SENSITIVITY AND STATE-VARIABLE FEEDBACK

Prepared Under Grant NGR-03-002-115
National Aeronautics and Space Administration

by

Robert C. White

Nuclear Engineering Department
The University of Arizona
Tucson, Arizona
July 1967

Engineering Experiment Station
The University of Arizona
College of Engineering
Tucson, Arizona

TABLE OF CONTENTS

LIST OF FIGURES

Figure

LIST OF FIGURES

Figure

ABSTRACT

Two new time-domain sensitivity measures, integral sensitivity and peak sensitivity, are defined in terms of the sensitivity function. A relation between integral sensitivity and classical frequency-domain sensitivity is established, and the generation of classical sensitivities, sensitivity functions, peak sensitivity, and integral sensitivity is discussed. Classical sensitivity is employed in a comparison of the sensitivity properties of linear control systems designed by two methods: series compensation and state-variable feedback. It is shown that under certain conditions the system designed by feeding back all of the state variables may be expected to be less sensitive than the series compensated system. A modification of state-variable feedback, the H-equivalent system, is considered in further attempt to reduce sensitivity to parameter changes. Several examples are presented to illustrate the theory.

CHAPTER I

INTRODUCTION

The need to consider the sensitivity properties of a control system arises from two general sources. While the system is in operation, there may be variations in components because of aging, environmental changes, etc. Secondly, it may be necessary to design a controller for a system without having an accurate knowledge of the parameters of the fixed plant. These problems have motivated a search for design methods that yield systems for which the performance is insensitive to variations in system parameters.

In order to evaluate these design methods, it is necessary to have quantitative sensitivity measures, many of which have been defined in the literature. The first definition of "classical sensitivity" was given in early work on the theory of feedback systems by Bode (1945). In fact, reduction of the effects of component variations on system performance was a primary motivation for the use of feedback. Variations of Bode's frequency domain definition of sensitivity have been used in further studies by Horowitz (1963) and Haddad and Truxal (1964). Kalman (1964) has used classical sensitivity to demonstrate a link between the theory of optimal control and classical control theory. Sensitivity in terms of pole and zero variations is discussed by Horowitz (1963), and has been used in the analysis of high order systems by Van Ness, et. al. (1965). A time-domain measure of sensitivity and its application to control systems analysis is discussed by Tomovic (1964).

This thesis is an attempt to study the sensitivity properties of a class of linear systems. The systems to be considered are non-time varying and have a single input R(s) and a single output Y(s). A vector differential equation of the form

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,r(t) \tag{1.1}$$

may be used to characterize the dynamics of the system. However, the sensitivity properties of a system depend on its topology, which is not described by Eq. (1.1). Therefore, the systems to be studied are defined in terms of block diagrams.

The problem to be solved is of the following form. A given fixed plant, which is unalterable internally, is specified by a transfer function $G_p(s)$. It is assumed that the state variables of $G_p(s)$ are measurable. Also specified is a closed-loop transfer function, Y(s)/R(s) = W(s), for the desired system. The general problem is to find a method for compensating the plant so as to yield W(s) in such a way that the sensitivity of the system performance with respect to changes in the parameters of the system is a minimum.

The design procedure to be investigated here is the method of obtaining W(s) by feeding back all of the state variables. A detailed discussion of this method is presented by Schultz and Melsa (1967). Here, the state-variable feedback system is compared to the system which realizes the same W(s) by series compensation. The use of series compensation to realize a specific W(s) is known as the Guillemin-Truxal method, which is described in Chapter 5 of Truxal (1955). Thus, given a fixed plant $G_p(s)$ any specified closed-loop transfer function W(s) may be obtained by either of the two methods. In this work the sensitivity properties of the resulting systems are examined. An extension of the state-variable feedback design is also investigated.

It is desired to find a general method of synthesis which yields W(s) with minimum sensitivity of the system performance with respect to parameter variations. Hence, a single measure of sensitivity and a single criterion of system performance must be defined. Then the solution based strictly on these definitions may be sought. However, such a procedure may lead to solutions which are impractical. To illustrate, a system may be designed such that the sensitivity of its performance with respect to a differential change in some parameter is a minimum (in some sense). But a finite change in the same parameter may result in instability. Such a case is demonstrated in Chapter V. Therefore, while attempting to find a design method based on precise definitions of sensitivity and performance criteria, the engineer must keep in mind an overall view of the nature of the system.

In Chapter II several definitions of sensitivity from the literature are discussed, and two new sensitivity measures are defined. The generation of sensitivity measures is the subject of Chapter III. Chapter IV is a general discussion of the sensitivity properties of systems designed by cascade compensation, and by feeding back the state variables. In Chapter V several numerical examples are presented, and some conclusions are stated in Chapter VI.

It is found that a system designed by feeding back all of the state variables may be expected to be less sensitive to parameter changes than the series compensated system.

CHAPTER II

SENSITIVITY MEASURES

In this chapter several sensitivity measures are discussed in relation to the type of systems to be studied here. A sensitivity measure should incorporate two features. It should be mathematically tractable, in order that its usefulness is not limited by computational problems. Also, it must be physically meaningful in relation to the performance of the system. In particular, the sensitivity measure should relate to the performance criteria which are used to design the system. The systems to be discussed in this thesis are designed for a specific closed-loop transfer function, $W(s) = Y(s)/R(s)$. Since $W(s)$ is usually chosen so as to yield a desired response to a step input, a meaningful sensitivity measure for this type of system should indicate how the step response is affected by parameter changes.

2.1 Root Sensitivity

A sensitivity measure which has been used frequently in the analysis of control systems and circuits is root sensitivity. This measure estimates the effect of a change in a system parameter on the positions of the poles of the closed-loop system. The interpretation of the results of an analysis using root sensitivity depends on the correspondence between closed-loop pole locations and the characteristics of the transient response. The control engineer gains by experience an intuitive notion of this correspondence, but for a complicated system, where many pole locations change with variations in a parameter, this correspondence may not be clear. Also, except in the simplest cases, the relation between the changes in pole locations and transient response, which one can obtain by inspection, is only qualitative. For these reasons root sensitivity was not used for the problems considered here.

2.2 Classical Sensitivity

The expression given here for classical sensitivity is the definition from Truxal (1955). The (classical) sensitivity of a function T(s, λ) with respect to a parameter λ may be defined as:

$$S_{\lambda}^{T} = S_{\lambda}^{T}(s) = \frac{d\ \ell nT}{d\ \ell n\lambda} \tag{2.1}$$

$$= \frac{dT/T}{d\lambda/\lambda}$$

$$= \frac{\lambda}{T}\frac{dT}{d\lambda} \tag{2.2}$$

For $Y(s)/R(s) = W(s)$, $S_{\lambda}^{W}(s)$ is a measure of the percentage change in W(s) for a percentage change in a parameter λ. A physical interpretation of S_{λ}^{W} is difficult, because S_{λ}^{W} is a function of the complex variable s. However, it is shown that $S_{\lambda}^{W}(j\omega)$ is related to a sensitivity measure which is used extensively in this study. Therefore, some formulas for classical sensitivity are presented here.

Consider the single-loop feedback system of Fig. 2.1. The (classical) sensitivity of the closed-loop transfer function with respect to G is:

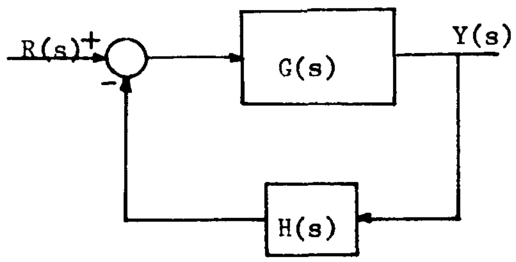


Figure 2.1 A single-loop control system.

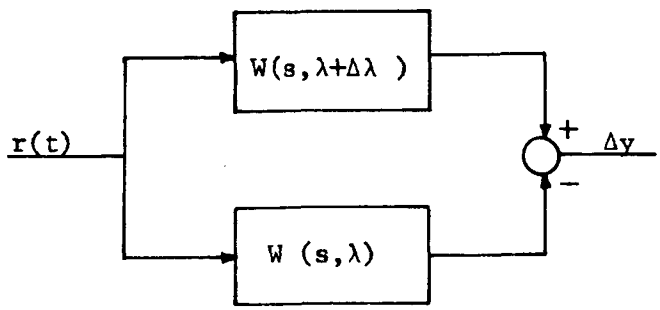


Figure 2.2 An experiment to illustrate the definition of the sensitivity function.

$$S_G^W = \frac{G}{W}\frac{dW}{dG}$$

$$= \frac{G}{W}\frac{d}{dG}\left[\frac{G}{1 + GH}\right]$$

$$= \frac{1}{1 + GH} \tag{2.3}$$

$$\simeq \frac{1}{GH} \text{ if } |GH| >> 1.$$

This result expresses the well-known fact that increasing the loop gain of a system reduces the effects of variations of elements in the forward path. This fact provides a precise link between classical control theory and the theory of optimal control. For the system of Fig. 2.1, the quantity $F(s) = 1 + GH(s)$ is called the return difference. Kalman (1964) has shown that the control law for a wide class of linear systems is optimal if and only if $|F(j\omega)| > 1$ for all real ω. Thus, it might be said that an optimal system is an insensitive system, and vice versa.

The sensitivity of W(s) with respect to H(s) is:

$$S_H^W = \frac{H}{W}\frac{dW}{dH}$$

$$= \frac{GH}{1 + GH} \tag{2.4}$$

It is seen that for a loop gain much greater than unity component variations in the feedback path are undiminished in their effect on W(s).

Suppose λ is a parameter which appears only in a component block G.

$$S_\lambda^W = \frac{\lambda}{W}\frac{dW}{d\lambda}$$

$$= \frac{\lambda}{W}\frac{dW}{dG}\frac{dG}{d\lambda}\frac{G}{G}$$

$$= \frac{G}{W}\frac{dW}{dG} \times \frac{\lambda}{G}\frac{dG}{d\lambda}$$

$$= S_G^W\, S_\lambda^G.$$

Consider the function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

Then $S_K^G = 1$ (2.5)

$$S_{p_i}^G = \frac{-p_i}{s + p_i} \tag{2.6}$$

$$S_{z_i}^G = \frac{z_i}{s + z_i} \tag{2.7}$$

It is clear from the above calculations that classical sensitivities are relatively easy to evaluate. This feature, along with the fact that they are related to another sensitivity measure which is closely connected with the step response of the system, makes classical sensitivity a useful tool in the analysis to follow.

2.3 Sensitivity Functions

The sensitivity measure discussed here is defined by Tomovic (1964). Let λ be a system parameter with a nominal value λ_o. Let $y(t, \lambda)$ be the response of the system to a step input. Then for a change in the parameter λ, the step response may be expanded in a Taylor series.

$$y(t, \lambda_o + \Delta\lambda) = y(t, \lambda_o) + \left.\frac{dy(t, \lambda)}{d\lambda}\right|_{\lambda_o} \Delta\lambda + \left.\frac{d^2y(t, \lambda)}{d\lambda^2}\right|_{\lambda_o} \frac{(\Delta\lambda)^2}{2!} + \ldots$$

$\left.\frac{dy(t, \lambda)}{d\lambda}\right|_{\lambda_o}$, which is a function of time, is a linear approximation of the change in $y(t, \lambda)$, at the time t, resulting from a change $\Delta\lambda$ in the parameter λ from its nominal value λ_o. Usually it is desired to have an estimate of the change in $y(t, \lambda)$ for a percentage change in λ. Therefore, the sensitivity of the system with respect to the parameter λ is defined as:

$$u_\lambda (t) = \frac{dy(t, \lambda)}{\frac{d\lambda}{\lambda}} \tag{2.8}$$

$u_\lambda(t)$ is called the sensitivity function for the parameter λ. The physical meaning of $u_\lambda(t)$ may become more concrete if the situation pictured in Fig. 2.2 is considered. A step input is applied simultaneously to two systems. In one system the parameter under consideration has a value λ, while in the other system the parameter has a value $\lambda + \Delta\lambda$. The difference between the outputs of the systems is:

$$\Delta y = y(t, \lambda + \Delta\lambda) - y(t, \lambda)$$

Division by the normalized change in the parameter yields:

$$\frac{\Delta y}{\Delta\lambda/\lambda} = \frac{y(t, \lambda + \Delta\lambda) - y(t, \lambda)}{\Delta\lambda/\lambda}$$

Under the assumption that the following limit exists,

$$\lim_{\Delta\lambda\to 0} \frac{\Delta y}{\Delta\lambda/\lambda} = \frac{dy(t, \lambda)}{d\lambda/\lambda} = u_\lambda (t).$$

A simple example illustrates the interpretation of sensitivity functions. Fig. 2.3 shows the block diagram for a control system for

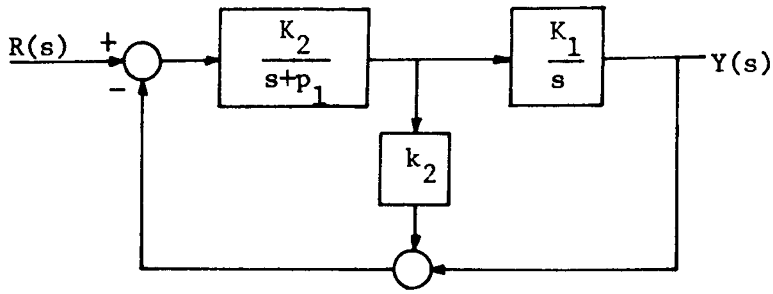


Figure 2.3 A second order control system.

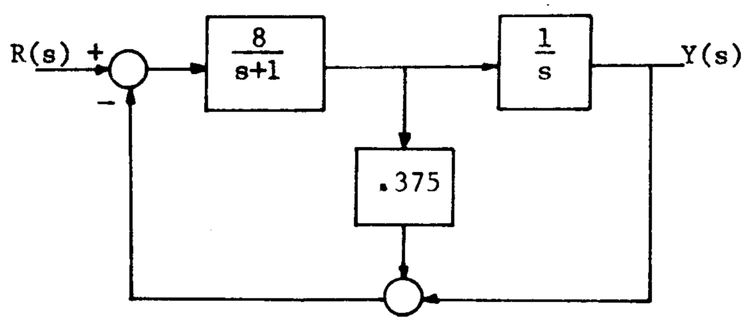


Figure 2.4 A second order control system.

which W(s) is required to be:

$$W(s) = \frac{8}{s^2 + 4s + 8}$$

Fig. 2.4 is one possible realization of W(s). The response (y(t)) of this system for a step input and the sensitivity functions ($u_{K_1}(t)$, $u_{K_2}(t)$, $u_{k_2}(t)$) are plotted in Fig. 2.5. Since the sensitivity functions approach zero as $t \to \infty$, K_1, K_2, and k_2 have no effect on the final value of y(t). From the fact that the magnitudes of $u_{K_1}(t)$ and $u_{K_2}(t)$ are largest during the time when the output is rising toward its final value, it may be concluded that K_1 and K_2 affect the rise time of the system, with an increase in K_1 or K_2 decreasing the rise time. Also, a change in K_2 has a smaller effect on the response than does a change in K_1. The curve of $u_{k_2}(t)$ indicates that k_2 affects the response in the region close to its peak value, so that an increase in k_2 decreases the overshoot. This behavior should be expected, since k_2 is the coefficient of rate feedback. Fig. 2.6 shows the actual affects of 20% increases in K_1 and k_2 for the particular system of Fig. 2.4. From this figure it is seen that the qualitative effects of changes in K_1 and k_2 are as predicted.

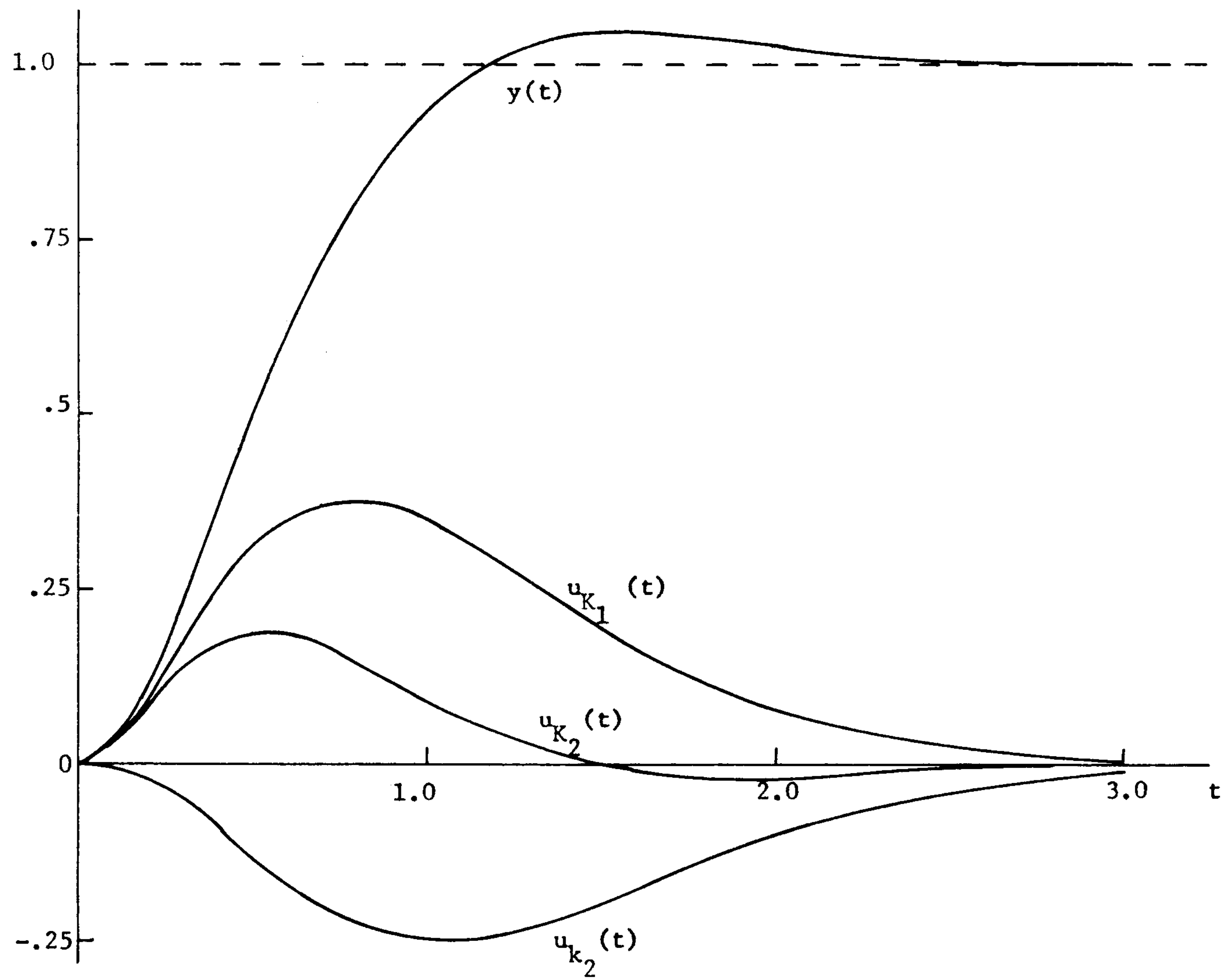


Figure 2.5 Sensitivity functions for the example.

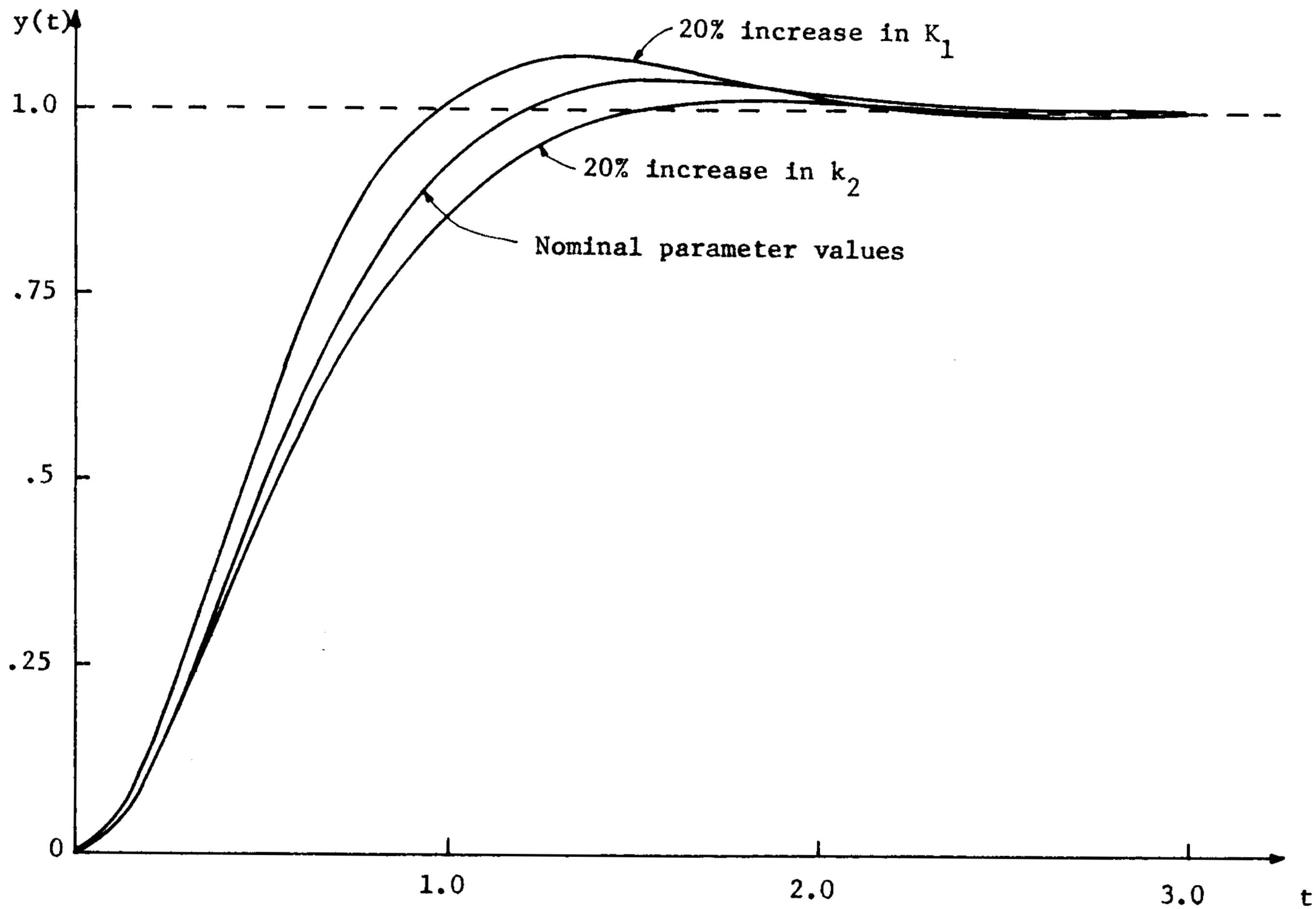


Figure 2.6 The effects of parameter variations on the step response.

2.4 Peak Sensitivity and Integral Sensitivity

The sensitivity functions have the desirable features of relating directly to transient response and indicating just how much each part of the response is affected by the parameters. However, this wealth of information is not in a compact form, since the sensitivity functions are functions of time. In an attempt to find a measure of sensitivity which relates directly to transient response and yet is more concise in form, two new sensitivity measures are defined here.

The peak sensitivity of the system with respect to a parameter is defined as

$$u^*_\lambda = u_\lambda (T) \tag{2.9}$$

where T = the value of t such that $|u_\lambda (t)|$ is a maximum. u_λ gives an estimate of the maximum change in the response (at time T) for a + 1% change in λ.

The integral sensitivity of the system with respect to a parameter λ is defined as

$$S_\lambda = \int_0^\infty u_\lambda^2 (t)\, dt \tag{2.10}$$

when this integral exists. Unless λ is a parameter affecting the final value of y(t), $u_\lambda (t)$ approaches zero as $t \to \infty$. It is shown in Chapter III that $u_\lambda (t)$ is the response of a linear system. Then if $u_\lambda (t) \to 0$ as $t \to \infty$, it approaches zero in an exponential fashion. In such a case $u_\lambda^2(t)$ is the sum of decaying exponentials, so that the above integral does exist. Therefore, it is concluded that S_λ exists if λ does not affect the final value of y(t). If the final value of y(t) does depend on λ, the integral sensitivity with respect to λ is not defined. The sensitivity of the system with respect to such a parameter might be characterized by the peak sensitivity and the final value of the sensitivity function $u_\lambda(t)$.

The definition given for integral sensitivity was chosen as a measure of the overall influence of a parameter λ on the step response. For the integrand, $u_\lambda^2(t)$ was preferred over $|u_\lambda(t)|$ for two reasons. The squared quantity weights large values of u (t) more heavily than small values. Also, the integrand $u_\lambda^2(t)$ allows the use of Parseval's Theorem in the evaluation of the integral. This is discussed in the next chapter.

Clearly, in obtaining more concise sensitivity measures, some information as to the way in which λ affects the response is lost. The sensitivity functions are useful in particular cases where this information is important.

CHAPTER III

GENERATION OF SENSITIVITY MEASURES

The purpose of this chapter is to show how sensitivity functions, peak sensitivities, and integral sensitivities may be found. To generate these sensitivity measures, an analog or digital computer is required, while classical sensitivities can be found easily from a block diagram of the system. It is shown that classical sensitivity and integral sensitivity are connected by a relationship which enables one to predict the nature of sensitivity functions and integral sensitivity from a knowledge of classical sensitivity.

3.1 The Relation between S_λ and S_λ^W.

From the definition of the sensitivity function,

$$u_\lambda(t) = \frac{dy(t)}{\frac{d\lambda}{\lambda}}$$

$$L\{u_\lambda(t)\} = U_\lambda(s) = \frac{dY(s)}{\frac{d\lambda}{\lambda}}$$

$$= R(s)\,\frac{d(Y(s)/R(s))}{d\lambda/\lambda}$$

for R(s) not a function of λ. Since the sensitivity functions are defined in Chapter II for a step input, $R(s) = \frac{1}{s}$. Then,

$$U_\lambda(s) = \frac{1}{s}\,\frac{dW(s)}{d\;\lambda/\lambda}\,\frac{W(s)}{W(s)}$$

$$= \frac{1}{s}\,W(s)\,\frac{\lambda}{W(s)}\,\frac{dW(s)}{d\lambda}$$

$$= \frac{1}{s}\,W(s)\;S_\lambda^W \tag{3.1}$$

For $s = j\omega$,

$$|U_\lambda(j\omega)| = \frac{|W(j\omega)|}{|\omega|}\;|S_\lambda^W\;(j\omega)|.$$

Now,

$$S_\lambda = \int_0^\infty u_\lambda^2(t)\;dt$$

$$= \int_{-\infty}^{\infty} u_\lambda^2(t)\;dt$$

since $u(t) = 0$ for $t < 0$. Then using Parseval's Theorem,

$$S_\lambda = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} U(s)\;U(-s)\;ds \tag{3.2}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(j\omega)|^2\;d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|W(j\omega)|^2}{\omega^2}\;\left|S_\lambda^W\;(j\omega)\right|^2\;d\omega \tag{3.3}$$

Eq. (3.3) shows the relation between integral sensitivity S_λ and classical sensitivity S_λ^W. Clearly, reducing $|S_\lambda^W(j\omega)|$ reduces S_λ.

In this thesis the systems to be studied have identical transfer functions W(s), but different classical sensitivities with respect to the same parameter. Then from Eq. (3.3) it is seen that the differences between integral sensitivities for such systems are determined by differences in their classical sensitivities. This link between classical and integral sensitivity is important, because classical sensitivities are easily found from a block diagram of the system, while the generation of S_λ requires a computer. For this reason it is desirable to have a method for finding classical sensitivities.

3.2 Generation of Classical Sensitivities

The procedure given here for finding classical sensitivities from the system block diagram is essentially the same as the method described by Tomovic (1964). The block diagram of a control system is shown in Fig. 3.1, where the component blocks of particular interest are $G_i(s)$ and $H_i(s)$. For the case where all of the $G_j(s)$ are first order and the $H_j(s) = k_j$ (constants), Fig. 3.1 is a block diagram of a system where all of the state variables are fed back. However, the expressions derived here for classical sensitivities are valid for $G_j(s)$ and $H_j(s)$ of any order. Fig. 3.2 shows a reduction of the block diagram for the purpose of calculating $S^W_{G_i}$ and $S^W_{H_i}$. L(s) is the transfer function from E_i to B_i. (These variables are defined in Fig. 3.1.) M(s) represents the sum of the feedback through the paths containing $H_1, H_2, \ldots, H_{i-1}$ when these paths are referred to the output. N(s) is the transfer function from the output of G_i to the system output. These quantities are defined by the following equations.

$$L(s) = \frac{B_i(s)}{E_i(s)} = \frac{\prod_{j=i+1}^{n} G_j}{1 + \sum_{j=i+1}^{n} \left(H_j \prod_{\ell=j}^{n} G_\ell\right)} \tag{3.4}$$

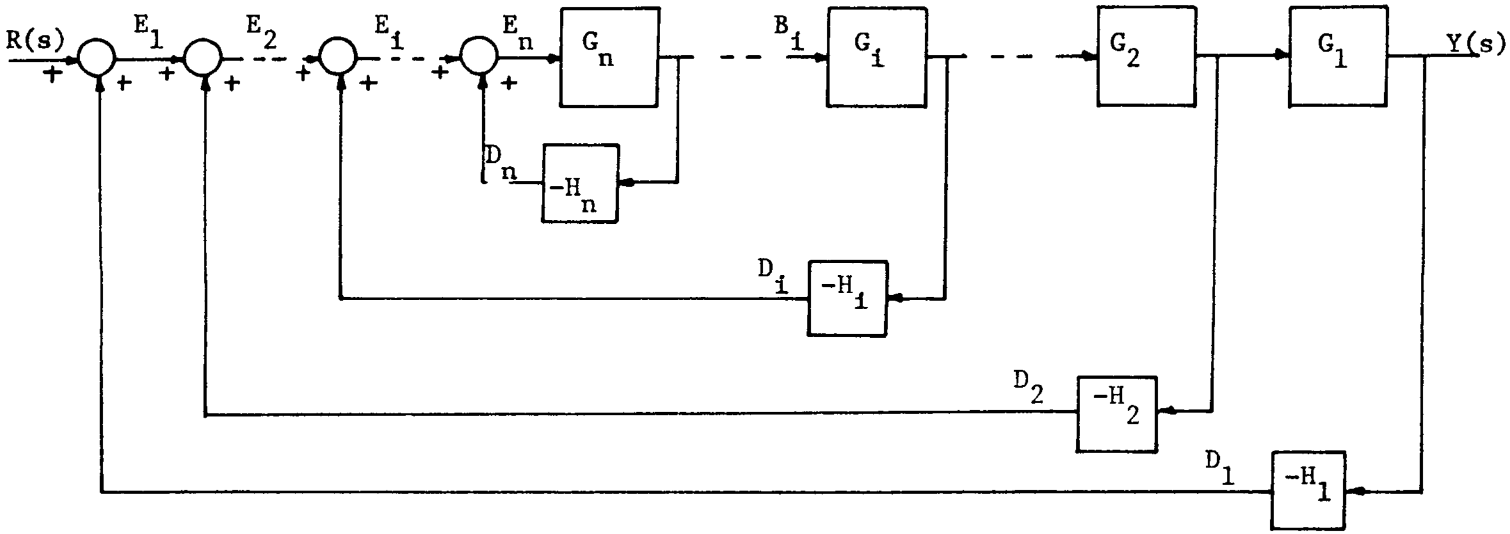


Figure 3.1 The block diagram of a control system.

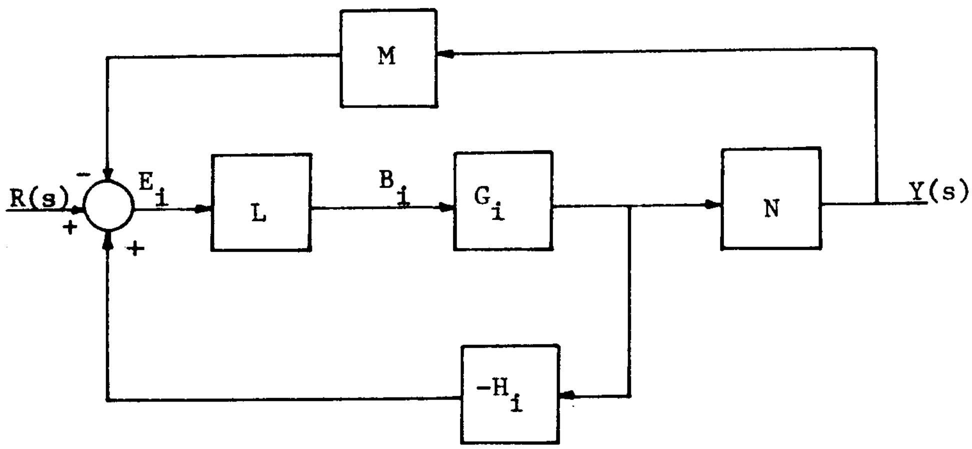


Figure 3.2 A reduced block diagram.

$$M(s) = H_1 + \sum_{j=2}^{i-1} (H_j \prod_{\ell=1}^{j-1} G_\ell) \tag{3.5}$$

$$N(s) = \prod_{j=1}^{i-1} G_j \tag{3.6}$$

Then,

$$\frac{Y(s)}{R(s)} = W(s) = \frac{G_i LN}{1 + G_i L \; [H_i + NM]} \tag{3.7}$$

$$= \frac{G_i LN}{1 + G_i LF}$$

where $F = H_i + NM$.

Then the sensitivity of $W(s)$ with respect to $G_i(s)$ is

$$S_{G_i}^{W} = \frac{G_i(s)}{W(s)} \frac{dW(s)}{dG_i(s)}$$

$$= \frac{G_i}{W} LN \frac{1 + G_i LF - G_i LF}{[1 + G_i LF]^2}$$

$$= \frac{1}{1 + G_i LF} .$$

The transfer function from the input to $E_i(s)$ is:

$$\frac{E_i(s)}{R(s)} = \frac{1}{1 + G_i LN \; [M + \frac{H_i}{N}]} \tag{3.8}$$

$$= \frac{1}{1 + G_i L \; [H_i + NM]}$$

$$= \frac{1}{1 + G_i LF}$$

$$= S_{G_i}^{W} \tag{3.9}$$

Thus, the classical sensitivity of the system with respect to $G_i(s)$ is just the transfer function from the input to $E_i(s)$. The sensitivity of W(s) with respect to $H_i(s)$ is:

$$S_{H_i}^{W} = \frac{H_i(s)}{W(s)} \frac{dW(s)}{dH_i(s)}$$

In order to simplify calculations, let G(s) be defined as:

$$G(s) = \frac{LG_i}{1 + LG_i\frac{M}{N}}.$$

Then, $W(s) = N \frac{G}{1 + GH_i}$

$$S_{H_i}^{W} = \frac{H_i}{W} NG \left[\frac{-G}{(1+ GH_i)^2}\right]$$

$$= \frac{-GH_i}{1 + GH_i}$$

$$= \frac{D_i(s)}{R(s)} \tag{3.10}$$

The classical sensitivity of the system with respect to $H_i(s)$ is the transfer function from the input to $D_i(s)$.

Eqs. (3.9) and (3.10) for classical sensitivities only apply to the system of Fig. 3.1. However, the series compensated system is easily treated as a special case. A unity feedback system with a fixed plant $G_p(s)$ and a series compensator $G_c(s)$ is shown in Fig. 3.3. Since there is no feedback from the output of $G_c(s)$, the transfer functions in the forward path may be combined. Let $G_1(s) = G_c(s)\ G_p(s)$. Then the

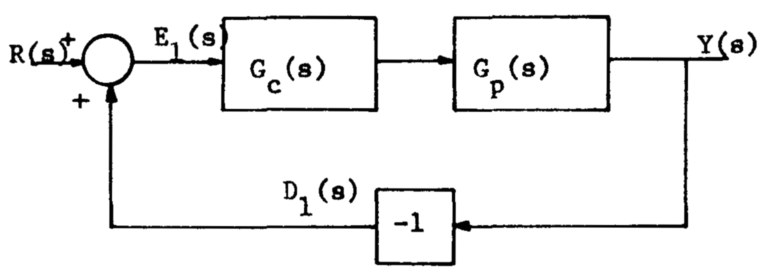


Figure 3.3 A series compensated system.

series compensated system of Fig. 3.3 is a special case of the system of Fig. 3.1, with only one block in the forward path ($G_1(s)$) and with $H_1(s) = 1$. Now, from Eqs. (3.9) and (3.10),

$$S_{G_1}^{W} = \frac{E_1(s)}{R(s)} = \frac{1}{1 + G_1(s)}$$

$$= \frac{1}{1 + G_c(s)G_p(s)}$$

$$S_{H_1}^{W} = \frac{D_1(s)}{R(s)} = \frac{-G_1(s)}{1 + G_1(s)}$$

$$= \frac{-G_c(s)\ G_p(s)}{1 + G_c(s)\ G_p(s)}$$

For system configurations which are not special cases of the diagram in Fig. 3.1, the classical sensitivities can be found by direct application of the definition (Eq. (2.2)).

3.3 Generation of Sensitivity Functions, Peak Sensitivity, and Integral Sensitivity.

In Section 3.1 S_λ was expressed as an integral in the form of Eq. (3.2). For the case where U(s) is a ratio of polynomials, the integral has been tabulated as a function of the coefficients of the polynomials (Newton, et.al. (1957)). However, the expressions for this integral become cumbersome rapidly as the order of U(s) increases. Since for an nth order system the order of U(s) is 2n, the evaluation of S_λ by Eq. (3.2) is impractical.

The method presented in Section 3.2 for finding classical sensitivities and Eq. (3.1) for $U_\lambda(s)$ indicate how sensitivity functions may be generated. Eq. (3.1) is repeated here:

$$U_\lambda(s) = \frac{1}{s} W(s)\ S_\lambda^W$$

If α is a parameter only of $G_i(s)$, then

$$U_\alpha(s) = \frac{1}{s} W(s)\ S_{G_i}^W\ S_\alpha^{G_i} \tag{3.11}$$

If β is a parameter only of $H_i(s)$, then

$$U_\beta(s) = \frac{1}{s} W(s)\ S_{H_i}^W\ S_\beta^{H_i} \tag{3.12}$$

The generation of $U_\alpha(s)$ and $U_\beta(s)$ is shown in Fig. 3.4. A step input is applied to a system with the transfer function W(s). The output is applied to the input of a second system (with transfer function W(s)) whose sensitivity is to be studied.

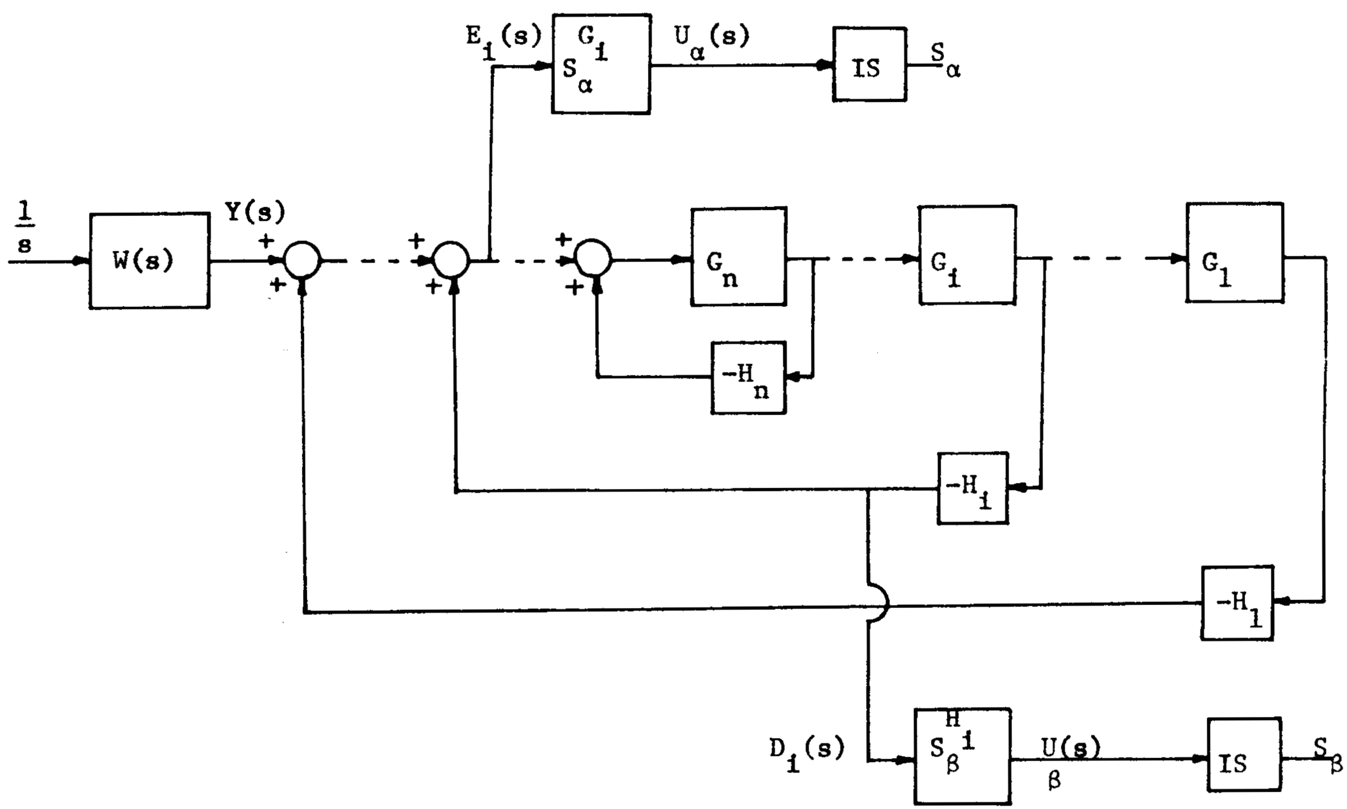


Figure 3.4 The generation of sensitivity functions and integral sensitivities.

The transfer functions $\frac{E_i(s)}{Y(s)}$ and $\frac{D_i(s)}{Y(s)}$ provide the terms $S^W_{G_i}$ and $S^W_{H_i}$ in Eqs. (3.11) and (3.12). The blocks labeled $S^{G_i}_\alpha$ and $S^{H_i}_\beta$ provide the corresponding terms in Eqs. (3.11) and (3.12) to complete the generation of $U_\alpha(s)$ and $U_\beta(s)$. For the cases where the parameters α and β are gains, poles, or zeros, $S^{G_i}_\alpha$ and $S^{H_i}_\beta$ are simple functions, as shown in Chapter II. Finally, the blocks labeled I. S. (Integral Squared) square the time functions $u_\alpha(t)$ and $u_\beta(t)$ and integrate to yield S_α and S_β. (The generation of the sensitivity functions is carried out in the time domain by a computer, but for convenience, the method is discussed using the transformed variables.)

For the example systems of Chapter V, a digital computer is used to generate the sensitivity functions, peak sensitivities, and integral sensitivities. For the 5th order system of Example 3 in Chapter V, the sensitivities with respect to eight parameters are found. The generation of the sensitivity functions, peak sensitivities, and integral sensitivities for each parameter leads to a system of equations of order 23. The computer time required for the solution is approximately 4 minutes.

It has been pointed out that the evaluation of S_λ from tables of the integral (Eq. (3.2)) is usually impractical. However, for the third order system of Example 1 in Chapter V, the integral sensitivities were found by this method. These results were compared to those obtained from a digital computer program, which approximately solves the differential equations for the integral sensitivities. The values obtained by the two methods agreed to within 0.3%.

CHAPTER IV

SENSITIVITY AND STATE-VARIABLE FEEDBACK

The sensitivity measures which have been discussed are used in this chapter and in Chapter V to study the sensitivity of some linear control systems. In the present chapter a slightly general discussion of the problem is attempted. Because sensitivity analysis in terms of sensitivity functions and integral sensitivity is practically limited to specific cases, much use is made of classical sensitivity.

4.1 Series Compensation and State Variable Feedback

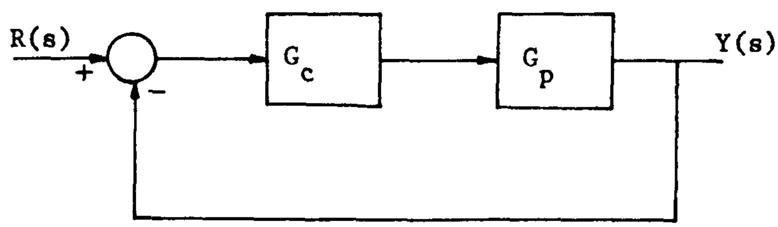
It is assumed that a given fixed plant is to be compensated in order to yield a desired closed-loop response. Figs. 4.1 and 4.2 indicate two approaches which may be used to solve the problem. The fixed plant is of order m, and has a transfer function

$$G_p(s) = G_1(s)G_2(s)\ldots G_m(s)$$

where the $G_i(s)$ are first order. In Fig. 4.1 a cascade compensator $G_c(s)$ has been used to realize the required W(s), which is of order n.

$$W(s) = \frac{G_c G_p}{1 + G_c G_p} \tag{4.1}$$

$G_c(s)$ may be found by the Guillemin-Truxal method discussed in Truxal (1955). In Fig. 4.2 W(s) is obtained by feeding back the state variables of the fixed plant and, if necessary, by adding first order series



$G_p(s) = G_1(s)G_2(s)\ldots G_m(s)$ (order m)

$G_{eq}(s) = G_c(s)G_p(s)$ (order n)

Figure 4.1 The series compensated system.

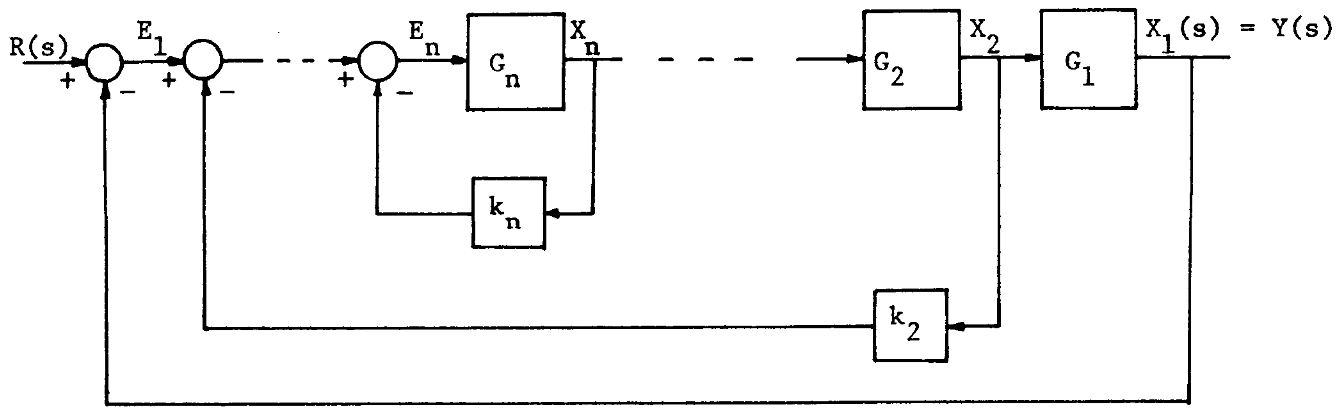


Figure 4.2 The state-variable feedback system.

compensating elements, whose state variables are also fed back. This method of design is described in detail by Schultz and Melsa (1967). The resulting system has the same nth order, closed-loop transfer function W(s) as the series compensated system. The following expression for W(s) of the state-variable feedback system is derived in the Appendix.

$$W(s) = \frac{G_1 G_2 \ldots G_n}{1 + k_1 G_1 G_2 \ldots G_n + k_2 G_2 \ldots G_n + \ldots + k_n G_n} \tag{4.2}$$

$$= \frac{\prod_{i=1}^{n} G_i}{1 + \sum_{\ell=1}^{n} \left[k_\ell \prod_{j=\ell}^{n} G_j \right]} \tag{4.3}$$

4.2 Sensitivities

The sensitivities of the two systems with respect to parameters in both the forward paths and the feedback paths are studied in this section. However, more attention is focused on the parameters in the forward path, expecially those in the fixed plant. This is because in most cases the designer is able to select compensating components with production tolerences which are small enough to avoid problems of sensitivity with respect to these components. Sensitivities with respect to the compensating elements should still be checked, however, in order to avoid a situation where the tolerances required are impractical.

Consider the state variable feedback system. Using Eq. (3.9),

$$S_{G_i}^{W} = \frac{E_i(s)}{R(s)}$$

It is shown in the Appendix that

$$S_{G_i}^{W} = \frac{1 + \sum_{j=1}^{n-i} \left[k_{i+j} \prod_{\ell = i+j}^{n} G_\ell \right]}{1 + \sum_{j=1}^{n} \left[k_j \prod_{\ell = j}^{n} G_\ell \right]} \tag{4.4}$$

For example in a third order system these sensitivities are:

$$S_{G_1}^{W} = \frac{1 + k_2 G_2 G_3 + k_3 G_3}{1 + k_1 G_1 G_2 G_3 + k_2 G_2 G_3 + k_3 G_3} \tag{4.5a}$$

$$S_{G_2}^{W} = \frac{1 + k_3 G_3}{1 + k_1 G_1 G_2 G_3 + k_2 G_2 G_3 + k_3 G_3} \tag{4.5b}$$

$$S_{G_3}^{W} = \frac{1}{1 + k_1 G_1 G_2 G_3 + k_2 G_2 G_3 + k_3 G_3} \tag{4.5c}$$

The denominators of $S^W_{G_i}$ do not depend on i, so the magnitudes of the $S^W_{G_i}$ may be compared by examining the numerators. For this discussion let

$$S^W_{G_i} = \frac{A_i(s)}{B(s)}$$

$$= \frac{A_i(j\omega)}{B(j\omega)} \qquad \text{for } s = j\omega.$$

For frequencies less than the system bandwidth, and if all $k_i > 0$, it may be expected that $|A_i(j\omega)|$ is smaller for larger values of i. In this case, from the discussion of the relation between classical sensitivity and integral sensitivity, it is expected that the S_{G_i} are smaller for larger values of i. Intuitively, one might predict this behavior from noticing that the $G_i(s)$ are more imbedded in feedback loops for larger values of i. For all of the examples studied with $k_i > 0$, it was found that S_{G_i} decreased as i increased. However, it is not always true that all of the k_i are positive. If one or more of the feedback coefficients are negative, it may be expected that for some value of i, $S_{G_{i+1}} > S_{G_i}$. An example of this situation is shown in Chapter V.

Consider now the series compensated system. Let $G_{eq}(s) = G_c(s)\, G_p(s)$. Then using the fact that the sensitivities for all blocks in cascade are equal,

$$S^W_{G_{eq}} = S^W_{G_c} = S^W_{G_i} = \frac{E(s)}{R(s)} \text{ for all i.}$$

Since the closed-loop transfer functions for the two systems are the same,

$$S^W_{G_{eq}} = \frac{E(s)}{R(s)} = \frac{E_1(s)}{R(s)} = S^W_{G_1}$$

where $E_1(s)$ is defined in Fig. 4.2, and $S^W_{G_1}$ refers to the state feedback system. Thus, the sensitivity of W(s) with respect to any block in the forward path of the series compensated system is equal to the sensitivity of W(s) with respect to G_1 in the system using state-variable feedback. Then for most cases $S^W_{G_i}$ is smaller for the state variable feedback system, since $S^W_{G_i}$ decreases as i increases in that system.

The sensitivities of the state-variable feedback system with respect to the feedback coefficients, k_i, are now considered. In the Appendix it is shown that

$$S^W_{k_i} = \frac{-k_i \prod\limits_{j=i}^{n} G_j}{1 + \sum\limits_{j=1}^{n} \left[k_j \prod\limits_{\ell=j}^{n} G_\ell \right]} \tag{4.6}$$

$$= \frac{-k_i \prod\limits_{j=i}^{n} G_j}{B(s)} \tag{4.7}$$

For the case of a third order system these sensitivities are:

$$S^W_{k_1} = \frac{-k_1 G_1 G_2 G_3}{B(s)}$$

$$= \frac{-G_1 G_2 G_3}{B(s)} = \frac{-Y(s)}{R(s)} \tag{4.8a}$$

for $k_1 = 1$

$$S_{k_2}^{W} = \frac{-k_2 G_2 G_3}{B(s)} \tag{4.8b}$$

$$S_{k_3}^{W} = \frac{-k_3 G_3}{B(s)} \,. \tag{4.8c}$$

For the series compensated system let $k = 1$ be the unity gain of the single feedback path. Then,

$$S_{k}^{W} = \frac{-G_c G}{1 + G_c G} = \frac{-Y(s)}{R(s)}$$

$$= S_{k_1}^{W} \qquad \text{for } k_1 = 1$$

Thus, the sensitivity of the state-variable feedback system with respect to the unity feedback gain from the output is the same as for the series compensated system. The relative magnitudes of $S_{k_i}^{W}(j\omega)$, for different values of i, depend on the magnitudes of the $G_i(j\omega)$. If $|G_i(j\omega)| > 1$ and if the k_i are of the same order of magnitude, it would appear that $|S_{k_i}^{W}(j\omega)|$ decreases as i increases. In such cases the state-variable feedback system would not be more sensitive with respect to changes in the feedback coefficients than would the series compensated sytem with respect to a change in the single unity feedback gain.

4.3 Restrictions Imposed by the Fixed Plant and the Closed-Loop Transfer Function.

From the comparisons made above between the series compensated system and the state-variable feedback system, it is seen that decreased sensitivity may be obtained by a change in the system configuration. However, it appears that the minimum sensitivity that can be achieved is limited by the fact that the fixed plant and the closed-loop response are specified. An example which illustrates this is the system of Fig. 4.3. The closed loop transfer function is

$$W(s) = \frac{K_1K_2K_3}{s^3 + (p_2 + p_3 + k_3K_3)s^2 + (p_2p_3 + p_2k_3K_3 + k_2K_2K_3)s + K_1K_2K_3}$$

If k_2 and k_3 are positive, it may be expected that G_3 is the least sensitive block. From Eq. (4.5c),

$$S_{G_3}^{W} = \frac{s(s + p_2)\ (s+p_3)}{s^3 + (p_2 + p_3 + k_3K_3)s^2 + (p_2p_3 + p_2k_3K_3 + k_2K_2K_3)s + K_1K_2K_3}$$

The examples of Chapter V show that the low frequency asymptote of S_λ^W is important in determining S_λ. Here, for small values of ω,

$$|S_{G_3}^{W}\ (j\omega)| \approx \frac{p_2p_3\omega}{K_1K_2K_3}$$

The product p_2p_3 is determined by the fixed plant, while the product $K_1K_2K_3$ is specified by the closed-loop transfer function. Decreasing $S_{G_3}^W$ by specifying a new closed loop response with a larger constant term, $K_1K_2K_3$, is usually not feasible, since this constant term determines the loop gain of the system; the loop gain is usually

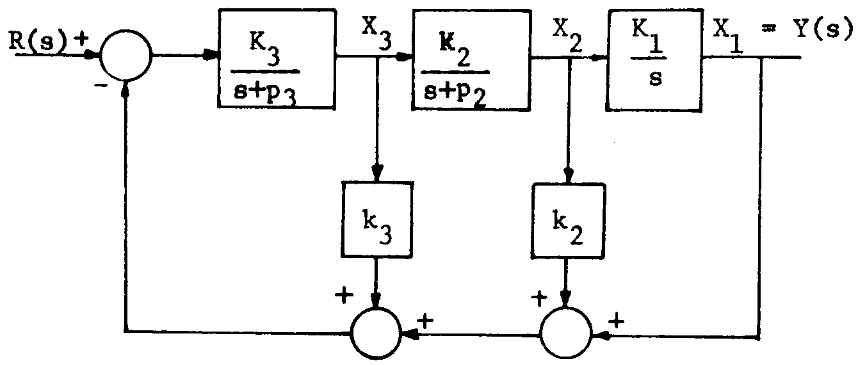


Figure 4.3 A third order control system.

restricted so that the system remains in a linear region of operation for some expected input.

The dependence of sensitivity on the fixed plant and the closed-loop transfer function is currently being investigated by Dial (1967).

4.4 Modifications of State Variable Feedback

Earlier in this chapter it was found that, under certain conditions, one would expect the system using state-variable feedback to be the least sensitive to the $G_i(s)$ nearest the input. In an attempt to extend this minimum value of sensitivity to the other $G_i(s)$, modifications of the feedback structure are investigated.

If in the system of Fig. 4.2 all the feedback paths are referred to the output, then the resulting system has the form of Fig. 4.4 where

$$H_{eq}(s) = 1 + \frac{k_2}{G_1} + \frac{k_3}{G_1 G_2} + \ldots + \frac{k_n}{G_1 \ldots G_{n-1}} \tag{4.9}$$

The system in this form is referred to as the "H-equivalent" system. The transfer function Y(s)/R(s) is unchanged. The H-equivalent system is often used as a block diagram reduction of the state-variable feedback system for the purpose of calculating the closed-loop transfer function. However, the H-equivalent system here is intended as an actual physical system; that is, the output is fed back through $H_{eq}(s)$, and no other state variables are fed back. For the H-equivalent system,

$$S_G^W = \frac{E'(s)}{R(s)} = \frac{1}{1 + GH_{eq}}$$

$$= \frac{1}{1 + G_1 G_2 \ldots G_n [1 + \frac{k_2}{G_1} + \ldots + \frac{k_n}{G_1 \ldots G_{n-1}}]}$$

$$= \frac{1}{1 + G_1 G_2 \ldots G_n + k_2 G_2 \ldots G_n + \ldots + k_n G_n}$$

$$= S_{G_n}^W \tag{4.10}$$

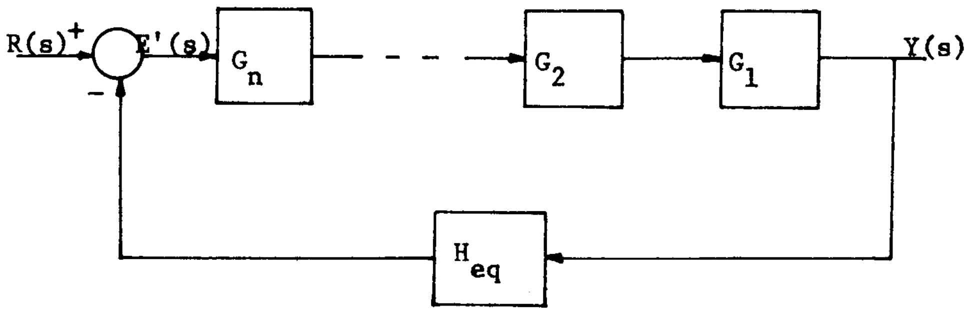


Figure 4.4 The H-equivalent system.

Thus, the sensitivity of the H-equivalent system with respect to any block in the forward path is equal to the sensitivity of the state-variable feedback system with respect to $G_n(s)$. This is the "minimum sensitivity" which was sought.

With regard to the construction of a system, the H-equivalent configuration has both advantages and disadvantages in comparison to the state-variable feedback system. For the H-equivalent system only the output is actually measured. This is an advantage when measurement of all of the state variables is difficult. However, unless the numerator and denominator of G(s) are of the same order, the numerator of $H_{eq}(s)$ is of higher order than the denominator. Then in order to realize $H_{eq}(s)$ approximately, poles must be added. This problem is treated in an example in Chapter V.

4.5 A Note on Integral Sensitivity and the Poles of the Fixed Plant

Consider again Eqs. (4.2) and (4.3) for W(s) of the state-variable feedback system. It is assumed that the functions $G_i(s)$ are of the form:

$$G_i(s) = \frac{K_i(s + z_i)}{s + p_i} \tag{4.11}$$

The factor $(s + z_i)$ is not always present. If the functions in the numerator and denominator of W(s) are cleared by multiplying by $\prod_{i=1}^{n} (s + p_i)$, W(s) may be written as:

$$W(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_o} \tag{4.12}$$

where the roots of Q(s), the characteristic polynomial, are the closed-loop poles of the system. Similarly, if the expressions for the classical sensitivities, $S_{G_i}^{W}$, are cleared of fractions, the sensitivities may be written as:

$$S_{G_i}^{W} = \frac{N_i(s)}{Q(s)}$$

$$= \frac{c_\ell s^\ell + c_{\ell-1} s^{\ell-1} + \ldots + c_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_o} \qquad (4.13)$$

From Eqs. (4.12) and (4.13) it is seen that the denominators of the $S_{G_i}^{W}$ are the characteristic polynomial, which is specified by the required closed-loop response.

The sensitivity with respect to $G_n(s)$ is:

$$S_{G_n}^{W} = \frac{\prod_{i=1}^{n} (s + p_i)}{Q(s)} \qquad (4.14)$$

Recall that the integral sensitivity, S_λ, depends on the <u>magnitude</u> of S_λ^W. Now,

$$\left| S_{G_n}^{W} \right|^2 = \frac{\prod_{i=1}^{n} (s + p_i)(-s + p_i)}{Q(s)\ Q(-s)} \qquad (4.15)$$

From Eq. (4.15) it is clear that the integral sensitivity, S_{G_n}, is the same for two systems which have the same closed-loop response, but whose open loop poles are symmetrical with respect to the $j\omega$-axis. Thus, one or more of the open loop poles could be located in the RHP, and S_{G_n} would remain the same. This emphasizes the fact that the sensitivity function, $u_\lambda(t)$, and therefore S_λ, are defined in terms of an incremental change in the parameter λ. Clearly, for sufficiently large changes in the gain K_n, a system with open-loop poles in the RHP behaves very differently from a system with only LHP open-loop poles.

The discussion above indicates that in addition to compensating the system for the desired closed-loop response and evaluating sensitivities, it is necessary to retain a wider view of the system design - for example, in terms of a root locus.

4.6 Summary

From the analysis in section 4.2 it is seen that, under certain conditions, the state-variable feedback system is less sensitive to parameter changes as compared to the series compensated system with the same closed-loop transfer function W(s). However, it appears that the minimum sensitivity attainable is restricted by the fixed plant and by the required W(s). The H-equivalent system, or a system using an approximation to $H_{eq}(s)$, might be used to extend this minimum value of sensitivity to all of the blocks in the forward path. Chapter V consists of a series of examples which illustrate the ideas discussed here.

CHAPTER V

EXAMPLES

This chapter consists of several examples to illustrate the sensitivity properties of systems designed by the methods discussed in Chapter IV. In Example 1 a fixed plant is compensated by feeding back all of the state variables and by the Guillemin-Truxal method. The sensitivities of the two resulting systems are compared. The same fixed plant is compensated with H-equivalent feedback in Example 2., and a system with an approximation of $H_{eq}(s)$ is discussed in Example 3. In Example 4, a zero, which is not desired in W(s), is included in the fixed plant. Sensitivity analysis is used to determine the parameters of a cascade compensator which provides for cancellation of the zero in the closed-loop response.

Example 1. Figure 5.1 shows the fixed plant of a control system which is required to have the following closed-loop transfer function.

$$W(s) = \frac{80}{s^3 + 14s^2 + 48s + 80}$$

$$= \frac{80}{(s + 10)(s^2 + 4s + 8)}$$

W(s) is obtained in two ways. One system is syntehsized using state-variable feedback, while the Guillemin-Truxal method is used to design

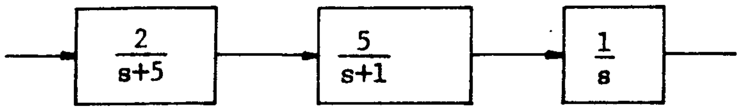
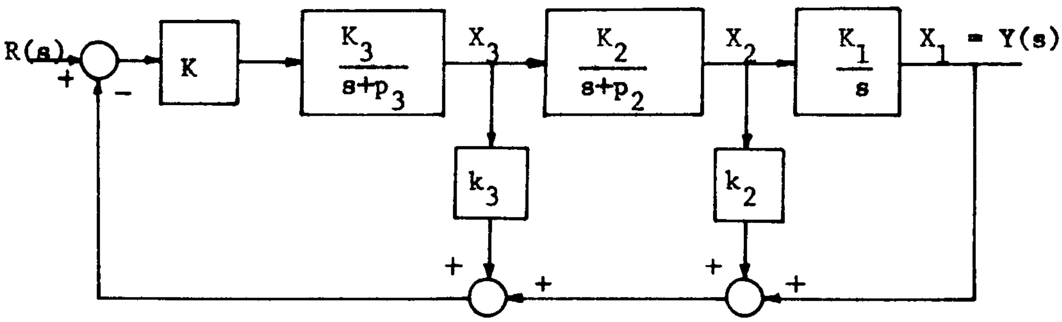


Figure 5.1 The fixed plant of Example 1.



$K_1 = 1$ $p_2 = 1$ $K = 8$

$K_2 = 5$ $p_3 = 5$ $k_2 = 35/80$

$K_3 = 2$ $k_3 = 1/2$

Figure 5.2 The compensated system of Example 1.

the second system. For both systems in this example classical sensitivities and sensitivity functions, as well as peak sensitivities and integral sensitivities, are found in order to show the connection between the different sensitivity measures.

For the state variable feedback system (Fig. 5.2) the sensitivities with respect to the blocks in the forward path are given by Eqs. (4.5). For this example the equations become:

$$S^W_{G_1} = \frac{s^3 + 14s^2 + 48s}{s^3 + 14s^2 + 48s + 80}$$

$$= \frac{3}{5} \frac{s(\frac{s}{6} + 1)(\frac{s}{8} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

$$S^W_{G_2} = \frac{s^3 + 14s^2 + 13}{s^3 + 14s^2 + 48s + 80}$$

$$= \frac{13}{80} \frac{s(\frac{s}{13} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

$$S^W_{G_3} = \frac{s^3 + 6s^2 + 5s}{s^3 + 14s^2 + 48s + 80}$$

$$= \frac{1}{16} \frac{s(s + 1)(\frac{s}{5} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

Asymptotic Bode plots for these sensitivities are shown in Fig. 5.3. There is also in Fig. 5.3 a Bode plot of $G_{eq}(s)$, which is included in order to indicate the bandwidth of the system. It may be noted that

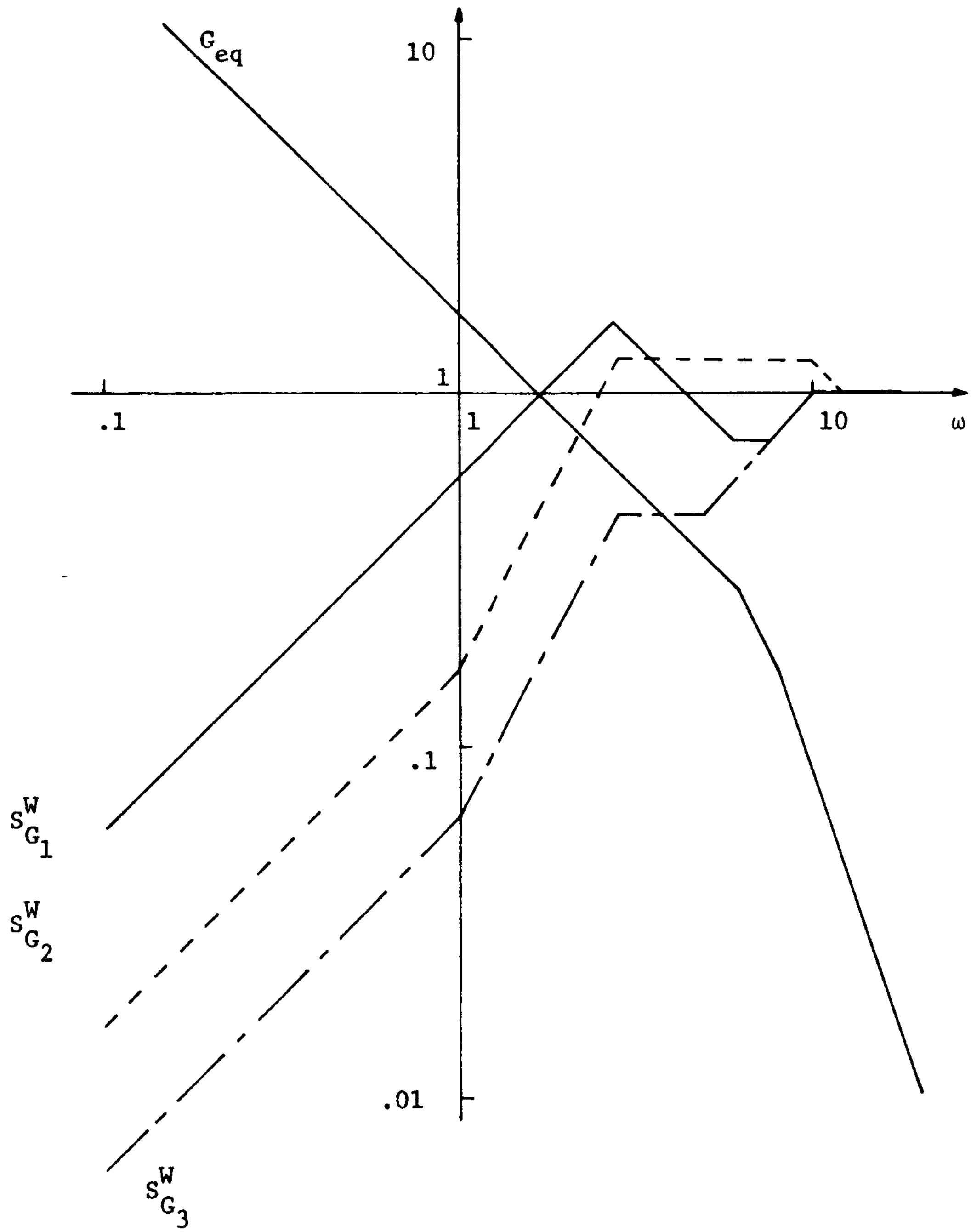


Figure 5.3 Gain sensitivities for the state-variable feedback system of Example 1.

for frequencies less than the gain crossover frequency, $S^W_{G_3} < S^W_{G_2} < S^W_{G_1}$. The classical sensitivities with respect to the specific parameters in the forward path are:

$$S^W_{K_1} = S^W_{G_1}$$

$$S^W_{K_2} = S^W_{G_2}$$

$$S^W_{K_3} = S^W_K = S^W_{G_3}$$

$$S^W_{p_2} = S^W_{G_2} \frac{-p_2}{s + p_2}$$

$$= \frac{-13}{80} \frac{s(\frac{s}{13} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

$$S^W_{p_3} = S^W_{G_3} \frac{-p_3}{s + p_3}$$

$$= \frac{-1}{16} \frac{s(s + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

Asymptotic Bode plots for $S^W_{p_2}$ and $S^W_{p_3}$ are in Fig. 5.4.

The sensitivities with respect to the feedback coefficients are given by Eqs. (4.8).

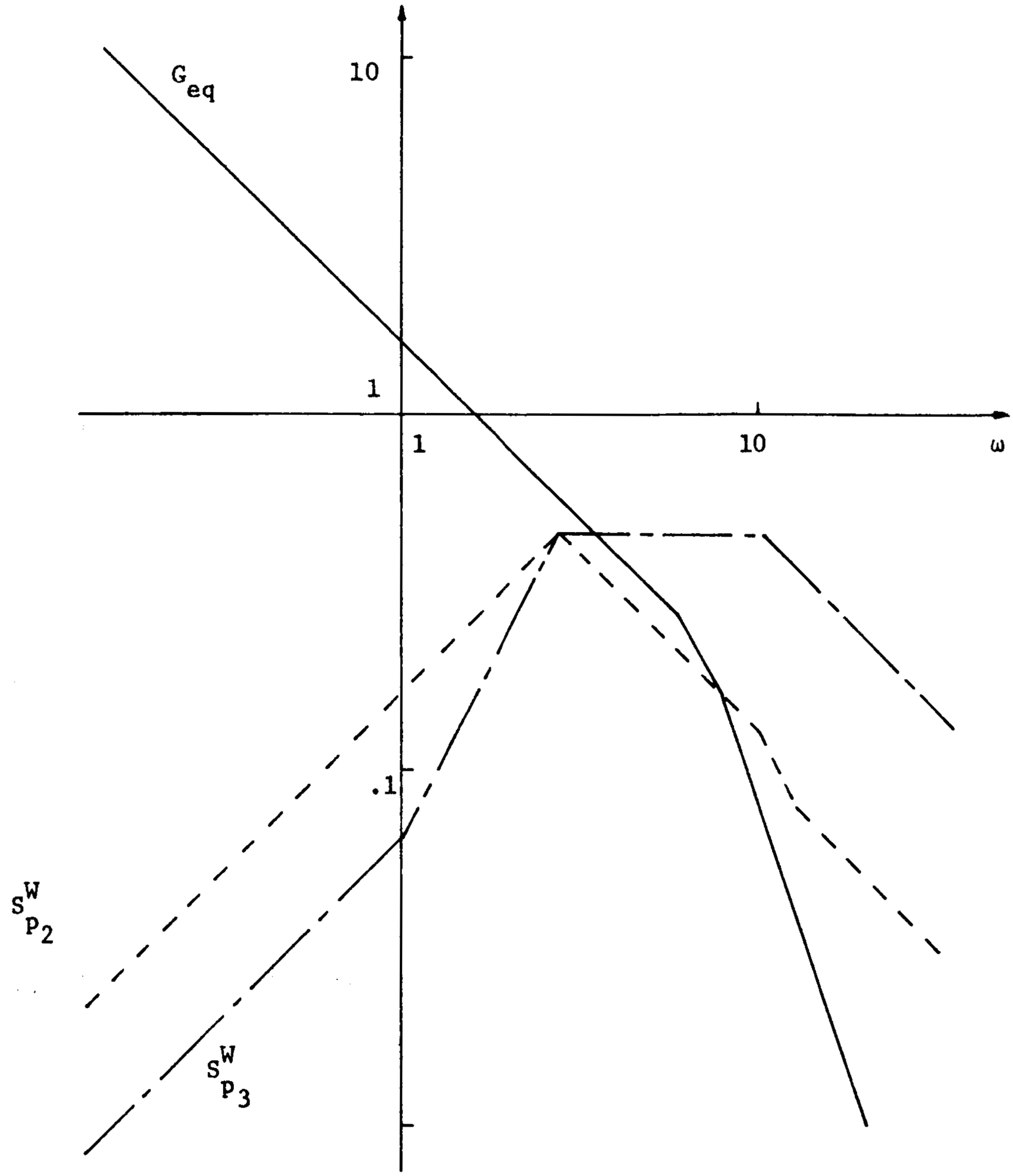


Figure 5.4 Pole sensitivities for the state-variable feedback system of Example 1.

$$S^W_{k_1} = \frac{-1}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

$$S^W_{k_2} = \frac{-7}{16} \frac{s}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

$$S^W_{k_3} = \frac{-1}{10} \frac{s(s + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)}$$

Fig. 5.5 shows Bode plots of these functions.

If the Guillemin-Truxal method is used to compensate the plant, the final closed-loop system is as shown in Fig. 5.6. For this system S^W_K, the sensitivity with respect to any gain in the forward path, is equal to $S^W_{K_1}$ for the state-variable feedback system. Similarly, the sensitivity of the series compensated system with respect to the unity feedback coefficient is equal to $S^W_{k_1}$ for the state-variable feedback system. The sensitivities of the series compensated system with respect to the poles of the fixed plant are:

$$S^W_{p_2} = S^W_K \left[\frac{-p_2}{s + p_2}\right]$$

$$= \frac{-3}{5} \frac{s(\frac{s}{6} + 1)(\frac{s}{8} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)(s + 1)}$$

$$S^W_{p_3} = S^W_K \left[\frac{-p_3}{s + p_3}\right]$$

$$= \frac{-3}{5} \frac{s(\frac{s}{6} + 1)(\frac{s}{8} + 1)}{(\frac{s}{10} + 1)(\frac{s^2}{8} + \frac{s}{2} + 1)(\frac{s}{5} + 1)}$$

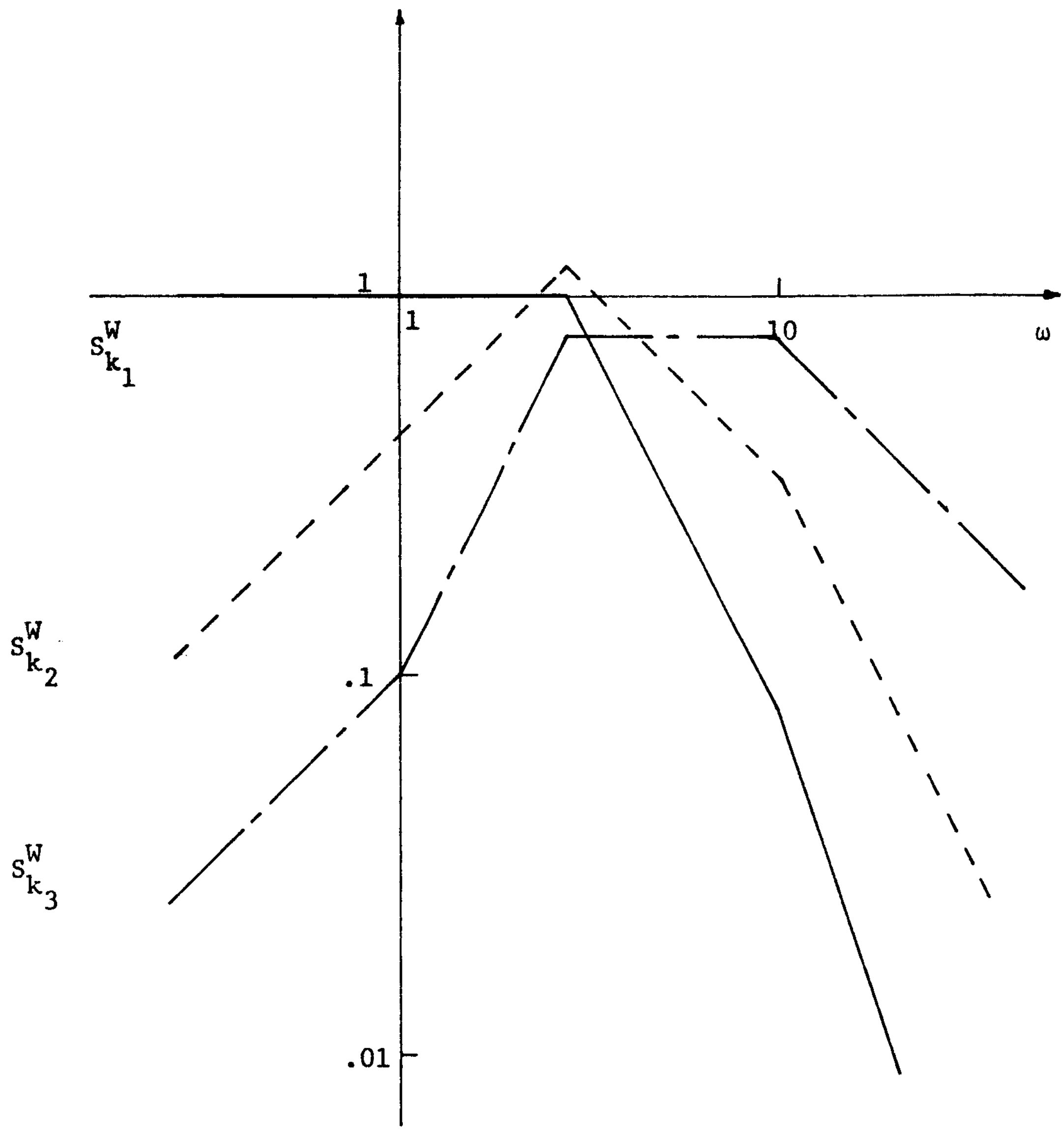


Figure 5.5 Feedback coefficient sensitivities for the state-variable feedback system of Example 1.

The Bode plots for these functions are shown in Fig. 5.7.

A comparison of the Bode plots of the classical sensitivities of the two systems shows that for all parameters, the magnitudes of the classical sensitivities for the state-variable feedback system are less than or equal to those for the series compensated system for frequencies less than the gain crossover frequency.

A similar comparison may be made in terms of sensitivity functions and integral sensitivities.

Figs. 5.8 and 5.9 show block diagrams for the generation of sensitivity functions for both systems. Plots of the sensitivity functions are shown in Figs. 5.10, 5.11, and 5.12, and a table listing peak sensitivities and integral sensitivities is in Fig. 5.18.

From these results it is clear that a reduction in sensitivity with respect to the parameters K_2, K_3, p_2, and p_3 has been obtained using the state-variable feedback method of design.

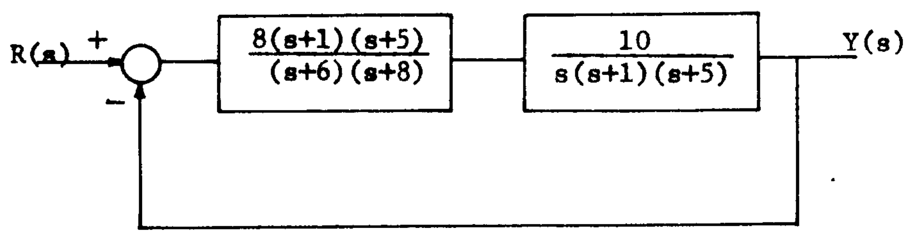


Figure 5.6 The system of Example 1 compensated by the Guillemin-Truxal method.

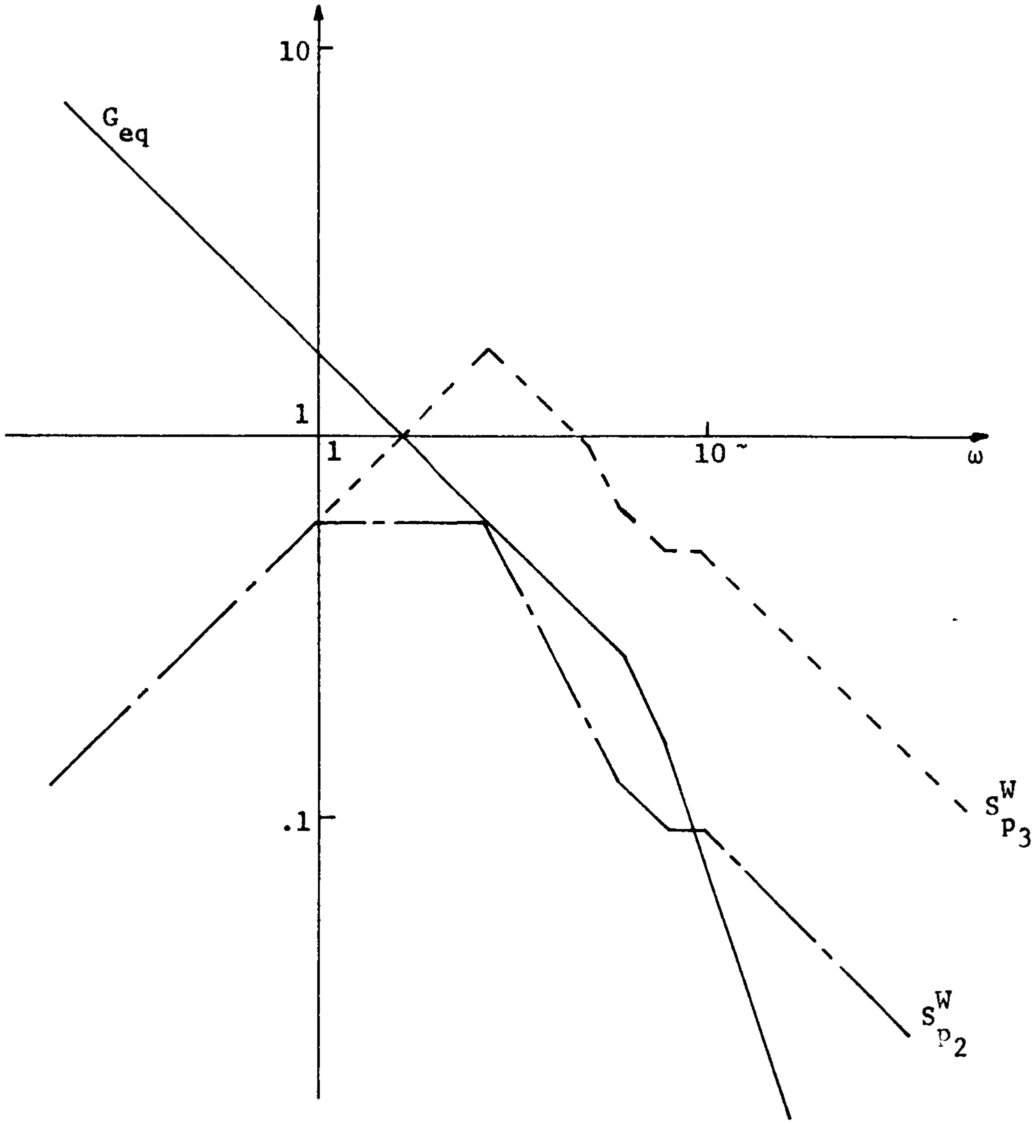


Figure 5.7 Pole sensitivities for the series compensated system of Example 1.

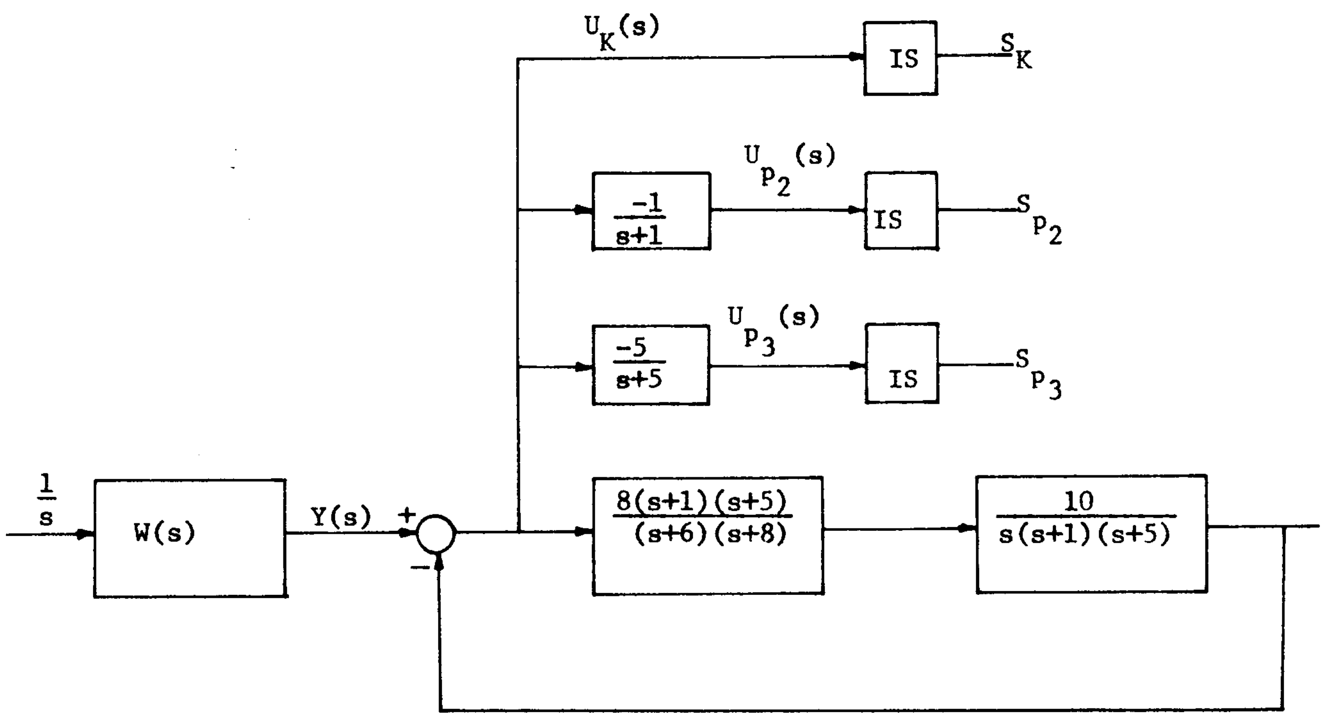


Figure 5.8 Generation of sensitivity functions and integral sensitivities for the series compensated system.

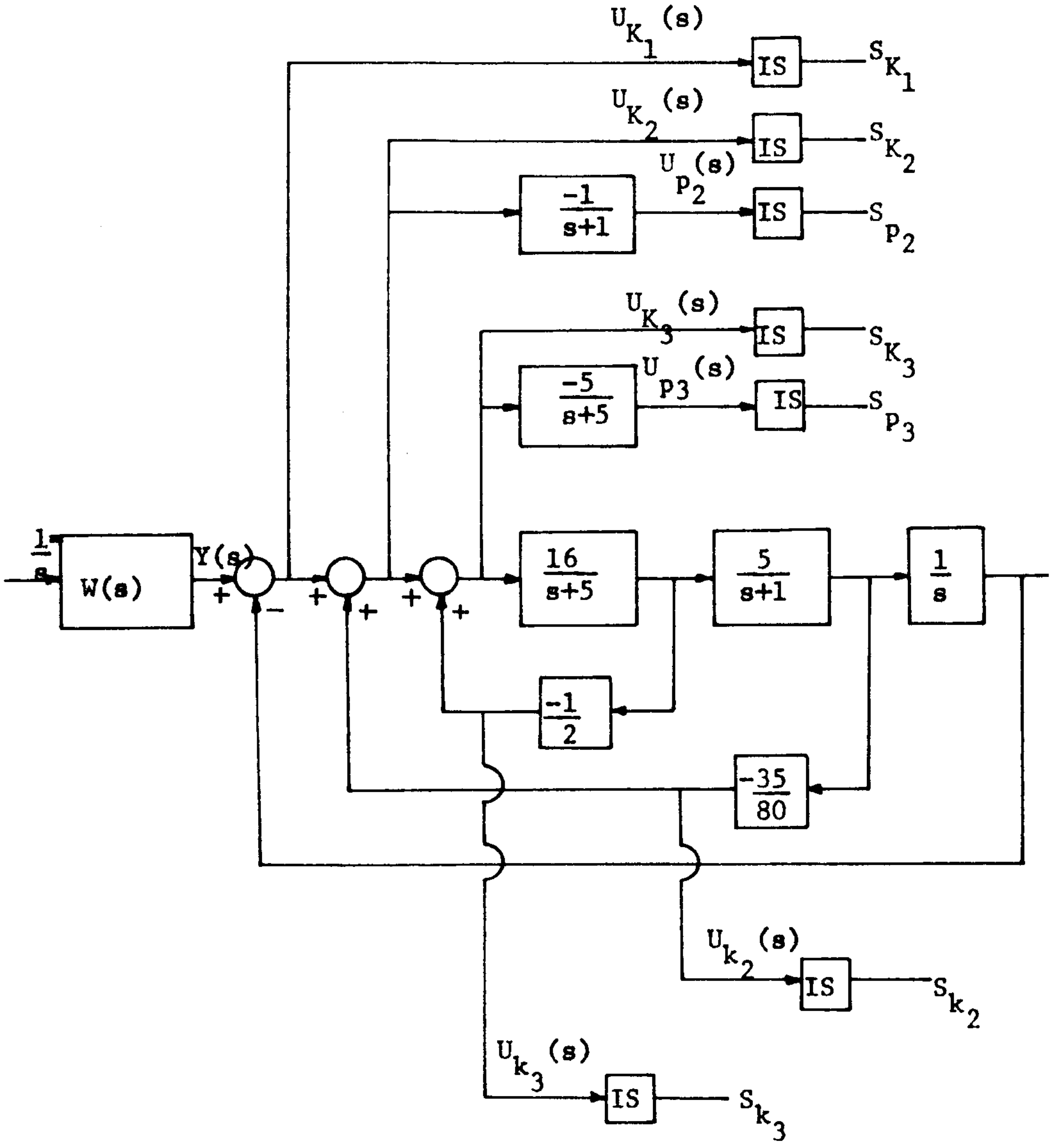


Figure 5.9 Generation of sensitivity functions and integral sensitivities for the state-variable feedback system.

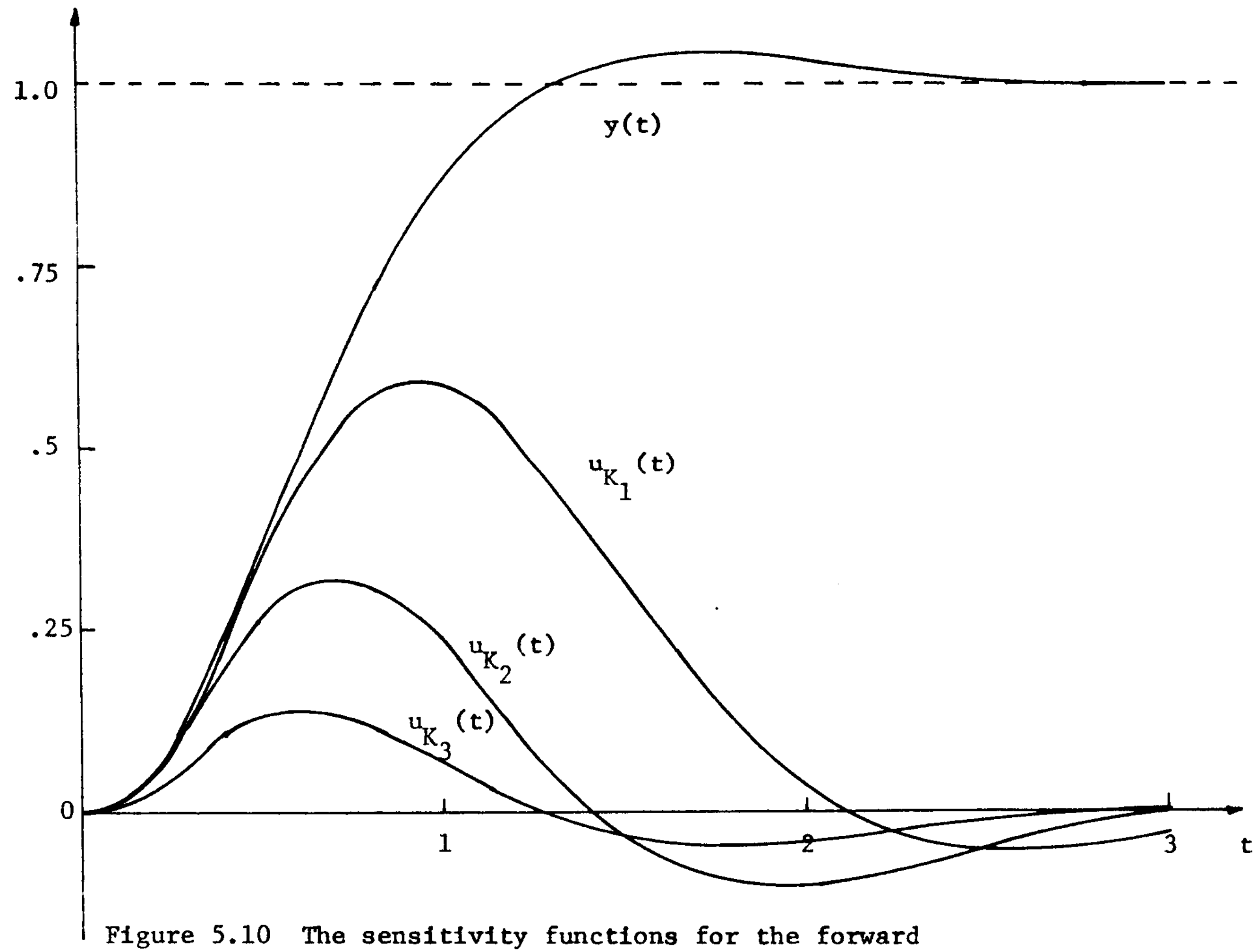


Figure 5.10 The sensitivity functions for the forward gains of the state-variable feedback system.

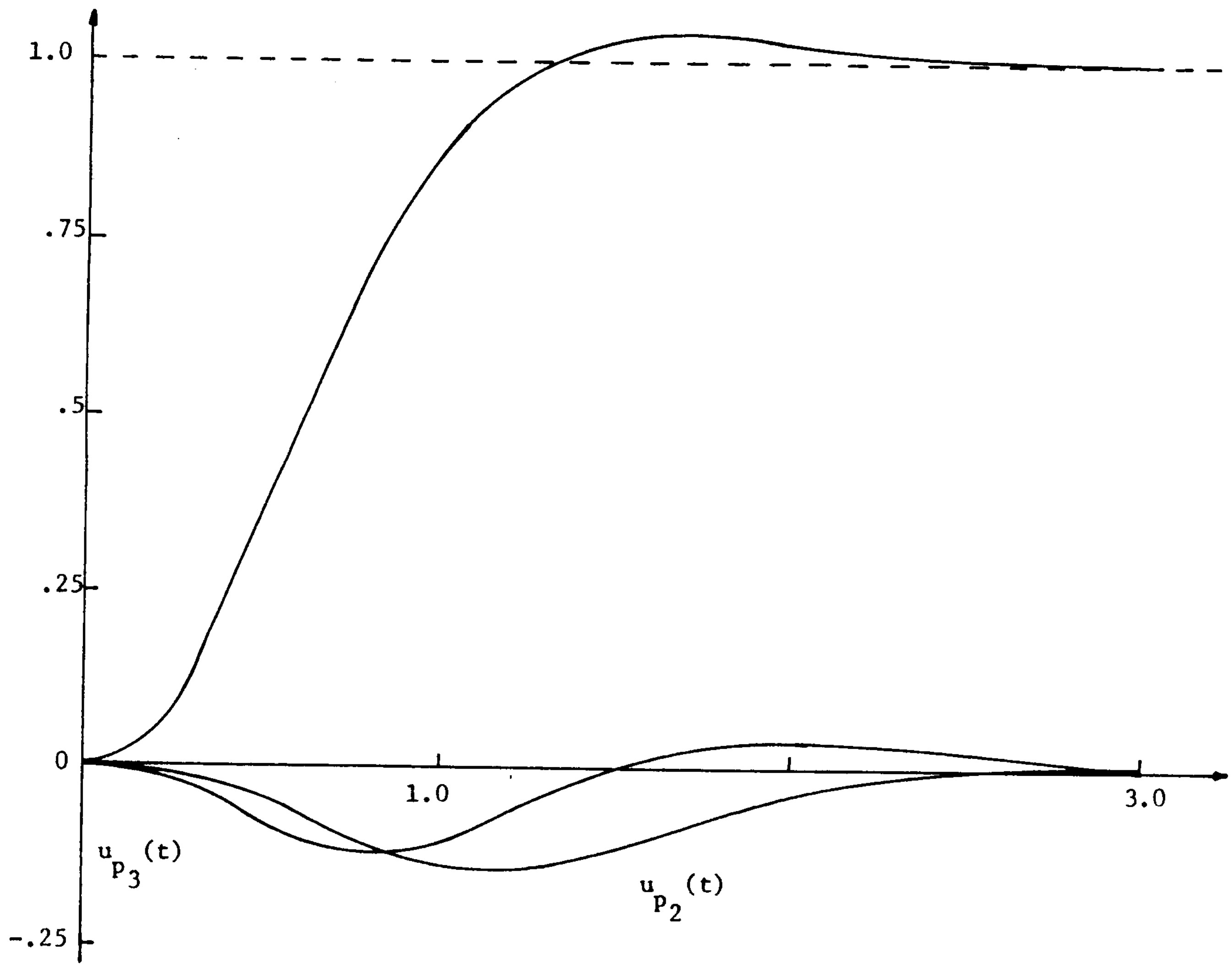


Figure 5.11 The sensitivity functions for the poles of the fixed plant in the state-variable feedback system.

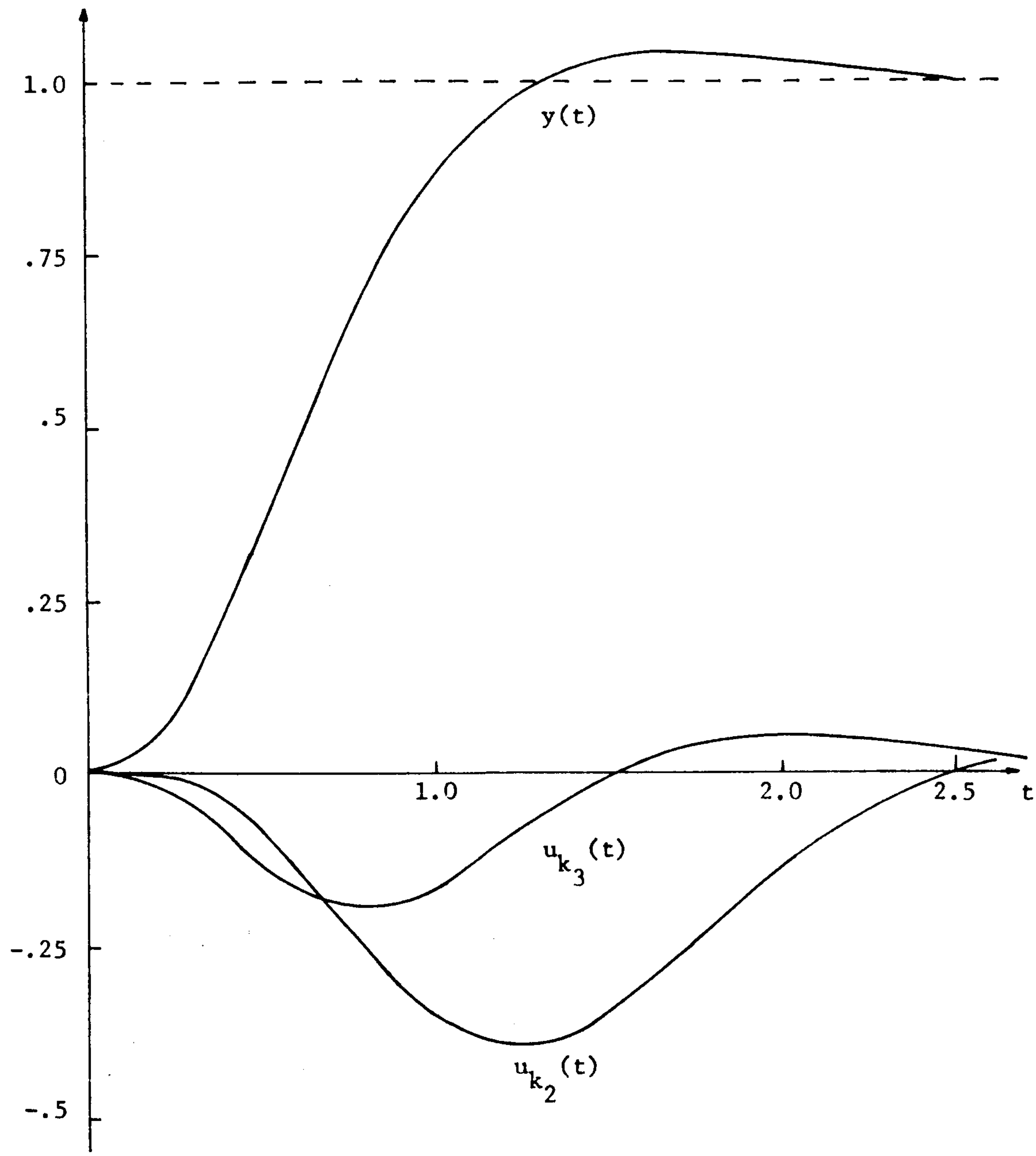


Figure 5.12 The sensitivity functions for the feedback coefficients of the state-variable feedback system.

Example 2. For the state-variable feedback system of Example 1. the sensitivities with repect to K_1, K_2, and p_2 may be reduced by using H-equivalent feedback. The $H_{eq}(s)$ system is shown in Fig. 13.

From Eq. (4.10) the sensitivity of the $H_{eq}(s)$ system with respect to any block in the forward path is:

$$S_G^W = S_{G_n}^W = S_{G_3}^W$$

where $S_{G_n}^W$ is the sensitivity of the state-variable feedback system with respect to G_n $(n = 3)$. The sensitivities with repect to p_2 and p_3 are:

$$S_{p_2}^W = S_G^W \left(\frac{-1}{s + 1}\right)$$

$$S_{p_3}^W = S_G^W \left(\frac{-5}{s + 5}\right)$$

$S_{p_3}^W$ is the same as for the state-variable feedback system. The sensitivity with respect to p_2 has been reduced, since $|S_G^W| < |S_{G_2}^W|$, where $S_{G_2}^W$ is the sensitivity with respect to G_2 for the state-variable feedback system.

The peak sensitivities and integral sensitivities for the H-equivalent system are listed in the table of Fig. 5.18.

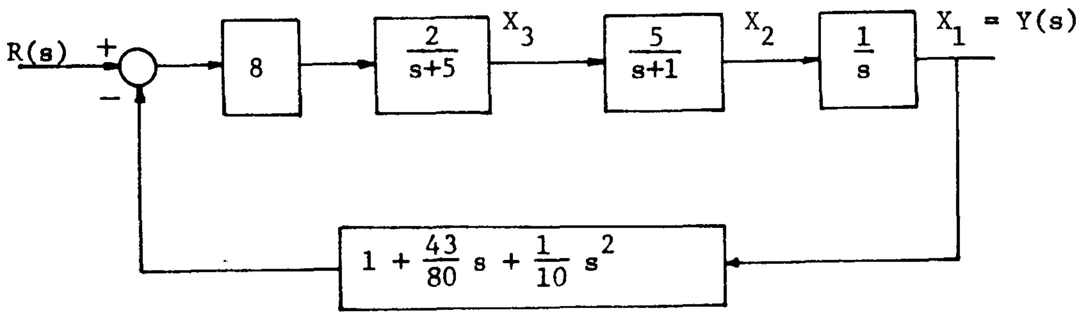


Figure 5.13 The H-equivalent system of Example 2.

Example 3. In Example 2. a reduction in sensitivity was obtained by using H-equivalent feedback. However, $H_{eq}(s)$ is not in a form which is easily realizable. It is desirable to approximate $H_{eq}(s)$ by a transfer function which is realizable by RC elements and a gain factor.

$$H_{eq}(s) = 1 + \frac{43}{80} s + \frac{1}{10} s^2$$

$$= 1 + \frac{s}{10}(s + 5.38)$$

In order to make the second term realizable, poles are added at $s = -40$ and $s = -50$, while preserving the low frequency gain.

$$H'_{eq}(s) = 1 + \frac{200\ s(s + 5.38)}{(s + 40)(s + 50)} \tag{5.1}$$

There are several factors to be considered in choosing the approximatinn of $H_{eq}(s)$. The large gain of $H_{eq}(s)$ at high frequencies is undesirable if there is noise at the system output. The addition of low frequency poles to $H_{eq}(s)$ alleviates this problem. However, two other considerations make the use of high frequency poles desirable. The poles of $H'_{eq}(s)$, which become zeros of $W(s)$, have less effect on $y(t)$ if they are placed at high frequencies. Secondly, the addition of poles in the manner shown in Eq. (5.1) causes the zeros of $H'_{eq}(s)$ to be different from those of $H_{eq}(s)$. This error in zero locations, which also affects $y(t)$, is smaller for high frequency poles. Thus, a compromise must be made between the filtering of output noise and the approximation of $H_{eq}(s)$. Another possiblity is to approximate $H_{eq}(s)$ by:

$$H'_{eq}(s) = \frac{2000\ (1 + .538s + .1s^2)}{(s + 40)(s + 50)}$$

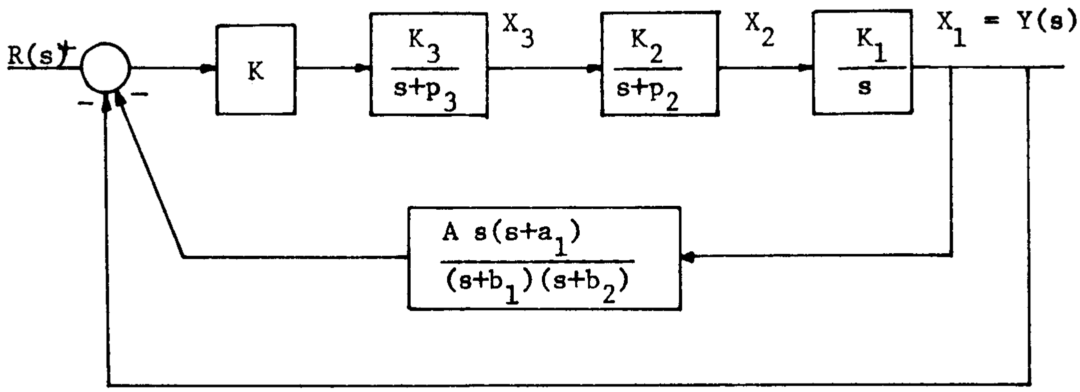
However, with this approximation a change in the pole location or the gain constant of H'_{eq} (s) results in a steady-state error at the output.

One other idea in the approximate realization of $H_{eq}(s)$ is to obtain a system which has zero steady-state error for a ramp input. For such a system the velocity error constant, K_ν, is infinite. K_ν may be expressed (Truxal, 1955) as:

$$\frac{1}{K_\nu} = \sum_{j=1}^{n} \frac{1}{p_j} - \sum_{j=1}^{n} \frac{1}{z_j}$$

where the p_j and z_j are the poles and zeros of the closed-loop transfer function. Since K_ν is determined by the closed-loop poles and zeros, the poles added to $H_{eq}(s)$ might be placed in such a way that $K_\nu = \infty$. This is a topic for further investigation.

The structure of the system with $H'_{eq}(s)$ feedback is shown in Fig. 5.14, and a block diagram for the generation of sensitivity functions is in Fig. 5.15. The table of Fig. 5.18 lists the peak sensitivities and integral sensitvities. For the parameters in the fixed plant, the sensitivities are approximately equal to those of the $H_{eq}(s)$ system. The sensitivities with respect to the parameters of $H'_{eq}(s)$ are reasonably small (less than S_{K_1} for the state variable feedback system).



$K_1 = 1$	$p_2 = 1$	$K = 8$
$K_2 = 5$	$p_3 = 5$	$A = 200$
$K_3 = 2$	$a_1 = 5.38$	
	$b_1 = 40$	
	$b_2 = 50$	

Figure 5.14 The H'-equivalent system of Example 3.

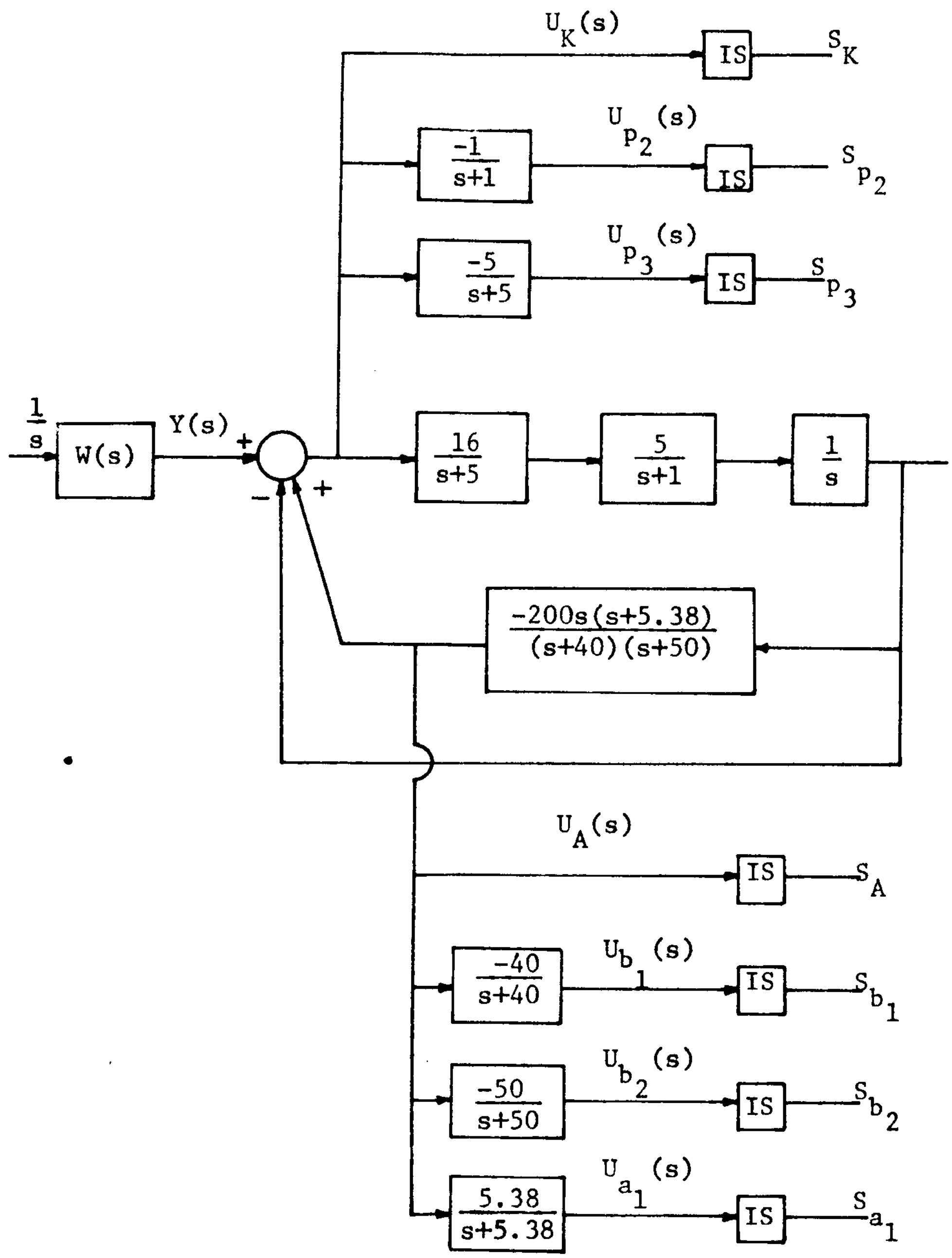


Figure 5.15 Generation of sensitivity functions and integral senstivities for the $H'_{eq}(s)$ system.

For the $H'_{eq}(s)$ system the new closed-loop transfer function is

$$W(s) = \frac{80(s + 40)(s + 50)}{(s + 61.9)(s^2 + 29.4s + 287)(s^2 + 4.81s + 9.05)}$$

The pole and zero locations are shown in Fig. 5.16, and a graph of y(t) is in Fig. 5.17. It is seen that the addition of poles in the feedback structure has altered the step response. This example demonstrates that while a system using an approximation to $H_{eq}(s)$ may show an improvement in sensitivity over a system with state-variable feedback, two new problems are introduced. The addition of poles to $H_{eq}(s)$ affects the closed-loop response, and the high gain of $H_{eq}'(s)$ at high frequencies is undesirable if there is noise at the output.

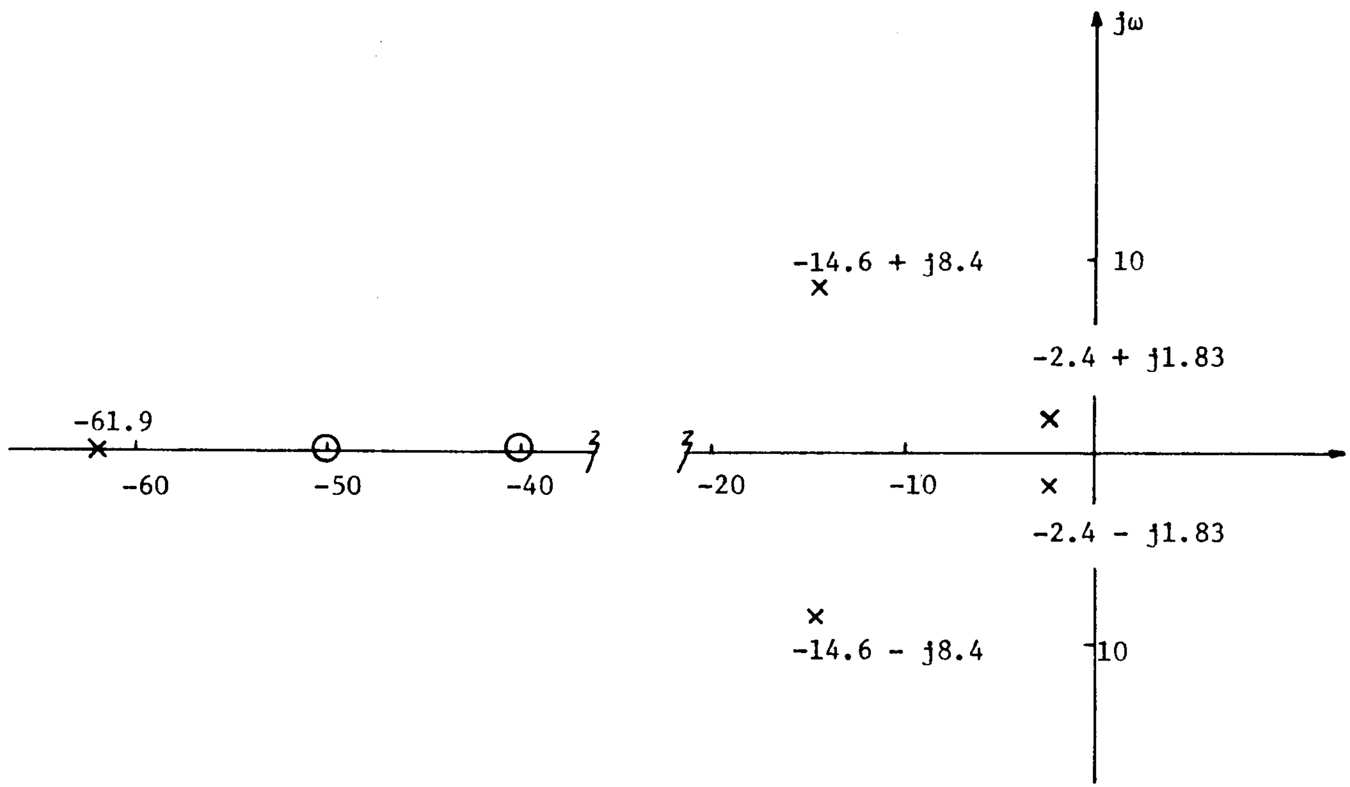


Figure 5.16 The closed loop poles and zeros for the $H'_{eq}(s)$ system.

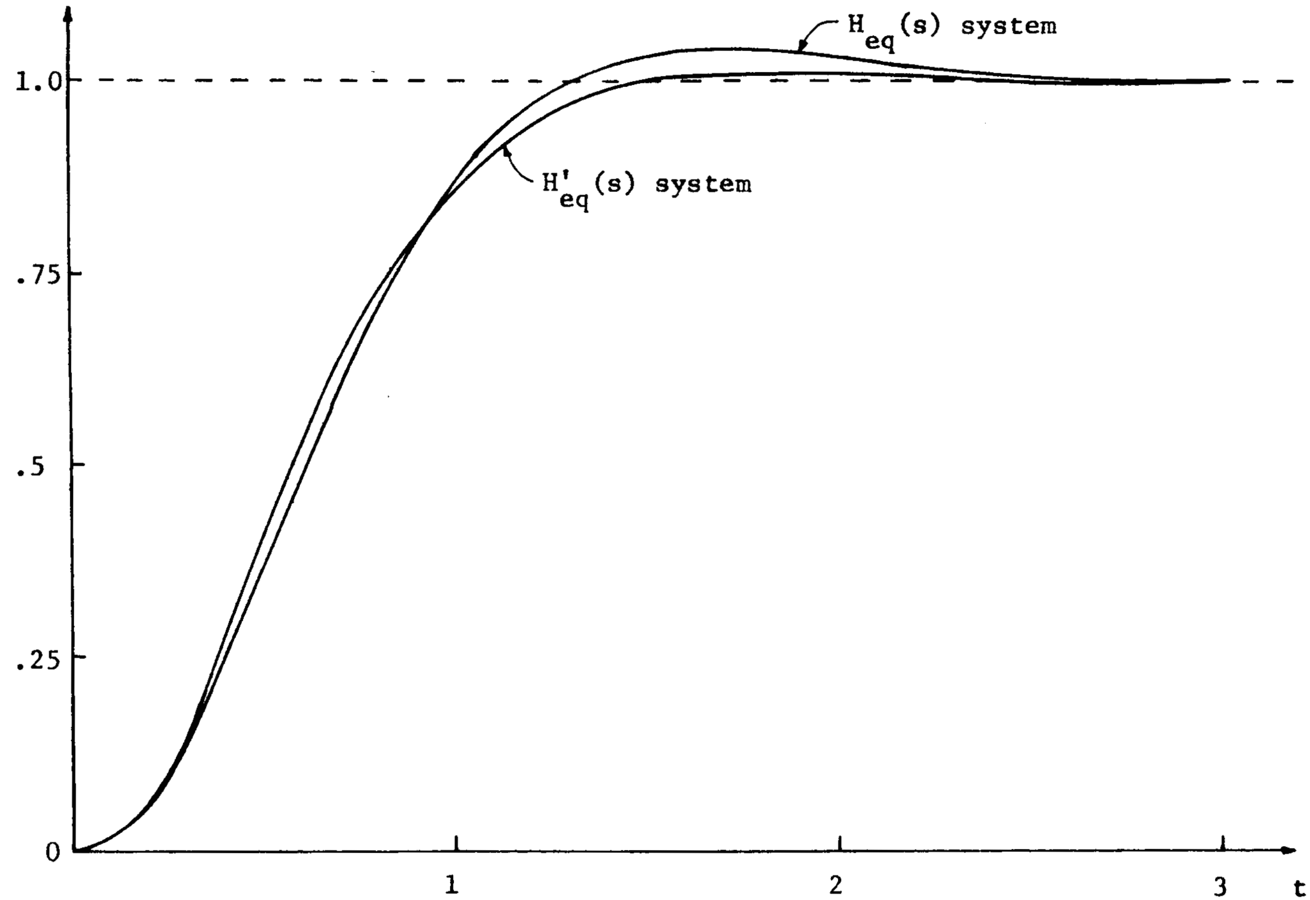


Figure 5.17 The step response of the $H'_{eq}(s)$ system.

parameter	series compensated system u^*_λ	series compensated system S_λ	state-variable feedback system u^*_λ	state-variable feedback system S_λ	$H_{eq}(s)$ system u^*_λ	$H_{eq}(s)$ system S_λ	$H'_{eq}(s)$ system u^*_λ	$H'_{eq}(s)$ system S_λ
K_1	0.593	0.286	0.593	0.286	0.140	0.0117	0.149	0.0112
K_2	0.593	0.286	0.321	0.0669	0.140	0.0117	0.149	0.0112
K_3	0.593	0.286	0.140	0.0117	0.140	0.0117	0.149	0.0112
P_2	-0.329	0.133	-0.148	0.0190	-0.0610	0.0031	-0.0595	0.0028
P_3	-0.548	0.260	-0.120	0.0096	-0.120	0.0096	-0.124	0.0089
k_2			-0.396	0.136				
k_3			-0.193	0.0245				
A							-0.505	0.205
a_1							-0.469	0.189
b_1							0.505	0.205
b_2							0.505	0.205

Figure 5.18 A table of sensitivities for Examples 1., 2., and 3.

Example 4. Fig. 5.19 shows the block diagram of a fixed plant for which the transfer function is

$$G_p(s) = \frac{5(s + 2)}{s(s + 1)(s + 5)}$$

The desired closed-loop transfer function is

$$W(s) = \frac{80}{(s + 10)(s^2 + 4s + 8)}$$

In order to realize W(s), the zero of the fixed plant must be cancelled, and it is assumed that it is impossible to insert a pole immediately preceding this zero.

Since direct series cancellation is impossible, the zero appears as a zero of W(s). Thus, W(s) is also required to have a pole at $s = -2$. That is,

$$W(s) = \frac{80(s + 2)}{(s + 2)(s + 10)(s^2 + 4s + 8)}$$

To accomplish this, the order of the system is increased by inserting a series compensator as shown in Fig. 5.20, and the new state variable x_4 is fed back. The parameters k_2, k_3, k_4, K, and p_4 are then chosen so as to realize W(s). The values of k_2, k_3, and K are found to be:

$$K = 16, \qquad k_2 = 7/16, \qquad k_3 = -3/4$$

To obtain the specified W(s), the values of k_4 and p_4 must be chosen such that the transfer function $\frac{x_4(s)}{z(s)}$, as defined in Fig. 5.20, is:

$$\frac{x_4(s)}{z(s)} = \frac{16}{s + p_4 + 16k_4} = \frac{16}{s + 10}$$

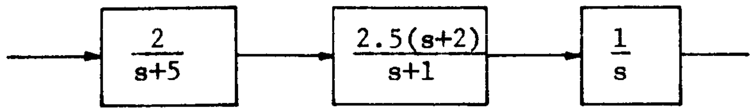
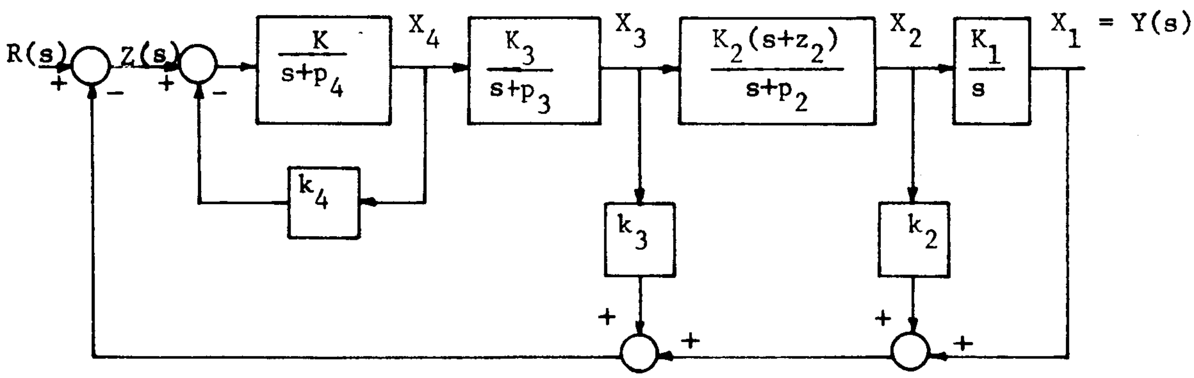


Figure 5.19 The fixed plant of Example 4.



$K_1 = 1$ $\quad$ $K = 16$ $\quad$ $p_2 = 1$ $\quad$ $k_2 = 7/16$

$K_2 = 2.5$ $\quad$ $Z_2 = 2$ $\quad$ $p_3 = 5$ $\quad$ $k_3 = -3/4$

$K_3 = 2$

Figure 5.20 The closed-loop system of Example 4.

Any values of k_4 and p_4 satisfying $p_4 + 16k_4 = 10$ produce the required pole at $s = -2$ in the closed loop transfer function.

Bounds on a desirable value of p_4 may be obtained from stability considerations (Schultz and Melsa 1967). If $p_4 = 0$, the system has two open-loop poles at the origin, and the root locus, as a function of the gain K, is in the RHP for small values of K. Another possible choice is $p_4 = 10$, which requires $k_4 = 0$. Since the state variable x_4 is not fed back when $k_4 = 0$, a zero of $H_{eq}(s)$ is lost. Therefore, as $K \to \infty$, two closed-loop poles (instead of only one) approach infinity. This is a disadvantage with regard to stability for high gain.

An intermediate value of p_4 may be obtained by considering the sensitivity of W(s) with respect to p_4 and k_4. From Eq. (4.14),

$$S_{G_4}^{W} = \frac{(s + p_4)(s + 5)(s + 1)s}{Q(s)}$$

Therefore,

$$S_{p_4}^{W} = S_{G_4}^{W} \frac{-p_4}{s + p_4}$$

$$= \frac{-p_4(s + 5)(s + 1)s}{Q(s)} \tag{5.2}$$

From Eq. (4.7),

$$S_{k_4}^{W} = \frac{-k_4G_4}{B(s)}$$

$$= \frac{-k_4K(s + 5)(s + 1)s}{Q(s)} \tag{5.3}$$

It is seen that $|S^{W}_{p_4}|$ and $|S^{W}_{k_4}|$ are proportional to $|p_4|$ and $|k_4|$ respectively. Usually it is desirable to decrease the sensitivity with respect to elements in the forward path and to accept higher sensitivities for the feedback coefficients, because the tolerances for the k_i's may be controlled. However, in this case the series compensator is also selected by the designer. A possible solution is to choose p_4 such that the sensitivities with respect to p_4 and k_4 are equal. From Eqs. (5.2) and (5.3) this requires $16k_4 = p_4$. We have

$$p_4 + 16k_4 = 10 \tag{5.4}$$

Therefore,

$$p_4 = 5, \qquad k_4 = 5/16$$

It should be noted that the sensitivities with respect to the other parameters of the system do not depend on the values of k_4 and p_4, as long as these values satisfy Eq. (5.4). This is seen from the fact that the transfer functions used to calculate the sensitivities for the other parameters involve p_4 and k_4 only through the function $\frac{x_4(s)}{z(s)}$.

A block diagram for the generation of sensitivity functions for this example is shown in Fig. 5.21, and the results are listed below.

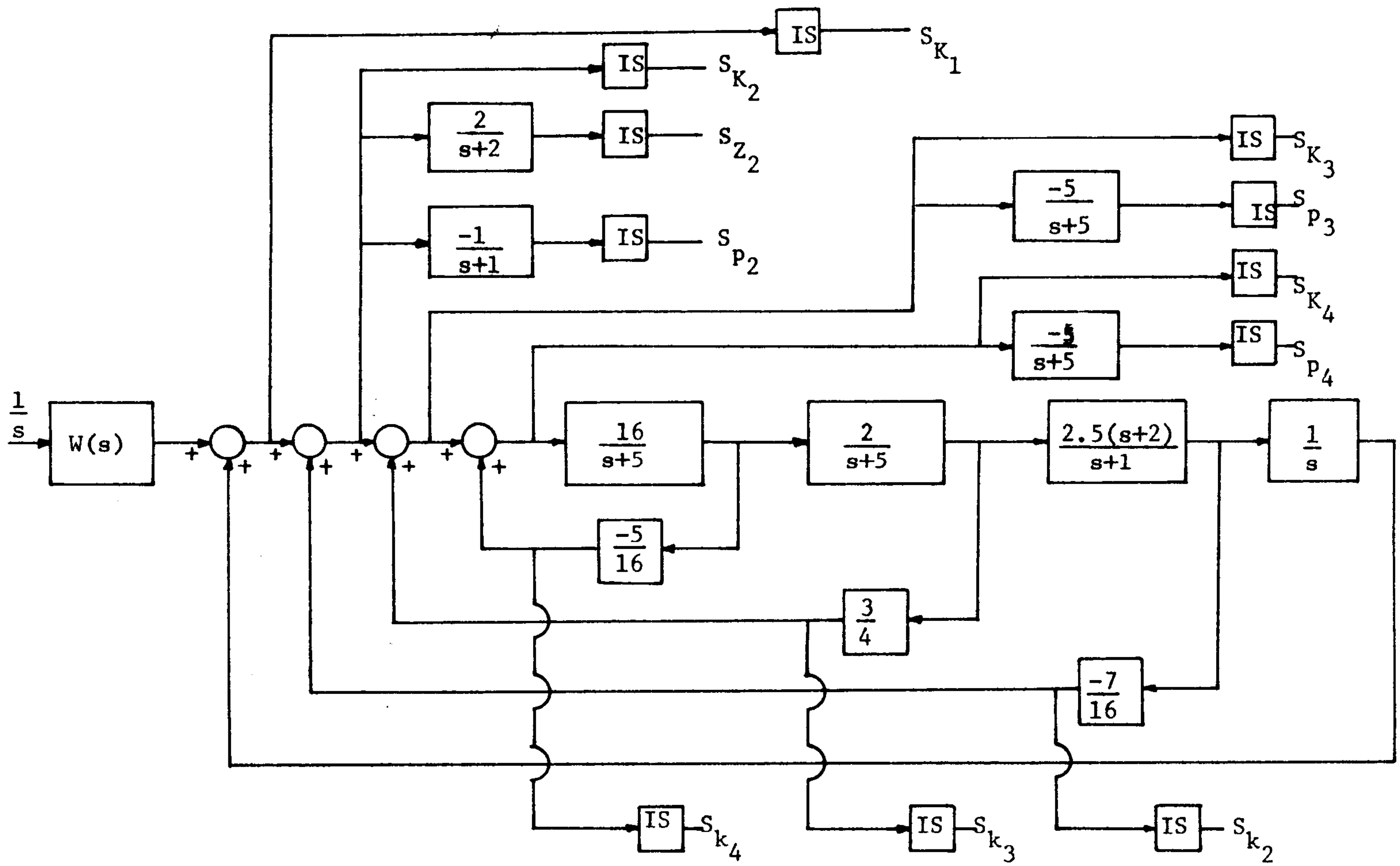


Figure 5.21 Generation of sensitivity functions and integral sensitivities for the system of Example 4.

Parameter	Peak Sensitivity	Integral Sensitivity
K_1	0.593	0.286
K_2	0.321	0.0670
K_3	0.458	0.152
K	0.248	0.0424
z_2	0.213	0.0354
p_2	-0.148	0.0191
p_3	-0.415	0.132
p_4	-0.223	0.0366
k_2	-0.396	0.1366
k_3	0.194	0.0295
k_4	-0.223	0.0366

It is seen that the peak and integral sensitivities with respect to p_4 and k_4 are equal, which follows from the equality of their classical sensitivities. It should also be noted that $S_{K_3} > S_{K_2}$. This occurs because the feedback coefficient k_3 is negative.

It may be noted that $|S^W_{p_4}| \to 0$ as $p_4 \to 0$. Thus, for minimum sensitivity with respect to p_4, the best choice is $p_4 = 0$. However, as mentioned above, this value of p_4 leads to instability for small values of K. This illustrates the need to maintain an overall view of the system behavior when a solution for minimum sensitivity is being sought.

CHAPTER VI

CONCLUSIONS

In this thesis a new sensitivity measure, integral sensitivity (S_λ), has been defined in terms of the sensitivity function ($u_\lambda(t)$).

$$S_\lambda = \int_0^\infty u_\lambda^2 (t)\, dt$$

where $u_\lambda(t) = \dfrac{dy(t, \lambda)}{\frac{d\lambda}{\lambda}}$, and $y(t, \lambda)$ is the response of the system to a step input. Although the integral sensitivity contains less information than the sensitivity function, it does, along with the peak sensitivity (u_λ^*), provide a quantitative measure of sensitivity in a concise form. Peak sensitivity is defined as:

$$u_\lambda^* = u_\lambda(T)$$

where T = the value of t such that $|u_\lambda(t)|$ is a maximum. Integral sensitivity is a measure of the overall effect on the system step response of a parameter variation, while the peak sensitivity is an estimate of the maximum change in y(t) for a + 1% change in the parameter. Part of the value of integral sensitivity is derived from its close connection to classical sensitivity ($S_\lambda^W = \frac{\lambda}{W}\frac{dW}{d\lambda}$) by the equation

$$S_\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|W(j\omega)|}{\omega^2} \left|S_\lambda^W(j\omega)\right| d\omega$$

From this relation the relative magnitudes of classical sensitivities, which may be found without the use of a computer, can be used to predict the relative magnitudes of integral sensitivities. Furthermore, integral sensitivities can be computed for practical cases only in a numerical fashion, while classical sensitivities can be evaluated in terms of the literal parameters of the system. In this way sensitivity considerations are included early in the design process.

In Chapter IV a comparison is made between the sensitivity properties of state-variable feedback systems and series compensated systems. It is seen that, under certain conditions, the sensitivities with repect to most of the system parameters may be expected to be smaller for the state-variable feedback system, and that the sensitivities with respect to blocks in the state feedback system are less for the blocks closer to the system input. This behavior is demonstrated by the examples of Chapter V.

The use of H-equivalent feedback is seen to be advantageous with regard to sensitivities for parameters in the forward path. However, in order to make the feedback transfer function realizable, it is necessary to add poles to $H_{eq}(s)$. The locations of these poles must be chosen with attention to their effects on W(s) and the filtering of output noise. There is also the possibility of choosing the poles such that the resulting system has zero steady-state error for a ramp input. The judicious choice of these pole locations as an integral part of the system design appears to be a subject for future work.

The following observations seem to indicate another topic for further research. By feeding back the state variables, a reduction in sensitivity for parameters in the forward path is obtained, but the feedback coefficients which are introduced represent a new source of sensitivity. Also, it was seen by an example calculation in section 4.3 that the sensitivity value of the least sensitive component depends entirely on the given fixed plant and the specified closed-loop response. These considerations lead to the conjecture that, given a fixed plant which constitutes the forward path, and a specified closed-loop response, there may exist a law of "conservation of sensitivity" for the system. That is, reduction of the sensitivity with respect to certain parameters may lead to increased sensitivity due to other parameters, and the total sensitivity is, in some sense, a constant.

APPENDIX

For the system of Fig. 4.2, Eqs. (4.4) and (4.6) are given for the sensitivities with respect to $G_i(s)$ and k_i respectively. The closed loop transfer function W(s) is given by Eq. (4.3). These expressions are derived here.

The system of Fig. 3.1 is the same as that of Fig. 4.2 for the case where $H_j = k_j$ for all j. Consider the reduced block diagram of Fig. 3.2. An expression for $S_{G_i}^{W} = E_i(s)/R(s)$ is given by Eq. (3.8).

$$S_{G_i}^{W} = \frac{1}{1 + G_i LN\ [M + k_i/N]} = \frac{1}{1 + G_i L\ [NM + k_i]}$$

Substitution for L(s), M(s) and N(s) from Eqs. (3.4), (3.5) and (3.6), and multiplication of the numerator and denominator of $S_{G_i}^{W}$ by the denominator of L(s) yields:

$$S_{G_i}^{W} = \frac{1 + k_{i+1}G_{i+1}\cdots G_n + k_{i+2}G_{i+2}\cdots G_n + \ldots + k_nG_n}{1 + k_{i+1}G_{i+1}\cdots G_n + \ldots + k_nG_n + [k_1G_1\cdots G_n + k_2G_2\cdots G_n + \ldots + k_iG_i\cdots G_n]}$$

$$= \frac{1 + k_{i+1}G_{i+1}\cdots G_n + k_{i+2}G_{i+2}\cdots G_n + \ldots + k_nG_n}{1 + k_1G_1\cdots G_n + k_2G_2\cdots G_n + \ldots + k_nG_n}$$

$$= \frac{1 + \sum\limits_{j=1}^{n} k_{i+j} \prod\limits_{\ell=i+j}^{n} G_\ell}{1 + \sum\limits_{j=1}^{n} k_j \prod\limits_{\ell=j}^{n} G_\ell}$$

This is Eq. (4.4). W(s) may be found from $S_{G_n}^{W}$.

$$W(s) = \frac{Y(s)}{R(s)} = \frac{E_n(s)}{R(s)} G_1(s) \ldots G_n(s) = S^W_{G_n} G_1 \ldots G_n$$

$$= \frac{G_1 \ldots G_n}{1 + k_1 G_1 \ldots G_n + k_2 G_2 \ldots G_n + \ldots + k_n G_n}$$

$$= \frac{\prod_{i=1}^{n} G_i}{1 + \sum_{\ell=1}^{n} k_\ell \prod_{j=\ell}^{n} G_j}$$

This is Eq. (4.3).

From Eq. (3.10) and with reference to Fig. (3.2),

$$S^W_{k_i} = \frac{D_i(s)}{R(s)} = -k_i \frac{1}{N(s)} \frac{Y(s)}{R(s)}$$

$$= \frac{-k_i}{G_1 G_2 \ldots G_{i-1}} W(s)$$

$$= \frac{-k_i G_i G_{i+1} \ldots G_n}{1 + k_1 G_1 \ldots G_n + k_2 G_2 \ldots G_n + \ldots + k_n G_n}$$

$$= \frac{-k_i \prod_{j=1}^{n} G_j}{1 + \sum_{j=1}^{n} k_j \prod_{\ell=j}^{n} G_\ell}$$

This is Eq. (4.6).

REFERENCES

H.W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand, 1945.

J.H. Dial, "The Specification and Synthesis of High-Order Control Systems," M.S. Thesis, The University of Arizona, June 1967.

R.A. Haddad and J.G. Truxal, "Sensitivity and Stability in Multiloop Systems," Joint Automatic Control Conference, 1964.

I.M. Horowitz, Synthesis of Feedback Systems, Academic Press, 1963.

R.E. Kalman, "When Is a Linear Control System Optimal?" Journal of Basic Engineering, March 1964.

G.C. Newton, L.A. Gould and J.F. Kaiser, Analytical Design of Linear Feedback Controls, Wiley, 1957.

D.G. Schultz and J.L. Melsa, State Functions and Linear Control Systems, McGraw-Hill, 1967.

R. Tomovic, Sensitivity Analysis of Dynamic Systems, McGraw-Hill, 1964.

J.G. Truxal, Automatic Feedback Control System Synthesis, McGraw-Hill, 1955.

J.E. Van Ness, J.M. Boyle and F.P. Imad, Sensitivities of Large, Multiple-Loop Control Systems, IEEE Transactions on Automatic Control, July 1965.