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ESTIMATION USING SAMPLED-DATA CONTAINING SEQUENTIALLY CORRELATED NOISE

by

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ESTIMATION USING SAMPLED-DATA CONTAINING

SEQUENTIALLY CORRELATED NOISE

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ABSTRACT

This paper presents improved filtering, prediction, and smoothing procedures for multi-stage linear dynamic systems when the measured quantities are linear combinations of the state variables with additive sequentially correlated noise.* The "augmented state" procedure suggested by Kalman** may lead to ill-conditioned computations in constructing the data processing filter. The design procedure described here eliminates these ill-conditioned computations and reduces the dimension of the filter required. The results include explicit relations for prediction, filtering, and smoothing procedures and the associated covariance matrices.

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* Other names for sequentially correlated noise are "colored noise," "correlated noise" and "noise with serial correlation."

** Kalman, R. E., "New Methods in Wiener Filtering Theory," Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability, John Wiley and Sons, J. L. Bogdanoff and F. Kozin, Ed., pp. 270-388, 1963.

I. Introduction

The problem considered is that of estimating the state variables of a multi-stage linear dynamic system based on measurements of linear combinations of the state variables containing additive sequentially correlated noise. A design procedure for the data processing estimation filters is developed which eliminates the ill-conditioned computations of the augmented state approach, and which is of a lower dimension than the augmented state filters.

The present design procedure was suggested by the work of Bryson and Johansen [Ref. 1] on the related problem for continuous linear dynamic systems. Considering the measurement vector as a set of constraints among the augmented state variables, a measurement differencing scheme is used to reduce the dimension of the estimation problem. The estimation theory of Kalman [Ref. 2] is then applied to this reduced problem.

II. The Problem

For simplicity of presentation, a constant coefficient system will be studied. Results for more general systems with time-varying coefficients are presented at the end of the paper. With this restriction, a fairly general system of the type we are considering is described by:

State:	$x_{i+1} = \Phi x_i + w_i$	$w_i : (0,Q)$	$x : (n \times 1)$
Measurement:	$z_i = Hx_i + \varepsilon_i$	$u_i : (0,\overline{Q})$	$z : (m \times 1)$
			$\varepsilon : (m \times 1)$ (1)
Measurement Noise:	$\varepsilon_{i+1} = \Psi\varepsilon_i + u_i$	w_i and u_i independent with the assumption that $HQH^T + \overline{Q} \triangleq R$ is nonsingular	

Here w_i and u_i are gaussian purely random vector sequences ("white noise") with zero means and covariances Q and $\overline{Q}$, respectively. The more general case where the dimension of ε_i is greater than the dimension of z_i (i.e., $z_i = Hx_i + G\varepsilon_i$) where there is cross-coupling between x_i and ε_i, and where some measurements contain purely random noise, is treated in Ref. 4.

The problem is to obtain the maximum likelihood estimate of x_i from the measurements up to and including z_k. If $k < i$, the estimate is called a prediction; if $k = i$, the estimate is called filtering; and if $k > i$, the estimate is called smoothing. (See the Appendix for a basic estimation problem and solution.)

III. The Augmented State Approach

The method of optimal filtering developed by Kalman [Ref. 2] would be applied to the problem as follows. The state is first augmented to include ε_i:

$$x_i^a \triangleq \begin{bmatrix} x_i \\ \hline \varepsilon_i \end{bmatrix} \quad \text{and} \quad H^a \triangleq [H \,\vdots\, I]$$
$$\phi^a = \left[\begin{array}{c|c} \Phi & 0 \\ \hline 0 & \Psi \end{array}\right] \qquad Q^a = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & \overline{Q} \end{array}\right] \tag{2}$$

The system description is then:

$$\underline{\text{State}}: \quad x_{i+1}^a = \left[\begin{array}{c|c} \Phi & 0 \\ \hline 0 & \Psi \end{array}\right] x_i^a + \begin{bmatrix} w_i \\ \hline u_i \end{bmatrix}$$
$$\underline{\text{Measurement}}: \quad z_i = H^a x_i^a \tag{3}$$

For this augmented system the measurements are "perfect," i.e., contain no noise. Calling P_i^a the covariance of the best estimate of x_i^a after measurement z_i, and M_i^a the covariance of the best estimate of x_i^a before measurement z_i (see Appendix), the relation between P_i^a and M_i^a for this case can be written as (c.f. Ref. 3)

$$
\begin{aligned}
P_i^a &= M_i^a - M_i^a H^{aT}(H^a M_i^a H^{aT})^{-1} H^a M_i^a \\
M_{i+1}^a &= \Phi^a P_i^a \Phi^{aT} + Q^a
\end{aligned}
\tag{4}
$$

Now P_i^a must be singular, since linear combinations of the components of x_i^a are known perfectly. In fact, it follows easily from (4) that

$$H^a P_i^a H^{aT} = 0 \tag{5}$$

Thus, if Φ^a is near unity* and Q^a is small, the covariance updating may become ill-conditioned (i.e., $M_{i+1}^a \to P_i^a$).

Another way to look at the estimation problem is to observe that the measurements represent m linear constraints among the augmented state variables. Thus, although the augmented state vector is of dimension $n + m$, there are only n linearly independent variables in the estimation problem. This implies that the $(n+m)$ by $(n+m)$ matrix P_i^a is singular (of rank $\leq n$), and also points to the fact that the estimation filter need only be of dimension n, rather than of dimension $n + m$ as it is for this augmented state filter.

IV. The Measurement Differencing Approach

In this section we develop estimation filters of dimension n for

---

* For example, this could be the case of sampling a continuous system at instants of time close together relative to the time constants of the system.

the system (1). In particular, since ε_i is often not of interest, we design estimation filters dealing only with the original state vector x_i.

Using the state transition relations for x_i and ε_i, z_{i+1} can be expressed in terms of x_i, ε_i and the purely random vectors w_i and u_i. Having done this, it is possible to use the constraint relations (i.e., the measurements) to eliminate ε_i. In effect, we determine a linear combination of z_{i+1} and z_i (two measurement vectors in sequence) which does not contain ε_i. From (1) the proper linear combination is easily seen to be:

$$\zeta_i \triangleq z_{i+1} - \Psi z_i = (H\Phi - \Psi H)x_i + Hw_i + u_i \tag{6}$$

The "measurement" ζ_i now contains only the purely random sequence $u_i + Hw_i$ instead of the sequentially correlated sequence ε_i. Since ζ_i is based on z_{i+1}, it will prove convenient to state the problem as:

<u>State</u>: $x_i = \Phi x_{i-1} + w_{i-1}$	$w_{i-1} : (0,Q)$	
<u>Measurement</u>: $\zeta_{i-1} = H^r x_{i-1} + u_{i-1} + Hw_{i-1}$ with $H^r = H\Phi - \Psi H$ $(\zeta_{i-1} = z_i - \Psi z_{i-1})$	$u_{i-1} : (0,\bar{Q})$ u_{i-1} and w_{i-1} independent	(7)

Note that the process noise w_{i-1} and the measurement noise $u_{i-1} + Hw_{i-1}$ are correlated. The formal problem (7) can be solved with the basic solutions given in the Appendix. However, the fact that ζ_{i-1} is calculated from z_i requires further consideration.

IV-1. The Filtering Solution

At first glance it would appear that the problem in the form (7) is immediately solved by application of the basic estimation results in the Appendix. Thus, based on ζ_{i-1} one would obtain an "estimate" $\hat{x}_{i-1}$ of x_{i-1} and a "prediction" $\bar{x}_i$ of x_i. However, ζ_{i-1} is based on z_i; this means that the "prediction" of x_i based on ζ_{i-1} is, in fact, the best estimate of x_i based on z_i. Another way of stating this is that the mean of x_i conditioned on ζ_{i-1} is the mean of x_i conditioned on z_i, which is the desired optimal estimate of x_i.

By stating the problem in the form (7), the dimension of the problem has been reduced from $n + m$ in form (3) to n, and the basic estimation solution of the Appendix formally applies. However, the formal "filtering" and "prediction" solutions are actually "single stage smoothing" and "filtering" solutions, respectively. In order to distinguish between the formal and the actual estimates, the following notation will be adopted for the actual estimates:

$$\hat{x}_{i/k} = \text{optimal estimate of } x_i \text{ given measurements up to and including } z_k.$$

$$P_{i/k} = \text{covariance of } \hat{x}_{i/k} = E\{(x_i-\hat{x}_{i/k})(x_i-\hat{x}_{i/k})^T\} \tag{8}$$

The $\bar{x}_i$, $\hat{x}_i$ notation will be reserved for the formal application of the basic solutions in the Appendix to the problem in the form (7), and results in the following equivalences:

Actual	Formal
$\hat{x}_{i/i}$ =	$\bar{x}_i$
$\hat{x}_{i-1/i}$ =	$\hat{x}_{i-1}$

(9)

where $\bar{x}_i$ and $\hat{x}_{i-1}$ are the formal "prediction" and "estimate" based on ζ_{i-1}. Note that a single stage smoothing estimate is obtained automatically if it is desired.

Using the equivalences in (9), the filtering solution can be written by the formal application of Eq. (A-2) to the problem in the form (7) as:

$$\hat{x}_{i-1/i} = \hat{x}_{i-1/i-1} + K_{i-1}(\zeta_{i-1} - H^r \hat{x}_{i-1/i-1})$$

$$\hat{x}_{i/i} = \Phi \hat{x}_{i-1/i} + D(\zeta_{i-1} - H^r \hat{x}_{i-1/i})$$

where

$$D = SR^{-1} \qquad R = \bar{Q} + HQH^T$$

$$S = QH^T \qquad H^r = H\Phi - \Psi H$$

$$\zeta_{i-1} = z_i - \Psi z_{i-1} \tag{10}$$

$$K_{i-1} = M_{i-1}H^{rT}(H^r M_{i-1} H^{rT} + R)^{-1}$$

$$P_{i-1} = (I - K_{i-1}H^r)M_{i-1}(I - K_{i-1}H^r)^T + K_{i-1}RK_{i-1}^T$$

$$M_i = (\Phi - DH^r)P_{i-1}(\Phi - DH^r)^T + Q - DRD^T$$

$$P_{i/i} = M_i$$

$$P_{i-1/i} = P_{i-1}$$

If the single stage smoothing estimate $\hat{x}_{i-1/i}$ is not explicitly desired, the filter can be written as:

$$\hat{x}_{i/i} = \Phi\hat{x}_{i-1/i-1} + [D + (\Phi-DH^r)K_{i-1}](\zeta_{i-1}-H^r\hat{x}_{i-1/i-1}) \quad (11)$$

IV-1.1. Starting Procedure

After the first measurement there is not yet sufficient information to calculate ζ_1, so the augmented state $x_1^a = \begin{bmatrix} x_1 \\ -- \\ \varepsilon_1 \end{bmatrix}$ approach must be used to obtain the best estimate of x_1^a and the associated covariance based on the "perfect measurement" $z_1 = H^a x_1^a$ and single stage estimation theory.* After the second measurement ζ_1 can be calculated, and the filter (10) can be used with the estimate of x_1 from $\hat{x}_1^a$ and its covariance as the "a priori" starting statistics.

IV-1.2. Estimate of ε_i

Since $z_i = Hx_i + \varepsilon_i$, an estimate of ε_i can be obtained any time after the first measurement by:

$$\begin{aligned} \hat{\varepsilon}_{i/i} &= z_i - H\hat{x}_{i/i} \\ \text{cov}\{\hat{\varepsilon}_{i/i}\} &= HP_{i/i}H^T \end{aligned} \quad (12)$$

The reduced filter (10) requires the storage of one measurement vector (z_{i-1} is needed in addition to z_i to calculate ζ_{i-1}), but it has two distinct advantages over the augmented state filter. First, the dimension is n instead of $n + m$. Second, the potentially ill-conditioned inversion $(H^a M_1^a H^{aT})^{-1}$ is eliminated.

---

* Note the analogy with the continuous linear dynamic system problem [Ref. 1] where a "starting procedure" is also required. In fact, the present problem helps to understand that requirement.

IV-2. The Prediction Solution

The prediction of x_{i+1} given the estimate $\hat{x}_{i/i}$ follows immediately from the state equation

$$x_{i+1} = \Phi x_i + w_i \tag{13}$$

and from single stage estimation theory as

$$\boxed{\begin{aligned} \hat{x}_{i+1/i} &= \Phi\hat{x}_{i/i} \\ P_{i+1/i} &= \Phi P_{i/i}\Phi^T + Q \end{aligned}} \tag{14}$$

Similarly, if prediction of ε_{i+1} given $\hat{\varepsilon}_{i/i}$ is desired, use of

$$\varepsilon_{i+1} = \Psi\varepsilon_i + u_i \tag{15}$$

and (12) yields

$$\begin{aligned} \hat{\varepsilon}_{i+1} &= \Psi\hat{\varepsilon}_i \\ \text{cov}\{\hat{\varepsilon}_{i+1/i}\} &= \Psi HP_{i/i}H^T\Psi^T + \overline{Q} \end{aligned} \tag{16}$$

IV-3. The Smoothing Solution

The smoothing solution also follows from the formulation (7) and the basic solution, Eq. (A-2), again noting that one must be careful of the nomenclature. When smoothing backwards from the N^{th} stage, the formal solution smooths backwards from ζ_{N-1}. In other words, the smoothing estimate at the $(N-1)^{st}$ stage has already been obtained from the "filtering" estimate (10). In terms of the formal smoothed estimate, $\hat{x}(i/\zeta_{N-1})$, obtained from applying Eq. (A-2) to the formulation (7), the actual estimate is

$$\hat{x}_{i/N} = \hat{x}(i/\zeta_{N-1}) \tag{17}$$

With these observations, the smoothing solution for x_i is

$$\hat{x}_{i/N} = \hat{x}_{i/i+1} - C_i(\hat{x}_{i+1/i+1} - \hat{x}_{i+1/N}) \quad ; \quad \hat{x}_{N-1/N} \text{ given}$$

$$P_{i/N} = P_{i/i+1} - C_i(P_{i+1/i+1} - P_{i+1/N})C_i^T \quad ; \quad P_{N-1/N} \text{ given} \tag{18}$$

where

$$C_i = P_{i/i+1}(\Phi - DH^r)^T P^{-1}_{i+1/i+1}$$

If a smoothed estimate of ε_i is desired, it is given directly from the constraint relations by

$$\hat{\varepsilon}_{i/N} = z_i - H\hat{x}_{i/N}$$

$$\text{cov}\{\hat{\varepsilon}_{i/N}\} = HP_{i/N}H^T \tag{19}$$

IV-3. Generalization to Systems with Time Varying Coefficients

A fairly general system of this kind can be described by

$$\begin{array}{llll} \underline{\text{State}}: & x_{i+1} = \Phi_i x_i + w_i & w_i : (0, Q_i) & x : (n \times 1) \\ & & & z : (m \times 1) \\ \underline{\text{Measurement}}: & z_i = H_i x_i + \varepsilon_i & u_i : (0, \overline{Q}_i) & \varepsilon : (m \times 1) \\ \underline{\text{Measurement Noise}}: & \varepsilon_{i+1} = \Psi_i \varepsilon_i + u_i & w_i \text{ and } u_i \text{ independent} & \end{array} \tag{20}$$

As mentioned in Section II, further generalizations can be found in Ref. 4. The technique of determining filtering, prediction, and smoothing solutions for (20) is the same as that used above.

In (1) it was assumed that R was nonsingular. In (20) the corresponding quantity R_i is the covariance of $u_i + H_{i+1}w_i$. If R_i is nonsingular, the elimination of the ill-conditioning in constructing the filters is guaranteed. However, even if R_i is singular, there will be cases where there is no ill-conditioning and the reduction in dimension of the data processing filters is desirable. For this reason the basic solution, Eq. (A-3) is used to obtain the solution to (20) as it is valid independent of the rank of R_i. (If R_i is singular, a further reduction in the dimension of the filters is possible; see Ref. 4.) In fact, given the reduced problem (7), any set of filtering equations may be used, but the results, which are based on ζ_{i-1}, must be "interpreted" in terms of z_i.

Using Eq. (A-3), the estimation filters for (20) are:

<u>Filtering</u>
$$\hat{x}_{i-1/i} = \hat{x}_{i-1/i-1} + K_{i-1}(\zeta_{i-1} - H^r_{i-1}\hat{x}_{i-1/i-1})$$
$$\hat{x}_{i/i} = \Phi_{i-1}\hat{x}_{i-1/i} + S_{i-1}(H^r_{i-1}M_{i-1}H^{rT}_{i-1}+R_{i-1})^{-1}(\zeta_{i-1} - H^r_{i-1}\hat{x}_{i-1/i-1})$$

<u>Prediction</u>
$$\hat{x}_{i+1/i} = \Phi_i\hat{x}_{i/i}$$

<u>Smoothing</u>
$$\hat{x}_{i/N} = \hat{x}_{i/i+1} - C_i(\hat{x}_{i+1/i+1} - \hat{x}_{i+1/N}) \quad ; \quad \hat{x}_{N-1/N} \text{ given}$$

where
$$S_{i-1} = Q_{i-1}H_i^T$$
$$R_{i-1} = \overline{Q}_{i-1} + H_iQ_{i-1}H_i^T$$
$$H^r_{i-1} = H_i\Phi_{i-1} - \Psi_{i-1}H_{i-1} \tag{21}$$
$$\zeta_{i-1} = z_i - \Psi_{i-1}z_{i-1}$$
$$K_{i-1} = M_{i-1}H^{rT}_{i-1}(H^r_{i-1}M_{i-1}H^{rT}_{i-1}+R_{i-1})^{-1}$$
$$C_i = (P_{i/i+1}\Phi_i^T - K_iS_i^T)P^{-1}_{i+1/i+1}$$
$$P_{i-1/i} = P_{i-1} = (I-K_{i-1}H^r_{i-1})M_{i-1}(I-K_{i-1}H^r_{i-1})^T + K_{i-1}R_{i-1}K^T_{i-1}$$
$$P_{i/i} = M_i = \Phi_{i-1}P_{i-1}\Phi^T_{i-1} + Q_{i-1} - S_{i-1}(H^r_{i-1}M_{i-1}H^{rT}_{i-1}+R_{i-1})^{-1}S^T_{i-1} -$$
$$- \Phi_{i-1}K_{i-1}S^T_{i-1} - S_{i-1}K^T_{i-1}\Phi^T_{i-1}$$
$$P_{i+1/i} = \Phi_iP_{i/i}\Phi_i^T + Q_i$$
$$P_{i/N} = P_{i/i+1} - C_i(P_{i+1/i+1} - P_{i+1/N})C_i^T \quad ; \quad P_{N-1/N} \text{ given}$$

Note that the starting procedure using the augmented state must be used as described in Section IV-1.1.

V. Summary and Conclusions

This paper has considered the estimation problem for multi-stage linear dynamic systems based on measurements of linear combinations of the state variables with additive sequentially correlated noise. By using a weighted first difference of the present and previous measurements, a filter, predictor, and smoother have been developed of lower dimension than those obtained from the augmented state approach. Further, the potential ill-conditioning of the augmented state approach is eliminated (R_i nonsingular) or reduced (R_i singular).

The results include explicit relations for prediction, filtering, and smoothing procedures and the associated covariances. These are summarized in Eq. (21). These improved methods should be useful in orbit determination, guidance, control, navigation, and flight testing.

APPENDIX

Basic Estimation Solution

The results of basic estimation theory are summarized here for use in this paper (see Refs. 2, 3, and 4).

The general problem may be stated as:

$$\text{State:} \quad x_{i+1} = \Phi_i x_i + w_i \qquad w_i : (\bar{w}_i, Q_i) \qquad x : (n \times 1)$$

$$\text{Measurement:} \quad z_i = H_i x_i + v_i \qquad v_i : (0, R_i) \qquad z : (m \times 1) \qquad \text{(A-1)}$$

$$E\{w_i v_j^T\} = S_i \delta_{ij}$$

where w_i and v_i are gaussian purely random vector sequences. δ_{ij} is the Kronecker delta function.

For the estimation solution the following definitions are used:

$\bar{x}_i$ = estimate of x_i using measurements up to z_{i-1} (single-stage prediction)

$\hat{x}_i$ = estimate of x_i using measurements up to z_i (filtering)

$\hat{x}(i/z_N)$ = estimate of x_i using measurements up to z_N $(N > i)$ (smoothing)

M_i = covariance of $\bar{x}_i = E\{(x_i - \bar{x}_i)(x_i - \bar{x}_i)^T\}$

P_i = covariance of $\hat{x}_i = E\{(x_i - \hat{x}_i)(x_i - \hat{x}_i)^T\}$

$P(i/N)$ = covariance of $\hat{x}(i/z_N) = E\{[x_i - \hat{x}(i/z_N)][x_i - \hat{x}(i/z_N)]^T\}$

With these definitions the estimation solution for the problem (A-1) will be given in two separate forms:

Form 1 (R_i non-singular)

$$
\begin{aligned}
\hat{x}_i &= \bar{x}_i + K_i(z_i - H_i\bar{x}_i) \\
\bar{x}_{i+1} &= \Phi_i\hat{x}_i + D_i(z_i - H_i\hat{x}_i) + \bar{w}_i \\
\hat{x}(i/z_N) &= \hat{x}_i - C_i(\bar{x}_{i+1} - \hat{x}_{i+1/N}) \quad ; \quad \hat{x}_{(N/z_N)} = \hat{x}_N \\
K_i &= M_iH_i^T(H_iM_iH_i^T + R_i)^{-1} \\
C_i &= P_i(\Phi_i - D_iH_i)^T M_{i+1}^{-1} \\
D_i &= S_iR_i^{-1} \\
D_iR_iD_i^T &= S_iR_i^{-1}S_i^T \\
P_i &= (I - K_iH_i)M_i(I - K_iH_i)^T + K_iR_iK_i^T \\
M_{i+1} &= (\Phi_i - D_iH_i)P_i(\Phi_i - D_iH_i)^T + Q_i - D_iR_iD_i^T \\
P(i/N) &= P_i - C_i(M_{i+1} - P_{i+1/N})C_i^T \quad ; \quad P_{N/N} = P_N
\end{aligned}
\tag{A-2}
$$

Form 2 (R_i singular)

$$
\begin{aligned}
\hat{x}_i &= \bar{x}_i + K_i(z_i - H_i\bar{x}_i) \\
\bar{x}_{i+1} &= \Phi\hat{x}_i + S_i(H_iM_iH_i^T + R_i)^{-1}(z_i - H_i\bar{x}_i) + \bar{w}_i \\
\hat{x}(i/z_N) &= \hat{x}_i - C_i(\bar{x}_{i+1} - \hat{x}_{i+1/N}) \quad ; \quad \hat{x}_{(N/z_N)} = \hat{x}_N \\
K_i &= M_iH_i^T(H_iM_iH_i^T + R_i)^{-1} \\
C_i &= (P_i\Phi_i^T - K_iS_i^T)M_{i+1}^{-1} \\
P_i &= (I - K_iH_i)M_i(I - K_iH_i)^T + K_iR_iK_i^T \\
M_{i+1} &= \Phi_iP_i\Phi_i^T + Q_i - S_i(H_iM_iH_i^T + R_i)^{-1}S_i^T - \Phi_iK_iS_i^T - S_iK_i^T\Phi_i^T \\
P(i/N) &= P_i - C_i(M_{i+1} - P_{i+1/N})C_i^T \quad ; \quad P_{N/N} = P_N
\end{aligned}
\tag{A-3}
$$

It is assumed that the a priori statistics M_1 and $\bar{x}_1$ are given.

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13. ABSTRACT

This paper presents improved filtering, prediction, and smoothing procedures for multi-stage linear dynamic systems when the measured quantities are linear combinations of the state variables with additive sequentially correlated noise. The "augmented state" procedure suggested by Kalman may lead to ill-conditioned computations in constructing the data processing filter. The design procedure described here eliminates these ill-conditioned computations and reduces the dimension of the filter required. The results include explicit relations for prediction, filtering, and smoothing procedures and the associated covariance matrices.

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