained by reversing the steps in this argument.

The following examples illustrate the theorems developed in this section.

Example 1. In Theorem 5.1, set $\mu = 1-2m$ with $m = 0, 1, 2, \ldots$. Then $h_0 = m$. Choose $g(y) = (\sigma y^2)^m e^{\sigma y^2}$, with $\sigma > 0$. Then $g(y)$ is an entire function of growth $(2,\sigma)$. In the notation of formula (5.9)

$$a_k = \frac{(4\sigma)^k \Gamma(k-m+1/2)\Gamma(k+1)}{\Gamma(1/2)\Gamma(k-m+1)} .$$

It follows that a solution of equation (2.6) has the series expansion

$$(5.31) \quad u(r,t) = \sum_{k=m}^{\infty} \frac{(4\sigma)^k \Gamma(k-m+1/2)\Gamma(k+1)}{\Gamma(1/2)\Gamma(k-m+1)} u(\mu,k,r,t),$$

which is convergent for $0 < t < 1/4\sigma$. An integral representation for this solution is given by

$$u(r,t) = \frac{r^{m+1/2}\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(m+1/2)} \int_0^\infty e^{-y^2/4t} y^{1/2-m} K_{m+1/2}(ry/2t) g(y)dy.$$

From [1], p. 444, (10.2.17) and (10.2.18), it follows that if $z \geq 0$, $K_{m+1/2}(z) = (\pi/2z)^{1/2} e^z P_m(1/z)$, where $P_m(y)$ is a polynomial of degree m in y. Then,

$$(5.32) \qquad u(r,t) = \frac{(\sigma r)^m}{t^{1/2}\Gamma(m+1/2)} \int_0^\infty e^{-(r+y)^2/4t} y^m P_m(2t/ry) e^{\sigma y^2} dy.$$

From this series (5.31), we have

$$(5.33) \qquad u(0,t) = \sum_{k=m}^{\infty} \frac{(4\sigma)^k \Gamma(k-m+1/2)}{\Gamma(1/2)\Gamma(k-m+1)} t^k = (4\sigma t)^m (1-4\sigma t)^{-1/2}.$$

This series converges for $0 < |t| < 1/4\sigma$. If $\sigma < 0$, $g(y)$ has growth $(2, |\sigma|)$ and the integral (5.32) converges for all $t > 0$ while the series (5.33) converges only when $0 < |t| < 1/4|\sigma|$. This example shows that the integral (5.8) and the series (5.7) may converge on different intervals, each of which contains the interval $(0,T)$.

Example 2. Set $m = 0$ in example 1. Then $\mu = 1$ and $g(y) = \exp(\sigma y^2)$. Moreover $\mu(0,t) = (1-4\sigma t)^{-1/2}$ and the integral (5.32) reduces to $u(r,t) = (\pi t)^{-1/2} \int_0^\infty e^{-(r+y)^2/4t} e^{\sigma y^2} dy$.

Introducing the change of variable,

$$\omega = (1-4\sigma t)^{1/2} \left[\frac{r+y}{2\sqrt{t}} + \frac{2r\sigma\sqrt{t}}{1-4\sigma t} \right],$$ it follows that

$$u(r,t) = (1-4\sigma t)^{-1/2} \exp(\sigma r^2/(1-4\sigma t)) \operatorname{erfc}(r[4t(1-4\sigma t)]^{-1/2}).$$

6

EXPANSION THEOREMS FOR THE INFINITE TIME INTERVAL

We now develop theorems which determine what solutions of equation (2.6) can be represented by the series (5.2) for $0 < t < \infty$. This is achieved by introducing suitable restrictions on the growth of the coefficients a_k and b_k.

Theorem 6.1. Let $\mu < 2$ and let h_0 be the least integer such that $h_0 + \mu/2 > 0$. Let $\{a_k\}_{k=h_0}^{\infty}$ be a sequence with the property that

$$(6.1) \qquad \limsup_{n \to \infty} |a_n|^{1/n} = q^2.$$

A function $u(r,t)$ has an expansion of the form (5.7), which converges for $0 < t < \infty$, if and only if

$$(6.2) \qquad u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} T_\mu(r,t,y)g(y)dy.$$

Here, $T_\mu(r,t,y)$ is given by (5.10) and $g(y)$ is an even entire function of growth $(1,q)$ defined by (5.9).

Proof. Substitute the series (5.9) into (6.2). Applying Theorem 3.3 with $m = 2k$, we formally obtain the series (5.7). The term-by-term integration needed to derive the series (5.7) is valid if

$$(6.3) \qquad J = \int_0^{\infty} T_\mu(r,t,y)G(y)dy < \infty.$$

Here $G(y)$ is obtained from $g(y)$ by replacing

a_k by $|a_k|$ in (5.9). Write (6.3) in the form

$$(6.4) \qquad J = \int_0^m T_\mu(r,t,y)G(y)dy + \int_m^M T_\mu(r,t,y)G(y)dy$$

$$+ \int_M^\infty T_\mu(r,t,y)G(y)dy.$$

Denote the integrals in (6.4) by J_1, J_2, and J_3 respectively. Then, a comparison of formula (6.4) with (5.12) shows that $I_1 = J_1$ and $I_2 = J_2$, hence the integrals J_1 and J_2 are finite. The function $g(y)$ and, hence, $G(y)$ has growth $(1,q)$. From formula (5.3), it follows that for M sufficiently large $G(y) < e^{qy/\theta}$ for $y > M$, where θ is any number between 0 and 1. Also, for y sufficiently large, if $r \neq 0$ and $t > 0$ are fixed, then $y^{\mu/2}r^{1-\mu/2}K_{\mu/2-1}(ry/2t)$ is bounded by some positive constant C. Thus, for M sufficiently large,

$$(6.5) \qquad J_3 < C \int_M^\infty \exp\left(\frac{-y^2}{4t} + \frac{\sigma y}{\theta}\right)dy.$$

Clearly, the integral in (6.5) is finite for $0 < t < \infty$. It follows that the inequality (6.3) is valid.

Necessity. Assume that formula (6.1) is valid. Use formula (5.6) to examine the growth of $g(y)$, as given by (5.9). Since only even powers of y are involved, replace n by $2n$ and c_{2n} by $a_n\Gamma(n+1/2)/\Gamma(n+\mu/2)$ in formula (5.6). Simplifying, we have

$$\limsup_{n \to \infty} |a_n|^{1/2n} \leq \sigma. \tag{6.6}$$

A comparison of (6.1) and (6.6) shows that $g(y)$ and, hence, $G(y)$ have growth $(1,q)$. It follows that the integral in (6.2) converges absolutely for $0 < t < \infty$. Therefore, the interchange of summation and integration signs needed to obtain (6.2) from (5.7) is valid. This completes the proof.

For expansions in terms of the set $\{u(\mu,k+1/2,r,t\}_{k=0}^{\infty}$, we have

Theorem 6.2. Let $\mu < 2$ and let $h_1 \geq 0$ be the least integer such that $h_1 + (\mu+1)/2 > 0$. Let $\{b_n\}_{n=h_1}^{\infty}$ be a sequence with the property that

$$\limsup_{n \to \infty} |b_n|^{1/n} = q^2. \tag{6.7}$$

A function $u(r,t)$ has an expansion of the form (5.18), which converges for $0 < t < \infty$, if and only if

$$u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} T_\mu(r,t,y)h(y)dy. \tag{6.8}$$

Here, $T_\mu(r,t,y)$ is given by (5.10) and $h(y)$ is an odd entire function of growth $(1,q)$ defined by (5.20).

The proof of this theorem, being similar to that of Theorem 6.1, is omitted.

The main result of this section is obtained by combining Theorems 6.1 and 6.2.

Theorem 6.3. Let $\mu < 2$ and let h_0 and $h_1 \geq 0$ be the least integers such that $h_0 + \mu/2 > 0$ and $h_1 + (\mu+1)/2 > 0$. Let $\{a_n\}_{n=h_0}^{\infty}$ and $\{b_n\}_{n=h_1}^{\infty}$ be sequences with the property that

$$(6.9) \qquad \limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} |b_n|^{1/n} = q^2 .$$

A function $u(r,t)$ has an expansion of the form

$$(6.10) \qquad u(r,t) = \sum_{k=h_0}^{\infty} a_k u(\mu,k,r,t) + \sum_{k=h_1}^{\infty} b_k u(\mu,k+1/2,r,t),$$

which converges for $0 < t < \infty$, if and only if

$$(6.11) \qquad u(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}\,\Gamma(1-\mu/2)} \int_0^{\infty} T_\mu(r,t,y)f(y)dy .$$

Here, $T_\mu(r,t,y)$ is defined by (5.10) and $f(y)$ is an entire function of growth $(1,q)$ defined by (5.28).

Clearly, the function defined by the series (6.10) satisfies equation (2.6) for $0 < t < \infty$. Moreover, $\lim_{t \to 0+} u(r,t) = 0$ if $r \neq 0$ and

$$u(0,t) = \sum_{k=h_0}^{\infty} a_k t^k/k! + \sum_{k=h_1}^{\infty} b_k t^{k+1/2}/\Gamma(k+3/2).$$

When $\mu > 2$, we have

Theorem 6.4. Let $\omega > 2$ and define $\mu = 4 - \omega$. Let h_0 and $h_1 \geq 0$ be the least integers such that $h_0 + \mu/2 > 0$ and $h_1 + (\mu+1)/2 > 0$. Let $\{a_k\}_{k=h_0}^{\infty}$ and $\{b_k\}_{k=h_1}^{\infty}$ be sequences which satisfy condition (6.9). A function $v(r,t)$ has an expansion of the form

$$v(r,t) = \sum_{k=h_0}^{\infty} a_k v(\omega,k,r,t) + \sum_{k=h_1}^{\infty} b_k v(\omega,k+1/2,r,t), \quad \text{which}$$

converges for $0 < t < \infty$, if and only if the function $u(r,t) = r^{\omega-2}v(r,t)$ has an expansion of the form (6.10) convergent for $0 < t < \infty$.

The proof of this theorem is identical to that of Theorem 5.4.

The following examples illustrate the theorems developed in this section.

Example 1. Let $0 < \mu < 2$. Then $h_0 = 0$ in Theorem 6.1. Write

$$(6.12) \qquad u(r,t) = \sum_{k=0}^{\infty} u(\mu,k,r,t).$$

A comparison of (6.12) with (5.7) shows that $a_k = 1$, $k = 0, 1, 2, \ldots$. Hence $\limsup_{n \to \infty} |a_n|^{1/n} = 1$. From formula (5.9) it follows that

$$g(y) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(k+\mu/2)} \frac{y^{2k}}{(2k)!} = \sqrt{\pi} \quad (y/2)^{1-\mu/2} I_{\mu/2-1}(y).$$

Since $g(y)$ has growth $(1,1)$, the series (6.12) converges to a solution of (2.6) for $0 < t < \infty$. Moreover $u(0,t) = e^t$. From (6.2), an integral representation for $u(r,t)$ is given by

$$(6.13) \qquad u(r,t) = \left(\frac{r}{2}\right)^{1-\mu/2} \frac{\exp(-r^2/4t)}{t\,\Gamma(1-\mu/2)}$$

$$\int_0^{\infty} [\exp(-y^2/4t)] y K_{\mu/2-1}(ry/2t) I_{\mu/2-1}(y) dy.$$

The integral in (6.13) cannot be evaluated in closed form. However, using the identity $K_a(z) = (\pi/2) \csc (a\pi)[I_{-a}(z) + I_a(z)]$, substitute into (6.13) and use [3], p. 197, (22) to evaluate the integral containing the product $I_{\mu/2-1}(ry/2t)I_{\mu/2-1}(y)$. We obtain

$$(6.14) \quad u(r,t) = e^t\Gamma(\mu/2)(r/2)^{1-\mu/2}I_{\mu/2-1}(r)$$

$$- (r/2)^{1-\mu/2} \frac{\Gamma(\mu/2) \exp (-r^2/4t)}{2t}$$

$$\int_0^\infty e^{-y^2/4t} y I_{1-\mu/2}(ry/2t)I_{\mu/2-1}(y)dy.$$

In this form, $u(r,t)$ is expressed as the difference of positive functions.

<u>Example 2.</u> Let $\mu = 1$ in example 1. Since $u(0,t) = e^t$, an integral representation for $u(r,t)$ is given by formula (2.15) and we have

$u(r,t) = \dfrac{r}{2\sqrt{\pi}} \displaystyle\int_0^t \frac{\exp (-r^2/4y) \exp (t-y)}{y^{3/2}} dy$. From [5], p. 45, (9), it follows that

$$u(r,t) = \frac{e^t}{2} \left(e^{-r} \operatorname{erfc} \left[\frac{r}{2\sqrt{t}} - \sqrt{t}\right] + e^r \operatorname{erfc} \left[\frac{r}{2\sqrt{t}} + \sqrt{t}\right]\right).$$

7

EXTENSION OF RESULTS TO THE CASE $\mu = 2$

We now extend the results of the previous sections to solutions of equation (2.6) when $\mu = 2$. In this case, solutions have a logarithmic singularity in the neighborhood of $r = 0$. Provided that appropriate modifications are made, we can use the techniques developed in earlier sections to study these solutions.

7.1 Integral Representations.

In Theorem 2.1, conditions (2.7a) and (2.7b) each reduce to $a = -1$, $b = 0$, when $\mu = 2$. Making these choices, set $c = 1$ in formula (2.5). Then (2.5) defines a solution of equation (2.6) for $\mu = 2$. We denote this solution by

$$(7.1.1)\qquad Y(r,t,f) = 4\pi[S_2(r,t)*f(t)],$$

where

$$(7.1.2)\qquad S_2(r,t) = (4\pi t)^{-1}\exp(-r^2/4t).$$

An integral form, formula (7.1.1) becomes

$$(7.1.3)\qquad Y(r,t,f) = \int_0^t y^{-1}\exp(-r^2/4y)f(t-y)dy.$$

Introducing the change of variable $z = r^2/4y$, it follows that

$$(7.1.4) \qquad Y(r,t,f) = \int_{r^2/4t}^{\infty} e^{-z} z^{-1} f(t-r^2/4z)\,dz.$$

<u>Theorem 7.1.1.</u> Let $f(t)$ be absolutely integrable on $(0,T)$ with $T \leq \infty$. Then,

$$(7.1.5) \qquad \lim_{t\to 0+} Y(r,t,f) = 0, \quad \text{if} \quad r > 0.$$

<u>Proof.</u> Introducing absolute values on both sides of (7.1.4), it follows that

$$(7.1.6) \qquad |Y(r,t,f)| \leq \int_{r^2/4t}^{\infty} (e^{-z} z) z^{-2} |f(t-r^2/4z)|\,dz.$$

Since $\lim_{z\to\infty} e^{-z} z = 0$, the function $e^{-z} z$ is bounded by some positive constant M for sufficiently large values of z. Hence

$$(7.1.7) \qquad \lim_{t\to 0+} |Y(r,t,f)| \leq M \lim_{t\to 0+} \int_{r^2/4t}^{\infty} z^{-2} |f(t-r^2/4z)|\,dz.$$

With the change of variable $y = t - r^2/4z$, the inequality (7.1.7) becomes

$$(7.1.8) \qquad \lim_{t\to 0+} |Y(r,t,f)| \leq M \lim_{t\to 0+} \int_0^t |f(y)|\,dy = 0.$$

We now examine the behavior of $Y(r,t,f)$ as $r \to 0$.

<u>Lemma 7.1.1.</u> If $\overline{Y}(r,t,f) = \int_{r^2/4t}^{\infty} e^{-y} y^{-1} f(t)\,dy$, then $\lim_{r\to 0} \dfrac{\overline{Y}(r,t,f)}{-\ln r^2} = f(t)$, if $t > 0$.

Proof. From the definition of $\overline{Y}(r,t,f)$ and [1], p. 229, (5.1.11), it follows that

$$(7.1.9) \quad \overline{Y}(r,t,f) = f(t)[-\gamma-\ln(r^2/4t)-\sum_{n=1}^{\infty}(-1)^n(r^2/4t)^n/n(n!)],$$

where γ is Euler's constant.

Since $\left|\sum_{n=1}^{\infty}(-1)^n(r^2/4t)^n/n(n!)\right| < \exp(r^2/4t)$, we find that

$$\lim_{r\to 0}\frac{\overline{Y}(r,t,f)}{-\ln r^2} = f(t), \quad \text{if} \quad t > 0.$$

Theorem 7.1.2. Let the function $f(t)$ be Lipschitz continuous with exponent a, $0 < a < 1$, on the interval $[0,T]$ with $T < \infty$. Then $\lim\limits_{r\to 0}\frac{Y(r,t,f)}{-\ln r^2} = f(t)$, if $0 < t \leq T$.

Proof. Let $\overline{Y}(r,t,f)$ be the function defined in Lemma 7.1.1. Let $Y(r,t,f)$ be given by (7.1.3). Define the function $J(r,t)$ by

$$(7.1.10) \quad J(r,t) = \overline{Y}(r,t,f) - Y(r,t,f).$$

The argument used in Theorem 2.4 to prove that $\lim\limits_{r\to 0} I(r,t) = 0$ can now be used to prove that $\lim\limits_{r\to 0} J(r,t) = 0$. Then, from definition (7.1.10), it follows that

$$(7.1.11) \quad \lim_{r\to 0}\frac{\overline{Y}(r,t,f)}{-\ln r^2} = \lim_{r\to 0}\frac{Y(r,t,f)}{-\ln r^2}.$$

By Lemma 7.1.1, the limit of the right member is $f(t)$. This completes the proof.

By repeating the arguments of Theorems 2.5 and 2.6, it can be shown that the conclusion of Theorem 7.1.2 is valid when $f(t)$ is continuous from the left and absolutely integrable on $[0,T]$ with $T < \infty$. Corresponding to Corollary 2.1, we have

<u>Corollary 7.1.1.</u> Let $f(t)$ be continuous from the left and absolutely integrable on $[0,\infty)$. Then $\lim_{r\to 0} \dfrac{Y(r,t,f)}{-\ln r^2} = f(t)$, if $0 < t < \infty$.

<u>7.2 Expansion Theorem Preliminaries.</u>

A comparison of definition (7.1.1) with (2.13) shows that

$$(7.2.1) \qquad Y(r,t,f) = [\Gamma(1-\mu/2)U_\mu(r,t,f)]\Big|_{\mu=2}.$$

Define the function $y(h,r,t)$ by

$$(7.2.2) \qquad y(h,r,t) = Y(r,t,t^h/\Gamma(h+1)), \quad \text{for} \quad h > -1.$$

It follows from (7.2.1) that

$$\begin{aligned}(7.2.3) \qquad y(h,r,t) &= [\Gamma(1-\mu/2)U_\mu(r,t,t^h/\Gamma(h+1))]\Big|_{\mu=2} \\ &= [\Gamma(1-\mu/2)u(\mu,h,r,t)]\Big|_{\mu=2} \\ &\ t^h e^{-r^2/4t}\ \psi(h+1,1,r^2/4t).\end{aligned}$$

In obtaining the [illegible] member of (7.2.3), we have used Theorem 5.1. The final member of (7.2.3) is defined for all real h. It provides an extension of the definition of $y(h,r,t)$ given by (7.2.2).

Theorem 7.2.1. If, for h real,

$$y(h,r,t) = t^h e^{-r^2/4t}\,\psi(h+1,1,r^2/4t), \tag{7.2.4}$$

then

(A) $y(h,r,t)$ satisfies equation (2.6), when $\mu = 2$,

(B) $\lim_{t\to 0+} y(h,r,t) = 0$, if $r \neq 0$,

and

(C) $\lim_{r\to 0} \dfrac{y(h,r,t)}{-\ln r^2} = \dfrac{t^h}{\Gamma(h+1)}$, if $t > 0$.

Proof of (A). This result follows from the remark following the proof of Theorem 3.2, part (A).

Proof of (B). From (7.2.4) and (A-14), it follows that

$$\lim_{t\to 0+} y(h,r,t) = \lim_{t\to 0+} t^h e^{-r^2/4t}\,\psi(h+1,1,r^2/4t) = 0, \quad \text{if} \quad r \neq 0.$$

Proof of (C). From (7.2.4), we have

$$\lim_{r\to 0} \frac{y(h,r,t)}{-\ln r^2} = t^h \lim_{r\to 0} \frac{e^{-r^2/4t}\,\psi(h+1,1,r^2/4t)}{-\ln r^2}.$$

For $t > 0$, use Table 2, (d), with $a = h + 1$, to evaluate the limit on the right. We then obtain result (C).

In order to determine which solutions of equation (2.6) can be represented in a series of the form $u(r,t) = \sum_{k=0}^{\infty} a_k y(k/2,r,t)$, we need the following integral representation for $y(m/2,r,t)$.

Theorem 7.2.2. If $m = 0, 1, 2, \ldots,$ then

$$(7.2.5)\qquad y(m/2,r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}} \int_0^\infty e^{-z^2/4t} zK_0(rz/2t) \left[\frac{\Gamma(m/2+1/2)}{\Gamma(m/2+\mu/2)} \; \frac{z^m}{m!}\right] dz.$$

Proof. Combining definition (7.2.2) and the result (7.2.1), it follows that $y(m/2,r,t) = [\Gamma(1-\mu/2)u(\mu,m/2,r,t)]\Big|_{\mu=2}$. Formula (7.2.5) now follows from formula (3.15).

7.3 Expansion Theorems.

When $\mu = 2$, the following result corresponds to Theorem 5.1.

Theorem 7.3.1. A function $y(r,t)$ has an expansion of the form

$$(7.3.1)\qquad y(r,t) = \sum_{k=0}^{\infty} a_k y(k,r,t),$$

which converges for $0 < t < T$, if and only if

$$(7.3.2)\qquad y(r,t) = \frac{\exp(-r^2/4t)}{t\sqrt{\pi}} \int_0^\infty e^{-z^2/4t} zK_0(rz/2t)g(z)dz.$$

Here, $g(z)$ is an even entire function of growth $(2,1/4T)$ defined by

$$(7.3.3)\qquad g(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1/2)}{\Gamma(k+1)} \; \frac{z^{2k}}{(2k)!}\,.$$

Proof. As in Theorem 5.1, the proof depends upon the validity of the term-by-term integration of the series (7.3.3) after multiplication by $e^{-z^2/4t}zK_0(rz/2t)$. Since

$\lim_{z \to \infty} zK_0(rz/2t) = \lim_{z \to 0} zK_0(rz/2t) = 0$, if $r \neq 0$, this term-by-term integration is valid for $0 < t < T$ if $g(z)$ has growth $(2, 1/4T)$. In the proof of necessity, this is part of the hypothesis; in the proof of sufficiency, this fact is proved by applying Stirling's formula to the coefficients in the series (7.3.1).

The companion result, for expansion valid for $0 < t < \infty$, is given by

<u>Theorem 7.3.2.</u> Let $\{a_k\}_{k=0}^{\infty}$ be a sequence with the property that $\limsup_{n \to \infty} |a_n|^{1/n} = q^2$. A function $y(r,t)$ has an expansion of the form (7.3.1), convergent for $0 < t < \infty$, if and only if (7.3.2) holds where $g(z)$ is an even entire function of growth $(1,q)$ defined by (7.3.3).

In Theorem 7.3.1 (Theorem 7.3.2), it is clear the function defined by the series (7.3.1) satisfies equation (2.6) when $\mu = 2$ for $0 < t < T$, $(0 < t < \infty)$. Moreover, $\lim_{t \to 0+} y(r,t) = 0$ and $\lim_{r \to 0} \frac{y(r,t)}{-\ln r^2} = \sum_{k=0}^{\infty} a_k t^k/k!$ in each theorem.

Analogous results are valid for expansions in terms of the set $\{y(k+1/2,r,t)\}_{k=0}^{\infty}$.

<u>7.4 Asymptotic Behavior.</u>

When $\mu = 2$, the method used in Theorem 4.3 to examine the asymptotic behavior of solutions of (2.6) does not apply. The difficulty arises in the following way. From definition (7.1.1) and [7], p. 146, (29), it follows that

$$(7.4.1) \qquad L[Y(r,t,f)] = 2K_0(r\sqrt{s}\,)\bar{f}(s),$$

where $\bar{f}(s)$ is the Laplace transform of $f(t)$. From (7.4.1) we find $L[Y(r,t,f)] \sim 2\bar{f}(s)[-\ln(r\sqrt{s}\,)]$, as $s \to 0+$. Since $L[Y]$ has logarithmic behavior as $s \to 0+$, Theorem 4.2 cannot be used to describe the asymptotic behavior of $\int_0^t Y(r,\tau,f)d\tau$, as $t \to \infty$. However, analogous to Theorem 4.4, we have

<u>Theorem 7.4.1.</u> Let $f(t)$ be absolutely integrable on $(0,T)$ for each $T \in (0,\infty)$. If $h \geq 0$ and $f(t) \sim t^h/\Gamma(h+1)$ as $t \to \infty$, then

$$(7.4.2) \qquad Y(r,t,f) \sim y(h,r,t), \quad r > 0, \quad \text{as} \quad t \to \infty.$$

<u>Proof.</u> By hypothesis $f(t) = t^h/\Gamma(h+1) + \epsilon(t)t^h$ where $\epsilon(t) \to 0$ as $t \to \infty$. Let $\epsilon_0 > 0$ be given. Choose $M > 0$ such that $|\epsilon(t)| < \epsilon_0/\Gamma(h+1)$ if $t \geq M$. Set $K(r,t) = t^{-1}e^{-r^2/4t}$. For $t > M$ we have

$$\begin{aligned}(7.4.2) \qquad Y(r,t,f) &= \int_0^M f(\tau)K(r,t-\tau)d\tau \\ &\quad + \int_M^t \left[\frac{\tau^h}{\Gamma(h+1)} + \epsilon(\tau)\tau^h\right]K(r,t-\tau)d\tau \\ &= \int_0^M \left[f(\tau) - \frac{\tau^h}{\Gamma(h+1)}\right]K(r,t-\tau)d\tau \\ &\quad + \int_0^t \frac{\tau^h}{\Gamma(h+1)}K(r,t-\tau)d\tau + \int_M^t \epsilon(\tau)\tau^h K(r,t-\tau)d\tau\end{aligned}$$

$$= y(h,r,t) + \int_0^M \left[f(\tau) - \frac{\tau^h}{\Gamma(h+1)} \right] K(r,t-\tau)d\tau$$

$$+ \int_M^t \varepsilon(\tau)\tau^h K(r,t-\tau)d\tau .$$

Consider $K(r,t)$ as a function of t for fixed r. It has a maximum when $t = r^2/4$, and this maximum is $4/er^2$. For $r > 0$, it follows that

$$\left|\int_0^M \left[f(\tau) - \frac{\tau^h}{\Gamma(h+1)}\right] K(r,t-\tau)d\tau\right| < C_1 \int_0^M \left[|f(\tau)| + \frac{\tau^h}{\Gamma(h+1)}\right] d\tau < C_2,$$

where C_1 and C_2 are positive constants depending on r. Moreover, $|\varepsilon(t)| < \varepsilon_0/\Gamma(h+1)$ when $t \geq M$ so that

$$\left|\int_M^t \varepsilon(\tau)\tau^h K(r,t-\tau)d\tau\right| < \varepsilon_0 \int_0^t \frac{\tau^h}{\Gamma(h+1)} K(r,t-\tau)d\tau = \varepsilon_0 y(h,r,t).$$

Substituting these results into the last line of (7.4.2), we find $|Y(r,t,f(t)) - y(h,r,t)| < C_2 + \varepsilon_0 y(h,r,t)$. Since $y(h,r,t) > 0$, it follows that

$$(7.4.3) \qquad \left| \frac{Y(r,t,f(t)) - y(h,r,t)}{y(h,r,t)} \right| < \frac{C_2}{y(h,r,t)} + \varepsilon_0 .$$

From formula (7.2.4) and Table 2, (d), we have

$$\lim_{t\to\infty} y(h,r,t) = \frac{1}{\Gamma(h+1)} \lim_{t\to\infty} t^h(-\ln\,(r^2/4t)), \quad \text{if} \quad r > 0.$$

Hence, if $h \geq 0$, it follows from (7.4.3) that

$$(7.4.4) \qquad \lim_{t\to\infty} \left| \frac{Y(r,t,f(t)) - y(h,r,t)}{y(h,r,t)} \right| < \varepsilon_0 .$$

Since ε_0 is arbitrary, the limit in (7.4.4) is zero. This completes the proof.

8

A BASIC FORM FOR SPECIAL SOLUTIONS

In a remark following the proof of Theorem 3.2, part (A), the point was made that a solution of equation (2.6) is given by

(8.1) $$G(r,t) = ct^h e^{-z} F(h+\mu/2, \mu/2, z).$$

In this formula, $z = r^2/4t$, c is any parameter independent of r and t, and $F(a,b,z)$ is any confluent hypergeometric function with parameters a and b. In sections 5 and 6, it was shown that certain solutions of the radial heat equation had valid expansions in terms of the set of radial time functions $\{u(\mu,m/2,r,t)\}_{m=0}^{\infty}$. Each element in this set has the form (8.1). There are other expansion theories for solutions of (2.6). Each theory uses one or more sets of special solutions. In this section, we introduce several of these sets, and indicate the major results of the theory associated with each set. We then show that the elements of these several sets all have the form (8.1).

When $\mu = 1$, equation (2.6) reduces to the one-dimensional heat equation. P.C. Rosenbloom and D.V. Widder [12] have made a detailed study of the validity of expansions of solutions of this equation in terms of two sets of special solutions. The first is the set of heat polynomials $\{v_n(r,t)\}_{n=0}^{\infty}$ defined by

(8.2) $$v_n(r,t) = (-t)^{n/2}H_n(r(-4t)^{-1/2}), \quad -\infty < r < \infty,$$
$$-\infty < t < \infty,$$

where $H_k(z)$ is the Hermite polynomial of degree k. The second is the set of associated functions $\{w_n(r,t)\}_{n=0}^{\infty}$ where

(8.3) $$w_n(r,t) = (4\pi t)^{-1/2}\exp(-r^2/4t)v_n(r/t,-1/t),$$
$$-\infty < r < \infty, \quad 0 < t < \infty.$$

The authors develop the following basic theorems regarding expansions for solutions of the heat equation:

(I) Expansions in terms of heat polynomials are valid in a time strip, $|t| < \sigma$, in which the solution $u(r,t)$ satisfies a Huygen's principle. That is,

$$u(r,t) = \int_{-\infty}^{\infty} [4\pi(t-t')]^{-1/2} \exp[-(r-y)^2/4(t-t')]u(y,t')dy, \quad \text{for all}$$

t and t' such that $-\sigma < t' < t < \sigma$.

(II) Expansions in terms of associated functions are valid in a half-plane $t > \sigma \geq 0$ in which the solution has certain entireness properties.

Several rules for determining the coefficients in these expansions are given. The L^2 theory of such expansions is also examined.

L.R. Bragg [2] has developed results analogous to (I) and (II) for expansions of solutions of (2.6) when $\mu > 1$. Again, the theory is developed in terms of two sets of special solutions: (a) the set $\{R_k^{\mu}(r,t)\}_{k=0}^{\infty}$ of radial heat polynomials and (b) the set $\{\tilde{R}_k^{\mu}(r,t)\}_{k=0}^{\infty}$ of associated functions. The elements of the first set are defined by

$$(8.4) \qquad R_k^{\mu}(r,t) = k!(4t)^k L_k^{(\mu/2-1)}(-r^2/4t),$$

where $L_n^{(a)}(z)$ is the generalized Laguerre polynomial of order n with exponent a. The associated function $\tilde{R}_k^{\mu}(r,t)$ is defined to be the Appell transform of $R_k^{\mu}(r,t)$. That is

$$(8.5) \qquad \tilde{R}_k^{\mu}(r,t) = A[R_k^{\mu}(r,t)] = S_{\mu}(r,t)R_k^{\mu}(r/t,-1/t).$$

Underlying the expansion theorems for the radial heat polynomials is an integral representation for solutions of the initial value problem.

$$(8.6) \qquad \begin{cases} [D_r^2 + (\frac{\mu-1}{r})D_r]u(r,t) = D_t u(r,t), & u > 1 \\ u(r,0) = \varphi(r), & r > 0. \end{cases}$$

In subsequent papers, L.R. Bragg has related the particular form of solution of (8.6) used in [2] to Laplace transforms and their inverses [3]. These results have been used [4] to examine the character of solutions of (8.6) when the function $\varphi(r)$ has a pole at $r = 0$, but is otherwise entire.

The expansion theorems given in [2] were discovered independently by D.T. Haimo [8] who also developed analogous results for the L^2 theory of expansions when $\mu > 1$ [9]. In [8] and [9] the special sets of solution functions are denoted $\{P_{n,(\mu-1)/2}(r,t)\}_{n=0}^{\infty}$ and $\{W_{n,(\mu-1)/2}(r,t)\}_{n=0}^{\infty}$. These special solutions are related to Bragg's radial heat polynomials and associated functions as follows:

(8.7) $$P_{n,(\mu-1)/2}(r,t) = R_n^{\mu}(r,t)$$

and

(8.8) $$W_{n,(\mu-1)/2}(r,t) = \tilde{R}_n^{\mu}(r,t).$$

We now review some expansion theories for solutions of the initial value problem

(8.9) $$\begin{cases} D_r^2 u(r,t) = D_t u(r,t) \\ u(r,0) = 0,\ r > 0;\ \ u(0,t) = f(t),\ \ t > 0. \end{cases}$$

H. Poritsky and R.A. Powell [11] studied the set of solution functions $\{T_n(r,t)\}_{n=0}^{\infty}$ where

(8.10) $$T_n(r,t) = 2\int_0^t S_1(r,t-y)y^n/n!\ dy.$$

Here $T_n(r,t_0)$ represents the temperature in the infinite rod $0 \leq r < \infty$ in which $T_n(r,0) = 0$ if $r > 0$ and heat is liberated at the rate $t^n/n!$ per unit time from $t = 0$ to $t = t_0$ at the point $r = 0$. The authors show that $T_n(r,t) = t^{n+1/2}/\Gamma(n+3/2)$. Moreover,

(8.11) $$T_n(r,t) = \int_0^t r(t-y)^{-1}S_1(r,t-y)T_n(0,y)dy.$$

D.V. Widder [14] defined a related set of solution functions $\{U_n(r,t)\}_{n=0}^{\infty}$ by setting

(8.12) $$U_n(r,t) = \int_0^t r(t-y)^{-1}S_1(r,t-y)y^n/n!\ dy.$$

He then examined what solutions of the problem (8.9) have expansions of the form $u(r,t) = \sum_{n=0}^{\infty} [a_n U_n(r,t) + b_n T_n(r,t)]$. The main results of [14] are obtained by setting $\mu = 1$ in Theorem 5.3 and Theorem 6.3.

Let us now examine how formula (8.1) relates to the special solutions which have been mentioned in this section.

Theorem 8.1. For appropriate choices of c, h, μ, and F (see Table 1), the special solutions mentioned in this section have the form (8.1).

TABLE 1

$G(r,t) = ct^h e^{-z} F(h+\mu/2,\mu/2,z)$, where $z = r^2/4t$.

	Author	$G(r,t)$	μ	h	c	$F(h+\mu/2,\mu/2,z)$
(a)	P.C.Rosenbloom and D.V.Widder	$v_n(r,t)$	1	$n/2$	$(-4)^{n/2}$	$e^z\psi(-h,\mu/2,-z)$
(b)		$w_n(r,t)$	1	$-(\frac{n+1}{2})$	$(4\pi)^{\frac{1}{2}}4^{n/2}$	$\psi(h+\mu/2,\mu/2,z)$
(c)	L.R. Bragg	$R_n^\mu(r,t)$	$\mu > 1$	n	$(-4)^n$	$e^z\psi(-h,\mu/2,-z)$
(d)		$\tilde{R}_n^\mu(r,t)$	$\mu > 1$	$-(n+\frac{\mu}{2})$	$(4\pi)^{\mu/2}4^n$	$\psi(h+\mu/2,\mu/2,z)$
(e)	H.Poritsky & R.A. Powell; D.V. Widder	$T_n(r,t)$	1	$n+1/2$	$\pi^{-1/2}$	$\psi(h+\mu/2,\mu/2,z)$
(f)	D.V. Widder	$U_n(r,t)$	1	n	$\pi^{-1/2}$	$\psi(h+\mu/2,\mu/2,z)$

NOTE: $\psi(h+\mu/2,\mu/2,z)$ and $e^z\psi(-h,\mu/2,-z)$ are confluent hypergeometric functions with parameters $h+\mu/2$ and $\mu/2$. They are linearly independent functions for all choices of h and μ (see the Appendix, formula (A-8)).

<u>Proof of Table 1, (a).</u> We must show that

$$v_n(r,t) = (-1)^{n/2}4^{n/2}t^{n/2}\psi(-n/2,1/2,-z). \tag{8.13}$$

Case I. Let $n = 2m$, with $m = 0, 1, 2, \ldots$. Formula (8.13) becomes

$$(8.14) \qquad v_{2m}(r,t) = (-1)^m 4^m t^m \psi(-m,1/2,-z).$$

Using the identity (A-12) and the fact that

$L_n^{(-1/2)}(z) = \frac{(-1)^n}{n!4^n} H_{2n}(\sqrt{z})$, it follows that

$$(8.15) \qquad v_{2m} = (-t)^m H_{2m}((-z)^{1/2}).$$

Since $z = r^2/4t$, formulas (8.15) and (8.2) are now identical.

Case II. Let $n = 2m + 1$, with $m = 0, 1, 2, \ldots$. Formula (8.13) becomes

$$(8.16) \qquad v_{2m+1}(r,t) = (-1)^{m+1/2} 4^{m+1/2} t^{m+1/2} \psi(-m-1/2,1/2,-z).$$

Applying (A-9) with $a = -m - 1/2$, and $b = 1/2$, it follows that

$$(8.17) \qquad v_{2m+1}(r,t) = (-1)^{m+1/2} 4^{m+1/2} t^{m+1/2} (-z)^{1/2} \psi(-m,3/2,-z).$$

Using identity (A-12) and the fact that

$L_n^{(1/2)}(z) = \frac{(-1)^n}{n!2^{2n+1}z^{1/2}} H_{2n+1}(z^{1/2})$, we obtain

$$(8.18) \qquad v_{2m+1}(r,t) = (-t)^{m+1/2} H_{2m+1}((-z)^{1/2}).$$

Now, formulas (8.2) and (8.18) are identical. This completes the proof of Table 1, (a).

Proof of Table 1, (b). Substitute (8.13) into (8.3), the definition of $w_n(r,t)$. It follows that

$$(8.19) \qquad w_n(r,t) = (4\pi t)^{-1/2}\exp\,(-r^2/4t)v_n(r/t,-1/t)$$

$$= (4\pi)^{-1/2}4^n t^{-(\frac{n+1}{2})} e^{-z}\psi(-n/2,1/2,z).$$

This is exactly the result given in Table 1,(b).

Proof of Table 1, (c). We must show that

$$(8.20) \qquad R_n^{\mu}(r,t) = (-4)^n t^n \psi(-n,\mu/2,-z).$$

It follows from the identity (A-12) that

$$(8.21) \qquad R_n^{\mu}(r,t) = (4t)^n n!\; L_n^{(\mu/2-1)}(-z)$$

This is the definition for $R_n^{\mu}(r,t)$ given by (8.4).

Proof of Table 1, (d). This result follows from definition (8.5) and the representation for $R_n^{\mu}(r,t)$ given in Table 1, (c).

Proof of Table 1, (e) and (f). The functions $T_n(r,t)$ and $U_n(r,t)$ are defined in (8.11) and (8.12) respectively.
Comparing these definitions with formula (2.13) and using formula (3.1), we find

$$(8.22) \qquad T_n(r,t) = U_1(r,t,t^{n+1/2}/\Gamma(n+3/2)) = u(1,n+1/2,r,t)$$

and

$$(8.23) \qquad U_n(r,t) = U_1(r,t,t^n/n!) = u(1,n,r,t).$$

It follows from Theorem 3.1, that

$$(8.24) \qquad T_n(r,t) = \pi^{-1/2}t^{n+1/2}e^{-z}\psi(n+1,1/2,z),$$

and

$$(8.25) \qquad U_n(r,t) = \pi^{-1/2} t^n e^{-z} \psi(n+1/2, 1/2, z).$$

These are exactly the entries in Table 1, (e) and (f).

If the set $\{u(\mu,m/2,r,t)\}$ appeared in Table 1, the entry would have the following form

	$G(r,t)$	μ	h	c	$F(h+\mu/2,\mu/2,z)$
(8.26)	$u(\mu,m/2,r,t)$	$\mu < 2$	$m/2$	$[\Gamma(1-\mu/2)]^{-1}$	$\psi(h+\mu/2,\mu/2,z)$

Comparing (8.26) with Table 1, (b) and (d), one is led to suspect that $u(\mu,m/2,r,t)$ is also an "associated" function. Here, the word "associated" is used in the sense that there exists a function $W(\mu,m/2,r,t)$ such that the Appell transform of W (denoted $A[w]$) is equal to $u(\mu,m/2,r,t)$. This is, in fact, the case. In the notation of Table 1, define the set $\{W(\mu,m/2,r,t)\}_{m=0}^{\infty}$ as follows

	$G(r,t)$	μ	h	c	$F(h+\frac{\mu}{2},\frac{\mu}{2},z)$
(8.27)	$W(\mu,\frac{m}{2},r,t)$	$\mu<2$	$-(\frac{m}{2}+\frac{\mu}{2})$	$(4\pi)^{\frac{\mu}{2}}(-1)^h[\Gamma(1-\frac{\mu}{2})]^{-1}$	$e^z\psi(-h,\frac{\mu}{2},-z)$

Then,

$$(8.28) \qquad W(\mu,m/2,r,t) = \frac{(4\pi)^{\mu/2}(-t)^{-(\frac{m+\mu}{2})}}{\Gamma(1-\mu/2)} \psi(\tfrac{m+\mu}{2},\mu/2,-z),$$

and

$$(8.29) \qquad A[w] = S_\mu(r,t)W(\mu,m/2,r/t,-1/t) = u(\mu,m/2,r,t,).$$

There is an important difference between the pair of functions

$W(\mu,m/2,r,t)$, $u(\mu,m/2,r,t)$ and the pair $R_n^{\mu}(r,t)$, $\tilde{R}_n^{\mu}(r,t)$. The latter pair are both real valued functions while the function $W(\mu,m/2,r,t)$ is always complex valued if $t > 0$. When $\mu \neq 0, -2, -4, \ldots$, this fact follows from formula (A-7). When $\mu = 0, -2, -4, \ldots$, formulas [1], p. 504, (13.1.6) and (13.1.7) show that $W(\mu,m/2,r,t)$ is still complex valued.

SUMMARY

D.V. Widder [12] has developed necessary and sufficient conditions for the validity of expansions of solutions of the one-dimensional heat equation.

In this thesis, we have developed comparable expansion theorems for the functions $U_\mu(r,t,f)$ and $V_\mu(r,t,f)$, which are solutions of the radial heat equation $[D_r^2 + \frac{\mu - 1}{r} D_r]u(r,t) = D_t u(r,t)$, for $\mu < 2$ and $\mu > 2$ respectively. Necessary and sufficient conditions were developed under which the function $U_\mu(r,t,f)$ could be represented in terms of the set of radial time functions $\{u(\mu,h,r,t)\}$. The asymptotic behavior of $U_\mu(r,t,f)$ and $t^{-1}\int_0^t U_\mu(r,y,f)dy$ was examined and related to the behavior of $f(t)$, its Laplace transform, and to certain radial time functions. Using the identity $U_{2-a}(r,t,f) = r^{-a}V_{2+a}(r,t,f)$, $a > 0$, these results were extended to the function $V_\mu(r,t,f)$.

When $\mu = 2$, solutions of the radial heat equation with logarithmic singularities in a neighborhood of $r = 0$ were obtained by modifying the definition of $U_\mu(r,t,f)$. The methods used to examine series representations and asymptotic behavior for $U_\mu(r,t,f)$ when $\mu < 2$ were also modified to obtain similar results when $\mu = 2$.

Finally, it was proved that the elements of several sets of special solutions of the radial heat equation, including the radial time functions, have a common form.

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APPENDIX

CONFLUENT HYPERGEOMETRIC FUNCTIONS

We collect here, for reference, the results about confluent hypergeometric functions which have been used earlier. These results are taken from [1], Chapter 13, and [6], Chapter 6.

The confluent hypergeometric equation is given by

$$z \frac{d^2y}{dz^2} + (b-z)\frac{dy}{dz} - ay = 0. \tag{A-1}$$

This equation has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. Any solution of (A-1) is defined to be a confluent hypergeometric function with parameters a and b. For the purposes of this paper, the parameters a and b are restricted to real values. The literature on confluent hypergeometric functions is developed in terms of two solutions of (A-1) which are denoted by

$$y_1 = \Phi(a,b,z) \tag{A-2}$$

and

$$y_2 = \psi(a,b,z). \tag{A-3}$$

Another solution of equation (A-1) is given by

$$y_3 = e^z\psi(b-a,b,-z). \tag{A-4}$$

The solution functions y_1 and y_2 can be represented by generalized hypergeometric series as follows:

(A-5) $$\Phi(a,b,z) = {}_1F_1(a;b;z), \quad z > 0,$$

and

(A-6) $$\psi(a,b,z) = z^{-a}{}_2F_0(a,1+a-b;-1/z), \quad z > 0.$$

The function $\psi(a,b,z)$ is a multi-valued function of z. We usually consider its principal branch in the plane cut along the negative real axis. However, define $f(-z+i0)$ as the limit of $f(-z+i\eta)$ as $\eta \to 0$ through positive values, and define $f(-z-i0)$ similarly. Then

(A-7) $$\psi(a,b,-z\pm i0) = e^{-z}\left[\frac{\Gamma(1-b)}{\Gamma(1-b+a)}\,\Phi(b-a,b,z) - \frac{\Gamma(b-1)}{\Gamma(a)}\,e^{\mp i\pi b}\,z^{\,1-b}\,\Phi(1-a,2-b,z)\right], \quad b\neq 0,\pm 1,\pm 2,\ldots,$$

where $z > 0$ and either upper or lower signs are to be taken throughout.

The solutions (A-3) and (A-4) are linearly independent for all values of the parameters a and b, since the wronskian of the pair is given by

(A-8) $$W(y_2,y_3) = e^{-\pi i(b-a)}z^{-b}e^{z}.$$

The Kummer transformation for $\psi(a,b,z)$ is given by the identity

(A-9) $$\psi(a,b,z) = z^{1-b}\psi(1+a-b,2-b,z).$$

There are numerous integral representations for $\psi(a,b,z)$.

The two that we require are given by

$$\text{(A-10)} \qquad \psi(a,b,z) = \frac{2z^{(1-b)/2}}{\Gamma(a)\Gamma(a-b+1)}\int_0^\infty e^{-t}t^{a-b/2-1/2}K_{b-1}[2(zt)^{1/2}]dt,$$

for $a > 0$, $a - b > -1$, and

$$\text{(A-11)} \qquad \psi(a,b,z) = \frac{e^z}{\Gamma(a)}\int_1^\infty e^{-zt}(t-1)^{a-1}t^{b-a-1}dt, \quad \text{for} \quad a > 0.$$

Let $L_a^{(b)}(z)$ denote the generalized Laguerre polynomial of order a with exponent b. Then

$$\text{(A-12)} \qquad \psi(-n,a+1,z) = (-1)^n n! L_n^{(a)}(z), \quad a > -1 \text{ and } n=0,1,2,\ldots .$$

Moreover, combining formulas (A-9) and (A-12), it follows that

$$\text{(A-13)} \qquad \psi(a,a+n+1,z) = z^{-a-n}(-1)^n n! L_n^{(-a-n)}(z), -(a+n)>-1 \text{ and } n = 0, 1, 2, \ldots .$$

From (A-6), we find that

$$\text{(A-14)} \qquad \psi(a,b,z) = z^{-a}[1+O|z|^{-1}], \quad \text{as} \quad z \to \infty.$$

For small z, the behavior of $\psi(a,b,z)$ is indicated in Table 2.

TABLE 2

$\psi(a,b,z) = F(a,b,z) + O(g(z))$, for small z.

	b	F(a,b,z)	g(z)
(a)	$b > 2$	$\frac{z^{1-b}\Gamma(b-1)}{\Gamma(a)}$	$\lvert z\rvert^{b-2}$
(b)	$b = 2$		$\lvert \ln z\rvert$
(c)	$1 < b < 2$		1
(d)	$b = 1$	$-\ln z/\Gamma(a)$	$\lvert z \ln z\rvert$
(e)	$0 < b < 1$	$\frac{\Gamma(1-b)}{\Gamma(a-b+1)}$	$\lvert z\rvert^{1-b}$
(f)	$b = 0$		$\lvert z \ln z\rvert$
(g)	$b < 0$		$\lvert z\rvert$