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APPLICATION OF LYAPUNOV STABILITY THEORY TO SOME NONLINEAR PROBLEMS IN HYDRODYNAMICS

by Gabe R. Buis and William G. Vogt

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Summary

Lyapunov stability theory is used to analyze the stability properties of equilibrium solutions to some linear and nonlinear partial differential equations. Stability conditions are established for systems which have, as a linear part, a self-adjoint differential operator or one which can be transformed into a self-adjoint differential operator.

Although the techniques described are applied to systems occurring in hydrodynamics the methods can be used for similar problems in other fields as well.

1. Introduction

Frequently the analysis of systems describing physical processes gives rise to a stability problem of a system of partial differential equations. Often a stability analysis is carried out on an approximate system model having a finite number of degrees of freedom, usually obtained by a spatial discretization or a modal truncation method. The stability conditions so derived are sometimes not sufficient for stability except in the case of infinitesimally small perturbations.

This approach is followed in one of the most recent contributions to the stability theory of hydrodynamical systems by Eckhaus[1]. His theory for analyzing the stability properties of the solutions to nonlinear partial differential equations is based on asymptotic expansions with respect to suitably defined small parameters and series expansions in terms of the eigenfunctions. This method becomes cumbersome for more complex systems.

Some of the systems studied by Eckhaus[1] are investigated here using general Lyapunov stability theory.[2,3] The context of the approach is the same as in[1], that is, an investigation of the formal properties of certain mathematical relationships without a rigorous justification of these formal properties. It turns out, for the examples cited, that considerable improvement in stability conditions is obtained by the procedure recommended in this report, even to the extent of allowing nonlinearities.

The stability of the equilibrium solution is defined in terms of the norm induced by the inner product of the Hilbert space on which the solutions of the system are defined. For a certain class of differential operators the stability conditions of the equilibrium solutions can be derived in a straightforward way. This class of differential operators can be considerably extended by a transformation of the differential operator which involves a modification of the inner product. The general approach, even to systems containing nonlinearities, is illustrated by examples.

2. Statement of the Problem

Many physical systems are formally described by partial differential equations of the form

$$\frac{\partial \underline{u}(t,\underline{x})}{\partial t} + \underline{L}\,\underline{u}(t,\underline{x}) = \underline{0} \quad (\underline{x}\varepsilon\Omega;\ t\,\varepsilon\ [0,\infty)) \tag{2.1}$$

where $\underline{u}(t,\underline{x})$ is an n-vector function and $\underline{L}$ is a matrix whose elements are linear or nonlinear differential operators specified on a bounded connected open subset Ω of an m-dimensional Euclidean space, E^m. The parameters of $\underline{L}$ can be space dependent but not time dependent. In order to uniquely specify solutions of (2.1) a set of additional constraints or boundary conditons must be given, generally by a relation of the form

$$\underline{H}\ \underline{u}\ (t,\underline{x}') = \underline{0} \quad (\underline{x}'\ \varepsilon\ \partial\Omega;\ t\ \varepsilon[0,\infty)) \tag{2.2}$$

where $\underline{H}$ is a matrix whose elements are formally specified differential operators and $\partial\Omega$ is the boundary of Ω. Furthermore any solution will depend on some initial function $\underline{\theta}_0\ (\underline{x})$ belonging to the n-dimensional space of functions, Θ which we will assume is a Hilbert space with elements smooth enough to assure that solutions to (2.1) and (2.2) exist and belong to Θ.

A solution to (2.1) and (2.2) will be designated as $\underline{u}(t,\underline{x};\ \underline{\theta}_0)$, that is, the solution starting at $t=0$ and with initial condition $\underline{\theta}_0\ (\underline{x})\ \varepsilon\ \Theta$, $\underline{u}(0,\ \underline{x};\ \underline{\theta}_0) = \underline{\theta}_0(\underline{x})$. The solution of particular interest is the equilibrium solution $\underline{u}_{eq}(\underline{x})\ \varepsilon\ \Theta$, which is assumed to be $\underline{u} = \underline{0}$.

In the following sections the stability of the equilibrium solution or the trivial solution, $\underline{u} = \underline{0}$, will be formulated and for some particular systems further investigated.

To formulate the concept of stability we suppose that Θ, as a Hilbert space, has an inner product $<\underline{v},\underline{u}>$ and the norm, $||\cdot||$, induced by the inner product $||\underline{u}|| = (<\underline{u},\underline{u}>)^{1/2}$. Then the distance at any time between $\underline{u}_1$ and $\underline{u}_2$ in Θ is given by $||\underline{u}_1 - \underline{u}_2||$. As used here a general inner product of two functions $\underline{u}, \underline{v} \in \Theta$ is denoted by:

$$<\underline{v},\underline{u}> = \int_\Omega \underline{v}^T \underline{W}(\underline{x}) \; \underline{u} \; d\Omega \tag{2.3}$$

where $(^T)$ denotes the transpose, $\underline{W}(\underline{x})$ is a "weighting" matrix. Its elements can be chosen as continuous functions in $\underline{x}$ such that

$$\underline{W}(\underline{x}) = \underline{W}^T(\underline{x}) \tag{2.4}$$

and

$$\beta_1 \underline{u}^T \underline{u} \geqq \underline{u}^T \underline{W}(\underline{x}) \; \underline{u} \geqq \beta_2 \underline{u}^T \underline{u}, \quad \infty > \beta_1 > \beta_2 > 0 \tag{2.5}$$

for all $\underline{x} \in \bar{\Omega} = \Omega + \partial \Omega$.

The conditions (2.4) and (2.5) assure that the norm induced by (2.3) and the norm induced by $<\underline{v}, \underline{u}> = \int_\Omega \underline{v}^T \underline{u} \; d\Omega$ are equivalent.

As will be shown, an advantageous choice of $\underline{W}(\underline{x})$ depends on the form of the linear part of the operator $\underline{L}$ in (2.1).

3. Stability of the Equilibrium Solution

The concept of stability can be defined in many different ways. Here stability will refer to stability in the sense of Lyapunov, that is, a system is stable if for a sufficiently small perturbation from equilibrium, the solution will remain close to equilibrium for all future time. The stability of the trivial solution $\underline{u} = \underline{0}$ can now more precisely be defined in terms of the norm given on Θ.

Definition. The trivial solution $\underline{u} = \underline{0}$ of (2.1) is said to be stable in the sense of Lyapunov if for every real number $\varepsilon>0$, there exists a real number $\delta>0$ such that $||\underline{\theta}_0(\underline{x})|| < \delta$ implies $||\underline{u}(t, \underline{x}; \underline{\theta}_0)|| < \varepsilon$ for all $t \geq 0$.

Definition. The trivial solution $\underline{u} = \underline{0}$ of (2.1) is said to be asymptotically stable if it is stable and in addition $||\underline{u}(t,\underline{x}; \underline{\theta}_0)|| \to 0$ as $t\to\infty$.

The stability properties of the trivial solution $\underline{u} = \underline{0}$ of (2.1) can often be determined by consideration of the rate of change of the functional:

$$V(\underline{u}) = ||\underline{u}||^2 = \langle \underline{u},\underline{u}\rangle = \int_\Omega \underline{u}^T \underline{W}(\underline{x}) \underline{u} \, d\Omega \tag{3.1}$$

The time variation of $V(\underline{u})$ along solutions is given by

$$V(t) = V(\underline{u}(t, \underline{x}; \underline{\theta}_0(\underline{x})) \qquad t \in [0,\infty)$$

with

$$V(0) = V(\underline{\theta}_0(\underline{x}))$$

and its time derivative by

$$\frac{dV(t)}{dt} = \lim_{h\to 0} \left[\frac{1}{h} (V(t+h) - V(t))\right] \tag{3.2}$$

for $\underline{u}(t,\underline{x}; \underline{\theta}_0(\underline{x}))$ if the limit exists.

Formally, from (3.2), it follows that since

$$\langle \underline{u}(t + h), \underline{u}(t + h)\rangle - \langle \underline{u}(t), \underline{u}(t)\rangle =$$

$$= \langle \underline{u}(t + h) + \underline{u}(t), \underline{u}(t + h) - \underline{u}(t)\rangle \tag{3.3}$$

$$\frac{dV(t)}{dt} = \lim_{h \to 0} \left[< \underline{u}(t+h) + \underline{u}(t), \frac{\underline{u}(t+h) - \underline{u}(t)}{h} \right] \tag{3.4}$$

or
$$\frac{dV(t)}{dt} = 2 < \underline{u}(t, \underline{x}; \underline{\theta}_0(\underline{x})), \frac{\partial \underline{u}(t, \underline{x}; \underline{\theta}_0(x)}{\partial t} >$$

yielding
$$\frac{dV(u)}{dt} = -2 <\underline{u}, \underline{L}\, \underline{u}> \text{[1)]} \tag{3.5}$$

It is clear that if $\frac{dV(u)}{dt}$ can be reduced to the form:

$$\frac{dV(\underline{u})}{dt} \leqq 2\alpha \, ||\underline{u}(t, \underline{x}; \underline{\theta}_0)||^2 = 2\alpha \, V(\underline{u}) \tag{3.6}$$

where α is some real constant, it follows by integration that

$$V(\underline{u}) \leqq V(\underline{\theta}_0) \exp 2\alpha t$$

or
$$||\underline{u}(t, \underline{x}; \underline{\theta}_0)|| \leqq ||\underline{\theta}_0(\underline{x})|| \exp \alpha t \tag{3.7}$$

and the equilibrium solution $\underline{u} = \underline{0}$ will be asymptotically stable for $\alpha<0$ and stable for $\alpha \leqq 0$ for all $t \geqq 0$. The functional $V(\underline{u})$ with these properties is a Lyapunov Functional[3]. Thus, with the proper choice of inner product or norm, the square of the norm becomes the Lyapunov functional which establishes asymptotic stability.

In the following section $\frac{dV(\underline{u})}{dt}$ will be evaluated for a certain class of linear differential operators $\underline{L}$. The section is followed by examples in which these methods are extended to certain nonlinear systems.

4. Self-Adjoint Linear Differential Operators

Consider the system as given by (2.1) and (2.2) and let $\underline{L}$ be a linear operator. The extent to which one is able to reduce $\frac{dV(u)}{dt}$ to the form

1) In a completely functional analytic presentation, this equation would only be defined on the domain of $\underline{L}$, $D(\underline{L}) \subseteq \Theta$, but provided this domain is dense in Θ, the conditions derived later are sufficient for stability[3].

(3.6) depends intrinsically on the operator $\underline{L}$. With respect to an inner product, $\underline{L}^*$, the adjoint operator of $\underline{L}$ is formally defined by

$$<\underline{v}, \underline{L}\, \underline{u}> = <\underline{L}^* \underline{v}, \underline{u}> \tag{4.1}$$

for all $\underline{u}$ in the domain of $\underline{L}$. An important class of operators is formed by the self-adjoint operators. An operator is formally self-adjoint if $\underline{L}^* = \underline{L}$.

If one restricts oneself to the inner product

$$<\underline{v},\underline{u}> = \int_\Omega \underline{v}^T \underline{u}\, d\Omega \tag{4.2}$$

it is clear that the corresponding class of self-adjoint operators is very limited. The introduction of the matrix, $\underline{W}(\underline{x})$, often permits $\underline{L}$ to be made self-adjoint with respect to the generalized inner product (2.3). This is shown in the following example.

Example. Let the differential operator $\underline{L}$ be given by

$$L\, u \equiv a(x) \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} + c(x)\, u, \tag{4.3}$$

$0 \leqq x \leqq 1$, $a(x) \geqq \delta > 0$, $b(x) \neq 0$ and boundary conditions $u(0) = u(1) = 0$.

Then with the inner product (4.2):

$$<v, L\, u> = \int_0^1 v\, L\, u\, dx = \int_0^1 \{a(x) v \frac{\partial^2 u}{\partial x^2} + b(x) v \frac{\partial u}{\partial x} + c(x)\, vu\} dx.$$

After integration by parts and substitution of the boundary conditons, assuming $v(0) = v(1) = 0$ one gets

$$<v, L\, u> = \int_0^1 \{u \frac{\partial^2}{\partial x^2}(a(x)v) - u \frac{\partial}{\partial x}(b(x)v) + c(x) vu\} dx$$

$$= <L^* v, u>$$

where L^* is given by

$$L^*v \equiv \frac{\partial^2}{\partial x^2}(a(x)v) - \frac{\partial}{\partial x}(b(x)v) + c(x)v,$$

$0 \leqq x \leqq 1$ and boundary conditions $v(0) = v(1) = 0$. Clearly $L^* \neq L$ and L is not self-adjoint.

But L is equivalent to L_e as given by

$$L_e u \equiv \frac{1}{w(x)} \frac{\partial (p(x) \frac{\partial u}{\partial x})}{\partial x} + q(x)\, u, \tag{4.4}$$

$0 \leqq x \leqq 1$, where $q(x) = c(x)$

$$p(x) = \exp \int_0^x \frac{b(x)}{a(x)}\, dx \qquad w(x) = (a(x))^{-1} \exp \int_0^x \frac{b(x)}{a(x)}\, dx$$

and boundary conditons $u(0) = u(1) = 0$. This can be verified directly. Application of the general inner product (2.3) gives

$$\langle v, L_e\, u\rangle = \int_0^1 v\, L_e\, u\, w\, dx = \int_0^1 \{v \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) + vq(x)uw\}dx$$

Integration by parts of the expression in the integrand and substitution of the boundary conditions, with the assumption $v(0) = v(1) = 0$ gives

$$\langle v,\ L_e\ u\rangle = \int_0^1 \{u \frac{\partial}{\partial x}(p(x) \frac{\partial v}{\partial x}) + u\, q(x) v\, w\}\, dx = \langle L^*{}_e v,\ u\rangle =$$

$$= \langle L_e v,\ u\rangle.$$

Thus L_e will be self-adjoint with respect to the inner product

$$\langle v,\ u\rangle = \int_0^1 v\, u\, w\, dx. \tag{4.5}$$

The significance of the choice of inner product becomes apparent on evaluation of $\frac{dV(\underline{u})}{dt}$ given by (3.5) where $V(\underline{u})$ is given by (3.1), for self-adjoint operators. If $\underline{L}$ is a self-adjoint operator with a lower semi-bounded spectrum and λ_{min} is the smallest eigenvalue of $\underline{L}$, then

$$\lambda_{min} \langle \underline{u}, \underline{u} \rangle \leqq \langle \underline{u}, \underline{L}\, \underline{u} \rangle \tag{4.6}$$

To show how this property can be used, consider the linear system

$$\frac{\partial \underline{u}}{\partial t} + \underline{L}\, \underline{u} = \underline{0} \tag{4.7}$$

and let $\underline{L}$ be a differential operator with possibly space dependent coefficients. Let a set of boundary conditions be specified and let $\underline{L}$ be self-adjoint with respect to the general inner product

$$\langle \underline{v}, \underline{u} \rangle = \int_{\Omega} \underline{v}^T\, \underline{W}(\underline{x})\, \underline{u}\, d\Omega \tag{4.8}$$

From (3.5) and the fact that $\underline{L}$ is self-adjoint with a lower semi-bounded spectrum and λ_{min} is the smallest eigenvalue of $\underline{L}$, it follows that

$$\frac{dV(\underline{u})}{dt} \leqq -2\lambda_{min} \langle \underline{u}, \underline{u} \rangle \tag{4.9}$$

Hence if $\lambda_{min} > 0$ we have (3.6) and thus asymptotic stability.

A similar approach can be followed when $\underline{L}$ is time varying by introducing $\underline{W}(\underline{x}, t)$. The elements of $\underline{W}(\underline{x}, t)$ must be continuous in both $\underline{x}$ and t and continuously differentiable in t and such that (2.4) and (2.5) are satisfied. The derivative $\frac{dV(\underline{u})}{dt}$ can be evaluated as in Section 3. This yields an additional term in (4.9) which can be estimated to yield a sufficient condition for stability.

The use of this general inner product has the advantage of relating the stability properties directly to the eigenvalues of the differential operator for a large class of operators. As will be shown in the examples in the next section in order to reduce $\frac{d\,V(\underline{u})}{dt}$ to the form $\alpha \langle \underline{u}, \underline{u} \rangle$ one can apply well-known integral inequalities rather than calculating the eigenvalues. The use of the general inner product representation facilitates the application of these inequalities and improves on the resulting stability condition. Of course these inequalities[4], although standard, are to some extent based on estimates of the eigenvalues.

When applied to nonlinear differential operators the advantages become even more apparent since this permits an estimate for the set of initial functions in Θ for which the trivial solution is asymptotically stable. In the next section this will be illustrated by applying the above methods to Burgers' model of turbulence and some of its modifications as studied by Eckhaus[1].

5. Applications.

Example 1. Burgers' model to describe turbulence as studied by Eckhaus[1] is given by

$$\frac{\partial u_1}{\partial t} - u_1 - \frac{1}{R}\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1^2}{\partial x} - u_1 u_2 = 0$$

$$\frac{\partial u_2}{\partial t} + \frac{1}{R} u_2 + \int_0^1 u_1^2\, dx = 0 \qquad (5.1)$$

$0 \leqq x \leqq 1$ and boundary conditions $u_1(0) = u_1(1) = 0$. To illustrate the foregoing we will make a slight generalization of the above problem and in the sequel establish the results of Eckhaus.

(a) Consider first the linear system

$$\frac{\partial \underline{u}}{\partial t} + \underline{L}\,\underline{u} = \underline{0}$$

where $$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} ; \quad \underline{L} = \begin{bmatrix} -1 - \frac{1}{R}\frac{\partial^2}{\partial x^2} & , \; 0 \\ 0 & , \; \frac{1}{R} \end{bmatrix} \tag{5.2}$$

with $R = R(x) > 0$ for $x \varepsilon [0,1]$ and boundary conditions $u_1(0) = u_1(1) = 0$. This assumption complicates the problem, because with constant R the operator $\underline{L}$ is self-adjoint with respect to the inner product

$$<\underline{v}, \underline{u}> = \int_0^1 (v_1 u_1 + v_2 u_2) dx \tag{5.3}$$

and the condition for asymptotic stability of the trivial solution $\underline{u} = \underline{0}$ can be derived from the smallest eigenvalue of $\underline{L}$ which is easily determined.

However, since R(x) is assumed to be space dependent, some sort of variational technique would be required to obtain λ_{min}. This can be avoided as shown below by making use of an integral inequality.

For the system as given by (5.2) choose as $V(\underline{u})$ functional:

$$V(\underline{u}) = ||\underline{u}||^2 = <\underline{u}, \underline{u}> = \int_0^1 \underline{u}^T \, \underline{W}(x) \, \underline{u} \, dx \tag{5.4}$$

where $$\underline{W}(x) = \begin{bmatrix} R(x) & , \; 0 \\ 0 & , \; 1 \end{bmatrix} . \tag{5.6}$$

$V(\underline{u})$ is now the required Lyapunov Functional[3] if $\frac{dV(\underline{u})}{dt} < 0$. $\underline{L}$ is self-adjoint for the inner product (5.4), however determination of the eigenvalues is not as immediate as in the case of constant R. The derivative of $V(\underline{u})$ is:

$$\frac{dV(\underline{u})}{dt} = -2 \langle \underline{u}, \underline{L}\ \underline{u} \rangle \tag{5.7}$$

Substituting $\underline{L}$ and $\underline{W}(\underline{x})$ gives:

$$\frac{dV(\underline{u})}{dt} = -2 \int_0^1 \{-R(x)u_1^2 - u_1 \frac{\partial^2 u_1}{\partial x^2} + \frac{1}{R(x)} u_2^2\}dx. \tag{5.8}$$

Integration by parts of the second term in the integrand of (5.8) and substitution of the boundary conditions gives

$$\frac{dV(\underline{u})}{dt} = -2 \int_0^1 \{-R(x)\ u_1^2 + (\frac{\partial u_1}{\partial x})^2 + \frac{1}{R(x)}\ u_2^2\}\ dx \tag{5.9}$$

Now the following integral inequality[4] holds for u_1

$$\int_0^1 (\frac{\partial u_1}{\partial x})^2\ dx \geqq \pi^2 \int_0^1 u_1^2\ dx. \tag{5.10}$$

Thus
$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 \{(\pi^2 - R(x))u_1^2 + \frac{1}{R(x)}\ u_2^2\}dx \tag{5.11}$$

or

$$\frac{dV(\underline{u})}{dt} \leqq -\min_{x\varepsilon\ [0,1]} (\frac{1}{R(x)}\ (\pi^2 - R(x)), \frac{1}{R(x)})\ ||\underline{u}||^2 \tag{5.12}$$

The equilibrium solution $\underline{u} = \underline{0}$ will be asymptotically stable if

$$\min_{x\varepsilon\ [0,1]} (\frac{1}{R(x)}\ (\pi^2 - R(x)),\frac{1}{R(x)}) > 0 \tag{5.13}$$

Since $R(x) > 0$ for $x \varepsilon\ [0,1]$ condition (5.13) reduces to

$$\pi^2 > \max_{x\varepsilon\ [0,1]} R(x)\ . \tag{5.14}$$

With constant $R>0$, for $\underline{u} = \underline{0}$ to be asymptotically stable, it follows immediately from (5.14) that a sufficient condition is that

$$0 < R < \pi^2 \tag{5.15}$$

This condition is identical to that obtained by requiring that λ_{min} of $\underline{L}$ be positive, since it is shown in [1] that.

$$\lambda_{1n} = \frac{\pi^2}{R}(n+1) - 1 \qquad (n=0,1,2,\ldots)$$

$$\lambda_{20} = \frac{1}{R} \tag{5.16}$$

(b) Next consider the nonlinear case and suppress (as in [1]) the $\frac{\partial u_2}{\partial t}$ term in (5.1). This gives:

$$\frac{\partial u}{\partial t} - u - \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial x} + R\left[\int_0^1 u^2 dx\right] u = 0 \tag{5.17}$$

$0 \leqq x \leqq 1$ and boundary conditions $u(0) = u(1) = 0$. For simplicity, assume in addition that R is a positive constant here. Taking as V(u) functional

$$V(u) = ||u||^2 = \int_0^1 u^2\, dx \tag{5.18}$$

its time derivative becomes:

$$\frac{dV(u)}{dt} = -2\int_0^1 \left\{ -u^2 - \frac{1}{R}u\left(\frac{\partial^2 u}{\partial x^2}\right) + u\frac{\partial u^2}{\partial x} + R\left(\int_0^1 u^2 dx\right) u^2\right\}dx. \tag{5.19}$$

Integration by parts and substitution of the boundary conditions **gives:**

$$\frac{dV(u)}{dt} = -2\int_0^1 \left\{-u^2 + \frac{1}{R}\left(\frac{\partial u}{\partial x}\right)^2 + R\left(\int_0^1 u^2 dx\right) u^2\right\}dx \tag{5.20}$$

Applying the integral inequality (5.10), the fact that $\int_0^1 u^2 dx \geqq 0$ and $R > 0$ gives:

$$\frac{dV(u)}{dt} \leqq -2\int_0^1 \left(\frac{\pi^2}{R} - 1\right) u^2\, dx. \tag{5.21}$$

Thus the modified nonlinear system also has an asymptotically stable equilibrium solution u = 0 for

$$0 < R < \pi^2 \tag{5.22}$$

This verifies the previously obtained result in [1] suppressing the $\frac{\partial u_2}{\partial t}$ term.

(c) Next it will be shown that a stability analysis of the nonlinear system (5.1) can be made with the techniques described above without making the modification under (b), a result not obtained in [1]. Consider the system as given by (5.1) and let R be constant. Since R is constant take as $V(\underline{u})$ functional:

$$V(\underline{u}) = \langle \underline{u}, \underline{u} \rangle = \int_0^1 (u_1^2 + u_2^2)\, dx \tag{5.23}$$

The time derivative of $V(\underline{u})$ becomes after substituting (5.1):

$$\frac{dV(\underline{u})}{dt} = -2 \int_0^1 [-u_1^2 - \frac{1}{R} u_1 \frac{\partial^2 u_1}{\partial x^2} - u_1^2 u_2 + \frac{1}{R} u_2^2 + u_2 \int_0^1 u_1^2 dx + u_1 \frac{\partial u_1^2}{\partial x}]\, dx \tag{5.24}$$

Integration by parts and substitution of the boundary conditions and inequality (5.10) gives:

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 [(\frac{\pi^2}{R} -1)\, u_1^2 + \frac{1}{R} u_2^2 - u_2(u_1^2 - \int_0^1 u_1^2 dx)]dx. \tag{5.25}$$

The inequality (5.25) can also by written as:

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 [(\frac{\pi^2}{R} -1)u_1^2 + \frac{1}{R} u_2^2]dx \left[1 - \frac{\int_0^1 u_2(u_1^2 - \int_0^1 u_1^2\, dx)dx}{\int_0^1 [(\frac{\pi^2}{R} -1)u_1^2 + \frac{1}{R} u_2^2]\, dx} \right] \tag{5.26}$$

and certainly:

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_2^1 [(\frac{\pi^2}{R} -1)u_1^2 + \frac{1}{R} u_2^2]dx \left[1 - \frac{\int_0^1 |u_2(u_1^2 + \int_0^1 u_1^2 dx)|\, dx}{\int_0^1 [(\frac{\pi^2}{R} -1)\, u_1^2 + \frac{1}{R} u_2^2]dx} \right] \tag{5.27}$$

Repeated application of the Buniakovsky - Schwartz inequality and the fact that $\int_0^1 u_1^2dx \leqq ||\underline{u}||^2$ and $\int_0^1 u_2^2dx \leqq ||\underline{u}||^2$ yields from (5.27):

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 [(\frac{\pi^2}{R} -1)u_1^2 + \frac{1}{R} u_2^2]dx \left[1- \frac{2||\underline{u}||^3}{\min(\frac{\pi^2}{R} -1, \frac{1}{R})\ ||\underline{u}||^2}\right] \quad (5.28)$$

or

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 [(\frac{\pi^2}{R} -1)u_1^2 + \frac{1}{R} u_2^2]dx \left[1- \frac{2}{\min(\frac{\pi^2}{R} -1,\frac{1}{R})} ||\underline{u}||\right]. \quad (5.29)$$

A sufficient condition for the trivial solution $\underline{u}=\underline{0}$ of (5.1) to be asymptotically stable is thus that

$$0 < R < \pi^2 \quad (5.30)$$

and the disturbances be bounded in norm by

$$||\underline{u}|| < \frac{1}{2} \min \ (\frac{\pi^2}{R} -1, \frac{1}{R}) \ . \quad (5.31)$$

Notice that (5.30 - 5.31) requires the linear approximation of the system (5.1) to be asymptotically stable, which should be expected.

The conditions (5.30 - 5.31) are derived without any prior knowledge about the solutions, an important advantage of the Lyapunov approach. However an examination of (5.1) shows that u_2 is independent of x. Consequently u_2 can be taken outside the integral sign in (5.26). Upon integration there follows

$$\frac{dV(\underline{u})}{dt} \leqq -2 \int_0^1 [(\frac{\pi^2}{R} -1) \ u_1^2 + \frac{1}{R} u_2^2] \ dx \quad (5.26a)$$

A sufficient condition for the asymptotic stability of the trivial solution $\underline{u} = \underline{0}$ of (5.1) is thus $0 < R < \pi^2$, the same as the linear approximation case.

This example shows that the Lyapunov stability theory not only enables us to verify the results of[1] but to extend these results considerably.

Example 2. Next we consider a second example from [1]

$$\frac{\partial u}{\partial t} - (x^2 + \frac{2}{\sqrt{R}})\, u - \frac{2}{\sqrt{R}}\, x\, \frac{\partial u}{\partial x} - \frac{1}{R}\,\frac{\partial u^2}{\partial x} + R^2 \left[\int_0^1 u^2 dx\right] u = 0 \qquad (5.32)$$

$0 \leqq x \leqq 1$, boundary conditions $u(0) = u(1) = 0$ and R a positive constant. The linearized system is given by:

$$\frac{\partial u}{\partial t} - (x^2 + \frac{2}{\sqrt{R}})\, u - \frac{2}{\sqrt{R}}\, x\, \frac{\partial u}{\partial u} - \frac{1}{R}\,\frac{\partial^2 u}{\partial x^2} = 0 \qquad (5.33)$$

$0 \leqq x \leqq 1$ and boundary conditions $u(0) = u(1) = 0$. The linear differential operator L of (5.33) is not self-adjoint for the inner product

$$\langle v,u\rangle = \int_0^1 vu\, dx. \qquad (5.34)$$

Hence on the basis of the preceding, we might expect that the stability condition derived with this inner product would not be the best possible one. To illustrate this take as the functional V(u):

$$V(u) = \int_0^1 u^2 dx.$$

The time derivative follows after integration by parts as

$$\frac{dV(u)}{dt} = -2 \int_0^1 \{\frac{1}{R}\,(\frac{\partial u}{\partial x})^2 - (\frac{1}{\sqrt{R}} + x^2)\, u^2\} dx. \qquad (5.36)$$

Applying the inequality (5.10) gives

$$\frac{dV(u)}{dt} \leqq -2 \int_0^1 (\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - x^2) u^2 dx. \qquad (3.37)$$

And a sufficient condition for asymptotic stability of the solution u=0 is:

$$\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \max_{x \varepsilon [0,1]} \{x^2\} > 0 \tag{5.38}$$

or

$$0 < R < \frac{1}{2} (1 + 2\pi^2 - \sqrt{1 + 4\pi^2}). \tag{5.39}$$

One can improve greatly on condition (5.39) as an evaluation of the eigenvalues of L suggests (see [1]) by observing that L is equivalent to the operator L_e as given by:

$$L_e u \equiv - \frac{1}{w(x)} \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) + q(x)u \tag{5.40}$$

with

$$p = \exp \sqrt{R}\, x^2$$

$$w = R \exp \sqrt{R}\, x^2$$

$$q = -\frac{2}{\sqrt{R}} - x^2$$

and boundary conditions u(0) = u(1) = 0.

The functional V(u) will be taken as

$$V(u) = < u,u > = \int_0^1 u^2 \; w(x)dx. \tag{5.41}$$

After integration by parts and substitution of the boundary conditions its time derivatives becomes:

$$\frac{dV(u)}{dt} \leqq -2 \int_0^1 \{e^{\sqrt{R}\, x^2} (\frac{\partial u}{\partial x})^2 - Re^{\sqrt{R}\, x^2} (x^2 + \frac{2}{\sqrt{R}}) u^2\} dx. \tag{5.42}$$

Here the inequality (5.10) can again be applied, now however to $e^{\frac{1}{2}\sqrt{R}\, x^2} u$, rather than to u. Substituting the result into (5.42) gives:

$$\frac{dV(u)}{dt} \leqq -2 \int_0^1 \left(\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} \right) u^2 \, w(x) \, dx. \tag{5.43}$$

The condition for asymptotic stability of the equilibrium solution u=0 of (5.33) follows from (5.43) as

$$0 < R < \pi^4. \tag{5.44}$$

This condition is identical to that found by evaluating the eigenvalues of L as should be expected. Further, comparison of (5.39) and (5.44) shows the intrinsic dependence of the stability condition on the V(u) functional chosen. The modification of the functional V(u) based on a transformation of the differential operator L in this case results in a significant improvement of the bounds of the system parameters to assure stability. However, as indicated in Example 1, the results are even more important when dealing with nonlinear systems. To show this consider the nonlinear system (5.32). Take as V(u) functional (5.41), thus

$$V(u) = \int_0^1 u^2 \, w(x) \, dx. \tag{5.45}$$

The time derivative of V(u) becomes:

$$\frac{dV(u)}{dt} = -2 \int_0^1 \left\{ e^{\sqrt{R}\, x^2} \left(\frac{\partial u}{\partial x}\right)^2 - Re^{\sqrt{R}\, x^2} \left(x^2 + \frac{2}{\sqrt{R}}\right) u^2 + \frac{1}{2} R\, e^{\sqrt{R}\, x^2} u \frac{\partial u^2}{\partial x} + \right.$$

$$\left. + R^3 e^{\sqrt{R}\, x^2} \left[\int_0^1 u^2 dx \right] u^2 \right\} dx. \tag{5.46}$$

Integration by parts, substitution of the boundary conditions and the inequality (5.10) together with the fact $\int_0^1 u^2 dx \geqq 0$ and $R > 0$ give

$$\frac{dV(u)}{dt} \leqq -2 \int_0^1 \left[\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \frac{2}{3} \sqrt{R}\, x \, u \right] u^2 w(x) \, dx. \tag{5.47}$$

Since $0 \leqq x \leqq 1$, $\frac{dV(u)}{dt}$ will certainly be negative definite for

$$\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \frac{2}{3}\sqrt{R}\,|u| > 0 \tag{5.48}$$

for all $x \varepsilon [0,1]$.

Hence the equilibrium solution u=0 of (5.32) will be asymptotically stable for all disturbances bounded by

$$\max_{x \varepsilon [0,1]} |u| < \frac{3}{2}\frac{1}{R}\left(\frac{\pi^2}{\sqrt{R}} - 1\right). \tag{5.49}$$

This result is again similar to that obtained by Eckhaus [1], however the above procedure enables one to obtain it in a straightforward way without making many complicated calculations as is the case when using asymptotic expansions.

6. Conclusions

Some recent results in the stability theory of partial differential equations have been obtained by means of Lyapunov stability theory. Although the application of this theory might not be as unified as the approach outlined by Eckhaus [1] it is felt that with some sophistication many results can be obtained in a less cumbersome way. It is also gratifying that many of the results of Eckhaus are verified using Lyapunov stability theory. Among the useful tools that can be applied we have demonstrated a transformation of the differential operators and the use of integral inequalities.

The feasibility of Lyapunov stability theory in the analysis of solutions to partial differential equations is clearly established by the ease with which sufficient conditions for stability are determined for the linear and nonlinear systems discussed in the examples.

7. References

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