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PLANE-STRESS ANALYSIS OF AN EDGE-STIFFENED RECTANGULAR PLATE SUBJECTED TO BOUNDARY LOADS, BOUNDARY DISPLACEMENTS, AND TEMPERATURE GRADIENTS

by Chuan-jui Lin and Charles Libove

Prepared by
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Page 59, equation (B36): Change

$$(-1)^n a_n'' \quad \text{to} \quad (-1)^m a_n''$$

Page 64, equation (B61): Change

$$(-1)^n B_n'' \quad \text{to} \quad (-1)^m B_n''$$

Page 177: The equation for $\beta_1(n)$ should read as follows:

$$\beta_1(n) = AE \frac{\lambda_1}{2} \sum_{m=1}^{M} \frac{(-1)^m m^2}{[m^2 + n^2\beta^2]^2}$$

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ABSTRACT

A plane-stress analysis is presented for an edge-stiffened isotropic or orthotropic elastic rectangular plate subjected to prescribed loads and prescribed temperature distributions. Along one or more of the edges the stiffeners are assumed to rigidly maintain a prescribed shape (e.g., straight), while the stiffeners along the remaining edges are assumed to have negligible flexural stiffness. All four stiffeners are assumed to be uniform and to possess finite axial stiffness. The plate edges are assumed to be integrally attached to the stiffeners along the originally straight centroidal axes of the stiffeners. This work represents a generalization of earlier studies in which only boundary loadings (rather than boundary displacements) were prescribed.

The analysis is by means of Fourier series. Numerical results are presented for the specific cases of an isotropic square stiffened plate with one edge, two opposite edges, or all four edges held straight. In all three of these cases the plate was assumed to have a pillow-shaped temperature distribution. In the second case stretching forces parallel to the non-straight edges were also considered.

ACKNOWLEDGMENTS

This investigation was conducted at Syracuse University under Grant NsG-385 from the National Aeronautics and Space Administration. The computational portions of this work were also supported, in part, by the National Science Foundation under Grant GP-1137.

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INTRODUCTION

In reference 1 a plane-stress analysis, by means of Fourier series, was presented for an elastic rectangular plate with four edge stiffeners, subjected to any equilibrium system of boundary loads and any prescribed temperature distribution. The plate was assumed to be isotropic or orthotropic, with elastic constants independent of position, and, if orthotropic, with axes of elastic symmetry parallel to its edges. The stiffeners were assumed to have finite extensional stiffness but negligible bending stiffness, and the attachment between plate edge and adjoining stiffener was assumed to be along the originally straight centroidal axis of the stiffener. The stiffeners were assumed to be either uniform in cross section or tapered in such a way as to result in any prescribed variation of stiffener cross-sectional stress along the length of stiffeners. This structure is shown schematically in figure 1.

In this earlier work, just described, the boundary conditions were entirely those of prescribed normal and shearing loadings along the outer periphery of the stiffeners. The purpose of the present paper is to generalize this previous work by assuming that along one or more edges the boundary condition of prescribed normal loading is replaced by one of prescribed normal displacement - i.e. along one or more of the edges the stiffener is assumed to be rigidly held in a prescribed shape. We can imagine that such a stiffener has been endowed with infinite flexural stiffness after being bent to its prescribed shape, while its axial stiffness remains finite. Along such an edge, one may still prescribe a distribution of external normal forces along the stiffener, but only the resultant force T and moment M of this distribution are significant, since the stiffener shape has been assumed to be rigidly fixed. On the remaining edges the boundary conditions continue to be those of prescribed normal and shear loading applied to perfectly flexible stiffeners. Only the case of constant - area stiffeners is considered.

By this generalization the range of applicability of the analysis is widened to include cases in which, for example, certain edges of the plate are forced to remain straight. As an illustration, one can cite an interior bay of an airplane shear web, bounded by spar caps along two opposite edges (the top and bottom) and upright along the other two edges. Continuity of the plate across the uprights suggests that when the adjacent bays are identical to the one under consideration and similarly heated or loaded the plate edges along the uprights will tend to remain straight. Thus such a bay would correspond to an edge-stiffened rectangular plate with two opposite edges kept straight. Similarly, an interior skin panel of a multi-spar multi-rib wing could correspond to the case of all four edges held straight.

The most general results of the present analysis are in the form of equations for stresses in the case of shape, resultant external normal force, and resultant external moment prescribed along (a) one edge, (b) two opposite edges, and (c) all four edges. The use of these equations is described in detail. In addition, numerical examples are given corresponding to each of these cases. These numerical examples are for doubly symmetric square plates with one edge, two opposite edges, or all four edges held straight, and, for the most part, a prescribed "pillow-shaped" temperature distribution, i.e. a temperature distribution in which the stiffener temperatures are constant at one value and the plate experiences a temperature rise, relating to the stiffeners, which varies as a half sine wave in both directions across the plate. For the case of two opposite edges held straight on overall tension loading is also considered which stretches the plate in the direction of the unconstrained edges, while the constrained edges remain straight and parallel to each other but move apart.

For the sake of simplicity, the term "temperature distribution" has been and will often again be used in this paper. It should be understood, however, that what is meant is a distribution of temperature rise with respect to some datum temperature distribution at which the structure is assumed to be stress-free in the absence of applied loads. Usually, the datum temperature distribution is a uniform one, and then it becomes correct to speak of thermal stresses due to a temperature distribution rather than to a temperature-rise distribution.

The structure, loading, boundary conditions, and results are described in more detail in the following sections. The symbols are defined where they are first used, and the definitions are also compiled in appendix A for convenience of reference. Those details of the analysis not essential for the understanding and use of the results are given in appendixes B through E.

DETAILED DESCRIPTION OF STRUCTURE

Geometry and Coordinate System

The combination of the plate and edge stiffeners is as shown schematically in figure 1. The plate has a length a and a width b. The position of any point in the plate is given by its coordinates x and y in a Cartesian coordinate system whose origin is at a corner of the plate and whose axes coincide with two adjacent edges, as shown in the figure. The cross-sectional areas of the stiffeners are denoted by A_1, A_2, A_3 and A_4 for the stiffeners located at $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively. In the analysis the stiffener axes are assumed to coincide with the plate edges, but in figure 1 the stiffener axes are shown slightly offset from the plate edges for clarity.

Thermal Strains

We assume that a datum temperature distribution exists for which the unloaded structure is free of internal stress. When the structure is in this datum condition the strains will be considered zero. For any other temperature distribution, those strains that would be produced if the thermal expansions of every infinitesimal element were permitted to occur without restraint from neighboring elements will be called the "thermal strains." The temperature distribution and coefficients of expansion are assumed to be known throughout the stiffeners and plate, hence the thermal strains are assumed to be everywhere known.

The notation for thermal strains is indicated in figure 2 and is as follows: $e_1(y)$, $e_2(y)$, $e_3(x)$ and $e_4(x)$ are the thermal strains in the stiffeners located at $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively; $e_x(x,y)$ and $e_y(x,y)$ are the thermal strains in the plate in x and y directions, respectively. The strains are positive for elongation. Because the analysis allows for an orthotropic plate, e_x and e_y need not equal, but since the x and y axes are parallel to the directions of elastic symmetry, there is no thermal shear strain e_{xy}.

Stress-Strain Relationships and Elastic Constants

The notation for the internal forces is indicated in figure 3. $P_1(y)$, $P_2(y)$, $P_3(x)$, $P_4(x)$ denote the cross-sectional tensions and $\varepsilon_1(y)$, $\varepsilon_2(y)$, $\varepsilon_3(x)$, $\varepsilon_4(x)$ the total strains (thermal plus elastic) in the stiffeners located at $x = 0$, $x = a$, $y = 0$, $y = b$, respectively. The plate stress resultants (force per unit length) are represented by $N_x(x,y)$ and $N_y(x,y)$ for normal stress and $N_{xy}(x,y)$ for shear stress, as shown in figure 3. The corresponding total strains are symbolized by $\varepsilon_x(x,y)$, $\varepsilon_y(x,y)$ and $\gamma_{xy}(x,y)$.

In terms of this notation, the assumed stress-strain relations for the stiffeners are

$$\varepsilon_i = e_i + \frac{P_i}{A_i E_i} \qquad (i = 1,2,3,4) \tag{1}$$

with Young's moduli E_1 and E_2 independent of y, E_3 and E_4 independent of x. The stress-strain relations of the plate are taken as:

$$\varepsilon_x = e_x + C_1 N_x - C_3 N_y$$

$$\varepsilon_y = e_y + C_2 N_y - C_3 N_x \qquad (2)$$

$$\gamma_{xy} = C_4 N_{xy}$$

where the elastic constants C_1, C_2, C_3 and C_4 are independent of x and y. If the plate is homogeneous and isotropic, with thickness h, Young's modulus E, and Poisson's ratio ν then

$$C_1 = C_2 = (Eh)^{-1}$$

$$C_3 = \nu(Eh)^{-1} \qquad (3)$$

$$C_4 = 2(1+\nu)(Eh)^{-1}$$

The assumption that the elastic constants are independent of spatial coordinates also implies the assumption that they are independent of temperature.

Boundary Conditions

Figure 1 shows the boundary conditions purely of prescribed loading which were employed in reference 1. This loading consists of end forces $P_1(0)$, $P_1(b)$ etc. applied to the centroids of the end cross sections of the stiffeners, and distributed external shear flows $q_1(y)$, $q_2(y)$, $q_3(x)$, $q_4(x)$ and distributed external normal forces

$N_1(y)$, $N_2(y)$, $N_3(x)$, $N_4(x)$ applied to the stiffeners. The distributed loadings have the dimensions of force per unit length. The shear flows are assumed to be acting along the centroidal axes of the stiffeners, although for clarity they are shown displaced from these axes in figure 1. Because the edge stiffeners have zero bending stiffness, the normal force is transmitted directly through them into the edge of the plate.

In the present paper the boundary condition of prescribed normal-force distribution is replaced by that of prescribed and rigidly maintained stiffener shape and prescribed external normal-force resultants along one edge, the opposite edges, or all four edges. (Along the remaining edges two stiffeners are still assumed to be perfectly flexible.) These cases are shown schematically in figure 4, in which the cross-hatched stiffeners are those whose shapes are rigidly prescribed. T_1 through T_4 are the prescribed external normal force resultants, and M_1 through M_4 the prescribed external moment resultants, acting on these stiffeners. Letting $u(x,y)$ and $v(x,y)$ denote the displacement components in the x and y directions, respectively, we can describe the boundary condition of prescribed shape mathematically as follows for case (a) of figure 4:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} \quad \text{is a prescribed function of } y.$$

Similarly, the shape boundary conditions for case (b) are:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} \text{ and } \left(\frac{\partial^2 u}{\partial y^2}\right)_{x=a} \quad \text{are prescribed functions of } y.$$

And those for case (c) are:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} \text{ and } \left(\frac{\partial^2 u}{\partial y^2}\right)_{x=a} \text{ are prescribed functions of } y \text{ ;}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} \text{ and } \left(\frac{\partial^2 v}{\partial x^2}\right)_{y=b} \text{ are prescribed functions of } x.$$

Note that the prescribed shape of an edge is described by means of its curvatures. If an edge is held straight the prescribed curvatures are zero.

The boundary condition of prescribed shape along an edge can be achieved, hypothetically, by forcibly bending the edge stiffener (in its attached condition) to the prescribed curvatures, and then endowing it with infinite flexural stiffness while its axial stiffness remains finite. If the now flexurally-rigid stiffener is free of all external forces, the values of T and M shown in figure 4 for that stiffener became zero; the system of internal normal forces acting between that stiffener and rest of the structure will then constitute a self-equilibrating system, and a mathematical statement to that effect must be included in the analysis as a boundary condition. On the other hand, if the stiffener is not completely free of external force then the internal normal forces along the edge of the stiffener plus the external T and M applied to that stiffener must together be self-equilibrating.

Of course, the total system of external forces must at the outset constitute an equilibrium system.

Corner Conditions

The stiffeners will be assumed to be hinged where they meet at the corners. This assumption is really superfluous in the case of figure 4a or 4b, for in these cases one of the two stiffeners meeting at every corner is a perfectly flexible one, incapable of developing any bending moment. The assumption is significant only in the case of 4c, where at each corner two rigid stiffeners meet.

If the loading and temperature distribution for the case of figure 4c are such that there is no tendency for the stiffeners to undergo relative rotation at the corners (as was true in the numerical examples considered in this paper), then, of course, the plate stresses and stiffener tensions are the same for rigid joints as for hinged joints.

FOURIER SERIES EXPANSIONS FOR PRESCRIBED BOUNDARY LOADS, PRESCRIBED BOUNDARY CURVATURES, AND PRESCRIBED THERMAL STRAINS

The results of the present analysis, to be discussed shortly, consist of formulas for the stiffener and plate stresses in terms of the given loading, the given boundary curvatures, and the known thermal strains. However, these prescribed quantities do not appear explicitly in these formulas; it is rather the Fourier coefficients of these quantities which are required. In anticipation of this requirement it is assumed that the given quantities can be expanded as Fourier series, with known coefficients, in the forms given below.

Prescribed Boundary Loadings

When the prescribed normal-force distributions $N_1(y)$, $N_2(y)$, $N_3(x)$, $N_4(x)$ exist, that is when their existence is not pre-empted by boundary conditions of prescribed shape, then they will be assumed known in the form of the following series:

$$
\begin{aligned}
N_1(y) &= \sum_{n=1}^{N} B'_n \sin \frac{n\pi y}{b} && (0 < y < b) \\
N_2(y) &= \sum_{n=1}^{N} B''_n \sin \frac{n\pi y}{b} && (0 < y < b) \qquad (4) \\
N_3(x) &= \sum_{m=1}^{M} B'''_m \sin \frac{m\pi x}{a} && (0 < x < a) \\
N_4(x) &= \sum_{m=1}^{M} B''''_m \sin \frac{m\pi x}{a} && (0 < x < a)
\end{aligned}
$$

where

$$B'_n = \frac{2}{b} \int_0^b N_1(y) \sin \frac{n\pi y}{b} \, dy, \quad \text{etc.} \tag{5}$$

And the shear flows will be assumed to be given in the following form:

$$
\begin{aligned}
q_1(y) &= \sum_{n=0}^{N} Q'_n \cos \frac{n\pi y}{b} \\
q_2(y) &= \sum_{n=0}^{N} Q''_n \cos \frac{n\pi y}{b} \\
q_3(x) &= \sum_{m=0}^{M} Q'''_m \cos \frac{m\pi x}{a} \\
q_4(x) &= \sum_{m=0}^{M} Q''''_m \cos \frac{m\pi x}{a}
\end{aligned} \tag{6}
$$

where

$$Q'_n = \frac{2-\delta_{n0}}{b} \int_0^b q_1(y) \cos \frac{n\pi y}{b} \, dy, \quad \text{etc.} \tag{7}$$

and δ_{n0} is Kronecker's delta.

When any of the loadings $N_1(y)$, $N_2(y)$, $N_3(x)$, or $N_4(x)$ exists, it is transmitted, through the assumedly perfectly flexible stiffener, into the plate. Thus the corresponding Fourier coefficients B'_n, B''_n, B'''_m, or B''''_m describe not only the externally applied normal loading but also the internal distributed tension $N_x(0,y)$, $N_x(a,y)$, $N_y(x,0)$, or $N_y(x,b)$, respectively, acting mutually between the stiffener and the edge of the plate.

Finite upper limits M and N are shown for the summation indexes in these (and subsequent) series in expectation of the fact that, for practical computational reasons, it will normally be necessary to use truncated rather than infinite series.

Prescribed Boundary Curvatures

When curvatures are prescribed, they will be assumed to be given by one or more of the following series, depending on which edges have the prescribed curvatures:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} = \sum_{n=1}^{N} K'_n \sin\frac{n\pi y}{b} \qquad (0 < y < b) \tag{8}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=a} = \sum_{n=1}^{N} K''_n \sin\frac{n\pi y}{b} \qquad (0 < y < b) \tag{9}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} = \sum_{m=1}^{M} K'''_m \sin\frac{m\pi x}{a} \qquad (0 < x < a) \tag{10}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=b} = \sum_{m=1}^{M} K''''_m \sin\frac{m\pi x}{a} \qquad (0 < x < a) \tag{11}$$

where

$$K'_n = \frac{2}{b}\int_0^b \left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} \sin\frac{n\pi y}{b}\, dy \;, \quad \text{etc.} \tag{12}$$

If an edge is held straight, then all the coefficients in the Fourier series expansion for the curvatures along that edge will, of course, be zero.

Prescribed Thermal Strains

If there are any discontinuities in thermal strain between the stiffeners and the edges of the plate, these will be represented by the following Fourier series with known coefficients:

$$
\begin{aligned}
e_1(y) - e_y(o,y) &= \sum_{n=1}^{N} T'_n \sin \frac{n\pi y}{b} \qquad (0 < y < b) \\
e_2(y) - e_y(a,y) &= \sum_{n=1}^{N} T''_n \sin \frac{n\pi y}{b} \qquad (0 < y < b) \\
e_3(x) - e_x(x,o) &= \sum_{m=1}^{M} T'''_m \sin \frac{m\pi x}{a} \qquad (0 < x < a) \\
e_4(x) - e_x(x,b) &= \sum_{m=1}^{M} T''''_m \sin \frac{m\pi x}{a} \qquad (0 < x < a)
\end{aligned}
\tag{13}
$$

where

$$T'_n = \frac{2}{b} \int_0^b [e_1(y) - e_y(0,y)] \sin \frac{n\pi y}{b}, \quad \text{etc.} \tag{14}$$

The quantity $\partial^2 e_y/\partial x^2 + \partial^2 e_x/\partial y^2$ is assumed known throughout the plate and representable by the following Fourier series in the open region $0 < x < a$, $0 < y < b$:

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} = \sum_{m=1}^{M} \sum_{n=1}^{N} T_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{15}$$

where

$$T_{mn} = \frac{4}{ab} \int_0^b \int_0^a [\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2}] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy \tag{16}$$

Integration by parts in the above equation gives the following alternate form which permits T_{mn} to be evaluated from the first partial derivatives of e_x and e_y instead of the second partial derivatives:

$$T_{mn} = -\frac{m\pi}{a}\frac{4}{ab}\int_0^b\int_0^a \frac{\partial e_y}{\partial x}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy$$

$$-\frac{n\pi}{b}\frac{4}{ab}\int_0^a\int_0^b \frac{\partial e_x}{\partial y}\cos\frac{n\pi y}{b}\sin\frac{m\pi x}{a}\,dydx \qquad (17)$$

Equation (17) may be used for piecewise continuous e_y or e_x with finite discontinuities provided that $\partial e_y/\partial x$ and $\partial e_x/\partial y$ are regarded to be infinite in the manner of the Dirac delta function at points of discontinuity. <u>If e_y and e_x are continuous</u> in the closed region $0 \leq x \leq a$, $0 \leq y \leq b$, further integration by parts gives

$$T_{mn} = -\frac{m\pi}{a}\frac{4}{ab}\int_0^b [e_y(a,y)\cos m\pi - e_y(o,y)]\sin\frac{n\pi y}{b}\,dy$$

$$-\left(\frac{m\pi}{a}\right)^2\frac{4}{ab}\int_0^b\int_0^a e_y(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy$$

$$-\frac{n\pi}{b}\frac{4}{ab}\int_0^a [e_x(x,b)\cos n\pi - e_x(x,o)]\sin\frac{m\pi x}{a}\,dx$$

$$-\left(\frac{n\pi}{b}\right)^2\frac{4}{ab}\int_0^a\int_0^b e_x(x,y)\sin\frac{n\pi y}{b}\sin\frac{m\pi x}{a}\,dydx \qquad (18)$$

Finally, the known quantities $(\partial e_y/\partial x)_{x=0}$, etc. are assumed to be expandable in the following Fourier series:

$$\left(\frac{\partial e_y}{\partial x}\right)_{x=0} = \sum_{n=1}^{N} V'_n \sin \frac{n\pi y}{b} \qquad (0 < y < b) \tag{19}$$

$$\left(\frac{\partial e_y}{\partial x}\right)_{x=a} = \sum_{n=1}^{N} V''_n \sin \frac{n\pi y}{b} \qquad (0 < y < b) \tag{20}$$

$$\left(\frac{\partial e_x}{\partial y}\right)_{y=0} = \sum_{m=1}^{M} V'''_m \sin \frac{m\pi x}{a} \qquad (0 < x < a \tag{21}$$

$$\left(\frac{\partial e_x}{\partial y}\right)_{y=b} = \sum_{m=1}^{M} V''''_m \sin \frac{m\pi x}{a} \qquad (0 < x < a) \tag{22}$$

where

$$V'_n = \frac{2}{b} \int_0^b \left(\frac{\partial e_y}{\partial x}\right)_{x=0} \sin \frac{n\pi y}{b}\, dy\ , \quad \text{etc.} \tag{23}$$

GENERAL RESULTS

The analyses are carried out in appendixes B through E. The results are of two kinds: (a) general results valid for any set of geometric, material and loading parameters, and (b) numerical results for specific cases. The former are presented in this section, the latter in the next section.

The general results of analyses are in the form of Fourier series for computing the plate stresses and stiffener tensions. In the subsections below the pertinent Fourier series for each of the various stresses will be pointed out. In the final subsection (Evaluation of Series Coefficients) the procedures for evaluating the Fourier coefficients in these series will be summarized. Enough detail will be given to enable the reader to make calculations based on the general results without having to study the derivations in the appendixes B through E.

Series for the Interior Plate Stresses and Interior Stiffener Tensions

The stress resultants N_x, N_y and N_{xy} at interior points of the plate are given by the double Fourier series in equations (B19), (B22) and (B25). The stiffener tensions $P_1(y)$, etc. are given by the single Fourier series in equations (B16) for all but the end sections of the stiffeners.

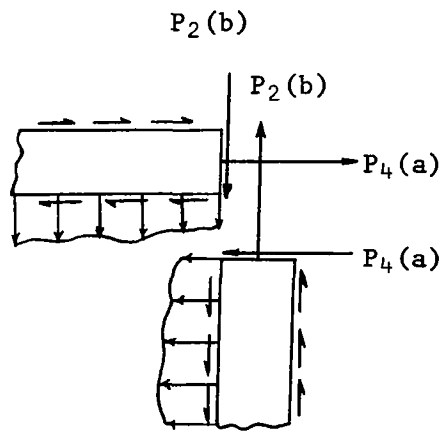
Series for Stresses along the Plate Edges

The shear stress resultant N_{xy} for points along the plate edges can be obtained from the same Fourier series, namely equation (B25), as for interior points. However the normal stresses along the plate edges require special single Fourier series, which will now be given. Equations (B20) and (B21) can be used for evaluating N_x along the edges $y = 0$ and $y = b$; equations (B23) and (B24) similarly give N_y along the edges $x = 0$ and $x = a$. The stresses N_x along the edges $x = 0$ and $x = a$ are given by the series for $N_1(y)$ and $N_2(y)$, respectively, in equations (4); and similarly N_y along the edges $y = 0$ and $y = b$ are given by $N_3(x)$ and $N_4(x)$ series, respectively, in equations (4).

The series which are referred to above for the normal stresses N_x and N_y along the plate edges are valid at all points except the corners. Special formulas to be used for the corner values of N_x and N_y will be given subsequently.

Series for the End Tensions of the Stiffeners

Where two perfectly flexible stiffeners meet, as at the point (a,o) of figure 4a, the tension at the end of each is merely equal to the externally applied load, such as $P_3(a)$ or $P_2(0)$. Where a perfectly rigid and perfectly flexible stiffener meet, as at the point (0,b) of figure 4a, the tension at the end of the rigid stiffener is again equal to the applied load, such as $P_1(b)$; but the tension at the end of the flexible stiffener, which is provided by the rigid stiffener, is an unknown rather than a prescribed quantity. Similarly, where two rigid stiffeners meet, as at the corner (a,b) of figure 4c, each stiffener provides an unknown tension to the end of the other stiffener. The tension which one stiffener exerts on the end cross-section of the adjacent stiffener is experienced as a shear force on the end section of the first stiffener. This mutual action between the stiffeners at the corner (a,b) of figure 4c is illustrated in the sketch below, in which the stiffeners are shown disjointed for the sake of clarity.



The equations for computing the unknown end tensions are as follows:

i) Equations (C14) and (C15) for the case of figure 4a.

ii) Equations (D13) through (D16) for the case of figure 4b.

iii) Equations (E11) through (E13) for the case of figure 4c.

Formulas for the Plate Normal Stresses at the Corners

None of the series for the plate stress resultants N_x and N_y thus far presented are valid at the corners of the plate. However, once the stiffener end tensions have been determined, these can be used to obtain the corner values of N_x and N_y. The procedure is as follows for the corner $x = 0$, $y = 0$: From equations (1) and (2) the stiffener strains at this corner are

$$\varepsilon_x = e_3(0) + P_3(0)/A_3E_3$$

$$\varepsilon_y = e_1(0) + P_1(0)/A_1E_1$$

while the plate strains are

$$\varepsilon_x = e_x(0,0) + C_1N_x(0,0) - C_3N_y(0,0)$$

$$\varepsilon_y = e_y(0,0) + C_2N_y(0,0) - C_3N_x(0,0)$$

Equating the stiffener strains and the corresponding plate strains (by virtue of the assumed continuity between stiffeners and plate) and solving the resulting equations simultaneously for $N_x(0,0)$ and $N_y(0,0)$, one obtains

$$N_x(0,0) = \frac{C_2[e_3(0)-e_x(0,0)] + C_3[e_1(0)-e_y(0,0)] + C_2P_3(0)/A_3E_3 + C_3P_1(0)/A_1E_1}{C_1C_2 - C_3^2}$$

$$N_y(0,0) = \frac{C_1[e_1(0)-e_y(0,0)] + C_3[e_3(0)-e_x(0,0)] + C_1P_1(0)/A_1E_1 + C_3P_3(0)/A_3E_3}{C_1C_2 - C_3^2} \tag{24}$$

In a similar manner, one obtains the following formulas for calculating the plate normal stresses at the corners (0,b), (a,0), and (a,b):

$$N_x(0,b) = \frac{C_2[e_4(0)-e_x(0,b)] + C_3[e_1(b)-e_y(0,b)] + C_2P_4(0)/A_4E_4 + C_3P_1(b)/A_1E_1}{C_1C_2 - C_3^2}$$

$$N_y(0,b) = \frac{C_1[e_1(b)-e_y(0,b)] + C_3[e_4(0)-e_x(0,b)] + C_1P_1(b)/A_1E_1 + C_3P_4(0)/A_4E_4}{C_1C_2 - C_3^2} \tag{25}$$

$$N_x(a,0) = \frac{C_2[e_3(a)-e_x(a,0)] + C_3[e_2(0)-e_y(a,0)] + C_2P_3(a)/A_3E_3 + C_3P_2(0)/A_2E_2}{C_1C_2 - C_3^2} \tag{26}$$

$$N_y(a,0) = \frac{C_1[e_2(0)-e_y(a,0)] + C_3[e_3(a)-e_x(a,0)] + C_1P_2(0)/A_2E_2 + C_3P_3(a)/A_3E_3}{C_1C_2 - C_3^2}$$

$$N_x(a,b) = \frac{C_2[e_4(a)-e_x(a,b)] + C_3[e_2(b)-e_y(a,b)] + C_2P_4(a)/A_4E_4 + C_3P_2(b)/A_2E_2}{C_1C_2 - C_3^2} \tag{27}$$

$$N_y(a,b) = \frac{C_1[e_2(b)-e_y(a,b)] + C_3[e_4(a)-e_x(a,b)] + C_1P_2(b)/A_2E_2 + C_3P_4(a)/A_4E_4}{C_1C_2 - C_3^2}$$

When the plate is isotropic and there is no discontinuity of thermal strains between stiffeners and plate at the corners, then the above formulas are reduced to

$$N_x(0,0) = \frac{Eh}{1-\nu^2}\left[\frac{P_3(0)}{A_3\ E_3} + \nu\,\frac{P_1(0)}{A_1\ E_1}\right]$$
$$N_y(0,0) = \frac{Eh}{1-\nu^2}\left[\frac{P_1(0)}{A_1\ E_1} + \nu\,\frac{P_3(0)}{A_3E_3}\right] \tag{28}$$

with similar expressions for the other three corners.

Evaluation of Series Coefficients

In order to use the series referred to in the above sections for numerical calculation of stresses, one must first evaluate the coefficients c_n', c_n'', g_m', g_m'', B_n' etc. (if these coefficients are unknown), c_{mn}, g_{mn}, j_{mn}, s_n', s_n'', s_m''', and s_m'''' appearing in them. The procedure for evaluating these coefficients will now be outlined in detail. It will be seen that the first four groups of coefficients, namely c_n', c_n'', g_m', g_m'', are the key to all the others.

i) <u>The case of figure 4a</u>. - The c_n', c_n'', g_m', and g_m'' are defined by the system of equations (C29) to (C32) and can be determined by solving these 2N + 2M equations simultaneously for the $\bar{c}_n'$, $\bar{c}_n''$, $\bar{g}_m'$, $\bar{g}_m''$ and noting the definitions in equations (B66). As an alternative, equations (C46) may be solved simultaneously for the $\bar{g}_m'$ and $\bar{g}_m''$; the $\bar{c}_n'$ and $\bar{c}_n''$ are then obtained directly from equations (C44) and (C45). The alternative is preferable because it requires the solution of only 2M simultaneous equations, regardless of how large a value is selected for N. With the c_n', c_n'', g_m', and g_m'' known, equations (C9) will give the values of B_n'.

If the structure,loading and thermal strains are symmetrical about the centerline $y = b/2$, considerable simplification results. The c_n', c_n'', g_m', and g_m'' are then defined by equations (C50) through (C52). Only the M equations (C51) need to be solved simultaneously. They give the $\bar{g}_m'$, after which the g_m'', c_n', and c_n'' are obtained directly from equations (C50) and (C53). Equations (C54) and (C55) may then be used to determine the B_n'. In this case the size of N again does not influence the number of equations that have to be solved simultaneously.

Once the c_n', c_n'', g_m', g_m'' and B_n' known, equations (B34), (B35), and (B57) through (B61) will furnish the remaining coefficients directly, for either the symmetrical or unsymmetrical case.

ii) <u>The case of figure 4b</u>. - The c_n', c_n'', g_m', g_m'' are defined by the system of equations (D28) through (D31) and can be determined by solving these 2N + 2M equations simultaneously for the $\bar{c}_n'$, $\bar{c}_n''$, $\bar{g}_m'$, and $\bar{g}_m''$. As an alternative, equations (D49) may be solved simultaneously for $\bar{g}_m'$ and $\bar{g}_m''$; the $\bar{c}_n'$ and $\bar{c}_n''$ are then obtained directly from equations (D42) and (D43). The alternative requires the solution of only 2M simultaneous equations, regardless how large a value is selected for N. With the c_n', c_n'', g_m', g_m'' known, equations (D9) wil furnish the values of B_n' and B_n''.

If the structure, loading and thermal strains are symmetrical about both centerlines, considerable simplification results. The g_m' and g_m'' are then defined by equations (D52) and the simultaneous system of (M+1)/2 equations (D57). The latter may be solved simultaneously for the $\bar{g}_m'$, after which the $\bar{c}_n'$, the $\bar{c}_n''$, B_n', B_n'' are obtained directly from equations (D56), (D53), and (D52). The size of N again does not influence the number of equations that have to be solved simultaneously.

With the c_n', c_n'', g_m', g_m'', B_n', and B_n'' known, equations (B34), (B35), and (B57) through (B61) will furnish the remaining coefficients directly, for either the symmetrical or unsymmetrical case.

iii) <u>The case of figure 4c</u>. - The B_n', B_n'', B_m''', B_m'''', c_n', c_n'', g_m', and g_m'' are defined by the system of equations (E23) through (E30) and can be determined by solving these 4N + 4M equations simultaneously for B_n', B_n'', B_m''', B_m'''', $\bar{c}_n'$, $\bar{c}_n''$, $\bar{g}_m'$, $\bar{g}_m''$ and noting the definitions in equations (B66).

For the case of a square plate (b=a) with structure, loading and thermal strains symmetrical about both centerlines (x = a/2 and y = b/2) and diagonals, considerable simplification results. The B_n', B_n'', B_m''', B_m'''', c_n', c_n'' g_m', and g_m'' are then defined by equations (E31), (E32), and (E33), and the M + 1 equations (E34) and (E35). Only equations (E34) and (E35) have to be solved simultaneously.

With these known, equations (B34), (B35), and (B57) through (B61) will furnish the remaining coefficients directly, for either the symmetrical or unsymmetrical case.

Limiting case of large stiffener area. - In the appendixes various limiting cases are considered in which some or all of the stiffener cross-sectional areas are allowed to approach infinity by comparison with the plate cross-sectional area. The results of these limiting cases will now be described. It will be seen that the calculation procedures for these limiting cases are much simpler than for the general case. Except for two of the limiting cases considered, it is no longer necessary to solve simultaneous equations in order to determine the Fourier coefficients in the series expansions for the stresses.

(a) The case of figure 4a: For this case all four stiffener cross-sectional areas were assumed to approach infinity simultaneously while maintaining constant ratios with each other. The results are contained in equations (C61) to (C64), which give explicit expressions for c_n'', g_m', g_m'' and c_n', correct to terms of the first degree in $1/(a^3E_{11}A_1E_1)$. More accurate results, correct to terms of the second degree in this quantity, are represented by equations (C66), (C67), (C68), and (C70).

(b) The case of figure 4b: For the configuration of figure 4b attention was restricted to the case in which structure, loading, and thermal strains are symmetrical about both centerlines ($x = a/2$, $y = b/2$), and five different types of limiting case were considered. These five cases are described below together with the results obtained form them.

(1) $A_3 = A_4 \to \infty$, $A_1 = A_2$ remaining finite. The quantities c_n', c_n'', g_m', and g_m'' are defined explicitly by equations (D52), (D59), and (D60).

(2) $A_3 = A_4 \to \infty$, followed by $A_1 = A_2 \to \infty$. The results for this iterated limiting process are represented by equations (D52), (D62), and (D64)

(3) $A_1 = A_2 \to \infty$, $A_3 = A_4$ remaining finite. Equations (D52), (D66), and (D68) result. In this case a system of simultaneous equations (D66) has to be solved.

(4) $A_1 = A_2 \to \infty$, followed by $A_3 = A_4 \to \infty$. Equations (D52), (D69), and (D70) are the result of this limiting process.

(5) $A_1 (= A_2)$ and $A_3 (= A_4)$ approaching infinity simultaneously while the ratio of A_1 to A_3 remains constant. The results are given by equations (D52), (D72) and (D73).

In the above-cited results, c_n', c_n'', g_m' and g_m'' are given correct to terms of the first degree in $1/(a^3 E_{11} A_1 E_1)$. Conditions (2), (4), and (5), which are physically identical, also turn out be mathematically identical, as an examination of the cited equations will show.

(c) The case of figure 4c: Here attention was restricted to the square case (b = a), with symmetry about both centerlines and diagonals, and a particular loading consisting only of prescribed thermal strains. The stiffener areas, all equal, were then allowed to approach infinity. In view of the highly specialized assumptions no general results will be cited for this case. The reader may refer to appendix E for a detailed description of the procedure. In this limiting case it is also necessary to solve simultaneous equations.

NUMERICAL RESULTS

The foregoing analytical results were used to obtain numerical stress data for twelve illustrative problems, ten of them thermal-stress problems involving non-uniform temperature distribution without any applied loads, and the other two prescribed-force problems corresponding to $T_1 = T_2 = T \neq 0$ in figure 4b, with all other loads vanishing and temperature uniform.

In all twelve problems the plate was square ($b = a$) and isotropic, with Young's modulus E, Poisson's ratio ν, and thickness h. In each problem the two x-wise stiffeners were assumed to be identical to each other (i.e. $A_3 = A_4$) and the y-wise stiffeners were similarly assumed to be identical (i.e. $A_1 = A_2$). Furthermore, in all but one of the problems the x-wise stiffeners were taken to be identical to the y-wise stiffeners; in presenting the results for those cases the symbol A will designate the common value of A_1, A_2, A_3 and A_4, and the symbol λ the common value of the area-ratio parameters λ_1 and λ_2, defined as follows: $\lambda_1 = 4ah/\pi^2 A_1$, $\lambda_2 = 4bh/\pi^2 A_3$. The stiffeners were also assumed to have the same Young's modulus as the plate.

In the thermal stress problems, the stiffeners were considered to be at a uniform temperature, while the plate was assumed to have a pillow-shaped temperature rise, relative to the stiffeners, of the form $\theta \sin(\pi x/a) \sin(\pi y/b)$; thus θ denotes the temperature rise of the plate center relative to the edges. The symbol α will denote the coefficient of thermal expansion of the plate material.

The table below summarizes the twelve problems for which calculations were made. The first column indicates for each problem which of the cases is being considered, that is, whether one edge, two edges, or all four edges are held straight. The second column indicates the loading condition. In this column "PSTD" stands for the pillow-shaped temperature distribution described above and connotes a purely thermal-stress problem, with no applied forces. The notation $T_1 = T_2 = T$ which appears for two of the problems associated with figure 4b denotes a pair of stretching loads applied in the x-direction to a structure of uniform temperature. The third column shows the value of Poisson's ratio used in the calculations; except for one calculation Poisson's ratio was taken as 0.3. The fourth and fifth columns give the values of the area-ratio parameters, defined as follows: $\lambda_1 = 4ah/\pi^2 A_1$, $\lambda_2 = 4bh/\pi^2 A_3$. The value of zero for these parameters refers to the limiting case in which the stiffener cross-sectional area approaches infinity by comparison with the plate cross-sectional area. Column 6 gives the equations used to compute the basic unknowns leading to the stresses. Column 7 tells in which figures of the present paper the results can be found. The results are in the form of dimensionless plots of plate stress and stiffener tension, represented by the solid curves of figures 6 through 17. The dashed curves which appear on some of the figures are selected results from reference 1 for the case in which all four stiffeners are perfectly flexible, included for the sake of comparison.

Additional information of interest concerning the calculations is appended in columns 8, 9, and 10. Columns 8 and 9 give the upper limits

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
No. of edges held straight	Loading	ν	λ_1	λ_2	Equations employed	Results	M	N	IBM 7074 computing time (min.)
One (fig.4a)	PSTD*	.3	2	2	(C50)-(C55)	Fig. 6	30	59	21
"	"	"	1	1	"	7	"	"	21
"	"	"	0	0	(C61)-(C64)	8	79	79	3.5
"	"	0	0	0	"	9	"	"	3.5
Two (fig.4b)	PSTD*	.3	2	2	(D52), (D53), (D56), (D57)	10	59	59	11
"	"	"	1	1	"	11	"	"	11
"	"	"	1	0	(D52), (D59), (D60)	12	"	"	2.5
"	"	"	0	0	(D52), (D62), (D64)	13	79	79	2.5
"	$T_1=T_2=T$	"	1	1	(D52), (D53), (D56), (D57)	14	59	59	10
"	"	"	0	0	(D52), (D62), (D64)	15	"	"	2.5
Four(fig. 4c)	PSTD*	"	1	1	(E34), (E35)	16	"	"	9
"	"	"	0	0	(E52)-(E56)	17	"	"	5.5

* PSTD stands for "pillow-shaped temperature distribution"

of the summation indexes employed in the assumed Fourier series for stress function and stiffener tensions; M pertains to the x-direction and N to y-direction. Only the highest M and N values used for each problem are shown. In general, calculations were also made for smaller M and N combinations in order that the convergence of the calculations could be observed. Column 10 shows the IBM 7074 computing time that was required to obtain, for the given M and N, all the results plotted in the respective figure. The stresses were computed at x/a and y/b intervals of 0.1. Because of symmetry it was sufficient to make calculations for only one-half or one-fourth of the structure.

Discussion of Numerical Results for the Thermal-Stress Problems

Figures 6 through 13, 16 and 17 present the computed results for the thermal stresses due to the pillow-shaped temperature distribution. The primary effect of assuming some of the stiffeners to be held straight is the creation of a running normal stress between these stiffeners and the adjacent edge of the plate. These stresses, which would be zero if the stiffeners were perfectly flexible, are depicted by $N_x(0,y)$ in figures 6a, 7a, 8a, 9a, 10a, 11a, 12a, 13a, 16a, and 17a, and by $N_y(x,0)$ in figures 16b and 17b. The maximum running normal stress between a rigid stiffener and adjacent plate edge is seen to range from about 35% to about 80% of the normal stresses produced in the middle of the plate.

An interesting discontinuity occurs in the mutual normal stress between a plate edge and an adjacent perfectly flexible stiffener where that stiffener meets a rigid stiffener, e.g. the stress $N_y(x,0)$ at the point $x = 0$ in figure 6b and the point $x = 0$ in figure 10b. This stress is zero everywhere along the edge of the plate except at the corner where the rigid and perfectly flexible stiffeners meet. At that point the plate stress must jump to a value consistent with the strains at the edges of the two meeting stiffeners. The strain at the end of the rigid stiffener is zero, but the strain at the end of the flexible stiffener is not zero. It is the latter strain, in conjunction with Poisson's ratio, that gives rise to the non-zero plate stress right at the corner.

Comparison of the dashed curves with the corresponding solid curves shows that a further effect of holding one or more edges straight is to increase at least one of the normal stresses at the plate center. In figure 17a, for example, the normal stress at the plate center is seen to be increased almost 80% as a result rigidizing the stiffeners. In figure 7a and 7b, it is seen that the effect of keeping the left side straight is to increase x-wise compressive stress and decrease the y-wise compressive stress at the plate center due to a pillow-shaped temperature rise. A similar result is shown in figures 11a and 11b when both the left and right sides are forced to remain straight.

On the other hand, comparison of dashed and solid curves for N_{xy} shows that the maximum shear stress associated with the x and y directions is generally reduced by the existence of one or more straight edges. In the case of all four edges held straight, the reduction is seen to be quite drastic (figs. 16c and 17c). The corner shear stress, in particular, is reduced to zero for this case, as one should expect. The non-zero corner shear stresses in the other cases are a consequence of the zero bending stiffness for at least one of the two stiffeners meeting at every corner. Finite stiffener bending stiffness plus rigid joints at the corners would, in an actual situation, tend to reduce the corner shear stresses virtually to zero.

In the case of perfectly flexible stiffeners (ref. 1), the stiffener tensions approach zero at the ends of the stiffeners. In the present case, however, a finite end tension is produced in a perfectly flexible stiffener where it meets a rigid stiffener, or in each of two rigid stiffeners where they meet. This is shown in figures 6f, 7f, 8f, 9f, 10e, 11e, 12e, 13e, 16d and 17d. However, the maximum tension in a stiffener still occurs away from the end.

As is to be expected, the ratio of stiffener cross-sectional area to plate cross-sectional area has a noticeable influence on the plate normal stresses. Increasing this ratio (i.e., diminishing λ) tends to increase the plate normal stresses; this can be seen, for example, by comparing the central value of N_y in figures 6b and 8b.

Discussion of Numerical Results for the Applied-Force Problems

In figures 14 and 15 are plotted the plate stresses and stiffener tensions produced by the pulling apart of two opposite stiffeners of infinite bending stiffness (fig. 4b with $T_1 = T_2 = T$ and all other loads and temperatures zero). From figure 14a and e or 15a and e it is seen that this loading produces a nearly uniform tensile stress in the direction of the stretching (note that the N_x scale in figure 14a does not start at zero). Deviations from uniformity are most pronounced near the ends ($x = 0$). The plate stresses in the direction transverse to the stretching are seen to be quite small (figs. 14b and 15b) except near the ends, where the Poisson contraction of the plate is partially suppressed (fig. 14b) or completely suppressed (fig. 15b) by the end stiffener.

For the case $\lambda \to 0$, the plate cross-sectional area becomes negligible compared to the stiffener cross-sectional areas, and one should therefore expect that the applied load T would be transmitted from one side to the other entirely through x-wise stiffeners, causing $P_3(x)/T$ to be uniform at the value 0.5. Figure 15e shows that this predicted behavior was very nearly achieved by the calculations, which led to values of 0.5 at $x = 0$ and .495 at $x = 0.5a$. The largest deviation from uniform stiffener tension occurred at $x \simeq 0.1a$, where a value of .483 was obtained as a result of what appears to be a Gibbs phenomenon. For the tension $P_1(y)$ in the other two stiffener (fig. 15d), the calculation led to zero, which is value one should expect for this limiting case.

The behavior of N_{xy} in the neighborhood of the plate corner is interesting. It seems to become infinite for all values of λ as the corner is approached (see figs. 14c and 15c). This singularity was not proved mathematically but the evidence for its existence is almost unmistakable from the computed values of $N_{xy}(0,0)$ as the upper summation limits M and N in the assumed series are increased. This is shown in figure 15f, where the computed dimensionless corner shear stress is plotted as a function of M on semi-log paper for the case $M = N$. The linearity of these graphs indicates rather convincingly that the corner shear stress would be infinite for the presumably exact solution ($M = N \rightarrow \infty$).

This behavior of the corner shear stress must be construed as a consequence of the assumption of perfect flexibility for one of the two stiffeners meeting at each corner. A similar behavior is obtained when all four stiffeners are perfectly flexible, for the case of a step-like temperature discontinuity between stiffeners and plate or tensions applied to the stiffener ends, as is noted in references 1, 2 and 3. In an actual situation, finite flexural stiffness in the stiffeners which were here assumed to have none would tend to eliminate the shear stress singularity at the plate corners.

APPLICABILITY OF RESULTS

There are many plane thermal-stress analyses in the literature for rectangular plates with free edges. In aircraft and spacecraft applications, however, the plate elements that one is concerned with are almost always attached to stiffening members (e.g., spar caps, rib caps, or shear-web uprights) along their edges. It was with such plate elements in mind that the work in references 1, 2, and 3 was done. In these references the stiffeners were assumed to have finite axial stiffness but negligible flexural stiffness and the boundary conditions were entirely those of prescribed load. Thus, while references 1, 2 and 3 represent a step toward more realistic detailed analysis of rectangular-plate plane-stress problems, they are not directly applicable to problems in which the boundary conditions are those of prescribed shape along one or more of the edges. The most obvious examples of such prescribed-shape boundary conditions arise when one considers plates that do not end at the stiffeners but are continuous across the stiffeners, forming a one-dimensional array of panels, as in a shear web, or a two-dimensional array, as in the cover of a multi-web multi-spar wing or the skin of a multi-ring multi-stringer fuselage. In these cases it may be reasonable to assume that an edge which is common to two adjacent panels is forced to remain straight. It is with such applications in mind that the present analysis and calculations were performed. The assumption of perfect flexibility for the stiffeners was retained along some edges; along the other edges this assumption was replaced by one of prescribed stiffener shapes plus prescribed resultants applied externally to the stiffeners of prescribed shape. Although the prescribed shape that one would use in most applications is that of straightness, the prescribed shape was left arbitrary in the present analysis for the sake of generality and because such generality entailed very little additional complication.

Figure 18 illustrates some of the types of structure to which the present paper and references 1, 2 and 3 may be applicable. Part a of figure 18 represents a single-bay structure made up of non-coplanar flat rectangular plates with stiffeners at their junctions and ends. The non-coplanarity of the sides suggests that there is little restraint against deformation of the plate edges. The assumption that any panel, such as 1234, is a plate with four perfectly flexible edge stiffeners may therefore be valid, and consequently the method of references 1, 2 and 3 may be applicable for the detailed stress analysis of such a panel.

Part b of figure 18 depicts a two-bay structure. If the loading and temperature distributions are approximately symmetrical about the central bulkhead or ring, then a panel such as 1234 may be considered as a plate bounded by the perfectly flexible stiffeners (12, 23, and 34) and one stiffener (41) that is forced to remain straight. Then appendix C and figures 4a, 6, 7, 8, and 9 of the present paper may be applicable.

Part c of figure 18 denotes a multi-bay wing. In the shear web of such a wing, an interior panel, such as 1234, may be approximately like a rectangular plate with two opposite edges held straight (14 and 23) and with perfectly flexible stiffeners along the other two edges (12 and

34). Appendix D and figures 4b, 10, 11, 12, 13 of the present paper may therefore be applicable to such a panel.

Part d of figure 18 represents the cover of a multi-spar multi-rib wing or a plate with integral waffle-like stiffening. An interior panel, such as 1234, may be approximately a plate with four edge held straight, and appendix E and figures 4c, 16, 17 of the present paper may be applicable to its analysis.

In the above cases, wherever a stiffener is common to two adjacent plate, such as stiffener 12 in figure 18a or stiffener 14 in 18b, its cross sectional area should be divided in a reasonable way between the two plates which it serves. For example, if in figures 18a, b and d the adjacent plates are nearly identical in geometry, loading, and temperature distribution, then it is reasonable to divide each shared stiffener equally between the two plates which it serves. The same can be said about stiffeners 14 and 23 in figure 18c if the adjacent panels of the shear are similar in geometry, heating, and loading. However, the two plates which share stiffener 34 in figure 18c (one in the shear web and one in the top cover) could be quite different. In such a case it might not be easy to decide on the effective cross-sectional area to be used for stiffener 34 when analyzing the plate 1234. When such ambiguities result in a significant uncertainty in the computed results, then an analysis of the present kind, based on isolating one plate and its surrounding stiffeners as a unit, might not be applicable. In the unlikely event that the cover plates are missing in figure 18c, or have negligible cross-sectional area, then the ambiguity referred to above would, of course, not exist.

The above remarks concern the applicability of the present analysis in general. Some brief comments are, perhaps, also in order on the applicability of the computed thermal-stress data plotted in figures 6 to 13, 16, and 17. The geometry and temperature distributions to be expected in actual design situations are, of course, manifold. It is therefore not suggested that the geometries and temperature distributions assumed in the thermal-stress calculations will correspond exactly to a broad spectrum of particular design problems. Nevertheless, it is felt that the calculations made and the results obtained serve two purposes. First, they attest to the feasibility of the calculation procedure, thereby encouraging its use for other specific geometries and temperature distributions which might correspond more closely to a particular application. Second, they provide qualitative information on the thermal plane stresses that may be encountered during aerodynamic (or other) heating as a result of a temperature lag between a thin plate and the more massive or better insulated stiffening elements to which it is attached. The pillow-shaped temperature distribution $\theta \sin (\pi x/a) \sin (\pi y/a)$ employed in the calculations is not too much different from the experimental temperatures obtained in a square plate that was heated by lamps and cooled by water circulating through hollow edge stiffeners (fig. 11 of ref. 4), although the experimental distribution is somewhat flatter in the central region of the plate and might be better represented by the temperature distributions used in reference 3. In the latter reference the temperatures were assumed constant in a central plateau region of the plate, zero at the stiffeners, and varying as one-quarter of a sine wave in the transition zone between the stiffeners and the central plateau.

In the thermal-stress calculations (fig. 6 to 13, 16, and 17), the rigid stiffeners were assumed to be entirely free of external restraint. That is, the external resultants, such as T_1 and T_2 in figure 4b, were assumed to be zero along with all the other force loadings. Thus, although certain edges were forced to remain straight, these edges were free to translate. In an actual situation, there might exist some restraint against this translation due to the presence of surrounding structure. The effect of such restraint would be to introduce non-zero values of the external resultants T_1, T_2, etc., with magnitudes depending upon the relative stiffness of the plate-stiffener combination and the surrounding structure. The stresses due to such externally developed restraining forces would have to be superimposed as corrections upon those shown in figures 6 to 13, 16 and 17. Figures 14 and 15 show the nature of these correction stresses for the case in which restraining forces $T_1 = T_2 = T$ are developed in the structure of figure 4b. From figures 14a and 15a, it can be seen that one effect of preventing the separation of the end stiffeners (through the development of a negative value of T) is to superimpose a nearly uniform compressive stress in the x-direction. From figures 14c and 15c, it is seen that another effect is, possibly, the introduction of high plate shear stresses near the corners. However, as mentioned earlier, the shear stress singularity at the plate corners would be eliminated in an actual structure by the presence of some flexural stiffness in the x-wise stiffeners.

To illustrate the use of data of the type shown in figures 14 and 15, let us consider the thermal-stress problem whose results are given in figure 11, but this time assume that constraining forces $T_1 = T_2 = T$ exist, which are of appropriate (negative) magnitude to prevent completely the overall thermal expansion of the structure in the x-direction. In order to determine T, it is necessary to determine the elongation Δ_1 due to thermal stress from the data of figure 11, the elongation Δ_2 due to T from the data of figure 14, and equate their sum to zero and solve the resulting equation for T. The two required elongations can be obtained by integration of the stiffener strains $\varepsilon_3(x)$ or of the plate strains $\varepsilon_x(x,y)$ along any line parallel to the x-axis. The stiffener strains are slightly more convenient in this case, because of the absence of the thermal contribution $e_3(x)$; they give rise to the following expressions for Δ_1 and Δ_2:

$$\Delta_1 = \int_0^a \frac{P_3(x)}{AE}\, dx = 2\theta\lambda\alpha a \int_0^{1/2} \frac{P_3(x)}{\theta\lambda AE\alpha}\, d\left(\frac{x}{a}\right)$$

$$\Delta_2 = \int_0^a \frac{P_3(x)}{AE}\, dx = 2\,\frac{Ta}{AE} \int_0^{1/2} \frac{P_3(x)}{T}\, d\left(\frac{x}{a}\right)$$

Using numerical integration (trapezoidal rule) in conjunction with figures 11e and 14e to evaluate the right-hand sides of these expressions, one obtains

$$\Delta_1 = 0.2393\ \theta\lambda\alpha a$$

$$\Delta_2 = 0.2185\ Ta/(AE)$$

The condition $\Delta_1 + \Delta_2 = 0$ then gives

$$T = -\ .4439\ \theta\alpha ahE$$

The plate and stiffener stresses due to this value of T can now be obtained from figure 14. Superposition of these stresses and the ones in figure 11 will give the stresses produced by the pillow-shaped temperature rise when the overall expansion of the plate in the x-direction is prevented. The results are shown by the dotted curve of figure 19. The solid curves are the data of figure 11, repeated for comparison. It is seen from figures 19a and e that the normal stresses in the x-direction are, of course, appreciably altered by the suppression of the overall thermal expansion in the x-direction. Figures 19b and d show only minor alterations in the normal stresses in the y-direction. Figure 19c shows that the shear stresses also are only slightly affected, except in the neighborhood of the corner (x and y < 0.1a), where the effect of the singularity in shear stress due to T is felt. Accepting this singularity as illusory for reasons already cited, it can be concluded that in this example the suppression of the x-wise expansion has a significant effect only on the x-wise normal stresses.

CONCLUDING REMARKS

Many detailed plane-stress analyses are available in the literature for rectangular plates with unstiffened edges, subjected to boundary forces or non-uniform temperature distributions. Rectangular plates with stiffened edges have been treated to a lesser extent in the literature, mainly in "shear-lag" analyses. These are generally restricted to uniform temperatures and unidirectional loading and are characterized by simplifying assumptions regarding the stresses or elastic constants.

Rectangular plates in practice generally have stiffening members along their edges and may be subjected to non-uniform temperature rises and multidirectional loading. In addition it may be important to have some of the detailed information about the stresses which is lost in the usual type of shear-lag analysis. Therefore in reference 1 a unified analysis of the edge-stiffened rectangular plate was presented, including both non-uniform temperature distributions and prescribed boundary loadings of a very general nature. The plate material was assumed to be homogeneous, linearly elastic, and orthotropic, but no further simplifying assumptions were introduced regarding the stress distributions or the elastic properties of the plate. The stiffeners were assumed to have finite extensional stiffness but negligible flexural stiffness. The boundary conditions considered were entirely those of prescribed running normal and shear loading along the outer periphery of the stiffeners and prescribed forces at the stiffener ends.

The present paper represents a generalization of reference 1. It permits the boundary condition of prescribed normal loading along some of the edges to be replaced by the boundary condition of prescribed shape of the stiffener axis. Generally, the prescribed shape that one would encounter in practice is that of straightness due to symmetry, continuity, or rigid fixation. However, for the sake of generality, and because it involved no great complication, the present analysis also permits prescribed edge shapes other than straight. Along the remaining edges the stiffeners have been assumed to have finite extensional stiffness but negligible bending stiffness, as in reference 1. Through this generalization of the boundary conditions, certain problems can be handled which are not directly solvable by the method of reference 1 alone.

The present method and that of reference 1 are based on Fourier series - double series for the plate stress function, single series for the stiffener tensions and certain other quantities. In some cases many terms of the series may be needed for sufficient accuracy, and therefore the feasibility of these methods depends on the availability of high-speed computers. This feasibility has been tested in the present paper through twelve specific numerical examples, solved with the aid of an IBM 7074 computer. The machine time required to obtain detailed stress surveys in one quadrant or in one half of the structure, depending on the type of symmetry, ranged from 2.5 minutes to 21 minutes in these examples.

The assumption of negligible bending stiffness for those stiffeners whose shape is not prescribed sometimes produces infinite plate shear stress at a corner where such a stiffener meets a stiffener of prescribed shape. This was noted in figures 14 and 15 of the present paper. A similar anomaly was observed in references 1, 2 and 3. Its elimination would require the incorporation of finite stiffener flexural stiffness into the analysis.

APPENDIX A

SYMBOLS

Remarks: (i) The subscript 1, 2, 3 or 4 on a symbol for a stiffener-related quantity identifies the stiffener location as $x = 0$, $x = a$, $y = 0$, or $y = b$, respectively. (ii) The Fourier coefficients of known quantities (loads, thermal strains), and combinations of such coefficients, are generally represented by capital letters, while the Fourier coefficients of initially unknown quantities (e.g., internal stresses) are denoted by small letters. (iii) The symbols for convenience represent the known quantities in illustrative problems are not listed here.

a	plate dimension in x direction; see figure 1.
a_{mn}	Fourier coefficients in series expansion for the stress function $F(x,y)$; see equations (B15), (B17) and (B50).
$a_n', a_n'', a_m''', a_m''''$	Fourier coefficients in series expansions for $F(0,y)$, $F(a,y)$, $F(x,0)$, $F(x,b)$ respectively; see equations (B13) and (B14).
A_1, A_2, A_3, A_4	stiffener cross-sectional areas.
A	common value of the above when all four stiffeners are identical and uniform.
b	plate dimension in y direction; see figure 1.
$B_n', B_n'', B_m''', B_m''''$	Fourier coefficients in series expansions for N_1, N_2, N_3, N_4 respectively; see equations (4) and (5).
B	a/b
c_{mn}	Fourier coefficients in series expansion for $N_y(x,y)$; see equations (B22) and (B35).

c'_n, c''_n — Fourier coefficients in series expansions for $N_y(0,y)$ and $N_y(a,y)$ respectively; see equations (B23) and (B24).

$\bar{c}'_n, \bar{c}''_n$ — $c'_n\ C_2\ n\pi/b$, $c''_n\ C_2\ n\pi/b$.

C_1, C_2, C_3, C_4 — plate compliances defined by equations (3).

C — A_3/A_1, if $A_1 = A_2$ and $A_3 = A_4$.

d_{mn} — Fourier coefficients in series expansion for $\partial^3F/\partial x^3$; see equations (B41) and (B46).

D_n — $(\gamma_n^{(2)})^2 - (\gamma_n^{(1)})^2$

D'_n — defined by equation (D11).

D''_n — defined by equation (D12).

e_{mn} — Fourier coefficients in series expansion for $\partial^4F/\partial x^4$; see equations (B26) and (B37).

$e_1(y), e_2(y), e_3(x), e_4(x)$ — stiffener thermal strains; see figure 2.

$e_x(x,y), e_y(x,y)$ — plate thermal strains; see figure 2.

E_{mn} — $C_2(m\pi/a)^4 + (C_4 - 2C_3)(m\pi/a)^2(n\pi/b)^2 + C_1(n\pi/b)^4$.

E_{11} — value of above with $m = 1$ and $n = 1$.

E_1, E_2, E_3, E_4 — Young's modulus for stiffeners.

E — Young's modulus for stiffeners and isotropic plate when all have the same Young's modulus.

$F(x,y)$ — stress function for plate; see equation (B4).

g_{mn} — Fourier coefficients in series expansion for $N_x(x,y)$; see equations (B19) and (B34).

g'_m, g''_m — Fourier coefficients in series expansions for $N_x(x,0)$ and $N_x(x,b)$ respectively; see equations (B20) and (B21).

$\bar{g}'_m, \bar{g}''_m$	$g'_m\ C_1\ m\pi/a$, $g''_m\ C_1\ m\pi/a$
h	thickness when plate is isotropic.
h_{mn}	Fourier coefficients in series expansions for $\partial^3F/\partial y^3$; see equation (B42) and (B47).
H_{mn}	$(1/E_{mn})(2/a)(n\pi/b)^3C_1 - (2/a)(b/n\pi)$
i	1, 2, 3, or 4.
i_{mn}	Fourier coefficients in series expansion $\partial^4F/\partial y^4$; see equations (B27) and (B38).
j_{mn}	Fourier coefficients in series expansion for N_{xy}; see equation (B25); see equation (B56) for value of j_{00}.
K_{mn}	combinations of known Fourier coefficients, defined by equation (B69).
$K'_n, K''_n, K'''_m, K''''_m$	Fourier coefficients in series expansions for prescribed boundary curvatures; see equations (8) to (12).
ℓ_{mn}	Fourier coefficients in series expansion for $(\partial^3F/\partial x^2\partial y)$; see equations (B43) and (B48).
m,n,p,q	summation indexes (integers).
M	upper limit on m, p, and i.
M_1, M_2, M_3, M_4	loading resultants; see figures 4.
n	summation index (integer).
N	upper limit on n and q.
$N_1(y), N_2(y), N_3(x), N_4(x)$	external running tensions, force per unit length; see figure 1.
N_x, N_y, N_{xy}	plate stress-resultants, force per unit length; see figure 3.
p	summation index (integer).
p_{mn}	Fourier coefficients in series expansion for $\partial^4F/\partial x^2\partial y^2$; see equations (B28) and (B39).

$P_1(y), P_2(y), P_3(x), P_4(x)$

stiffener cross-sectional tensions; see figure 3.

$P_1(0), P_1(b), P_2(0), P_2(b), P_3(0), P_3(a), P_4(0), P_4(a)$

stiffener end loads; see figure 1.

P, Q — integers appearing in sinusoidal temperature distribution expansion; see equation (C71).

q — summation index (integer).

$q_1(y), q_2(y), q_3(x), q_4(x)$

external shear-flow loadings; see figure 1.

$Q_n', Q_n'', Q_m''', Q_m''''$ — Fourier coefficients in series expansions for q_1, q_2, q_3, q_4 respectively; see equations (6) and (7).

$R_n', R_n'', R_m''', R_m''''$ — combinations of known Fourier coefficients, defined by equations (B68) and (B69).

$s_n', s_n'', s_m''', s_m''''$ — Fourier coefficients in series expansions for the stiffener cross sectional tensions; see equations (B16) and (B60).

$S_n^{(1)}, S_n^{(2)}, S_m^{(3)}, S_m^{(4)}$

completely known loading terms; defined by equations (C17) through (C20).

$S_n^{(5)}, S_n^{(6)}, S_m^{(7)}, S_m^{(8)}$

completely known loading terms; defined by equations (D18) through (D21).

$S_n', S_n'', S_m''', S_m''''$ — completely known loading terms; defined by equations (E15) through (E18).

$S_{mn}^{(1)}$ — known quantities; defined by equation (C21).

$S_{mn}^{(2)}$ — known quantities; defined by equation (D22).

$\bar{S}_m^{(3)}, \bar{S}_m^{(4)}$ — known loading terms, defined by equations (C58b) and (C59a).

$t'_n, t''_n, t'''_m, t''''_m$	Fourier coefficients in series expansions for the derivatives of the stiffener cross-sectional tensions; see equations (B29) and (B33).
T_{mn}	Fourier coefficients in series expansion for $\partial^2 e_y/\partial x^2 + \partial^2 e_x/\partial y^2$; see equation (15), also equations (16), (17) and (18).
$T'_n, T''_n, T'''_m, T''''_m$	Fourier coefficients in series expansions for thermal-strain discontinuities between stiffeners and plate edges; see equations (13) and (14).
T_1, T_2, T_3, T_4	loading resultants; see figure 4.
T_o	stiffener temperature for the sinusoidal plate temperature distribution.
u,v	x and y components of displacements in plate.
U_n	known quantities, defined by equation (C57a).
v	plate displacement component in y-direction.
V_{mp}	known quantities, defined by equation (C58a).
V'_{mp}	known quantities, defined by equation (D58a).
V''_{mn}	known quantities, defined by equation (D58c).
$V'_n, V''_n, V'''_m, V''''_m$	Fourier coefficients in series expansions for $(\partial e_y/\partial x)_{x=0}$ etc.; see equations (19) through (23).
w_{mn}	Fourier coefficients in series expansion for $(\partial^3 F/\partial x \partial y^2)$; see equations (B44) and (B49).
x,y	Cartesian coordinates; see figure 1
y	Cartesian coordinate; see figure 1
y', y''	dummy variables representing y
$Y'_n, Y''_n, Y'''_m, Y''''_m$	known loading terms; see equations (C70a),(C66a),(C67a), and (C68a).

Z'_n, Z''_m known quantities; defined by equations (D64a) and (D58b).

α coefficient of thermal expansion of plate and stiffeners in numerical examples.

$\alpha_1(n), \alpha_2(n), \alpha_3(m), \alpha_4(m)$

known quantities; defined by equations (B67).

$\beta_1(n), \beta_2(m)$ known quantities, defined by equations (B67).

$\gamma'_n, \gamma''_n, \gamma'''_n, \gamma''''_n$ known quantities, defined by equations (C22),(C23),(D23) and (C56d).

$\gamma_n^{(1)}, \gamma_n^{(2)}$ known quantities, given by equations (C10) and (D6).

$\Gamma'_m, \Gamma''_m, \Gamma'''_m$ known quantities, given by equations (E19) through (E21).

$\Gamma_m^{(1)}, \Gamma_m^{(2)}$ known quantities, given by equations (E25a) and (E25b).

$\Gamma''_{mp}, \Gamma'''_{mn}, \Gamma''''_{mn}$ known quantities, given by equation (C36).

$\Gamma_{mp}^{(2)}, \Gamma_{mn}^{(3)}, \Gamma_{mn}^{(4)}$ known quantities, given by equation (D35).

δ_n known loading terms, defined by equations (C11).

$\delta_n^{(1)}, \delta_n^{(2)}$ known loading terms, defined by equations (D7) and (D8).

$\delta'_n, \delta''_n, \delta'''_m, \delta''''_m$ known loading terms, defined by equations (E23a), (E24a), (E25c) and (E26a).

δ_{ij} Kronecker's delta, unity when both subscripts are equal, zero otherwise.

Δ_n completely known loading terms, defined by equation (C64a).

$\Delta'_n, \Delta''_n, \Delta'''_{mn}$ known quantities, defined by equations (C56a), (C56b) and (C56c).

$\Delta_n^{(1)}, \Delta_{pn}^{(2)}$ known quantities, defined by equations (D61b) and (D63a).

$\epsilon_x(x,y), \epsilon_y(x,y), \gamma_{xy}(x,y)$

plate total strains; see equations (2).

$\epsilon_1(y), \epsilon_2(y), \epsilon_3(x), \epsilon_4(x)$

stiffener total strains; see equations (1)

ζ_n	known quantities, defined by equations (D56b).
ζ_n', ζ_n''	known quantities, defined by equations (C34).
$\zeta_n^{(1)}, \zeta_n^{(2)}$	known quantities, defined by equations (D33).
η_{mn}', η_{mn}''	known quantities, defined by equations (C41).
$\eta_{mn}^{(1)}, \eta_{mn}^{(2)}$	known quantities, defined by equations (D39).
$\theta_{mp}', \theta_{mp}'', \theta_{mp}''', \theta_{mp}''''$	known quantities, given by equations (C40), (C42).
$\theta_{mp}^{(1)}, \theta_{mp}^{(2)}, \theta_{mp}^{(3)}, \theta_{mp}^{(4)}$	known quantities, given by equations (D38) and (D40).
$\theta_{mp}', \theta_{mn}''$	known quantities, defined by equations (D57a) and (D57b).
λ_1, λ_2	area-ratio parameter used in numerical example, defined by equations (C80)
λ	common value of λ_1 and λ_2 when they are equal.
$\Lambda_{mn}', \Lambda_{mn}''$	known quantities; defined by equations (E23b) and (E25d).
μ_{mn}', μ_{mn}''	known quantities, defined by equations (C12) and (E25e).
ν_{mn}', ν_{mn}''	known quantities, defined by equations (C13) and (E25f).
ν	Poisson's ratio for isotropic plate.
ξ_{mn}', ξ_{mn}''	known quantities, defined by equations (C35).
$\xi_{mn}^{(1)}, \xi_{mn}^{(2)}$	known quantities, defined by equations (D34).
ξ_{pn}	known quantities, defined by equations (D56c).
Ξ_n	known quantities, defined by equations (C56e).
Ξ_n', Ξ_n''	known quantities, defined by equations (C56f) and (D61a).
Π_{mn}	known quantities, defined by equations (E22).
ρ_m', ρ_m''	known quantities, defined by equations (C43).
$\rho_m^{(1)}, \rho_m^{(2)}$	known quantities, defined by equations (D41).

$\sigma'_{mp}, \sigma'''_{mn}, \sigma''''_{mn}$	known quantities, defined by equations (C37).
ϕ_n	known quantities, defined by equations (D56a).
$\phi'_n, \phi''_n, \phi'''_n, \phi''''_n$	known quantities, defined by equations (C33).
$\phi_n^{(1)}, \phi_n^{(2)}, \phi_n^{(3)}, \phi_n^{(4)}$	known quantities, defined by equations (D32).
Φ_n	$\phi'_n\phi''''_n - \phi''_n\phi'''_n$.
Φ'_n	$\phi_n^{(1)}\phi_n^{(4)} - \phi_n^{(2)}\phi_n^{(3)}$
$\Phi_n^{(1)}$	$1 - \Xi''_n$
ψ_{mn}	known quantities, defined by equations (C48).
ψ'_{mn}	known quantities, defined by equations (D51).
ψ_{mnp}	known quantities, defined by equations (C47).
ψ'_{mnp}	known quantities, defined by equations (D50).

APPENDIX B

ANALYSIS FOR THE CASE IN WHICH THE BOUNDARY CONDITIONS ARE ENTIRELY THOSE OF PRESCRIBED LOADING (ref. 1)

The case in which curvatures are prescribed along one or more edges can best be analyzed by making appropriate modifications in the basic analysis of reference 1, in which the boundary conditions are entirely those of prescribed loading. Therefore the analysis of reference 1 for the case of constant-area stiffeners is summarized in this appendix, and portions of it will be used as needed in the subsequent appendices when various cases of prescribed boundary curvature are considered.

Basic Equations

With $u(x,y)$ and $v(x,y)$ denoting the x- and y-components, respectively, of infinitesimal displacement, the strain-displacement relations for the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{B1}$$

Equstions (B1) imply the following compatibility condition on the strains

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \epsilon_y}{\partial x^2} = 0 \tag{B2}$$

The plate equilibrium equations, namely

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{B3}$$

imply the existence of a stress function $F(x,y)$ such that

$$N_x = \frac{\partial^2 F}{\partial y^2} \qquad N_y = \frac{\partial^2 F}{\partial x^2} \qquad N_{xy} = \frac{-\partial^2 F}{\partial x \partial y} \tag{B4}$$

Eliminating ϵ_x, ϵ_y and γ_{xy} in equation (B2) by means of the stress-strain relations, equations (2), and then N_x, N_y and N_{xy} through equations (B4), yields the following form of the compatibility equation:

$$C_2 \frac{\partial^4 F}{\partial x^4} + (C_4 - 2C_3) \frac{\partial^4 F}{\partial x^2 \partial y^2} + C_1 \frac{\partial^4 F}{\partial y^4} + \frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} = 0 \tag{B5}$$

Considering infinitesimal lengths of the stiffeners as free bodies, and utilizing the third of equations (B4) to express N_{xy} at the plate edges in terms of $F(x,y)$, one obtains the following equilibrium equations governing the longitudinal variations of the stiffener axial forces.

$$\left.\begin{aligned}
dP_1/dy - (\partial^2 F/\partial x \partial y)_{x=0} - q_1(y) &= 0 \\
dP_2/dy + (\partial^2 F/\partial x \partial y)_{x=a} + q_2(y) &= 0 \\
dP_3/dx - (\partial^2 F/\partial x \partial y)_{y=0} - q_3(x) &= 0 \\
dP_4/dx + (\partial^2 F/\partial x \partial y)_{y=b} + q_4(x) &= 0
\end{aligned}\right\} \tag{B6}$$

The stiffeners and plate edges, being integrally attached, must have equal longitudinal strains along all four edges. Using the stress-strain relations (eqs. (1) and (2)), the stress function defined by equations (B4), and the assumption of perfect stiffener flexibility (i.e., $(N_x)_{x=0} = N_1(y)$, etc.), these conditions lead to the following equations:

$$
\left.
\begin{aligned}
&\frac{P_1(y)}{A_1E_1} + [e_1(y) - e_y(0,y)] - C_2\left(\frac{\partial^2F}{\partial x^2}\right)_{x=0} + C_3N_1(y) = 0 \\
&\frac{P_2(y)}{A_2E_2} + [e_2(y) - e_y(a,y)] - C_2\left(\frac{\partial^2F}{\partial x^2}\right)_{x=a} + C_3N_2(y) = 0 \\
&\frac{P_3(x)}{A_3E_3} + [e_3(x) - e_x(x,0)] - C_1\left(\frac{\partial^2F}{\partial y^2}\right)_{y=0} + C_3N_3(x) = 0 \\
&\frac{P_4(x)}{A_4E_4} + [e_4(x) - e_x(x,b)] - C_1\left(\frac{\partial^2F}{\partial y^2}\right)_{y=b} + C_3N_4(x) = 0
\end{aligned}
\right\}
\qquad \text{(B7)}
$$

Boundary Values of Stress Function $F(x,y)$

The fact that the distributed boundary normal loadings $N_1(y)$, $N_2(y)$, $N_3(x)$, and $N_4(x)$ are transmitted directly to the plate means that the second derivative of $F(x,y)$ in the direction of the edge is known. Therefore two integrations will give the variation of F along each edge in terms of the unknown corner values and the known N_1, N_2, N_3, N_4. As an illustration, along the edge $x = 0$, $\partial^2F(0,y)/\partial y^2 = N_1(y)$, and two integrations lead to

$$
F(0,y) = F(0,0) + \frac{y}{b}\left[F(0,b) - F(0,0) - \int_0^b\int_0^{y'} N_1(y'')dy''dy'\right] + \int_0^y\int_0^{y'} N_1(y'')dy''dy' \qquad \text{(B8)}
$$

(Here F has been assumed continuous along the perimeter of the plate, including the corners.) Replacing $N_1(y'')$ by its series expansion, equation (4), and carrying out the integrations indicated in equation (B8), one obtains

$$F(0,y) = F(0,0) + \frac{y}{b}\,[F(0,b) - F(0,0)] - \sum_{n=1}^{N} B_n'\left(\frac{b}{n\pi}\right)^2 \sin\frac{n\pi y}{b} \tag{B9}$$

Similarly, the variation of F(x,y) along the other plate edges is as follows:

$$F(a,y) = F(a,0) + \frac{y}{b}[F(a,b) - F(a,0)] - \sum_{n=1}^{N} B_n''\left(\frac{b}{n\pi}\right)^2 \sin\frac{n\pi y}{b} \tag{B10}$$

$$F(x,0) = F(0,0) + \frac{x}{a}[F(a,0)-F(0,0)] - \sum_{m=1}^{M} B_m'''\left(\frac{a}{m\pi}\right)^2 \sin\frac{m\pi x}{a} \tag{B11}$$

$$F(x,b) = F(0,b) + \frac{x}{a}[F(a,b)-F(0,b)] - \sum_{m=1}^{M} B_m''''\left(\frac{a}{m\pi}\right)^2 \sin\frac{m\pi x}{a} \tag{B12}$$

For the later use it will be necessary to expand the boundary values of F in single Fourier series of the following form:

$$\left.\begin{aligned}
F(0,y) &= \sum_{n=1}^{N} a_n' \sin(n\pi y/b) && (0 < y < b)\\
F(a,y) &= \sum_{n=1}^{N} a_n'' \sin(n\pi y/b) && (0 < y < b)\\
F(x,0) &= \sum_{m=1}^{M} a_m''' \sin(m\pi x/a) && (0 < x < a)\\
F(x,b) &= \sum_{m=1}^{M} a_m'''' \sin(m\pi x/a) && (0 < x < a)
\end{aligned}\right\} \tag{B13}$$

Determining the coefficients in these series through the formula

$$a_n' = (2/b) \int_0^b F(0,y) \sin (n\pi y/b)\, dy,$$

etc., with F(0,y), etc. replaced by the right-hand sides of equations (B9) to (B12), one obtains

$$\left.\begin{aligned}
a_n' &= \frac{2}{n\pi} [F(0,0) - (-1)^n F(0,b)] - \left(\frac{b}{n\pi}\right)^2 B_n' \\
a_n'' &= \frac{2}{n\pi} [F(a,0) - (-1)^n F(a,b)] - \left(\frac{b}{n\pi}\right)^2 B_n'' \\
a_m''' &= \frac{2}{m\pi} [F(0,0) - (-1)^m F(a,0)] - \left(\frac{a}{m\pi}\right)^2 B_m''' \\
a_m'''' &= \frac{2}{m\pi} [F(0,b) - (-1)^m F(a,b)] - \left(\frac{a}{m\pi}\right)^2 B_m''''
\end{aligned}\right\} \qquad (B14)$$

Series Assumptions for $F(x,y)$ and $P_1(y)$, $P_2(y)$, $P_3(x)$, and $P_4(x)$

The stress function F(x,y) will be assumed to be representable in the interior of the plate (i.e. in the open region $0 < x < a$, $0 < y < b$) by the double Fourier series

$$F(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad (B15)$$

with as yet unknown coefficients. Equation (B15) is not valid at the plate edges; however there the values of F are already expressed in series form by equations (B13) and (B14). Similarly the unknown stiffener axial forces will be assumed in the form

$$
\left.\begin{aligned}
P_1(y) &= \sum_{n=1}^{N} s_n' \sin (n\pi y/b) && (0 < y < b) \\
P_2(y) &= \sum_{n=1}^{N} s_n'' \sin (n\pi y/b) && (0 < y < b) \\
P_3(x) &= \sum_{m=1}^{M} s_m''' \sin (m\pi x/a) && (0 < x < a) \\
P_4(x) &= \sum_{m=1}^{M} s_m'''' \sin (m\pi x/a) && (0 < x < a)
\end{aligned}\right\} \quad \text{(B16)}
$$

At the end cross sections the stiffener forces are already known from the given loading (see fig. 1).

The coefficients in the series in equations (B15) and (B16) are related to the left-hand sides through the usual formulas

$$
a_{mn} = \frac{4}{ab} \int_0^b \int_0^a F(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \tag{B17}
$$

$$
s_n' = \frac{2}{b} \int_0^b P_1(y) \sin \frac{n\pi y}{b} \, dy, \text{ etc.} \tag{B18}
$$

<u>Series for the Derivatives of $F(x,y)$ and P_1, P_2, P_3, P_4</u>

The derivatives appearing in equations (B4) to (B7) will be assumed representable by the following series:

$$
N_x = \partial^2 F/\partial y^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix} (0 < x < a) \\ (0 < y < b) \end{matrix} \tag{B19}
$$

$$(N_x)_{y=0} = (\partial^2 F/\partial y^2)_{y=0} = \sum_{m=1}^{M} g'_m \sin (m\pi x/a) \qquad (0 < x < a) \tag{B20}$$

$$(N_x)_{y=b} = (\partial^2 F/\partial y^2)_{y=b} = \sum_{m=1}^{M} g''_m \sin (m\pi x/a) \qquad (0 < x < a) \tag{B21}$$

$$N_y = \partial^2 F/\partial x^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{B22}$$

$$(N_y)_{x=0} = (\partial^2 F/\partial x^2)_{x=0} = \sum_{n=1}^{N} c'_n \sin (n\pi y/b) \qquad (0 < y < b) \tag{B23}$$

$$(N_y)_{x=a} = (\partial^2 F/\partial x^2)_{x=a} = \sum_{n=1}^{N} c''_n \sin (n\pi y/b) \qquad (0 < y < b) \tag{B24}$$

$$N_{xy} = -\partial^2 F/\partial x \partial y = -\sum_{m=0}^{M} \sum_{n=0}^{N} j_{mn} \cos (m\pi x/a) \cos (n\pi y/b) \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le b)\end{matrix} \tag{B25}$$

$$\partial^4 F/\partial x^4 = \sum_{m=1}^{M} \sum_{n=1}^{N} e_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{B26}$$

$$\partial^4 F/\partial y^4 = \sum_{m=1}^{M} \sum_{n=1}^{N} i_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{B27}$$

$$\partial^4 F/\partial x^2 \, \partial y^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} p_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{B28}$$

$$dP_1/dy = \sum_{n=0}^{N} t'_n \cos (n\pi y/b) \qquad (0 \le y \le b)$$

$$dP_2/dy = \sum_{n=0}^{N} t''_n \cos (n\pi y/b) \qquad (0 \le y \le b)$$

$$dP_3/dx = \sum_{m=0}^{M} t'''_m \cos (m\pi x/a) \qquad (0 \le x \le a) \qquad \text{(B29)}$$

$$dP_4/dx = \sum_{m=0}^{M} t''''_m \cos (m\pi x/a) \qquad (0 \le x \le a)$$

where

$$g_{mn} = \frac{4}{ab} \int_0^b \int_0^a (\partial^2 F/\partial y^2) \sin (m\pi x/a) \sin (n\pi y/b)\, dx\, dy, \text{ etc.} \qquad \text{(B30)}$$

$$g'_m = \frac{2}{a} \int_0^a (\partial^2 F/\partial y^2)_{y=0} \sin (m\pi x/a)\, dx, \text{ etc.} \qquad \text{(B31)}$$

$$j_{mn} = \frac{(2-\delta_{m0})(2-\delta_{n0})}{ab} \int_0^b \int_0^a (\partial^2 F/\partial x \partial y) \cos(m\pi x/a) \cos(n\pi y/b)\, dx\, dy \qquad \text{(B32)}$$

$$t'_n = \frac{2 - \delta_{n0}}{b} \int_0^b (dP_1/dy) \cos (n\pi y/b)\, dy, \text{ etc.} \qquad \text{(B33)}$$

The coefficients appearing in the series for the derivatives (eqs. (B19) to (B29)) are not independent of the coefficients in the series for the basic quantities (eqs. (B15) and (B16)). The former can be related to the latter by using integration-by-parts** in the right-hand sides of equations (B30) to (B33). For example, two partial integrations with

**Such a technique was employed for plate bending problems by A. E. Green (ref. 5), who ascribes its earlier use to S. Goldstein.

respect to y in equation (B30) give

$$g_{mn} = \frac{n\pi}{b}\frac{2}{b}\left[a_m''' - (-1)^n a_m''''\right] - \left(\frac{n\pi}{b}\right)^2 a_{mn} \qquad \text{(B34)}$$

Similarly,

$$c_{mn} = \frac{m\pi}{a}\frac{2}{a}\left[a_n' - (-1)^m a_n''\right] - \left(\frac{m\pi}{a}\right)^2 a_{mn} \qquad \text{(B35)}$$

$$j_{mn} = \frac{(2-\delta_{m0})(2-\delta_{n0})}{ab}\left[(-1)^{m+n} F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)\right]$$

$$+ \frac{m\pi}{a}\frac{2-\delta_{n0}}{b}\left[(-1)^n a_m'''' - a_m'''\right] + \frac{n\pi}{b}\frac{2-\delta_{m0}}{a}\left[(-1)^m a_n'' - a_n'\right] +$$

$$+ \frac{m\pi}{a}\frac{n\pi}{b} a_{mn} \qquad \text{(B36)}$$

$$e_{mn} = \frac{m\pi}{a}\frac{2}{a}\left[c_n' - (-1)^m c_n''\right] - \left(\frac{m\pi}{a}\right)^3 \frac{2}{a}\left[a_n' - (-1)^m a_n''\right] + \left(\frac{m\pi}{a}\right)^4 a_{mn}$$

$$\text{(B37)}$$

$$i_{mn} = \frac{n\pi}{b}\frac{2}{b}\left[g_m' - (-1)^n g_m''\right] - \left(\frac{n\pi}{b}\right)^3 \frac{2}{b}\left[a_m''' - (-1)^n a_m''''\right] + \left(\frac{n\pi}{b}\right)^4 a_{mn}$$

$$\text{(B38)}$$

$$p_{mn} = \frac{4}{ab}\frac{m\pi}{a}\frac{n\pi}{b}\left[(-1)^{m+n}F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)\right]$$

$$+ \frac{2}{b}\left(\frac{m\pi}{a}\right)^2 \frac{n\pi}{b}\left[(-1)^n a_m'''' - a_m'''\right] + \frac{2}{a}\left(\frac{n\pi}{b}\right)^2 \frac{m\pi}{a}\left[(-1)^m a_n'' - a_n'\right]$$

$$+ \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 a_{mn} \qquad \text{(B39)}$$

$$\left.\begin{aligned}
t'_n &= \frac{2-\delta_{no}}{b}\,[(-1)^n\, P_1(b) - P_1(0)] + \frac{n\pi}{b}\, s'_n \\
t''_n &= \frac{2-\delta_{no}}{b}\,[(-1)^n\, P_2(b) - P_2(0)] + \frac{n\pi}{b}\, s''_n \\
t'''_m &= \frac{2-\delta_{mo}}{a}\,[(-1)^m\, P_3(a) - P_3(0)] + \frac{m\pi}{a}\, s'''_m \\
t''''_m &= \frac{2-\delta_{mo}}{a}\,[(-1)^m\, P_4(a) - P_4(0)] + \frac{m\pi}{a}\, s''''_m
\end{aligned}\right\} \qquad \text{(B40)}$$

Reference 1 did not require series expansions for $\partial^3F/\partial x^3$, $\partial^3F/\partial y^3$, $\partial^3F/\partial x^2\partial y$, and $\partial^3F/\partial x\partial y^2$, but the present paper will require such expansions in the subsequent appendices. These derivatives will therefore be assumed expandable in the following series:

$$\partial^3F/\partial x^3 = \sum_{m=0}^{M}\sum_{n=1}^{N} d_{mn} \cos\,(m\pi x/a)\,\sin\,(n\pi y/b) \qquad \begin{matrix}(0 \le x \le a)\\ (0 < y < b)\end{matrix} \qquad \text{(B41)}$$

$$\partial^3F/\partial y^3 = \sum_{m=1}^{M}\sum_{n=0}^{N} h_{mn} \sin\,(m\pi x/a)\,\cos\,(n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\ (0 \le y \le b)\end{matrix} \qquad \text{(B42)}$$

$$\partial^3F/\partial x^2\partial y = \sum_{m=1}^{M}\sum_{n=0}^{N} \ell_{mn} \sin\,(m\pi x/a)\,\cos\,(n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\ (0 \le y \le b)\end{matrix} \qquad \text{(B43)}$$

$$\partial^3F/\partial x\partial y^2 = \sum_{m=0}^{M}\sum_{n=1}^{N} w_{mn} \cos\,(m\pi x/a)\,\sin\,(n\pi y/b) \qquad \begin{matrix}(0 \le x \le a)\\ (0 < y < b)\end{matrix} \qquad \text{(B44)}$$

where

$$d_{mn} = \frac{(2-\delta_{mo})}{a}\,\frac{2}{b}\int_0^b\!\!\int_0^a (\partial^3F/\partial x^3)\,\cos\,(m\pi x/a)\,\sin\,(n\pi y/b)\,dxdy, \text{ etc.} \qquad \text{(B45)}$$

By using integration-by-parts in the right-hand sides of above equations, one obtains

$$d_{mn} = \frac{2-\delta_{mo}}{a} [(-1)^m c_n'' - c_n'] - \frac{2}{a}(\frac{m\pi}{a})^2 [(-1)^m a_n'' - a_n'] - (\frac{m\pi}{a})^3 a_{mn} \tag{B46}$$

Similarly,

$$h_{mn} = \frac{2-\delta_{no}}{b} [(-1)^n g_m'' - g_m'] - \frac{2}{b}(\frac{n\pi}{b})^2 [(-1)^n a_m'''' - a_m'''] - (\frac{n\pi}{b})^3 a_{mn} \tag{B47}$$

$$\begin{aligned} \ell_{mn} &= - \frac{m\pi}{a} j_{mn} \\ &= - \frac{2}{a} \frac{2-\delta_{no}}{b} (\frac{m\pi}{a})[(-1)^{m+n} F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)] \\ &\quad - \frac{2-\delta_{no}}{b} (\frac{m\pi}{a})^2 [(-1)^n a_m'''' - a_m'''] - \frac{2}{a} \frac{m\pi}{a} \frac{n\pi}{b} [(-1)^m a_n'' - a_n'] - (\frac{m\pi}{a})^2 \frac{n\pi}{b} a_{mn} \end{aligned} \tag{B48}$$

$$\begin{aligned} w_{mn} &= - \frac{n\pi}{b} j_{mn} \\ &= - \frac{(2-\delta_{mo})}{a} \frac{2}{b}(\frac{n\pi}{b})[(-1)^{m+n} F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)] \\ &\quad - \frac{m\pi}{a} \frac{2}{b} \frac{n\pi}{b} [(-1)^n a_m'''' - a_m'''] - \frac{2-\delta_{mo}}{a} (\frac{n\pi}{b})^2 [(-1)^m a_n'' - a_n'] \\ &\quad - \frac{m\pi}{a} (\frac{n\pi}{b})^2 a_{mn} \end{aligned} \tag{B49}$$

Through equations (B34) to (B40) and (B46) to (B49) all the unknown coefficients in the derivative series are expressed in terms of the basic unknowns a_{mn}; c_n', c_n'', g_m', g_m''; s_n', s_n'', s_m''', s_m''''; $F(0,0)$, $F(0,b)$, $F(a,0)$ and $F(a,b)$

Series Solution for Basic Equations (B5) to (B7)

Considering first equation (B5), substituting into it the series expansions for equations (15) and (B27) to (B29), eliminating e_{mn}, i_{mn}, and p_{mn} through (B37) to (B39), eliminating a_n', a_n'', a_m''', and a_m'''' through equations (B14), and then solving for a_{mn}, one obtains

$$a_{mn} = \frac{4}{mn\pi^2} [(-1)^{m+n} F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)]$$

$$- \frac{1}{E_{mn}} \{T_{mn} + \frac{2}{a} \frac{m\pi}{a} [c_n' - (-1)^m c_n''] C_2 + \frac{2}{b} \frac{n\pi}{b} [g_m' - (-1)^n g_m''] C_1$$

$$+ \frac{2}{a} \frac{m\pi}{a} (\frac{b}{n\pi})^2 [B_n' - (-1)^m B_n''][(\frac{m\pi}{a})^2 C_2 + (\frac{n\pi}{b})^2 (C_4 - 2C_3)]$$

$$+ \frac{2}{b} \frac{n\pi}{b} (\frac{a}{m\pi})^2 [B_m''' - (-1)^n B_m''''][(\frac{n\pi}{b})^2 C_1 + (\frac{m\pi}{a})^2 (C_4 - 2C_3)]\} \quad \text{(B50)}$$

where

$$E_{mn} = C_2(\frac{m\pi}{a})^4 + (C_4 - 2C_3)(\frac{m\pi}{a})^2 (\frac{n\pi}{b})^2 + C_1(\frac{n\pi}{b})^4 \quad \text{(B51)}$$

Thus the unknown a_{mn} have been obtained in terms of the smaller class of unknowns, c_n', c_n'', g_m', g_m'', through the compatibility equation.

Turning now to the stiffener equilibrium equations (B6), substituting the series from equations (B25), (B29) and (6), and utilizing equations (B40), one obtains the relationships

$$\frac{2-\delta_{no}}{b} [(-1)^n P_1(b) - P_1(0)] + \frac{n\pi}{b} s_n' - Q_n' - \sum_{m=0}^{M} j_{mn} = 0$$

$$(n=0,1,2,\ldots,N) \quad \text{(B52)}$$

$$\frac{2-\delta_{no}}{b} [(-1)^n \, p_2(b) - p_2(0)] + \frac{n\pi}{b} s_n'' + Q_n'' + \sum_{m=0}^{M} (-1)^m \, j_{mn} = 0$$

$$(n=0,1,2,\ldots,N) \qquad \text{(B53)}$$

$$\frac{2-\delta_{mo}}{a} [(-1)^m \, p_3(a) - p_3(0)] + \frac{m\pi}{a} s_m''' - Q_m''' - \sum_{n=0}^{N} j_{mn} = 0$$

$$(m=0,1,2,\ldots,M) \qquad \text{(B54)}$$

$$\frac{2-\delta_{mo}}{a} [(-1)^m \, p_4(a) - p_4(0)] + \frac{m\pi}{a} s_m'''' + Q_m'''' + \sum_{n=0}^{N} (-1)^n \, j_{mn} = 0$$

$$(m=0,1,2,\ldots,M) \qquad \text{(B55)}$$

From equations (B36) and (B14), it is noted that

$$j_{00} = \frac{1}{ab} [F(a,b) - F(a,0) - F(0,b) + F(0,0)] \qquad \text{(B56)}$$

$$j_{0n} = \frac{1}{a} \, \frac{b}{n\pi} (B_n' - B_n'') \qquad \text{for } n \neq 0 \qquad \text{(B57)}$$

$$j_{mo} = \frac{1}{b} \, \frac{a}{m\pi} (B_m''' - B_m'''') \qquad \text{for } m \neq 0 \qquad \text{(B58)}$$

Using these results in the n=0 and m=0 equations (B52) to (B55), one obtains four expansions for j_{00}, of which three are redundant because the structure as a whole in equilibrium (see ref. 1). Selecting the first as non-redundant,

$$j_{00} = -Q_0' + \frac{1}{b} \{P_1(b) - P_1(0) + \sum_{m=1}^{M} \frac{a}{m\pi} (B_m'''' - B_m''')\} \qquad \text{(B59)}$$

Finally, substituting into equations (B7) the series expansions from equations (4), (13), (B16), (B20), (B21), (B23), and (B24), one obtains

$$\left.\begin{aligned}
&s_n'/(A_1E_1) + T_n' - C_2\, c_n' + C_3B_n' = 0\\
&s_n''/(A_2E_2) + T_n'' - C_2\, c_n'' + C_3B_n'' = 0\\
&s_m'''/(A_3E_3) + T_m''' - C_1\, g_m' + C_3B_m''' = 0\\
&s_m''''/(A_4E_4) + T_m'''' - C_1\, g_m'' + C_3B_m'''' = 0
\end{aligned}\right\} \quad (B60)$$

Reduction in the Number of Simultaneous Equations

Essentially the problem has now been reduced to the solution of equations (B50), (B52) to (B55) and (B60) for the unknowns a_{mn}, s_n', s_n'', s_m''', s_m'''', c_n', c_n'', g_m' and g_m''. Since equation (B50) explicitly expresses a_{mn} in terms of c_n', c_n'', g_m', and g_m'', the solution of equations (B52) to (B55), and (B60) for s_n', s_n'', s_m''', s_m'''', c_n', c_n'', g_m' and g_m'' will be sufficient. However, further reduction in the number of equations to be solved may still be attained by using equations (B60) to express s_n', s_n'', s_m''' and s_m'''' in terms of c_n', c_n'', g_m' and g_m''. Equations (B52) to (B55), with the n=0 and m=0 equations excluded, can then be used to obtain a system of simultaneous equations in which the c_n', c_n'', g_m' and g_m'' are the only unknowns. This is accomplished by eliminating s_n', s_n'', s_m''', and s_m'''' with the aid of equations (B60), and j_{mn} by means of the following expression

$$\begin{aligned}
\underset{\substack{(m\neq 0)\\(n\neq 0)}}{j_{mn}} = &- \frac{mn\pi^2}{abE_{mn}} \{T_{mn} + \frac{2}{a}\frac{m\pi}{a}[c_n' - (-1)^m c_n'']C_2 + \frac{2}{b}\frac{n\pi}{b}[g_m' - (-1)^n g_m'']C_1\}\\
&+ \frac{1}{E_{mn}} \{\frac{2}{b}(\frac{m\pi}{a})^3 [B_m''' - (-1)^n B_m'''']C_2 + \frac{2}{a}(\frac{n\pi}{b})^3 [B_n' - (-1)^m B_n'']C_1\}
\end{aligned}$$

(B61)

which is obtained from equations (B36), (B14), and (B50). The resulting form of equations (B52) to (B55) is:

$$\bar{c}'_n\,\alpha_1(n) - \bar{c}''_n\,\beta_1(n) = R'_n - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} \frac{\bar{g}'_m - (-1)^n\,\bar{g}''_m}{E_{mn}} \qquad (n=1,2,\ldots,N) \tag{B62}$$

$$-\bar{c}'_n\,\beta_1(n) + \bar{c}''_n\,\alpha_2(n) = R''_n + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} (-1)^m \frac{\bar{g}'_m - (-1)^n\bar{g}''_m}{E_{mn}} \qquad (n=1,2,\ldots,N) \tag{B63}$$

$$\bar{g}'_m\,\alpha_3(m) - \bar{g}''_m\beta_2(m) = R'''_m - \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{\bar{c}'_n - (-1)^m\,\bar{c}''_n}{E_{mn}} \qquad (m=1,2,\ldots,M) \tag{B64}$$

$$-\bar{g}'_m\beta_2(m) + \bar{g}''_m\,\alpha_4(m) = R''''_m + \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} (-1)^n\,\frac{\bar{c}'_n - (-1)^m\,\bar{c}''_n}{E_{mn}} \qquad (m=1,2,\ldots,M) \tag{B65}$$

where

$$\bar{c}'_n = c'_n\,C_2(n\pi/b), \quad \bar{c}''_n = c''_n\,C_2(n\pi/b), \quad \bar{g}'_m = g'_m C_1(m\pi/a), \quad \bar{g}''_m = g''_m\,C_1(m\pi/a) \tag{B66}$$

$$\left.\begin{aligned}
\alpha_1(n) &= A_1E_1 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}} \\
\alpha_2(n) &= A_2E_2 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}} \\
\alpha_3(m) &= A_3E_3 + \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2}{E_{mn}} \\
\alpha_4(m) &= A_4E_4 + \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2}{E_{mn}} \\
\beta_1(n) &= \frac{2}{a}\sum_{m=1}^{M}\frac{(-1)^m(m\pi/a)^2}{E_{mn}} \\
\beta_2(m) &= \frac{2}{b}\sum_{n=1}^{N}\frac{(-1)^n(n\pi/b)^2}{E_{mn}}
\end{aligned}\right\} \tag{B67}$$

and R_n', R_n'', R_m''', R_m'''' are the following combinations of known thermal and loading quantities:

$$R_n' = Q_n' + \frac{2}{b}[P_1(0) - (-1)^n P_1(b)] + \frac{b}{an\pi}(B_n'-B_n'') + A_1E_1 \frac{n\pi}{b}(C_3B_n' + T_n') - \sum_{m=1}^{M} K_{mn}$$

$$R_n'' = -Q_n'' + \frac{2}{b}[P_2(0) - (-1)^n P_2(b)] - \frac{b}{an\pi}(B_n'-B_n'') + A_2E_2 \frac{n\pi}{b}(C_3B_n'' + T_n'')+ \sum_{m=1}^{M} (-1)^m K_{mn}$$

$$R_m''' = Q_m''' + \frac{2}{a}[P_3(0)-(-1)^m P_3(a)]+ \frac{a}{bm\pi}(B_m'''-B_m'''') + A_3E_3 \frac{m\pi}{a}(C_3B_m'''+T_m''') - \sum_{n=1}^{N} K_{mn}$$

$$R_m'''' = -Q_m'''' + \frac{2}{a}[P_4(0)-(-1)^m P_4(a)]- \frac{a}{bm\pi}(B_m'''-B_m'''')+A_4E_4 \frac{m\pi}{a}(C_3B_m''''+T_m'''')+ \sum_{n=1}^{N} (-1)^n K_{mn}$$

(B68)

with

$$K_{mn} = \frac{1}{E_{mn}}\{\frac{mn\pi^2}{ab}T_{mn} - \frac{2}{b}(\frac{m\pi}{a})^3 C_2[B_m''' - (-1)^n B_m''''] - \frac{2}{a}(\frac{n\pi}{b})^3 C_1[B_n' - (-1)^m B_n'']\}$$

(B69)

Equations (B62) to (B65) can be solved simultaneously for c_n', c_n'', g_m', and g_m''. With these known, equations (B60) will furnish the values of s_n', s_n'', s_m''', s_m'''', and equations (B57) to (B59) and (B61) the values of the j_{mn}. Equations (B16) will then give the stiffener stresses, equations (B19) to (B25) the plate stresses.

APPENDIX C

ANALYSIS FOR THE CASE OF ONE STIFFENER WITH PRESCRIBED DISPLACEMENT CONDITIONS

In this appendix the case of figure 4a is considered. In this case the edge $x = 0$ of the plate is assumed to be forced into a prescribed shape by means of an attached rigid stiffener (shown shaded in fig. 4a) which also has this prescribed shape. The shape is defined by $(\partial^2 u/\partial y^2)_{x=0}$, which is assumed to be a given function of y and expandable in the form of a Fourier series, equation (8), with known Fourier coefficients K_n' $(n = 1, 2, \ldots, N)$.

By virtue of the new conditions along the edge $x = 0$, certain quantities which were considered to be known or given in the previous appendix are now unknown. These are (a) the N Fourier coefficients B_n' which, through the first of equations (4), described the running tension $N_x(0,y)$ acting mutually between the stiffener and the edge of the plate, and (b) the tension $P_3(0)$ and $P_4(0)$ existing at the left ends of the horizontal stiffeners. Because the stiffener along $x = 0$ is now rigid, these N+2 quantities, needed in equations (B68) and (B69), are no longer known from the given external loading, but must be regarded as additional unknowns along with the c_n', c_n'', g_m', g_m''.

What makes the present case still solvable in the face of this increase in the number of unknowns is the fact that there are exactly N+2 new conditions which must be imposed on the problem. N of these new conditions state that the curvature $\partial^2 u/\partial y^2$ along the edge x=0 of the plate, produced by the stresses in the plate, must, when expanded in a

Fourier series, be consistent with the known Fourier coefficients K'_n (n=1,2,...,N) associated with the prescribed shape of the edge. The remaining two of the new conditions express the fact that the rigid stiffener at x=0 must be in equilibrium under the action of the unknown $P_3(0)$, $P_4(0)$, $N_x(0,y)$ and the known T_1 and M_1 (see fig. 5). It should be noted that $P_3(0)$ and $P_4(0)$ are no longer applied by an agent that is external to the entire structure, but are applied by the rigid stiffener along x=0. They are forces that now act mutually between the rigid vertical stiffener at x=0 and the two horizontal stiffeners.

The explicit mathematical formulation of these N+2 new conditions and their incorporation into the analysis of the previous appendix will constitute the bulk of the present appendix.

Formulation of Boundary Condition of Prescribed Curvature

Differentiating the last of the strain-displacement equations, (B1), with respect to y, one obtains

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \epsilon_y}{\partial x}$$

Eliminating the strains in terms of the stresses by means of equations (2), and then the stresses in terms of the stress function through equations (B4), this becomes

$$\frac{\partial^2 u}{\partial y^2} = (C_3 - C_4)\frac{\partial^3 F}{\partial x \partial y^2} - C_2\frac{\partial^3 F}{\partial x^3} - \frac{\partial e_y}{\partial x} \qquad \text{(C1)}$$

Thus the curvatures $\partial^2 u/\partial y^2$ of the edge x=0 of the plate are

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} = (C_3 - C_4)\left(\frac{\partial^3 F}{\partial x \partial y^2}\right)_{x=0} - C_2\left(\frac{\partial^3 F}{\partial x^3}\right)_{x=0} - \left(\frac{\partial e_y}{\partial x}\right)_{x=0} \tag{C2}$$

The terms on the right-hand side of this equation can be expressed in series form with the aid of equations (19), (B41) and (B44). The result is

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} = \sum_{n=1}^{N} [(C_4 - C_3) \frac{n\pi}{b} \sum_{m=0}^{M} j_{mn} - C_2 \sum_{m=0}^{M} d_{mn} - V_n'] \sin \frac{n\pi y}{b} \tag{C3}$$

Comparing equations (C3) and (8), one obtains the following N equations representing the condition of prescribed curvature along the edge x=0:

$$K_n' = (C_4 - C_3) \frac{n\pi}{b} \sum_{m=0}^{M} j_{mn} - C_2 \sum_{m=0}^{M} d_{mn} - V_n' \qquad (n=1,2,\ldots,N) \tag{C4}$$

The unknowns j_{mn} and d_{mn} in this system of equations can be expressed in terms of the basic unknowns B_n', c_n', c_n'', g_m', g_m''. To accomplish this it is first observed that equations (B57) and (B61) can both be represented by the following single equation, in which any undefined quantities are to be considered zero:

$$j_{mn} = -\frac{mn\pi^2}{abE_{mn}}\{T_{mn} + \frac{2}{a}\frac{m\pi}{a}[c_n' - (-1)^m c_n'']\, C_2 + \frac{2}{b}\frac{n\pi}{b}[g_m' - (-1)^n g_m'']C_1\}$$

$$+ \frac{1}{E_{mn}}\{\frac{2}{b}\left(\frac{m\pi}{a}\right)^3[B_m''' - (-1)^n B_m'''']C_2 + \frac{2-\delta_{mo}}{a}\left(\frac{n\pi}{b}\right)^3[B_n' - (-1)^m B_n'']C_1\} \tag{C5}$$

Furthermore, from equations (B46), (B14), (B50), and (B51), one obtains

$$d_{on} = \frac{1}{a}[c_n'' - c_n'] \tag{C6}$$

$$E_{mn} d_{mn} = \left(\frac{m\pi}{a}\right)^3 T_{mn} - \frac{2}{a} [c_n' - (-1)^m c_n''][C_1 \left(\frac{n\pi}{b}\right)^4 + (C_4 - 2C_3)\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2]$$
$(m \neq 0)$
$(n \neq 0)$

$$+ \frac{2}{b}\left(\frac{m\pi}{a}\right)^3 \left(\frac{n\pi}{b}\right) [g_m' - (-1)^n g_m''] C_1$$

$$- \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 [B_n' - (-1)^m B_n''] C_1$$

$$+ \frac{2}{b} \frac{n\pi}{b} \frac{m\pi}{a} [B_m''' - (-1)^n B_m''''][\left(\frac{n\pi}{b}\right)^2 C_1 + \left(\frac{m\pi}{a}\right)^2 (C_4 - 2C_3)] \tag{C7}$$

and these two equations can be represented by the following single equation in which, once again, undefined quantities are to be regarded as zero:

$$E_{mn} d_{mn} = \left(\frac{m\pi}{a}\right)^3 T_{mn} - \frac{2-\delta_{mo}}{a} \left(\frac{n\pi}{b}\right)^2 [c_n' - (-1)^m c_n''][C_1 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2 (C_4 - 2C_3)]$$
$(n \neq 0)$

$$+ \frac{2}{b} \left(\frac{m\pi}{a}\right)^3 \left(\frac{n\pi}{b}\right) [g_m' - (-1)^n g_m'']C_1$$

$$- \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 [B_n' - (-1)^m B_n'']C_1$$

$$+ \frac{2}{b} \frac{m\pi}{a} \frac{n\pi}{b} [B_m''' - (-1)^n B_m''''][\left(\frac{n\pi}{b}\right)^2 C_1 + \left(\frac{m\pi}{a}\right)^2 (C_4 - 2C_3)] \tag{C8}$$

Substituting from equations (C5) and (C8) into equation (C4), and separating B_n' terms, one obtains

$$K_n' = -V_n' - B_n'\left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M} \frac{1}{E_{mn}} [(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+\sum_{m=0}^{M} \frac{1}{E_{mn}}\{T_{mn}\left(\frac{m\pi}{a}\right)[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2]$$

$$+C_1\left(\frac{n\pi}{b}\right)[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2][\frac{2}{b} \frac{m\pi}{a}(g_m' - (-1)^n g_m'') + \frac{2-\delta_{mo}}{a}\left(\frac{n\pi}{b}\right)(-1)^m B_n'']$$

$$-C_2\left(\frac{n\pi}{b}\right)[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2][\frac{2}{b} \frac{m\pi}{a}(B_m''' - (-1)^n B_m'''') - \frac{2-\delta_{mo}}{a} \frac{n\pi}{b}(c_n' - (-1)^m c_n')]\}$$

This expression can be solved for each unknown B_n' in terms of the corresponding K_n', V_n', c_n', c_n'' and all the g_m' and g_m'', with the following result:

$$B_n' = \frac{\delta_n}{\gamma_n^{(1)}} + \frac{1}{\gamma_n^{(1)}} \sum_{m=1}^{M} \mu'_{mn}[\bar{g}_m' - (-1)^n \bar{g}_m''] + \frac{1}{\gamma_n^{(1)}} \sum_{m=0}^{M} \nu'_{mn}[\bar{c}_n' - (-1)^m \bar{c}_n''] \qquad (C9)$$

where $\gamma_n^{(1)}$, δ_n, μ'_{mn} and ν'_{mn} are known quantities and are defined as follows:

$$\gamma_n^{(1)} = \left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M} \frac{1}{E_{mn}} \left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right] \frac{2-\delta_{mo}}{a} \qquad (C10)$$

$$\delta_n = -K_n' - V_n' + \sum_{m=0}^{M} \frac{1}{E_{mn}} \{[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2][T_{mn}\left(\frac{m\pi}{a}\right) + C_1\left(\frac{n\pi}{b}\right)^2\left(\frac{2-\delta_{mo}}{a}\right)(-1)^m B_n'']$$

$$- C_2[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2] \frac{n\pi}{b}\frac{2}{b}\frac{m\pi}{a}[B_m''' - (-1)^n B_m'''']\} \qquad (C11)$$

$$\mu'_{mn} = \frac{1}{E_{mn}} \left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right] \frac{n\pi}{b}\frac{2}{b} \qquad (C12)$$

$$\nu'_{mn} = \frac{1}{E_{mn}}\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right] \frac{n\pi}{b} \frac{2-\delta_{mo}}{a} \qquad (C13)$$

Thus the unknown B_n' have, in effect, through equation (C9), been replaced by an equal number of known K_n'. If the edge x=0 is forced to remain straight, then the K_n' are all zero.

Formulation of Boundary Condition of Equilibrium

The normal forces acting on the rigid stiffener (see fig. 5) must be self-equilibrating. Therefore

$$P_3(0) + P_4(0) + \int_0^b N_x(0,y)\,dy = T_1$$

and

$$P_4(0) b + \int_0^b y\, N_x(0,y)\, dy = \frac{T_1 b}{2} + M_1$$

Substituting

$$N_x(0,y) = \sum_{n=1}^{N} B_n' \sin\left(\frac{n\pi y}{b}\right)$$

and solving for $P_3(0)$ and $P_4(0)$, one obtains

$$P_3(0) = - \sum_{n=1}^{N} \frac{b}{n\pi} B_n' + \frac{T_1}{2} - \frac{M_1}{b} \tag{C14}$$

$$P_4(0) = \sum_{n=1}^{N} (-1)^n \frac{b}{n\pi} B_n' + \frac{T_1}{2} + \frac{M_1}{b} \tag{C15}$$

Thus, in effect, the unknown $P_3(0)$ and $P_4(0)$ have been expressed in terms of the known T_1 and M_1.

Separating B_n' Terms in R_n', R_n'', R_m''', R_m''''

Equations (C14), (C15), and (B69) can be used to eliminate $P_3(0)$, $P_4(0)$, and K_{mn} from equations (B68). If the B_n' terms are then written separately from the rest, equations (B68) become

$$\begin{aligned} R_n' &= S_n^{(1)} + \gamma_n' B_n' \\ R_n'' &= S_n^{(2)} - \gamma_n'' B_n' \\ R_m''' &= S_m^{(3)} + \sum_{n=1}^{N} B_n' H_{mn} \\ R_m'''' &= S_m^{(4)} - \sum_{n=1}^{N} (-1)^n B_n' H_{mn} \end{aligned} \tag{C16}$$

where $S_n^{(1)}$, $S_n^{(2)}$, $S_m^{(3)}$ and $S_m^{(4)}$ are completely known loading terms defined as follows:

$$S_n^{(1)} = Q_n' + \frac{2}{b}[P_1(0)-(-1)^n P_1(b)] - \frac{b}{an\pi} B_n'' + A_1E_1 \frac{n\pi}{b} T_n' - \sum_{m=1}^{M} S_{mn}^{(1)} \tag{C17}$$

$$S_n^{(2)} = -Q_n'' + \frac{2}{b}[P_2(0) - (-1)^n P_2(b)] + \frac{b}{an\pi} B_n'' + A_2E_2 \frac{n\pi}{b}(C_3B_n'' + T_n'') + \sum_{m=1}^{M}(-1)^m S_{mn}^{(1)} \tag{C18}$$

$$S_m^{(3)} = Q_m''' - (-1)^m \frac{2}{a} P_3(a) + \frac{a}{bm\pi}(B_m''' - B_m'''') + A_3E_3 \frac{m\pi}{a}(C_3B_m''' + T_m''') - \sum_{n=1}^{N} S_{mn}^{(1)} + \frac{T_1}{a} - \frac{2M_1}{ab} \tag{C19}$$

$$S_m^{(4)} = -Q_m'''' - \frac{2}{a}(-1)^m P_4(a) - \frac{a}{bm\pi}(B_m''' - B_m'''') + A_4E_4 \frac{m\pi}{a}(C_3B_m'''' + T_m'''') + \sum_{n=1}^{N}(-1)^n S_{mn}^{(1)} + \frac{T_1}{a} + \frac{2M_1}{ab} \tag{C20}$$

with

$$S_{mn}^{(1)} = \frac{1}{E_{mn}} \{\frac{mn\pi^2}{ab} T_{mn} - \frac{2}{b}(\frac{m\pi}{a})^3 C_2[B_m''' - (-1)^n B_m''''] + (-1)^m \frac{2}{a}(\frac{n\pi}{b})^3 C_1 B_n''\} \tag{C21}$$

H_{mn}, γ_n' and γ_n'' are also known quantities and are defined by the following equations:

$$\gamma_n' = \frac{b}{an\pi} + A_1E_1 \frac{n\pi}{b} C_3 + \sum_{m=1}^{M} \frac{1}{E_{mn}} \frac{2}{a} (\frac{n\pi}{b})^3 C_1 \tag{C22}$$

$$\gamma_n'' = \frac{b}{an\pi} + \sum_{m=1}^{M} (-1)^m \frac{1}{E_{mn}} \frac{2}{a} \left(\frac{n\pi}{b}\right)^3 C_1 \tag{C23}$$

and

$$H_{mn} = \frac{1}{E_{mn}} \frac{2}{a} \left(\frac{n\pi}{b}\right)^3 C_1 - \frac{2}{a} \frac{b}{n\pi} \tag{C24}$$

Revision of Equations (B62) to (B65)

Substituting from equations (C16) into equations (B62) to (B65), one obtains

$$\bar{c}_n' \alpha_1(n) - \bar{c}_n'' \beta_1(n) = S_n^{(1)} + \gamma_n' B_n' - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} \frac{\bar{g}_m' - (-1)^n \bar{g}_m''}{E_{mn}}$$

$$(n=1,2,\ldots,N) \tag{C25}$$

$$-\bar{c}_n' \beta_1(n) + \bar{c}_n'' \alpha_2(n) = S_n^{(2)} - \gamma_n'' B_n' + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} (-1)^m \frac{\bar{g}_m' - (-1)^n \bar{g}_m''}{E_{mn}}$$

$$(n=1,2,\ldots,N) \tag{C26}$$

$$\bar{g}_m' \alpha_3(m) - \bar{g}_m'' \beta_2(m) = S_m^{(3)} + \sum_{n=1}^{N} B_n' H_{mn} - \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{\bar{c}_n' - (-1)^m \bar{c}_n''}{E_{mn}}$$

$$(m=1,2,\ldots,M) \tag{C27}$$

$$-\bar{g}_m' \beta_2(m) + \bar{g}_m'' \alpha_4(m) = S_m^{(4)} - \sum_{n=1}^{N} (-1)^n B_n' H_{mn} + \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} (-1)^n \frac{\bar{c}_n' - (-1)^m \bar{c}_n''}{E_{mn}}$$

$$(m=1,2,\ldots,M) \tag{C28}$$

If equation (C9) is now used to eliminate the B_n', one obtains the following system of simultaneous equations in which c_n', c_n'', g_m' and g_m'' are the only unknowns:

$$\bar{c}'_n \phi'_n - \bar{c}''_n \phi''_n = \zeta'_n + \sum_{m=1}^{M} \xi'_{mn} \, [\bar{g}'_m - (-1)^n \, \bar{g}''_m]$$

$$(n = 1, 2, \ldots, N) \qquad \text{(C29)}$$

$$-\bar{c}'_n \phi'''_n + \bar{c}''_n \phi''''_n = \zeta''_n - \sum_{m=1}^{M} \xi''_{mn} \, [\bar{g}'_m - (-1)^n \, \bar{g}''_m]$$

$$(n = 1, 2, \ldots, N) \qquad \text{(C30)}$$

$$\sum_{p=1}^{M} \bar{g}'_p \, [\alpha_3(m)\delta_{mp} - \sum_{n=1}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn}] - \sum_{p=1}^{M} \bar{g}''_p \, \Gamma''_{mp}$$

$$+ \sum_{n=1}^{N} \bar{c}'_n \, \Gamma'''_{mn} - \sum_{n=1}^{N} \bar{c}''_n \, \Gamma''''_{mn} = S_m^{(3)} + \sum_{n=1}^{N} \frac{\delta_n}{\gamma_n^{(1)}} H_{mn}$$

$$(m=1,2,3,\ldots,M) \qquad \text{(C31)}$$

$$-\sum_{p=1}^{M} \bar{g}'_p \, \sigma'_{mp} + \sum_{p=1}^{M} \bar{g}''_p \, [\alpha_4(m)\delta_{mp} - \sum_{n=1}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn}]$$

$$- \sum_{n=1}^{N} \bar{c}'_n \, \sigma'''_{mn} + \sum_{n=1}^{N} \bar{c}''_n \, \sigma''''_{mn} = S_m^{(4)} - \sum_{n=1}^{N} (-1)^n \frac{\delta_n}{\gamma_n^{(1)}} H_{mn}$$

$$(m=1,2,3,\ldots,M) \qquad \text{(C32)}$$

where

$$\phi'_n = \alpha_1(n) - \frac{\gamma'_n}{\gamma_n^{(1)}} \sum_{m=0}^{M} \nu'_{mn}$$

$$\phi''_n = \beta_1(n) - \frac{\gamma'_n}{\gamma_n^{(1)}} \sum_{m=0}^{M} (-1)^m \, \nu'_{mn} \qquad \text{(C33)}$$

$$\phi'''_m = \beta_1(n) - \frac{\gamma''_n}{\gamma_n^{(1)}} \sum_{m=0}^{M} \nu'_{mn}$$

$$\phi_n'''' = \alpha_2(n) - \frac{\gamma_n''}{\gamma_n^{(1)}} \sum_{m=0}^{M} (-1)^m \nu_{mn}' \tag{C33}$$

$$\left.\begin{aligned} \zeta_n' &= S_n^{(1)} + \frac{\gamma_n' \delta_n}{\gamma_n^{(1)}} \\ \zeta_n'' &= S_n^{(2)} - \frac{\gamma_n'' \delta_n}{\gamma_n^{(1)}} \end{aligned}\right\} \tag{C34}$$

$$\left.\begin{aligned} \xi_{mn}' &= \frac{\gamma_n'}{\gamma_n^{(1)}} \mu_{mn}' - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \frac{1}{E_{mn}} \\ \xi_{mn}'' &= \frac{\gamma_n''}{\gamma_n^{(1)}} \mu_{mn}' - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \frac{(-1)^m}{E_{mn}} \end{aligned}\right\} \tag{C35}$$

$$\left.\begin{aligned} \Gamma_{mp}'' &= \beta_2(m)\delta_{mp} - \sum_{n=1}^{N} (-1)^n H_{mn} \frac{1}{\gamma_n^{(1)}} \mu_{pn}' \\ \Gamma_{mn}''' &= \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{1}{\gamma_n^{(1)}} H_{mn} \sum_{p=0}^{M} \nu_{pn}' \\ \Gamma_{mn}'''' &= (-1)^m \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{1}{\gamma_n^{(1)}} H_{mn} \sum_{p=0}^{M} (-1)^p \nu_{pn}' \end{aligned}\right\} \tag{C36}$$

and

$$\sigma'_{mp} = \delta_{mp}\beta_2(m) - \sum_{n=1}^{N} (-1)^n H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn}$$

$$\sigma'''_{mn} = (-1)^n \left[\frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - H_{mn} \frac{1}{\gamma_n^{(1)}} \sum_{p=0}^{M} \nu'_{pn}\right] \tag{C37}$$

$$\sigma''''_{mn} = (-1)^n \left[(-1)^m \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - H_{mn} \frac{1}{\gamma_n^{(1)}} \sum_{p=0}^{M} (-1)^p \nu'_{pn}\right]$$

Some uncoupling of unknowns in equations (C31) and (C32) can be effected by adding and subtracting equations corresponding to the same value of m. By adding, one obtains

$$\sum_{p=1}^{M} \bar{g}'_p \theta'_{mp} + \sum_{p=1}^{M} \bar{g}''_p \theta''_{mp} + \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} \bar{c}'_n \eta'_{mn} - \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} \bar{c}''_n \eta''_{mn} = \rho'_m \tag{C38}$$

By subtracting, one obtains

$$\sum_{p=1}^{M} \bar{g}'_p \theta'''_{mp} - \sum_{p=1}^{M} \bar{g}''_p \theta''''_{mp} + \sum_{\substack{n=2,4\ldots \\ (n\ \text{even})}}^{N} \bar{c}'_n \eta'_{mn} - \sum_{\substack{n=2,4\ldots \\ (n\ \text{even})}}^{N} \bar{c}''_n \eta''_{mn} = \rho''_m \tag{C39}$$

where

$$\theta'_{mp} = \left[\alpha_3(m) - \beta_2(m)\right]\delta_{mp} - 2 \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn}$$

$$\theta''_{mp} = \left[\alpha_4(m) - \beta_2(m)\right] \delta_{mp} - 2 \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn} \tag{C40}$$

$$\eta'_{mn} = \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{2}{\gamma_n^{(1)}} H_{mn} \sum_{p=0}^{M} \nu'_{pn}$$

$$\eta''_{mn} = (-1)^m \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{2}{\gamma_n^{(1)}} H_{mn} \sum_{p=0}^{M} (-1)^p \nu'_{pn} \qquad \text{(C41)}$$

$$\theta'''_{mp} = [\alpha_3(m) + \beta_2(m)]\,\delta_{mp} - 2 \sum_{\substack{n=2,4\ldots \\ (n\ \text{even})}}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn}$$

$$\theta''''_{mp} = [\alpha_4(m) + \beta_2(m)]\,\delta_{mp} - 2 \sum_{\substack{n=2,4\ldots \\ (n\ \text{even})}}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn} \qquad \text{(C42)}$$

$$\rho'_m = S_m^{(3)} + S_m^{(4)} + 2 \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} \frac{\delta_n}{\gamma_n^{(1)}} H_{mn}$$

$$\rho''_m = S_m^{(3)} - S_m^{(4)} + 2 \sum_{\substack{n=2,4\ldots \\ (n\ \text{even})}}^{N} \frac{\delta_n}{\gamma_n^{(1)}} H_{mn} \qquad \text{(C43)}$$

Equation (C38) involves only the odd subscript c'_n and c''_n, and equation (C39) only the even-subscript c'_n and c''_n. These equations may replace equations (C31) and (C32).

Reduction in the Number of Simultaneous Equations

Equations (C29) and (C30), written for the same value of n, can be solved for $\bar{c}'_n$ and $\bar{c}''_n$ in terms of all the $\bar{g}'_m$ and $\bar{g}''_m$. The result is

$$\bar{c}'_n = \frac{1}{\Phi_n} \{\zeta'_n \phi''''_n + \zeta''_n \phi''_n + \sum_{p=1}^{M} (\phi''''_n \xi'_{pn} - \phi''_n \xi''_{pn}) [\bar{g}'_p - (-1)^n \bar{g}''_p]\} \tag{C44}$$

$$\bar{c}''_n = \frac{1}{\Phi_n} \{\zeta'_n \phi'''_n + \zeta''_n \phi'_n + \sum_{p=1}^{M} (\phi'''_n \xi'_{pn} - \phi'_n \xi''_{pn}) [\bar{g}'_p - (-1)^n \bar{g}''_p]\} \tag{C45}$$

where

$$\Phi_n = \phi'_n \phi''''_n - \phi''_n \phi'''_n .$$

Utilizing equations (C44) and (C45) to eliminate the $\bar{c}'_n$ and $\bar{c}''_n$ in equations (C38) and (C39), and combining like terms, one obtains the following simultaneous equations involving only the $\bar{g}'_m$ and $\bar{g}''_m$ as unknowns:

$$\sum_{p=1}^{M} \bar{g}'_p \{\theta'_{mp} + \sum_{\substack{n=1,3\ldots \\ (n \text{ odd})}}^{N} \psi_{mnp}\} + \sum_{p=1}^{M} \bar{g}''_p \{\theta''_{mp} + \sum_{\substack{n=1,3\ldots \\ (n \text{ odd})}}^{N} \psi_{mnp}\} = \rho'_m - \sum_{\substack{n=1,3\ldots \\ (n \text{ odd})}}^{N} \psi_{mn}$$

(C46)

$$\sum_{p=1}^{M} \bar{g}'_p \{\theta'''_{mp} + \sum_{\substack{n=2,4\ldots \\ (n \text{ even})}}^{N} \psi_{mnp}\} - \sum_{p=1}^{M} \bar{g}''_p \{\theta''''_{mp} + \sum_{\substack{n=2,4\ldots \\ (n \text{ even})}}^{N} \psi_{mnp}\} = \rho''_m - \sum_{\substack{n=2,4\ldots \\ (n \text{ even})}}^{N} \psi_{mn}$$

$$(m = 1, 2, \ldots, M) \tag{C46}$$

where

$$\psi_{mnp} = \frac{\eta'_{mn}}{\Phi_n}(\phi''''_n \xi'_{pn} - \phi''_n \xi''_{pn}) - \frac{\eta''_{mn}}{\Phi_n}(\phi'''_n \xi'_{pn} - \phi'_n \xi''_{pn}) \tag{C47}$$

$$\psi_{mn} = \frac{\eta'_{mn}}{\Phi_n}(\zeta'_n \phi''''_n + \zeta''_n \phi''_n) - \frac{\eta''_{mn}}{\Phi_n}(\zeta'_n \phi'''_n + \zeta''_n \phi'_n) \tag{C48}$$

The advantage of this reduction is evident: whereas the original simultaneous equations system, equations (C29) to (C32) requires the solution of 2N+2M simultaneous equations, the reduced system, equations (C46), contains only

2M simultaneous equations. Thus N may be taken arbitrarily large without increasing the number of simultaneous equations that have to be solved.

Procedure for Use of Equations

The procedure for using the foregoing analysis will now be summarized: Equations (C46) are first solved for the $\bar{g}'_p$ and $\bar{g}''_p$. With these known, equations (C44) and (C45) give directly the $\bar{c}'_n$ and $\bar{c}''_n$, and equation (C9) the B'_n. Equations (B60) then give the s'_n, s''_n, s'''_m, s''''_m, and equations (B57) to (B49) and (B61) the j_{mn}. Finally, equations (B16) and (B19) to (B25) give the stiffener and plate stresses.

Special Case: Symmetry About y = b/2

When the structure and loading are symmetrical about the line $y = b/2$ considerable simplification of the foregoing equations is possible. The symmetry implies that $A_3 = A_4$, $e_3(x) = e_4(x)$, $P_1(0) = P_1(b)$, $P_2(0) = P_2(b)$, $P_3(0) = P_4(0)$, $P_3(a) = P_4(a)$, $N_3(x) = N_4(x)$, $q_3(x) = -q_4(x)$, T_1 and $M_1 = 0$. It also implies that $B'''_m = B''''_m$, $Q'''_m = -Q''''_m$, $T'''_m = T''''_m$, and $\alpha_3(m) = \alpha_4(m)$. In addition, as a result of the symmetry the following quantities all vanish for n even: K'_n, T_{mn}, B''_n, Q'_n, Q''_n, T'_n, T''_n, and V'_n.

Consequently the following quantities vanish for n even:

$S_n^{(1)}$, $S_n^{(2)}$ (see eqs. (C17) and (C18)); δ_n (eq. (C11)); ζ'_n, ζ''_n (eqs. (C34)), Ψ_{mn} (eq. (C48)).

and the following equalities hold:

$$S_m^{(3)} = S_m^{(4)} \qquad \text{(eqs. (C19) and (C20))}$$

$$\theta'_{mp} = \theta''_{mp} \qquad \text{(eqs. (C40))}$$

$$\theta'''_{mp} = \theta''''_{mp} \qquad \text{(eqs. (C42))}$$

$$\rho''_m = 0 \qquad \text{(the second equation of eqs. (C43))}$$

Therefore, equations (C46) become

$$\left.\begin{aligned}
&\sum_{p=1}^{M} (\bar{g}'_p + \bar{g}''_p)\Big(\theta'_{mp} + \sum_{\substack{n=1,3\ldots \\ (n \text{ odd})}}^{N} \psi_{mnp}\Big) = \rho'_m - \sum_{\substack{n=1,3\ldots \\ (n \text{ odd})}}^{N} \psi_{mn} \\
&\sum_{p=1}^{N} (\bar{g}'_p - \bar{g}''_p)\Big(\theta'''_{mp} + \sum_{\substack{n=2,4\ldots \\ (n \text{ odd})}}^{N} \psi_{mnp}\Big) = 0 \qquad (m = 1,2,\ldots,M)
\end{aligned}\right\} \quad \text{(C49)}$$

Whence

$$\bar{g}'_p = \bar{g}''_p \qquad (p = 1, 2, \ldots, M) \qquad \text{(C50)}$$

and

$$\sum_{p=1}^{M} \bar{g}'_p \Big(\theta'_{mp} + \sum_{\substack{n=1,3,\ldots \\ (n \text{ odd})}}^{N} \psi_{mnp}\Big) = \tfrac{1}{2}\Big(\rho'_m - \sum_{n=1,3,\ldots}^{N} \psi_{mn}\Big) \qquad \text{(C51)}$$

$$(m = 1, 2, \ldots, M)$$

From equation (C44) and (C45), together with (C50) and the earlier consequences of symmetry, there results

$$\left.\begin{aligned}
\bar{c}'_n &= 0 \\
\bar{c}''_n &= 0
\end{aligned}\right\} \quad \text{for } n \text{ even,} \qquad \text{(C52)}$$

and, for n odd,

$$\left.\begin{aligned}
\bar{c}'_n &= \frac{1}{\Phi_n}\Big\{\zeta'_n\phi''''_n + \zeta''_n\phi''_n + 2\sum_{p=1}^{M} (\phi''''_n\xi'_{pn} - \phi''_n\xi''_{pn})\bar{g}'_p\Big\} \\
\bar{c}''_n &= \frac{1}{\Phi_n}\Big\{\zeta'_n\phi'''_n + \zeta''_n\phi'_n + 2\sum_{p=1}^{M} (\phi'''_n\xi'_{pn} - \phi'_n\xi''_{pn})\,\bar{g}'_p\Big\}
\end{aligned}\right\} \quad \text{(C53)}$$

Similarly, from equation (C9)

$$B'_n = 0 \qquad \text{for n even,} \tag{C54}$$

and, for n odd,

$$B'_n = \frac{\delta_n}{\gamma_n^{(1)}} + \frac{2}{\gamma_n^{(1)}} \sum_{m=1}^{M} \mu'_{mn}\, \bar{g}'_m + \frac{1}{\gamma_n^{(1)}} \sum_{m=0}^{M} \nu'_{mn}\, [\bar{c}'_n - (-1)^m\, \bar{c}''_n] \tag{C55}$$

The procedure for the symmetrical case can now be summarized as follows: Solve equations (C51) simultaneously for the $\bar{g}'_p$, then use equations (C53) and (C55) to compute the odd-subscript $\bar{c}'_n$, $\bar{c}''_n$ and B'_n. With these known, equations (B60) will furnish the values of s'_n, s''_n, s'''_m, s''''_m (n odd), and equations (B57) and (B61) the values of the j_{mn} (n odd). Equations (B16) and (B19) to (B25), with the only odd values of n included, will then give the stiffener and plate stresses.

Limiting Case of Large Stiffener Areas

The case in which the stiffener cross-sectional areas are large compared with the plate cross-sectional area is of practical and theoretical interest. In order to study this case, let it be assumed that A_1E_1, A_2E_2, A_3E_3 and A_4E_4 will approach infinity while maintaining constant ratios with each other. Then equations (C29) to (C32) can be simplified through the following steps: First, divide equation (C29) by A_1E_1, (C30) by A_2E_2, (C31) by A_3E_3, and (C32) by A_4E_4 and rearrange them to obtain

$$\bar{c}'_n[1-\Xi_n + \frac{\Delta'_n}{A_1E_1}] + \bar{c}''_n[\Xi'_n - \frac{\Delta''_n}{A_1E_1}] = \frac{\zeta'_n}{A_1E_1} + \sum_{m=1}^{M} [\bar{g}'_m-(-1)^n\bar{g}''_m][\frac{\mu'_{mn}}{\gamma_n^{(1)}} \; \frac{n\pi}{b}C_3 + \frac{\Delta'''_{mn}}{A_1E_1}] \tag{C56}$$

$$-\bar{c}'_n \frac{\phi'''_n}{A_2E_2} + \bar{c}''_n\left[1 + \frac{U_n}{A_2E_2}\right] = \frac{\zeta''_n}{A_2E_2} - \frac{1}{A_2E_2}\sum_{m=1}^{M}\left[\bar{g}'_m - (-1)^n\,\bar{g}''_m\right]\xi''_{mn} \tag{C57}$$

$$\sum_{p=1}^{M} \bar{g}'_p\left[\delta_{mp} + \frac{V_{mp}}{A_3E_3}\right] - \frac{1}{A_3E_3}\sum_{p=1}^{M} \bar{g}''_p\,\Gamma''_{mp}$$

$$+ \frac{1}{A_3E_3}\sum_{n=1}^{N} \bar{c}'_n\,\Gamma'''_{mn} - \frac{1}{A_3E_3}\sum_{n=1}^{N} \bar{c}''_n\,\Gamma''''_{mn} = \frac{\bar{S}'''_m}{A_3E_3} \tag{C58}$$

$$- \frac{1}{A_4E_4}\sum_{p=1}^{M} \bar{g}'_p\,\sigma'_{mp} + \sum_{p=1}^{M} \bar{g}''_p\left[\delta_{mp} + \frac{V_{mp}}{A_4E_4}\right]$$

$$- \frac{1}{A_4E_4}\sum_{n=1}^{N} \bar{c}'_n\,\sigma'''_{mn} + \frac{1}{A_4E_4}\sum_{n=1}^{N} \bar{c}''_n\,\sigma''''_{mn} = \frac{\bar{S}''''_m}{A_4E_4} \tag{C59}$$

where

$$\Delta'_n = \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}} - \frac{\gamma''''_n}{\gamma^{(1)}_n}\sum_{m=0}^{M}\nu'_{mn} \tag{C56a}$$

$$\Delta''_n = \beta_1(n) - \frac{\gamma''''_n}{\gamma^{(1)}_n}\sum_{m=0}^{M}(-1)^m\,\nu'_{mn} \tag{C56b}$$

$$\Delta'''_{mn} = \frac{\mu'_{mn}}{\gamma^{(1)}_n}\gamma''''_n - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\frac{1}{E_{mn}} \tag{C56c}$$

with

$$\gamma''''_n = \frac{b}{an\pi} + \sum_{m=1}^{M}\frac{1}{E_{mn}}\,\frac{2}{a}\left(\frac{n\pi}{b}\right)^3 C_1 \tag{C56d}$$

$$\Xi_n = \frac{1}{\gamma^{(1)}_n}\sum_{m=0}^{M}\nu'_{mn}\,\frac{n\pi}{b}\,C_3 \tag{C56e}$$

$$\Xi'_n = \frac{1}{\gamma_n^{(1)}} \sum_{m=0}^{M} (-1)^m \, \nu'_{mn} \, \frac{n\pi}{b} \, C_3 \tag{C56f}$$

$$U_n = \frac{2}{a} \sum_{m=1}^{M} \frac{(m\pi/a)^2}{E_{mn}} - \frac{\gamma''_n}{\gamma_n^{(1)}} \sum_{m=0}^{M} (-1)^m \, \nu'_{mn} \tag{C57a}$$

$$V_{mp} = \delta_{mp} \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2}{E_{mn}} - \sum_{n=1}^{N} H_{mn} \frac{1}{\gamma_n^{(1)}} \mu'_{pn} \tag{C58a}$$

$$\bar{S}_m^{(3)} = S_m^{(3)} + \sum_{n=1}^{N} \frac{\delta_n}{\gamma_n^{(1)}} H_{mn} \tag{C58b}$$

$$\bar{S}_m^{(4)} = S_m^{(4)} - \sum_{n=1}^{N} (-1)^n \frac{\delta_n}{\gamma_n^{(1)}} H_{mn} \tag{C59a}$$

Examining the coefficients of the unknowns $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$ and $\bar{g}''_m$ in equations (C56) through (C59), it is observed that some of these coefficients are of the order of 1, while the others are of the order of $1/(a^2 E_{11} A_1 E_1)$. Retaining only terms of the order of 1 in these coefficients, one reduces equations (C56) to (C59) to the following system:

$$\bar{c}'_n \, [1 - \Xi_n] = \frac{\zeta'_n}{A_1 E_1} - \bar{c}''_n \, \Xi'_n + \sum_{m=1}^{M} [\bar{g}'_m - (-1)^n \, \bar{g}''_m] \, \frac{\mu'_{mn}}{\gamma_n^{(1)}} \, \frac{n\pi}{b} \, C_3 \tag{C60}$$

$$\bar{c}''_n = \frac{\zeta''_n}{A_2 E_2} \tag{C61}$$

$$\bar{g}'_m = \frac{\bar{S}_m^{(3)}}{A_3 E_3} \tag{C62}$$

$$\bar{g}''_m = \frac{\bar{S}_m^{(4)}}{A_4 E_4} \tag{C63}$$

Using equations (C61) through (C63) to simplify equation (C60), the latter becomes

$$\bar{c}'_n = \frac{\Delta_n}{A_1E_1} \tag{C64}$$

where

$$\Delta_n = \frac{1}{1 - \Xi_n} \{\zeta'_n - \frac{A_1E_1}{A_2E_2} \zeta''_n \Xi'_n + A_1E_1 \sum_{m=1}^{M} [\frac{\bar{S}_m^{(3)}}{A_3E_3} - (-1)^n \frac{\bar{S}_m^{(4)}}{A_4E_4}] \frac{\mu'_{mn}}{\gamma_n^{(1)}} \frac{n\pi}{b} c_3\} \tag{C64a}$$

Thus a solution (eqs. (C61) through (C64)) is obtained which gives the unknowns $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$, $\bar{g}''_m$ explicitly without the necessity of solving simultaneous equations. This solution can be seen to be correct to terms of the first degree in $1/(a^3E_{11}A_1E_1)$. With $\bar{c}'_n$, etc., known, the procedure for computing stresses is the same as described earlier for the general case.

A solution correct to terms of the second degree in $1/(a^3E_{11}A_1E_1)$ can be obtained by the following procedure: First substitute from equations (C62) to (C64) into (C57) to obtain

$$\bar{c}''_n = \frac{1}{[1 + \frac{U_n}{A_2E_2}]} \{ \frac{\zeta''_n}{A_2E_2} + \frac{1}{A_2E_2} \frac{\Delta_n}{A_1E_1} \phi'''_n - \frac{1}{A_2E_2} \sum_{m=1}^{M} [\frac{\bar{S}_m^{(3)}}{A_3E_3} - (-1)^n \frac{\bar{S}_m^{(4)}}{A_4E_4}] \xi''_{mn}\} \tag{C65}$$

Expanding

$$1/[1 + \frac{U_n}{A_2E_2}]$$

in a power series of the form

$$\frac{1}{1 + \epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \dots$$

and neglecting the terms which are powers of $1/(a^3E_{11}A_2E_2)$ higher than second in equations (C65) gives

$$\bar{c}''_n = \frac{\zeta''_n}{A_2E_2} + \frac{Y''_n}{(A_2E_2)^2} \tag{C66}$$

where

$$Y''_n = -U_n\zeta''_n + A_2E_2 \frac{\Delta_n}{A_1E_1} \phi'''_n - A_2E_2 \sum_{m=1}^{M} \left[\frac{\bar{S}^{(3)}_m}{A_3E_3} - (-1)^n \frac{\bar{S}^{(4)}_m}{A_4E_4}\right] \xi''_{mn} \tag{C66a}$$

By further application of this technique, using the first order approximations given by equations (C61) to (C64), the second order approximations for $\bar{g}'_m$ and $\bar{g}''_m$ (from eqs. (C58) and (C59)) can be expressed by following equations:

$$\bar{g}'_m = \frac{\bar{S}^{(3)}_m}{A_3E_3} + \frac{Y'''_m}{(A_3E_3)^2} \tag{C67}$$

$$\bar{g}''_m = \frac{\bar{S}^{(4)}_m}{A_4E_4} + \frac{Y''''_m}{(A_4E_4)^2} \tag{C68}$$

where

$$Y'''_m = -V_{mm}\bar{S}^{(3)}_m + \sum_{\substack{p=1\\(p\neq m)}}^{M} \sum_{n=1}^{N} \bar{S}^{(3)}_p H_{mn} \frac{1}{\gamma^{(1)}_n} \mu'_{pn} + A_3E_3 \sum_{p=1}^{M} \frac{\bar{S}^{(4)}_p}{A_4E_4} \Gamma''_{mp}$$

$$-A_3E_3 \sum_{n=1}^{N} \frac{\Delta_n}{A_1E_1} \Gamma'''_{mn} + A_3E_3 \sum_{n=1}^{N} \frac{\zeta''_n}{A_2E_2} \Gamma''''_{mn} \tag{C67a}$$

$$Y'''_m = -V_{mm}\bar{S}^{(4)}_m + A_4E_4 \sum_{p=1}^{M} \frac{\bar{S}^{(3)}_p}{A_3E_3} + \sum_{\substack{p=1\\(p\neq m)}}^{M} \sum_{n=1}^{N} \bar{S}^{(4)}_p H_{mn} \frac{1}{\gamma^{(1)}_n} \mu'_{pn}$$

$$+ A_4E_4 \sum_{n=1}^{N} \frac{\Delta_n}{A_1E_1} \sigma'''_{mn} - A_4E_4 \sum_{n=1}^{N} \frac{\zeta''_n}{A_2E_2} \sigma''''_{mn} \qquad \text{(C68a)}$$

Substituting from equations (C66), (C67), and (C68) into equation (C56), one obtains

$$\bar{c}'_n = \frac{1}{[1 - \Xi_n + \frac{\Delta'_n}{A_1E_1}]} \{ \frac{\zeta'_n}{A_1E_1} - [\frac{\zeta''_n}{A_2E_2} + \frac{Y''_n}{(A_2E_2)^2}][\Xi'_n - \frac{\Delta''_n}{A_1E_1}]$$

$$+ \sum_{m=1}^{M} [(\frac{\bar{S}^{(3)}_m}{A_3E_3} + \frac{Y'''_m}{(A_3E_3)^2}) - (-1)^n(\frac{\bar{S}^{(4)}_m}{A_4E_4} - \frac{Y''''_m}{(A_4E_4)^2})]$$

$$[\frac{\mu'_{mn}}{\gamma^{(1)}_n} \frac{n\pi}{b} C_3 + \frac{\Delta'''_{mn}}{A_1E_1}]\} \qquad \text{(C69)}$$

Expanding

$$1/[1 - \Xi_n + \frac{\Delta'_n}{A_1E_1}]$$

in a power series of the form

$$\frac{1}{1 - K + \epsilon} = \frac{1}{1 - K} \frac{1}{1 + \epsilon'} = \frac{1}{1 - K}(1 - \epsilon' + \epsilon'^2 - \epsilon'^3 + \ldots)$$

where

$$\epsilon' = \frac{\epsilon}{1 - K}$$

and neglecting powers of $1/(a^3E_{11}A_1E_1)$ higher than second in equation (C69) gives

$$\bar{c}'_n = \frac{\Delta_n}{A_1E_1} + \frac{1}{1 - \Xi_n} \frac{Y'_n}{(A_1E_1)^2} \qquad \text{(C70)}$$

where

$$Y'_n = -\frac{1}{1-\Xi_n}\Delta'_n\zeta'_n - \frac{1}{1-\Xi_n}\Delta'_n A_1E_1\sum_{m=1}^{M}\left[\frac{\bar{S}_m^{(3)}}{A_3E_3} - (-1)^n\frac{\bar{S}_m^{(4)}}{A_4E_4}\right]\frac{\mu'_{mn}}{\gamma_n^{(1)}}\frac{n\pi}{b}C_3$$

$$+ A_1E_1\sum_{m=1}^{M}\left[\frac{\bar{S}_m^{(3)}}{A_3E_3} - (-1)^n\frac{\bar{S}_m^{(4)}}{A_4E_4}\right]\Delta'''_{mn} + (A_1E_1)^2\sum_{m=1}^{M}\left[\frac{Y'''_m}{(A_3E_3)^2} - (-1)^n\frac{Y''''_m}{(A_4E_4)^2}\right]\frac{\mu'_{mn}}{\gamma_n^{(1)}}\frac{n\pi}{b}C_3$$

$$+ \frac{1}{1-\Xi_n}A_1E_1\Delta'_n\frac{\zeta''_n}{A_2E_2}\Xi'_n + A_1E_1\frac{\zeta''_n}{A_2E_2}\Delta''_n - (A_1E_1)^2\frac{Y''_n}{(A_2E_2)^2}\Xi'_n \tag{C70a}$$

Equations (C66), (C67), (C68) and (C70) constitute a solution correct to the second degree in $1/(a^3E_{11}A_1E_1)$, in which the necessity of solving simultaneous equations is once again obviated.

Illustrative Thermal-Stress Problem

In order to illustrate the details involved in the application of the foregoing analytical results, a particular example will be considered which has the following characteristics:

a) Edge x=0 kept straight; therefore the K'_n in equation (8) are all zero.

b) Plate isotropic, therefore elastic constants are given by equations (3).

c) Plate and stiffeners have the same Young's modulus E.

d) $A_1 = A_2, \cdot A_3 = A_4$.

e) No force loading.

f) Stiffener temperature constant at the value T_o.

g) Plate temperatures $T(x,y)$ symmetrical about both centerlines ($x = a/2$, $y = b/2$) and varying sinusoidally in accordance with the following equation:

$$T(x,y) = T_o + \theta \sin\left(\frac{P\pi x}{a}\right) \sin\left(\frac{Q\pi y}{b}\right) \qquad \begin{matrix}(0 \leq x \leq a)\\ (0 \leq y \leq b)\end{matrix} \tag{C71}$$

where θ is a constant, representing the temperature rise of the plate center relative to the stiffeners, and P and Q are odd integers.

h) Plate and stiffeners have the same coefficient of expansion α.

These are the only specializations to be presented later, the problem was further specialized to the case of a square plate (b=a), with all stiffeners identical ($A_1 = A_2 = A_3 = A_4$), and subjected to a "pillow-shaped" temperature distribution ($P = Q = 1$).

<u>Reduction of general equations to special case</u>. - From the given temperature distribution one obtains the following plate and stiffener thermal strains:

$$e_x = e_y = \alpha[T_o + \theta \sin \frac{P\pi x}{a} \sin \frac{Q\pi y}{b}] \tag{C72}$$

$$e_i = \alpha T_o \qquad i = 1, 2, 3, 4 \tag{C73}$$

Therefore

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = -\alpha\theta \left[\left(\frac{P\pi}{a}\right)^2 + \left(\frac{Q\pi}{b}\right)^2\right] \sin \frac{P\pi x}{a} \sin \frac{Q\pi y}{b}$$

$$\left(\frac{\partial e_y}{\partial x}\right)_{x=0} = \alpha\theta \frac{P\pi}{a} \sin \frac{Q\pi y}{b} \tag{C74}$$

and

$$e_1(y) - e_y(0,y) = \alpha T_o - \alpha T_o = 0$$

$$e_2(y) - e_y(a,y) = 0$$

$$e_3(x) - e_x(x,0) = 0 \qquad (C75)$$

$$e_4(x) - e_x(x,b) = 0$$

Equations (C75) reflect the absence of any temperature discontinuity between stiffeners and plate in this example. Substituting from equations (C74) and (C75) into the right-hand sides of equations (14), (16), and (23), one obtains

$$T_n' = T_n'' = T_m''' = T_m'''' = 0 \qquad (C76)$$

$$T_{mn} = -\delta_{mP}\delta_{nQ}\,\alpha\Theta\left(\frac{\pi}{a}\right)^2 (P^2 + Q^2B^2) \qquad (C77)$$

and

$$V_n' = \alpha\Theta\,\frac{P\pi}{a}\,\delta_{nQ} \qquad (C78)$$

where

$$B = a/b \qquad (C79)$$

Due to the absence of prescribed forces, the following quantities are all zero:

$P_1(0), P_1(b), P_2(0), P_2(b), P_3(a), P_4(a)$ (fig. 4a)

T_1, M_1 (fig. 4a)

B_n'', B_m''', B_m'''' (see eqs. (4))

$Q_n', Q_n'', Q_m''', Q_m''''$ (see eqs. (6))

(It should be noted that $P_3(0)$, $P_4(0)$ and B_n' do not necessarily vanish.)

It will be convenient to introduce additional dimensionless parameters λ_1, λ_2, and C, defined as follows:

$$\lambda_1 = 4ah/(\pi^2 A_1) = 4ah/(\pi^2 A_2)$$
$$\lambda_2 = 4bh/(\pi^2 A_3) = 4bh/(\pi^2 A_4) \tag{C80}$$
$$C = A_3/A_1$$

and to note, from equations (B51) and (3), that

$$E_{mn} = (\pi^4/a^4 Eh)(m^2 + n^2B^2)^2 \tag{C81}$$

Because in this example the structure and loading are symmetrical about y = b/2, the simplified system of equations, namely equations (C50) through (C55), will be used for the determination of the $\bar{g}'_p$, $\bar{c}'_n$, $\bar{c}''_n$ and B'_n. The quantities needed in order to use these equations will now be evaluated.

Substituting from equations (3), (C77), (C78), and (C81) into equations (C10) to (C13), one obtains

$$\gamma_n^{(1)} = -(n^2B^2/aEh) \sum_{i=0}^{M} \frac{[i^2 + (2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2} \tag{C10'}$$

$$\delta_n = \alpha\theta \frac{P\pi}{a} \left\{ \frac{[P^2 + (2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1 \right\} \delta_{nQ} \tag{C11'}$$

$$\mu'_{mn} = -\frac{2nB^2[m^2 + (2+\nu)n^2B^2]}{\pi\,[m^2 + n^2B^2]^2} \tag{C12'}$$

$$\nu'_{mn} = \frac{nB[n^2B^2 - \nu m^2](2 - \delta_{mo})}{\pi[m^2 + n^2B^2]^2} \tag{C13'}$$

Substituting into equations (C17) to (C24) from equations (C77) and (C81), one obtains

$$S_n^{(1)} = \delta_{nQ}\ \alpha\Theta Eh\ \frac{PQB}{[P^2 + Q^2B^2]} \qquad (C17')$$

$$S_n^{(2)} = \delta_{nQ}\ \alpha\Theta Eh\ \frac{PQB}{(P^2 + Q^2B^2)} \qquad (C18')$$

$$S_m^{(3)} = \delta_{mP}\ \alpha\Theta Eh\ \frac{PQB}{(P^2 + Q^2B^2)} \qquad (C19')$$

$$S_m^{(4)} = \delta_{mP}\ \alpha\Theta Eh\ \frac{PQB}{(P^2 + Q^2B^2)} \qquad (C20')$$

$$\gamma_n' = \frac{1}{n\pi B} + \nu\ \frac{4nB}{\pi\lambda_1} + \frac{2n^3B^3}{\pi} \sum_{i=1}^{M} \frac{1}{[i^2 + n^2B^2]^2} \qquad (C22')$$

$$\gamma_n'' = \frac{1}{n\pi B} + \frac{2n^3B^3}{\pi} \sum_{i=1}^{M} \frac{(-1)^i}{[i^2 + n^2B^2]^2} \qquad (C23')$$

$$H_{mn} = \frac{2n^3B^3}{\pi[m^2 + n^2B^2]^2} - \frac{2}{n\pi B} \qquad (C24')$$

Substituting from equations (B67), (C10'), (C13'), (C22') and (C23') into equations (C33), one obtains

$$\begin{aligned} \phi_n' &= A_1E\ \bar{\phi}_n' \\ \phi_n'' &= A_1E\ \bar{\phi}_n'' \\ \phi_m''' &= A_3E\ \bar{\phi}_m''' \\ \phi_n'''' &= A_3E\ \bar{\phi}_m'''' \end{aligned} \qquad (C33')$$

where $\bar{\phi}_n'$ through $\bar{\phi}_m''''$ are numerical constants defined as follows:

$$\bar{\phi}'_n = 1 + \frac{\lambda_1}{2}\sum_{i=1}^{M}\frac{i^2}{[i^2 + n^2B^2]^2}$$

$$+ \frac{\{\lambda_1 + 4\nu n^2B^2 + 2\lambda_1 n^4B^4 \sum_{i=1}^{M}\frac{1}{[i^2 + n^2B^2]^2}\}}{4n^2B^2 \sum_{i=0}^{M}\frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}} \sum_{i=0}^{M}\frac{\{n^2B^2-\nu i^2\}(2-\delta_{i0})}{[i^2 + n^2B^2]^2} \tag{C33'a}$$

$$\bar{\phi}''_n = \frac{\lambda_1}{2}\sum_{i=1}^{M}\frac{(-1)^i\, i^2}{[i^2 + n^2B^2]^2}$$

$$+ \frac{\{\lambda_1 + 4\nu n^2B^2 + 2\lambda_1 n^4B^4 \sum_{i=1}^{M}\frac{1}{[i^2+n^2B^2]^2}\}}{4n^2B^2 \sum_{i=0}^{M}\frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}} \sum_{i=0}^{M}\frac{(-1)^i[n^2B^2 - \nu i^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2} \tag{C33'b}$$

$$\bar{\phi}'''_n = \frac{\lambda_1}{2}\sum_{i=1}^{M}\frac{(-1)^i\, i^2}{[i^2 + n^2B^2]^2}$$

$$+ \frac{\lambda_1\{1 + 2n^4B^4 \sum_{i=1}^{M}\frac{(-1)^i}{[i^2 + n^2B^2]^2}\}}{4n^2B^2 \sum_{i=0}^{M}\frac{[i^2 + (2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}} \sum_{i=1}^{M}\frac{[n^2B^2 - \nu i^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2} \tag{C33'c}$$

$$\bar{\phi}_n'''' = 1 + \frac{\lambda_1}{2}\sum_{i=1}^{M} \frac{i^2}{[i^2 + n^2B^2]^2}$$

$$+ \frac{\lambda_1\{1 + 2n^4B^4 \sum_{i=1}^{M} \frac{(-1)^i}{[i^2 + n^2B^2]^2}\}}{4n^2B^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2]\ (2 - \delta_{i0})}{[i^2 + n^2B^2]^2}} \sum_{i=0}^{M} \frac{(-1)^i[n^2B^2-\nu i^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2} \qquad \text{(C33'd)}$$

Substituting from equations (C10') (C11') (C17'), (C18'), (C22'), and (C23') into equations (C34), one obtains

$$\begin{aligned} \zeta_n' &= \alpha\Theta Eh\ \bar{\zeta}_n'\ \delta_{nQ} \\ \zeta_n'' &= \alpha\Theta Eh\ \bar{\zeta}_n''\ \delta_{nQ} \end{aligned} \qquad \text{(C34')}$$

where $\bar{\zeta}_n'$ and $\bar{\zeta}_n''$ are known quantities given as follows

$$\bar{\zeta}_n' = \frac{PQB}{[P^2 + Q^2B^2]} - \frac{P\{1 + \frac{4\nu Q^2B^2}{\lambda_1} + 2Q^4B^4 \sum_{i=1}^{M} \frac{1}{[i^2+Q^2B^2]^2}\}\{\frac{[P^2 + (2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{Q^3B^3 \sum_{i=0}^{M} \frac{[i^2 + (2+\nu)Q^2B^2](2 - \delta_{i0})}{[i^2 + Q^2B^2]^2}} \qquad \text{(C34'a)}$$

$$\bar{\zeta}_n'' = \frac{PQB}{[P^2 + Q^2B^2]} + \frac{P\{1 + 2Q^4B^4 \sum_{i=1}^{M} \frac{(-1)^i}{[i^2+Q^2B^2]^2}\}\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{Q^3B^3 \sum_{i=0}^{M} \frac{[i^2 + (2+\nu)\ Q^2B^2](2 - \delta_{i0})}{[i^2 + Q^2B^2]^2}} \qquad \text{(c34'b)}$$

Substituting from equations (C81), (C12'), (C22') and (C23') into equations (C35), one obtains

$$\xi'_{pn} = A_1 E\, \bar{\xi}'_{pn}$$

$$\xi''_{pn} = A_1 E\, \bar{\xi}''_{pn} \qquad \text{(C35')}$$

where $\bar{\xi}'_{pn}$ and $\bar{\xi}''_{pn}$ are known quantities given by the following equations:

$$\bar{\xi}'_{pn} = \frac{[p^2+(2+\nu)n^2B^2]\{\lambda_1 + 4\nu n^2B^2+2\lambda_1 n^4B^4\sum_{i=1}^{M} \frac{1}{[i^2 + n^2B^2]^2}\}}{2n^2B\,[p^2+n^2B^2]^2 \sum_{i=0}^{M} \frac{[i^2 + (2+\nu)n^2B^2](2 - \delta_{i0})}{[i^2 + n^2B^2]^2}} - \frac{\lambda_1\, n^2B^3}{2[p^2 + n^2B^2]^2} \qquad \text{(C35'a)}$$

$$\bar{\xi}''_{pn} = \frac{[p^2+(2+\nu)n^2B^2]\,\lambda_1\{1 + 2n^4B^4 \sum_{i=1}^{M} \frac{(-1)^i}{[i^2 + n^2B^2]^2}\}}{2n^2B\,[p^2+n^2B^2]^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}} - \frac{(-1)^p\,\lambda_1\, n^2B^3}{2[p^2 + n^2B^2]^2} \qquad \text{(C35'b)}$$

Substituting from (C81) into the third and last of equations (B67), one obtains

$$\alpha_3(m) = A_3E + A_3E\,\frac{\lambda_2}{2}\,B^4 \sum_{n=1}^{N} \frac{n^2}{[m^2 + n^2B^2]^2}$$

and

$$\beta_2(m) = A_3E\,\frac{\lambda_2}{2}\,B^4 \sum_{n=1}^{N} \frac{(-1)^n\, n^2}{[m^2 + n^2B^2]^2}$$

therefore

$$\alpha_3(m) - \beta_2(m) = A_3E\{1 + \lambda_2 B^4 \sum_{n=1,3,\ldots}^{N} \frac{n^2}{[m^2 + n^2B^2]^2}\} \qquad \text{(C82)}$$

Substituting from equations (C82), (C10'), (C12') (C24') into the first of equations (C40), one obtains

$$\theta'_{mp} = A_3 E \, [1 + \lambda_2 B^4 \sum_{\substack{n=1,3\ldots \\ (n\ odd)}}^{N} \frac{n^2}{[m^2 + n^2B^2]^2}] \, \delta_{mp}$$

$$-2A_3E\lambda_2B^4 \sum_{n=1,3\ldots}^{N} \{\frac{n^2}{[m^2+n^2B^2]^2} - \frac{1}{n^2B^4}\} \frac{[p^2 + (2+\nu)\ n^2B^2]}{[p^2+n^2B^2]^2 \displaystyle\sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}}$$

(C40')

Substituting from equations (C81), (C10'), (C13'), (C24') into equations (C41), one obtains

$$\begin{aligned} \eta'_{mn} &= A_1 E \, \bar{\eta}'_{mn} \\ \eta''_{mn} &= A_1 E \, \bar{\eta}''_{mn} \end{aligned} \qquad \text{(C41')}$$

where

$$\bar{\eta}'_{mn} = \frac{\lambda_1 m^2}{[m^2+n^2B^2]^2} + \frac{\lambda_1 [\frac{n^3B^3}{[m^2+n^2B^2]^2} - \frac{1}{nB}]}{nB \displaystyle\sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2]\ (2-\delta_{i0})}{[i^2 + n^2B^2]^2}} \sum_{i=0}^{M} \frac{[n^2B^2-\nu i^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}$$

(C41'a)

$$\bar{\eta}''_{mn} = \frac{(-1)^m \lambda_1 m^2}{[m^2+n^2B^2]^2} + \frac{\lambda_1 [\frac{n^3B^3}{[m^2+n^2B^2]^2} - \frac{1}{nB}]}{nB \displaystyle\sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2]\ (2-\delta_{i0})}{[i^2+n^2B^2]^2}} \sum_{i=0}^{M} \frac{(-1)^i [n^2B^2-\nu i^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}$$

(C41'b)

Substituting from equations (C10'), (C11'), (C19'), (C20'), (C24') into the first of equations (C43), one obtains

$$\rho_m' = \alpha\Theta Eh\ \{\delta_{mP}\ \frac{2PQB}{[P^2+Q^2B^2]} - \frac{4P[\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1][\frac{Q^3B^3}{[m^2+Q_1^2B^2]^2} - \frac{1}{QB}]}{Q^2B^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)Q^2B^2](2-\delta_{iO})}{[i^2 + Q^2B^2]^2}}\}$$

(C43')

Substituting from equations (C33'), (C35'), and (C41') into equation (C47) one obtains

$$\psi_{mnp} = \frac{A_1E[\bar{\eta}_{mn}'(C\bar{\phi}_n'''' \bar{\xi}_{pn}' - \bar{\phi}_n''\bar{\xi}_{pn}'') - \bar{\eta}_{mn}''(C\bar{\phi}_n''' \bar{\xi}_{pn}' - \bar{\phi}_n' \bar{\xi}_{pn}'')]}{C(\bar{\phi}_n' \bar{\phi}_n'''' - \bar{\phi}_n'' \bar{\phi}_n''')}$$

(C47')

Substituting from equations (C33'), (C34'), (C41') into equation (C48), one obtains

$$\psi_{mn} = \frac{\alpha\Theta Eh\ [\bar{\eta}_{mn}'(C\bar{\zeta}_n' \bar{\phi}_n'''' + \bar{\zeta}_n'' \bar{\phi}_n'') - \bar{\eta}_{mn}''(C\bar{\zeta}_n' \bar{\phi}_n''' + \bar{\zeta}_n'' \bar{\phi}_n')]\delta_{nQ}}{C(\bar{\phi}_n' \bar{\phi}_n'''' - \bar{\phi}_n'' \bar{\phi}_n''')}$$

(C48')

Substituting into equation (C51) from equations (C40'), (C43'), (C47') and (C48'), one finally obtains the following system of simultaneous equations for the unknown g_m's

$$\{\sum_{n=1,3...}^{N} \frac{\bar{\eta}_{mn}'(C\bar{\phi}_n''''\bar{\xi}_{mn}' - \bar{\phi}_n''\bar{\xi}_{mn}'') - \bar{\eta}_{mn}''(C\bar{\phi}_n'''\bar{\xi}_{mn}' - \bar{\phi}_n'\bar{\xi}_{mn}'')}{C(\bar{\phi}_n' \bar{\phi}_n'''' - \bar{\phi}_n'' \bar{\phi}_n''')} + C[1 + \lambda_2 B^4 \sum_{n=1,3..}^{N} \frac{n^2}{(m^2+n^2B^2)^2}]$$

$$-2\lambda_2 B^4 C \sum_{n=1,3..}^{N} [\frac{n^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^4}] \frac{[m^2 + (2+\nu)n^2B^2]}{[m^2+n^2B^2]^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{iO})}{[i^2 + n^2B^2]^2}}\}G(m)$$

$$= \delta_{mP} \frac{\lambda_1 \pi QB}{4[P_1^2+Q^2B^2]} - \frac{\lambda_1 \pi}{2m} \frac{P[\frac{[P^2+(2+\nu)Q^2B^2]}{(P^2 + Q^2B^2)} - 1][\frac{Q^3B^3}{(m^2+Q^2B^2)^2} - \frac{1}{QB}]}{Q^2B^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)Q^2B^2](2 - \delta_{i0})}{[i^2 + Q^2B^2]^2}}$$

$$- \frac{\lambda_1 \pi}{8} \frac{[\bar{\eta}'_{mQ}(C\bar{\zeta}'_Q\bar{\phi}''''_Q + \bar{\zeta}''_Q\bar{\phi}''_Q) - \bar{\eta}''_{mQ}(C\bar{\zeta}'_Q\bar{\phi}'''_Q + \bar{\zeta}''_Q\bar{\phi}'_Q)]}{mC(\bar{\phi}'_Q\bar{\phi}''''_Q - \bar{\phi}''_Q\,\bar{\phi}'''_Q)}$$

$$- \sum_{p=1}^{M} \{ \sum_{n=1,3...}^{N} \frac{[\bar{\eta}'_{mn}(C\bar{\phi}''''_n\bar{\xi}'_{pn} - \bar{\phi}''_n\bar{\xi}''_{pn}) - \bar{\eta}''_{mn}(C\bar{\phi}'''_n\bar{\xi}'_{pn} - \bar{\phi}'_n\bar{\xi}''_{pn})]}{C(\bar{\phi}'_n\bar{\phi}''''_n - \bar{\phi}''_n\bar{\phi}'''_n)}$$

$$-2\lambda_2 CB^4 \sum_{n=1,3...}^{N} [\frac{n^2}{[m^2+n^2B^2]^2} - \frac{1}{n^2B^4}] \frac{[p^2+(2+\nu)n^2B^2]}{[p^2+n^2B^2]^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2}} \}$$

$$(1 - \delta_{mp})\, G(p)\, \frac{p}{m} \qquad \text{(C51')}$$

$$(m = 1,\ 2,\ \ldots,\ M)$$

where

$$G(p) = \frac{g'_p}{\alpha \Theta E h}$$

<u>Procedure for numerical solution</u>. - It will be observed that equations (C51') are in a suitable form to be solved by the Gauss-Seidel iterative procedure (ref. 6), since the equation for any particular value of m has G(m) as its only unknown on the left-hand side and all the G's except

G(m) as unknowns on the right-hand side. Solution by the Gauss-Seidel procedure involves the initial assumption that all of the G(p) on the right-hand side of the m=1 equation are equal to zero. This permits the m=1 equation to be solved for an approximate value of G(1). Substituting this approximate value, together with G(3) = G(4) = ... = 0 into the m=2 equation permits that equation to be solved for an approximate value of G(2). Continuing in such a fashion it is possible to obtain a set of approximate values for G(1) through G(M). This set is called the first-iteration solution to the system of equations. A second-iteration solution is obtained in the same manner as the first except that the initial values of G(2), G(3), etc. are those given by the first-iteration solution. Third and higher iterations can be performed in a similar way. As one generates more sets of solutions to the system of equations there should appear a general observable trend whereby each individual G(p) tends to approach a certain value. The calculation is stopped when changes in all the G(p) values appear to be negligible (less than .000001 in the present calculations) from one iteration to the next.

With the G(p) known, equations (C33') (C34'), (C35'), and (C53) yield

$$C'(n) = \frac{(\pi/4)\delta_{nQ}[\lambda_1 \bar{\zeta}'_n \bar{\phi}''''_n/B + \lambda_2 \bar{\zeta}''_n \bar{\phi}''_n] + 2\sum_{p=1}^{M} [C\bar{\phi}''''_n \bar{\xi}'_{pn} - \bar{\phi}''_n \bar{\xi}''_{pn}] \frac{p}{BC} G(p)}{n(\bar{\phi}'_n \bar{\phi}''''_n - \bar{\phi}''_n \bar{\phi}'''_n)}$$

$$C''(n) = \frac{(\pi/4)\delta_{nQ}[\lambda_1 \bar{\zeta}'_n \bar{\phi}'''_n/B + \lambda_2 \bar{\zeta}''_n \bar{\phi}'_n] + 2\sum_{p=1}^{M} [C\bar{\phi}'''_n \bar{\xi}'_{pn} - \bar{\phi}'_n \bar{\xi}''_{pn}] \frac{p}{BC} G(p)}{n(\bar{\phi}'_n \bar{\phi}''''_n - \bar{\phi}''_n \bar{\phi}'''_n)} \qquad (C53')$$

where

$$C'(n) = \frac{c'_n}{\alpha\Theta Eh}$$

$$C''(n) = \frac{c''_n}{\alpha\Theta Eh}$$

With c'_n, c''_n and g'_m known, equations (C10') to (C13'), and (C55) yield:

$$B'(n) = -\frac{\delta_{nQ} P\pi \left\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\right\}}{Q^2B^2 \sum_{i=0}^{M} \frac{[i^2+(2+\nu)Q^2B^2](2-\delta_{i0})}{[i^2 + Q^2B^2]^2}} + \frac{4\sum_{m=1}^{M} \frac{m[m^2+(2+\nu)n^2B^2]}{[m^2+n^2B^2]^2} G(m)}{n \sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2)^2}}$$

$$- \frac{\sum_{m=0}^{M} \frac{[n^2B^2 - \nu m^2](2-\delta_{mo})}{[m^2 + n^2B^2]^2} [C'(n) - (-1)^m C''(n)]}{\sum_{i=0}^{M} \frac{[i^2+(2+\nu)n^2B^2](2-\delta_{i0})}{[i^2+n^2B^2]^2}} \qquad (C55')$$

where

$$B'(n) = \frac{B'_n}{\alpha\Theta Eh}$$

Note that

$$G(m) = g'_m/(\alpha\Theta Eh)$$

$$C'(n) = c'_n/(\alpha\Theta Eh)$$

$$C''(n) = C''_n/(\alpha\Theta Eh)$$

With B'_n as known, from equations (C14) and (C15), one obtains the following tensions at the left ends of the x-wise stiffeners where they join the rigid vertical stiffener:

$$P_3(0) = P_4(0) = -\alpha\theta A_3 E\lambda_2 \sum_{n=1,3\ldots}^{N} \frac{\pi}{4n} B'(n) \qquad (C15')$$

With the coefficients c_n', c_n'' g_m', g_m'' and B_n' as known quantities, equations (B60) will furnish the values of s_n', s_n'', s_m''', s_m'''', and equations (B57) to (B59) and (B61) the values of the j_{mn}. Equations (B16) will then give the stiffener force, equations (B19) to (B25) the plate stresses. One obtains the following equations for these quantities.

$$P_1(y) = \theta A_1 E\alpha \sum_{n=1,3\ldots}^{N} [C'(n) - \nu B'(n)] \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \qquad (C83)$$

$$P_2(y) = \theta A_1 E\alpha \sum_{n=1,3\ldots}^{N} C''(n) \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \qquad (C84)$$

$$P_3(x) = P_4(x) = \theta A_3 E\alpha \sum_{m=1}^{M} G(m) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \qquad (C85)$$

$$N_x = \alpha\theta E h \sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} G(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \qquad (C86)$$

$$(N_x)_{y=0} = (N_x)_{y=b} = \alpha\theta E h \sum_{m=1}^{M} G(m) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \qquad (C87)$$

$$N_y = \alpha\theta E h \sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} C(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \qquad (C88)$$

$$(N_y)_{x=0} = \alpha\Theta Eh \sum_{n=1,3\ldots}^{N} C'(n) \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{C89}$$

$$(N_y)_{x=a} = \alpha\Theta Eh \sum_{n=1,3\ldots}^{N} C''(n) \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{C90}$$

$$N_{xy} = -\alpha\Theta Eh\left\{\sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} J(m,n) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) + \sum_{n=1,3\ldots}^{N} \frac{B'(n)}{nB\pi} \cos\left(\frac{n\pi y}{b}\right)\right\}$$

$$(0 \le x \le a)$$
$$(0 \le y \le b) \tag{C91}$$

where $G(m,n)$, $C(m,n)$ and $J(m,n)$ are known quantities and are defined as follows

$$G(m,n) = \frac{2n^2B^2}{\pi[m^2 + n^2B^2]^2}\{m[C'(n) - (-1)^m C''(n)] + 2nB^2 G(m)\}$$

$$+ \frac{2m[m^2 + 2n^2B^2]B'(n)}{\pi[m^2 + n^2B^2]^2} - \frac{Q^2B^2}{(P^2+Q^2B^2)} \delta_{mP} \delta_{nQ}$$

$$C(m,n) = \frac{2mB'(n)}{n^2B^2\pi}\left\{\frac{m^2[m^2+2n^2B^2]}{[m^2+n^2B^2]^2} - 1\right\} - \frac{P^2}{(P^2+Q^2B^2)} \delta_{mP}\delta_{nQ}$$

$$+ \frac{2m^2}{\pi[m^2+n^2B^2]^2}\{m[C'(n) - (-1)^m C''(n)] + 2nB^2 G(m)\}$$

$$J(m,n) = \frac{2nB}{\pi[m^2+n^2B^2]^2}\{n^2B^2B'(n) - m^2[C'(n) - (-1)^m C''(n)] - 2mnB^2 G(m)\}$$

$$+ \frac{PQB}{(P^2 + Q^2B^2)} \delta_{mP} \delta_{nQ} \tag{C83}$$

And from the first of equations (4), one obtains the following running tension between the rigid stiffener and the plate edge at x=0:

$$N_x(0,y) = \sum_{n=1,3\ldots}^{N} B_n' \sin\left(\frac{n\pi y}{b}\right) = \alpha\Theta Eh \sum_{n=1,3\ldots}^{N} B'(n) \sin\left(\frac{n\pi y}{b}\right)$$

$$(0 < y < b) \qquad \text{(C84)}$$

Numerical results for $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. - The numerical procedure and equations described above were applied to the special case of a square plate (B = 1), with all stiffener areas equal ($A_1 = A_2 = A_3 = A_4$), a pillow-shaped temperature distribution (P = Q = 1), and Poisson's ratio ν equals to 0.3. The assumption that B = 1 and all stiffener areas are equal implies that $\lambda_1 = \lambda_2$ (see eqs. (C71)), and the common symbol λ will therefore be used for both λ_1 and λ_2.

The results obtained for the stiffener tensions and plate stress are shown in dimensionless form in figure 6 for $\lambda = 2.0$ and figure 7 for $\lambda = 1.0$. In general, stresses were computed at x/a and y/a interval of 0.1.

Limiting case of large stiffener areas. - For the case in which the stiffener cross-sectional areas are very large compared to the plate cross-sectional area, equations (C60) to (C63) may be employed as approximations which become more and more accurate as the ratios of stiffener to plate cross-sectional areas approach infinity. The quantities Ξ_n and Ξ_n' needed in these equations are defined by equations (C56e) and (C56f) which can be reduced to the following expressions if use is made of equations (3), (C10'), and (C13'):

$$\Xi_n = -\nu \frac{\displaystyle\sum_{m=0}^{M} \frac{(n^2B^2 - \nu m^2)(2 - \delta_{mo})}{(m^2 + n^2B^2)^2}}{\displaystyle\sum_{m=0}^{M} \frac{[m^2+(2+\nu)n^2B^2](2 - \delta_{mo})}{(m^2 + n^2B^2)^2}} \tag{C56'e}$$

$$\Xi'_n = -\nu \frac{\displaystyle\sum_{m=0}^{M} (-1)^m \frac{(n^2B^2 - \nu m^2)(2-\delta_{mo})}{(m^2 + n^2B^2)^2}}{\displaystyle\sum_{m=0}^{M} \frac{[m^2 + (2+\nu)n^2B^2](2 - \delta_{mo})}{(m^2 + n^2B^2)^2}} \tag{C56'f}$$

Substituting from equations (C10'), (C11'), (C12'), (C19'), (C20'), (C24') and (C34') into equations (C61) and (C62), one obtains

$$\bar{c}''_n = \alpha\Theta \frac{h}{A_1} \bar{\bar{c}}''_n \delta_{nQ} \tag{C61'}$$

$$\bar{g}'_m = \bar{g}''_m = \alpha\Theta \frac{h}{A_3} \bar{\bar{g}}_m \tag{C62'}$$

where

$$\bar{\bar{c}}''_n = \frac{PQB}{(P^2+Q^2B^2)} + \frac{\{1 + 2Q^4B^4 \displaystyle\sum_{m=1}^{M} \frac{(-1)^m}{[m^2 + Q^2B^2]^2}\}\{\frac{P[P^2 + (2+\nu)\ Q^2B^2]}{[P^2 + Q^2B^2]} - P\}}{Q^3B^3 \displaystyle\sum_{m=0}^{M} \frac{[m^2+(2+\nu)Q^2B^2]\ (2-\delta_{mo})}{[m^2 + Q^2B^2]^2}} \tag{C61'a}$$

$$\bar{\bar{g}}_m = \frac{PQB}{(P^2+Q^2B^2)} \delta_{mP} - \frac{2P\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}\{\frac{QB}{[m^2+Q^2B^2]^2} - \frac{1}{Q^3B^3}\}}{\displaystyle\sum_{i=0}^{M} \frac{[i^2 + (2+\nu)\ Q^2B^2]\ (2-\delta_{io})}{[i^2 + Q^2B^2]^2}} \tag{C62'a}$$

With $\bar{c}''_n, \bar{g}'_m$ and $\bar{g}''_m$ known, $\bar{c}'_n$ (eq. (C60)) can be reduced to the following expressions if use is made of equations (3), (C10'), (C12'), (C34'), (C61'), (C62'), (C56'e), (C56'f)

$$\bar{c}'_n = \alpha\Theta \frac{h}{A_1} \frac{c^*_n}{1 - \Xi_n} - \frac{1}{1 - \Xi_n} \frac{\alpha\Theta}{b} \bar{\bar{c}}'_n \delta_{nQ} \qquad (C60')$$

where

$$\bar{\bar{c}}'_n = \frac{\nu PQ\, \pi^2 \{\frac{[P^2 + (2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{Q^2B^2 \sum_{m=0}^{M} \frac{[m^2+(2+\nu)Q^2B^2](2-\delta_{mo})}{[m^2 + Q^2B^2]^2}} \qquad (C60'a)$$

$$c^*_n = \bar{S}'_n\delta_{nQ} - \bar{\bar{c}}''_n \Xi'_n\delta_{nQ} + \frac{4\nu B}{C} \{\sum_{m=1}^{M} \frac{\bar{\bar{g}}_m[m^2+(2+\nu)n^2B^2]}{[m^2+n^2B^2]^2}\}/\{\sum_{m=0}^{M} \frac{[m^2+(2+\nu)n^2B^2](2-\delta_{mo})}{[m^2 + n^2B^2]^2}\}$$

(C60'b)

with

$$\bar{S}'_n = \frac{PQB}{(P^2+Q^2B^2)} - \frac{\{1+2Q^4B^4\sum_{m=1}^{M}\frac{1}{[m^2+Q^2B^2]^2}\}\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\}\, P}{Q^3B^3 \sum_{m=0}^{M} \frac{[m^2+(2+\nu)Q^2B^2](2 - \delta_{mo})}{[m^2 + Q^2B^2]^2}} \qquad (C60'c)$$

Substituting the above expressions for $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$ and $\bar{g}''_m$ into equation (C55), one obtains

$$B'_n = \alpha\Theta Eh\, \bar{B}'_n\, \delta_{nQ} + \alpha\Theta Eh\, \lambda_1 B^*_n \qquad (C55'')$$

where

$$\bar{B}'_n = \frac{\bar{\bar{c}}'_n \sum_{m=0}^{M} \frac{[Q^2B^2 - \nu m^2](2-\delta_{mo})}{[m^2+Q^2B^2]^2}}{(1-\Xi_n)Q\pi \sum_{m=0}^{M} \frac{[m^2+(2+\nu)Q^2B^2](2-\delta_{mo})}{[m^2 + Q^2B^2]^2}} - \frac{P\pi\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{Q^2B^2 \sum_{m=0}^{M} \frac{[m^2+(2+\nu)Q^2B^2](2-\delta_{mo})}{[m^2 + Q^2B^2]^2}}$$

(C55"a)

and

$$B^*_n = \frac{1}{nB \sum_{m=0}^{M} \frac{[m^2+(2+\nu)n^2B^2](2-\delta_{mo})}{[m^2 - n^2B^2]^2}}$$

$$\{ \frac{B\pi}{C} \sum_{m=1}^{M} \frac{[m^2+(2+\nu)n^2B^2]\bar{\bar{g}}_m}{[m^2 + n^2B^2]^2} - \frac{\pi}{4} \frac{c^*_n}{1 - \Xi_n} \sum_{m=0}^{M} \frac{[n^2B^2-\nu m^2](2-\delta_{mo})}{[m^2+n^2B^2]^2}$$

$$-\delta_{nQ} \frac{\pi\bar{\bar{c}}''_n}{4} \sum_{m=0}^{M} \frac{(-1)^m [n^2B^2 - \nu m^2](2-\delta_{mo})}{[m^2 + n^2B^2]^2} \}$$

(C54"b)

With the B'_n known, equations (C14) and (C15) yield the following tensions at the left ends of the x-wise stiffeners where they join the rigid vertical stiffener, when terms of degree higher than 1 in λ_1 and λ_2 are neglected:

$$P_3(0) = P_4(0) = -\alpha\Theta A_3E\lambda_2\delta_{nQ} \frac{\pi\bar{B}'_n}{4Q} \qquad \text{(C15")}$$

With the coefficients c'_n, c''_n, g'_m and g''_m known, the stiffener stresses and the plate stresses are given as follows:

$$P_1(y) = \alpha\Theta A_1E\lambda_1 \sum_{n=1,3...}^{N} [\frac{\pi}{4nB} \frac{c^*_n}{1-\Xi_n} - \nu B^*_n] \sin(\frac{n\pi y}{b}) \qquad (0 < y < b)$$

(C85)

$$P_2(y) = \alpha\Theta A_1 E\lambda_1 \sum_{n=1,3\ldots}^{N} \delta_{nQ} \frac{\pi\bar{\bar{c}}''_n}{4QB} \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{C86}$$

$$P_3(x) = P_4(x) = \alpha\Theta A_3 E\lambda_2 \sum_{m=1}^{M} \frac{B\pi\bar{\bar{g}}_m}{4m} \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{C87}$$

$$N_x = \alpha\Theta Eh \sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} \delta_{nQ}\, \bar{g}_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{C88}$$

$$(N_x)_{y=0} = (N_x)_{y=b} = \alpha\Theta Eh\lambda_2 \sum_{m=1}^{M} \frac{B\pi\bar{\bar{g}}_m}{4m} \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{C89}$$

$$N_y = \alpha\Theta Eh \sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} \delta_{nQ}\, \bar{c}_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{C90}$$

$$(N_y)_{x=0} = -\alpha\Theta Eh \sum_{n=1,3\ldots}^{N} \delta_{nQ} \frac{1}{1 - \Xi_n} \frac{\bar{\bar{c}}'_n}{n\pi} \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{C91}$$

$$(N_y)_{x=a} = -\alpha\Theta Eh \left\{\sum_{m=1}^{M} \sum_{n=1,3\ldots}^{N} \delta_{nQ}\, \bar{j}_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\right.$$

$$\left. + \sum_{n=1,3\ldots}^{N} \frac{\bar{B}'_n}{Q\pi B} \delta_{nQ} \cos\left(\frac{n\pi y}{b}\right)\right\} \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le b)\end{matrix} \tag{C93}$$

where

$$\bar{g}_{mn} = -\frac{Q^2B^2}{(P^2+Q^2)}\delta_{mP} - \frac{2mQB^2\bar{\bar{c}}'_n}{\pi^2(1-\Xi_n)[m^2+Q^2B^2]^2} + \frac{2m[m^2 + 2Q^2B^2]\bar{B}'_n}{\pi[m^2 + Q^2B^2]^2}$$

$$\bar{c}_{mn} = \bar{B}'_n \frac{2m}{Q^2\pi}\frac{1}{B^2}\left\{\frac{m^2(m^2+2Q^2B^2)}{[m^2+Q^2B^2]^2} - 1\right\} - \delta_{mP}\frac{P^2}{(P^2 + Q^2B^2)}$$

$$-\frac{2m^3\bar{\bar{c}}'_n}{Q\pi^2(1 - \Xi_n)[m^2 + Q^2B^2]^2} \tag{C94}$$

$$\bar{j}_{mn} = \frac{PQB}{(P^2+Q^2B^2)}\delta_{mP} + \frac{2m^2B\bar{\bar{c}}'_n}{\pi^2[m^2+Q^2B^2]^2(1-\Xi_n)} + \frac{2Q^3B^3\bar{B}'_n}{\pi[m^2 + Q^2B^2]^2}$$

With the B'_n known, the first of equations (4) yields the following running tensions between the rigid stiffener and the plate edge at $x = 0$:

$$N_x(0,y) = \alpha\Theta Eh\sum_{n=1,3\ldots}^{N}\delta_{nQ}\,\bar{B}'_n \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{C95}$$

<u>Numerical results for limiting case of large stiffener areas</u>. - The numerical results for $\lambda_1 = \lambda_2 = \lambda \rightarrow 0$ for square plates (B=1) with all stiffeners identical, subjected to a pillow-shaped temperature distribution (P = Q = 1) are presented in dimensionless form in figures 8 and 9. The former represents for $\nu = 0.3$, the latter for $\nu = 0$.

APPENDIX D

ANALYSIS FOR THE CASE OF TWO OPPOSITE STIFFENERS WITH PRESCRIBED DISPLACEMENT CONDITIONS

This appendix considers the case of figure 4b, in which the stiffeners at $x = 0$ and $x = a$ are bent to prescribed shapes defined by known values of K_n' and K_n'' in equations (8) and (9). Correspondingly, the Fourier coefficients B_n' and B_n'', which describe the running tension between the stiffeners and the plate, are now unknowns. In addition the loading resultants T_1, M_1, T_2 and M_2 constitute four new knowns, supplanting $P_3(0)$, $P_4(0)$, $P_3(a)$ and $P_4(a)$, which are now unknowns. Further introductory remarks can be made for this case which are obvious generalizations of those in appendix C.

Formulation of Boundary Condition of Prescribed Curvature

The boundary curvatures $\partial^2 u/\partial y^2$ of the edges $x = 0$ and $x = a$ of the plate are (see appendix C)

$$\begin{aligned}
\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} &= (C_3 - C_4)\left(\frac{\partial^3 F}{\partial x \partial y^2}\right)_{x=0} - C_2\left(\frac{\partial^3 F}{\partial x^3}\right)_{x=0} - \left(\frac{\partial e_y}{\partial x}\right)_{x=0} \\
\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=a} &= (C_3 - C_4)\left(\frac{\partial^3 F}{\partial x \partial y^2}\right)_{x=a} - C_2\left(\frac{\partial^3 F}{\partial x^3}\right)_{x=a} - \left(\frac{\partial e_y}{\partial x}\right)_{x=a}
\end{aligned} \qquad \text{(D1)}$$

The terms on the right-hand side of these equations can be expressed in series form with the aid of equations (19),(20), B41), and (B44). The result is

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=0} = \sum_{n=1}^{N} [(c_4-c_3) \frac{n\pi}{b} \sum_{m=0}^{M} j_{mn} - c_2 \sum_{m=0}^{M} d_{mn} - V_n'] \sin \frac{n\pi y}{b}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{x=a} = \sum_{n=1}^{N} [(c_4-c_3) \frac{n\pi}{b} \sum_{m=0}^{M} (-1)^m j_{mn} - c_2 \sum_{m=0}^{M} (-1)^m d_{mn} - V_n''] \sin \frac{n\pi y}{b} \tag{D2}$$

Comparing equations (D2) with (8) and (9), one obtains the following 2N equations representing the conditions of prescribed curvatures along the edges x=0 and x=a:

$$K_n' = (c_4 - c_3) \frac{n\pi}{b} \sum_{m=0}^{M} j_{mn} - c_2 \sum_{m=0}^{M} d_{mn} - V_n'$$

$$K_n'' = (c_4 - c_3) \frac{n\pi}{b} \sum_{m=0}^{M} (-1)^m j_{mn} - c_2 \sum_{m=0}^{M} (-1)^m d_{mn} - V_n''$$

$$(n = 1, 2, \ldots, N) \tag{D3}$$

The unknown j_{mn} and d_{mn} in these equations can be expressed in terms of the basic unknowns $B_n', B_n'', c_n', c_n'', g_m', g_m''$ (see appendix C).

Substituting from equations (C5) and (C8) into equations (D3), and separating B_n' and B_n'' terms, one obtains

$$K_n' = -V_n' - B_n'\left(\frac{n\pi}{b}\right)^2 c_1 \sum_{m=0}^{M} \frac{1}{E_{mn}}[(c_3-c_4)\left(\frac{n\pi}{b}\right)^2 - c_2\left(\frac{m\pi}{2}\right)^2]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+B_n''\left(\frac{n\pi}{b}\right)^2 c_1 \sum_{m=0}^{M} (-1)^m \frac{1}{E_{mn}}[(c_3-c_4)\left(\frac{n\pi}{b}\right)^2 - c_2\left(\frac{m\pi}{a}\right)^2]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+ \sum_{m=0}^{M} \frac{1}{E_{mn}} \{T_{mn}\left(\frac{m\pi}{a}\right) [(c_3 - c_4)\left(\frac{n\pi}{b}\right)^2 - c_2\left(\frac{m\pi}{a}\right)^2]$$

$$-(\frac{n\pi}{b})c_2\ [c_1(\frac{n\pi}{b})^2 - c_3(\frac{m\pi}{a})^2][\frac{2}{b}\ \frac{m\pi}{a}(B_m^{'''}-(-1)^n B_m^{''''}) - \frac{2-\delta_{mo}}{a}\ \frac{n\pi}{b}(c_n^{'}-(-1)^m c_n^{''})]$$

$$+\ \frac{2}{b}(\frac{m\pi}{a})(\frac{n\pi}{b})c_1[(c_3-c_4)(\frac{n\pi}{b})^2 - c_2(\frac{m\pi}{a})^2][g_m^{'} - (-1)^n\ g_m^{''}]\}$$

(D4)

$$K_n^{''} = -V_n^{''}-B_n^{'}(\frac{n\pi}{b})^2\ c_1\sum_{m=0}^{M}(-1)^m\ \frac{1}{E_{mn}}\ [(c_3-c_4)(\frac{n\pi}{b})^2 - c_2(\frac{m\pi}{a})^2](\frac{2-\delta_{mo}}{a})$$

$$+B_n^{''}(\frac{n\pi}{b})^2\ c_1\sum_{m=0}^{M}\frac{1}{E_{mn}}\ [(c_3-c_4)(\frac{n\pi}{b})^2 - c_2(\frac{m\pi}{a})^2](\frac{2-\delta_{mo}}{a})$$

$$+\sum_{m=0}^{M}(-1)^m\ \frac{1}{E_{mn}}\ \{T_{mn}\ (\frac{m\pi}{a})[(c_3-c_4)(\frac{n\pi}{b})^2 - c_2(\frac{m\pi}{a})^2]$$

$$-(\frac{n\pi}{b})c_2[c_1(\frac{n\pi}{b})^2-c_3(\frac{m\pi}{a})^2][\frac{2}{b}\ \frac{m\pi}{a}\ (B_m^{'''}-(-1)^n B_m^{''''}) - \frac{2-\delta_{mo}}{a}\ \frac{n\pi}{b}(c_n^{'}-(-1)^m c_n^{''})]$$

$$+\ \frac{2}{b}(\frac{m\pi}{a})(\frac{n\pi}{b})c_1[(c_3-c_4)(\frac{n\pi}{b})^2-c_2(\frac{m\pi}{a})^2][g_m^{'}-(-1)^n g_m^{''}]\}$$

These equations can be solved for each unknown $B_n^{'}$ and $B_n^{''}$ in terms of the corresponding $K_n^{'}$, $K_n^{''}$, $c_n^{'}$, $c_n^{''}$, and all the $g_m^{'}$, $g_m^{''}$. Rearranging equations (D4), one obtains

$$B_n^{'}\gamma_n^{(1)} - B_n^{''}\gamma_n^{(2)} = \delta_n^{(1)} + \sum_{m=1}^{M}\mu_{mn}^{'}[\bar{g}_m^{'}-(-1)^n\bar{g}_m^{''}]+\sum_{m=0}^{M}\nu_{mn}^{'}[\bar{c}_n^{'} - (-1)^m\ \bar{c}_n^{''}]$$

$$B_n^{'}\gamma_n^{(2)}-B_n^{''}\gamma_n^{(1)} = \delta_n^{(2)} + \sum_{m=1}^{M}(-1)^m\mu_{mn}^{'}[\bar{g}_m^{'}-(-1)^n\bar{g}_m^{''}] + \sum_{m=0}^{M}(-1)^m\nu_{mn}^{'}[\bar{c}_n^{'} - (-1)^m\ \bar{c}_n^{''}]$$

(D5)

$\gamma_n^{(1)}$, $\mu_{mn}^{'}$ and $\nu_{mn}^{'}$ have been defined already in appendix C; where

$$\gamma_n^{(2)} = (\frac{n\pi}{b})^2 \, C_1 \sum_{m=0}^{M} \frac{(-1)^m}{E_{mn}} [(C_3-C_4)\,(\frac{n\pi}{b})^2 - C_2(\frac{m\pi}{a})^2]\,(\frac{2-\delta_{mo}}{a}) \tag{D6}$$

$$\delta_n^{(1)} = -K_n' - V_n' + \sum_{m=0}^{M} \frac{1}{E_{mn}} \{T_{mn}(\frac{m\pi}{a})[(C_3-C_4)(\frac{n\pi}{b})^2 - C_2(\frac{m\pi}{a})^2]$$

$$-C_2[C_1(\frac{n\pi}{b})^2 - C_3(\frac{m\pi}{a})^2]\,\frac{n\pi}{b}\,\frac{2}{b}\,\frac{m\pi}{a}\,[B_m''' - (-1)^n B_m'''']\} \tag{D7}$$

$$\delta_n^{(2)} = -K_n'' - V_n'' + \sum_{m=0}^{M} \frac{(-1)^m}{E_{mn}} \{T_{mn}(\frac{m\pi}{a})[(C_3-C_4)(\frac{n\pi}{b})^2 - C_2(\frac{m\pi}{a})^2]$$

$$-C_2[C_1(\frac{n\pi}{b})^2 - C_3(\frac{m\pi}{a})^2]\,\frac{n\pi}{b}\,\frac{2}{b}\,\frac{m\pi}{a}\,[B_m''' - (-1)^n B_m'''']\} \tag{D8}$$

Solving simultaneous equations (D5) for B_n' and B_n'', one obtains

$$B_n' = \frac{D_n'}{D_n}$$

$$B_n'' = \frac{D_n''}{D_n} \tag{D9}$$

where

$$D_n = (\gamma_n^{(2)})^2 - (\gamma_n^{(1)})^2 \tag{D10}$$

$$D_n' = \gamma_n^{(2)}\delta_n^{(2)} + \gamma_n^{(2)} \sum_{m=1}^{M} (-1)^m \mu_{mn}' [\bar{g}_m' - (-1)^n \bar{g}_m''] + \gamma_n^{(2)} \sum_{m=0}^{M} (-1)^m \nu_{mn}' [\bar{c}_n' - (-1)^m \bar{c}_n'']$$

$$-\gamma_n^{(1)}\delta_n^{(1)} - \gamma_n^{(1)} \sum_{m=1}^{M} \mu_{mn}' [\bar{g}_m' - (-1)^n \bar{g}_m''] - \gamma_n^{(1)} \sum_{m=0}^{M} \nu_{mn}' [\bar{c}_n' - (-1)^m \bar{c}_n''] \tag{D11}$$

and

$$D_n''=\gamma_n^{(1)}\delta_n^{(2)}+\gamma_n^{(1)}\sum_{m=1}^{M}(-1)^m\mu_{mn}'[\bar{g}_m'-(-1)^n\bar{g}_m'']+\gamma_n^{(1)}\sum_{m=0}^{M}(-1)^m\nu_{mn}'[\bar{c}_n'-(-1)^m\bar{c}_n'']$$

$$-\gamma_n^{(2)}\delta_n^{(1)}-\gamma_n^{(2)}\sum_{m=1}^{M}\mu_{mn}'[\bar{g}_m'-(-1)^n\bar{g}_m''] - \gamma_n^{(2)}\sum_{m=0}^{M}\nu_{mn}'[\bar{c}_n'-(-1)^m\bar{c}_n''] \tag{D12}$$

Thus the unknown B_n', B_n'' have, in effect, through equations (D9), been supplanted by an equal number of known K_n' and K_n''. If the edges x=0 and x=a are forced to remain straight, then the K_n' and K_n'' are all zero.

Formulation of Boundary Conditions of Equilibrium

The normal forces acting on the rigid stiffener at x=0 must be self-equilibrating. Therefore equations (C14) and (C15) of appendix C apply also in the present case. They are:

$$P_3(0) = -\sum_{n=1}^{N}\frac{b}{n\pi}B_n' + \frac{T_1}{2} - \frac{M_1}{b} \tag{D13}$$

$$P_4(0) = \sum_{n=1}^{N}(-1)^n\frac{b}{n\pi}B_n' + \frac{T_1}{2} + \frac{M_1}{b} \tag{D14}$$

Similarly, the normal forces acting on the rigid stiffener at x=a must be self-equilibrating, i.e.

$$P_3(a) + P_4(a) + \int_0^b N_x(a,y)\,dy = T_2$$

$$P_4(a)\cdot b + \int_0^b y\,N_x(a,y)\,dy = \frac{T_2 b}{2} + M_2$$

By substituting

$$N_x(a,y) = \sum_{n=1}^{N} B_n'' \sin\left(\frac{n\pi y}{b}\right)$$

and solving for $P_3(a)$ and $P_4(a)$, one obtains

$$P_3(a) = -\sum_{n=1}^{N} \frac{b}{n\pi} B_n'' + \frac{T_2}{2} - \frac{M_2}{b} \tag{D15}$$

$$P_4(a) = \sum_{n=1}^{N} (-1)^n \frac{b}{n\pi} B_n'' + \frac{T_2}{2} + \frac{M_2}{b} \tag{D16}$$

Thus, through (D13) - (D16), the unknowns $P_3(0)$, $P_4(0)$, $P_3(a)$ and $P_4(a)$ have been expressed in terms of the knowns T_1, M_1, T_2 and M_2.

Separating B_n' and B_n'' Terms in R_n', R_n'', R_m''', R_m''''

Equations (D13), (D14), (D15), (D16), and (B69) can be used to eliminate $P_3(0)$, $P_4(0)$, $P_3(a)$, $P_4(a)$, and K_{mn} from equations (B68). If the B_n' and B_n'' terms are then written separately from the rest, equations (B68) become

$$\begin{aligned}
R_n' &= S_n^{(5)} + \gamma_n' B_n' - \gamma_n'' B_n'' \\
R_n'' &= S_n^{(6)} - \gamma_n'' B_n' + \gamma_n''' B_n'' \\
R_m''' &= S_m^{(7)} + \sum_{n=1}^{N} B_n' H_{mn} - \sum_{n=1}^{N} (-1)^m B_n'' H_{mn} \\
R_m'''' &= S_m^{(8)} - \sum_{n=1}^{N} (-1)^n B_n' H_{mn} + \sum_{n=1}^{N} (-1)^{m+n} B_n'' H_{mn}
\end{aligned} \tag{D17}$$

where $S_n^{(5)}$, $S_n^{(6)}$, $S_m^{(7)}$, $S_m^{(8)}$ are completely known quantities defined as follows:

$$S_n^{(5)} = Q_n' + \frac{2}{b}[P_1(0) - (-1)^n P_1(b)] + A_1E_1 \frac{n\pi}{b} T_n' - \sum_{m=1}^{M} S_{mn}^{(2)} \qquad \text{(D18)}$$

$$S_n^{(6)} = -Q_n' + \frac{2}{b}[P_2(0)-(-1)^n P_2(b)] + A_2E_2 \frac{n\pi}{b} T_n'' + \sum_{m=1}^{M}(-1)^m S_{mn}^{(2)} \qquad \text{(D19)}$$

$$S_m^{(7)} = Q_m''' + \frac{a}{bm\pi}(B_m''' - B_m'''') + A_3E_3 \frac{m\pi}{a} (C_3B_m''' + T_m''')$$
$$- \sum_{n=1}^{N} S_{mn}^{(2)} + \frac{T_1}{a} - \frac{2M_1}{ab} - (-1)^m \left[\frac{T_2}{a} - \frac{2M_2}{ab}\right] \qquad \text{(D20)}$$

$$S_m^{(8)} = -Q_m'''' - \frac{a}{bm\pi}(B_m''' - B_m'''') + A_4E_4 \frac{m\pi}{a} (C_3B_m'''' + T_m'''')$$
$$+ \sum_{n=1}^{N}(-1)^n S_{mn}^{(2)} + \frac{T_1}{a} + \frac{2M_1}{ab} - (-1)^m \left[\frac{T_2}{a} + \frac{2M_2}{ab}\right] \qquad \text{(D21)}$$

with

$$S_{mn}^{(2)} = \frac{1}{E_{mn}} \left\{ \frac{mn\pi^2}{ab} T_{mn} - \frac{2}{b}\left(\frac{m\pi}{a}\right)^3 C_2[B_m''' - (-1)^n B_m''''] \right\} \qquad \text{(D22)}$$

H_{mn}, γ_n', and γ_n'' have been defined already in appendix C; γ_n''' are also known quantities and are defined by the following equation:

$$\gamma_n''' = \frac{b}{an\pi} + A_2E_2 \frac{n\pi}{b} C_3 + \sum_{m=1}^{M} \frac{1}{E_{mn}} \frac{2}{a} \left(\frac{n\pi}{b}\right)^3 C_1 \qquad \text{(D23)}$$

Revision of Equations (B62) to (B65)

Substituting equations (D17) into equations (B62) to (B65), one obtains

$$\bar{c}_n'\alpha_1(n)-\bar{c}_n''\beta_1(n)=S_n^{(5)}+\gamma_n'B_n' - \gamma_n''B_n'' - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} \frac{\bar{g}_m'-(-1)^n\bar{g}_m''}{E_{mn}}$$

$$(n=1,2,3,\ldots,N) \qquad (D24)$$

$$-\bar{c}_n'\beta_1(n) + \bar{c}_n''\alpha_2(n) = S_n^{(6)}-\gamma_n''B_n' + \gamma_n'''B_n''+\frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M}(-1)^m \frac{\bar{g}_m'-(-1)^n\bar{g}_m''}{E_{mn}}$$

$$(n=1,2,3\ldots,N) \qquad (D25)$$

$$\bar{g}_m'\alpha_3(m)-\bar{g}_m''\beta_2(m) = S_m^{(7)} + \sum_{n=1}^{N} B_n'H_{mn}-\sum_{n=1}^{N}(-1)^m B_n''H_{mn} - \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{\bar{c}_n'-(-1)^m\bar{c}_n''}{E_{mn}}$$

$$(m=1,2,3,\ldots,M) \qquad (D26)$$

$$-\bar{g}_m'\beta_2(m)+\bar{g}_m''\alpha_4(m) = S_m^{(8)}-\sum_{n=1}^{N}(-1)^n B_n'H_{mn} + \sum_{n=1}^{N}(-1)^{m+n} B_n''H_{mn}$$

$$+ \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N}(-1)^n \frac{\bar{c}_n'-(-1)^m\bar{c}_n''}{E_{mn}} \qquad (m=1,2,3,\ldots,M)$$

$$(D27)$$

Equations (D24) to (D27) can be used to obtain a system of simultaneous equations in which $\bar{c}_n'$, $\bar{c}_n''$, $\bar{g}_m'$, $\bar{g}_m''$ are the only unknown coefficients, by eliminating B_n' and B_n'' with the aid of equations (D9). The resulting system of simultaneous equations is:

$$\bar{c}_n'\phi_n^{(1)} - \bar{c}_n''\phi_n^{(2)} = \zeta_n^{(1)} + \sum_{m=1}^{M} \xi_{mn}^{(1)} [\bar{g}_m' - (-1)^n \bar{g}_m''] \qquad (n=1,2,\ldots,N)$$

$$(D28)$$

$$-\bar{c}_n'\phi_n^{(3)} + \bar{c}_n''\phi_n^{(4)} = \zeta_n^{(2)} -\sum_{m=1}^{M} \xi_{mn}^{(2)}[\bar{g}_m' - (-1)^n \bar{g}_m''] \qquad (n=1,2,\ldots,N)$$

$$(D29)$$

$$\sum_{p=1}^{M} \bar{g}_p' \{\alpha_3(m)\delta_{mp} - \sum_{n=1}^{N} \frac{H_{mn}}{D_n} \mu_{pn}' [\gamma_n^{(2)}((-1)^p + (-1)^m) - \gamma_n^{(1)}(1 + (-1)^{m+p})]\} - \sum_{p=1}^{M} \bar{g}_p'' \Gamma_{mp}^{(2)}$$

$$+ \sum_{n=1}^{N} \bar{c}_n' \Gamma_{mn}^{(3)} - \sum_{n=1}^{N} \bar{c}_n'' \Gamma_{mn}^{(4)}$$

$$= S_m^{(7)} + \sum_{n=1}^{N} \frac{H_{mn}}{D_n} [\gamma_n^{(2)}\delta_n^{(2)} - \gamma_n^{(1)}\delta_n^{(1)} - (-1)^m \gamma_n^{(1)}\delta_n^{(2)} + (-1)^m \gamma_n^{(2)}\delta_n^{(1)}]$$

$$(m=1,2,\ldots,M) \qquad \text{(D30)}$$

$$-\sum_{p=1}^{M} \bar{g}_p' \Gamma_{mp}^{(2)} + \sum_{p=1}^{M} \bar{g}_p'' \{\alpha_4(m)\delta_{mp} - \sum_{n=1}^{N} \frac{H_{mn}}{D_n} \mu_{pn}' [\gamma_n^{(2)}((-1)^p + (-1)^m) - \gamma_n^{(1)}(1 + (-1)^{m+p})]\}$$

$$-\sum_{n=1}^{N} (-1)^n \bar{c}_n' \Gamma_{mn}^{(3)} + \sum_{n=1}^{N} (-1)^n \bar{c}_n'' \Gamma_{mn}^{(4)}$$

$$= S_m^{(8)} - \sum_{n=1}^{N} (-1)^n \frac{H_{mn}}{D_n} [\gamma_n^{(2)}\delta_n^{(2)} - \gamma_n^{(1)}\delta_n^{(1)} - (-1)^m \gamma_n^{(1)}\delta_n^{(2)} + (-1)^m \gamma_n^{(2)}\delta_n^{(1)}]$$

$$(m=1,2,\ldots,M) \qquad \text{(D31)}$$

where $\phi_n^{(1)}$ through $\phi_n^{(4)}$, $\zeta_n^{(1)}$, $\zeta_n^{(2)}$, $\xi_{mn}^{(1)}$, $\xi_{mn}^{(2)}$, $\Gamma_{mp}^{(2)}$, $\Gamma_{mn}^{(3)}$, and $\Gamma_{mn}^{(4)}$ are defined by the following equations:

$$\phi_n^{(1)} = \alpha_1(n) - \frac{1}{D_n}[\gamma_n'\gamma_n^{(2)} - \gamma_n''\gamma_n^{(1)}] \sum_{m=0}^{M} (-1)^m \nu_{mn}' + \frac{1}{D_n} [\gamma_n'\gamma_n^{(1)} - \gamma_n''\gamma_n^{(2)}] \sum_{m=0}^{M} \nu_{mn}'$$

$$\phi_n^{(2)} = \beta_1(n) + \frac{1}{D_n}[\gamma_n'\gamma_n^{(1)} - \gamma_n''\gamma_n^{(2)}] \sum_{n=0}^{M} (-1)^m \nu_{mn}' - \frac{1}{D_n}[\gamma_n'\gamma_n^{(2)} - \gamma_n''\gamma_n^{(1)}] \sum_{m=0}^{M} \nu_{mn}'$$

$$\text{(D32)}$$

$$\phi_n^{(3)} = \beta_1(n) - \frac{1}{D_n}[\gamma_n''\gamma_n^{(2)}-\gamma_n'''\gamma_n^{(1)}]\sum_{m=0}^{M}(-1)^m\,\nu'_{mn} + \frac{1}{D_n}[\gamma_n''\gamma_n^{(1)}-\gamma_n'''\gamma_n^{(2)}]\sum_{m=0}^{M}\nu'_{mn}$$

$$\phi_n^{(4)} = \alpha_2(n) + \frac{1}{D_n}[\gamma_n''\gamma_n^{(1)}-\gamma_n'''\gamma_n^{(2)}]\sum_{m=0}^{M}(-1)^m\nu'_{mn} - \frac{1}{D_n}[\gamma_n''\gamma_n^{(2)}-\gamma_n'''\gamma_n^{(1)}]\sum_{m=0}^{M}\nu'_{mn}$$

(D32)

$$\zeta_n^{(1)} = S_n^{(5)} - \frac{\delta_n^{(1)}}{D_n}[\gamma_n'\gamma_n^{(1)}-\gamma_n''\gamma_n^{(2)}] + \frac{\delta_n^{(2)}}{D_n}[\gamma_n'\gamma_n^{(2)} - \gamma_n''\gamma_n^{(1)}]$$

$$\zeta_n^{(2)} = S_n^{(6)} + \frac{\delta_n^{(1)}}{D_n}[\gamma_n''\gamma_n^{(1)}-\gamma_n'''\gamma_n^{(2)}] - \frac{\delta_n^{(2)}}{D_n}[\gamma_n''\gamma_n^{(2)} - \gamma_n'''\gamma_n^{(1)}]$$

(D33)

$$\xi_{mn}^{(1)} = (-1)^m\frac{\gamma_n'\gamma_n^{(2)}\mu'_{mn}}{D_n} - \frac{\gamma_n'\gamma_n^{(1)}\mu'_{mn}}{D_n} - (-1)^m\frac{\gamma_n''\gamma_n^{(1)}\mu'_{mn}}{D_n} + \frac{\gamma_n''\gamma_n^{(2)}\mu'_{mn}}{D_n} - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\frac{1}{E_{mn}}$$

$$\xi_{mn}^{(2)} = (-1)^m\frac{\gamma_n''\gamma_n^{(2)}\mu'_{mn}}{D_n} - \frac{\gamma_n''\gamma_n^{(1)}\mu'_{mn}}{D_n} - (-1)^m\frac{\gamma_n'''\gamma_n^{(1)}\mu'_{mn}}{D_n} + \frac{\gamma_n'''\gamma_n^{(2)}\mu'_{mn}}{D_n} - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\frac{(-1)^m}{E_{mn}}$$

(D34)

$$\Gamma_{mp}^{(2)} = \beta_2(m)\delta_{mp} - \sum_{n=1}^{N}(-1)^n\frac{H_{mn}}{D_n}\mu'_{pn}\,\{\gamma_n^{(2)}[(-1)^p+(-1)^m] - \gamma_n^{(1)}[1+(-1)^{m+p}]\}$$

$$\Gamma_{mn}^{(3)} = \frac{2}{a}\left(\frac{m\pi}{a}\right)^2\frac{1}{E_{mn}} - \frac{H_{mn}}{D_n}\{\gamma_n^{(2)}\sum_{p=0}^{M}[(-1)^p+(-1)^m]\,\nu'_{pn} - \gamma_n^{(1)}\sum_{p=0}^{M}[1+(-1)^{m+p}]\,\nu'_{pn}\}$$

$$\Gamma_{mn}^{(4)} = (-1)^m\frac{2}{a}\left(\frac{m\pi}{a}\right)^2\frac{1}{E_{mn}} - \frac{H_{mn}}{D_n}\{\gamma_n^{(2)}\sum_{p=0}^{M}[1+(-1)^{m+p}]\nu'_{pn} - \gamma_n^{(1)}\sum_{p=0}^{M}[(-1)^p+(-1)^m]\,\nu'_{pn}\}$$

(D35)

Some uncoupling of unknowns in equations (D30) and (D31) can be effected by adding and subtracting these two equations corresponding to the same value of m. By adding, one obtains

$$\sum_{n=1,3,\ldots} \bar{c}'_n \eta^{(1)}_{mn} - \sum_{n=1,3,\ldots} \bar{c}''_n \eta^{(2)}_{mn} = \rho^{(1)}_m$$

$$(m=1,2,\ldots,M) \tag{D36}$$

By subtracting, one obtains

$$\sum_{p=1}^{M} \bar{g}'_p \theta^{(3)}_{mp} - \sum_{p=1}^{M} \bar{g}''_p \theta^{(4)}_{mp} + \sum_{n=2,4,\ldots}^{N} \bar{c}'_n \eta^{(1)}_{mn} - \sum_{n=2,4,\ldots}^{N} \bar{c}''_n \eta^{(2)}_{mn} = \rho^{(2)}_m$$

$$(m=1,2,\ldots,M) \tag{D37}$$

where

$$\theta^{(1)}_{mp} = \{\alpha_3(m)-\beta_2(m)\}\delta_{mp} - 2\sum_{n=1,3,\ldots}^{N} \frac{H_{mn}}{D_n} \mu'_{pn} \{\gamma^{(2)}_n[(-1)^p+(-1)^m] - \gamma^{(1)}_n[1+(-1)^{m+p}]\}$$

$$\theta^{(2)}_{mp} = \{\alpha_4(m)-\beta_2(m)\}\delta_{mp} - 2\sum_{n=1,3,\ldots}^{N} \frac{H_{mn}}{D_n} \mu'_{pn} \{\gamma^{(2)}_n[(-1)^p+(-1)^m] - \gamma^{(1)}_n[1+(-1)^{m+p}]\}$$

$$\eta^{(1)}_{mn} = \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{2H_{mn}}{D_n}\left\{ \gamma^{(2)}_n \sum_{p=0}^{M} [(-1)^p+(-1)^m]\nu'_{pn} - \gamma^{(1)}_n \sum_{p=0}^{M} [1+(-1)^{m+p}]\nu'_{pn} \right\} \tag{D38}$$

$$\eta^{(2)}_{mn} = (-1)^m \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{2H_{mn}}{D_n}\left\{\gamma^{(2)}_n \sum_{p=0}^{M} [1+(-1)^{m+p}]\nu'_{pn} - \gamma^{(1)}_n \sum_{p=0}^{M} [(-1)^p+(-1)^m]\, \nu'_{pn}\right\}$$

$$\theta^{(3)}_{mp} = \{\alpha_3(m)+\beta_2(m)\}\delta_{mp} - 2\sum_{n=2,4\ldots}^{N} \frac{H_{mn}}{D_n} \mu'_{pn} \{\gamma^{(2)}_n[(-1)^p+(-1)^m] - \gamma^{(1)}_n[1+(-1)^{m+p}]\} \tag{D39}$$

$$\theta^{(4)}_{mp} = \{\alpha_4(m)+\beta_2(m)\}\delta_{mp} - 2\sum_{n=2,4\ldots}^{N} \frac{H_{mn}}{D_n} \mu'_{pn} \{\gamma^{(2)}_n[(-1)^p+(-1)^m] - \gamma^{(1)}_n\, [1 + (-1)^{m+p}]\}$$

$$\tag{D40}$$

$$\rho_m^{(1)} = S_m^{(7)}+S_m^{(8)}+2\sum_{n=1,3...}^{N} \frac{H_{mn}}{D_n} [\gamma_n^{(2)}\delta_n^{(2)}-\gamma_n^{(1)}\delta_n^{(1)}-(-1)^m\gamma_n^{(1)}\delta_n^{(2)}+(-1)^m\gamma_n^{(2)}\delta_n^{(1)}]$$

$$\rho_m^{(2)} = S_m^{(7)}-S_m^{(8)}+2\sum_{n=2,4...}^{N} \frac{H_{mn}}{D_n} [\gamma_n^{(2)}\delta_n^{(2)}-\gamma_n^{(1)}\delta_n^{(1)}-(-1)^m\gamma_n^{(1)}\delta_n^{(2)}+(-1)^m\gamma_n^{(2)}\delta_n^{(1)}] \tag{D41}$$

Equation (D36) involves only the odd-subscript c_n' and c_n'', and equation (D37) only the even-subscript c_n' and c_n''. These equations may replace equations (D30) and (D31).

Reduction in the Number of Simultaneous Equations

Equations (D28) and (D29), written for the same value of n, can be solved for $\bar{c}_n'$ and $\bar{c}_n''$ in terms of all the $\bar{g}_m'$ and $\bar{g}_m''$. The result is

$$\bar{c}_n' = \frac{1}{\Phi_n'}\{\zeta_n^{(1)}\phi_n^{(4)}+\zeta_n^{(2)}\phi_n^{(2)}+\sum_{p=1}^{M}(\phi_n^{(4)}\xi_{pn}^{(1)}-\phi_n^{(2)}\xi_{pn}^{(2)})\ [\bar{g}_p' - (-1)^n\ \bar{g}_p'']\} \tag{D42}$$

$$\bar{c}_n'' = \frac{1}{\Phi_n'}\{\zeta_n^{(1)}\phi_n^{(3)}+\zeta_n^{(2)}\phi_n^{(1)}+\sum_{p=1}^{M}(\phi_n^{(3)}\xi_{pn}^{(1)}-\phi_n^{(1)}\xi_{pn}^{(2)})\ [\bar{g}_p'-(-1)^n\bar{g}_p'']\} \tag{D43}$$

where $\Phi_n' = \phi_n^{(1)}\phi_n^{(4)}-\phi_n^{(2)}\phi_n^{(3)}$

Utilizing equations (D42) and (D43) to eliminate the $\bar{c}_n'$ and $\bar{c}_n''$ in equations (D36) and (D37) and combining like terms, one can obtain two sets of simultaneous equations involving only the $\bar{g}_m'$ and $\bar{g}_m''$ as unknowns:

$$\sum_{p=1}^{M} \bar{g}_p'\{\Theta_{mp}^{(1)}+\sum_{n=1,3...}^{N}\psi_{mnp}'\} +\sum_{p=1}^{M} \bar{g}_p''\{\Theta_{mp}^{(2)}+\sum_{n=1,3..}^{N}\psi_{mnp}'\} = \rho_m^{(1)} - \sum_{n=1,3...}^{N}\psi_{mn}' \qquad (m=1,2,...,M) \tag{D49}$$

$$\left.\begin{aligned}\sum_{p=1}^{M} \bar{g}'_p\{\Theta^{(3)}_{mp}+\sum_{n=2,4\ldots}^{N} \psi'_{mnp}\} -\sum_{p=1}^{M} \bar{g}''_p\{\Theta^{(4)}_{mp}+\sum_{n=2,4\ldots}^{N} \psi'_{mnp}\} = \rho^{(2)}_m - \sum_{n=2,4\ldots}^{N} \psi'_{mn} \\ (m=1,2,\ldots,M) \qquad\qquad (D49)\end{aligned}\right\}$$

where

$$\psi'_{mnp} = \frac{\eta^{(1)}_{mn}}{\Phi'_n} (\phi^{(4)}_n \xi^{(1)}_{pn} - \phi^{(2)}_n \xi^{(2)}_{pn}) - \frac{\eta^{(2)}_{mn}}{\Phi'_n} (\phi^{(3)}_n \xi^{(1)}_{pn} - \phi^{(1)}_n \xi^{(2)}_{pn}) \qquad (D50)$$

$$\psi'_{mn} = \frac{\eta^{(1)}_{mn}}{\Phi'_n} (\zeta^{(1)}_n \phi^{(4)}_n + \zeta^{(2)}_n \phi^{(2)}_n) - \frac{\eta^{(2)}_{mn}}{\Phi'_n} (\zeta^{(1)}_n \phi^{(3)}_n + \zeta^{(2)}_n \phi^{(1)}_n) \qquad (D51)$$

Whereas the original simultaneous equations system, equations (D28) to (D31), requires the solution of 2N+2M simultaneous equations, the reduced system, equations (D49), contains only 2M simultaneous equations. Thus N may be taken arbitrarily large without increasing the number of simultaneous equations that have to be solved.

Procedure for Use of Equations

The procedure for using the foregoing analysis will now be summarized: Equations (D49) are first solved for the $\bar{g}'_p$ and $\bar{g}''_p$. With these known, equations (D42) and (D43) give directly the $\bar{c}'_n$ and $\bar{c}''_n$, and equations (D9) the B'_n and B''_n. Equations (B60) then give the s'_n, s''_n, s'''_m, s''''_m, and equations (B57) to (B59) and (B61) the j_{mn}. Finally, equations (B16) and (B19) to (B25) give the stiffener and plate stresses.

Special Case: Symmetry About x = a/2 and y = b/2

When the structure and loading are symmetrical about both centerlines $x = a/2$ and $y = b/2$, then $A_1 = A_2$, $A_3 = A_4$, $P_1(0) = P_1(b) = P_2(0) = P_2(b)$,

$P_3(0) = P_3(a) = P_4(0) = P_4(a)$; $q_3(x) = -q_4(x)$, $q_1(y) = -q_2(y)$, $e_1(y) = e_2(y)$, $e_3(x) = e_4(x)$, $V'_n = -V''_n$, $K'_n = -K''_n$. In this case, one may set

$$\left.\begin{array}{ll} B'_n = B''_n = 0 & \text{for n even} \\ \bar{c}'_n = \bar{c}''_n = 0 & \text{for n even} \\ \bar{g}'_m = \bar{g}''_m = 0 & \text{for m even} \\ B'_n = B''_n & \text{for n odd} \\ \bar{c}'_n = \bar{c}''_n & \text{for n odd} \\ \bar{g}'_m = \bar{g}''_m & \text{for m odd} \end{array}\right\} \qquad (D52)$$

We can simplify the simultaneous equations (D49) as follows. From equations (D5) and (D52), one obtains

$$B'_n = \frac{1}{\gamma_n^{(1)} - \gamma_n^{(2)}} \left[\delta_n^{(1)} + 2\sum_{m=1,3\ldots}^{M} \mu'_{mn}\bar{g}'_m + 2\left(\sum_{m=1,3\ldots}^{M} \nu'_{mn}\right) \bar{c}'_n\right] \qquad (D53)$$

$$(n=1,3,\ldots,N)$$

Equations (D24) to (D27) are replaced by

$$\bar{c}'_n[\alpha_1(n)-\beta_1(n)] = S_n^{(5)} + B'_n(\gamma'_n - \gamma''_n) - \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1,3\ldots}^{M} \frac{\bar{g}'_m}{E_{mn}} \qquad (D54)$$

$$(n=1,3,\ldots,N)$$

$$\bar{g}'_m[\alpha_3(m)-\beta_2(m)] = S_m^{(7)} + 2\sum_{n=1,3,\ldots}^{N} B'_n H_{mn} - \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1,3\ldots}^{N} \frac{\bar{c}'_n}{E_{mn}} \qquad (D55)$$

$$(m=1,3,\ldots,M)$$

Substituting from (D53) into (D54), one obtains

$$\bar{c}'_n = \frac{1}{\phi_n}\left[\zeta_n + \sum_{p=1,3\ldots}^{M} \bar{g}'_p \, \xi_{pn}\right] \qquad (n=1,3,\ldots,N) \qquad \text{(D56)}$$

where

$$\phi_n = \alpha_1(n) - \beta_1(n) - \frac{2(\gamma'_n - \gamma''_n)}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{m=1,3\ldots}^{M} \nu'_{mn} \qquad \text{(D56a)}$$

$$\zeta_n = S_n^{(5)} + (\gamma'_n - \gamma''_n)\frac{\delta_n^{(1)}}{\gamma_n^{(1)} - \gamma_n^{(2)}} \qquad \text{(D56b)}$$

$$\xi_{pn} = \frac{2(\gamma'_n - \gamma''_n)}{\gamma_n^{(1)} - \gamma_n^{(2)}}\,\mu'_{pn} - \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \frac{1}{E_{pn}} \qquad \text{(D56c)}$$

Substituting from equations (D53) and (D56) into equations (D55), one obtains

$$\sum_{\substack{p=1,3\ldots \\ (p\ \text{odd})}}^{M} \bar{g}'_p\left\{\Theta'_{mp} + \sum_{\substack{n=1,3\ldots \\ (n\ \text{odd})}}^{N} \frac{\xi_{pn}}{\phi_n}\Theta''_{mn}\right\} = S_m^{(7)} + \sum_{n=1,3\ldots}^{N} H_{mn}\frac{2\delta_n^{(1)}}{\gamma_n^{(1)} - \gamma_n^{(2)}} - \sum_{n=1,3\ldots}^{N} \frac{\zeta_n}{\phi_n}\Theta''_{mn}$$

$$(m=1,3,\ldots,M) \qquad \text{(D57)}$$

where

$$\Theta'_{mp} = [\alpha_3(m) - \beta_2(m)]\delta_{mp} - 4\sum_{n=1,3\ldots}^{N} \frac{H_{mn}}{\gamma_n^{(1)} - \gamma_n^{(2)}}\,\mu'_{pn} \qquad \text{(D57a)}$$

$$\Theta''_{mn} = \frac{4}{a}\left(\frac{m\pi}{a}\right)^2 \frac{1}{E_{mn}} - \frac{4H_{mn}}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{p=1,3\ldots}^{M} \nu'_{pn} \qquad \text{(D57b)}$$

The number of equations in the system (D57) is (M+1)/2, regardless of the value selected for N.

Equations (D57) can be solved simultaneously for $\bar{g}'_p$ (p odd). With $\bar{g}'_p$'s as known, equations (D56) and (D53) will give $\bar{c}'_n$ and B'_n directly (n odd). With these known, equations (B60) will furnish the values of

s'_n, s''_n, s'''_m, s''''_m, and equations (B57) to (B59) and (B61) the values of j_{mn}. Equations (B16) and (B19) to (B25) give the stiffener and plate stresses.

Limiting Cases of Large Stiffener Areas

Five different sets of limiting conditions will now be considered, all for the case in which the structure and loading are symmetrical about the lines x = a/2 and y = b/2. In terms of the notations defined by equations (C80), these limiting conditions are: (1) $\lambda_2 \rightarrow 0$, λ_1 finite; (2) $\lambda_2 \rightarrow 0$, followed by $\lambda_1 \rightarrow 0$; (3) $\lambda_1 \rightarrow 0$, λ_2 finite; (4) $\lambda_1 \rightarrow 0$, followed by $\lambda_2 \rightarrow 0$; (5) $\lambda_1 = \lambda_2 = \lambda$, followed by $\lambda \rightarrow 0$. The reduction of the general equations to these limiting cases results in considerable simplification; in particular it is no longer necessary to solve simultaneous equations, except for condition (3). Conditions (2), (4) and (5) which are physically identical also turn out to be mathematically identical. The details of the reduction follow.

Condition (1): $\lambda_2 \rightarrow 0$ while λ_1 maintains a finite value. - Use equations (D53) to eliminate the B'_n in equations (D55); and then divide equations (D55) by A_3E_3, and rearrange them to obtain

$$\sum_{p=1,3\ldots}^{M} \bar{g}'_p\left(\delta_{mp} + \frac{V'_{mp}}{A_3E_3}\right) = \frac{Z''_m}{A_3E_3} + \sum_{n=1,3\ldots}^{N} V''_{mn} \frac{\bar{c}'_n}{A_3E_3} \qquad (m=1,3\ldots,M) \qquad \text{(D58)}$$

where

$$V'_{mp} = \delta_{mp}\left[\frac{4}{b}\sum_{n=1,3\ldots}^{N} \frac{(n\pi/b)^2}{E_{mn}}\right] - 4\sum_{n=1,3\ldots}^{N} H_{mn} \frac{\mu'_{pn}}{\gamma_n^{(1)} - \gamma_n^{(2)}} \qquad \text{(D58a)}$$

$$Z''_m = S_m^{(7)} + 2 \sum_{n=1,3\ldots}^{N} H_{mn} \frac{\delta_n^{(1)}}{\gamma_n^{(1)} - \gamma_n^{(2)}} \tag{D58b}$$

$$V''_{mn} = 2H_{mn} \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{p=1,3\ldots}^{N} \nu'_{pn} - \frac{4}{a} \frac{(m\pi/a)^2}{E_{mn}} \tag{D58c}$$

Substituting from equation (D56) into equation (D58) to eliminate $\bar{c}'_n$, and then examining the coefficients of the unknown $\bar{g}'_m$ in equation (D58), it is observed that some of these coefficients are of the order of 1, while the others are of the order of $1/(a^3E_{11}A_3E_3)$. Retaining only terms of the order of 1 in these coefficients, one reduces equations (D58) to the following system:

$$\bar{g}'_m = \frac{Z''_m}{A_3E_3} + \sum_{n=1,3\ldots}^{N} V''_{mn} \frac{\zeta_n}{\phi_n \; A_3E_3} \qquad (m = 1,3,\ldots,M) \tag{D59}$$

Substituting from equation (D59) into equation (D56), one obtains the following equation for $\bar{c}'_n$

$$\bar{c}'_n = \frac{1}{\phi_n} \{\zeta_n + \sum_{p=1,3\ldots}^{M} \xi_{pn} \left[\frac{Z''_p}{A_3E_3} + \frac{1}{A_3E_3} \sum_{q=1,3\ldots}^{N} \frac{V''_{pq} \; \zeta_q}{\phi_q}\right]\} \tag{D60}$$

$$(n=1,3,\ldots,N)$$

Thus a solution (eqs. (D59) and (D60)) is obtained which gives the unknowns $\bar{g}'_m$ and $\bar{c}'_n$ explicitly without the necessity of solving simultaneous equations. This solution is seen to be correct to terms of the first in $1/(a^3E_{11}A_1E_1)$. With $\bar{c}'_n$ and $\bar{g}'_m$ known, then equation (D53) furnishes the values of B'_n, and the procedure for computing stresses is the same as described earlier for the special case of symmetry about $x = a/2$, $y = b/2$.

Condition (2): $\lambda_2 \to 0$ with λ_1 finite, followed by $\lambda_1 \to 0$. - Equations (D59) and (D60) can be used as a starting point for this case. It is first noted, from equations (D56a), (B67), (C22), and (C23) that

$$\phi_n = A_1 E_1 [1 - \Xi''_n + \frac{\Delta_n^{(1)}}{A_1 E_1}] \tag{D61}$$

where

$$\Xi''_n = \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} \frac{n\pi}{b} C_3 \sum_{m=1,3\ldots}^{M} \nu'_{mn} \tag{D61a}$$

$$\Delta_n^{(1)} = \frac{4}{a} \sum_{m=1,3\ldots}^{M} \frac{(m\pi/a)^2}{E_{mn}} - \frac{4}{\gamma_n^{(1)} - \gamma_n^{(2)}} [\sum_{m=1,3\ldots}^{M} \frac{1}{E_{mn}} \frac{2}{a} (\frac{n\pi}{b})^3 C_1](\sum_{p=1,3\ldots}^{M} \nu'_{pn}) \tag{D61b}$$

Substituting from equation (D61) into equations (D59) to eliminate ϕ_n, examining the coefficients of the unknown $\bar{g}'_m$ in these equations, and retaining only terms of the order of 1 in these coefficients, one reduces equations (D59) to the following system:

$$\bar{g}'_m = \frac{Z''_m}{A_3 E_3} \qquad (m = 1, 3, \ldots, M) \tag{D62}$$

A similar reduction of equation (D60) can be effected with the aid of the following form of ξ_{pn}, obtained from equations (D56c), (C22), and (C23):

$$\xi_{pn} = A_1 E_1 [\frac{n\pi}{b} C_3 \mu'_{pn} \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} + \frac{1}{A_1 E_1} \Delta_{pn}^{(2)}] \tag{D63}$$

where

$$\Delta_{pn}^{(2)} = \frac{4}{\gamma_n^{(1)}-\gamma_n^{(2)}} \left[\sum_{m=1,3\ldots}^{M} \frac{1}{E_{mn}} \frac{2}{a} \left(\frac{n\pi}{b}\right)^3 C_1\right] \mu'_{pn} - \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \frac{1}{E_{pn}} \tag{D63a}$$

Substituting from equations (D61) and (D63) into equation (D60), examining the coefficients of the unknown $\bar{c}'_n$, and retaining only terms of the order of 1 in these coefficients, one reduces equation (D60) to the following equation:

$$\bar{c}'_n = \frac{Z'_n}{A_1 E_1} \qquad (n = 1, 3, \ldots, N) \tag{D64}$$

where

$$Z'_n = \frac{1}{\Phi_n^{(1)}} \left[\zeta_n + \frac{A_1 E_1}{A_3 E_3} \frac{n\pi}{b} C_3 \frac{2}{\gamma_n^{(1)}-\gamma_n^{(2)}} \sum_{m=1,3\ldots}^{M} \mu'_{mn} Z''_m\right] \tag{D64a}$$

with

$$\Phi_n^{(1)} = 1 - \Xi''_n \tag{D64b}$$

Equations (D62) and (D64) are the pertinent results for this limiting condition.

<u>Condition (3): $\lambda_1 \to 0$ while λ_2 maintains a finite value</u>. - This limiting case (A_1E_1 and A_2E_2 approaching infinity) can be studied with equations (D57) as the starting point. These equations are written in the following form, in order to be more readily suitable for solution by the Gauss-Seidel iteration method:

$$\begin{aligned}\bar{g}'_m\left[\theta'_{mm} + \sum_{n=1,3\ldots}^{N} \frac{\xi_{mn}}{\phi_n} \theta''_{mn}\right] = S_m^{(7)} &+ \sum_{n=1,3\ldots}^{N} H_{mn} \frac{2\delta_n^{(1)}}{\gamma_n^{(1)}-\gamma_n^{(2)}} - \sum_{n=1,3\ldots}^{N} \frac{\zeta_n}{\phi_n} \theta''_{mn} \\ &+ \sum_{\substack{p=1,3\ldots \\ (p \neq m)}}^{M} \bar{g}'_p\left[4 \sum_{n=1,3\ldots}^{N} \frac{H_{mn}}{\gamma_n^{(1)}-\gamma_n^{(2)}} \mu'_{pn} - \sum_{n=1,3\ldots}^{N} \frac{\xi_{pn}}{\phi_n} \theta''_{mn}\right]\end{aligned}$$

$$(m = 1, 3, \ldots, M) \tag{D65}$$

Substituting from equations (D61) and (D63) into (D65), then examining the coefficients of the unknown $\bar{g}'_m$ in equations (D65), and retaining only terms of the order of 1 in these coefficients as A_1E_1 and A_2E_2 approach infinity, one reduces equations (D65) to the following system:

$$\bar{g}'_m \left\{\Theta'_{mn} + 2 \sum_{n=1,3\ldots}^{N} \frac{(n\pi/b)\ C_3\ \mu'_{mn}\ \Theta''_{mn}}{[1 - \Xi''_n][\gamma_n^{(1)} - \gamma_n^{(2)}]}\right\}$$

$$= S_m^{(7)} + \sum_{n=1,3\ldots}^{N} H_{mn} \frac{2\delta_n^{(1)}}{\gamma_n^{(1)} - \gamma_n^{(2)}} - \sum_{n=1,3\ldots}^{N} \frac{\zeta_n\ \Theta''_{mn}}{A_1E_1[1 - \Xi''_n]}$$

$$+ \sum_{\substack{p=1,3\ldots\\(p \neq m)}}^{M} \bar{g}'_p \left\{4 \sum_{n=1,3\ldots}^{N} \frac{H_{mn}\ \mu'_{pn}}{\gamma_n^{(1)} - \gamma_n^{(2)}} - 2 \sum_{n=1,3\ldots}^{N} \frac{(n\pi/b) C_3\ \mu'_{pn}\ \Theta''_{mn}}{[1 - \Xi''_n][\gamma_n^{(1)} - \gamma_n^{(2)}]}\right\}$$

$$(m = 1,\ 3,\ \ldots,\ M) \qquad \text{(D66)}$$

The solution of equations (D66) can be effected by the Gauss-Seidel iterative procedure (see appendix C - Procedure for numerical solution.).

Using equations (D53) to eliminate B'_n in equations (D54), and then dividing equations (D54) by A_1E_1, and rearranging terms, one obtains

$$\bar{c}'_n \left[1 - \Xi''_n + \frac{\Delta_n^{(1)}}{A_1E_1}\right] = \frac{\zeta_n}{A_1E_1} + \frac{n\pi}{b} C_3 \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{m=1,3\ldots}^{M} \mu'_{mn}\ \bar{g}'_m + \sum_{m=1,3\ldots}^{M} \frac{\Delta_{mn}^{(2)}}{A_1E_1}\ \bar{g}'_m$$

$$\text{(D67)}$$

in which the $\bar{g}'_m$ are defined by equations (D66). As $\lambda_1 \longrightarrow 0$ $(A_1E_1 \longrightarrow \infty)$, equations (D67) are simplified to the following:

$$\bar{c}'_n = \frac{1}{\Phi_n^{(1)}} \left[\frac{\zeta_n}{A_1E_1} + \frac{n\pi}{b} \; C_3 \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{m=1,3\ldots}^{M} \mu'_{mn} \; \bar{g}'_m\right] \tag{D68}$$

where $\Phi_n^{(1)}$ is defined by equation (D64b)

With $\bar{g}'_m$ and $\bar{c}'_n$ known, through equations (D66) and (D68), equations (D53) will give the values of B'_n. From that point on, the procedure for computing stresses is the same as described earlier for the special case of symmetry about $x = a/2$ and $y = b/2$.

Condition (4): $\lambda_1 \longrightarrow 0$ with λ_2 finite, followed by $\lambda_2 \longrightarrow 0$. - Dividing equations (D66) by A_3E_3 and examining the coefficients of the unknowns $\bar{g}'_m$, it is observed that these coefficients contain terms of the order of 1 and terms of the order of $1/(a^3E_{11}A_3E_3)$. Retaining only the terms of the order of 1, one reduces these equations to the following system:

$$\bar{g}'_m = \frac{Z''_m}{A_3E_3} \tag{D69}$$

where Z''_m is defined by equation (D58b). Substituting from equation (D69) into equation (D68), one obtains

$$\bar{c}'_n = \frac{Z'_n}{A_1E_1} \tag{D70}$$

where Z'_n is defined by equation (D64a).

With $\bar{g}'_m$ and $\bar{c}'_n$ known from equations (D69) and (D70), equation (D53) will then furnish the values of B'_n. The subsequent procedure for computing stresses is the same as described earlier for the special case of symmetry about $x = a/2$ and $y = b/2$.

Condition (5): $\lambda_1 = \lambda_2 = \lambda,\ \lambda \rightarrow 0$. - This case can be thought of as one in which all stiffener cross-sectional areas are approaching infinity simultaneously while maintaining fixed ratios with respect to each other. For the study of this case we return to equations (D54) and (D55), but rewritten as equations (D67) and (D58), respectively. Examining the coefficients of $\bar{c}'_n$ and $\bar{g}'_m$ in equations (D67) and (D58), it is observed that some of these coefficients are of the order of 1, while the others are of the order of $1/(a^3E_{11}A_1E_1)$. Retaining only terms of the order of 1 in these coefficients, one reduces equation (D67) and (D58) to the following system:

$$\bar{c}'_n = \frac{1}{\Phi_n^{(1)}} \left\{ \frac{\zeta_n}{A_1E_1} + \frac{n\pi}{b} C_3 \frac{2}{\gamma_n^{(1)} - \gamma_n^{(2)}} \sum_{m=1,3\ldots}^{M} \mu'_{mn} \bar{g}'_m \right\} \tag{D71}$$

$$\bar{g}'_m = \frac{Z''_m}{A_3E_3} \tag{D72}$$

Using equation (D72) to simplify equation (D71), the latter becomes

$$\bar{c}'_n = \frac{Z'_n}{A_1E_1} \tag{D73}$$

where Z'_n is defined by equation (D64a).

Thus a solution (eqs. (D72) and (D73)) is obtained which gives the basic unknowns $\bar{c}'_n$ and $\bar{g}'_m$ explicitly without the necessity of solving simultaneous equations. This solution is seen to be identical to those obtained for conditions (2) and (4).

Illustrative Thermal-Stress Problem

A particular example will be presented to illustrate the details involved in the application of the foregoing analytical results, this example has the following characteristics:

a) Two opposite edges of $x = 0$ and $x = a$ kept straight; therefore the K_n' and K_n'' in equations (8) and (9) are all zero.

b) Plate isotropic; therefore elastic constants are given by equations (3).

c) Plate and stiffeners have the same Young's modulus E.

d) $A_1 = A_2$, $A_3 = A_4$.

e) No force loading.

f) Stiffener temperature constant at the value T_o.

g) Plate temperatures $T(x,y)$ symmetrical about both centerlines ($x = a/2$, $y = b/2$) and varying sinusoidally in accordance with the following equation

$$T(x,y) = T_o + \theta \sin\left(\frac{P\pi x}{a}\right) \sin\left(\frac{Q\pi y}{b}\right) \qquad \begin{array}{c}(0 \leq x \leq a)\\(0 \leq y \leq b)\end{array}$$

where P and Q are odd integers.

h) Plate and stiffeners have the same coefficient of expansion α.

These are the only specializations to be employed at present. However, in the numerical example, to be presented later, the problem will be further specialized to the case of a square plate (b=a) with all stiffeners identical ($A_1 = A_2 = A_3 = A_4 = A$), subjected to a "pillow-shaped" temperature distribution (P=Q=1), and having $\nu = 0.3$.

Reduction of general equations to special case. - From the given temperature distribution, one obtains the following equations for the known coefficients in terms of temperature distribution and the coefficients of expansion α (see appendix C):

$$T_n' = T_n'' = T_m''' = T_m'''' = 0 \tag{D74}$$

$$T_{mn} = -\delta_{mP}\delta_{nQ}\,\alpha\Theta\,\left(\frac{\pi}{a}\right)^2\,[P^2 + Q^2B^2] \tag{D75}$$

$$V_n' = -V_n'' = \alpha\Theta\,\frac{P\pi}{a}\,\delta_{nQ} \tag{D76}$$

Due to the absence of prescribed forces, the following quantities are all zero:

$P_1(0),\ P_1(b),\ P_2(0),\ P_2(b)$	(fig. 4b)
$T_1,\ M_1,\ T_2,\ M_2$	(fig. 4b)
$B_m''',\ B_m''''$	(see eqs. (4))
$Q_n',\ Q_n'',\ Q_m''',\ Q_m''''$	(see eqs. (6))

(It should be noted that $P_3(0)$, $P_3(a)$, $P_4(0)$, $P_4(a)$, B_n' and B_n'' do not necessarily vanish.)

Because in this example the structure and loading are symmetrical about both centerlines, $x = a/2$ and $y = b/2$, the simplified system of equations, namely equations (D56) and (D57) will be used for the determination of the $\bar{c}_n'$ and $\bar{g}_m'$ with odd m and n. (It should be noted that $\bar{c}_n' = \bar{c}_n''$ for n odd, $\bar{g}_m' = \bar{g}_m''$ for m odd, $\bar{c}_n' = \bar{c}_n'' = 0$ for n even, and $\bar{g}_m' = \bar{g}_m'' = 0$ for m even). With $\bar{c}_n'$ and $\bar{g}_m'$ known, equation (D53) will furnish the values of odd subscripted B_n'. (It should be noted that $B_n' = B_n''$ for n odd, and $B_n' = B_n'' = 0$ for n even). Equations (D57) must first be solved simultaneously for $\bar{g}_m$. The quantities needed in order to use these equations will now be evaluated. Substituting from equations(3) and (C81) into equation (D6), one obtains

$$\gamma_n^{(2)} = - (n^2B^2/aEh) \sum_{i=0}^{M} \frac{(-1)^i[i^2 + (2+\nu)n^2B^2](2-\delta_{i0})}{[i^2 + n^2B^2]^2} \qquad \text{(D6')}$$

From equations (C10') and (D6'), one obtains

$$\frac{1}{\gamma_n^{(1)} - \gamma_n^{(2)}} = - \frac{aEh}{4n^2B^2 \sum\limits_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\ n^2B^2]}{[m^2 + n^2B^2]^2}} \qquad \text{(D77)}$$

Substituting equations (3), (C81), (D75) and (D76) into equation (D7), one obtains

$$\delta_n^{(1)} = \alpha\theta \frac{P\pi}{a} \{\frac{[P^2 + (2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\} \delta_{nQ} \qquad \text{(D7')}$$

Substituting into equations (D18) and (D20) from equations (D74) and (D75), one obtains

$$S_n^{(5)} = \delta_{nQ}\ \alpha\theta Eh \frac{PQB}{[P^2 + Q^2B^2]} \qquad \text{(D18')}$$

$$S_m^{(7)} = \delta_{mP}\ \alpha\theta Eh \frac{PQB}{[P^2 + Q^2B^2]} \qquad \text{(D20')}$$

Substituting from equations (B67), (D77), (C22'), (C23') and (C13'), into equation (D56a), one obtains

$$\phi_n = A_1 E\ \bar{\phi}_n \qquad \text{(D56'a)}$$

where

$$\bar{\phi}_n = 1 + \lambda_1 \sum_{m=1,3\ldots}^{M} \frac{m^2}{[m^2 + n^2B^2]^2}$$

$$+ \frac{[\nu + \lambda_1 n^2B^2 \sum\limits_{m=1,3\ldots}^{M} \frac{1}{(m^2 + n^2B^2)^2}]}{\sum\limits_{m=1,3\ldots}^{M} \frac{[m^2+(2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}} \sum_{m=1,3\ldots}^{M} \frac{[n^2B^2 - \nu m^2]}{[m^2 + n^2B^2]^2} \tag{D56"a}$$

Substituting from equations (D77), (D18'), (C22'), (C23') and (D7') into equation (D56b), one obtains

$$\zeta_n = \alpha\theta Eh \, \bar{\zeta}_n \, \delta_{nQ} \tag{D56'b}$$

where $\bar{\zeta}_n$ are known quantities defined as follows

$$\bar{\zeta}_n = \frac{PQB}{[P^2+Q^2B^2]} - \frac{P[\frac{\nu}{\lambda_1} + Q^2B^2 \sum\limits_{i=1,3\ldots}^{M} \frac{1}{(i^2+Q^2B^2)^2}] \{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{QB \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2 + (2+\nu)\ Q^2B^2]}{[i^2 + Q^2B^2]^2}} \tag{D56"b}$$

Substituting from equations (C81), (C12'), (C22'), (C23') and (D77) into equation (D56c), one obtains

$$\xi_{pn} = A_1 E \bar{\xi}_{pn} \tag{D56'c}$$

where $\bar{\xi}_{pn}$ are known quantities as expressed by the following equations:

$$\bar{\xi}_{pn} = \frac{B[p^2+(2+\nu)\, n^2B^2][\nu+\lambda_1 n^2B^2 \sum_{i=1,3...}^{M} \frac{1}{(i^2 + n^2B^2)^2}]}{(p^2 + n^2B^2)^2 \sum_{i=1,3...}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}} - \frac{\lambda_1\, n^2B^3}{(p^2 + n^2B^2)^2} \qquad \text{(D56"c)}$$

Substituting from equations (C82), (C24'), (C12') and (D77) into equation (D57a), one obtains

$$\Theta'_{mp} = A_3E \{[1 + \lambda_2B^4 \sum_{n=1,3...}^{N} \frac{n^2}{(m^2 + n^2B^2)^2}]\, \delta_{mp}$$

$$-\lambda_2B^4 \sum_{n=1,3...}^{N} \frac{[\frac{n^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^4}][p^2 + (2+\nu)\, n^2B^2]}{(p^2 + n^2B^2)^2 \sum_{i=1,3...}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}} \} \qquad \text{(D57'a)}$$

Substituting from equations (C81), (C13'), (C24'), and (D77) into equation (D57b), one obtains

$$\Theta''_{mn} = A_1E\, \bar{\Theta}''_{mn} \qquad \text{(D57'b)}$$

where $\bar{\Theta}''_{mn}$ are known quantities and are defined as follows

$$\bar{\Theta}''_{mn} = \frac{\lambda_1 m^2}{[m^2+n^2B^2]^2} + \frac{\lambda_1[\frac{n^3B^3}{(m^2+n^2B^2)^2} - \frac{1}{nB}]}{nB \sum_{i=1,3...}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}} \sum_{i=1,3...}^{M} \frac{[n^2B^2 - \nu i^2]}{[i^2+n^2B^2]^2} \qquad \text{(D57"b)}$$

Substituting from equations (C24'), (D7'), (D77), (D20'), (D56'a), (D56'b), (D56'c), (D57'a), and (D57'b) into equation (D57), and substituting $\bar{g}'_p = g'_p\, C_1 \frac{p\pi}{a}$, one obtains

$$\{\sum_{n=1,3\ldots}^{N} \frac{\bar{\xi}_{mn}}{\bar{\phi}_n} \bar{\Theta}''_{mn} + [1 + \lambda_2 B^4 \sum_{n=1,3\ldots}^{N} \frac{n^2}{(m^2 + n^2B^2)^2}]\, C$$

$$-\lambda_2 B^4 C \sum_{n=1,3\ldots}^{N} \frac{[\frac{n^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^4}][m^2 + (2+\nu)\, n^2B^2]}{(m^2+n^2B^2)^2 \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2 + (2+\nu)\, n^2B^2]}{[i^2 + n^2B^2]^2}}\} \quad G(m)$$

$$= \delta_{mP} \frac{\pi\lambda_1 \; Q\, B}{4(P^2 + Q^2B^2)} - \frac{\pi\lambda_1}{4m}\, \delta_{nQ} \frac{\bar{\zeta}_n}{\bar{\phi}_n} \; \bar{\Theta}''_{mn}$$

$$+ \frac{\pi\lambda_1}{4m} \frac{P[\frac{1}{Q^3B^3} - \frac{QB}{(m^2+Q_1^2B^2)^2}]\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\}}{\sum\limits_{i=1,3\ldots}^{M} \frac{[i^2 + (2+\nu)\; Q^2B^2]}{[i^2 + Q^2B^2]^2}}$$

$$- \sum_{p=1,3\ldots}^{M} [\sum_{n=1,3\ldots}^{N} \{ \frac{\bar{\xi}_{pn}}{\bar{\phi}_n} \bar{\Theta}''_{mn} - \lambda_2 B^4 C \frac{[\frac{n^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^4}][m^2+(2+\nu)n^2B^2]}{[p^2 + n^2B^2)^2 \sum\limits_{i=1,3}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}} \}] \cdot$$

$$\cdot\; (1 - \delta_{mp}) \frac{p}{m} G(p) \qquad \text{(D57')}$$

where

$$G(p) = \frac{g'_p}{\alpha\Theta Eh} \qquad \text{and} \qquad C = A_3/A_1$$

In the solution of this system of equations the Gauss-Seidel iterative procedure can be employed (see appendix C - Procedure for numerical solution).

After equations (D57') have been solved, the G(p) (therefore g'_p) will be known. The c'_n can then be determined from equation (D56). Substituting from equations (D56'a), (D56'b) and (D56'c) into equation (D56), and substituting $\bar{c}'_n = c'_n C_2 \frac{n\pi}{b}$, one obtains

$$C(n) = \frac{\pi\lambda_1}{4\bar{\phi}_n\ nB} \bar{\zeta}_n \delta_{nQ} + \frac{1}{\bar{\phi}_n\ nB} \sum_{p=1,3\ldots}^{M} G(p)\ p\ \bar{\xi}_{pn} \qquad \text{(D56')}$$

where

$$C(n) = \frac{c'_n}{\alpha\Theta Eh}$$

With C(n) and G(m) known, substituting from equations (D77), (D7'), (C12'), and (C13') into equation (D53), and substituting $\bar{c}'_n = c'_n C_2 \frac{n\pi}{b}$ and $\bar{g}'_m = g'_m C_1 \frac{m\pi}{a}$, one obtains

$$B(n) = -\frac{P\pi\left\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\right\}}{4Q^2B^2 \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2+(2+\nu)Q^2B^2]}{[i^2+Q^2B^2]^2}} \delta_{nQ} + \frac{\sum\limits_{m=1,3\ldots}^{M} \frac{m[m^2+(2+\nu)n^2B^2]}{[m^2+n^2B^2]^2} G(m)}{n \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}}$$

$$- \frac{\sum\limits_{m=1,3\ldots}^{M} \frac{[n^2B^2 - \nu m^2]}{[m^2+n^2B^2]^2} C(n)}{\sum\limits_{i=1,3\ldots}^{M} \frac{[i^2+(2+\nu)\ n^2B^2]}{[i^2+Q^2B^2]^2}} \qquad \text{(D53')}$$

where

$$B(n) = \frac{B'_n}{\alpha\Theta Eh}$$

With B'_n as known, one obtains

$$P_3(0) = P_4(0) = -\alpha\Theta A_3 E\lambda_2 \sum_{n=1,3\ldots}^{N} \frac{\pi}{4n} B(n) \qquad \text{(D78)}$$

from equations (D13) and (D14).

With C(n), G(m) and B(n) now known, equations (B60) will give the values of the odd-subscripted s'_n, s''_n, s'''_m, s''''_m, and equation (B61) the values of j_{mn} (m and n odd). Equations (B16) and (B19) to (B25) will give the stiffeners and plate stresses. One thus obtains the following results:

$$P_1(y) = P_2(y) = \Theta A_1 E\alpha \sum_{n=1,3\ldots}^{N} [C(n) - \nu B(n)] \sin\left(\frac{n\pi y}{b}\right)$$

$$(0 < y < b) \qquad \text{(D79)}$$

$$P_3(x) = P_4(x) = \Theta A_3 E\alpha \sum_{m=1,3\ldots}^{M} G(m) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \qquad \text{(D80)}$$

$$N_x = \alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} G(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$(0 < x < a)$$
$$(0 < y < b) \qquad \text{(D81)}$$

$$(N_x)_{y=0} = (N_x)_{y=b} = \alpha\Theta Eh \sum_{m=1,3\ldots}^{M} G(m) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \qquad \text{(D82)}$$

$$N_y = \alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} C(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$\begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \qquad \text{(D83)}$$

$$(N_y)_{x=0} = (N_y)_{x=a} = \alpha\Theta Eh \sum_{n=1,3\ldots}^{N} C(n) \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \qquad \text{(D84)}$$

$$N_{xy} = -\alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} J(m,n) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$\begin{matrix}(0 \leq x \leq a)\\(0 \leq y \leq b)\end{matrix} \qquad \text{(D85)}$$

where G(m,n), C(m,n), and J(m,n) are known quantities, and are defined by the following equations:

$$G(m,n) = \frac{4n^2B^2}{\pi[m^2+n^2B^2]^2} [m\, C(n) + nB^2G(m)]$$

$$+ \frac{4m[m^2+ 2n^2B^2]B(n)}{\pi[m^2 + n^2B^2]^2} - \frac{Q^2B^2}{(P^2 + Q^2B^2)} \delta_{mP}\delta_{nQ} \qquad \text{(D86)}$$

$$C(m,n) = \frac{4mB(n)}{n^2B^2\pi} \left[\frac{m^2(m^2 + 2n^2B^2)}{(m^2 + n^2B^2)^2} - 1\right] - \frac{P^2}{(P^2+Q^2B^2)} \delta_{mP}\delta_{nQ}$$

$$+ \frac{4m^2}{\pi[m^2+n^2B^2]^2} [m\, C(n) + nB^2\, G(m)] \qquad \text{(D87)}$$

$$J(m,n) = \frac{4nB}{\pi[m^2+n^2B^2]^2} \{n^2B^2B(n) - m^2\, C(n) - mnB^2\, G(m)\}$$

$$+ \frac{PQB}{(P^2+Q^2B^2)} \delta_{mP}\delta_{nQ} \qquad \text{(D88)}$$

And from the first two of equation (4), one obtains the following running tensions between the rigid stiffeners and plate edges at $x = 0$ and $x = a$.

$$N_x(0,y) = N_x(a,y) = \sum_{n=1,3\ldots}^{N} B_n' \sin\left(\frac{n\pi y}{b}\right) = \alpha\theta Eh \sum_{n=1,3\ldots}^{N} B(n) \sin\left(\frac{n\pi y}{b}\right)$$

$$(0 < y < b) \tag{D89}$$

<u>Numerical results for λ_1 and $\lambda_2 \neq 0$</u>. - The numerical procedure and equations described above were applied to the special case of a square plate (b=a), will all stiffeners identical ($A_1 = A_2 = A_3 = A_4$), and Poisson's ratio ν equal to 0.3. The above assumption implies $B = 1$ and $\lambda_1 = \lambda_2$, and the common symbol λ will therefore be used for both λ_1 and λ_2.

The results obtained for the stiffener tensions and the plate stresses are presented in dimensionless form in figure 10 for $\lambda = 2.0$ and figure 11 for $\lambda = 1.0$. The values of M and N employed in the calculation are indicated on the figures. In general, stresses are computed at x/a and y/a interval of 0.1.

<u>Limiting case of large stiffener areas, condition (1)</u>. - Five different limiting conditions were considered in the earlier analysis. However numerical calculations were made for only two of these, conditions (1) and (2). The equations needed for condition (1) will now be presented.

Substituting from equations (D7'), (D20'), (D77) and (C24') into (D58b), one obtains

$$Z''_m = \alpha\Theta Eh \left[\frac{PQB}{(P^2+Q^2B^2)}\delta_{mP} - \frac{[\frac{QB}{(m^2+Q^2B^2)^2} - \frac{1}{Q^3B^3}]\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\}P}{\sum_{i=1,3...}^{M} \frac{[i^2+(2+\nu)Q^2B^2]}{[i^2+Q^2B^2]^2}} \right] \qquad \text{(D58'b)}$$

Substituting from equations (C13'), (C24'), (C81) and (D77) into (D58c), one obtains the following equations for V''_{mn}:

$$V''_{mn} = -\frac{4aEh}{\pi^2}\{[\frac{n^2B^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^2}] \frac{\sum_{p=1,3...}^{M} \frac{[n^2B^2-\nu p^2]}{[p^2+n^2B^2]^2}}{\sum_{p=1,3...}^{M} \frac{[p^2+(2+\nu)n^2B^2]}{[p^2+n^2B^2]^2}} + \frac{m^2}{[m^2+n^2B^2]^2}\} \qquad \text{(D58'c)}$$

Then substituting from equations (D58'b), (D58'c), (D56'a) and (D56'b) into (D59), one obtains

$$\bar{g}'_m = \alpha\Theta \frac{h}{A_3} \bar{G}(m) \qquad \text{(D59')}$$

where the $\bar{G}(m)$ are completely defined by the following equation:

$$\bar{G}(m) = \frac{PQB}{(P^2+Q^2B^2)}\delta_{mP} - \frac{[\frac{QB}{(m^2+Q^2B^2)^2} - \frac{1}{Q^3B^3}]\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\}P}{\sum_{i=1,3...}^{M} \frac{[i^2+(2+\nu)Q^2B^2]}{[i^2+Q^2B^2]^2}}$$

$$- \lambda_1\{[\frac{Q^2B^2}{(m^2+Q^2B^2)^2} - \frac{1}{Q^2B^2}] \frac{\sum_{p=1,3...}^{M} \frac{[Q^2B^2-\nu p^2]}{[p^2+Q^2B^2]^2}}{\sum_{p=1,3...}^{M} \frac{[p^2+(2+\nu)Q^2B^2]}{[p^2+Q^2B^2]^2}} + \frac{m^2}{[m^2+Q^2B^2]^2}\} \frac{\bar{\zeta}_Q}{\bar{\phi}_Q} \qquad \text{(D59'a)}$$

with $\bar{\phi}_Q$ and $\bar{\zeta}_Q$ are defined by equations (D56"a) and (D56"b).

With $\bar{g}'_m$ as known, substituting from equations (D59'), (D56'a), (D56'b) and (D56'c) into equation (D56), one obtains

$$\bar{c}'_n = \alpha\Theta \frac{h}{A_1} \frac{\bar{\zeta}_n}{\bar{\phi}_n} \delta_{nQ} + \alpha\Theta \frac{h}{A_3} \frac{1}{\bar{\phi}_n} \sum_{m=1,3\ldots}^{M} \bar{G}(m)\, \bar{\xi}_{mn} \qquad \text{(D56")}$$

Substituting from equations (D59'),(D56"), (D7'), (D77), (C12') and (C13') into (D53), one then obtains

$$B'_n = -\alpha\Theta Eh\, \lambda_1 \frac{\pi}{4QB} \bar{B}'_n\, \delta_{nQ} + \alpha\Theta Eh\, \lambda_2 \frac{\pi}{4n} B^*_n \qquad \text{(D53")}$$

where

$$\bar{B}'_n = \frac{P\left\{\dfrac{[P^2+(2+\nu)Q^2B^2]}{[P^2 + Q^2B^2]} - 1\right\}}{\lambda_1 QB \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2+(2+\nu)Q^2B^2]}{[m^2 + Q^2B^2]}} + \frac{\bar{\zeta}_n \displaystyle\sum_{m=1,3,\ldots}^{M} \frac{[Q^2B^2 - \nu m^2]}{[m^2 + Q^2B^2]^2}}{\bar{\phi}_n \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2+(2+\nu)Q^2B^2]}{[m^2 + Q^2B^2]}} \qquad \text{(D53"a)}$$

$$B^*_n = \frac{B \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2+(2+\nu)n^2B^2]\bar{G}(m)}{[m^2+n^2B^2]^2}}{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}} - \frac{\left[\displaystyle\sum_{p=1,3\ldots}^{M} \bar{G}(p)\bar{\xi}_{pn}\right] \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[n^2B^2-\nu m^2]}{[m^2+n^2B^2]^2}}{\bar{\phi}_n \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2}} \qquad \text{(D53"b)}$$

With $\bar{c}'_n$, $\bar{g}'_m$, and B'_n known through equations (D56"), (D59'), and (D53"), the Fourier coefficients of the stresses can now be evaluated. From equations (B60), one obtains

$$s_n' = s_n'' = \alpha\Theta A_1 E \lambda_1 \delta_{nQ} \frac{\pi}{4QB} [\frac{\bar{\zeta}_n}{\bar{\phi}_n} + \nu \bar{B}_n'] \qquad \text{(D90)}$$

$$s_m''' = s_m'''' = \alpha\Theta A_3 E \lambda_2 \frac{B\pi}{4m} \bar{G}(m)$$

From equation (B61), with $\bar{c}_n'$, $\bar{g}_m'$, and B_n' as known, one obtains

$$\underset{\substack{(m\neq 0)\\(n\neq 0)}}{j_{mn}} = \alpha\Theta Eh\, \delta_{nQ}\, \bar{j}_{mn} \qquad \text{(D91)}$$

where

$$\bar{j}_{mn} = \frac{PQB}{(P^2 + Q^2B^2)} \delta_{mP} - \frac{m^2\lambda_1}{[m^2 + Q^2B^2]^2} \frac{\bar{\zeta}_n}{\bar{\phi}_n} - \frac{\lambda_1 Q^2B^2\bar{B}_n'}{[m^2+Q^2B^2]^2} \qquad \text{(D90a)}$$

With $\bar{c}_n'$, $\bar{g}_m'$, and B_n' as known, from equations (B34) and (B35), one obtains

$$\left.\begin{aligned} g_{mn} &= \delta_{nQ}\, \alpha\Theta Eh\, \bar{g}_{mn} \\ c_{mn} &= \delta_{nQ}\, \alpha\Theta Eh\, \bar{c}mn \end{aligned}\right\} \qquad \text{(D91)}$$

where

$$\bar{g}_{mn} = - \frac{Q^2B^2\delta_{mP}}{(P^2+Q^2B^2)} + \frac{\lambda_1 B\ m\ Q_1}{[m^2+Q^2B^2]^2} \frac{\bar{\zeta}_n}{\bar{\phi}_n} - \frac{\lambda_1 m[m^2+2Q^2B^2]\bar{B}_n'}{[m^2 + Q^2B^2]^2} \qquad \text{(D91a)}$$

$$\bar{c}_{mn} = \frac{\lambda_1 m}{Q^3B^3} [1 - \frac{m^2(m^2+2Q^2B^2)}{(m^2+Q^2B^2)^2}] - \delta_{mP} \frac{P^2}{(P^2 + Q^2B^2)}$$

$$+ \frac{\lambda_1 m^3}{QB[m^2 + Q^2B^2]^2} \frac{\bar{\zeta}_n}{\bar{\phi}_n} \qquad \text{(D91b)}$$

Now with $\bar{c}_n'$, $\bar{g}_m'$, B_n', s_n', s_m''', c_{mn}, g_{mn}, and j_{mn} known, equations (B16) and (B19) to (B25) will give the following stiffener and plate stress equations:

$$P_1(y) = P_2(y) = \alpha\Theta A_1 E\lambda_1 \sum_{n=1,3...}^{N} \delta_{nQ} \frac{\pi}{4QB} [\frac{\bar{\zeta}_n}{\bar{\phi}_n} + \nu\bar{B}'_n] \sin (\frac{n\pi y}{b}) \qquad (0 < y < b)$$

(D92)

$$P_3(x) = P_4(x) = \alpha\Theta A_3 E \lambda_2 \sum_{m=1,3...}^{M} \frac{B\pi}{4m} \bar{G}(m) \sin (\frac{m\pi x}{a}) \qquad (0 < x < a)$$

(D93)

$$N_x = \alpha\Theta Eh \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} \delta_{nQ} \bar{g}_{mn} \sin (\frac{m\pi x}{a}) \sin (\frac{n\pi y}{b}) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix}$$

(D94)

$$(N_x)_{y=0} = (N_x)_{y=b} = \alpha\Theta Eh \lambda_2 \sum_{m=1,3...}^{M} \frac{\pi \bar{G}(m)}{4m} \sin (\frac{m\pi x}{a}) = 0 \qquad (0 < x < a)$$

(D95)

$$N_y = \alpha\Theta Eh \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} \delta_{nQ} \bar{c}_{mn} \sin (\frac{m\pi x}{a}) \sin (\frac{n\pi y}{b}) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix}$$

(D96)

$$(N_y)_{x=0} = (N_y)_{x=a} = \alpha\Theta Eh \sum_{n=1,3...}^{N} \delta_{nQ} \frac{\lambda_1\pi}{4QB} \frac{\bar{\zeta}_n}{\bar{\phi}_n} \sin (\frac{n\pi y}{b}) \qquad (0 < y < b)$$

(D97)

$$N_{xy} = -\alpha\Theta Eh \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} \delta_{nQ} \bar{j}_{mn} \cos (\frac{m\pi x}{a}) \cos (\frac{n\pi y}{b}) \qquad \begin{matrix}(0 \leq x \leq a)\\(0 \leq y \leq b)\end{matrix}$$

(D98)

Substituting from equation (D53") into equations (D13) to (D16) and neglecting terms involving λ_2^2, one obtains

$$P_3(0) = P_3(a) = P_4(0) = P_4(a) = \alpha\Theta A_3 E \lambda_2 \delta_{nQ} \frac{\lambda_1\pi^2}{16Q^2B} \bar{B}'_n$$

(D99)

With $B_n' = B_n''$ as known, from the first two equations (4), one obtains

$$N_x(0,y) = N_x(a,y) = -\alpha\Theta E h \delta_{nQ} \frac{\lambda_1 \pi}{4QB} \bar{B}_n' \sin\left(\frac{Q\pi y}{b}\right) \qquad (0 < y < b) \tag{D100}$$

<u>Numerical results for condition (1)</u>. - The numerical result for $\lambda_1 = 1$, $\lambda_2 \to 0$ for square plate (B=1) with all stiffeners identical, subjected to a pillow-shaped temperature distribution (P=Q=1) is shown in dimensionless form in figure 12 for Poisson's ratio ν equals to 0.3.

<u>Limiting case of large stiffener areas, condition (2)</u>. - In this case, equations (D62), (D64), and (D53) will be the governing equations. The additional quantities needed in order to use these equations will now be evaluated.

Substituting from equations (3), (C13'), and (D77) into equation (D64b), and with aid of equation (D61a), one obtains

$$\Phi_n^{(1)} = 1 + \nu \frac{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{[n^2B^2 - \nu m^2]}{[m^2 + n^2B^2]^2}}{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2}} \tag{D64'b}$$

Substituting from equation (D58'b) into (D62), one obtains

$$\bar{g}_m' = \alpha\Theta \frac{h}{A_3} G'(m) \tag{D62'}$$

where

$$G'(m) = \frac{PQB}{(P^2+Q^2B^2)} \delta_{mP} - \left[\frac{QB}{(m^2+Q^2B^2)^2} - \frac{1}{Q^3B^3}\right] \frac{\left\{\frac{[P + (2+\nu)\, Q^2B^2]}{[P^2 + Q^2B^2]} - 1\right\} P}{\displaystyle\sum_{i=1,3\ldots}^{M} \frac{[i^2 + (2+\nu)\, Q^2B^2]}{[i^2 + Q^2B^2]^2}} \tag{D62'a}$$

Substituting from equations (3), (C12'), (D7'), (D18'), (D77), (D56'b), (D58'b) and (D64'b) into equation (D64), one obtains

$$\bar{c}_n' = -\alpha\theta \frac{\pi^2}{4a} \frac{c_n^{(1)}}{\Phi_n^{(1)}} \delta_{nQ} + \alpha\theta \frac{h}{A_1} \frac{1}{\Phi_n^{(1)}} c_n^{(2)} \qquad \text{(D64')}$$

where

$$c_n^{(1)} = \frac{\nu P \left\{ \dfrac{[P^2 + (2+\nu)\, Q^2B^2]}{[P^2 + Q^2B^2]} - 1 \right\}}{QB \displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, Q^2B^2]}{[m^2 + Q^2B^2]^2}} \qquad \text{(D64")}$$

$$c_n^{(2)} = \left\{ \frac{PQB}{(P^2 + Q^2B^2)} - \frac{PQB\left[\dfrac{[P^2 + (2+\nu)Q^2B^2]}{(P^2 + Q^2B^2)} - 1\right] \displaystyle\sum_{m=1,3\ldots}^{M} \frac{1}{[m^2+Q^2B^2]^2}}{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, Q^2B^2]}{[m^2 + Q^2B^2]^2}} \right\} \delta_{nQ}$$

$$+\nu B \left\{ \sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2} G'(m) \right\} \Big/ \left\{ C \sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2} \right\} \qquad \text{(D64''')}$$

With $\bar{c}_n'$ and $\bar{g}_m'$ (m,n odd) known, and substituting from equations (D63'), (D62'), (D7'), (D77), (C12') and (C13') into equations (D53), there follows

$$B_n' = \alpha\theta E h\, B_n^{(1)}\, \delta_{nQ} + \alpha\theta E h\, \lambda_1\, B_n^{(2)} \qquad \text{(D53''')}$$

where

$$B_n^{(1)} = \frac{\pi}{4Q^2B^2}\left\{\frac{c_n^{(1)}\sum\limits_{m=1,3\ldots}^{M}\frac{[Q^2B^2 - \nu m^2]}{[m^2 + Q^2B^2]^2}}{\Phi_n^{(1)}\sum\limits_{m=1,3\ldots}^{M}\frac{[m^2+(2+\nu)Q^2B^2]}{[m^2 + Q^2B^2]^2}} - \frac{P\left[\frac{[P^2+(2+\nu)\ Q^2B^2]}{[P^2 + Q^2B^2]} - 1\right]}{\sum\limits_{m=1,3\ldots}^{M}\frac{[m^2+(2+\nu)Q^2B^2]}{[m^2+Q^2B^2]^2}}\right\} \tag{D53'"a}$$

$$B_n^{(2)} = \frac{B\pi\sum\limits_{m=1,3\ldots}^{M}\frac{[m^2+(2+\nu)n^2B^2]}{[m^2+n^2B^2]^2}G'(m)}{4nC\sum\limits_{m=1,3\ldots}^{M}\frac{[m^2 + (2+\nu)\ n^2B^2]}{[m^2 + n^2B^2]^2}} - \frac{\pi\ c_n^{(2)}\sum\limits_{m=1,3\ldots}^{M}\frac{[n^2B^2-\nu m^2]}{[m^2+ n^2B^2]^2}}{4nB\Phi_n^{(1)}\sum\limits_{m=1,3,\ldots}^{M}\frac{[m^2+(2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}} \tag{D53'"b}$$

With the basic quantities $\bar{c}_n'$, $\bar{g}_m'$, and B_n' known (equations (D64') (D62'), and (D53'")), the additional quantities needed in order to determine the stresses will now be evaluated. Equation (B60), (B61), (B34), and (B35) will give the values of the odd-subscripted s_n', s_n'', s_m''', s_m'''', j_{mn}, g_{mn}, c_{mn} as follows:

$$s_n' = s_n'' = \alpha\Theta A_1 E\lambda_1\left[\frac{\pi}{4n}\ \frac{c_n^{(2)}}{\Phi_n^{(1)}} - \nu B_n^{(2)}\right] \tag{D101}$$

$$s_m''' = s_m'''' = \alpha\Theta A_3 E\lambda_2\ \frac{B\pi}{4m}\ G'(m) \tag{D102}$$

$$\underset{\substack{(m\neq 0)\\(n\neq 0)}}{j_{mn}} = \alpha\Theta Eh\ \delta_{nQ}\ J_{mn} \tag{D103}$$

$$g_{mn} = \alpha\Theta Eh\, \delta_{nQ}\, G_{mn} \tag{D104}$$

$$c_{mn} = \alpha\Theta Eh\, \delta_{nQ}\, C_{mn} \tag{D105}$$

where

$$J_{mn} = \frac{PQB}{(P^2+Q^2B^2)}\,\delta_{mP} + \frac{m^2 C_n^{(1)}}{B[m^2+Q^2B^2]^2\,\Phi^{(1)}} + \frac{4\,Q^3E^3\,B_n^{(1)}}{\pi[m^2 + Q^2B^2]^2} \tag{D103a}$$

$$G_{mn} = -\frac{Q^2B^2}{(P^2+Q^2B^2)}\delta_{mP} - \frac{mQ\,C_n^{(1)}}{\Phi_n^{(1)}[m^2 + n^2B^2]^2} + \frac{4m[m^2+2n^2B^2]B_n^{(1)}}{\pi[m^2 + n^2B^2]^2} \tag{D104a}$$

$$C_{mn} = \frac{4m\,B_n^{(1)}}{\pi\,Q^2B^2}\left[\frac{m^2(m^2 + 2Q^2B^2)}{(m^2 + Q^2B^2)^2} - 1\right] - \frac{m^3C_n^{(1)}}{Q^2B^2[m^2 + Q^2B^2]^2\,\Phi_n^{(1)}} \tag{D105a}$$

$$- \frac{P^2}{(P^2 + Q^2B^2)}\,\delta_{mP}$$

Now with $\bar{c}_n'$, $\bar{g}_m'$, B_n', s_n', s_m''', c_{mn}, g_{mn}, and j_{mn} known, equations (B16), and (B19) to (B25) will give the following stiffener tensions and the plate stresses:

$$P_1(y) = P_2(y) = \alpha\Theta A_1 E\lambda_1 \sum_{n=1,3\ldots}^{N} \left[\frac{\pi}{4n}\,\frac{C_n^{(2)}}{\Phi_n^{(1)}} - \nu B_n^{(2)}\right] \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{D106}$$

$$P_3(x) = P_4(x) = \alpha\Theta A_3 E\lambda_2 \sum_{m=1,3\ldots}^{M} \frac{B\pi}{4m}\, G'(m)\, \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{D107}$$

$$N_x = \alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} G_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \delta_{nQ} \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix}$$

(D108)

$$(N_x)_{y=0} = (N_x)_{y=b} = \alpha\Theta Eh\, \lambda_2 \sum_{m=1,3\ldots}^{M} \frac{\pi\, G'(m)}{4m} \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a)$$

(D109)

$$N_y = \alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \delta_{nQ} \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix}$$

(D110)

$$(N_y)_{x=0} = (N_y)_{x=a} = -\alpha\Theta Eh \sum_{n=1,3\ldots}^{N} \frac{\pi}{4Q^2B^2} \frac{C_n^{(1)}}{\Phi_n^{(1)}} \sin\left(\frac{n\pi y}{b}\right)\delta_{nQ}$$

$$(0 < y < b) \qquad \text{(D111)}$$

$$N_{xy} = -\alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} J_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \delta_{nQ} \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le b)\end{matrix}$$

(D112)

With the $B_n' = B_n''$ known, equations (D13) to (D16) yield the following tensions at the ends of the x-wise stiffeners where they join the rigid vertical stiffeners:

$$P_3(0) = P_3(a) = P_4(0) = P_4(a) = \alpha\Theta A_3 E\lambda_2 \sum_{n=1,3\ldots}^{N} \frac{\pi B_n^{(1)}}{4n} \delta_{nQ}$$

(D113)

And from the first two of equations (4), one obtains the following running tensions between the rigid stiffeners and the plate edges at x=0 and x=a:

$$N_x(0,y) = N_x(a,y) = \alpha\Theta Eh \sum_{n=1,3...}^{N} B_n^{(1)} \sin\left(\frac{n\pi y}{b}\right) \delta_{nQ} \qquad (0 < y < b)$$

(D114)

Numerical results for condition (2). - The numerical result for $\lambda_2 \rightarrow 0$ and $\lambda_1 \rightarrow 0$ for square plate (B = 1) with all stiffeners identical, subjected to a "pillow-shaped" temperature (P = Q = 1) is shown in dimensionless form in figure 13 for Poisson's ratio ν equals to 0.3.

Illustrative Prescribed-Force Problem

As another illustration of the application of the general theory of this appendix, the case will now be considered in which the stiffeners at x = 0 and x = a are rigid and straight and the plate is stretched by means of forces applied perpendicular to these stiffeners. The loading in this case is that of figure 4b with $T_1 = T_2 = T \neq 0$ and all other loads vanishing. As in the previous example, the plate is assumed to be isotropic and to have the same Young's modulus as the stiffeners, while the stiffeners are assumed to be symmetric about the plate centerlines (that is $A_1 = A_2$, $A_3 = A_4$). The temperature is assumed to be uniform and will therefore produce no stresses. Later on, for the sake of numerical calculations, the problem will be further specialized to the case of a square plate (b=a) with all stiffeners identical ($A_1 = A_2 = A_3 = A_4 = A$).

Reduction of general equations to special case. - Due to the absence of prescribed normal forces, shearing flows, and prescribed moments, the following quantities are all zero:

$P_1(0),\ P_1(b),\ P_2(0),\ P_2(b)$ (fig. 4b)

$M_1,\ M_2$ (fig. 4a)

$B_m''',\ B_m''''$ (see eqs. (4))

$Q_n',\ Q_n'',\ Q_m''',\ Q_m''''$ (see eqs. (6))

(It should be noted that $P_3(0)$, $P_3(a)$, $P_4(0)$, $P_4(a)$, B_n' and B_n'' do not necessarily vanish.) And due to absence of thermal loading, the following quantities are also all zero:

$T_n',\ T_n'',\ T_m''',\ T_m''''$ (see eqs. (14))

T_{mn} (see eq. (16))

$V_n',\ V_n''$ (see eqs. (23))

Because in this example the structure and loading are symmetrical about both centerlines ($x = a/2$ and $y = b/2$) it follows that $\bar{c}_n' = \bar{c}_n''$, $\bar{g}_m' = \bar{g}_m''$, and $B_n' = B_n''$, and the simplified system of equations, namely equations (D53), (D56), and (D57), may be used for the determination of the B_n', $\bar{c}_n'$, and $\bar{g}_m'$.

From the given loading conditions the quantities $\delta_n^{(1)}$, defined by equation (D7), $S_n^{(5)}$, defined by equation (D18), and $S_m^{(7)}$, defined by equation (D20), can be reduced to the following

$$\delta_n^{(1)} = 0 \tag{D7''}$$

$$S_n^{(5)} = 0 \tag{D18''}$$

$$S_m^{(7)} = \frac{2T}{a} \qquad (M = 1, 3, \ldots, M) \tag{D20''}$$

Substituting from equation (D7"), (D18"), (C22'), (C23') and (D71) into equation (D56b), one obtains

$$\zeta_n = 0 \qquad \text{(D56'"b)}$$

Equations (D57), which define the quantities $\bar{g}'_m$, can be reduced to the following form if use is made of equations (C24'), (D7"), (D71), (D20"), (D56'a), (D56'b), (D56'c), (D57'a), and (D57'"b):

$$\Big\{ \sum_{n=1,3\ldots}^{N} \frac{\bar{\xi}_{mn}}{C\bar{\phi}_n} \bar{\Theta}_{mn} + \Big[1 + \lambda_2 B^4 \sum_{n=1,3\ldots}^{N} \frac{n^2}{(m^2 + n^2B^2)^2}\Big]$$

$$-\lambda_2 B^4 \sum_{n=1,3\ldots}^{N} \frac{\Big[\frac{n^2}{(m^2 + n^2B^2)^2} - \frac{1}{n^2B^4}\Big]\big[m^2 + (2+\nu)n^2B^2\big]}{(m^2 + n^2B^2)^2 \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2 + (2+\nu)n^2B^2]}{[i^2 + n^2B^2]^2}} \Big\} \; G(m)$$

$$= \frac{2}{\pi m} - \sum_{p=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \Big[\frac{\bar{\xi}_{pn}}{C\bar{\phi}_n} \bar{\Theta}_{mn} - \lambda_2 B^4 \frac{\Big[\frac{n^2}{(m^2+n^2B^2)^2} - \frac{1}{n^2B^4}\Big]\big[m^2+(2+\nu)n^2B^2\big]}{(m^2+n^2B^2)^2 \sum\limits_{i=1,3\ldots}^{M} \frac{[i^2+(2+\nu)n^2B^2]}{[i^2+n^2B^2]^2}} \Big] \cdot$$

$$\cdot \frac{(1 - \delta_{mP})\, p}{m} G(p) \quad (m = 1,3,\ldots,M) \qquad \text{(D57(1))}$$

where

$$G(p) = \frac{g'_p A_3}{T h}$$

With $\bar{g}'_m$ known from the solution of the above equations, the $\bar{c}'_n$ can be determined from equation (D56). If use is made of equations (D56'a), (D56'c), (D56'"b), and (B66), then equation (D56) can be put in the following form:

$$C(n) = \frac{1}{n\ BC\ \bar{\phi}_n} \sum_{m=1,3\ldots}^{M} m\ \bar{\xi}_{mn}\ G(m) \qquad (n = 1,\ 3,\ \ldots,\ M)$$

(D56(1))

where $$C(n) = \frac{c'_n A_1}{Th}$$

With the $\bar{c}'_n$ and $\bar{g}'_m$ known, equations (D53), (D77), (D7'), (C12'), (C13'), and (B66) lead to the following expression for the determination of B'_n:

$$B(n) = \frac{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{m[m^2+(2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2} G(m)}{\displaystyle nC \sum_{m=1,3\ldots}^{M} \frac{[m^2+(2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}} - \frac{\displaystyle C(n) \sum_{m=1,3\ldots}^{M} \frac{[n^2B^2 - \nu m^2]}{[m^2 + n^2B^2]^2}}{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\ n^2B^2]}{[m^2 + n^2B^2]^2}}$$

$(n = 1,\ 3,\ \ldots,\ N)$ (D53(1))

where

$$B(n) = \frac{B'_n A_1}{Th}$$

With the B'_n known, from equations (D13) and (D14) one obtains the following tensions at the ends of x-wise stiffeners where they join the rigid vertical stiffeners:

$$P_3(0) = P_3(a) = P_4(0) = P_4(a) = T[\tfrac{1}{2} - \sum_{n=1,3\ldots}^{N} \frac{\lambda_1 \pi}{4nB} B(n)] \qquad \text{(D115)}$$

And from the first two of equations (4), one obtains the following running tensions between the rigid stiffeners and the plate edges at $x = 0$ and $x = a$:

$$N_x(0,y) = N_x(a,y) = \frac{Th}{A_1} \sum_{n=1,3\ldots}^{N} B(n) \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{D116}$$

With $\bar{c}'_n$, $\bar{g}'_m$, B'_n as known, equations (B60) will give the values of the odd subscripted s'_n, s''_n, s'''_m, s''''_m, and equations (B61), (B34), and (B35) values of j_{mn}, g_{mn}, and c_{mn} with m and n both odd. The expressions obtained are:

$$s'_n = s''_n = T[C(n) - \nu B(n)] \tag{D117}$$

$$s'''_m = s''''_m = TG(m) \tag{D118}$$

$$j_{mn} = \frac{Th}{A_1} J(m,n) \tag{D119}$$

$$g_{mn} = \frac{Th}{A_1} G(m,n) \tag{D120}$$

$$c_{mn} = \frac{Th}{A_1} C(m,n) \tag{D121}$$

where

$$J(m,n) = \frac{4nB}{\pi[m^2 + n^2B^2]^2} [n^2B^2B(n) - m^2C(n) - mn\, B^2G(m)/C] \tag{D119a}$$

$$G(m,n) = \frac{4}{\pi[m^2+n^2B^2]^2} [mn^2B^2C(n) + n^3B^4G(m)/C + m(m^2 + 2n^2B^2)\, B(n)] \tag{D120a}$$

$$C(m,n) = \frac{4m^3C(n)}{\pi[m^2 + n^2B^2]^2} + \frac{4m^2n\, B^2G(m)}{\pi[m^2 + n^2B^2]^2\, C} + \frac{4m\, B(n)}{n^2B^2\pi} \left[\frac{m^2(m^2 + 2n^2B^2)}{(m^2 + n^2B^2)^2} - 1\right] \tag{D121a}$$

Equations (B16) and (B19) through (B25) then give the following stiffener tensions and plate stresses:

$$P_1(y) = P_2(y) = T \sum_{n=1,3...}^{N} [C(n) - \nu B(n)] \sin\left(\frac{n\pi y}{b}\right) \qquad (0 < y < b) \tag{D122}$$

$$P_3(x) = P_4(x) = T \sum_{m=1,3...}^{M} G(m) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{D123}$$

$$N_x = \frac{Th}{A_1} \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} G(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{D124}$$

$$(N_x)_{y=0} = (N_x)_{y=b} = \frac{Th}{A_1} \sum_{m=1,3...}^{M} [G(m)/C] \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{D125}$$

$$N_y = \frac{Th}{A_1} \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} C(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{D126}$$

$$(N_y)_{x=0} = (N_y)_{x=a} = \frac{Th}{A_1} \sum_{n=1,3...}^{N} C(n) \sin\frac{n\pi y}{b} \qquad (0 < y < b) \tag{D127}$$

$$N_{xy} = -\frac{Th}{A_1} \sum_{m=1,3...}^{M} \sum_{n=1,3...}^{N} J(m,n) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le b)\end{matrix} \tag{D128}$$

Numerical results for λ_1 and $\lambda_2 \neq 0$. - The equations described above were applied to the special case of a square plate (B = 1), with all stiffener areas equal ($A_1 = A_2 = A_3 = A_4 = A$), and Poisson's ratio

ν equals to 0.3. The assumption that B = 1 and that all stiffener areas are equal implies that $\lambda_1 = \lambda_2$ (see eqs. (C81)), and the common symbol λ will therefore be used for both λ_1 and λ_2. The Gauss-Seidel iteration procedure was used in solving equations (D57(1)).

The results obtained from the stiffener tensions and plate stresses are shown in dimensionless form in figure 14 for $\lambda = 1.0$. The values of M and N employed in the calculations are indicated on the figure. In general, stresses were computed at x/a and y/a interval of 0.1.

Limiting case of large stiffener areas. - For the case in which all the stiffener cross-sectional areas are very large compared to the plate cross-sectional area, equations (D53), (D71), and (D72) may be employed as approximations which become more and more accurate as the ratio of stiffener to plate cross-sectional areas approach infinity.

Equation (D72) can be reduced to the following equation by substituting equations (D7"), (D20"), (D77), and (C24') into equation (D72) with Z''_m defined by equation (D58b):

$$\bar{G}(m) = \frac{2}{m\pi} \tag{D72'}$$

where

$$\bar{G}(m) = \frac{g'_m A_3}{Th}$$

Substituting from equations (3), (C12'), (D7"), (D18"), (D77), and (D64'b) into equation (D71), one obtains

$$\bar{C}(n) = \frac{\nu}{nC\,\Phi_n^{(1)}} \frac{\sum_{m=1,3\ldots}^{M} \frac{m[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2}\, \bar{G}(m)}{\sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2}} \tag{D71'}$$

where

$$\bar{C}(n) = \frac{c'_n A_1}{Th}$$

and the $\bar{G}(m)$ are known from equations (D72').

With the $\bar{c}'_n$ and $\bar{g}'_m$ known, equation (D53) can be expressed in the following form if use is made of equations (D77), (D7'), (C12'), (C13'), and (B66):

$$\bar{B}(n) = \frac{\displaystyle\sum_{m=1,3\ldots}^{M} \frac{m[m^2+(2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}\bar{G}(m)}{\displaystyle nC \sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)n^2B^2]}{[m^2 + n^2B^2]^2}} - \frac{\displaystyle \bar{C}(n) \sum_{m=1,3\ldots}^{M} \frac{[n^2B^2 - \nu m^2]}{[m^2 + n^2B^2]^2}}{\displaystyle \sum_{m=1,3\ldots}^{M} \frac{[m^2 + (2+\nu)\, n^2B^2]}{[m^2 + n^2B^2]^2}} \tag{D53(2)}$$

where

$$\bar{B}(n) = \frac{B'_n A_1}{Th}$$

With the B'_n thus known, from equations (D13) and (D14) one obtains the following tensions at the ends of x-wise stiffeners where they join the rigid vertical stiffeners:

$$P_3(0) = P_3(a) = P_4(0) = P_4(a) = T[\tfrac{1}{2} - (\pi\lambda_2/4) \sum_{n=1,3\ldots}^{N} \bar{B}(n)/n] \tag{D129}$$

And from the first two of equations (4), one obtains the following running tensions between the rigid stiffeners and the plate edges at $x = 0$ and $x = a$:

$$N_x(0,y) = N_x(a,y) = \frac{Th}{A_1} \sum_{n=1,3\ldots}^{N} \bar{B}(n) \sin(n\pi y/b) \quad (0 < y < b) \tag{D130}$$

With $\bar{c}_n'$, $\bar{g}_m'$, B_n' known, equations (B60) will give the odd-subscripted s_n', s_n'', s_m''', s_m'''', and equation (B61), (B34), and (B35) the odd-subscripted j_{mn}, g_{mn}, c_{mn}, as follows:

$$s_n' = s_n'' = T[\bar{C}(n) - \nu\bar{B}(n)] \tag{D131}$$

$$s_m''' = s_m'''' = T\bar{G}(m) \tag{D132}$$

$$\underset{\substack{(m\neq 0)\\(n\neq 0)}}{j_{mn}} = \frac{Th}{A_1}\bar{J}(m,n) \tag{D133}$$

$$g_{mn} = \frac{Th}{A_1}\bar{G}(m,n) \tag{D134}$$

$$c_{mn} = \frac{Th}{A_1}\bar{C}(m,n) \tag{D135}$$

where

$$\bar{J}(m,n) = \frac{4nB}{\pi[m^2+n^2B^2]^2}[n^2B^2\bar{B}(n) - m^2\bar{C}(m) - mn\, B^2\bar{G}(m)/C] \tag{D133a}$$

$$\bar{G}(m,n) = \frac{4}{\pi[m^2+n^2B^2]^2}[mn^2B^2\bar{C}(n) + n^3B^4\bar{G}(m)/C + m(m^2 + 2n^2B^2)\bar{B}(n)] \tag{D134a}$$

$$\bar{C}(m,n) = \frac{4m^3\bar{C}(n)}{\pi[m^2 + n^2B^2]^2} + \frac{4m^2n\, B^2\bar{G}(m)}{\pi\,[m^2 + n^2B^2]C}$$

$$+ \frac{4m\bar{B}(n)}{n^2B^2\pi}[\frac{m^2(m^2 + 2n^2B^2)}{(m^2 + n^2B^2)^2} - 1] \tag{D135a}$$

Finally, equations (B16), and (B19) through (B25) will give the following stiffener tensions and plate stresses:

$$P_1(y) = P_2(y) = T \sum_{n=1,3\ldots}^{N} [\bar{C}(n) - \nu\bar{B}(n)] \sin (n\pi y/b) \qquad (0 < y < b) \tag{D136}$$

$$P_3(x) = P_4(x) = T \sum_{m=1,3\ldots}^{M} \bar{G}(m) \sin (m\pi x/a) \qquad (0 < x < a) \tag{D137}$$

$$N_x = \frac{Th}{A_1} \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{G}(m,n) \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{D138}$$

$$(N_x)_{y=0} = (N_x)_{y=b} = \frac{Th}{A_1} \sum_{m=1,3\ldots}^{M} [\bar{G}(m)/C] \sin (m\pi x/a) \qquad (0 < x < a) \tag{D139}$$

$$N_y = \frac{Th}{A_1} \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{C}(m,n) \sin (m\pi x/a) \sin (n\pi y/b) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{D140}$$

$$(N_y)_{x=0} = (N_y)_{x=a} = \frac{Th}{A_1} \sum_{n=1,3\ldots}^{N} \bar{C}(n) \sin (n\pi y/b) \qquad (0 < y < b) \tag{D141}$$

$$N_{xy} = - \frac{Th}{A_1} \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{J}(m,n) \cos (m\pi x/a) (n\pi y/b) \qquad \begin{matrix}(0 \leq x \leq a)\\(0 \leq y \leq b)\end{matrix} \tag{D142}$$

<u>Numerical results for the case of large stiffener areas</u>. - The numerical results for $\lambda_1 = \lambda_2 \rightarrow 0$, obtained from the above equations, are shown in figure 15 for square plate (B = 1) with all stiffeners identical and Poisson's ratio ν equals to 0.3.

APPENDIX E

ANALYSIS FOR ALL FOUR STIFFENERS
WITH PRESCRIBED DISPLACEMENT CONDITIONS

In the present appendix the case of figure 4c is considered. In this case all the edges of the plate are assumed to be forced into the prescribed shapes by means of attached rigid stiffeners (shown shaded in fig. 4c), which shapes are defined by known values of K_n', K_n'', K_m''', and K_m'''' in equations (8) through (11). Correspondingly, the Fourier coefficients B_n', B_n'', B_m''' and B_m'''', which describe the running tensions between the stiffeners and the plate edges, are now unknowns. In addition the loading resultants T_1, M_1, T_2, M_2, T_3, M_3, T_4 and M_4 constitute eight new knowns, supplanting $P_1(0)$, $P_1(b)$, $P_2(0)$, $P_2(b)$, $P_3(0)$, $P_3(a)$, $P_4(0)$, $P_4(a)$, which are now unknowns.

Formulation of Boundary Condition of Prescribed Curvature

Equations (D4) apply also in the present case. They express the requirement that at x=0 and x=a the curvatures of the stiffeners and the curvatures of the adjacent plate edges to which they are joined must be equal. These equations are rewritten in slightly rearranged form as follows:

$$K'_n = -V'_n - B'_n\left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M} \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+B''_n\left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M}(-1)^m \frac{1}{E_{mn}}\left[(C_3 - C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+ \sum_{m=0}^{M} \frac{1}{E_{mn}} \Big\{T_{mn}\left(\frac{m\pi}{a}\right)\left[(C_3 - C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]$$

$$+C_2\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{n\pi}{b}\right)^2\left[c'_n - (-1)^m c''_n\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+ \frac{2}{b}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_1\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left[g'_m - (-1)^n\, g''_m\right]\Big\}$$

$$- \frac{2}{b}\,\frac{n\pi}{b} \sum_{m=1}^{M} \frac{1}{E_{mn}}\,\frac{m\pi}{a}\, C_2\left[B'''_m - (-1)^n B''''_m\right]\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right] \qquad \text{(E1)}$$

$$K''_n = -V''_n - B'_n\left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M}(-1)^m \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+B''_n\left(\frac{n\pi}{b}\right)^2 C_1 \sum_{m=0}^{M} \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+ \sum_{m=0}^{M} (-1)^m \frac{1}{E_{mn}} \Big\{T_{mn}\left(\frac{m\pi}{a}\right)\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]$$

$$+C_2\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{n\pi}{b}\right)^2\left[c'_n - (-1)^n c''_n\right]\left(\frac{2-\delta_{mo}}{a}\right)$$

$$+ \frac{2}{b}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)C_1\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right]\left[g'_m - (-1)^n\, g''_m\right]\Big\}$$

$$- \frac{2}{b}\,\frac{n\pi}{b} \sum_{m=1}^{M} (-1)^m \frac{1}{E_{mn}}\,\frac{m\pi}{a}\, C_2\left[B'''_m - (-1)^n B''''_m\right]\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right]$$

Two more equations, analogous to these, will now be obtained from the conditions of prescribed curvature along the edges y = 0 and y = b. Differentiating the last of strain-displacement relations, equations (B1), with respect to x, one obtains

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}$$

Eliminating the strains in terms of the stresses by means of equations (2), and then the stresses in terms of the stress function through equations (B4), this becomes

$$\frac{\partial^2 v}{\partial x^2} = (C_3 - C_4)\frac{\partial^3 F}{\partial x^2 \partial y} - C_1 \frac{\partial^3 F}{\partial y^3} - \frac{\partial e_x}{\partial y} \qquad \text{(E2)}$$

The curvatures of $\partial^2 v/\partial x^2$ of the edges y = 0 and y = b are therefore

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} = (C_3 - C_4)\left(\frac{\partial^3 F}{\partial x^2 \partial y}\right)_{y=0} - C_1\left(\frac{\partial^3 F}{\partial y^3}\right)_{y=0} - \left(\frac{\partial e_x}{\partial y}\right)_{y=0}$$

$$\text{(E3)}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=b} = (C_3 - C_4)\left(\frac{\partial^3 F}{\partial x^2 \partial y}\right)_{y=b} - C_1\left(\frac{\partial^3 F}{\partial y^3}\right)_{y=b} - \left(\frac{\partial e_x}{\partial y}\right)_{y=b}$$

The terms on the right-hand side of the first of equations (E3) can be expressed in series form with the aid of equations (21), (B42), and (B43). The result is

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} = \sum_{m=1}^{M}\left\{(C_4 - C_3)\frac{m\pi}{a}\sum_{n=0}^{N} j_{mn} - C_1 \sum_{n=0}^{N} h_{mn} - V_m^{\prime\prime\prime}\right\} \sin\frac{m\pi x}{a}$$

Comparing this equation with equation (10), one obtains the following M equations representing the boundary condition of prescribed curvature along the edge y = 0:

$$K_m'' = (C_4-C_3)\frac{m\pi}{a}\sum_{n=0}^{N} j_{mn} - C_1\sum_{n=0}^{N} h_{mn} - V_m'' \qquad (m = 1, 2, \ldots, M) \tag{E4}$$

Similarly, the condition of prescribed curvature along the edge y=b is

$$K_m''' = (C_4-C_3)\frac{m\pi}{a}\sum_{n=0}^{N}(-1)^n j_{mn} - C_1\sum_{n=0}^{N}(-1)^n h_{mn} - V_m''' \qquad (m = 1, 2, \ldots, M) \tag{E5}$$

The unknowns j_{mn} and h_{mn} in these equations can be expressed in terms of basic unknowns c_n', c_n'', g_m', g_m'', B_n', etc. To accomplish this, we first observe that equations (B58) and (B61) can both be represented by the following single equation, in which any undefined quantities are to be considered zero:

$$\underset{(m\neq 0)}{j_{mn}} = -\frac{mn\pi^2}{abE_{mn}}\{T_{mn} + \frac{2}{a}\frac{m\pi}{a}[c_n'-(-1)^m c_n'']C_2 + \frac{2}{b}\frac{m\pi}{b}[g_m'-(-1)^n g_m'']C_1\}$$

$$+\frac{1}{E_{mn}}\{\frac{2-\delta_{no}}{b}(\frac{m\pi}{a})^3[B_m'''-(-1)^n B_m''']C_2 + \frac{2}{a}(\frac{n\pi}{b})^3[B_n'-(-1)^m B_n'']C_1\} \tag{E6}$$

Furthermore, from equation (B47), (B14), (B50), and (B51), one obtains

$$h_{mo} = \frac{1}{b}[g_m'' - g_m'] \tag{E7}$$

$$\underset{\substack{(m\neq 0)\\(n\neq 0)}}{E_{mn}h_{mn}} = (\frac{n\pi}{b})^3 T_{mn} + \frac{2}{a}\frac{m\pi}{a}(\frac{n\pi}{b})^3[c_n'-(-1)^m c_n'']C_2$$

$$-\frac{2}{b}[g_m'-(-1)^n g_m''][C_2(\frac{m\pi}{a})^4 + (C_4-2C_3)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2]$$

$$+\frac{2}{a}(\frac{m\pi}{a})(\frac{n\pi}{b})[B_n'-(-1)^m B_n''][(\frac{m\pi}{a})^2 C_2 + (\frac{n\pi}{b})^2(C_4-2C_3)]$$

$$-\frac{2}{b}(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2[B_m'''-(-1)^n B_m''']C_2 \tag{E8}$$

and these two equations can be represented by the following single equation in which, once again, undefined quantities are to be regarded as zero:

$$E_{mn}h_{mn} = (\frac{n\pi}{b})^3 T_{mn} + \frac{2}{a}\frac{m\pi}{a}(\frac{n\pi}{b})^3[c_n' - (-1)^m c_n'']C_2$$
$$(m \neq 0)$$

$$- \frac{2-\delta_{no}}{b}[g_m' - (-1)^n g_m''][C_2(\frac{m\pi}{a})^4 + (C_4 - 2C_3)(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2]$$

$$+ \frac{2}{a}(\frac{m\pi}{a})(\frac{n\pi}{b})[B_n' - (-1)^m B_n''][(\frac{m\pi}{a})^2 C_2 + (\frac{n\pi}{b})^2(C_4 - 2C_3)]$$

$$- \frac{2}{b}(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2[B_m''' - (-1)^n B_m'''']C_2 \qquad \text{(E9)}$$

Substituting from equations (E6) and (E9) into equations (E4) and (E5), one obtains the following equations, which are analogous of equations (E1):

$$K_m''' = -V_m''' + \sum_{n=0}^{N}\frac{1}{E_{mn}}T_{mn}\frac{n\pi}{b}[(C_3 - C_4)(\frac{m\pi}{a})^2 - C_1(\frac{n\pi}{b})^2]$$

$$+ \sum_{n=0}^{N}\frac{1}{E_{mn}}\{\frac{2}{a}(\frac{m\pi}{a})(\frac{n\pi}{b})[(C_3 - C_4)(\frac{m\pi}{a})^2 - C_1(\frac{n\pi}{b})^2][c_n' - (-1)^m c_n'']C_2$$

$$+ \frac{2-\delta_{no}}{b}(\frac{m\pi}{a})^2[g_m' - (-1)^n g_m''][C_2(\frac{m\pi}{a})^2 - C_3(\frac{n\pi}{b})^2]C_1\}$$

$$- \frac{2}{a}\frac{m\pi}{a}\sum_{n=1}^{N}\frac{1}{E_{mn}}(\frac{n\pi}{b})[B_n' - (-1)^m B_n''][C_2(\frac{m\pi}{a})^2 - C_3(\frac{n\pi}{b})^2]C_1$$

$$- B_m'''(\frac{m\pi}{a})^2\sum_{n=0}^{N}\frac{1}{E_{mn}}\frac{2-\delta_{no}}{b}[(C_3 - C_4)(\frac{m\pi}{a})^2 - C_1(\frac{n\pi}{b})^2]C_2$$

$$+ B_m''''(\frac{m\pi}{a})^2\sum_{n=0}^{N}(-1)^n\frac{1}{E_{mn}}\frac{2-\delta_{no}}{b}[(C_3 - C_4)(\frac{m\pi}{a})^2 - C_1(\frac{n\pi}{b})^2]C_2$$
$$\text{(E10)}$$

$$K_m'''' = -V_m'''' + \sum_{n=0}^{N} (-1)^n \frac{1}{E_{mn}} T_{mn} \frac{n\pi}{b} [(c_3 - c_4)(\frac{m\pi}{a})^2 - c_1(\frac{n\pi}{b})^2]$$

$$+ \sum_{n=0}^{N} (-1)^n \frac{1}{E_{mn}} \{\frac{2}{a}(\frac{m\pi}{a})(\frac{n\pi}{b})[(c_3 - c_4)(\frac{m\pi}{a})^2 - c_1(\frac{n\pi}{b})^2][c_n' - (-1)^m c_n'']C_2$$

$$+ \frac{2-\delta_{no}}{b} (\frac{n\pi}{a})^2 [g_m' - (-1)^n g_m''][c_2(\frac{m\pi}{a})^2 - c_3(\frac{n\pi}{b})^2]C_1\}$$

$$- \frac{2}{a} \frac{m\pi}{a} \sum_{n=1}^{N} (-1)^n \frac{1}{E_{mn}} (\frac{n\pi}{b})[B_n' - (-1)^m B_n''][c_2(\frac{m\pi}{a})^2 - c_3(\frac{n\pi}{b})^2]C_1$$

$$- B_m'''(\frac{m\pi}{a})^2 \sum_{n=0}^{N} (-1)^n \frac{1}{E_{mn}} \frac{2-\delta_{no}}{b} [(c_3 - c_4)(\frac{m\pi}{a})^2 - c_1(\frac{n\pi}{b})^2]C_2$$

$$+ B_m''''(\frac{m\pi}{a})^2 \sum_{n=0}^{N} \frac{1}{E_{mn}} \frac{2-\delta_{no}}{b} [(c_3 - c_4)(\frac{m\pi}{a})^2 - c_1(\frac{n\pi}{b})^2]C_2 \qquad (E10)$$

Formulation of Boundary Conditions of Equilibrium

Equilibrium considerations for the stiffeners at x=0 and x=a lead to the following relations (see appendix D):

$$P_3(0) = - \sum_{n=1}^{N} \frac{b}{n\pi} B_n' + \frac{T_1}{2} - \frac{M_1}{b}$$

$$P_3(a) = - \sum_{n=1}^{N} \frac{b}{n\pi} B_n'' + \frac{T_2}{2} - \frac{M_2}{b}$$

$$P_4(0) = \sum_{n=1}^{N} (-1)^n \frac{b}{n\pi} B_n' + \frac{T_1}{2} + \frac{M_1}{b} \qquad (E11)$$

$$P_4(a) = \sum_{n=1}^{N} (-1)^n \frac{b}{n\pi} B_n'' + \frac{T_2}{2} + \frac{M_2}{b}$$

Additional equations of the same type result from considering the equilibrium of the other two stiffeners. The equilibrium equations for the stiffener at $y = 0$ are

$$P_1(0) + P_2(0) + \int_0^a N_y(x,0)\,dx = T_3$$

$$a \cdot P_2(0) + \int_0^a x\,N_y(x,0)\,dx = \frac{T_3 a}{2} + M_3$$

By substituting

$$N_y(x,0) = \sum_{m=1}^{M} B_m^{\prime\prime\prime} \sin\left(\frac{m\pi x}{a}\right)$$

and solving these equations, one obtains

$$P_1(0) = -\sum_{m=1}^{M} \frac{a}{m\pi} B_m^{\prime\prime\prime} + \frac{T_3}{2} - \frac{M_3}{a}$$

$$P_2(0) = \sum_{m=1}^{M} (-1)^m \frac{a}{m\pi} B_m^{\prime\prime\prime} + \frac{T_3}{2} + \frac{M_3}{a} \qquad \text{(E12)}$$

Thus, in effect, the unknown $P_1(0)$ and $P_2(0)$ have been expressed in terms of the known T_3 and M_3. Similarly, from the equilibrium equations for the stiffener at $y=b$, one obtains

$$P_1(b) = -\sum_{m=1}^{M} \frac{a}{m\pi} B_m^{\prime\prime\prime\prime} + \frac{T_4}{2} - \frac{M_4}{a}$$

$$P_2(b) = \sum_{m=1}^{M} (-1)^m \frac{a}{m\pi} B_m^{\prime\prime\prime\prime} + \frac{T_4}{2} + \frac{M_4}{a} \qquad \text{(E13)}$$

Separating B'_n, B''_n, B'''_m, and B''''_m Terms in R'_n, R''_n, R'''_m, R''''_m

Substituting from equations (E11) to (E13) and (B69) into equation (B68), and isolating the B'_n, B''_n, B'''_m and B''''_m terms, one obtains

$$\begin{aligned}
R'_n &= S'_n + \sum_{m=1}^{M} \Pi_{mn} [B'''_m - (-1)^n B''''_m] + \gamma'_n B'_n - \gamma''_n B''_n \\
R''_n &= S''_n - \sum_{m=1}^{M} (-1)^m \Pi_{mn} [B'''_m - (-1)^n B''''_m] - \gamma''_n B'_n + \gamma'''_n B''_n \\
R'''_m &= S'''_m + \sum_{n=1}^{N} H_{mn} [B'_n - (-1)^m B''_n] + \Gamma'_m B'''_m - \Gamma''_m B''''_m \\
R''''_m &= S''''_m - \sum_{n=1}^{N} (-1)^n H_{mn} [B'_n - (-1)^m B''_n] - \Gamma''_m B'''_m + \Gamma'''_m B''''_m
\end{aligned} \tag{E14}$$

where S'_n, S''_n, S'''_m, S''''_m are known loading terms defined as follows:

$$S'_n = Q'_n + A_1 E_1 \frac{n\pi}{n} T'_n - \sum_{m=1}^{M} \frac{1}{E_{mn}} \frac{mn\pi^2}{ab} T_{mn} + \frac{T_3}{b} - \frac{2M_3}{ab} - (-1)^n [\frac{T_4}{b} - \frac{2M_4}{ab}] \tag{E15}$$

$$S''_n = -Q''_n + A_2 E_2 \frac{n\pi}{b} T''_n + \sum_{m=1}^{M} (-1)^m \frac{1}{E_{mn}} \frac{mn\pi^2}{ab} T_{mn} + \frac{T_3}{b} + \frac{2M_3}{ab} - (-1)^n \cdot$$

$$\cdot [\frac{T_4}{b} + \frac{2M_4}{ab}] \tag{E16}$$

$$S'''_m = Q'''_m + A_3 E_3 \frac{m\pi}{a} T'''_m - \sum_{m=1}^{M} \frac{1}{E_{mn}} \frac{mn\pi^2}{ab} T_{mn} + \frac{T_1}{a} - \frac{2M_1}{ab} - (-1)^m [\frac{T_2}{a} - \frac{2M_2}{ab}] \tag{E17}$$

$$S_m'''' = -Q_m'''' + A_4E_4 \frac{m\pi}{a} T_m'''' + \sum_{n=1}^{N} (-1)^n \frac{1}{E_{mn}} \frac{mn\pi^2}{ab} T_{mn} + \frac{T_1}{a} + \frac{2M_1}{ab} - (-1)^m \cdot$$

$$\cdot \left[\frac{T_2}{a} + \frac{2M_2}{ab}\right] \qquad \text{(E18)}$$

γ_n' and γ_n'' have been defined in equations (C10) and (D6).

Γ_m' through Γ_m''' and Π_{mn} are also known quantities and are defined by the following equations:

$$\Gamma_m' = \frac{a}{bm\pi} + A_3E_3 \frac{m\pi}{a} C_3 + \sum_{n=1}^{N} \frac{1}{E_{mn}} \frac{2}{b} \left(\frac{m\pi}{a}\right)^3 C_2 \qquad \text{(E19)}$$

$$\Gamma_m'' = \frac{a}{bm\pi} + \sum_{n=1}^{N} (-1)^n \frac{1}{E_{mn}} \frac{2}{b} \left(\frac{m\pi}{a}\right)^3 C_2 \qquad \text{(E20)}$$

$$\Gamma_m''' = \frac{a}{bm\pi} + A_4E_4 \frac{m\pi}{a} C_3 + \sum_{n=1}^{N} \frac{1}{E_{mn}} \frac{2}{b} \left(\frac{m\pi}{a}\right)^3 C_2 \qquad \text{(E21)}$$

and

$$\Pi_{mn} = \frac{2}{b} \frac{m\pi}{a} \left[\frac{1}{E_{mn}} \left(\frac{m\pi}{a}\right)^2 C_2 - \frac{a^2}{m^2\pi^2}\right] \qquad \text{(E22)}$$

System of Simultaneous Equations

Using $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$ as defined in equations (C10) and (D6), equations (E1) can be written in the following form:

$$B_n'\gamma_n^{(1)} - B_n''\gamma_n^{(2)} = \delta_n' - \sum_{m=1}^{M} \Lambda_{mn}' [B_m''' - (-1)^n B_m''''] + \sum_{m=1}^{M} \mu_{mn}' [\bar{g}_m' - (-1)^n \bar{g}_m'']$$

$$+ \sum_{m=0}^{M} \nu_{mn}' [\bar{c}_n' - (-1)^m \bar{c}_n''] \qquad (n = 1, 2, \ldots, N)$$

(E23)

$$B'_n\gamma_n^{(2)}-B''_n\gamma_n^{(1)} = \delta''_n - \sum_{m=1}^{M}(-1)^m\Lambda'_{mn}[B'''_m-(-1)^n B''''_m] + \sum_{m=1}^{M}(-1)^m\mu'_{mn}[\bar{g}'_m-(-1)^n\bar{g}''_m]$$

$$+\sum_{m=0}^{M}(-1)^m\nu'_{mn}[\bar{c}'_n-(-1)^m\,\bar{c}''_n] \quad (n = 1,\ 2,\ \ldots,\ N) \tag{E24}$$

where δ'_n, Λ'_{mn}, and δ''_n are known quantities, given as follows:

$$\delta'_n = -K'_n - V'_n + \sum_{m=1}^{M}\frac{1}{E_{mn}}T_{mn}\left(\frac{m\pi}{a}\right)\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right] \tag{E23a}$$

$$\Lambda'_{mn}= \frac{2}{b}\,\frac{n\pi}{b}\,\frac{1}{E_{mn}}\,\frac{m\pi}{a}\left[C_1\left(\frac{n\pi}{b}\right)^2 - C_3\left(\frac{m\pi}{a}\right)^2\right]C_2 \tag{E23b}$$

$$\delta''_n = -K''_n - V''_n + \sum_{m=1}^{M}(-1)^m\frac{1}{E_{mn}}T_{mn}\left(\frac{m\pi}{a}\right)\left[(C_3-C_4)\left(\frac{n\pi}{b}\right)^2 - C_2\left(\frac{m\pi}{a}\right)^2\right] \tag{E24a}$$

Similarly, equations (E10) can be written in the following form:

$$B'''_m\Gamma_m^{(1)}-B''''_m\Gamma_m^{(2)} = \delta'''_m - \sum_{n=1}^{N}\Lambda''_{mn}[B'_n - (-1)^m\,B''_n]$$

$$+\sum_{n=1}^{N}\mu''_{mn}[\bar{c}'_n-(-1)^m\bar{c}''_n] + \sum_{n=0}^{N}\nu''_{mn}[\bar{g}'_m-(-1)^n\bar{g}''_n]$$

$$(m = 1,\ 2,\ \ldots,\ M) \tag{E25}$$

$$B'''_m\Gamma_m^{(2)} - B''''_m\Gamma_m^{(1)} = \delta''''_m - \sum_{n=1}^{N}(-1)^n\Lambda''_{mn}\,[B'_n - (-1)^m B''_n]$$

$$+\sum_{n=1}^{N}(-1)^n\mu''_{mn}[\bar{c}'_n-(-1)^m\bar{c}''_n] + \sum_{n=0}^{N}(-1)^n\nu''_{mn}[\bar{g}'_m-(-1)^n\bar{g}''_m]$$

$$(m = 1,\ 2,\ \ldots,\ M) \tag{E26}$$

where

$$\Gamma_m^{(1)} = \left(\frac{m\pi}{a}\right)^2 \sum_{n=0}^{N} \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{m\pi}{a}\right)^2 - C_1\left(\frac{n\pi}{b}\right)^2\right]\left(\frac{2-\delta_{no}}{b}\right)C_2 \tag{E25a}$$

$$\Gamma_m^{(2)} = \left(\frac{m\pi}{a}\right)^2 \sum_{n=0}^{N} (-1)^n \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{m\pi}{a}\right)^2 - C_1\left(\frac{n\pi}{b}\right)^2\right]\left(\frac{2-\delta_{no}}{b}\right)C_2 \tag{E25b}$$

$$\delta_m^{'''} = -K_m^{'''} - V_m^{'''} + \sum_{n=1}^{N} \frac{1}{E_{mn}} T_{mn}\left(\frac{n\pi}{b}\right)\left[(C_3-C_4)\left(\frac{m\pi}{a}\right)^2 - C_1\left(\frac{n\pi}{b}\right)^2\right] \tag{E25c}$$

$$\Lambda_{mn}'' = \frac{2}{a}\,\frac{m\pi}{a}\,\frac{1}{E_{mn}}\left(\frac{n\pi}{b}\right)\left[C_2\left(\frac{m\pi}{a}\right)^2 - C_3\left(\frac{n\pi}{b}\right)^2\right]C_1 \tag{E25d}$$

$$\mu_{mn}'' = \frac{1}{E_{mn}}\left[(C_3-C_4)\left(\frac{m\pi}{a}\right)^2 - C_1\left(\frac{n\pi}{b}\right)^2\right]\frac{m\pi}{a}\,\frac{2}{a} \tag{E25e}$$

$$\nu_{mn}'' = \frac{1}{E_{mn}}\left[C_2\left(\frac{m\pi}{a}\right)^2 - C_3\left(\frac{n\pi}{b}\right)^2\right]\frac{m\pi}{a}\,\frac{2-\delta_{no}}{b} \tag{E25f}$$

$$\delta_m^{''''} = -K_m^{''''} - V_m^{''''} + \sum_{n=1}^{N} (-1)^n \frac{1}{E_{mn}} T_{mn}\left(\frac{n\pi}{b}\right)\left[(C_3-C_4)\left(\frac{m\pi}{a}\right)^2 - C_1\left(\frac{n\pi}{b}\right)^2\right] \tag{E26a}$$

Substituting from equations (E14) into equations (B62) to (B65) yields the following equations:

$$\bar{c}_n'\alpha_1(n)-\bar{c}_n''\beta_1(n) = S_n' + \sum_{m=1}^{M}\Pi_{mn}\left[B_m^{'''} - (-1)^n B_m^{''''}\right] + \gamma_n' B_n' - \gamma_n'' B_n''$$

$$- \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} \frac{\bar{g}_m' - (-1)^n \bar{g}_m''}{E_{mn}} \quad (n = 1, 2, \ldots, N) \tag{E27}$$

$$-\bar{c}'_n\beta_1(n) + \bar{c}''_n\alpha_2(n) = S''_n - \sum_{m=1}^{M}(-1)^m\, \Pi_{mn}[B'''_m - (-1)^n B''''_m] - \gamma''_n B'_n + \gamma'''_n B''_n$$

$$+ \frac{2}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M}(-1)^m \frac{\bar{g}'_m - (-1)^n\, \bar{g}''_m}{E_{mn}} \quad (n = 1, 2, \ldots, N) \tag{E28}$$

$$\bar{g}'_m\alpha_3(m) - \bar{g}''_m\beta_2(m) = S'''_m + \sum_{n=1}^{N} H_{mn}[B'_n - (-1)^m B''_n] + \Gamma'_m B'''_m - \Gamma''_m B''''_m$$

$$- \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{\bar{c}'_n - (-1)^m\, \bar{c}''_n}{E_{mn}} \quad (m = 1, 2, \ldots, M) \tag{E29}$$

$$-\bar{g}'_m\beta_2(m) + \bar{g}''_m\alpha_4(m) = S''''_m - \sum_{n=1}^{N}(-1)^n H_{mn}[B'_n - (-1)^m B''_n] - \Gamma''_m B'''_m + \Gamma'''_m B''''_m$$

$$+ \frac{2}{a}\left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N}(-1)^n \frac{\bar{c}'_n - (-1)^m\, \bar{c}''_n}{E_{mn}} \quad (m = 1, 2, \ldots, M) \tag{E30}$$

Equations (E23) through (E30) constitute 4N + 4M equations that have to be solved simultaneously for the 4N + 4M unknowns $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$, $\bar{g}''_n$, B'_n, B''_n, B'''_m, and B''''_m.

Procedure for Use of Equations

All the pertinent equations for this case have been presented above. The procedure for using them will now be summarized: Equations (E23) to (E30) are first solved simultaneously for the $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$, $\bar{g}''_m$, B'_n, B''_n, B'''_n, and B''''_n. With these known, equations (B60) then give the s'_n, s''_n, s'''_m, s''''_m, and equations (B57) to (B59) and (B61) the j_{mn}. Finally, equations (B16) and (B19) to (B25) give the stiffener and plate stresses.

Special Case: Square Plate, Structure and Loading Symmetrical About Centerlines and Diagonals

If the plate is square (b=a) and if the structure, loading, and thermal strains are symmetrical about the centerlines (x=a/2, y=b/2) and about the diagonals, considerable simplification of the simultaneous equations can be effected. Symmetry about centerlines and diagonals implies that $A_1 = A_2 = A_3 = A_4 = A$, $P_1(0) = P_1(b) = P_2(0) = P_2(b) = P_3(0) = P_3(a) = P_4(0) = P_4(a)$, $q_1(y) = -q_2(y) = q_3(x) = -q_4(x)$, $e_1(y) = e_2(y)$, $e_3(x) = e_4(x)$, $T_{mn} = 0$, for m even or n even, $T_1 = T_2 = T_3 = T_4 = T$, $M_1 = M_2 = M_3 = M_4 = M_o$,

$$\left.\begin{aligned} K_n' &= K_n'' = 0 \\ V_n' &= V_n'' = 0 \end{aligned}\right\} \quad \text{for n even}$$

$$\left.\begin{aligned} K_m''' &= K_m'''' = 0 \\ V_m''' &= V_m'''' = 0 \end{aligned}\right\} \quad \text{for m even}$$

$$\left.\begin{aligned} K_n' &= -K_n'' \\ V_n' &= -V_n'' \end{aligned}\right\} \longrightarrow \delta_n' = -\delta_n'' \qquad \text{for n odd}$$

$$\left.\begin{aligned} K_m''' &= -K_m'''' \\ V_m''' &= -V_m'''' \end{aligned}\right\} \longrightarrow \delta_m''' = -\delta_m'''' \qquad \text{for m odd}$$

By the property of symmetry about the centerlines, it follows that

$$\bar{c}_n' = \bar{c}_n'' = \bar{g}_m' = \bar{g}_m'' = B_n' = B_n'' = B_m''' = B_m'''' = 0 \qquad \text{(m,n even)} \tag{E31}$$

$$\left.\begin{aligned} B'_n &= B''_n \\ B'''_m &= B''''_m \\ \bar{c}'_n &= \bar{c}''_n \\ \bar{g}'_m &= \bar{g}''_m \end{aligned}\right\} \quad (m,n \text{ odd}) \qquad \text{(E32)}$$

In order to insure that the physical symmetry about the diagonals is manifested in the mathematical solution, M and N will be restricted to be equal. It can then be expected that

$$\left.\begin{aligned} B'''_m &= B'_m \\ \bar{g}'_m &= \bar{c}'_m \end{aligned}\right\} \qquad \text{(E33)}$$

Consequently, the original simultaneous equations system, equations (E23) to (E30), can be reduced to the following:

$$B'_n[(\gamma_n^{(1)}-\gamma_n^{(2)})+2\Lambda'_{nn}] = \delta'_n + 2\sum_{p=1,3\ldots}^{M} \mu'_{pn}\bar{c}'_p + 2\bar{c}'_n \sum_{p=1,3\ldots}^{M} \nu'_{pn}$$

$$-2\sum_{p=1,3\ldots}^{M} \Lambda'_{pn}B'_p\,(1-\delta_{np}) \qquad (n = 1,\ 3,\ \ldots,\ M) \qquad \text{(E34)}$$

$$\bar{c}'_n\{[\alpha_1(n)-\beta_1(n)] + \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \frac{1}{E_{nn}}\} = S'_n + B'_n(\gamma'_n-\gamma''_n) + 2\sum_{p=1,3\ldots}^{M} \Pi_{pn}\,B'_p$$

$$-\frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{p=1,3\ldots}^{M} \frac{\bar{c}'_p(1-\delta_{pn})}{E_{pn}}$$

$$(n = 1,\ 3,\ \ldots,\ M) \qquad \text{(E35)}$$

The procedure for the special case can be summarized as follows: Solve equations (E34) and (E35) simultaneously for the $\bar{c}'_n$ and B'_n $(n = 1, 3, \ldots, N)$. (This can be done by the Gauss-Seidel iterative method, which will be described in detail in a later section dealing with an illustrative thermal-stress problem.) With these known and using the relations $\bar{c}''_n = \bar{c}'_n$, $\bar{g}'_m = \bar{g}''_m = \bar{c}'_m$, $B''_n = B'_n$, $B'''_m = B''''_m = B'_n$ (m,n odd), equations (B60) will furnish the values of s'_n, s''_n, s'''_m, s''''_m (m,n odd), and (B61) the values of the j_{mn} (m,n odd). Equation (B16) and (B19) through (B25), in which only the odd values of n and m are included, will then give the stiffener and plate stresses.

Limiting Case of Large Stiffener Areas

Still referring to the special case of symmetry with respect to plate centerlines and diagonals, the limiting condition in which the stiffener cross-sectional areas A are large compared with the plate cross-sectional area will now be given brief consideration. This limiting condition can be analyzed by means of a first-order perturbation applied to equations (E34) and (E35), as follows: Assume $\bar{c}'_n = \bar{c}_n^{(0)} + \lambda\bar{c}_n^{(1)}$, $B'_n = B_n^{(0)} + \lambda B_n^{(1)}$, substitute these expansions into equations (E34) and (E35), and equate separately terms of the zeroth and first degree in λ. This will lead to four systems of simultaneous equations, two for $\bar{c}_n^{(0)}$ and $B_n^{(0)}$ and two more for $\bar{c}_n^{(1)}$ and $B_n^{(1)}$. The latter two equations will involve the $\bar{c}_n^{(0)}$ and $B_n^{(0)}$. Therefore the $\bar{c}_n^{(0)}$ and $B_n^{(0)}$ equations must be solved first, after which the $\bar{c}_n^{(1)}$ and $B_n^{(1)}$ equations can be solved.

This technique will be described in more detail in connection with a particular application in the next section.

Illustrative Thermal-Stress Problem

In order to illustrate the details involved in the application of the foregoing analytical results, a particular example will be considered which has the following characteristics:

a) All four edges kept straight; therefore the K_n' etc. in equations (8) through (11) are all zero.

b) Plate isotropic; therefore the elastic constants are given by equations (3).

c) Plate and stiffeners have the same Young's modulus E.

d) $A_1 = A_2 = A_3 = A_4 = A$.

e) Plate with the same dimension in x and y directions; therefore B in equation (C79) equals to 1 (a = b).

f) No force loading.

g) Stiffener temperature constant at the value T_o.

h) Plate temperature T(x,y) symmetrical about both centerlines (x = a/2, y = b/2) and varying sinusoidally in accordance with the following equation:

$$T(x,y) = T_o + \Theta \sin\left(\frac{P\pi x}{a}\right) \sin\left(\frac{Q\pi y}{b}\right) \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le b)\end{matrix}$$

where P and Q are odd integers.

i) P = Q

j) Plate and stiffeners have the same coefficient of expansion α.

It will be recognized that in this problem, as described above, there exists symmetry about the plate centerlines and diagonals. When numerical calculations are considered, the problem will be further specialized to the case of a "pillow-shaped" temperature distribution, namely P = Q = 1.

Reduction of general equations to special case. - From the given temperature distribution, one obtains the following equations for the known coefficients in terms of temperature distribution and the coefficient of expansion α (see appendix C):

$$T_n' = T_n'' = T_m''' = T_m'''' = 0 \tag{E36}$$

$$T_{mn} = -\delta_{mP}\delta_{nQ}\ \alpha\Theta\left(\frac{\pi}{a}\right)^2 [P^2 + Q^2B^2] \tag{E37}$$

$$V_n' = -V_n'' = \alpha\Theta\ \frac{P\pi}{a}\ \delta_{nQ} \tag{E38}$$

Similarly,

$$V_m''' = -\ V_m'''' = \alpha\Theta\ \frac{Q\pi}{a}\ \delta_{mP} \tag{E39}$$

Due to the absence of prescribed forces, the following quantities are all zero:

$T_1,\ M_1,\ T_2,\ M_2,\ T_3,\ M_3,\ T_4,\ M_4$ (fig. 4c)

$Q_n',\ Q_n'',\ Q_m''',\ Q_m''''$ (see eqs. (6))

(It should be noted that $P_1(0)$, $P_1(b)$, $P_2(0)$, $P_2(b)$, $P_3(0)$, $P_3(a)$, $P_4(0)$, $P_4(a)$, B_n', B_n'', B_m''', B_m'''' do not necessarily vanish.)

Because in this example the structure and loading are symmetrical about both centerlines and both diagonals, the simplified system of equations, namely equations (E34) and (E35), are the governing equations. From equations (E36) and (E37), (C81), and (E15):

$$S_n' = \delta_{nQ}\ \alpha\Theta Eh\ \frac{PQB}{(P^2 + Q^2B^2)} \tag{E15'}$$

Substituting from equations (3) and (C81) into equation (E22), one obtains

$$\Pi_{mn} = \frac{2mB}{\pi} \left[\frac{m^2}{(m^2 + n^2B^2)^2} - \frac{1}{m^2}\right] \qquad \text{(E22')}$$

Equation (E23a) can be reduced in the following expression by substituting from equations (3), (C81), (E37) and (E38):

$$\delta'_n = \alpha\Theta \left(\frac{P\pi}{a}\right) \left\{\frac{[P^2 + (2+\nu)\, Q^2B^2]}{[P^2 + Q^2B^2]} - 1\right\} \delta_{NQ} \qquad \text{(E23'a)}$$

Substituting from equations (C81) and (3) into equation (E23b), one obtains

$$\Lambda'_{mn} = \frac{2mn[n^2B^2 - \nu m^2]\, B}{b[m^2 + n^2B^2]^2\, Eh} \qquad \text{(E23'b)}$$

Substituting from equation (C81) into the first and fifth of equation (B67), one obtains

$$\alpha_1(n) = AE \left\{1 + \frac{\lambda_1}{2} \sum_{m=1}^{M} \frac{m^2}{[m^2 + n^2B^2]^2}\right\}$$

$$\beta_1(n) = AE \frac{\lambda_1}{2} \sum_{m=1,\ldots}^{M} \frac{(-1)^m m^2}{[m^2 + n^2\beta^2]^2}$$

Therefore,

$$\alpha_1(n) - \beta_1(n) = AE \left\{1 + \lambda_1 \sum_{m=1,3\ldots}^{M} \frac{m^2}{[m^2 + n^2B^2]^2}\right\} \qquad \text{(E40)}$$

Equation (E34) can be reduced to the following expression by substituting from equations (C81), (C12'), (C13'), (D77), (E23'a) and (E23'b):

$$\{ \frac{2(B^2 - \nu)B^2}{[1 + B^2]^2} - 2n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{[p^2 + (2+\nu)\, n^2B^2]}{[p^2 + n^2B^2]^2} \} B(n)$$

$$= \delta_{nQ} \frac{P\pi}{2} \{ \frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\} - 2nB^2 \sum_{p=1,3\ldots}^{M} \frac{p[p^2+(2+\nu)n^2B^2]\, C(p)}{[p^2 + n^2B^2]^2}$$

$$+2n^2B^2C(n) \sum_{p=1,3\ldots}^{M} \frac{[n^2B^2-\nu p^2]}{[p^2+n^2B^2]^2} - 2nB^2 \sum_{p=1,3\ldots}^{M} \frac{p[n^2B^2-\nu p^2]B(p)}{[(p^2+n^2B^2]^2}(1-\delta_{np})$$

$$(n = 1,\ 3,\ \ldots,\ M) \qquad \text{(E34')}$$

where

$$B(n) = \frac{B'_n}{\alpha\Theta Eh}, \qquad C(n) = \frac{c'_n}{\alpha\Theta Eh}$$

Substituting from equations (C81), (C22'), (C23'), (E15'), (E22'), and (E40) into equation (E35), one obtains

$$\{n[1 + \lambda_1 \sum_{p=1,3\ldots}^{M} \frac{p^2}{(p^2 + n^2B^2)^2}] + \frac{\lambda_1 B^3}{n(1 + B^2)^2} \} C(n)$$

$$=\delta_{nQ} \frac{\pi\lambda_1}{4} \frac{PQ}{[P^2+Q^2B^2]} + [\nu n + \lambda_1 n^3B^2 \sum_{p=1,3\ldots}^{M} \frac{1}{(p^2 + n^2B^2)^2}]B(n)$$

$$+ \lambda_1 \sum_{p=1,3\ldots}^{M} [\frac{p^3}{(p^2+n^2B^2)^2} - \frac{1}{p}]B(p) - \lambda_1 n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{p(1-\delta_{np})C(p)}{[p^2 + n^2B^2]^2}$$

$$(n = 1,\ 3,\ \ldots,\ M) \qquad \text{(E35')}$$

Procedure for numerical solution. - In the solution of this system of equations (E34') and (E35') the Gauss-Seidel iterative procedure of reference 6 was employed. Briefly this involved an initial assumption that all the B(p)'s and C(p)'s in equations (E35') were equal to zero except C(1). This allowed an approximate value of C(1) to be obtained.

Substituting this approximate value of C(1) into equations (E34') and setting B(3), B(5), etc. and all other C(p)'s equal to zero gave an approximate value of B(1). Substituting these approximate values of B(1) and C(1) into equation (E35') and setting all other C(p)'s except C(3) and all other B(p)'s equal to zero gave an approximate value of C(3). Substituting these approximate values of C(1), C(3), and B(1) into equations (E34') and setting all other C(p)'s and all other B(p)'s except B(3) equal to zero gave an approximate value of B(3). Continuing in such a fashion it was possible to obtain one set of approximate values for the B(p)'s and C(p)'s. This set is called the first iteration solution to the system of equations (E34') and (E35'). Additional iterations were obtained in the same manner as the first iteration except that initial values of unknowns are not assumed to be zero are taken from the results of the previous iteration. The procedure was stopped when no B(p) or C(p) changed more than 0.000001 from one iteration to the next. Alternatively, one can reverse the order in which equations (E34') and (E35'), solving for a B(p) in equations (E34') first, then for the corresponding C(p) in equations (E35'). The final results are the same as in the above procedure.

With the B_n' known, and $T_1 = 0$, $M_1 = 0$, from the first two of equations (E11), one obtains the end tensions in x-direction of stiffener at y=0:

$$P_3(0) = P_3(a) = -\alpha\theta AE\,\lambda \sum_{n=1,3\ldots}^{N} \frac{\pi}{4n}\,B(n) \qquad \text{(E11')}$$

Due to the properties of double symmetry and symmetry about the diagonals, with B_n' and c_n' as known, then $B_p'' = B_p''' = B_p'''' = B_p'$ and $c_p'' = g_p' = g_p'' = c_p'$. With these known, equations (B60) will furnish the values of s_n', s_n'', s_m''', s_m''''; (B61) the values of j_{mn}; (B34) the values of g_{mn}; and (B35) the values of c_{mn} (only the odd-subscripted quantities are needed). Equations (B16) and (B19) to (B25), with the only odd values of n and m included, will then give the following stiffener and plate stresses:

$$P_1(y) = P_2(y) = \theta AE\alpha \sum_{n=1,3\ldots}^{N} [C(n)-\nu B(n)] \sin\left(\frac{n\pi y}{a}\right) \qquad (0 < y < a) \tag{E41}$$

$$P_3(x) = P_4(x) = \theta AE\alpha \sum_{p=1,3\ldots}^{M} [C(p)-\nu B(p)] \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{E42}$$

$$N_x = \alpha\theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} G(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < a)\end{matrix} \tag{E43}$$

$$(N_x)_{y=0} = (N_x)_{y=a} = \alpha\theta Eh \sum_{p=1,3\ldots}^{M} C(p) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{E44}$$

$$N_y = \alpha\theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} C(m,n) \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{E45}$$

$$(N_y)_{x=0} = (N_y)_{x=a} = \alpha\theta Eh \sum_{n=1,3\ldots}^{N} C(n) \sin\left(\frac{n\pi y}{a}\right) \qquad (0 < y < a) \tag{E46}$$

$$N_{xy} = -\alpha\Theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} J(m,n) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right) \quad \begin{matrix}(0 \le x \le a)\\(0 \le y \le a)\end{matrix} \tag{E47}$$

where G(m,n), C(m,n), and J(m,n) are known quantities, and given by the following equations:

$$G(m,n) = \frac{g_{mn}}{\alpha\Theta Eh}$$

$$= \frac{4n^2B^2}{\pi[m^2 + n^2B^2]^2}[mC(n) + nB^2C(m)] + \frac{4m[m^2 + 2n^2B^2]B(n)}{\pi[m^2 + n^2B^2]^2}$$

$$+ \frac{4nB^2B(m)}{\pi m^2}\left\{\frac{n^2B^2[n^2B^2 + 2m^2]}{[m^2+n^2B^2]^2} - 1\right\} - \frac{Q^2B^2}{(P^2 + Q^2B^2)}\delta_{mP}\delta_{nQ} \tag{E48}$$

$$C(m,n) = \frac{c_{mn}}{\alpha\Theta Eh}$$

$$= \frac{4mB(n)}{n^2B^2\pi}\left[\frac{m^2(m^2+2n^2B^2)}{(m^2+n^2B^2)^2} - 1\right] + \frac{4nB^2B(m)}{\pi}\ \frac{(n^2B^2 + 2m^2)}{[m^2 + n^2B^2]^2}$$

$$+ \frac{4m^2}{\pi[m^2+n^2B^2]^2}[mC(n) + nB^2C(m)] - \frac{P^2}{(P^2+Q^2B^2)}\delta_{mP}\delta_{nQ} \tag{E49}$$

$$J(m,n) = \frac{j_{mn}}{\alpha\Theta Eh}$$

$$= \frac{PQB}{(P^2+Q^2B^2)}\delta_{mP}\delta_{nQ} - \frac{4m^2nBC(n)}{\pi[m^2 + n^2B^2]^2} - \frac{4mn^2B^3C(m)}{\pi[m^2 + n^2B^2]^2}$$

$$+ \frac{4m^3B}{\pi[m^2 + n^2B^2]^2}B(m) + \frac{4n^3B^3}{\pi[m^2+n^2B^2]^2}\cdot B(n) \tag{E50}$$

With $B_n' = B_n'' = B_n''' = B_n''''$ (n odd) known, from the first of equations (4), one obtains the following running tension between stiffeners and plate:

$$N_x(0,y) = \alpha\Theta Eh \sum_{n=1,3...}^{N} B(n) \sin\left(\frac{n\pi y}{a}\right) \tag{E51}$$

Numerical results for $\lambda = 1$. - The numerical procedure and equations described above were applied to square plate subjected to a "pillow-shaped" temperature distribution (P = Q = 1) and having stiffener and plate cross sections corresponding to $\lambda = 1.0$. The results obtained for the stiffener tensions and plate stresses are shown in dimensionless form in figure 16 for $\nu = 0.3$.

Limiting case of large stiffener areas. - Still referring to the special case of symmetry with respect to plate centerlines and diagonals, the first-order perturbation solution corresponding to stiffener cross-sectional area large in comparison with plate cross-sectional area ($\lambda \rightarrow 0$) will now be given. This solution is obtained by assuming that

$$\begin{aligned} C(n) &= C^{(0)}(n) + \lambda C^{(1)}(n) \\ B(n) &= B^{(0)}(n) + \lambda B^{(1)}(n) \end{aligned} \tag{E52}$$

where

$$C(n) = \frac{C_n'}{\alpha\Theta Eh}, \qquad B(n) = \frac{B_n'}{\alpha\Theta Eh}$$

Substituting from equations (E52) into equation (E34'), the following two equations are obtained from those terms which do not involve λ and those which do involve λ, respectively:

$$\left\{\frac{2(B^2-\nu)B^2}{[1+B^2]^2} - 2n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{[p^2+(2+\nu)n^2B^2]}{[p^2+n^2B^2]^2}\right\} B^{(0)}(n)$$

$$= \delta_{nQ}\frac{P\pi}{2}\left\{\frac{[P^2+(2+\nu)Q^2B^2]}{[P^2+Q^2B^2]} - 1\right\} - 2nB^2 \sum_{p=1,3\ldots}^{N} \frac{p[p^2+(2+\nu)n^2B^2]\, C^{(0)}(p)}{[p^2+n^2B^2]^2}$$

$$+2n^2B^2\, C^{(0)}(n) \sum_{p=1,3\ldots}^{M} \frac{[n^2B^2-\nu p^2]}{[p^2+n^2B^2]^2} - 2nB^2 \sum_{p=1,3\ldots}^{M} \frac{p[n^2B^2-\nu p^2]B^{(0)}(p)}{[p^2+n^2B^2]^2}(1-\delta_{np}) \qquad \text{(E53)}$$

$$\left\{\frac{2(B^2-\nu)B^2}{[1+B^2]^2} - 2n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{[p^2+(2+\nu)n^2B^2]}{[p^2+n^2B^2]^2}\right\} B^{(1)}(n)$$

$$= -2nB^2 \sum_{p=1,3\ldots}^{M} \frac{p[p^2+(2+\nu)n^2B^2]C^{(0)}(p)}{[p^2+n^2B^2]^2} + 2n^2B^2\, C^{(1)}(n) \sum_{p=1,3\ldots}^{M} \frac{[n^2B^2-\nu p^2]}{[p^2+n^2B^2]^2}$$

$$-2nB^2 \sum_{p=1,3\ldots}^{M} \frac{p[n^2B^2-\nu p^2]\, B^{(1)}(n)}{[p^2+n^2B^2]^2}(1-\delta_{np}) \qquad \text{(E54)}$$

Similarly, substituting from equations (E52) into equation (E35'), one obtains

$$C^{(0)}(n) = \nu B^{(0)}(n) \qquad (n = 1, 3, \ldots, N) \qquad \text{(E55)}$$

and

$$C^{(1)}(n) = \delta_{nQ} \frac{\pi}{4} \frac{P}{[P^2+Q^2B^2]} + B^{(0)}(n)\, n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{1}{[p^2 + n^2B^2]^2}$$

$$+ \sum_{p=1,3\ldots}^{M} \left[\frac{p^3}{(p^2+n^2B^2)^2} - \frac{1}{p}\right] \frac{B^{(0)}(p)}{n} - nB^2 \sum_{p=1,3\ldots}^{M} \frac{pC^{(0)}(p)}{[p^2+n^2B^2]^2}$$

$$- C^{(0)}(n) \sum_{p=1,3\ldots}^{M} \frac{p^2}{[p^2 + n^2B^2]^2} + \nu B^{(1)}(n)$$

The $C^{(1)}(n)$ and $B^{(1)}(n)$ are needed only in the combination $C^{(1)}(n) - \nu B^{(1)}(n)$ for later stress calculations; from the last equation, one obtains the following expression for the combination:

$$C^{(1)}(n)-\nu B^{(1)}(n) = \delta_{nQ} \frac{\pi}{4} \frac{P}{[P^2+Q^2B^2]} + B^{(0)}(n)\, n^2B^2 \sum_{p=1,3\ldots}^{M} \frac{1}{[p^2 + n^2B^2]^2}$$

$$+ \sum_{p=1,3\ldots}^{M} \left[\frac{p^3}{(p^2+n^2B^2)^2} - \frac{1}{p}\right] \frac{B^{(0)}(p)}{n} - nB^2 \sum_{p=1,3\ldots}^{M} \frac{pC^{(0)}(p)}{[p^2+n^2B^2]^2}$$

$$-C^{(0)}(n) \sum_{p=1,3\ldots}^{M} \frac{p^2}{[p^2 + n^2B^2]^2} \quad (n = 1, 3, \ldots, M)$$

(E56)

Equations (E53) through (E56) are to be solved for $B^{(0)}(n)$, $C^{(0)}(n)$, $B^{(1)}(n)$, and $C^{(1)}(n)$. Equations (E53) and (E55) must be solved simultaneously for the $B^{(0)}(n)$ and $C^{(0)}(n)$. The procedure for solving simultaneously for $B^{(0)}(n)$ and $C^{(0)}(n)$ is the same as described previously for the case of any λ. With $B^{(0)}(n)$ and $C^{(0)}(n)$ known, equations (E56) will furnish the values of $C^{(1)}(n)-\nu B^{(1)}(n)$ directly. By substituting from equations (E52) into equations (B16) and (B19) to (B25), and neglecting terms in λ for the plate stresses, one obtains the following equations for calculating stresses:

$$P_1(y) = P_3(y) = \theta AE\alpha\lambda \sum_{n=1,3\ldots}^{N} [C^{(1)}(n)-\nu B^{(1)}(n)] \sin\left(\frac{n\pi y}{a}\right) \qquad (0 < y < a) \tag{E57}$$

$$P_3(x) = P_4(x) = \theta AE\alpha\lambda \sum_{p=1,3\ldots}^{M} [C^{(1)}(p) - \nu B^{(1)}(p)] \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{E58}$$

$$N_x = \alpha\theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{g}_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < b)\end{matrix} \tag{E59}$$

$$(N_x)_{y=0} = (N_x)_{y=a} = \alpha\theta Eh \sum_{p=1,3\ldots}^{M} C^{(0)}(p) \sin\left(\frac{m\pi x}{a}\right) \qquad (0 < x < a) \tag{E60}$$

$$N_y = \alpha\theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{c}_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \qquad \begin{matrix}(0 < x < a)\\(0 < y < a)\end{matrix} \tag{E61}$$

$$(N_y)_{x=0} = (N_y)_{x=a} = \alpha\theta Eh \sum_{n=1,3\ldots}^{N} C^{(0)}(n) \sin\left(\frac{n\pi y}{a}\right) \qquad (0 < y < a) \tag{E62}$$

$$N_{xy} = -\alpha\theta Eh \sum_{m=1,3\ldots}^{M} \sum_{n=1,3\ldots}^{N} \bar{j}_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right) \qquad \begin{matrix}(0 \le x \le a)\\(0 \le y \le a)\end{matrix} \tag{E63}$$

where $\bar{g}_{mn}$, $\bar{c}_{mn}$, and $\bar{j}_{mn}$ are known quantities, and given by the following equations:

$$\bar{g}_{mn} = \frac{g_{mn}}{\alpha\theta Eh}$$

$$= \frac{4n^2B^2}{\pi[m^2 + n^2B^2]^2} [mC^{(0)}(n) + nB^2C^{(0)}(m)] + \frac{4m[m^2+2n^2B^2]B^{(0)}(n)}{\pi[m^2 + n^2B^2]^2}$$

$$+ \frac{4nB^2\ B^{(0)}(m)}{\pi m^2} \{\frac{n^2B^2[n^2B^2 + 2m^2]}{[m^2 + n^2B^2]^2} - 1\} - \frac{Q^2B^2}{(P^2 + Q^2B^2)} \delta_{mP}\delta_{nQ} \tag{E64}$$

$$\bar{c}_{mn} = \frac{c_{mn}}{\alpha\theta Eh}$$

$$= \frac{4mB^{(0)}(n)}{n^2B^2\pi} [\frac{m^2(m^2+2n^2B^2)}{(m^2+n^2B^2)^2} - 1] + \frac{4nB^2B^{(0)}(m)}{\pi} \frac{(n^2B^2 + 2m^2)}{[m^2 + n^2B^2]^2}$$

$$+ \frac{4m^2}{\pi[m^2 + n^2B^2]^2}[mC^{(0)}(n) + nB^2C^{(0)}(m)] - \frac{P^2}{(P^2+Q^2B^2)} \delta_{mP}\delta_{nQ} \tag{E65}$$

$$\bar{j}_{mn} = \frac{j_{mn}}{\alpha\theta Eh}$$

$$= \frac{PQB}{(P^2+Q^2B^2)} \delta_{mP}\delta_{nQ} - \frac{4m^2nBC^{(0)}(n)}{\pi[m^2 + n^2B^2]^2} - \frac{4mn^2B^3\ C^{(0)}(m)}{\pi[m^2 + n^2B^2]^2}$$

$$+ \frac{4m^3B}{\pi[m^2 + n^2B^2]^2} B^{(0)}(m) + \frac{4n^3B^3}{\pi[m^2 + n^2B^2)^2} B^{(0)}(n) \tag{E66}$$

With the B'_n known, from the first of equations (4), one obtains in a similar manner the following equation for the running tension between the rigid stiffener and plate edge at $x = 0$:

$$N_x(0,y) = \alpha\theta Eh \sum_{n=1,3\ldots}^{N} B^{(0)}(n) \sin(n\pi y/b) \qquad (0 < y < b) \tag{E67}$$

And from first two of equations (E11), one obtains the end tensions in x-wise direction of stiffener at y=0 as follows:

$$P_3(0) = P_3(a) = -\alpha\theta AE\lambda \sum_{n=1,3\ldots}^{N} \frac{\pi}{4n} B^{(0)}(n) \tag{E11"}$$

Numerical results for large stiffener areas. - The result for $\lambda \longrightarrow 0$ for a square plate (B=1) with all stiffeners identical subjected to a "pillow-shaped" temperature distribution is shown in dimensionless form in figure 17 for Poisson's ratio ν equals to 0.3. In general, stresses were computed at x/a and y/b interval of 0.1.

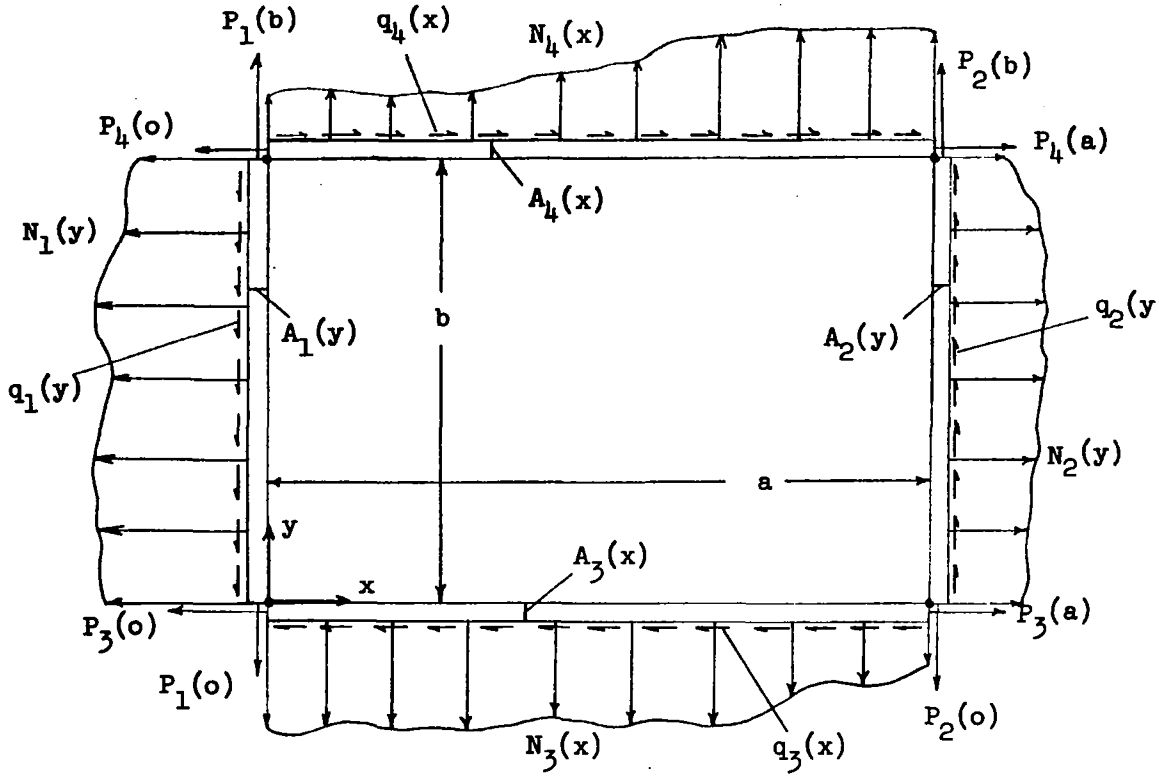


Figure 1. Structure and loading considered in reference 1.

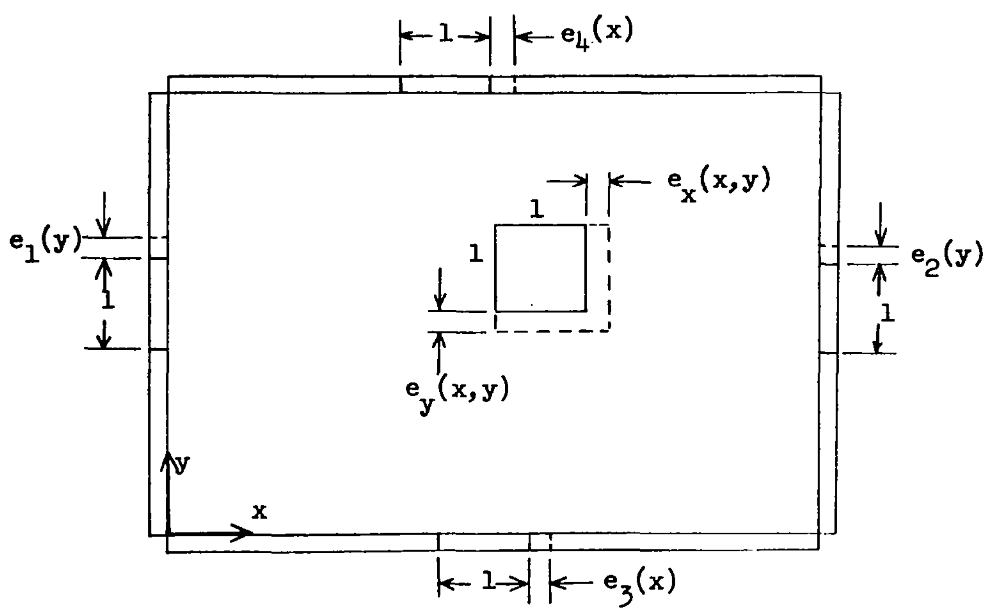


Figure 2. Notation for thermal strains.

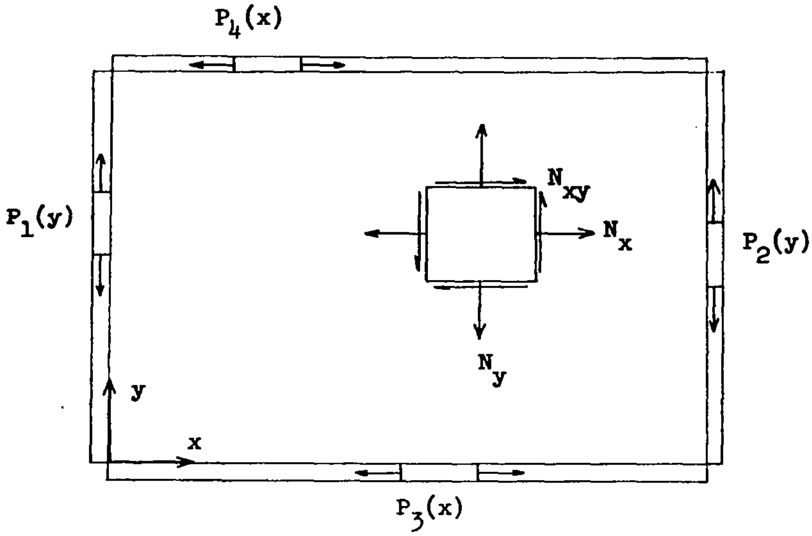
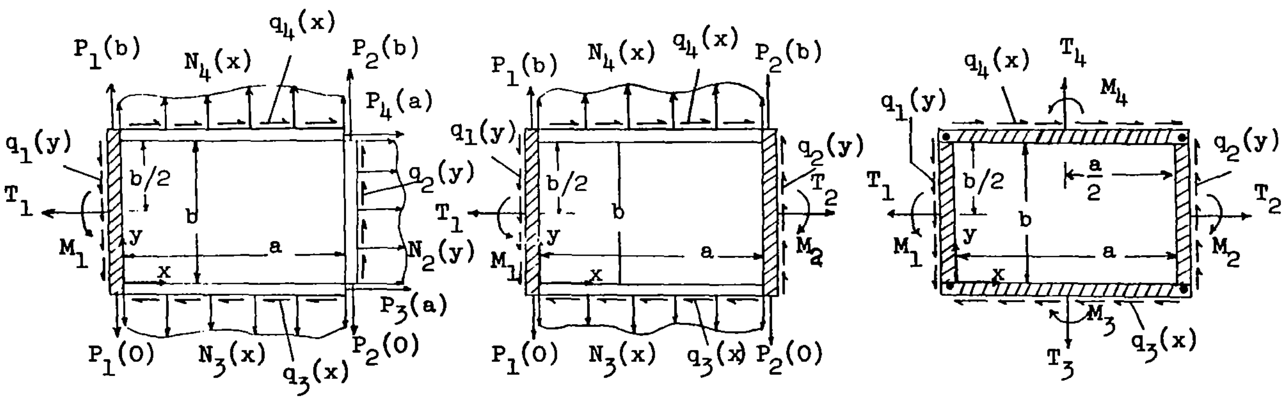


Figure 3. Notation for stiffener and plate forces.



(a) Stiffener shape prescribed along the edge x = 0.

(b) Stiffener shape prescribed along the edges x = 0, a.

(c) Stiffener shape prescribed along all four edges.

Figure 4. Structure and prescribed loadings considered in present paper.

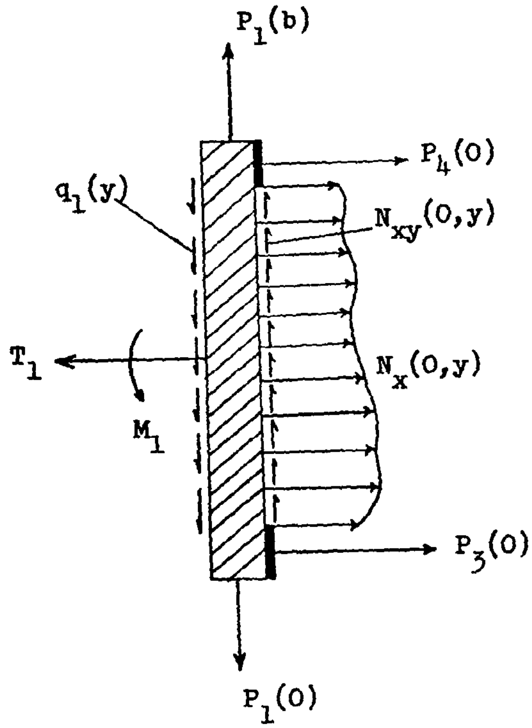
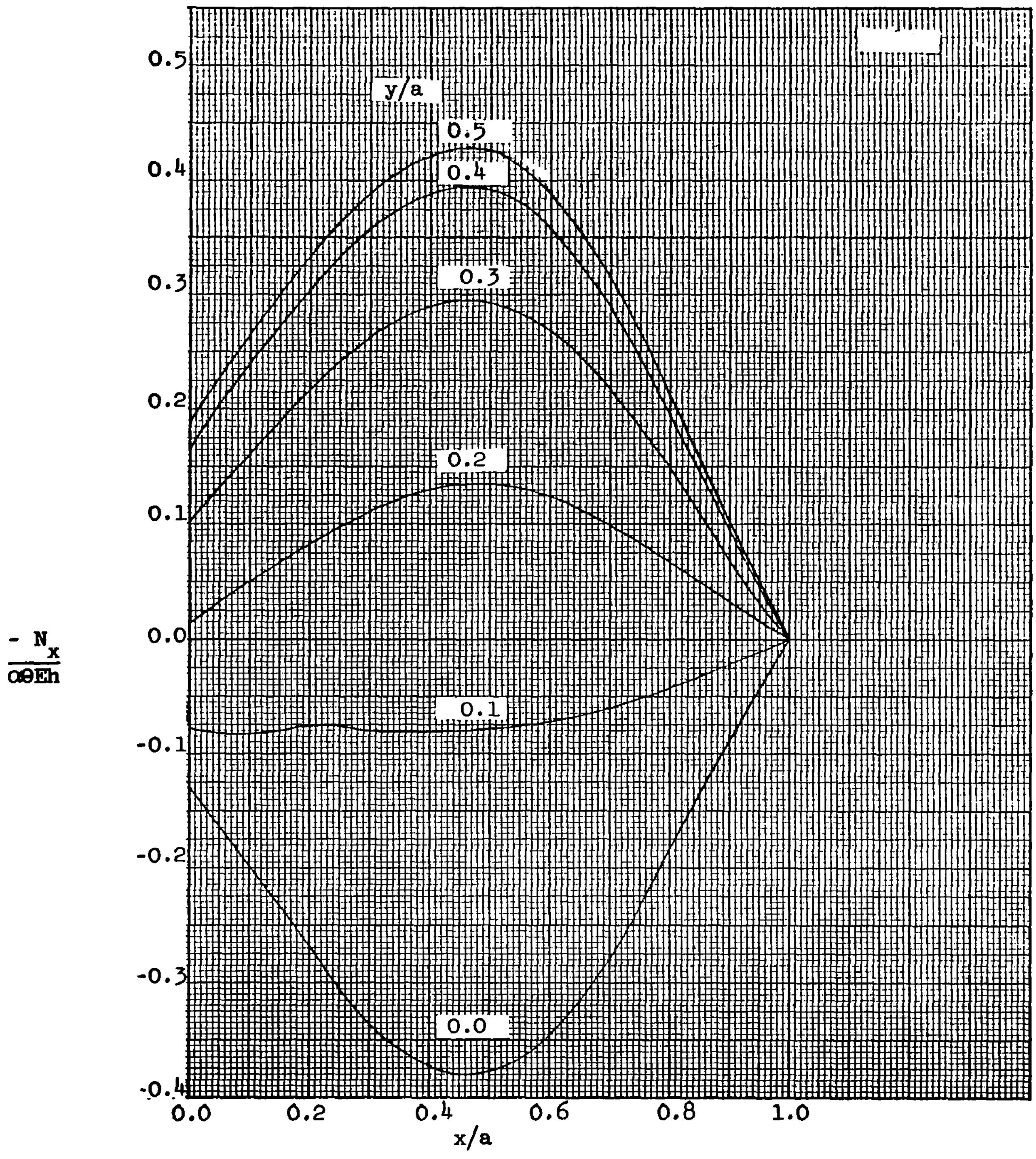
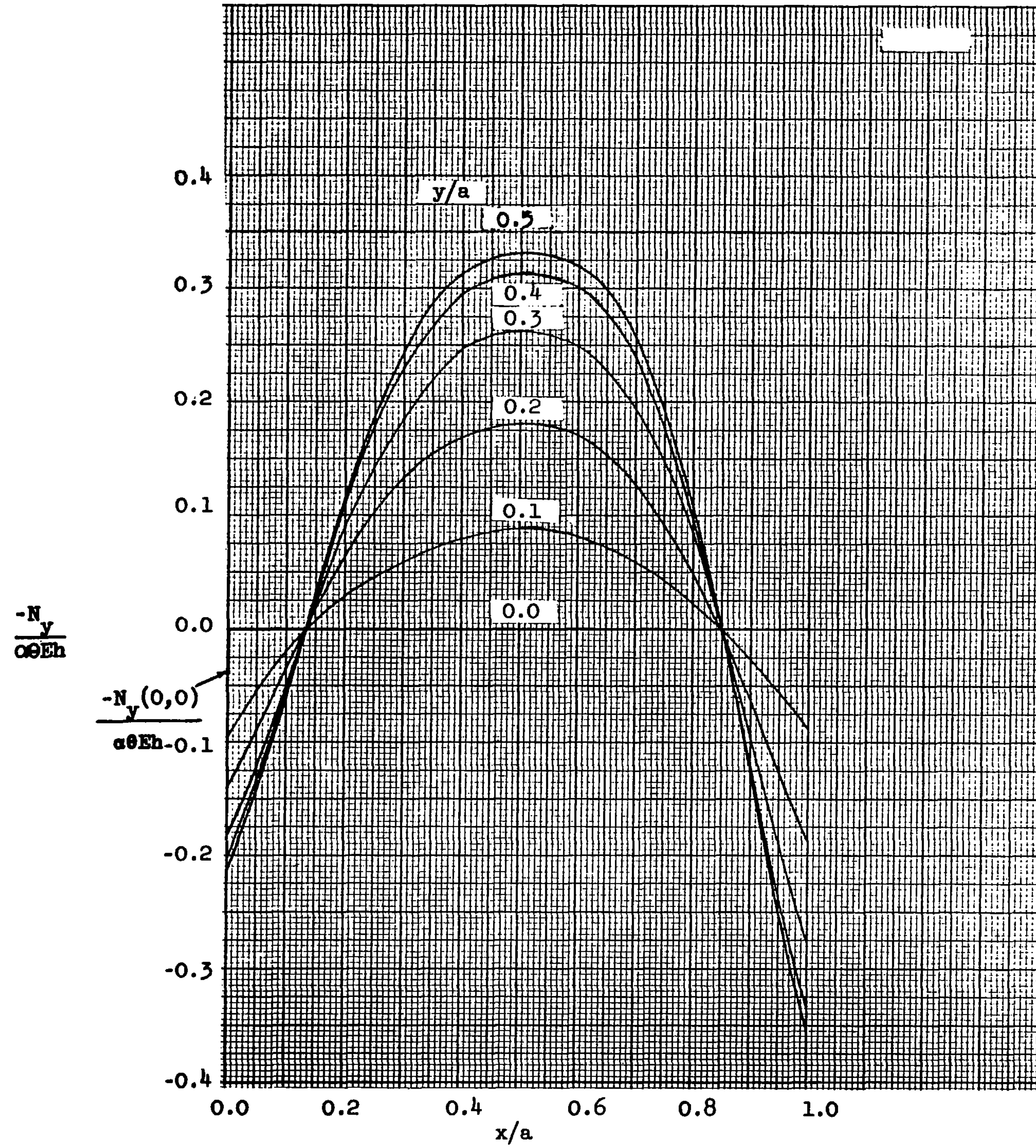


Figure 5. Free-body diagram of the prescribed-shape stiffener of figure 4a.



(a)

Figure 6. Plate stresses and stiffener tensions for the case of one edge held straight, pillow-shaped temperature distribution, $\lambda = 2.0$, and $\nu = 0.3$. (M = 30, N = 59)



(b)

Figure 6 (continued)

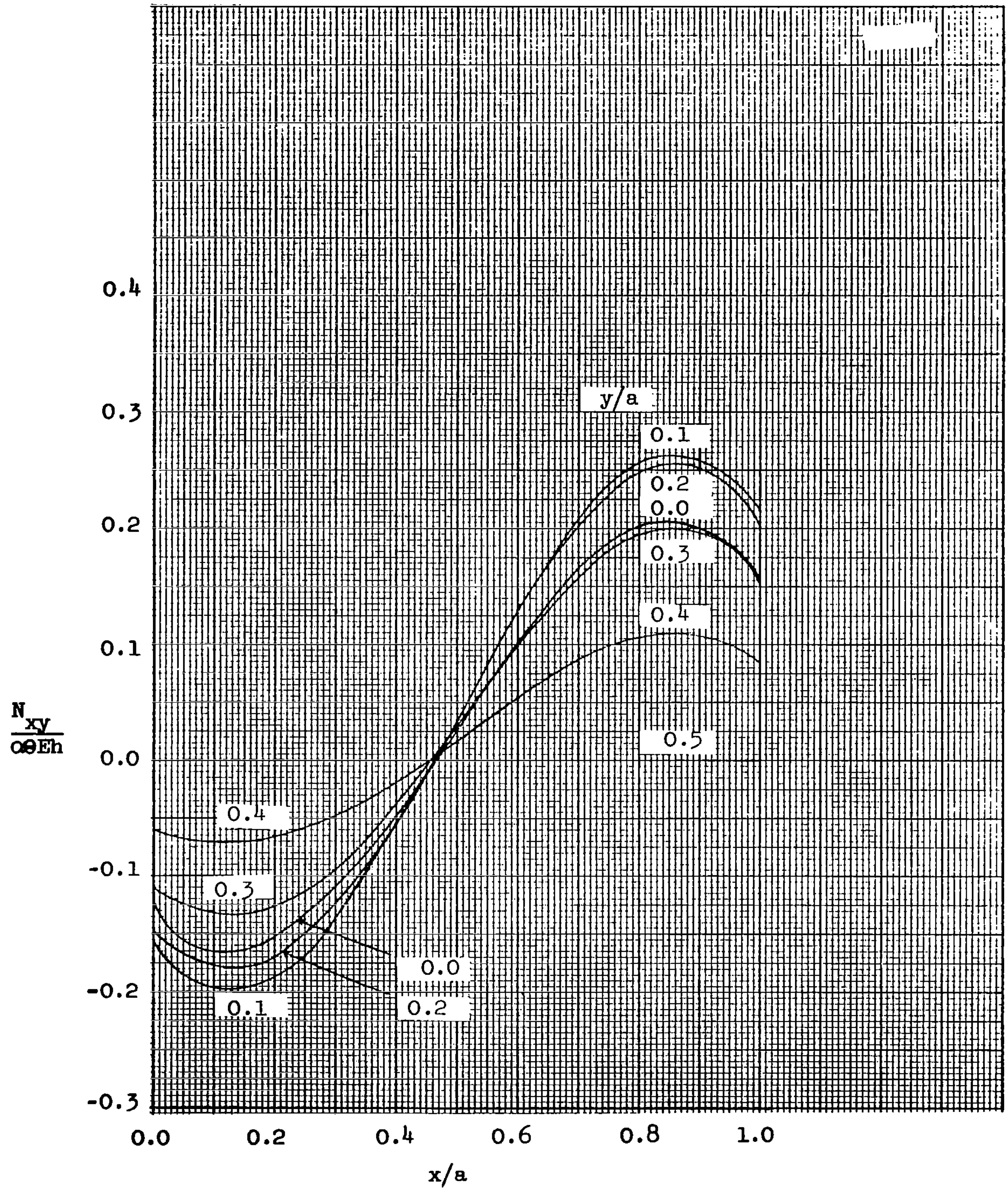


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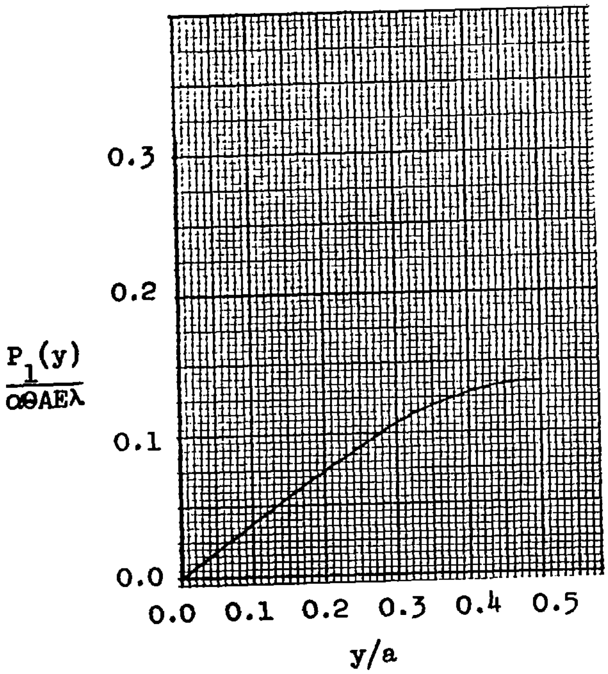
Figure 6. (continued)



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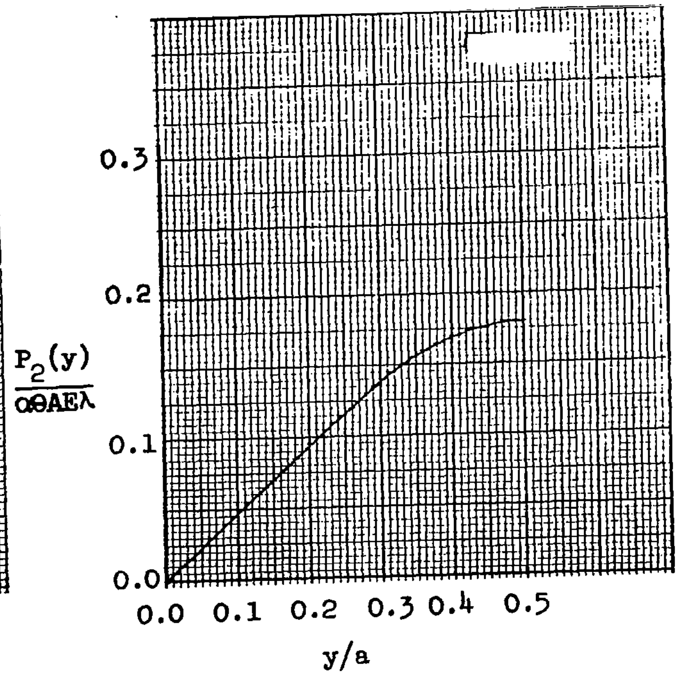

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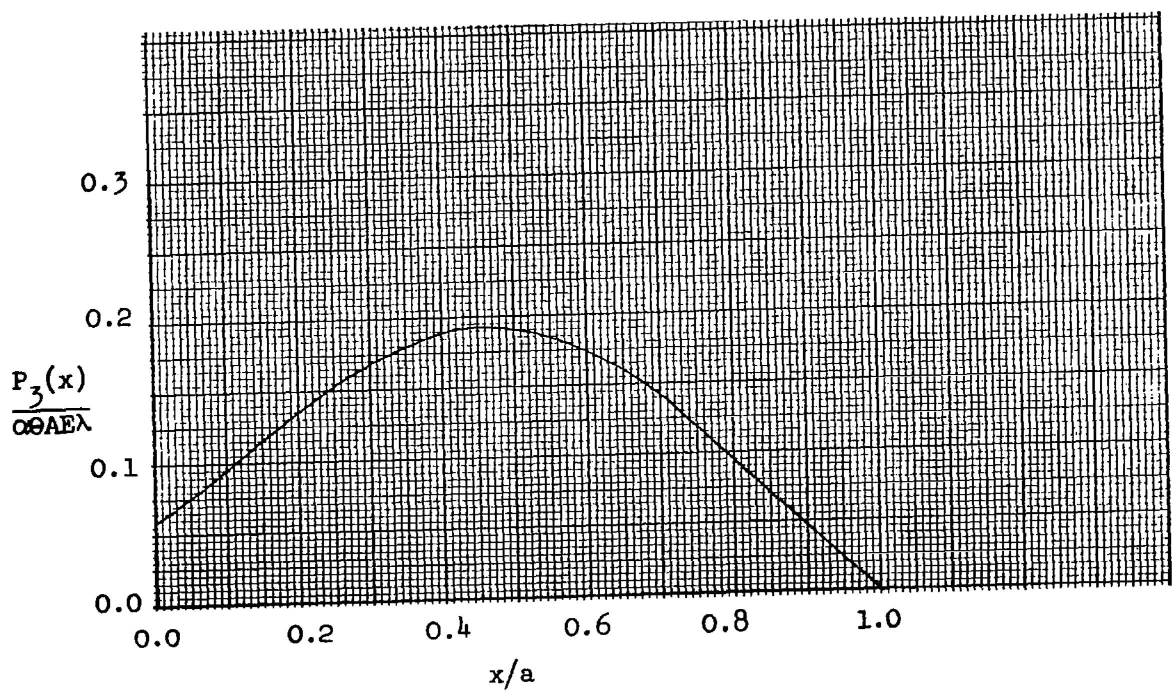
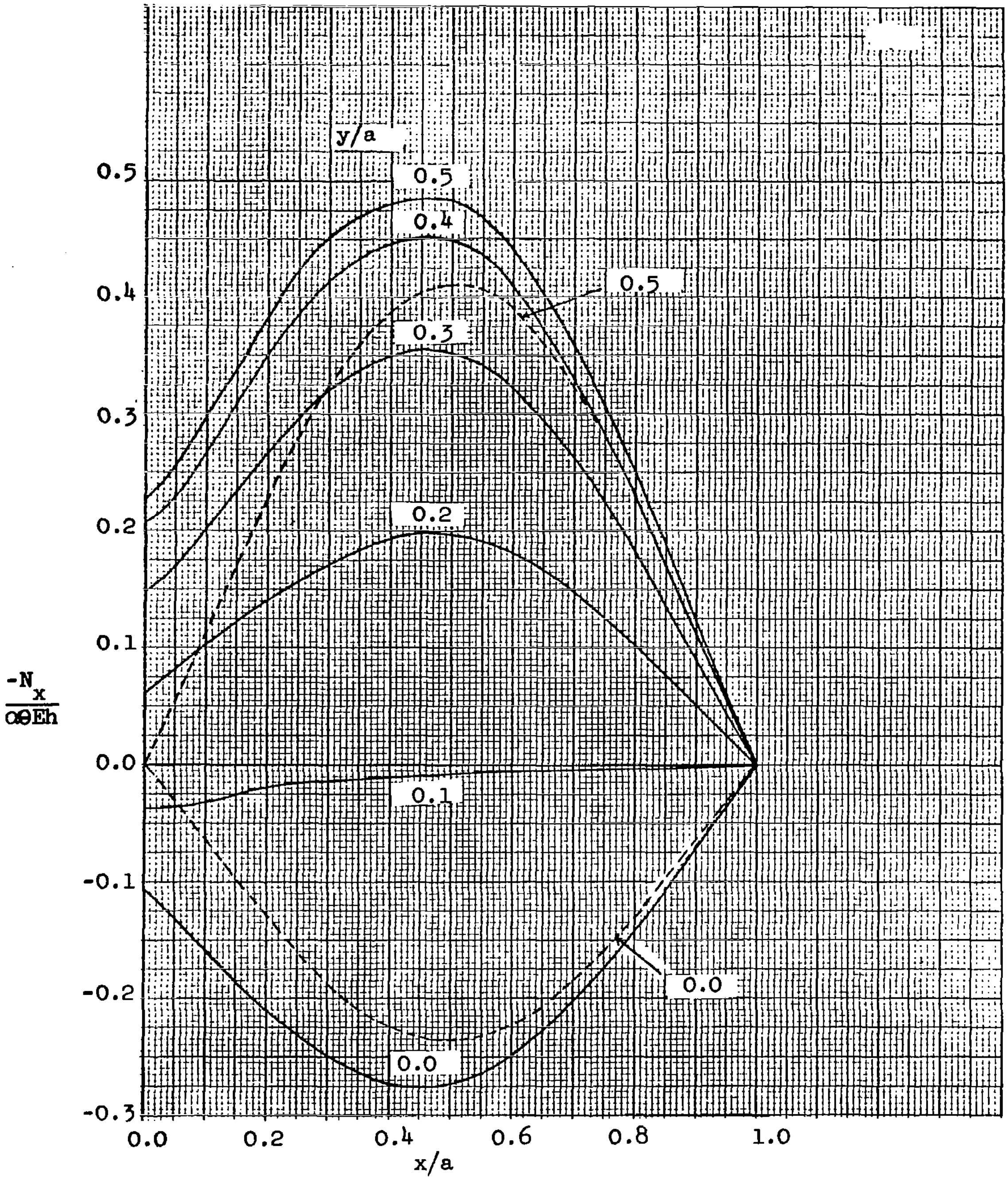


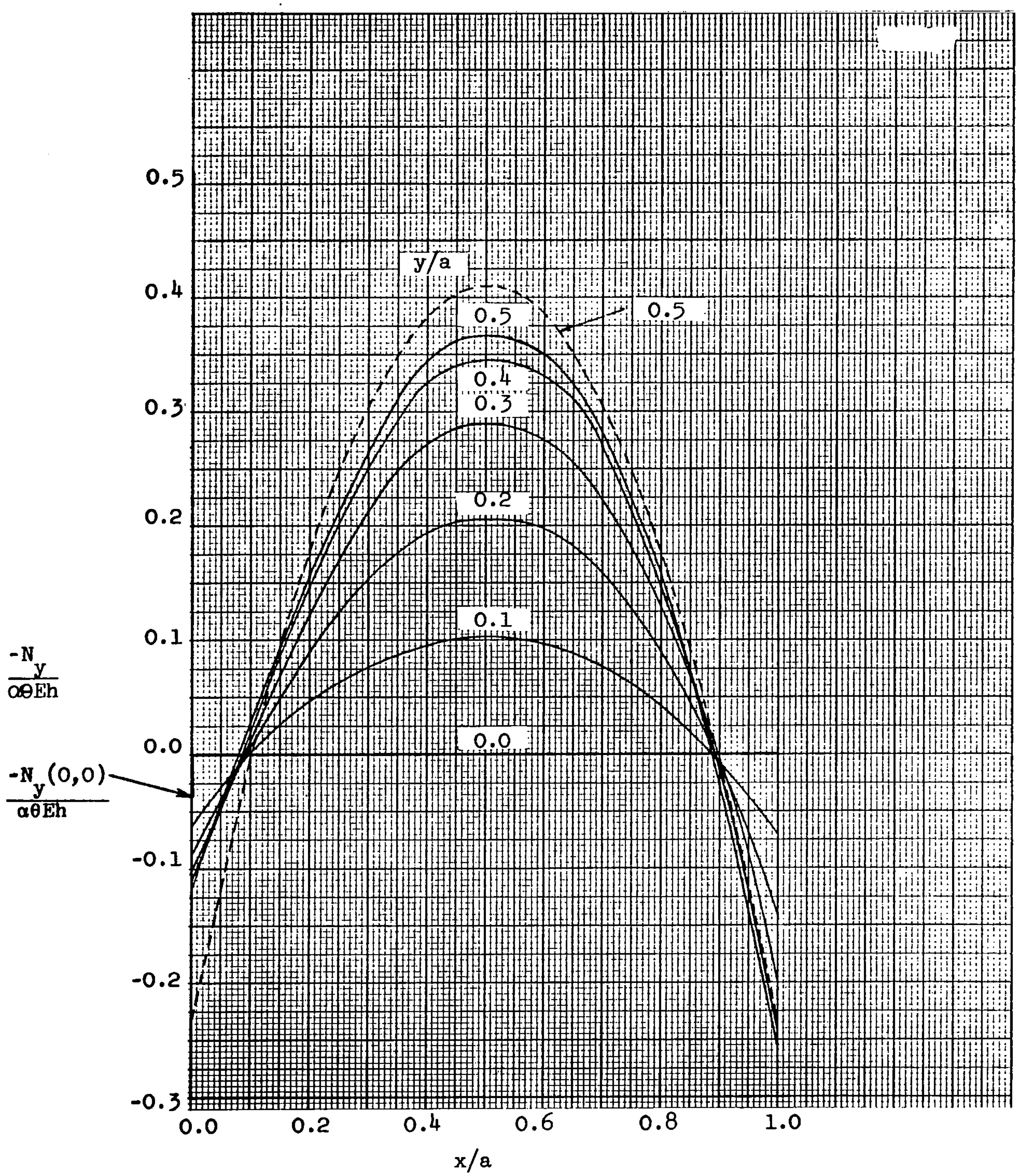
(f)

Figure 6. (continued)



(a)

Figure 7. Plate stresses and stiffener tensions for the case of one edge held straight, pillow-shaped temperature distribution, $\lambda = 1.0$ and $\nu = 0.3$. (Dashed curves, from fig. 5c of ref. 1, are for the case of all edge stiffeners perfectly flexible.) (M = 30, N = 59)



(b)

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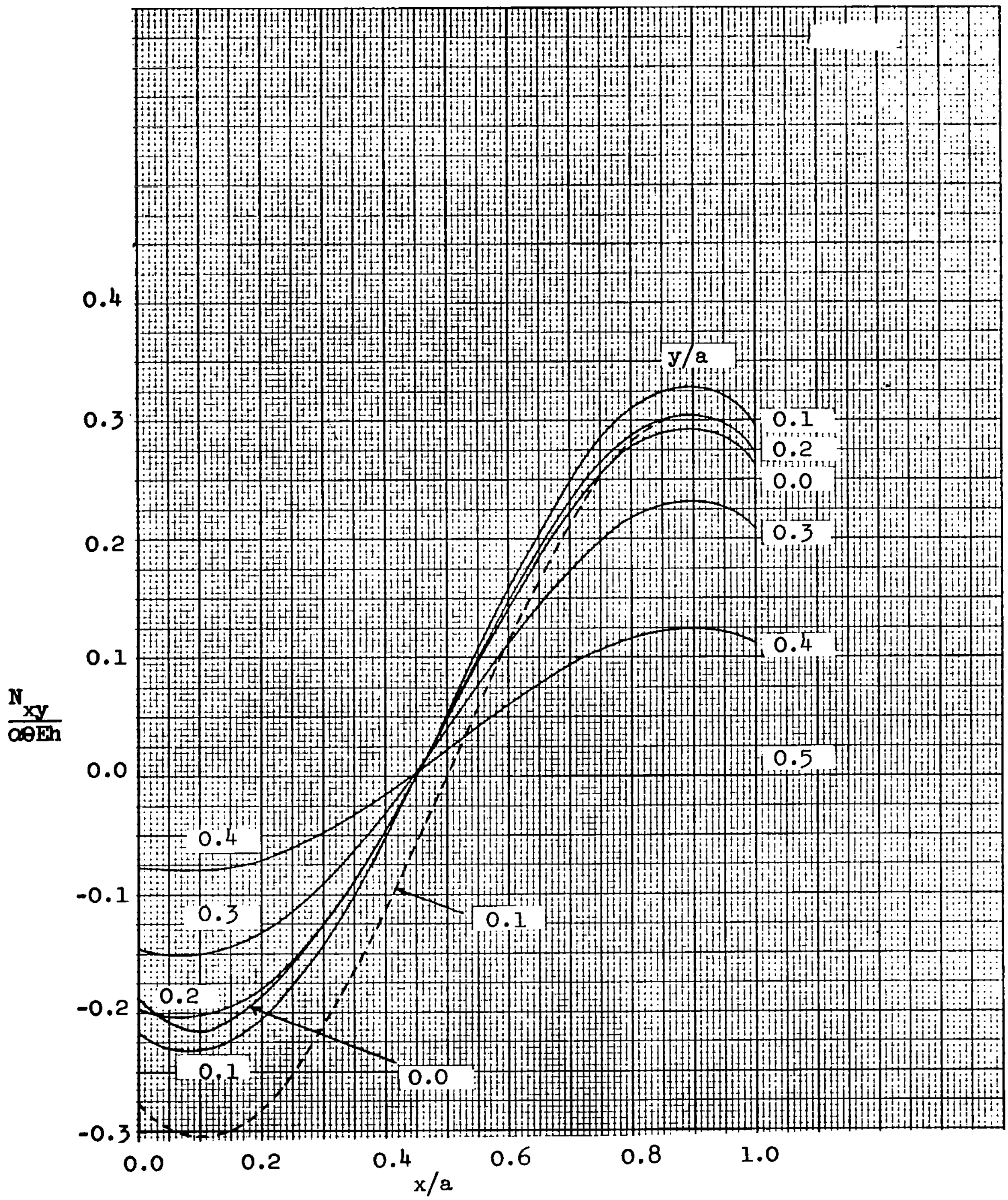
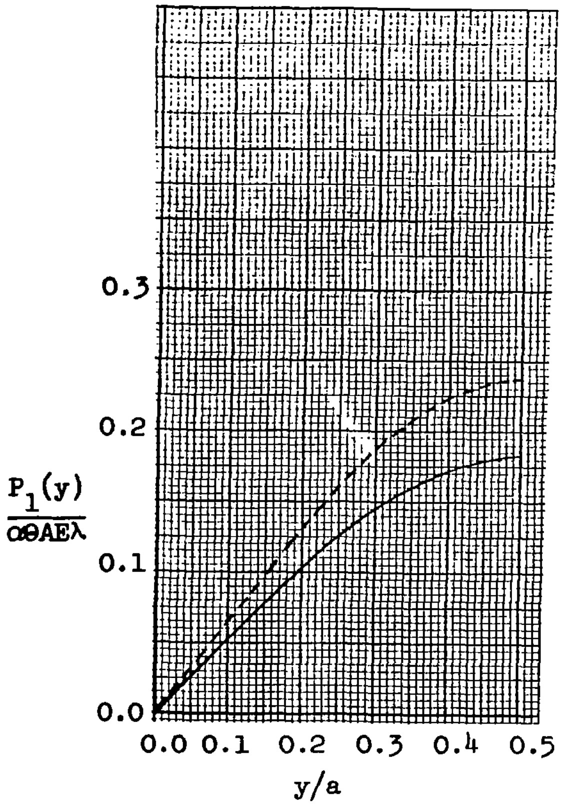


(c)

Figure 7. (continued)



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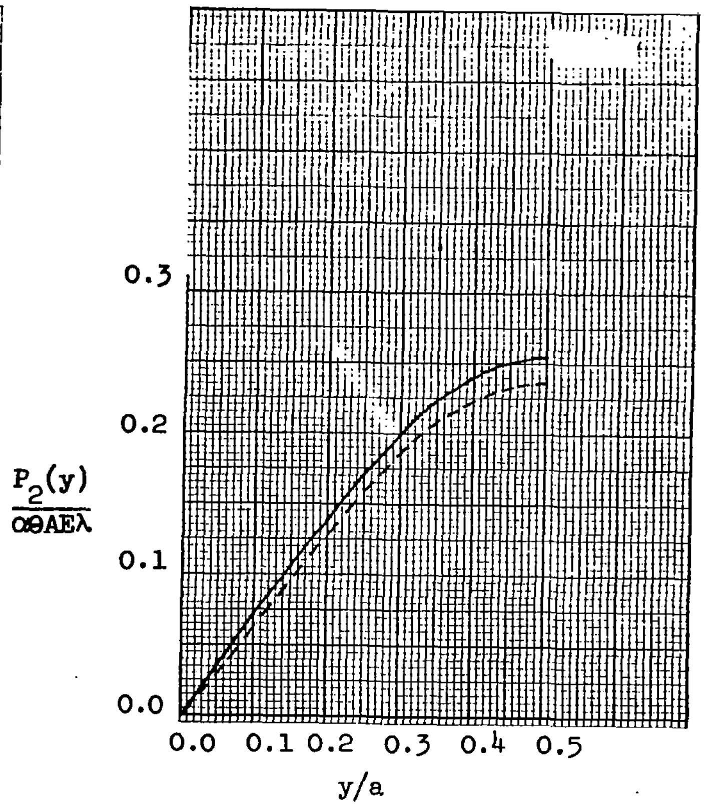

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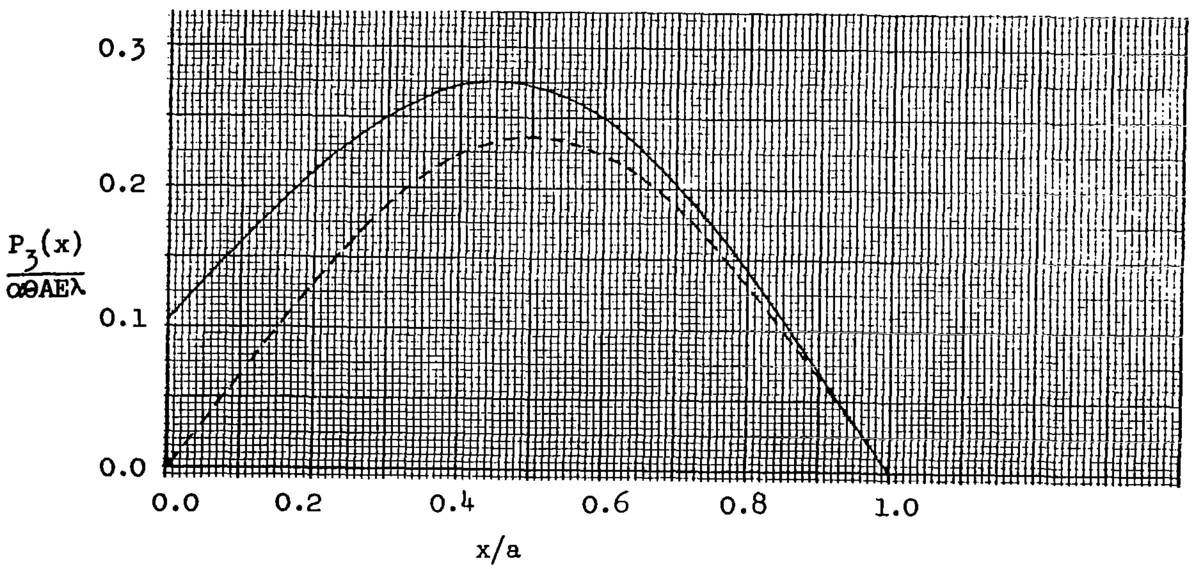
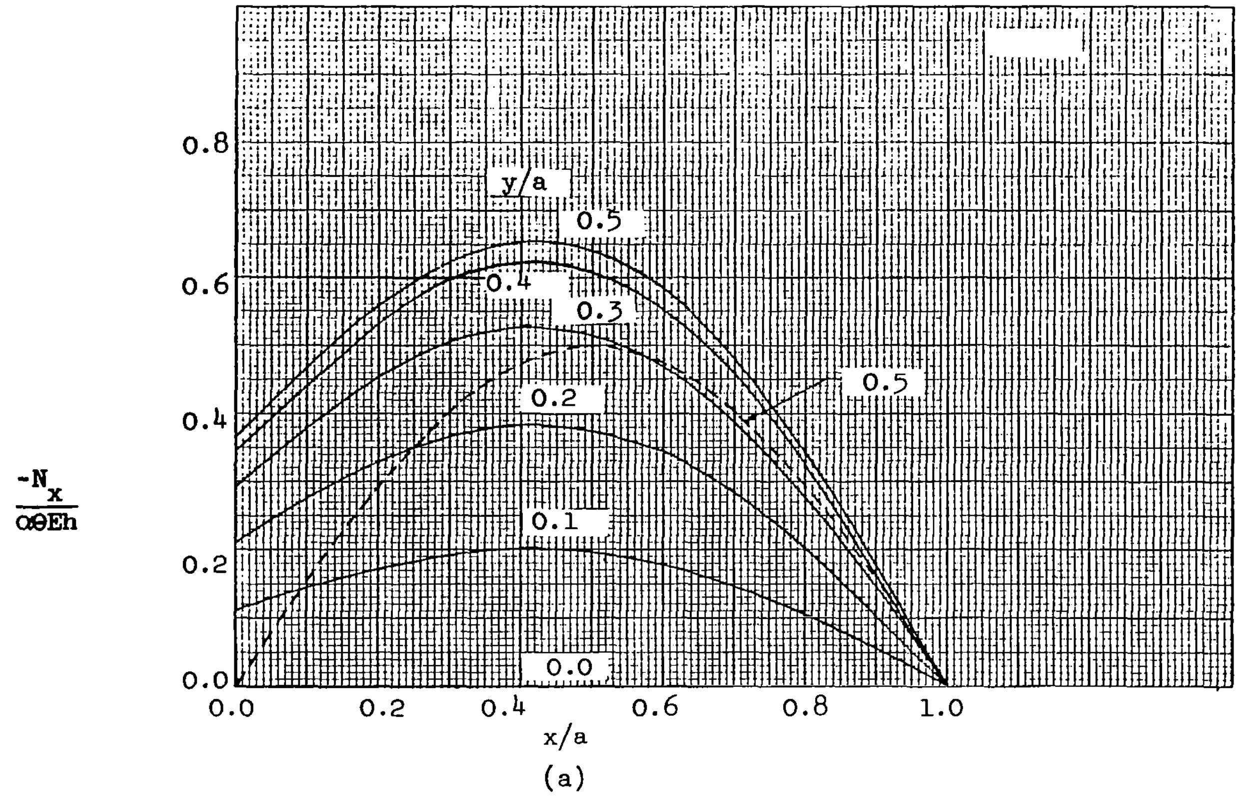
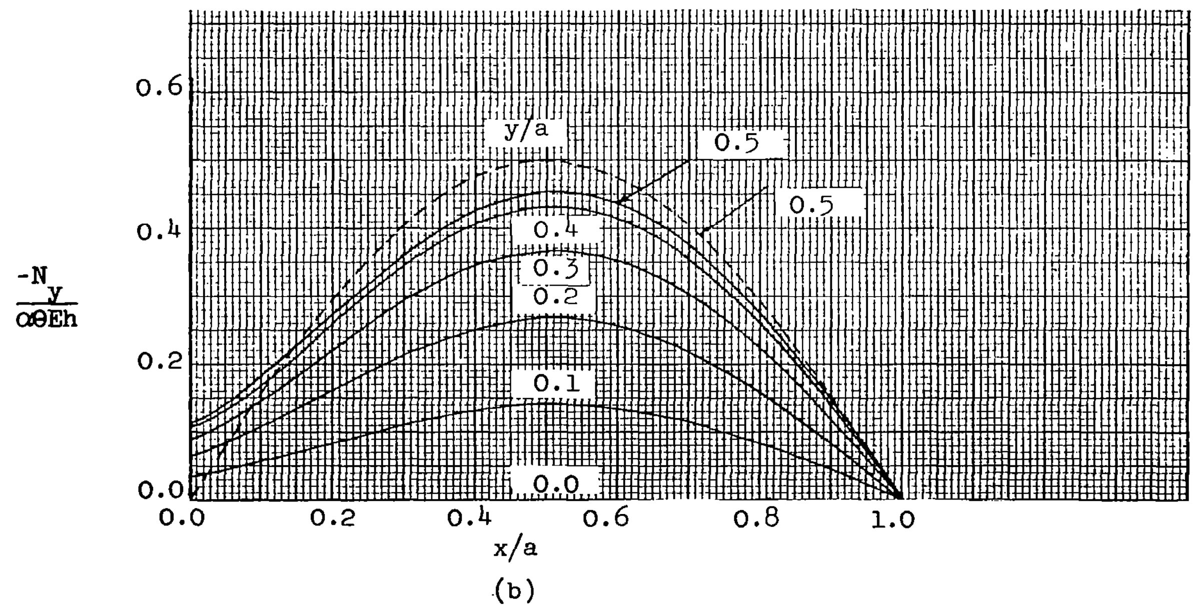


(f)

Figure 7. (continued)

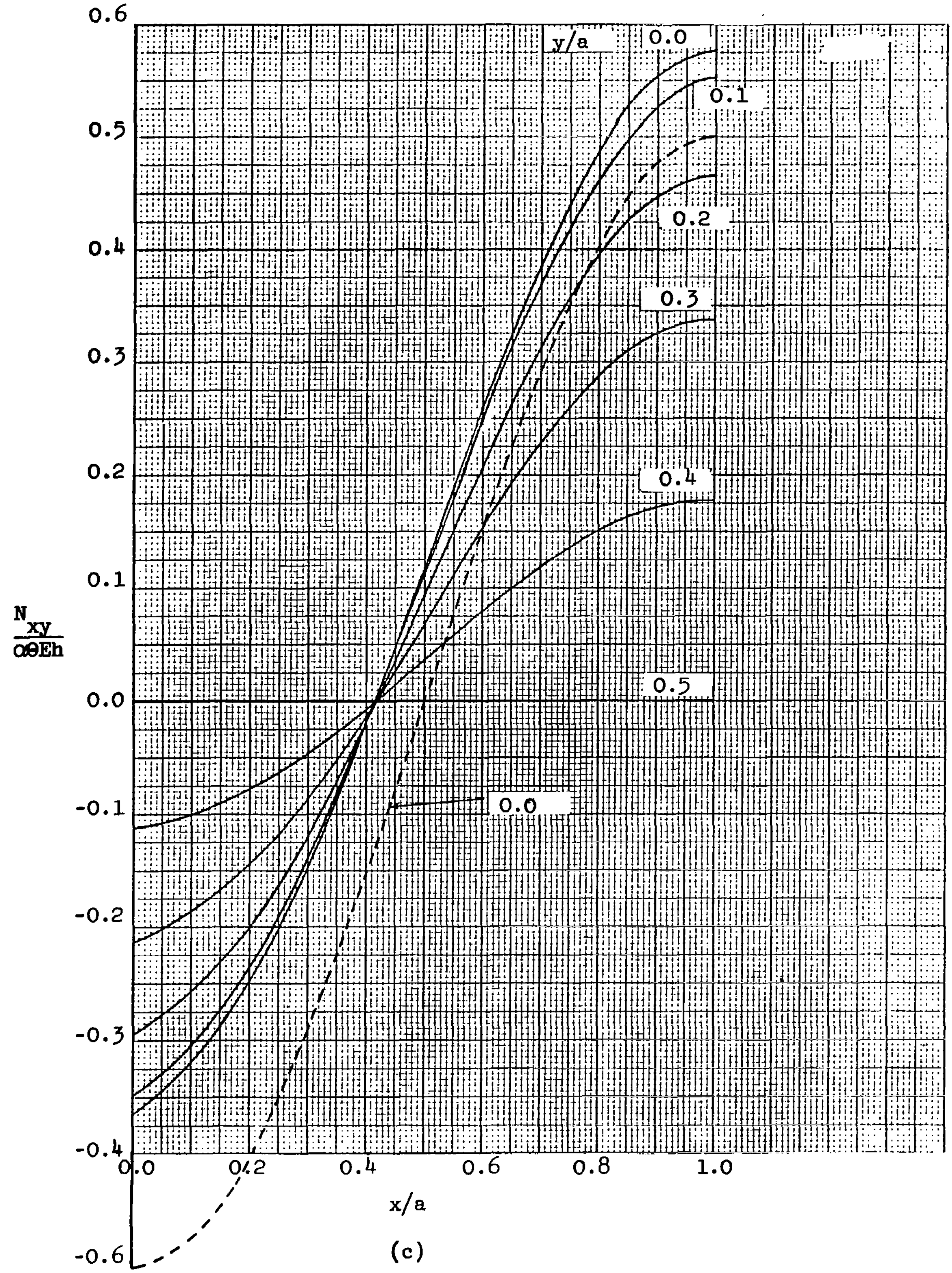


(a)



(b)

Figure 8. Plate stresses and stiffener tensions for the case of one edge held straight, pillow-shaped temperature distribution, $\lambda \to 0$, and $\nu = 0.3$. (Dashed curves, from fig. 5a of ref. 1, are for the case of all edge stiffeners perfectly flexible.) (M = 79, N = 79)



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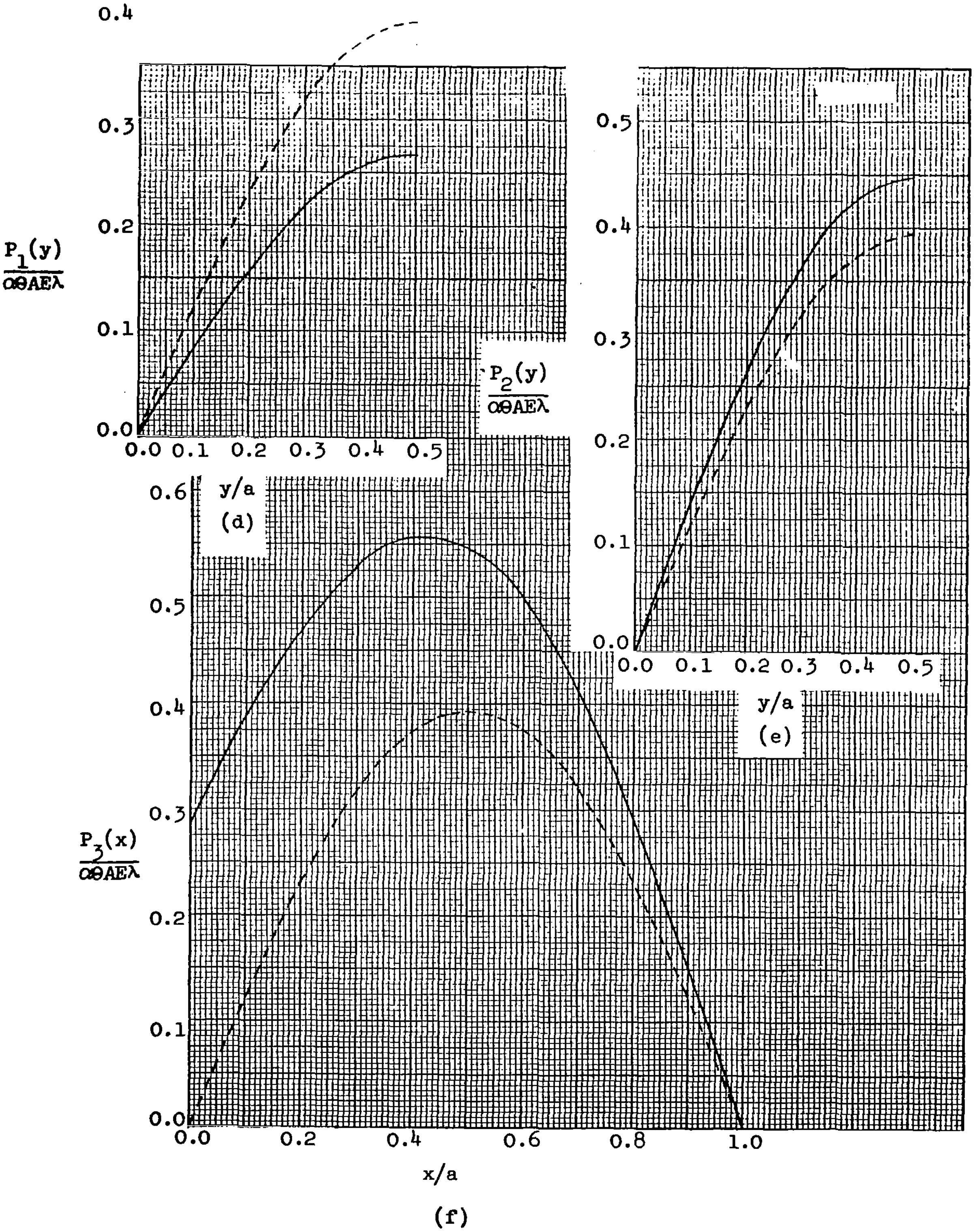

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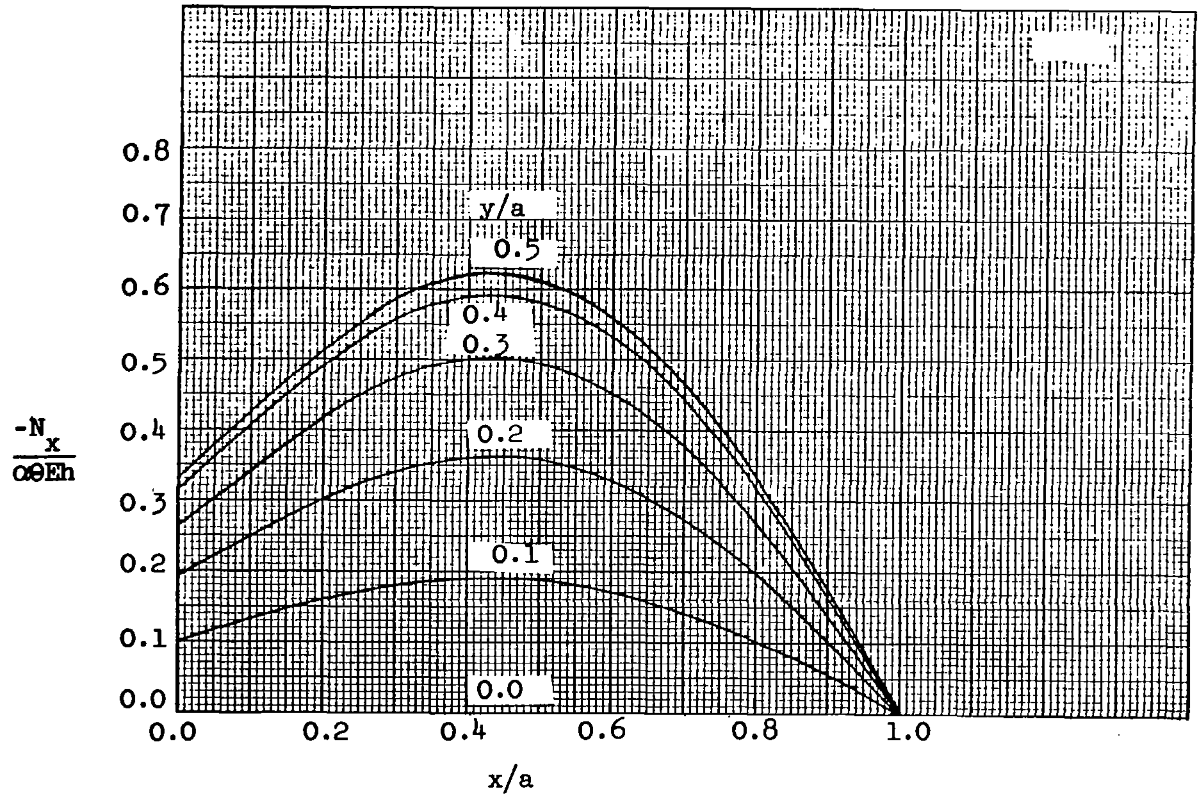
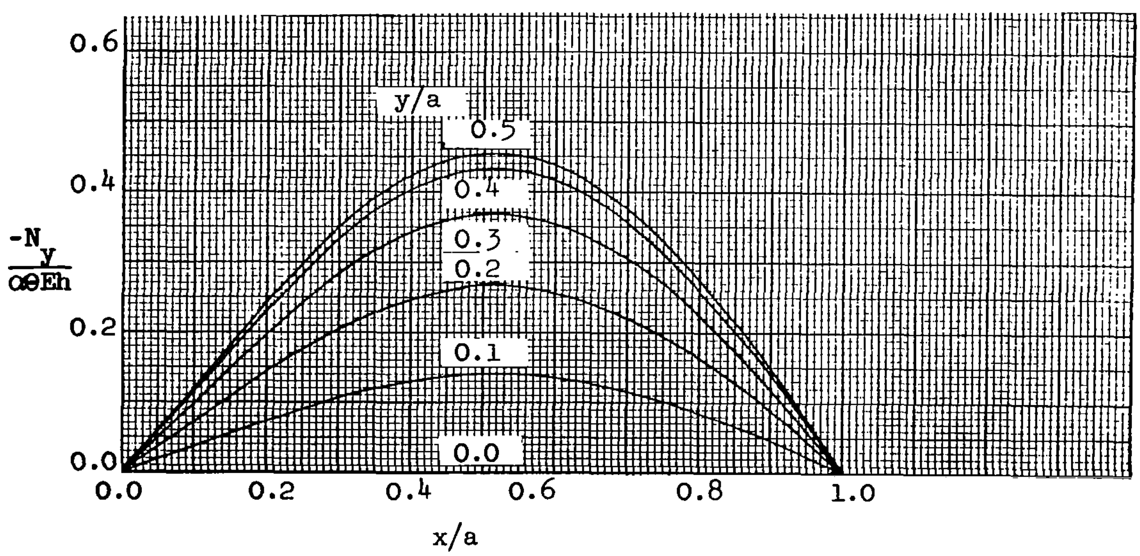
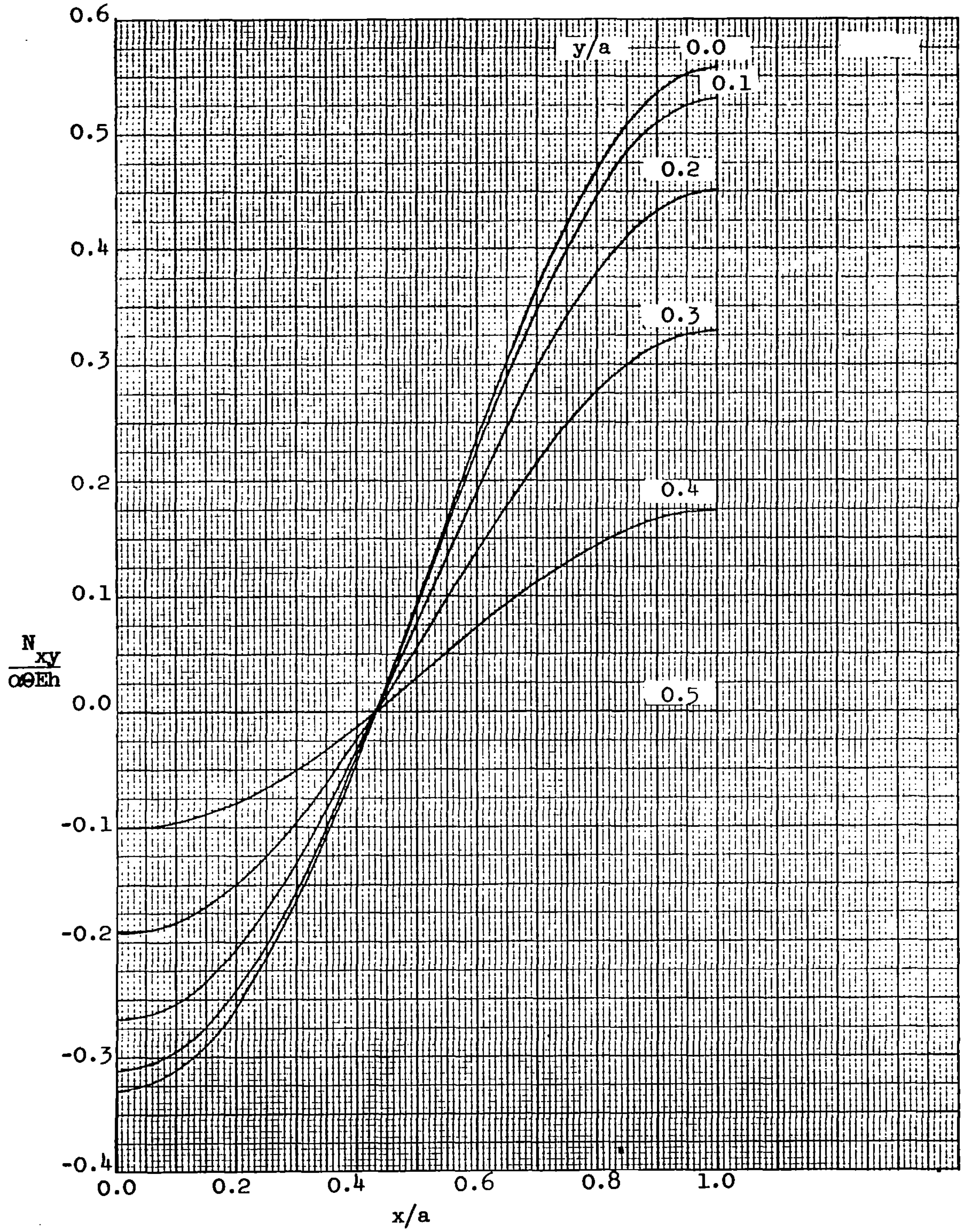


(a)



(b)

Figure 9. Plate stresses and stiffener tensions for the case of one edge held straight, pillow-shaped temperature distribution, $\lambda \rightarrow 0$, and $\nu = 0$. $(M = 79, N = 79)$.



(c)

Figure 9. (continued)

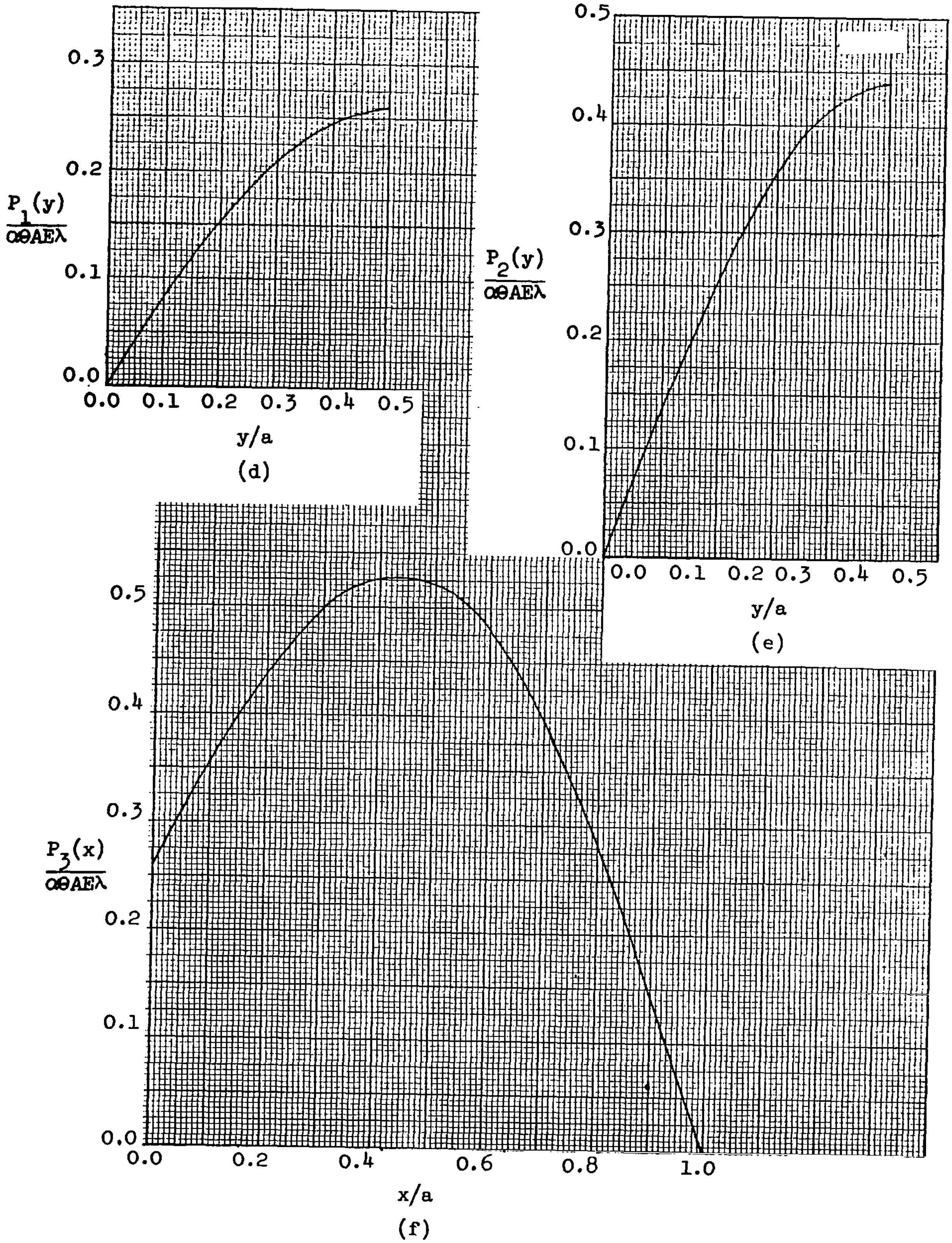


Figure 9. (continued)

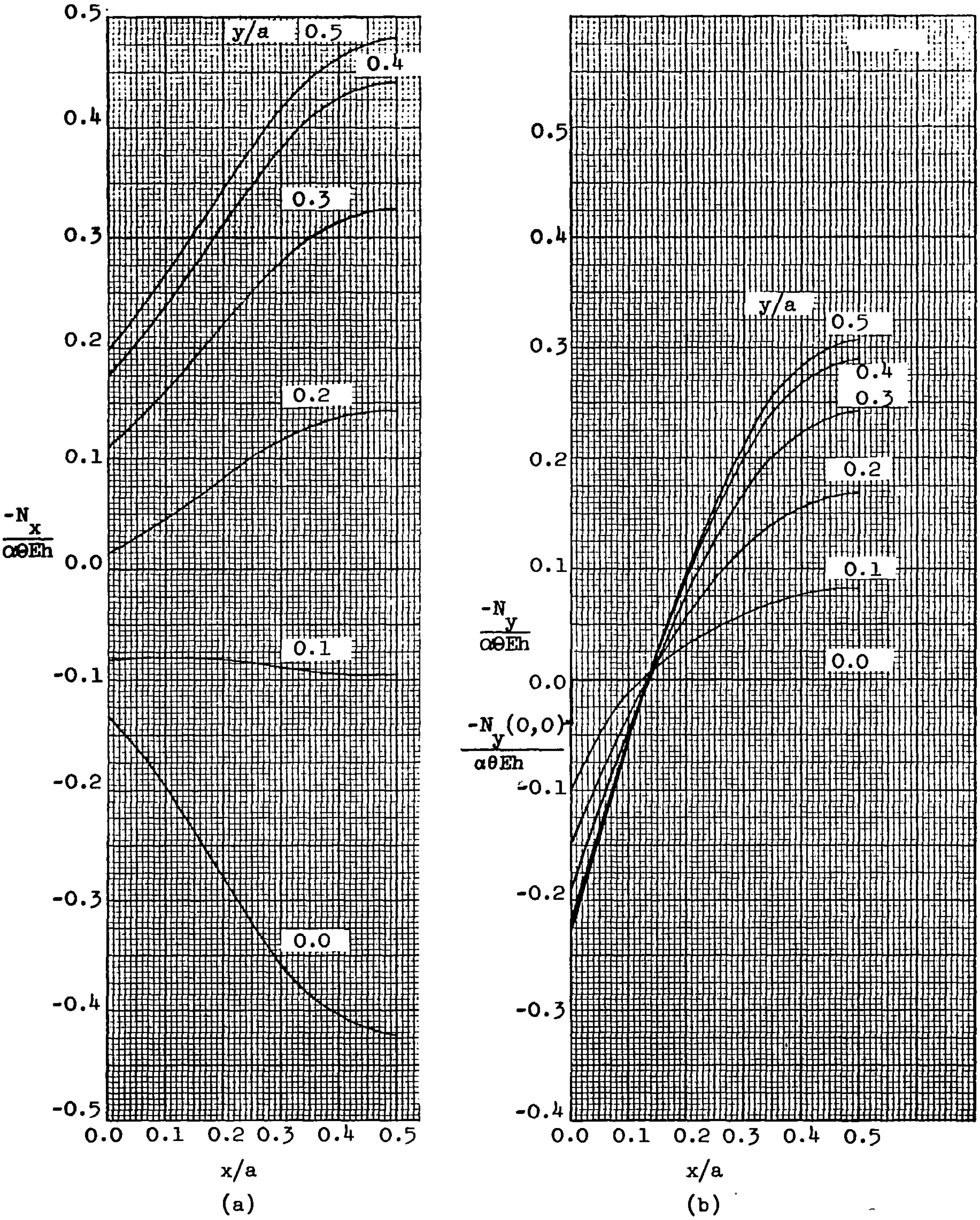


Figure 10. Plate stresses and stiffener tensions of the case of two opposite edges held straight, pillow-shaped temperature distribution, $\lambda = 2.0$, and $\nu = 0.3$. (M = 59, N = 59)

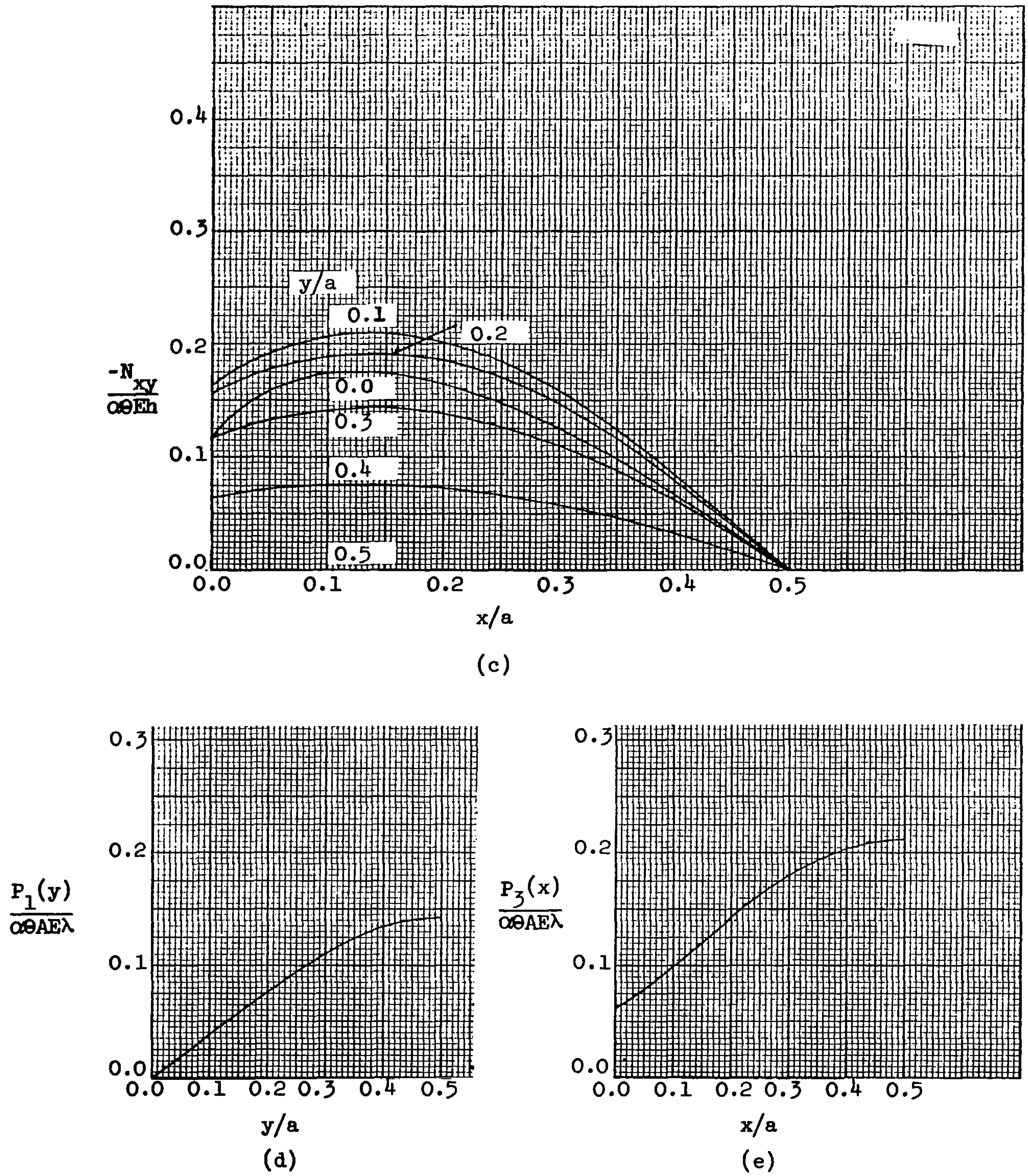


Figure 10. (continued)

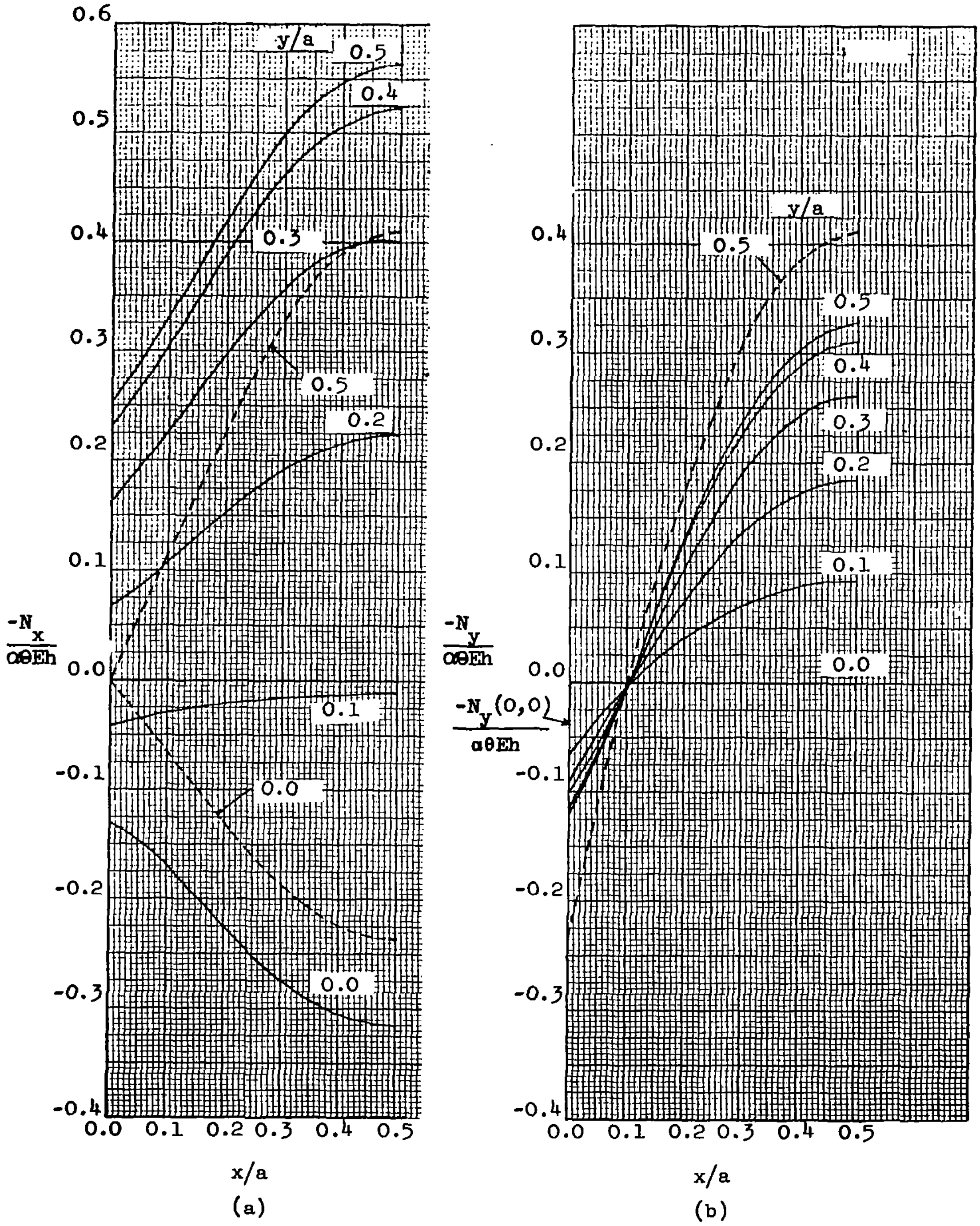
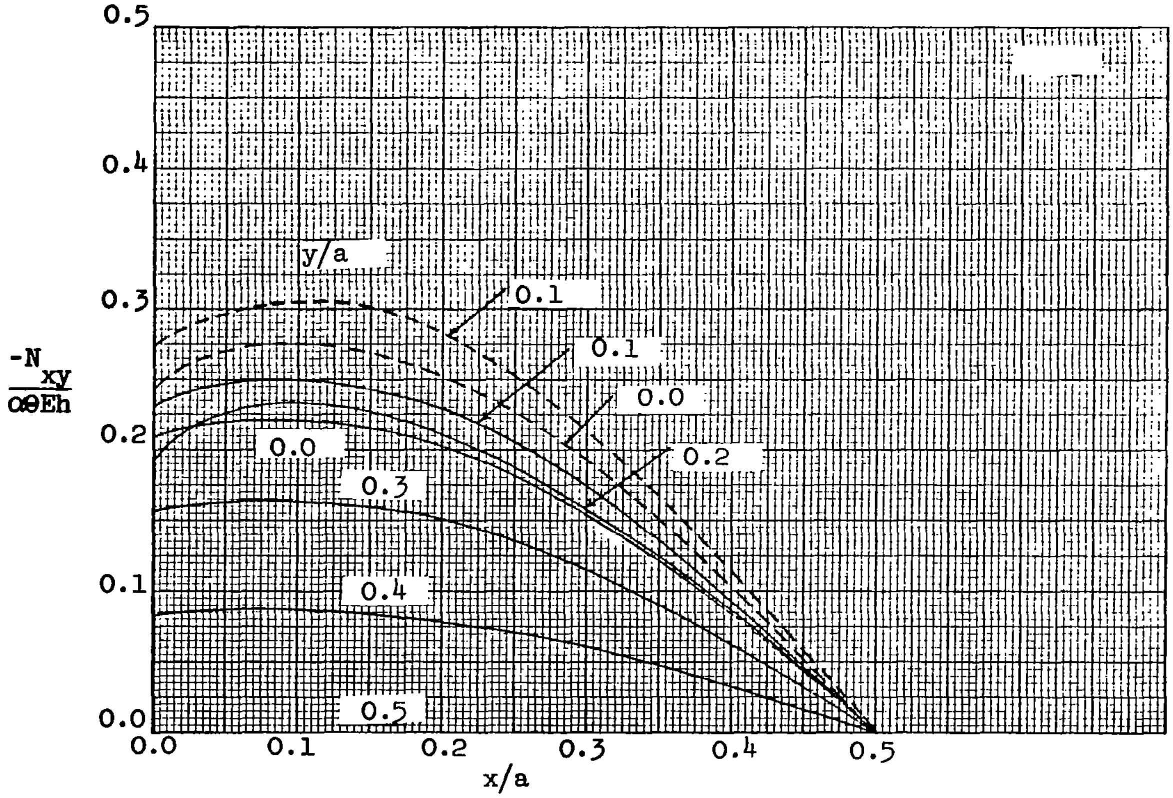
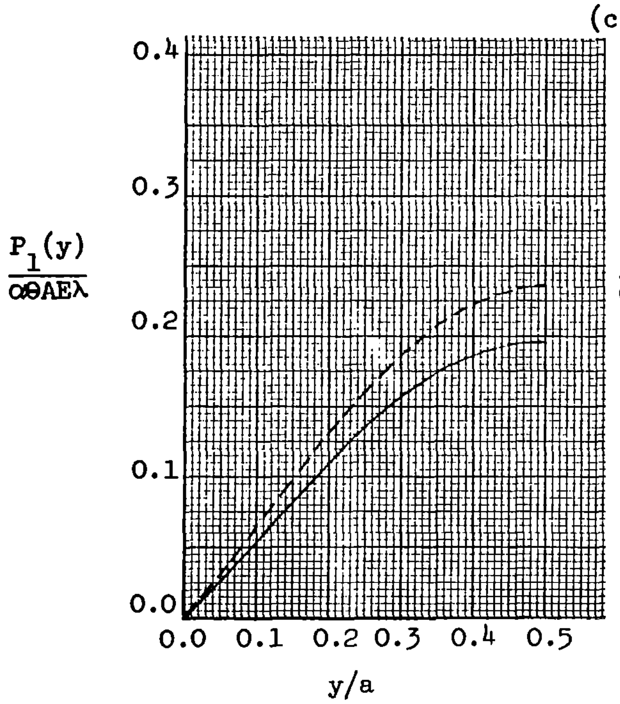


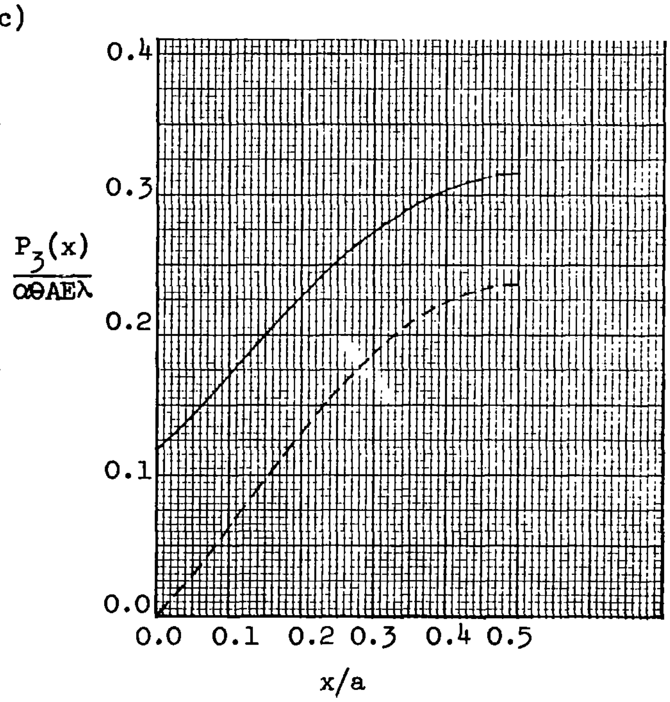
Figure 11. Plate stresses and stiffener tensions for the case of two opposite edges held straight, pillow-shaped temperature distribution, $\lambda = 1.0$, and $\nu = 0.3$. (Dashed curves, from fig. 5c of ref. 1, are for the case of all edge stiffeners perfectly flexible.) $(M = 59, N = 59)$



(c)



(d)



(e)

Figure 11. (continued)

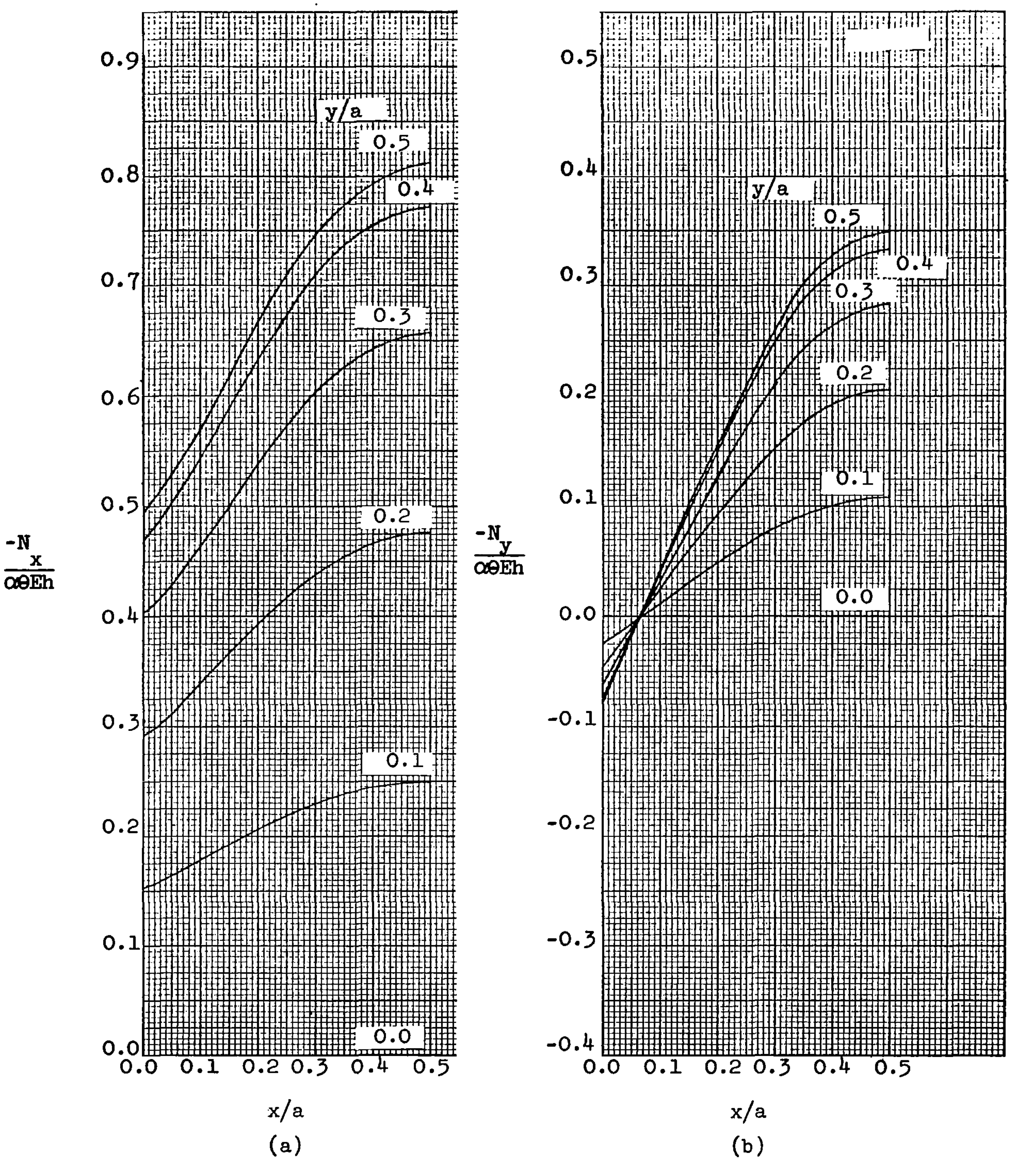
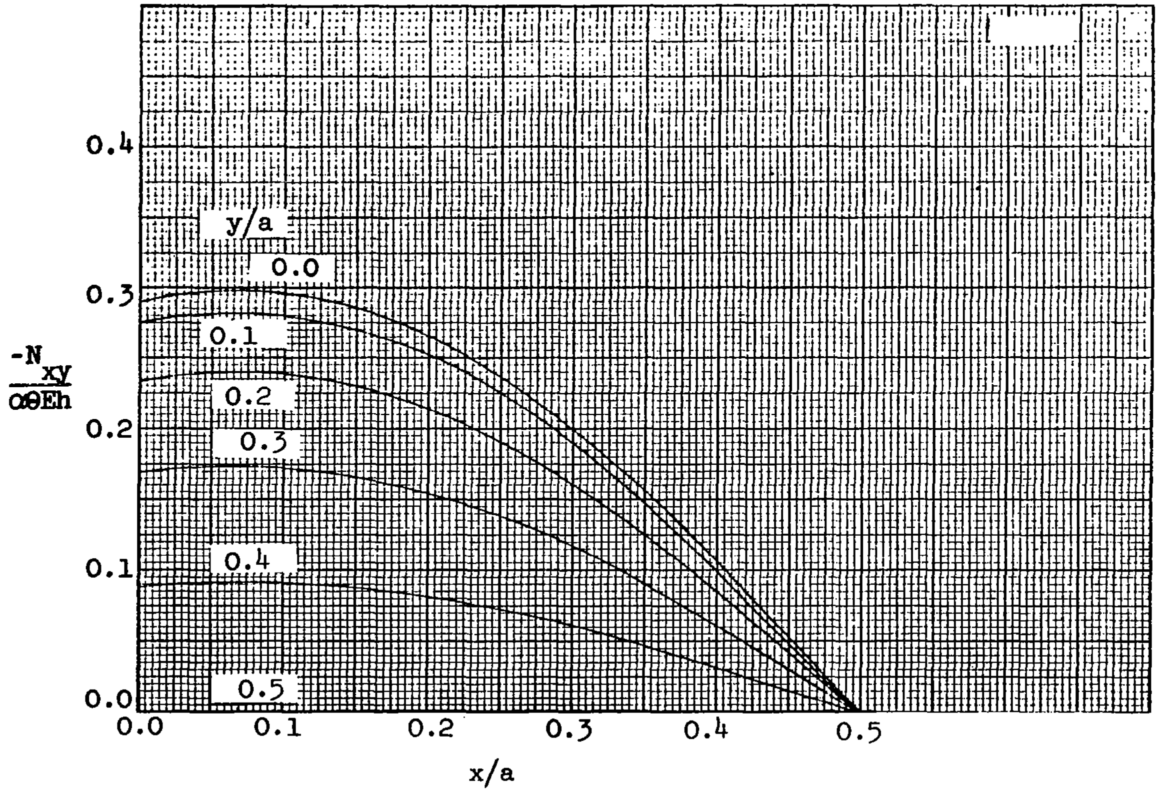


Figure 12. Plate stresses and stiffener tensions for the case of two opposite edges held straight, pillow-shaped temperature distribution, $\lambda_1 = 1$, $\lambda_2 \rightarrow 0$, and $\nu = 0.3$. (M = 59, N = 59)



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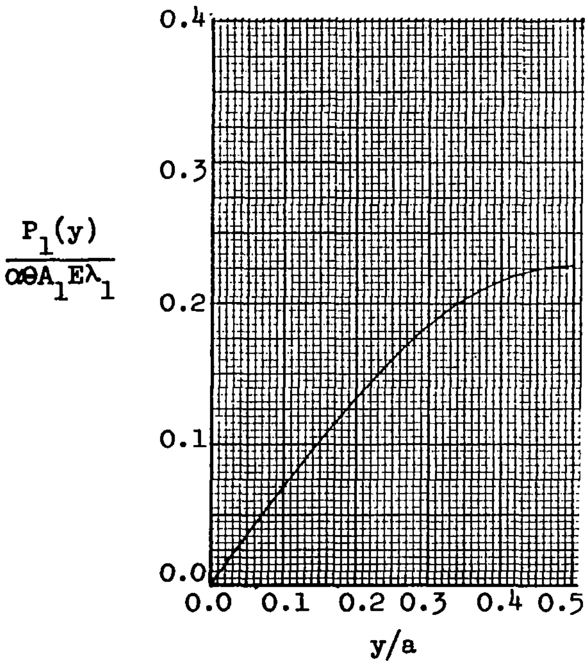

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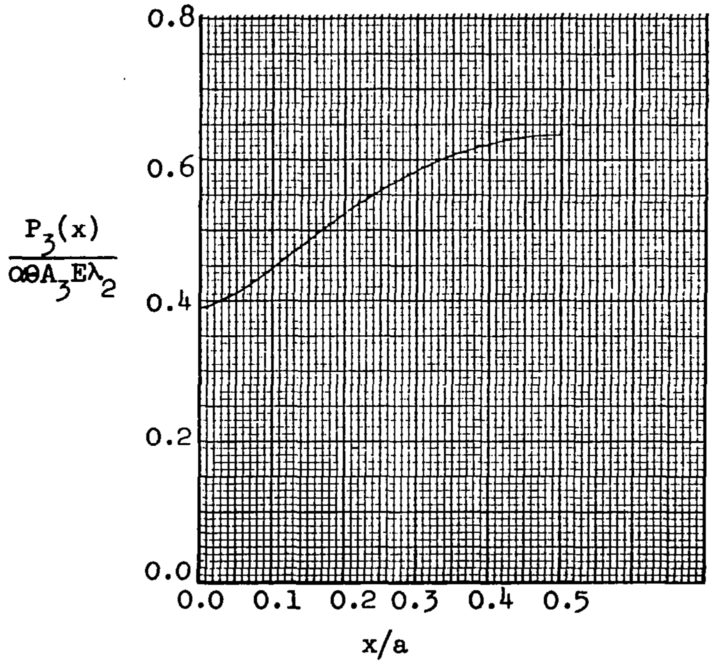


(e)

Figure 12. (continued)

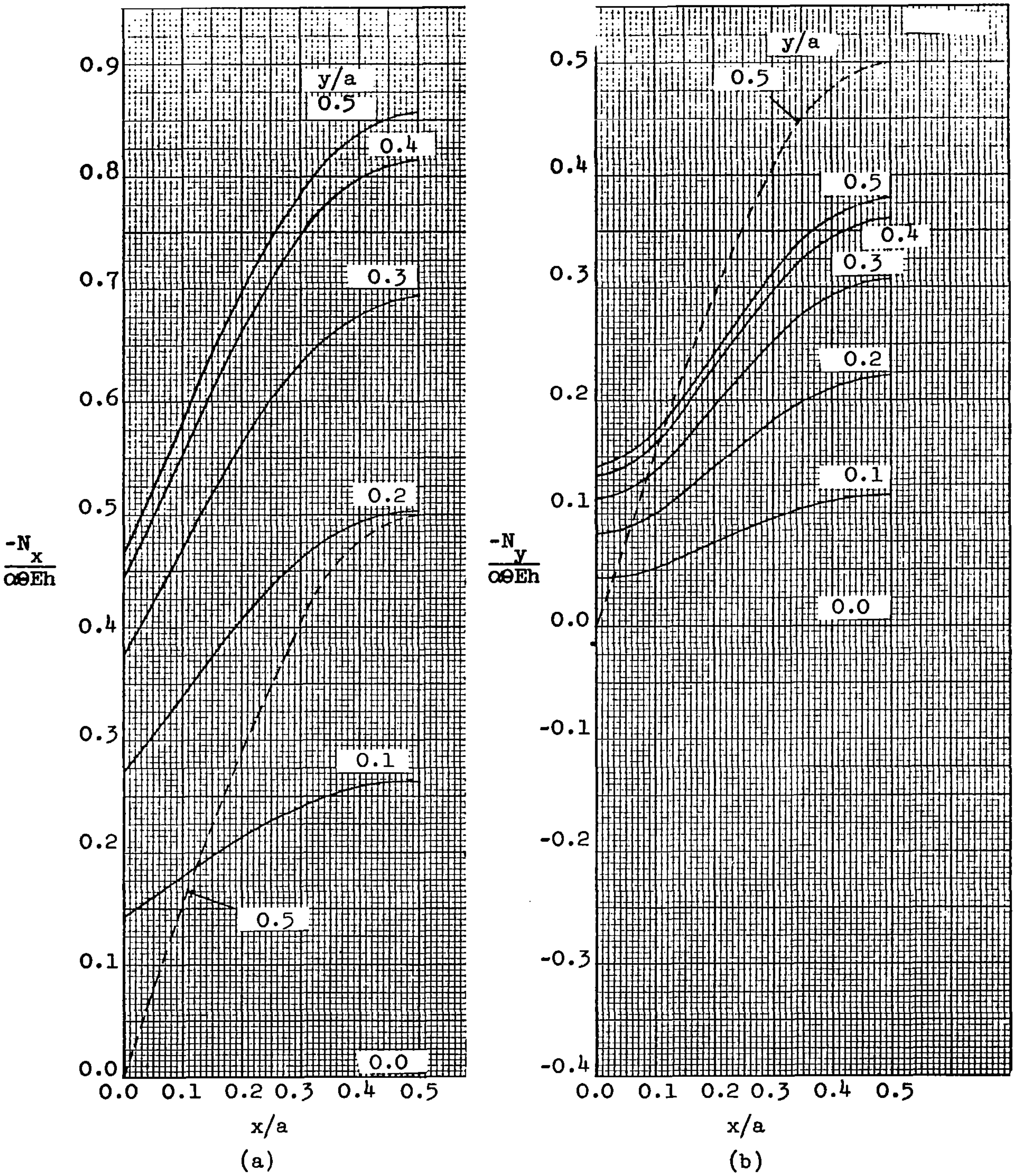


Figure 13. Plate stresses and stiffener tensions for the case of two opposite edges held straight, pillow-shaped temperature distribution, $\lambda_2 \longrightarrow 0$, $\lambda_1 \longrightarrow 0$, and $\nu = 0.3$. (Dashed curves, from fig. 5a of ref. 1, are for the case of all edge stiffeners perfectly flexible.) (M = 79, N = 79)

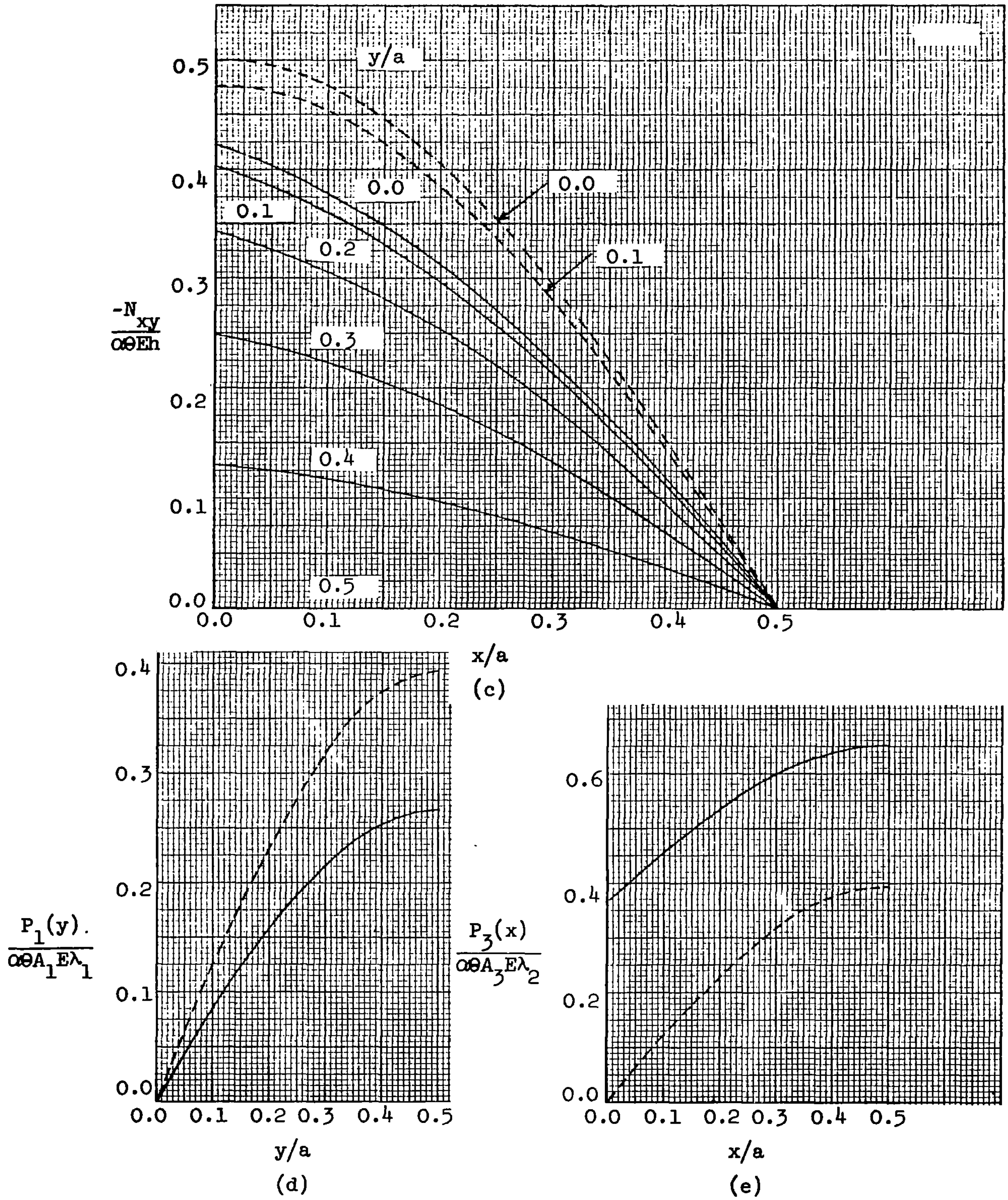
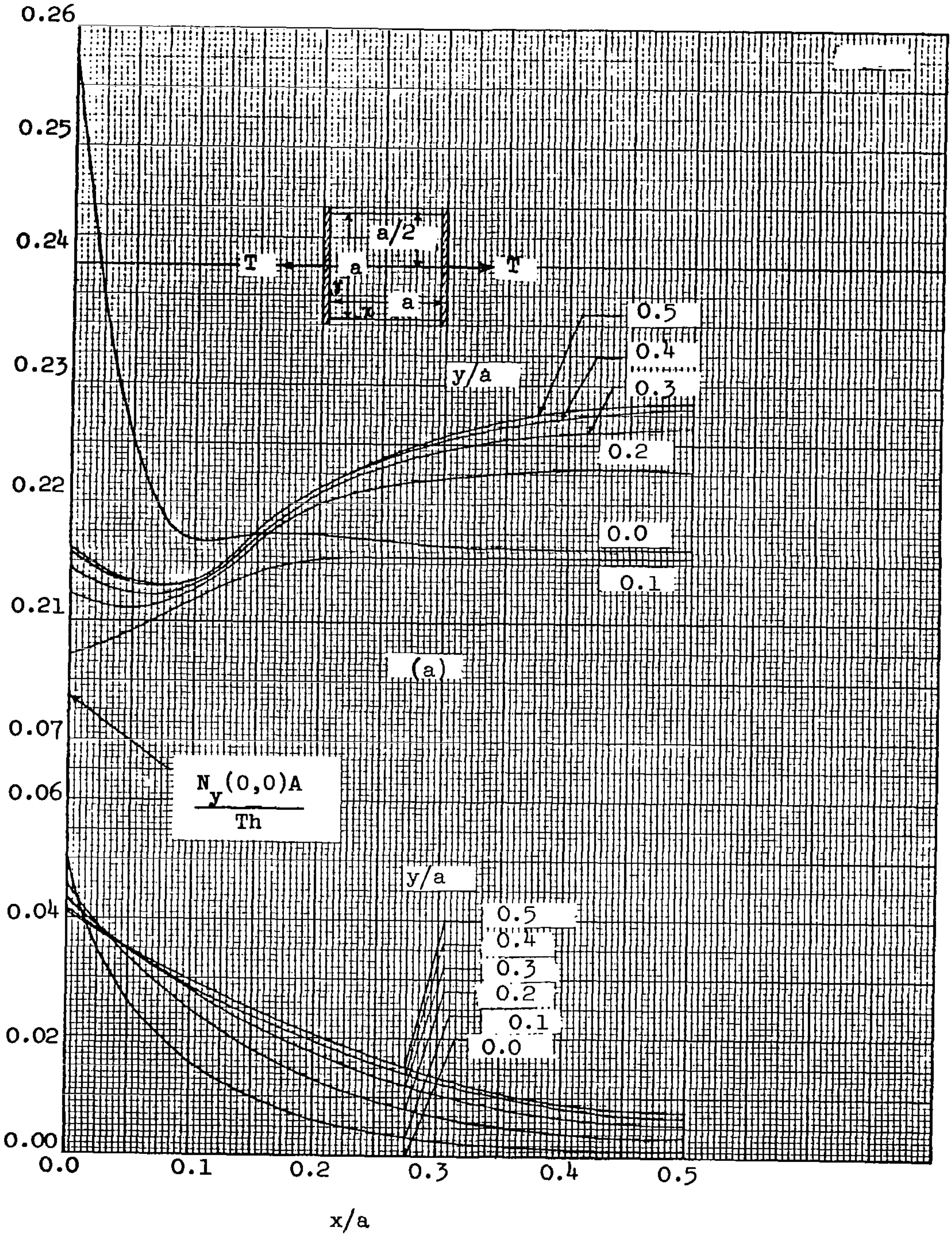
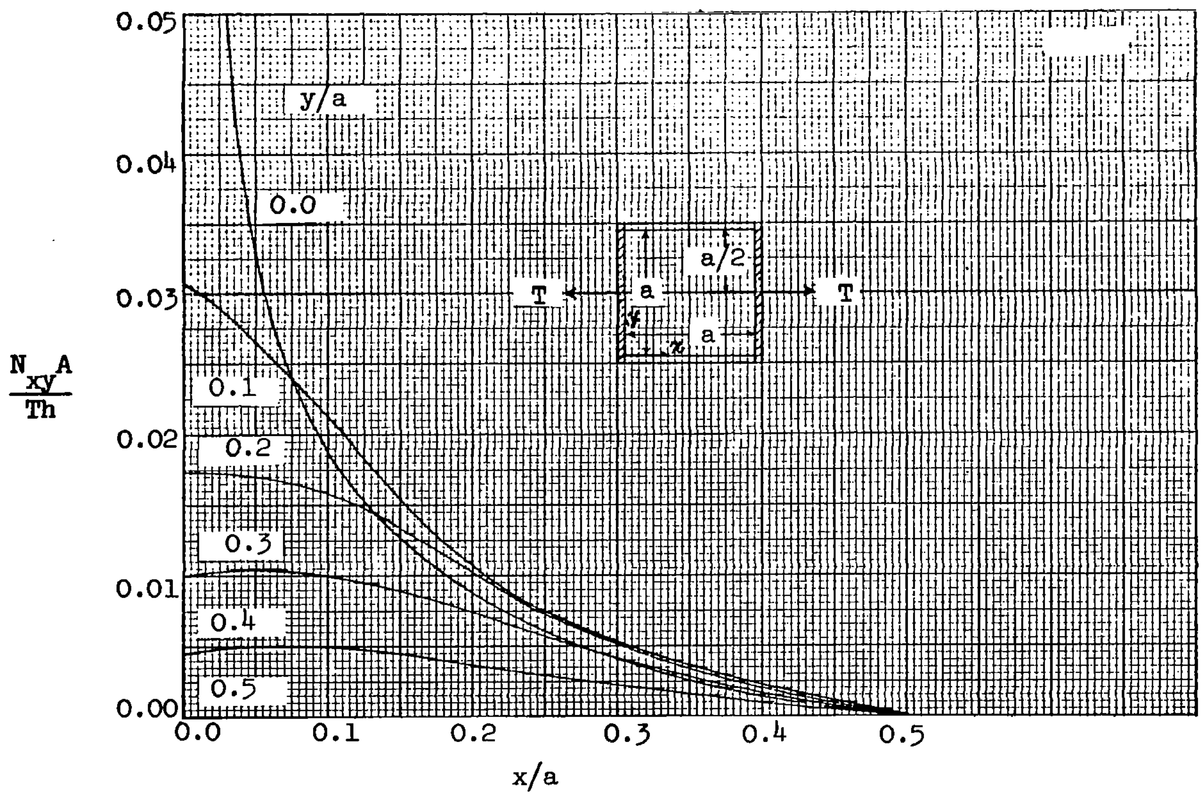


Figure 13. (continued)



(b)

Figure 14. Plate stresses and stiffener tensions for the case of two opposite edges held straight, stretching forces applied normal to the straight edges, $\lambda = 1.0$, and $\nu = 0.3$. ($M = 59$, $N = 59$)



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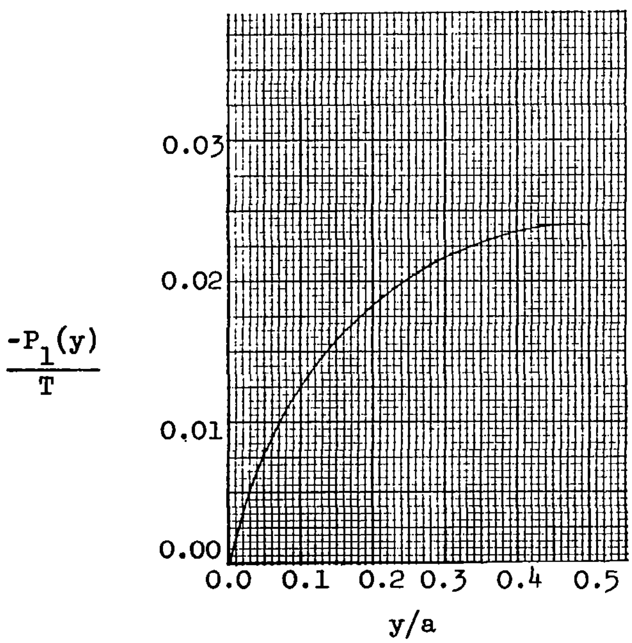

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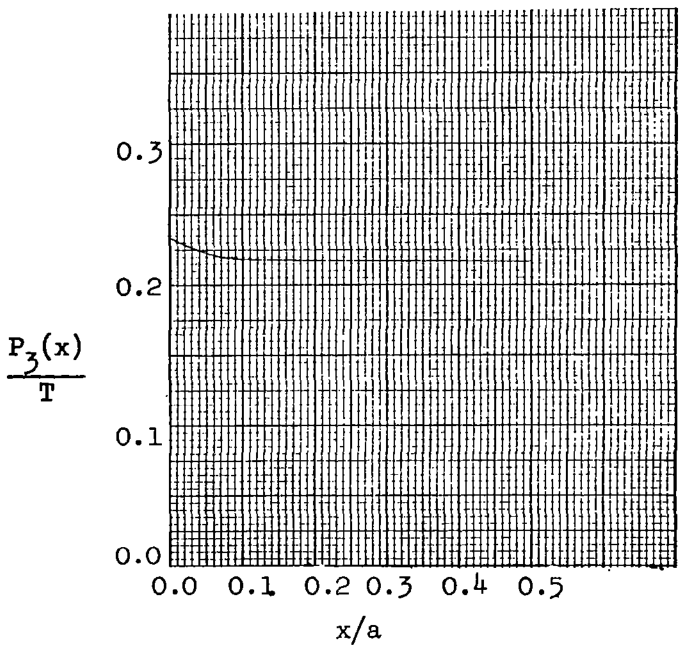


(e)

Figure 14. (continued)

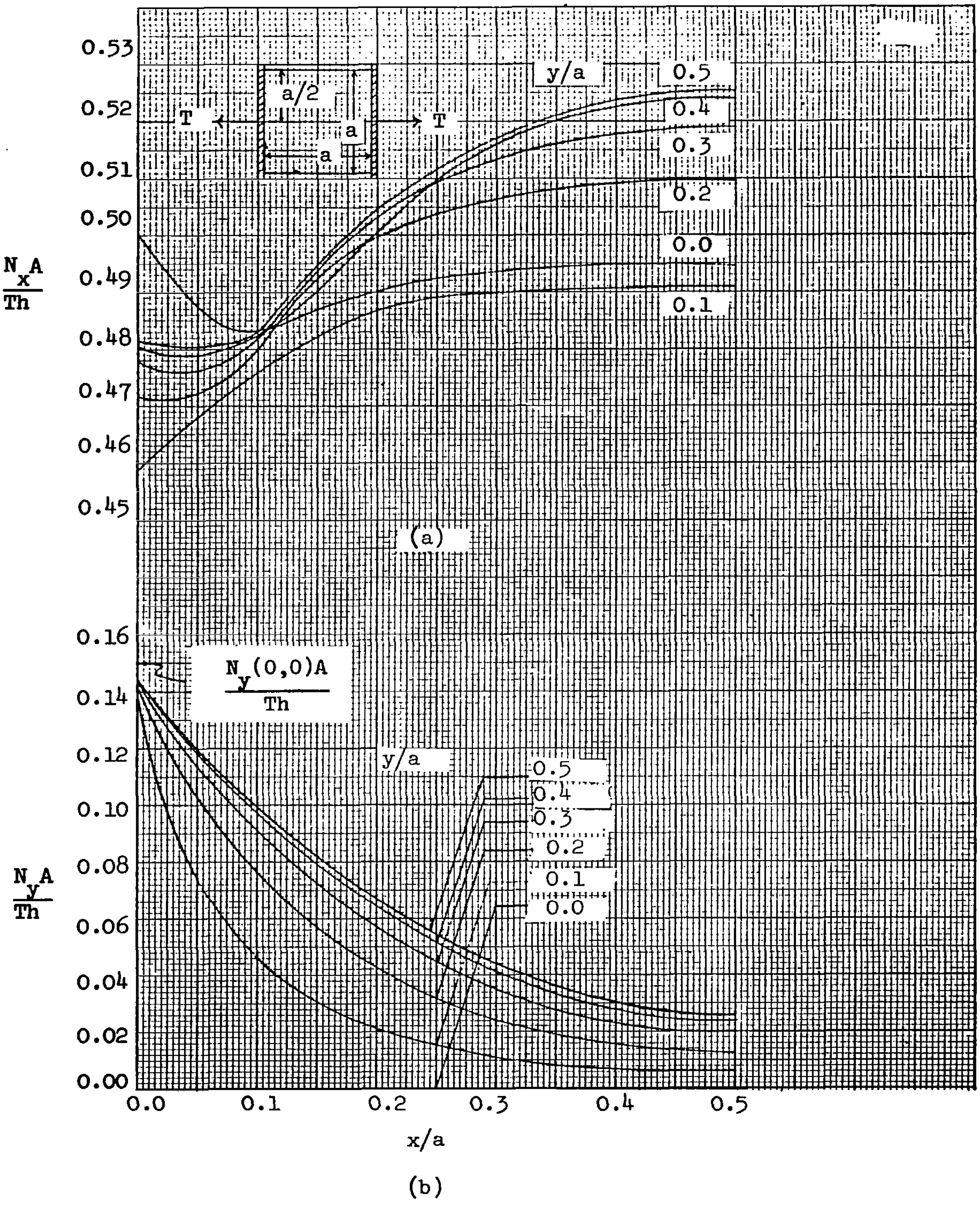
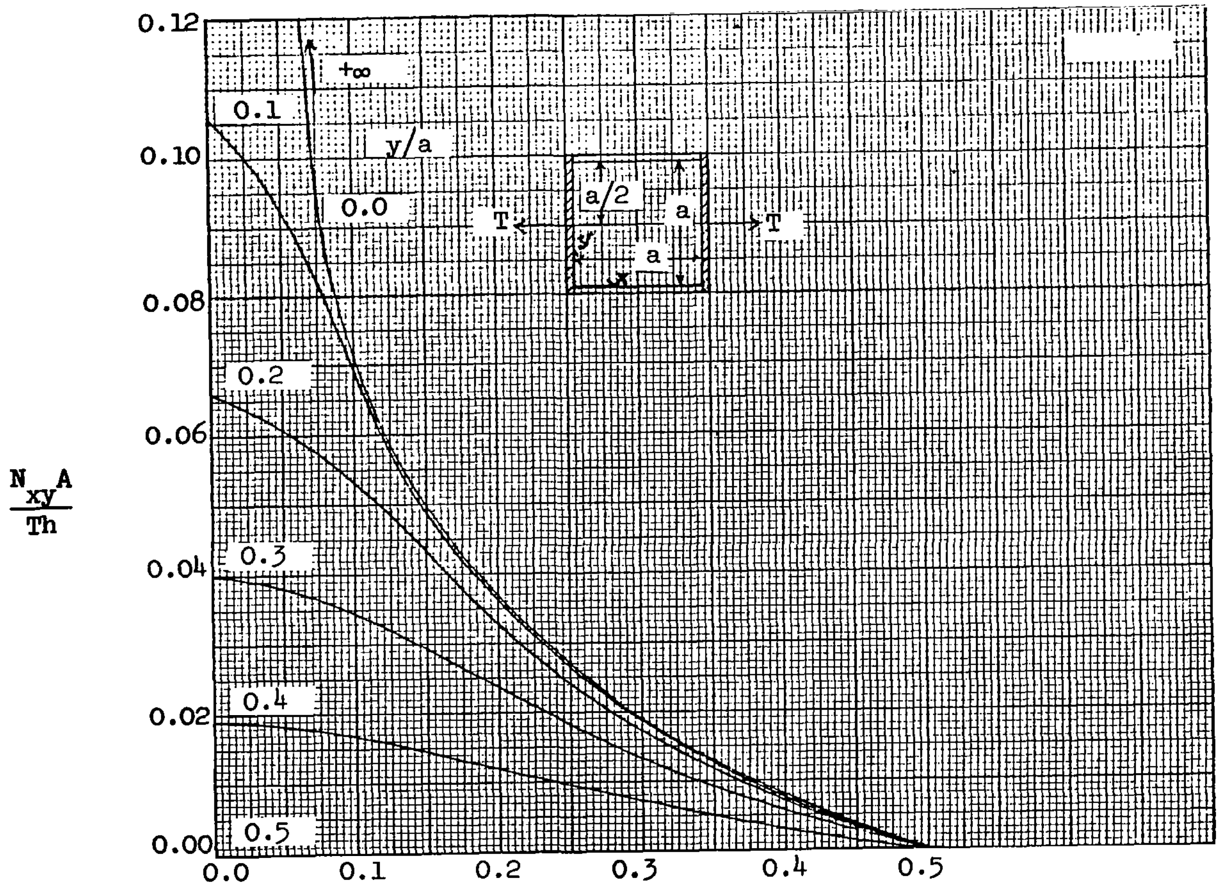


Figure 15. Plate stresses and stiffener tensions for the case of two opposite edges held straight, stretching forced applied normal to the straight edges, $\lambda \to 0$, and $\nu = 0.3$. ($M = 59$, $N = 59$)



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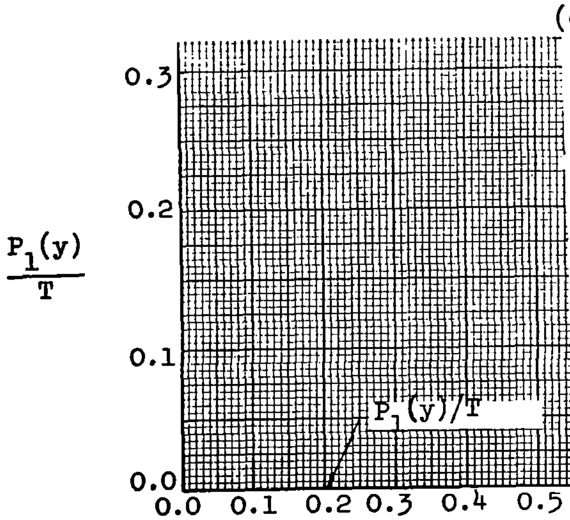
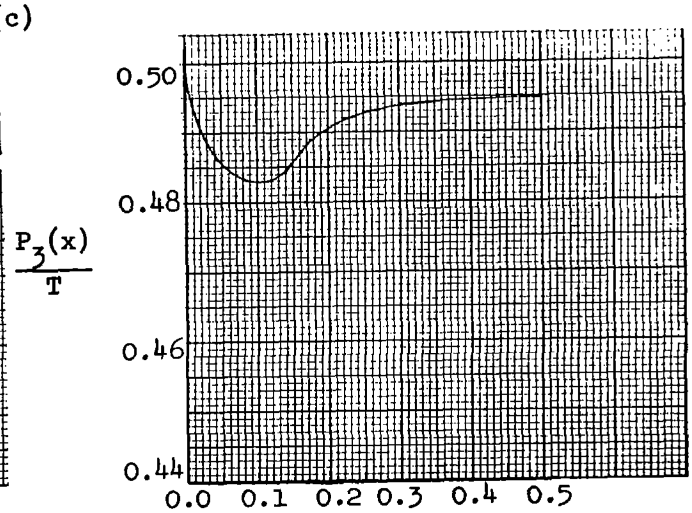
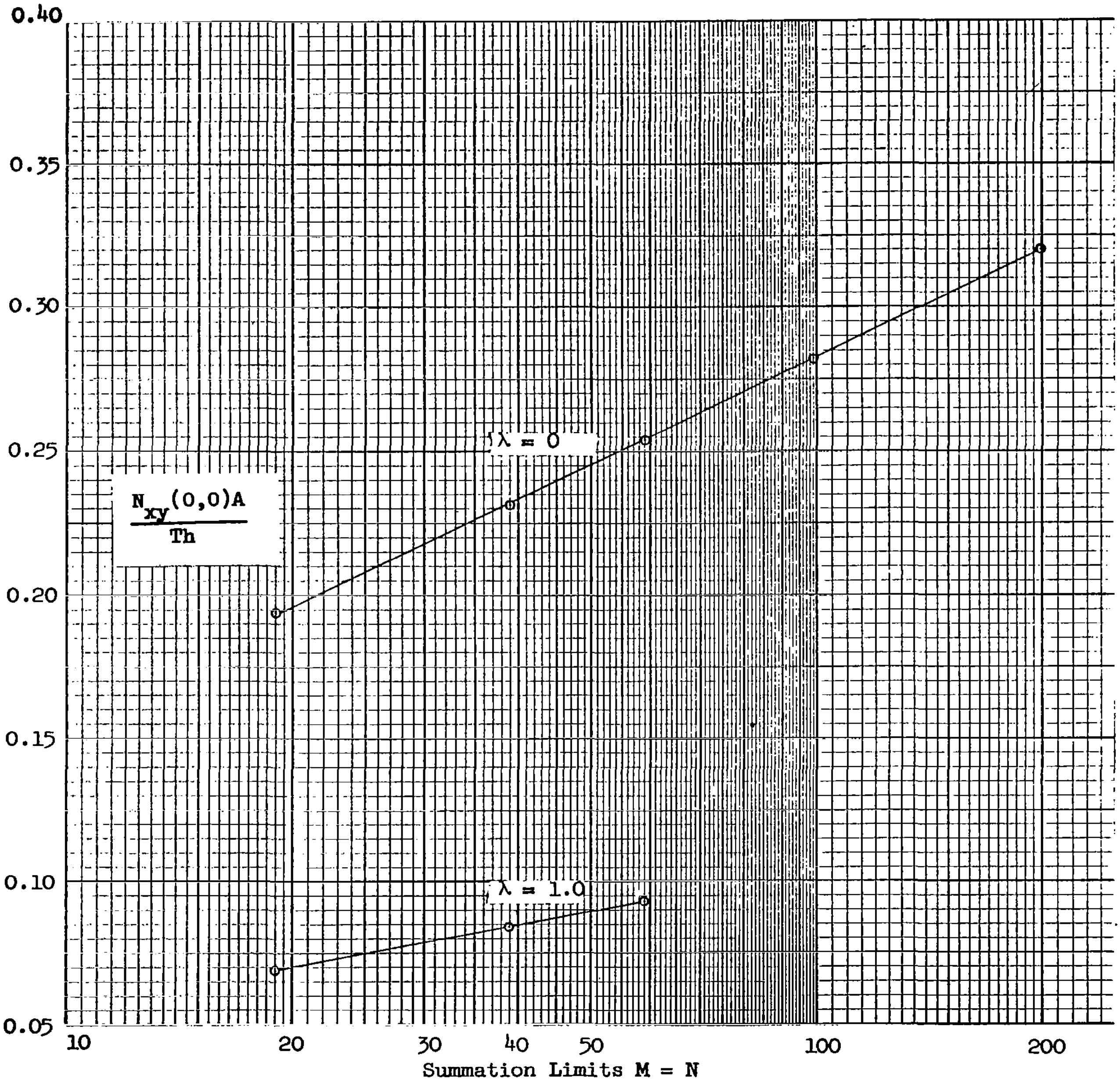


(d)



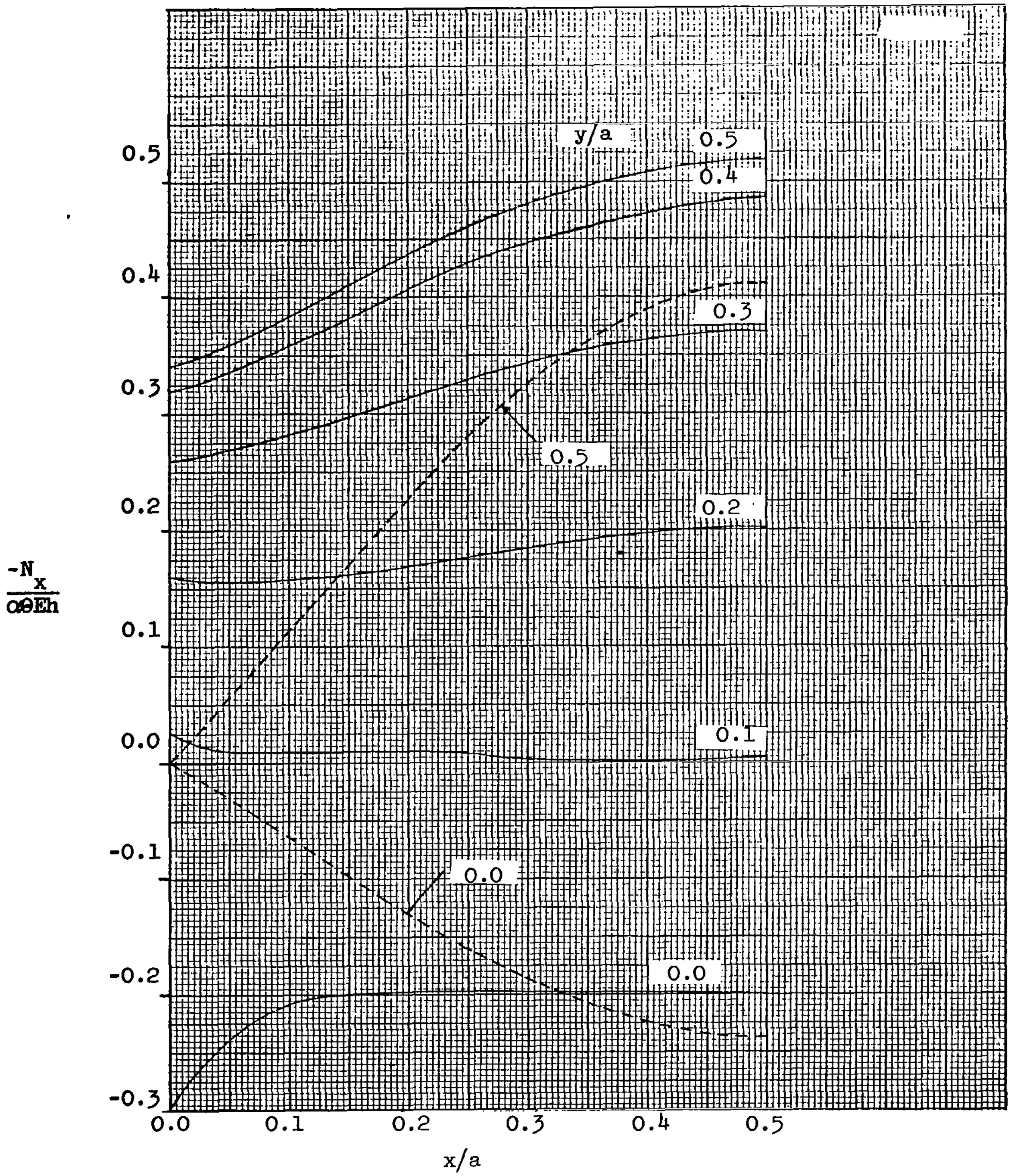
(e)

Figure 15. (Continued)



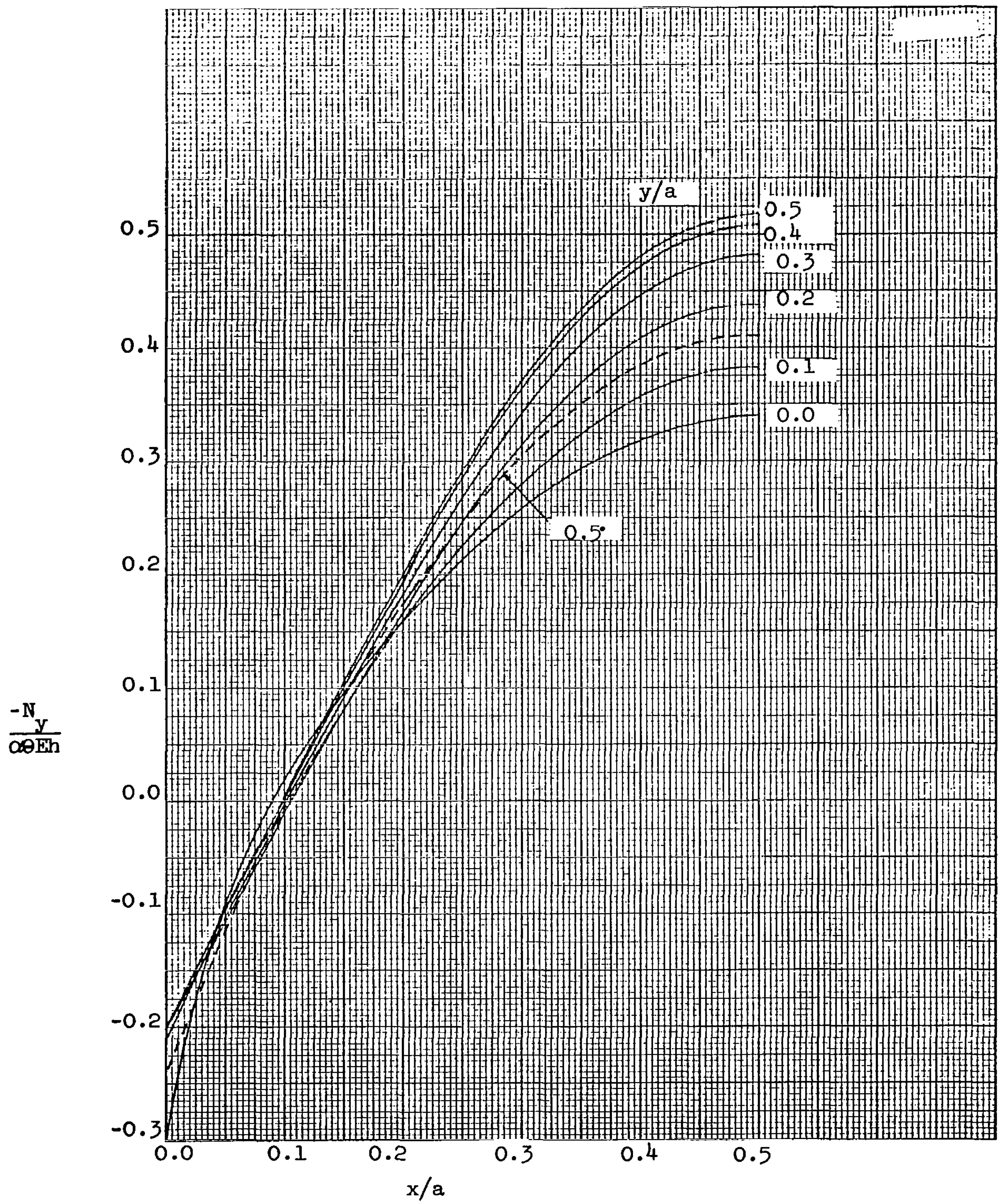
(f) Demonstration of singular corner shear stress behavior.

Figure 15. (continued)



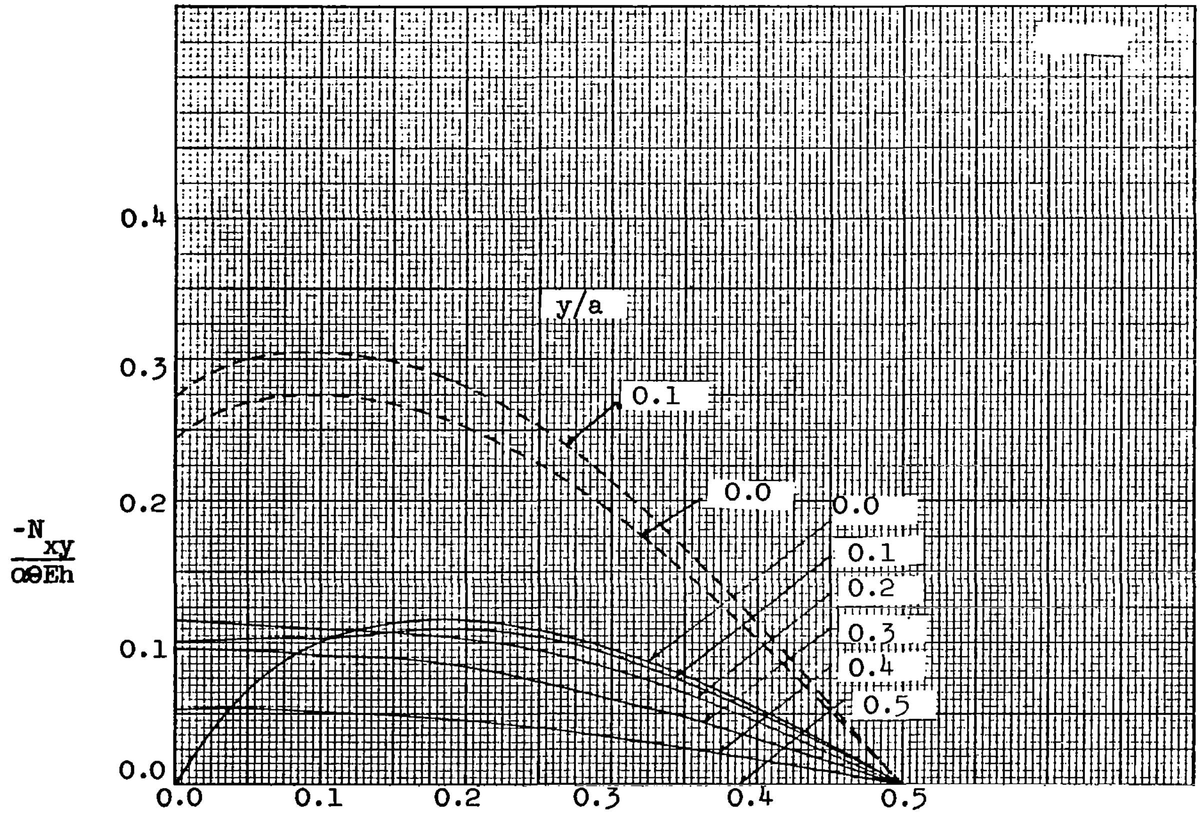
(a)

Figure 16. Plate stresses and stiffener tensions for the case of all four edges held straight, pillow-shaped temperature distribution, $\lambda = 1.0$, and $\nu = 0.3$. (Dashed curves, from fig. 5c of ref. 1, are for the case of all edge stiffeners perfectly flexible.) (M = 59, N = 59)



(b)

Figure 16. (continued)



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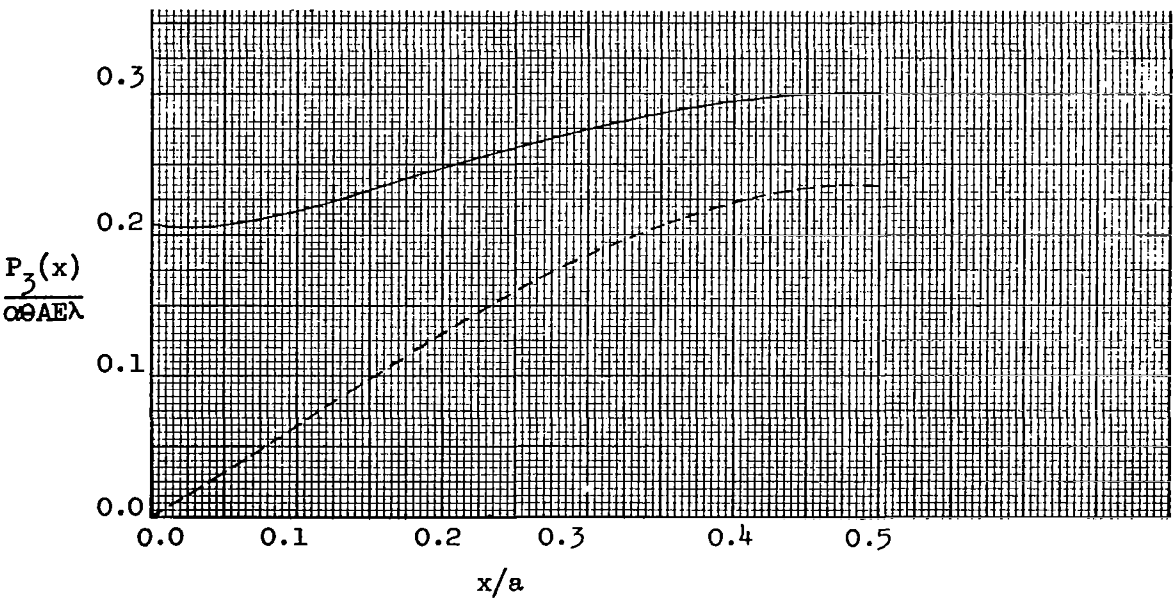


(d)

Figure 16. (continued)

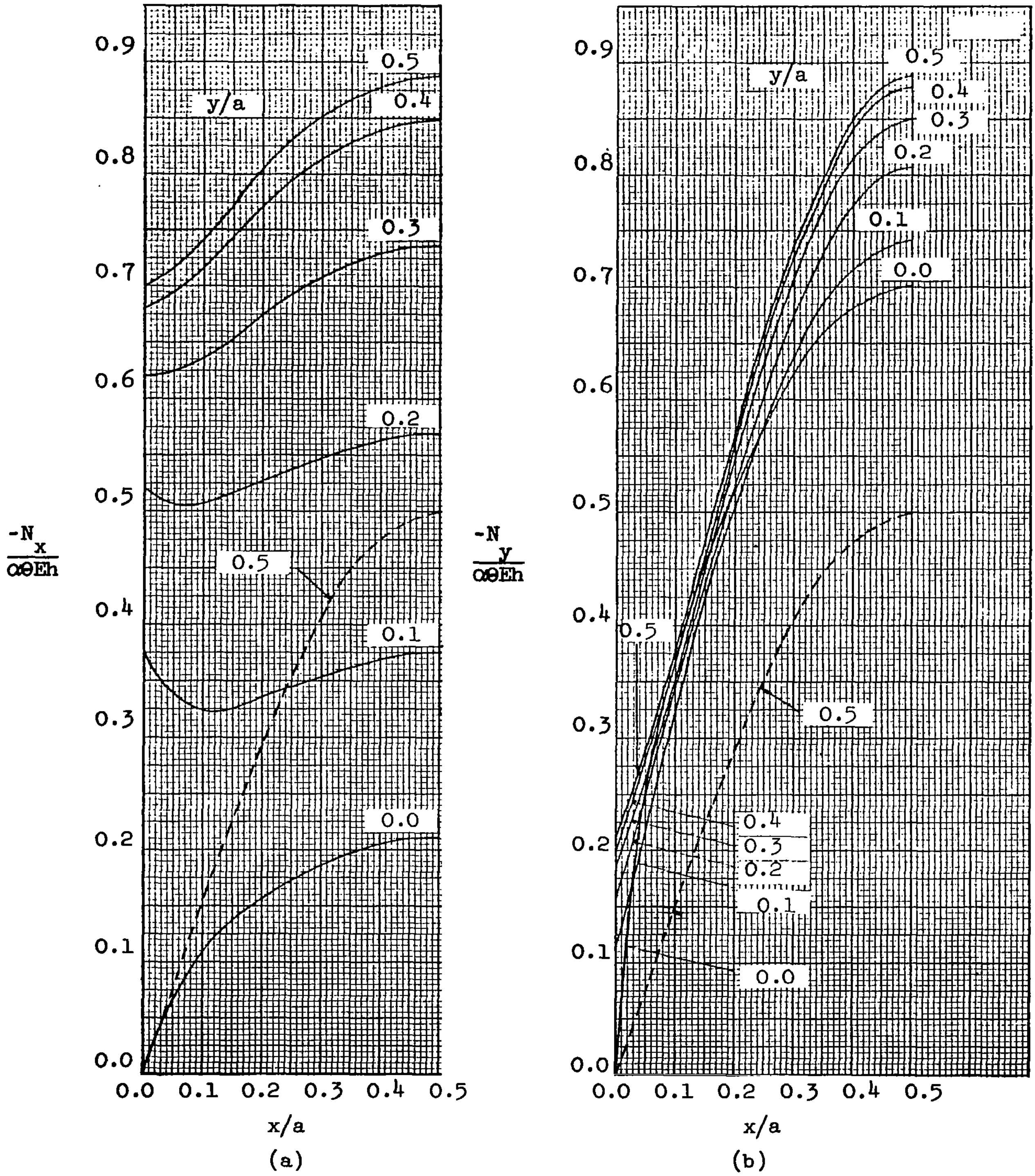
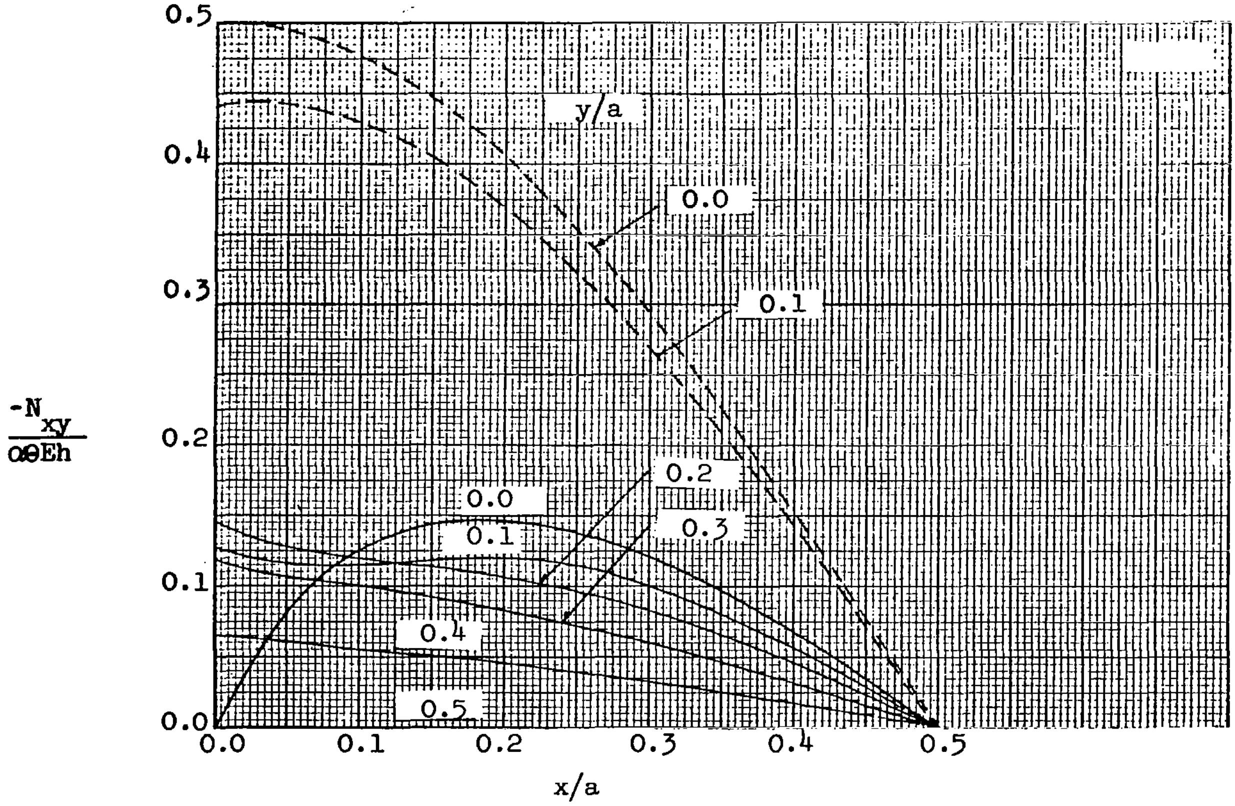


Figure 17. Plate stresses and stiffener tensions for the case of all four edges held straight, pillow-shaped temperature distribution, $\lambda \to 0$, and $\nu = 0.3$. (Dashed curves, from fig. 5a of ref. 1, are for the case of all edge stiffeners perfectly flexible.) ($M = 59$, $N = 59$).



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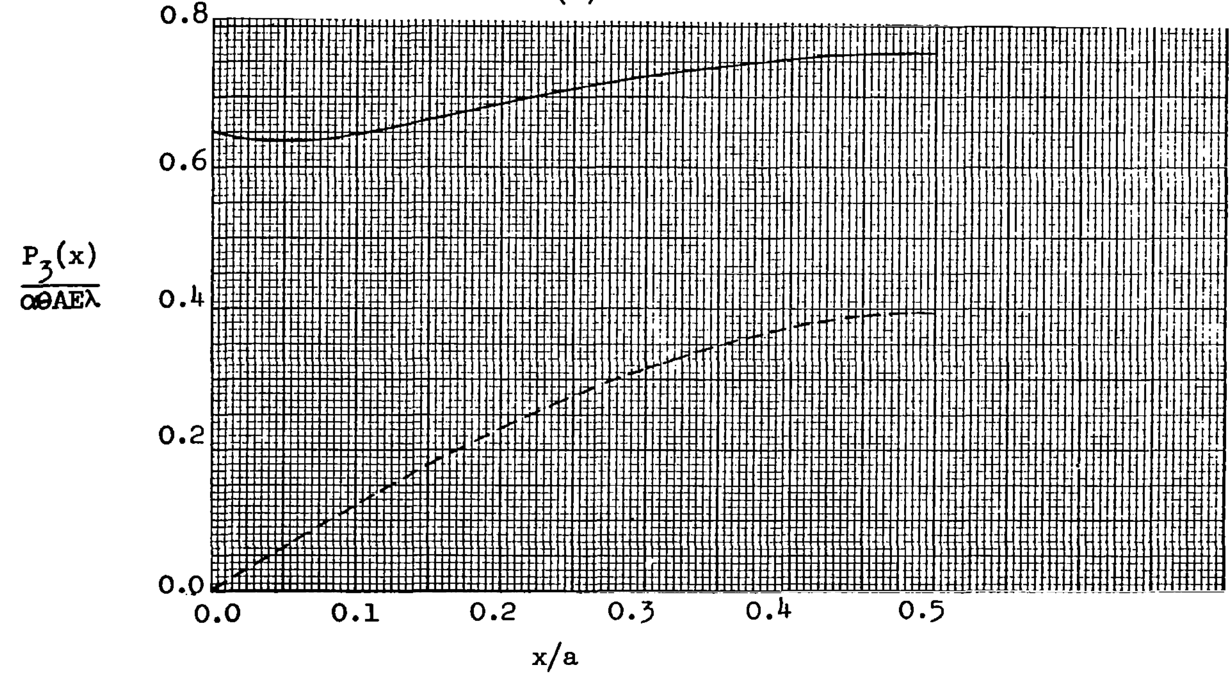

(d)

Figure 17. (continued)

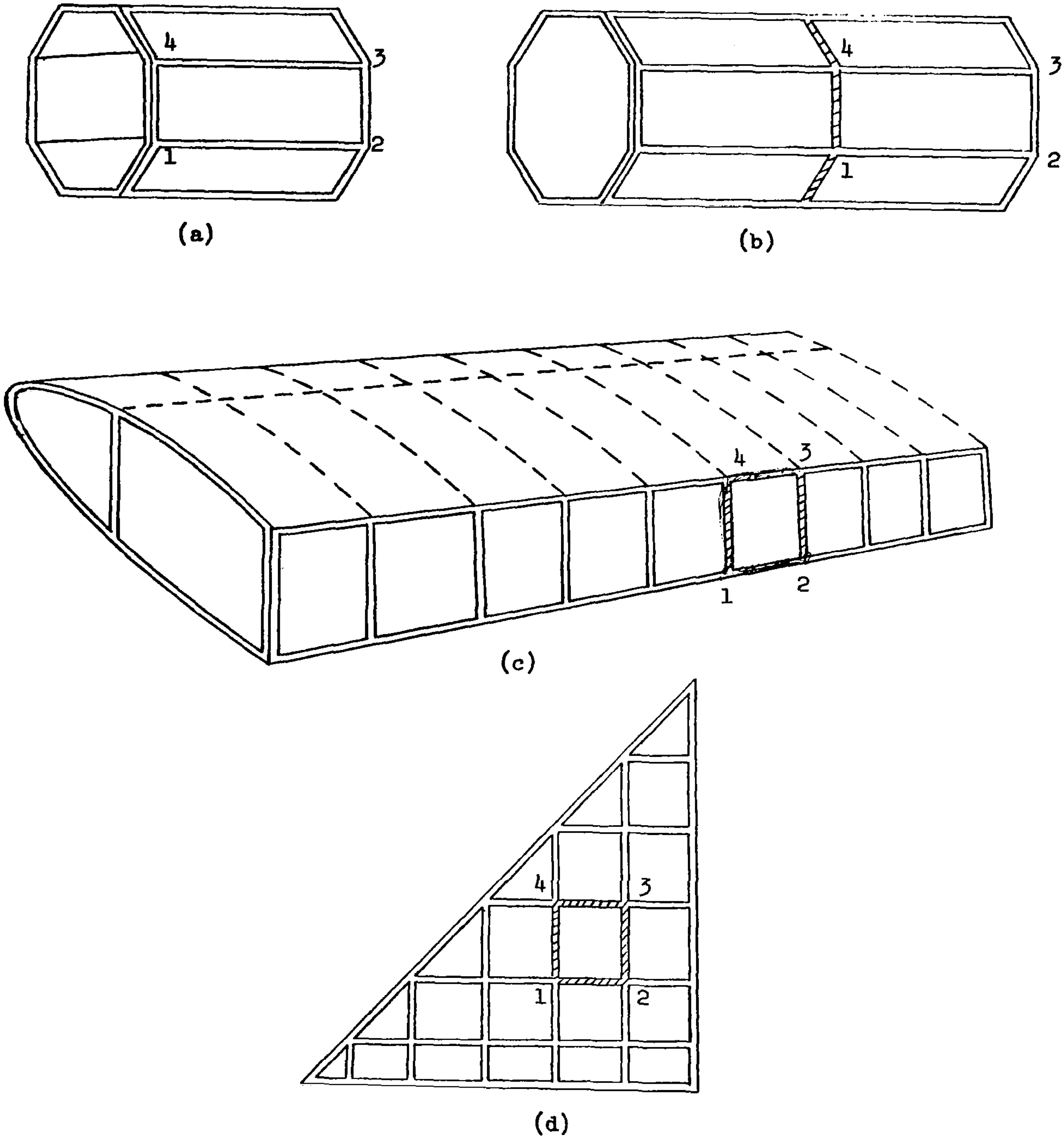

Figure 18. Types of structure to which the present analysis and references 1, 2 and 3 may be applicable.

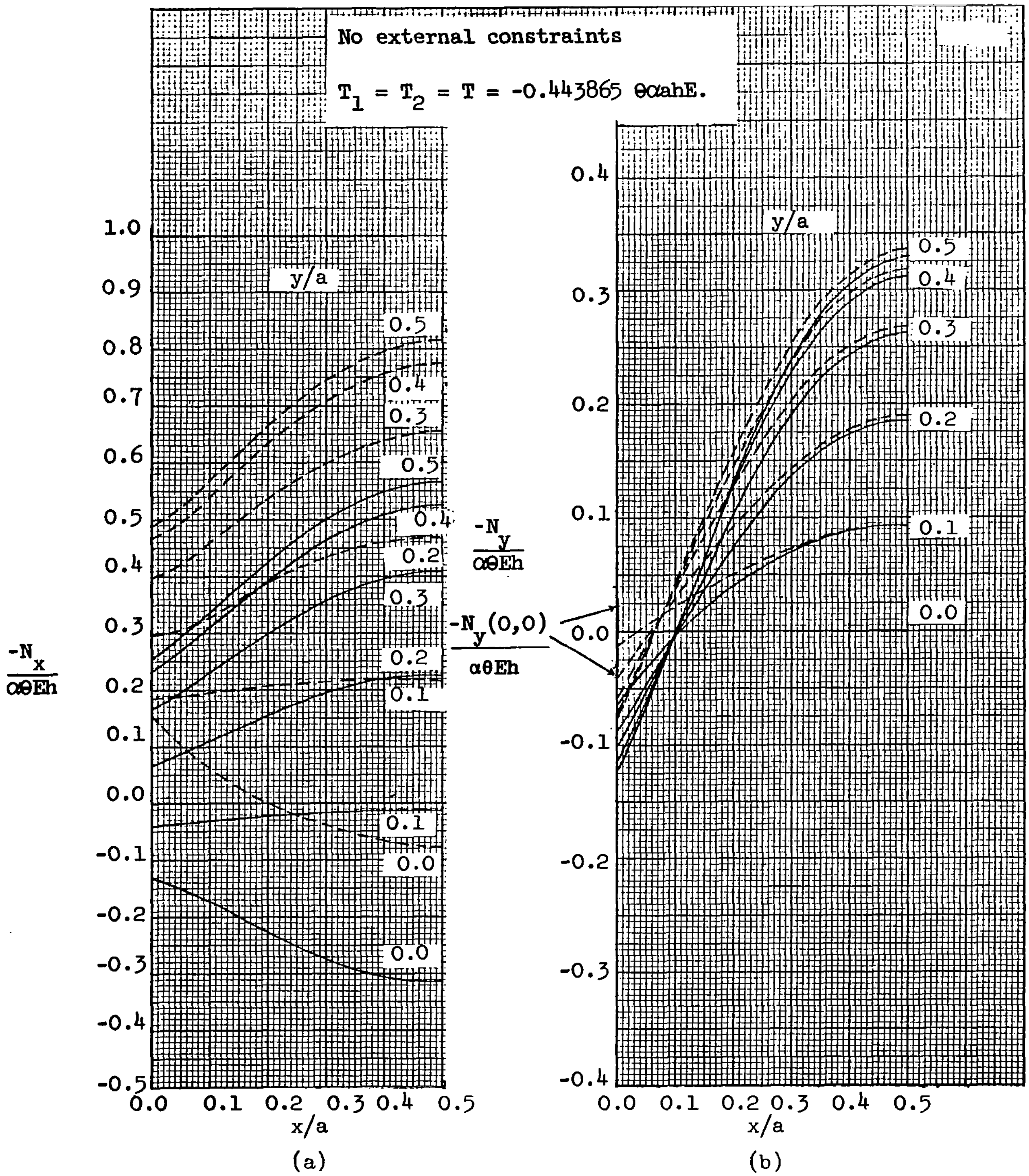

Figure 19. Results from figure 11 compared with those obtained when overall expansion in x-direction is completely prevented by the external constraining forces $T_1=T_2=-0.4439\ \theta\alpha ahE$.

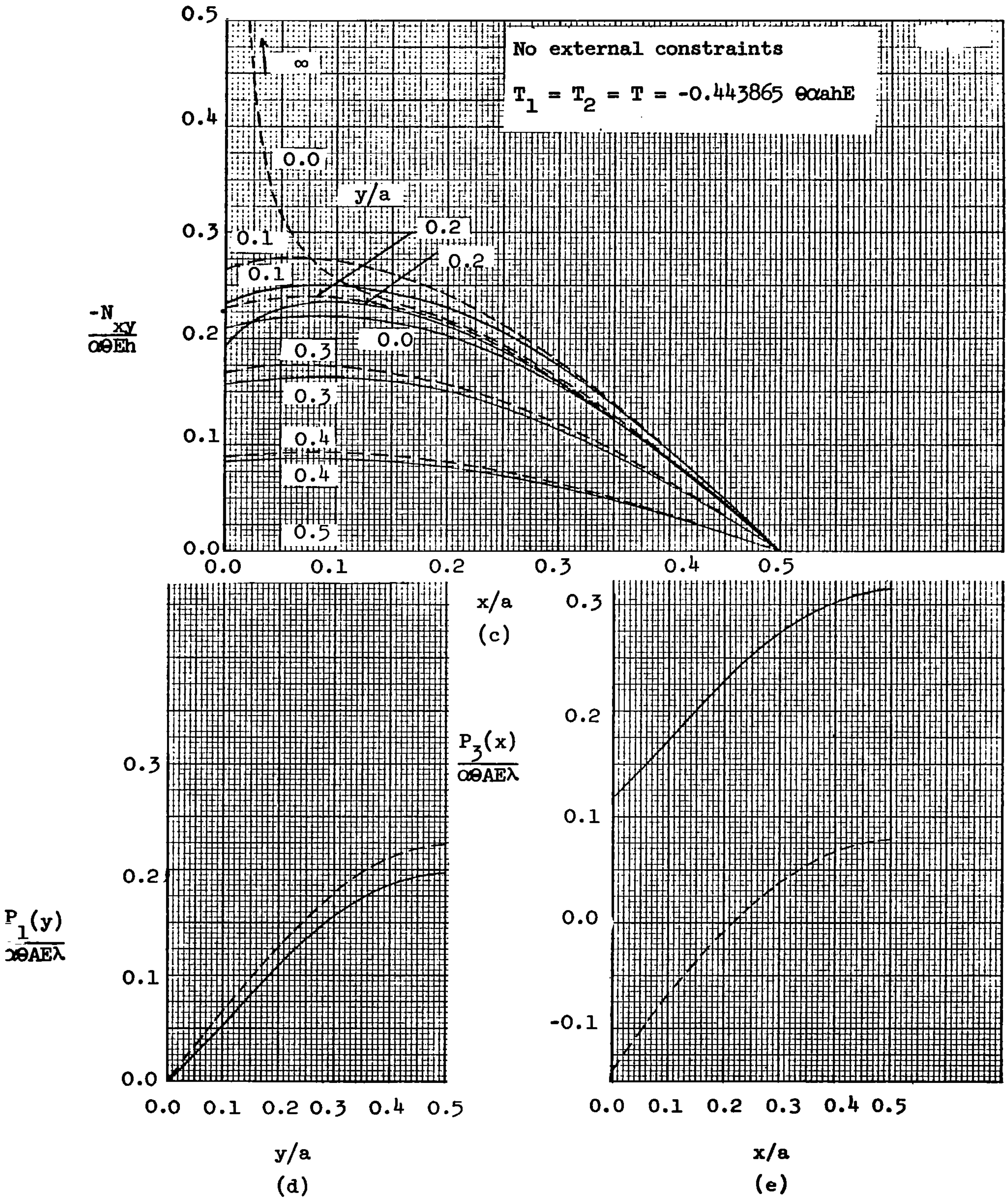

Figure 19. (continued)

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2. Dunn, F., "Exact Solution of a Thermal Stress Problem Using Double Fourier Series." Syracuse University Master's Thesis, January, 1965.

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5. Green, A. E., "Double Fourier Series and Boundary Value Problems." Proceedings of the Cambridge Philosophical Society, vol. 40, part 3, 1944, pp. 222-228.

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