The University of Michigan
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Technical Report

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# ABSTRACT

A kinetic equation for an inhomogeneous and non-isotropic plasma is derived in the plasma limit. The treatment is based on the joint solution of the first two members of the BBKGY hierarchy, and on a linearization procedure about the unperturbed state. The unperturbed distribution functions. $f_0(\eta)$ and $g_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$, are unspecified. Furthermore, the unperturbed pair correlation function, $g_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$, is eliminated in favor of the unperturbed one-particle distribution function, $f_0(\eta)$. This elimination is accomplished by solving the equation for $g_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$. The resulting kinetic equation is free from Bogolyubov's adiabatic hypothesis; therefore, it is valid for arbitrary frequency. In the limiting case when the frequency under consideration is much higher than the collision-frequency, a general expression for the high-frequency electric conductivity tensor is derived. From this general expression the results for the homogeneous and isotropic case previously derived by Klevans and Wu[18], as well as the results for the thermodynamic equilibrium case derived by Oberman, Ron, and Dawson[16] can be recovered.

# PART I. THE KINETIC EQUATION

## Chapter I. Derivation of a Kinetic Equation for a Plasma.

### 1. Introduction

It is well-known that the BBKGY[1] hierarchy of equations (which is obtained by successive integrations of the Liouville equation) is a systematic starting point to approach kinetic theory. Unfortunately, this hierarchy is a chain of coupled equations which, so far, has not been decoupled rigorously for any non-equilibrium situation. However, in case of the kinetic theory of plasmas, the equations have been expanded[2-5] in terms of the so-called plasma parameter, $\epsilon \equiv (n\lambda_D^3)^{-1}$, where $n$ is the particle density, and $\lambda_D$ is the Debye length (i.e., the inverse of the number of particles in a Debye sphere). For a high-temperature plasma, $\epsilon$ is small, so that quantities like the three particle correlation function which are of higher order in $\epsilon$ can be neglected. Based on this argument, the BBKGY hierarchy can be truncated after the second equation. This results in a set of two equations for f (the one-particle distribution function), and g (the pair correlation function). Symbolically, the set of these equations can be written as

$$V[f] = L[g] \quad , \tag{I-1}$$

$$M[g] = N[f] \quad , \tag{I-2}$$

where V, L, M, and N, are operators. In principle, Eq. (I-2) (together with suitable initial conditions) determine g in terms of f; substitution into Eq. (I-1) yields the desired kinetic equation. However, the problem of obtaining a general solution of Eq. (I-2) without further assumptions is too difficult to manage. Hence, many approximation schemes[2-8] have been contrived. Here, we only name two of them which have connection with this thesis.

(1) The BGL[3-5] (Balescu, Guernsey, and Lenard) approximation

(2) The Guernsey[6] "small-amplitude" approximation.

(1) The BGL approximation

Since the BGL-approximation is described extensively in the literature[1], we shall say only a few words about it. The BGL-approximation is based on the following assumptions: (a) the system is spatially uniform, and (b) Bogolubov's adiabatic hypothesis; this means that the one-particle distribution function remains stationary on a time scale during which the two-particle distribution function changes.

However, in high frequency phenomena (e.g., plasma oscillations, or microwave propagation in a plasma) Bogolubov's adiabatic hypothesis is not valid. Furthermore, the spatially inhomogeneous effects are lost through the assumption of spatial uniformity. Thus, an alternative treatment is desired. This lead to the development of Guernsey's "small-amplitude" approximation.

(2) Guernsey's "small-amplitude" approximation

Guernsey[6] has shown that, if Eqs. (I-1) and (I-2) are linearized about thermodynamic equilibrium, the resulting equation for the linearized pair correlation function can be solved exactly in terms of the one-particle distribution function. The one-particle distribution function is now allowed to vary with time in an arbitrary way, and is no longer "frozen" while the pair correlation function changes. The substitution of the resulting expression into the linearized version of Eq. (I-1) gives a so-called "small amplitude" kinetic equation which correctly describes mixed situations in which "collisional" and plasma-oscillation effects both play an important part.

In this approach, the assumptions (a) and (b) (in the BGL-approximation) are replaced by a condition of small deviations from thermodynamic equilibrium (electrons and ions have Maxwellian distributions with equal temperature, and the two-particle distribution function is Debye-Hückel). In reality the condition of thermodynamic equilibrium is usually not met. In this case a more general theory is desirable.

In this chapter we shall generalize Guernsey's "small-amplitude" kinetic equation; by that we mean the following:

(1) The Maxwellian distribution function is replaced by an arbitrary unperturbed one-particle distribution function, $f_0(\eta)$,

(2) The Debye-Hückel distribution function is replaced by an arbitrary unperturbed two-particle distribution function $g_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$, where $\eta \equiv (\underset{\sim}{v}, \sigma)$, and $\sigma$ denotes the $\sigma$-type ions.

This allows us to eliminate the condition that the unperturbed state is in thermodynamic equilibrium.

It should be mentioned that C.S. Wu[8] has used an operator method[7] to approach this approximation, and has developed a "useful" operator for the same purpose.

2a. Basic Equations

We consider a gas of charged particles interacting only through a Coulomb potential. An arbitrary number of ion types is assumed, with $N_\sigma$ ions of type $\sigma$ (charge $e_\sigma$, mass $m_\sigma$). A "small" electric field, $\underset{\sim}{E}_a(\underset{\sim}{r}, t)$, is applied to the system; by "small" we mean that under the influence of the field, the system has only "small" departures from its unperturbed state. The system is described in general by the Liouville equation, or the BBKGY hierarchy of equations derived from it by integrating over the coordinates and momenta of all but one particle, two particles, etc. To first order in the plasma parameter, $\epsilon \equiv (n \lambda_D^3)^{-1}$, the system may be described by the one-particle distribution function $f(y, t)$ and the pair correlation function $g(y, y', t)$ [with $y \equiv (\underset{\sim}{v}, \underset{\sim}{r}, \sigma)$]

which satisfy the following first two members of the truncated hierarchy equations[1]: (The hierarchy is truncated in the usual[2-4] way by keeping only terms which are formally of zeroth and first order in $\epsilon$.)

$$\left[\frac{\partial}{\partial t}+\underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}}+\frac{e_\sigma}{m_\sigma}\underset{\sim}{E}(\underset{\sim}{r},t)\cdot\frac{\partial}{\partial \underset{\sim}{v}}\right]f(y,t)$$
$$=\frac{e_\sigma}{m_\sigma}\int dy'\, e_{\sigma'}\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}}g(y,y',t) \qquad \text{(I-1-a)}$$

$$\left(\frac{\partial}{\partial t}+\underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}}+\underset{\sim}{v}'\cdot\frac{\partial}{\partial \underset{\sim}{r}'}\right)g(y,y',t)$$
$$=\left\{\frac{e_\sigma}{m_\sigma}\left[\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{e_{\sigma'}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}}f(y,t)f(y',t)-\underset{\sim}{E}(\underset{\sim}{r},t)\cdot\frac{\partial}{\partial \underset{\sim}{v}}g(y,y',t)\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial \underset{\sim}{v}}f(y,t)\int dy''\, e_{\sigma''}\,g(y',y'',t)\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}''|}\right)\right]\right\} \qquad \text{(I-2-a)}$$
$$+\{y\longleftrightarrow y'\}$$

where the symbol $\{y\longleftrightarrow y'\}$ means that the expression in the curved brace has the primed and un-primed quantities interchanged,

$$\int dy\equiv\int d\underset{\sim}{r}\int d\underset{\sim}{v}\sum_\sigma \quad ,$$

$$\underset{\sim}{E}(\underset{\sim}{r},t)\equiv\underset{\sim}{E}_a(\underset{\sim}{r},t)-\int dy''\, e_{\sigma''}f(y'',t)\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}''|}\right) , \qquad \text{(I-3)}$$

and the normalization has been chosen such that

$$\int d\underset{\sim}{v}\int d\underset{\sim}{r}\, f(\underset{\sim}{r},\underset{\sim}{v},\sigma)=N_\sigma \quad .$$

Under the assumption of small departure from the unperturbed state, Eqs. (I-1-a) and (I-2-a) may be linearized by setting

$$f(y,\ t) = f_0(\eta) + f_1(y,\ t) \quad ,$$

$$g(y,\ y',\ t) = g_0(\eta,\ \eta',\ \underset{\sim}{r} - \underset{\sim}{r}') + g_1(y,\ y',\ t) \quad , \qquad \text{(I-4)}$$

$$\eta \equiv (\underset{\sim}{v},\ \sigma) \quad ,$$

where $f_0$ and $g_0$ designate, respectively, the one-particle distribution function and pair correlation function of the unperturbed state, and $f_1$ and $g_1$ are their perturbed parts; with

$$|f_1| \ll |f_0| \quad , \quad |g_1| \ll |g_0|$$

Now, we substitute the expression (I-4) into Eqs. (I-1-a) and (I-2-a), and ignore second-order terms in $f_1$ and $g_1$ as well as terms like $\underset{\sim}{E} f_1$ and $\underset{\sim}{E} g_1$; then we obtain the following set of linear integro-differential equations for, $g_0$, $f_1$, and $g_1$:

$$\left(\underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}} + \underset{\sim}{v}'\cdot\frac{\partial}{\partial \underset{\sim}{r}'}\right) g_0(\eta, \eta', \underset{\sim}{r}-\underset{\sim}{r}')$$

$$= \left\{\frac{e_\sigma}{m_\sigma}\frac{\partial}{\partial \underset{\sim}{v}} f_0(\eta)\cdot\left[e_{\sigma'} f_0(\eta')\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) + \int dy'' e_{\sigma''}\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}''|}\right) g_0(\eta', \eta'', \underset{\sim}{r}'-\underset{\sim}{r}'')\right]\right\}$$

$$+ \{y \longleftrightarrow y'\} \qquad \text{(I-5)}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}}\right) f_1(y, t) + \frac{e_\sigma}{m_\sigma}\underset{\sim}{E}(\underset{\sim}{r}, t)\cdot\frac{\partial}{\partial \underset{\sim}{v}} f_0(\eta)$$

$$= \frac{e_\sigma}{m_\sigma}\int dy' e_{\sigma'}\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}} g_1(y, y', t) \qquad \text{(I-6)}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}+\mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{r}'}\right)g_1(\eta,\eta',t)$$

$$=\Bigg\{\frac{e_\sigma}{m_\sigma}\Bigg[\left(\frac{\partial}{\partial \mathbf{r}}\frac{e_{\sigma'}}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\left(f_0(\eta)f_1(\eta',t)+f_0(\eta')f_1(\eta,t)\right)-\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{v}}g_0(\eta,\eta',\mathbf{r}-\mathbf{r}')$$

$$+\frac{\partial}{\partial \mathbf{v}}f_1(\eta,t)\cdot\int d\eta''\, e_{\sigma''}\, g_0(\eta',\eta'',\mathbf{r}'-\mathbf{r}'')\left(\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}''|}\right)$$

$$+\frac{\partial}{\partial \mathbf{v}}f_0(\eta)\cdot\int d\eta''\, e_{\sigma''}\, g_1(\eta',\eta'',t)\left(\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}''|}\right)\Bigg]\Bigg\}+\left\{\eta\longleftrightarrow\eta'\right\}. \qquad \text{(I-7)}$$

Equation (I-5) contains only the unperturbed quantities; we shall use it to express $g_0$ in terms of $f_0$ in the next chapter. In this chapter, we shall concern ourselves with Eqs. (I-6) and (I-7).

## 2b. Reduction to Integral Equations

In order to deal with Eqs. (I-6) and (I-7), it is simpler to pass from the ($\mathbf{r}$, t) representation to the ($\mathbf{k}$, $\omega$) representation by taking Fourier-Laplace transforms. Defining

$$\mathcal{G}(\eta,\eta',\mathbf{k},\mathbf{k}',\omega)\equiv\int_0^\infty dt\iint d\mathbf{r}\,d\mathbf{r}'\,e^{i(\omega t+\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}')}\,g_1(\eta,\eta',t)\,, \qquad \text{(I-8)}$$

$$\mathcal{F}(\eta,\mathbf{k},\omega)\equiv\int_0^\infty dt\int d\mathbf{r}\,e^{i(\omega t+\mathbf{k}\cdot\mathbf{r})}\,f_1(\eta,t) \qquad \text{(I-9)}$$

and taking Fourier-Laplace transforms of Eqs. (I-6) and (I-7), we obtain the following set of equations:

$$(\omega+\mathbf{k}\cdot\mathbf{v})\mathcal{F}(\eta,\mathbf{k},\omega)=-i\frac{e_\sigma}{m_\sigma}\mathbf{E}(\mathbf{k},\omega)\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\eta)$$
$$-\frac{4\pi}{(2\pi)^3}\int d\mathbf{k}'\frac{\mathbf{k}'}{k'^2}\cdot\frac{\partial}{\partial\mathbf{v}}\int d\eta' e_{\sigma'}\mathcal{G}(\eta,\eta',\mathbf{k}+\mathbf{k}',-\mathbf{k}',\omega)+i\,\mathcal{F}_I(\eta,\mathbf{k}), \qquad \text{(I-10)}$$

$$(\omega+\mathbf{k}\cdot\mathbf{v}+\mathbf{k}'\cdot\mathbf{v}')\mathcal{G}(\eta,\eta',\mathbf{k},\mathbf{k}',\omega)=i\,\mathcal{G}_I(\eta,\eta',\mathbf{k},\mathbf{k}')$$
$$+\left\{4\pi\frac{e_\sigma}{m_\sigma}\frac{\mathbf{k}}{k^2}\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\eta)\int d\eta'' e_{\sigma''}\mathcal{G}(\eta',\eta'',\mathbf{k}',\mathbf{k},\omega)\right\}+\begin{Bmatrix}\eta\leftrightarrow\eta'\\ \mathbf{k}\leftrightarrow\mathbf{k}'\end{Bmatrix}$$
$$+\Big\{4\pi\frac{e_\sigma}{m_\sigma}e_{\sigma'}\Big[\frac{\mathbf{k}}{k^2}\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\eta)\mathcal{F}(\eta',\mathbf{k}+\mathbf{k}',\omega)-f_0(\eta')\frac{\mathbf{k}'}{k'^2}\cdot\frac{\partial}{\partial\mathbf{v}}\mathcal{F}(\eta,\mathbf{k}+\mathbf{k}',\omega)\Big]$$
$$-\frac{e_\sigma}{m_\sigma}\Big[i\mathbf{E}(\mathbf{k}+\mathbf{k}',\omega)\cdot\frac{\partial}{\partial\mathbf{v}}\mathcal{G}_0(\eta,\eta',-\mathbf{k}')+\frac{4\pi\mathbf{k}'}{k'^2}\cdot\frac{\partial}{\partial\mathbf{v}}\mathcal{F}(\eta,\mathbf{k}+\mathbf{k}',\omega)\int d\eta'' e_{\sigma''}\mathcal{G}_0(\eta',\eta'',\mathbf{k}')\Big]\Big\}$$
$$+\begin{Bmatrix}\eta\leftrightarrow\eta'\\ \mathbf{k}\leftrightarrow\mathbf{k}'\end{Bmatrix} \qquad \text{(I-11)}$$

where the symbol $\begin{Bmatrix}\eta\leftrightarrow\eta'\\ \mathbf{k}\leftrightarrow\mathbf{k}'\end{Bmatrix}$ means that the immediately preceding expression in the curved brace has the primed and unprimed quantities interchanged, and

$$\mathbf{E}(\mathbf{k},\omega)\equiv\mathbf{E}_a(\mathbf{k},\omega)+4\pi i\frac{\mathbf{k}}{k^2}\int d\eta'' e_{\sigma''}\mathcal{F}(\eta'',\mathbf{k},\omega), \qquad \text{(I-12)}$$

$$\left.\begin{aligned}\mathcal{G}_I(\eta,\eta',\mathbf{k},\mathbf{k}')&\equiv\iint d\mathbf{r}\,d\mathbf{r}'\,e^{i(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}')}g_1(y,y',0)\\ \mathcal{F}_I(\eta,\mathbf{k})&\equiv\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}f_1(y,0)\end{aligned}\right\}, \qquad \text{(I-13)}$$

Now, our task is to solve Eq. (I-11) by obtaining an expression of $\mathcal{G}$ in terms of $\mathcal{F}$, $\mathcal{G}[\mathcal{F}]$; then we substitute this $\mathcal{G}[\mathcal{F}]$ in Eq. (I-10) to obtain the desired kinetic equation. But by examining the form of Eqs. (I-10) and (I-11), we find the following: (a) Eq. (I-11) indicates that $\mathcal{G}$ is completely determined by

$$G(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) \equiv \int d\eta'\, e_{\sigma}' \mathcal{G}(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}', \omega) \quad , \qquad \text{(I-14)}$$

(b) from Eq. (I-10) it is this quantity, G, which is needed for the kinetic equation.

For convenience, we introduce the following notations:

$$G_0(\eta, \underset{\sim}{k}) \equiv \int d\eta''\, e_{\sigma}'' \mathcal{G}_0(\eta, \eta'', \underset{\sim}{k}) \quad , \qquad \text{(I-15)}$$

$$D_0(\eta, \underset{\sim}{k}) \equiv -4\pi^2 \frac{e_\sigma}{m_\sigma} \hat{\underset{\sim}{k}} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0(\eta) \quad , \qquad \text{(I-16)}$$

and

$$\begin{aligned}
\mathcal{P}(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}', \omega) \equiv\ & \mathcal{G}_I(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') \\
& + \Big\{ \frac{i e_{\sigma'}}{\pi k} D_0(\eta, \underset{\sim}{k}) \mathcal{F}(\eta', \underset{\sim}{k}+\underset{\sim}{k}', \omega) - \frac{e_\sigma}{m_\sigma} \underset{\sim}{E}(\underset{\sim}{k}+\underset{\sim}{k}', \omega) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \mathcal{G}_0(\eta, \eta', -\underset{\sim}{k}') \\
& + 4\pi i \frac{e_\sigma}{m_\sigma} \big[ e_{\sigma'} f_0(\eta') + G_0(\eta', \underset{\sim}{k}') \big] \frac{\underset{\sim}{k}'}{k'^2} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \mathcal{F}(\eta, \underset{\sim}{k}+\underset{\sim}{k}', \omega) \Big\} \\
& + \left\{ \begin{matrix} \eta \leftrightarrow \eta' \\ \underset{\sim}{k} \leftrightarrow \underset{\sim}{k}' \end{matrix} \right\} . \qquad \text{(I-17)}
\end{aligned}$$

Then Eqs. (I-10) and (I-11) can be expressed as follows:

$$(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v})\,\mathcal{F}(\eta,\underset{\sim}{k},\omega) = -i\frac{e_\sigma}{m_\sigma}\underset{\sim}{E}(\underset{\sim}{k},\omega)\cdot\frac{\partial}{\partial\underset{\sim}{v}}f_0(\eta)$$

$$-\frac{4\pi}{(2\pi)^3}\frac{e_\sigma}{m_\sigma}\int d\underset{\sim}{k}'\,\frac{\underset{\sim}{k}'}{k'^2}\cdot\frac{\partial}{\partial\underset{\sim}{v}}G(\eta,\underset{\sim}{k}+\underset{\sim}{k}',-\underset{\sim}{k}',\omega) + i\,\mathcal{F}_I(\eta,\underset{\sim}{k}), \qquad \text{(I-10-a)}$$

$$(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}')\,\mathcal{G}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega) = i\,\mathcal{P}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega)$$

$$-\frac{1}{\pi k}D_0(\eta,\underset{\sim}{k})G(\eta',\underset{\sim}{k}',\underset{\sim}{k},\omega) - \frac{1}{\pi k'}D_0(\eta',\underset{\sim}{k}')G(\eta,\underset{\sim}{k},\underset{\sim}{k}',\omega)\,. \qquad \text{(I-11-a)}$$

We further note that the Laplace transform defined in Eqs. (I-8) and (I-9) requires $\mathrm{Im}\,\omega > 0$. In order to simplify the calculations, we let $\omega$ approach the real axis from above (it is clear that the result can then be analytically continued). From now on, we shall concern ourselves with this limit. In this limit

$$(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}')^{-1} \longrightarrow -2\pi i\,\delta_+(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}')$$

where

$$\delta_\pm(x) \equiv \frac{1}{2}\delta(x) \pm \frac{i}{2\pi}P\left(\frac{1}{x}\right)$$
$$\equiv \pm\frac{i}{2\pi}\lim_{\epsilon\to 0^+}\left(\frac{1}{x \pm i\epsilon}\right), \qquad \text{(I-18)}$$

with $\delta(x)$ denoting the Dirac delta function and P the Cauchy principal value. Equation (I-11-a) can then formally be rewritten as

$$\mathcal{G}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega) = (\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}' + i\epsilon)^{-1}$$

$$\cdot\left[ i\mathcal{P}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega) - \frac{D_0(\eta,\underset{\sim}{k})}{\pi k} G(\eta',\underset{\sim}{k}',\underset{\sim}{k},\omega) - \frac{D_0(\eta',\underset{\sim}{k}')}{\pi k'} G(\eta,\underset{\sim}{k},\underset{\sim}{k}',\omega)\right]. \quad \text{(I-11-b)}$$

In order to reduce to an integral equation for G, we multiply Eq. (I-11-b) by $e_{\sigma'}$ and integrate it over $\eta'$; the result is

$$\left[1 + \frac{1}{\pi k'^2}\int d\eta' e_{\sigma'} \frac{k' D_0(\eta',\underset{\sim}{k}')}{(\underset{\sim}{k}'\cdot\underset{\sim}{v}' + \omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)}\right] G(\eta,\underset{\sim}{k},\underset{\sim}{k}',\omega)$$

$$= i\int d\eta' e_{\sigma'} \frac{\mathcal{P}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega)}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}' + i\epsilon)} - \frac{D_0(\eta,\underset{\sim}{k})}{\pi k k'}\int d\eta' e_{\sigma'} \frac{k' G(\eta',\underset{\sim}{k}',\underset{\sim}{k},\omega)}{(\underset{\sim}{k}'\cdot\underset{\sim}{v}' + \omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)} \quad . \quad \text{(I-19)}$$

For convenience, we defined:

$$\Delta(u,\underset{\sim}{k}) \equiv 1 + \frac{1}{\pi k^2}\int d\eta' e_{\sigma'} \frac{D_0(\eta',\underset{\sim}{k})}{(\underset{\sim}{k}\cdot\underset{\sim}{v}' - u - i\epsilon)} \quad , \quad \text{(I-20)}$$

$$q(\eta,\underset{\sim}{k},\underset{\sim}{k}',\omega) \equiv i\int d\eta' e_{\sigma'} \frac{\mathcal{P}(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}',\omega)}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{k}'\cdot\underset{\sim}{v}' + i\epsilon)} \quad , \quad \text{(I-21)}$$

where the superscript * denotes complex conjugate. Then Eq. (I-19) can be expressed (suppressing the argument, $\omega$ in G, q, $\mathcal{F}$ ) as

$$\overset{*}{\Delta}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = q(\eta,\underset{\sim}{k},\underset{\sim}{k}')$$

$$-\frac{D_0(\eta,\underset{\sim}{k})}{\pi k k'}\int d\eta' e_{\sigma'} \frac{k' G(\eta',\underset{\sim}{k}',\underset{\sim}{k})}{(\underset{\sim}{k}'\cdot\underset{\sim}{v}'+\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)} \quad . \qquad \text{(I-19-a)}$$

This is a singular integral equation for G; we shall find its solution by using a technique from the theory of singular integral equations developed by Muskhelishvili[9].

3. Solution to the Integral Equation

By examining Eq. (I-19-a) we find that the kernel of this integral equation depends on $\underset{\sim}{v}'$ only through the combination $\hat{\underset{\sim}{k}}' \cdot \underset{\sim}{v}'$. In fact Eq. (I-19-a) can be written as

$$\overset{*}{\Delta}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = q(\eta,\underset{\sim}{k},\underset{\sim}{k}')$$

$$-\frac{D_0(\eta,\underset{\sim}{k})}{\pi k k'}\int du' \frac{\overline{G}(u',\underset{\sim}{k}',\underset{\sim}{k})}{\left(u'+\frac{\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}+i\epsilon\right)} \quad , \qquad \text{(I-22)}$$

where the "barring" operation is defined as

$$\overline{G}(u', \underset{\sim}{k}', \underset{\sim}{k}) \equiv \int d\eta'\, e_{\sigma'}\, \delta(u' - \underset{\sim}{k}' \cdot \underset{\sim}{v}')\, G(\eta', \underset{\sim}{k}', \underset{\sim}{k}) \quad . \qquad \text{(I-23)}$$

Equation (I-22) shows that G is completely determined by $\overline{G}$. By performing the barring operation on Eq. (I-22), we find

$$\Delta^*\left(\frac{-\omega-ku}{k'},k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')$$

$$=\bar{q}(u,\underset{\sim}{k},\underset{\sim}{k}')-\frac{\bar{D}_0(u)}{\pi kk'}\int du'\,\frac{k'\,\bar{G}(u',\underset{\sim}{k}',\underset{\sim}{k})}{(k'u'+\omega+ku+i\epsilon)}\quad, \tag{I-24}$$

where $\bar{q}$, $\bar{D}_0$ are defined analogously to $\bar{G}$. Changing the dummy variable of integration from u' to - u', Eq. (I-24) becomes

$$\Delta^*\left(\frac{-\omega-ku}{k'},k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')$$

$$=\bar{q}(u,\underset{\sim}{k},\underset{\sim}{k}')+\frac{\bar{D}_0(u)}{\pi kk'}\int du\,\frac{k'\bar{G}(-u',\underset{\sim}{k}',\underset{\sim}{k})}{(k'u'-\omega-ku-i\epsilon)}\quad. \tag{I-24-a}$$

Since from Eq. (I-18)

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{x\mp i\epsilon}\right)=\pm i\pi\left[\delta(x)\mp\frac{i}{\pi}P\left(\frac{1}{x}\right)\right]\quad,$$

thus Eq. (I-24-a) may be rewritten as

$$\Delta^*\left(\frac{-\omega-ku}{k'},k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')=\bar{q}(u,\underset{\sim}{k},\underset{\sim}{k}')$$

$$+\frac{i\bar{D}_0(u)}{kk'}\left\{\bar{G}\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right)-\frac{i}{\pi}P\int du'\,\frac{\bar{G}(-u',\underset{\sim}{k}',\underset{\sim}{k})}{\left(u'-\frac{\omega+ku}{k'}\right)}\right\}. \tag{I-24-b}$$

Furthermore, from the definitions of $\Delta$ and $\bar{D}_0$ [cf. Eq. (I-20)], we find

$$\Delta=\Delta_1+i\Delta_2 \tag{I-20-a}$$

with

$$\Delta_1(u,k) \equiv 1 + \frac{1}{k^2}\,\frac{1}{\pi}\,P\!\int du'\,\frac{\bar{D}_0(u')}{(u'-u)} \quad , \tag{I-25-a}$$

$$\Delta_2(u,k) \equiv \frac{1}{k^2}\,\bar{D}_0(u) \quad , \tag{I-25-b}$$

and

$$\Delta_1(u,k) = 1 + \frac{1}{\pi}\,P\!\int du'\,\frac{\Delta_2(u';k)}{(u'-u)} \quad . \tag{I-25-c}$$

Hence, we may write Eq. (I-24-b) somewhat more explicitly as

$$\begin{aligned} &\Delta^*\!\left(\frac{-\omega-ku}{k'},k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}') \\ &-\frac{ik\Delta_2(u,k)}{k'}\left\{\bar{G}\!\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right)-\frac{i}{\pi}P\!\int du'\,\frac{\bar{G}(-u',\underset{\sim}{k}',\underset{\sim}{k})}{\left(u'-\frac{\omega+ku}{k'}\right)}\right\} \\ &=\bar{g}(u,\underset{\sim}{k},\underset{\sim}{k}'). \end{aligned} \tag{I-26}$$

Since Eq. (I-26) is to hold for arbitrary u, $\omega$, $\underset{\sim}{k}$, $\underset{\sim}{k}'$, we may interchange $\underset{\sim}{k}$ and $\underset{\sim}{k}'$, then replace u by $-[(\omega + ku)/k']$ (in this order) to obtain a second equation

$$\begin{aligned} &\Delta^*(u,k)\,\bar{G}\!\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right) \\ &-\frac{ik'}{k}\Delta_2\!\left(\frac{-\omega-ku}{k'},k'\right)\left\{\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')-\frac{i}{\pi}P\!\int du'\,\frac{\bar{G}(-u',\underset{\sim}{k},\underset{\sim}{k}')}{(u'+u)}\right\} \\ &=\bar{g}\!\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right). \end{aligned} \tag{I-27}$$

In Eq. (I-27) changing the dummy variable of integration from u' to - u', we obtain

$$\begin{aligned} &\Delta^*(u,k)\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) \\ &-\frac{ik'}{k}\Delta_2\left(\frac{-\omega-ku}{k'}, k'\right)\left\{\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')+\frac{i}{\pi}P\int du' \frac{\bar{G}(u',\underset{\sim}{k},\underset{\sim}{k}')}{(u'-u)}\right\} \\ &= \bar{g}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right). \end{aligned} \tag{I-28}$$

Introducing the Hilbert transform operator

$$\begin{aligned} &H[\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')] \equiv \frac{1}{\pi}P\int du' \frac{\bar{G}(u',\underset{\sim}{k},\underset{\sim}{k}')}{(u'-u)}, \\ &H\left[\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right] \equiv \frac{1}{\pi}P\int du' \frac{\bar{G}(-u',\underset{\sim}{k}',\underset{\sim}{k})}{\left(u'-\frac{\omega+ku}{k'}\right)} \end{aligned} \tag{I-29}$$

then we may write Eqs. (I-26) and (I-28) as

$$\begin{aligned} &\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}') \\ &-\frac{ik}{k'}\Delta_2(u,k)\left\{\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) - iH\left[\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right]\right\} = \bar{g}(u,\underset{\sim}{k},\underset{\sim}{k}'), \end{aligned} \tag{I-26-a}$$

$$\begin{aligned} &\Delta^*(u,k)\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) \\ &-\frac{ik'}{k}\Delta_2\left(\frac{-\omega-ku}{k'}, k'\right)\left\{\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')+iH[\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')]\right\} = \bar{g}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right). \end{aligned} \tag{I-28-a}$$

Multiplying Eq. (I-28-a) by $\frac{k}{k'}$ then subtracting the resulting expression from Eq. (I-26-a), we find the following integral equation relating $\bar{G}(u, \underset{\sim}{k}, \underset{\sim}{k}')$ and $\bar{G}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)$:

$$\Delta_1\left(\frac{-\omega-ku}{k'},k'\right)\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}') - \Delta_2\left(\frac{-\omega-ku}{k'},k'\right) H\left[\bar{G}(u,\underset{\sim}{k},\underset{\sim}{k}')\right]$$
$$- \frac{k}{k'}\left\{\Delta_1(u,k)\bar{G}\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right) + \Delta_2(u,k) H\left[\bar{G}\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right)\right]\right\} \tag{I-30}$$
$$= \bar{g}(u,\underset{\sim}{k},\underset{\sim}{k}') - \frac{k}{k'}\bar{g}\left(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k}\right).$$

The singular integral equation of type Eq. (I-30) has been studied in considerable detail by Russian mathematicians[9]. However, much of the formalism may be dispensed with in our case, due to the nature of the coefficients, $\Delta_1$, $\Delta_2$. We will need the following result from the theory of Cauchy integrals:

Let f(u) be any function satisfying a Hölder condition on the real axis. Then the complex function

$$\mathcal{F}(Z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \frac{f(u')}{u' - Z} \tag{I-31}$$

is analytic except on the real axis, which is a branch cut, and

$$\left.\begin{aligned} &\mathcal{F}^{(+)}(u) - \mathcal{F}^{(-)}(u) = f(u) \quad , \\ &\mathcal{F}^{(+)}(u) + \mathcal{F}^{(-)}(u) = - i\, H[f(u)] \quad , \\ &\text{where} \\ &\mathcal{F}^{(\pm)}(u) \equiv \lim_{\epsilon \to 0^+} \mathcal{F}(u \pm i\epsilon) \quad . \end{aligned}\right\} \tag{I-32}$$

Eq. (I-32) are known as Plemelj formulas. (cf. Muskhelishvili[11])

Before starting to find the solution of Eq. (I-30), we note that its coefficients $\Delta_1$, $\Delta_2$ have the following interesting and helpful properties:

(a) From the definition of $\Delta$[cf. to Eq. (I-25)]

$$\Delta_1 = 1 + H[\Delta_2]$$

(b) Since $f_0$ and its derivatives must vanish as $|\underset{\sim}{v}| \to \infty$

$$\lim_{|u| \to \infty} \Delta_1(u) = 1 \quad ; \quad \lim_{|u| \to \infty} \Delta_2(u) = 0$$

(c) It is assumed that $\Delta_1(u)$, $\Delta_2(u)$ are not simultaneously zero for any real u (this is certainly true if $f_0$ is Maxwellian).

We now turn to the solution of Eq. (I-30). We start by introducing the following complex functions, which are known as Cauchy's integrals:

$$\Gamma_1(z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \, \frac{\bar{G}(u', \underset{\sim}{k}, \underset{\sim}{k}')}{(u' - z)} \quad , \tag{I-33}$$

$$\Gamma_2(z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \, \frac{\frac{k}{k'}\bar{G}\left(\frac{-\omega - ku'}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)}{(u' - z)} \quad , \tag{I-34}$$

$$\Phi_1(z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \, \frac{\bar{g}(u', \underset{\sim}{k}, \underset{\sim}{k}')}{(u' - z)} \quad , \tag{I-35}$$

$$\Phi_2(z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \, \frac{\frac{k}{k'}\bar{g}\left(\frac{-\omega - ku'}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)}{(u' - z)} \quad , \tag{I-36}$$

$$\mathfrak{D}(Z,k) \equiv \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} du' \, \frac{\Delta_2(u',k)}{(u'-Z)} \quad . \tag{I-37}$$

Provided $\overline{G}$ and $\overline{q}$ satisfy Hölder conditions on the real axis these functions are analytic except on the real axis, which is a branch cut. As $|Z| \to \infty$ the $\Gamma$'s and the $\Phi$'s approach zero, and

$$\lim_{|Z| \to \infty} \mathfrak{D}(Z, k) = \frac{1}{2} \quad . \tag{I-38}$$

As Z approaches a point u on the real axis from above and below, we have the following Plemelj formulas:

$$\Gamma_1^{(+)}(u) - \Gamma_1^{(-)}(u) = \overline{G}(u, \underset{\sim}{k}, \underset{\sim}{k}') \quad , \tag{I-39}$$

$$\Gamma_1^{(+)}(u) + \Gamma_1^{(-)}(u) = - i H[\overline{G}(u, \underset{\sim}{k}, \underset{\sim}{k}')] \quad , \tag{I-40}$$

$$\Gamma_2^{(+)}(u) - \Gamma_2^{(-)}(u) = \frac{k}{k'} G\left(\frac{-\omega - ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) \quad , \tag{I-41}$$

$$\Gamma_2^{(+)}(u) + \Gamma_2^{(-)}(u) = - i \frac{k}{k'} H\left[\overline{G}\left(\frac{-\omega - ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right] . \tag{I-42}$$

The relations between the limiting values of the $\Phi$'s are the same as Eqs. (I-39) - (I-42) with $\Gamma$ replaced by $\Phi$ and $\overline{G}$ replaced by $\overline{q}$; further

$$\mathfrak{D}^{(+)}(u, k) - \mathfrak{D}^{(-)}(u, k) = i \Delta_2(u, k) \quad , \tag{I-43}$$

$$\begin{aligned} \mathfrak{D}^{(+)}(u, k) + \mathfrak{D}^{(-)}(u, k) &= 1 + H[\Delta_2(u, k)] \\ &= \Delta_1(u, k) \quad , \end{aligned} \tag{I-44}$$

Since these relations hold for arbitrary u, $\omega$, $\underset{\sim}{k}$, $\underset{\sim}{k}'$, we may interchange $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ and replace u by $-[(\omega + ku)/k']$ in Eqs. (I-43) and (I-44); then we take the complex conjugate of the resulting expressions to obtain the following relations:

$$\mathfrak{D}^{(+)*}\left(\frac{-\omega-ku}{k'}, k'\right) - \mathfrak{D}^{(-)*}\left(\frac{-\omega-ku}{k'}, k'\right) = -i\Delta_2\left(\frac{-\omega-ku}{k'}, k'\right) \qquad \text{(I-43-a)}$$

$$\mathfrak{D}^{(+)*}\left(\frac{-\omega-ku}{k'}, k'\right) + \mathfrak{D}^{(-)*}\left(\frac{-\omega-ku}{k'}, k'\right) = \Delta_1\left(\frac{-\omega-ku}{k'}, k'\right) \qquad \text{(I-44-a)}$$

Using the Plemelj formulas, Eq. (I-30) may be written as

$$\begin{aligned}&\left\{2\mathfrak{D}^{(+)*}\left(\frac{-\omega-ku}{k'}, k'\right)\Gamma_1^{(+)}(u) - 2\mathfrak{D}^{(+)}(u,k)\Gamma_2^{(+)}(u) + \Phi_2^{(+)}(u) - \Phi_1^{(+)}(u)\right\}\\ &-\left\{2\mathfrak{D}^{(-)*}\left(\frac{-\omega-ku}{k'}, k'\right)\Gamma_1^{(-)}(u) - 2\mathfrak{D}^{(-)}(u,k)\Gamma_2^{(-)}(u) + \Phi_2^{(-)}(u) - \Phi_1^{(-)}(u)\right\} = 0.\end{aligned} \qquad \text{(I-45)}$$

Now the complex function

$$\xi_1(Z) \equiv 2\mathfrak{D}^{*}\left(\frac{-\omega-kZ}{k'}, k'\right)\Gamma_1(Z) - 2\mathfrak{D}(Z,k)\Gamma_2(Z) + \Phi_2(Z) - \Phi_1(Z) \qquad \text{(I-46)}$$

is analytic everywhere except on the real axis and vanishes as $|Z| \to \infty$. But according to Eq. (I-45) the jump across the real axis is zero;

therefore $\xi_1(Z)$ is analytic "everywhere". It follows that $\xi_1(Z)$ must be identically zero. Thus

$$2\mathfrak{D}^*\left(\frac{-\omega-kZ}{k'}, k'\right)\Gamma_1(Z) - 2\mathfrak{D}(Z,k)\Gamma_2(Z) = \Phi_1(Z) - \Phi_2(Z) \tag{I-47}$$

The Plemelj formulas with Eq. (I-47) could be used to obtain an explicit relationship between $\overline{G}(u, \underset{\sim}{k}, \underset{\sim}{k}')$ and $\overline{G}[-(\omega + ku)/k', \underset{\sim}{k}', \underset{\sim}{k}]$; however, it proves to be more convenient to work with the $\Gamma$'s with the help of Eq. (I-47). In order to obtain the desired function, G, we proceed as follows:

(1) We go back to Eq. (I-22), and rewrite it in terms of the Hilbert transform operator. Then we use the Plemelj formulas to eliminate $\overline{G}$ and $H[\overline{G}]$ in favor of $\Gamma$'s. Thus we obtain

$$\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)G(\eta, \underset{\sim}{k}, \underset{\sim}{k}') = g(\eta, \underset{\sim}{k}, \underset{\sim}{k}')$$
$$+\frac{i}{k^2}D_0(\eta,\underset{\sim}{k})\left\{\left[\Gamma_2^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) - \Gamma_2^{(-)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\right] + \left[\Gamma_2^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) + \Gamma_2^{(-)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\right]\right\},$$

or

$$G(\eta, \underset{\sim}{k}, \underset{\sim}{k}') = \frac{g(\eta, \underset{\sim}{k}, \underset{\sim}{k}')}{\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)} + \frac{2i\,D_0(\eta,\underset{\sim}{k})\,\Gamma_2^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}{k^2\mathfrak{D}^{(+)*}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)}, \tag{I-48}$$

but from Eqs. (I-43-a) and (I-44-a)

$$\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) = 2\mathfrak{D}^{(+)*}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) \tag{I-49}$$

thus

$$G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = \frac{g(\eta,\underset{\sim}{k},\underset{\sim}{k}')}{\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)} + \frac{i}{k^2}\,\frac{D_0(\eta,\underset{\sim}{k})\,\Gamma_2^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}{\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)} \tag{I-50}$$

(2) To determine $\Gamma_2$, we return to Eq. (I-28-a) and use the Plemelj formulas to express $\overline{G}$, $\Delta$, in terms of $\mathfrak{D}$ and $\Gamma$'s

$$\begin{aligned} &2\mathfrak{D}^{(-)}(u,k)\left[\Gamma_2^{(+)}(u)-\Gamma_2^{(-)}(u)\right] + 2i\Delta_2\left(\frac{-\omega-ku}{k'}, k'\right)\Gamma_1^{(-)}(u) \\ &= \frac{k}{k'}\,\overline{g}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right); \end{aligned} \tag{I-51}$$

using Eq. (I-47) to eliminate $\Gamma_1$ in favor of $\Gamma_2$ in Eq. (I-51), we find (for detailed steps see Appendix)

$$\begin{aligned} &\frac{\Gamma_2^{(+)}(u)}{\mathfrak{D}^{(+)*}\left(\frac{-\omega-ku}{k'}, k'\right)} - \frac{\Gamma_2^{(-)}(u)}{\mathfrak{D}^{(-)*}\left(\frac{-\omega-ku}{k'}, k'\right)} \\ &= \frac{2}{\Delta^*(u,k)\,\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)}\left\{\frac{k}{k'}\,\overline{g}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right. \\ &\quad \left. - \frac{2i\Delta_2\left(\frac{-\omega-ku}{k'}, k'\right)\left[\Phi_1^{(-)}(u)-\Phi_2^{(-)}(u)\right]}{\Delta\left(\frac{-\omega-ku}{k'}, k'\right)}\right\} \equiv h(u). \end{aligned} \tag{I-52}$$

Defining

$$\mathcal{H}(Z) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} du' \frac{h(u')}{u'-Z} \qquad , \qquad . \tag{I-53}$$

and observing that

$$\mathcal{H}^{(+)}(u) - \mathcal{H}^{(-)}(u) = h(u) \qquad ; \tag{I-54}$$

then we may rewrite Eq. (I-52) as

$$\left[\frac{\Gamma_2^{(+)}(u)}{\mathcal{D}^{(+)*}\left(\frac{-\omega-ku}{k'}, k'\right)} - \mathcal{H}^{(+)}(u)\right] - \left[\frac{\Gamma_2^{(-)}(u)}{\mathcal{D}^{(-)*}\left(\frac{-\omega-ku}{k'}, k'\right)} - \mathcal{H}^{(-)}(u)\right] \tag{I-55}$$

$$= 0.$$

Now the complex function

$$\xi_2(Z) \equiv \frac{\Gamma_2(Z)}{\mathcal{D}^*\left(\frac{-\omega-kZ}{k'}, k'\right)} - \mathcal{H}(Z) \tag{I-56}$$

is analytic everywhere except on the real axis and vanishes as $|Z| \to \infty$. But Eq. (I-55) implies that the jump across the real axis is zero; therefore $\xi_2(Z)$ is analytic "everywhere". It follows that $\xi_2(Z)$ must be identically zero. Thus

$$\Gamma_2(Z) = \mathcal{D}^*\left(\frac{-\omega - kZ}{k'}, k'\right)\mathcal{H}(Z) \quad . \tag{I-57}$$

Letting Z approach the real axis from above, and using the definition of $\mathcal{H}$, we obtain

$$\Gamma_2^{(+)}(\hat{k}\cdot v) = \frac{1}{2\pi i} \mathcal{D}^{(+)*}\left(\frac{-\omega - k\cdot v}{k'}, k'\right) \int du \, \frac{h(u)}{u - \hat{k}\cdot v - i\epsilon} \quad . \tag{I-58}$$

Substituting Eqs. (I-58) and (I-52) into Eq. (I-50), we finally find

$$\begin{aligned} G(\eta, k, k') &= g(\eta, k, k')/\Delta^*\left(\frac{-\omega - k\cdot v}{k'}, k'\right) \\ &+ \frac{1}{\pi k^2} D_0(\eta, k) \int_C du \frac{\frac{k}{k'} \bar{g}[-(\omega - ku)/k', k', k]}{(u - \hat{k}\cdot v)\Delta^*(u,k)\Delta^*[-(\omega + ku)/k', k']} \\ &- \frac{1}{\pi k^2} D_0(\eta, k) \int_C du \frac{[\Phi_1^{(-)}(u) - \Phi_2^{(-)}(u)]}{(u - \hat{k}\cdot v)\Delta^*(u,k)} \left\{ 2i\Delta_2\left(\frac{-\omega - ku}{k'}, k'\right) \Big/ \left|\Delta\left(\frac{-\omega - ku}{k'}, k'\right)\right|^2 \right\}, \end{aligned} \tag{I-59}$$

where C is a path just below the real axis. Now, we proceed to simplify Eq. (I-59). By using the relations

$$\frac{2i\Delta_2}{|\Delta|^2} = \frac{1}{\Delta^*} - \frac{1}{\Delta} \quad ,$$

and

$$\frac{k}{k'} \bar{g}\left(\frac{-\omega - ku}{k'}, k', k\right) = \Phi_2^{(+)}(u) - \Phi_2^{(-)}(u)$$

we may write Eq. (I-59) as

$$\begin{aligned} G(\eta, k, k') &= \frac{g(\eta, k, k')}{\Delta^*\left(\frac{-\omega - k\cdot v}{k'}, k'\right)} + \frac{D_0(\eta, k)}{\pi k^2} \int_C du \, \frac{\Phi_2^{(+)}(u) - \Phi_1^{(-)}(u)}{(u - \hat{k}\cdot v)\Delta^*(u,k)\Delta^*\left(\frac{-\omega - ku}{k'}, k'\right)} \\ &+ \frac{D_0(\eta, k)}{\pi k^2} \int_C du \, \frac{\Phi_1^{(-)}(u) - \Phi_2^{(-)}(u)}{(u - \hat{k}\cdot v)\Delta^*(u,k)\Delta\left(\frac{-\omega - ku}{k'}, k'\right)} \quad . \end{aligned}$$

Since the integrand of the last integral has no singularity in the lower-half u-plane, and vanishes as $|u| \to \infty$; therefore, we may close its path of integration in the lower-half plane with a large semi-circle; then according to Cauchy's residue theorem, the last integral vanishes. Thus

$$G(\eta,\mathbf{k},\mathbf{k}') = \frac{q(\eta,\mathbf{k},\mathbf{k}')}{\Delta^{*}\left(\frac{-\omega-\mathbf{k}\cdot\mathbf{v}}{k'},\mathbf{k}'\right)} + \frac{D_0(\eta,\mathbf{k})}{\pi k^2}\int_C du\, \frac{\Phi_2^{(+)}(u) - \Phi_1^{(-)}(u)}{(u-\hat{\mathbf{k}}\cdot\mathbf{v})\,\Delta^{*}\left(\frac{-\omega-ku}{k'},\mathbf{k}'\right)\Delta^{*}(u,\mathbf{k})} \quad . \tag{I-60}$$

In order to simplify Eq. (I-60), we use the definitions of q and $\Phi$'s [cf. Eqs. (I-21), (I-35), and (I-36)]. As shown in Appendix we may write

$$q(\eta,\mathbf{k},\mathbf{k}') = \frac{-2\pi}{k'}\,\bar{p}^{(-)}\left(\eta, \frac{-\omega-\mathbf{k}\cdot\mathbf{v}}{k'},\mathbf{k},\mathbf{k}'\right) \quad , \tag{I-61}$$

$$\Phi_2^{(+)}(u) - \Phi_1^{(-)}(u) = \frac{2\pi}{k'}\,\bar{\bar{p}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'},\mathbf{k},\mathbf{k}'\right) \quad , \tag{I-62}$$

with

$$\bar{\bar{p}}^{(-,-)}\left(u,\frac{-\omega-ku}{k'},\mathbf{k},\mathbf{k}'\right) \equiv \left(\frac{1}{2\pi i}\right)^2 \iint du'\,du''\,\frac{1}{(u'-u+i\epsilon)\left(u''+\frac{\omega+ku}{k'}+i\epsilon\right)}$$

$$\cdot\left\{\iint d\eta\, e_\sigma\, d\eta'\, e_{\sigma'}\,\delta(u'-\hat{\mathbf{k}}\cdot\mathbf{v})\,\delta(u''-\hat{\mathbf{k}}'\cdot\mathbf{v}')\,p(\eta,\eta',\mathbf{k},\mathbf{k}')\right\} \quad , \tag{I-63}$$

and p as defined in Eq. (I-17). The substitution of Eqs. (I-61) and (I-62) into (I-60) yields

$$G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = -\frac{2\pi}{k'}\,\overline{p}^{(-)}\!\left(\eta, \frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k},\underset{\sim}{k}'\right)\Big/\Delta^{*}\!\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}'\right)$$

$$+\frac{2}{k'k^2}D_0(\eta,\underset{\sim}{k})\int_C du\,\frac{\overline{\overline{p}}^{(-,-)}\!\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right)}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\Delta^{*}\!\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}'\right)\Delta^{*}(u,\underset{\sim}{k})} \tag{I-64}$$

Equation (I-64) gives an exact formal solution for the perturbed correlation function; in order to simplify it further, it is necessary to introduce the explicit form of p [cf. Eq. (I-17)]. The simplification goes as follows:

By defining (note the argument $\omega$ has been suppressed)

$$\mathcal{E}(\eta, \underset{\sim}{k}, \underset{\sim}{k}') \equiv e_\sigma \mathcal{F}(\eta, \underset{\sim}{k} + \underset{\sim}{k}') \quad , \tag{I-65}$$

$$P(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}') \equiv \mathcal{G}_I(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}')$$

$$+\left\{\frac{e_\sigma}{m_\sigma}\left[4\pi i\left(e_{\sigma'}f_0(\eta')+G_0(\eta',\underset{\sim}{k}')\right)\frac{\underset{\sim}{k}'}{k'^2}\cdot\frac{\partial}{\partial\underset{\sim}{v}}\mathcal{F}(\eta,\underset{\sim}{k}+\underset{\sim}{k}')-\underset{\sim}{E}(\underset{\sim}{k}+\underset{\sim}{k}')\cdot\frac{\partial}{\partial\underset{\sim}{v}}\mathcal{G}_0(\eta,\eta',-\underset{\sim}{k}')\right]\right\}$$

$$+\begin{Bmatrix}\eta\longleftrightarrow\eta' \\ \underset{\sim}{k}\longleftrightarrow\underset{\sim}{k}'\end{Bmatrix}, \tag{I-66}$$

we may rewrite Eq. (I-17) as

$$p(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}') = P(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}')$$

$$+\frac{i}{\pi}\left[\frac{1}{k}D_0(\eta,\underset{\sim}{k})\,\mathcal{E}(\eta',\underset{\sim}{k}',\underset{\sim}{k})+\frac{1}{k'}D_0(\eta',\underset{\sim}{k}')\,\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')\right]. \tag{I-67}$$

Then

$$\bar{p}^{(-)}\left(\eta, \frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) = \bar{P}^{(-)}\left(\eta, \frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right)$$

$$+\frac{i}{\pi}\left[\frac{1}{k}D_0(\eta,\underset{\sim}{k})\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) + \frac{1}{k'}\bar{D}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}\right)\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')\right], \tag{I-68}$$

and

$$\bar{\bar{p}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) = \bar{\bar{P}}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right)$$

$$+\frac{i}{\pi}\left[\frac{1}{k}\bar{D}_0^{(-)}(u)\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) + \frac{1}{k'}\bar{D}_0^{(-)}\left(\frac{-\omega-ku}{k'}\right)\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}')\right],$$

or by using the relation

$$\bar{D}_0^{(-)}(x, \alpha) = \frac{\alpha^2}{2i}\left[\Delta^*(x, \alpha) - 1\right] \quad ,$$

$$\bar{\bar{p}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) = \bar{\bar{P}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right)$$

$$-\frac{1}{2\pi}\left[k\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right) + k'\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}')\right]$$

$$+\frac{1}{2\pi}\left[k\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\Delta^*(u,k) + k'\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}')\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)\right]. \tag{I-69}$$

The substitution of Eqs. (I-68) and (I-69) into Eq. (I-64), yields

$$G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = -\frac{k'}{\pi i}\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}') - \frac{2\pi}{k'}\left[\Delta^*\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)\right]^{-1}$$

$$\cdot\left[\bar{P}^{(-)}\left(\eta, \frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) + \frac{i}{\pi k}D_0(\eta,\underset{\sim}{k})\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) - \frac{k'}{2\pi}\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')\right]$$

$$+\frac{2D_0(\eta,\underset{\sim}{k})}{k'k^2}\int_C du\,\frac{\bar{\bar{P}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) - \frac{1}{2\pi}\left[k'\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}') + k\bar{\mathcal{E}}^{(-)}\left(\frac{-\omega-ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right]}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\Delta^*[-(\omega+ku)/k', k']\,\Delta^*(u,k)}$$

$$+\frac{D_0(\eta,\underset{\sim}{k})}{\pi k'k^2}\left\{\int_C du\,\frac{k'\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}')}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*(u,k)} + \int_C du\,\frac{k\bar{\mathcal{E}}^{(-)}[-(\omega+ku)/k', \underset{\sim}{k}', \underset{\sim}{k}]}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*[-(\omega+ku)/k', k']}\right\}. \tag{I-70}$$

Observing the last two integrals in Eq. (I-70), we note that the function

$$\frac{\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}')}{\Delta^*(u,k)} \quad , \quad \left[\frac{\bar{\mathcal{E}}^{(-)}(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k})}{\Delta^*(\frac{-\omega-ku}{k'},k')}\right]$$

has no singularity in the lower-half, (upper-half) u-plane and vanishes as $|u| \to \infty$; therefore, we may close the path of integration with a large semi-circle in the lower-half, (upper-half) plane. According to Cauchy's residue theorem the former vanishes and the latter is equal to

$$2\pi i k \frac{\bar{\mathcal{E}}^{(-)}(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},\underset{\sim}{k}')}{\Delta^*(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},k')} \quad .$$

Therefore, Eq. (I-70) reduces to

$$G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = -\frac{k'}{\pi i}\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')$$
$$-\frac{2\pi}{k'}\left[\bar{P}^{(-)}(\eta,\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},\underset{\sim}{k},\underset{\sim}{k}') - \frac{k'}{2\pi}\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')\right]\Big/\Delta^*(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},k')$$
$$+\frac{2D_0(\eta,\underset{\sim}{k})}{k'k^2}\int_C du \frac{\bar{\bar{P}}^{(-,-)}(u,\frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}') - \frac{1}{2\pi}\left[k'\bar{\mathcal{E}}^{(-)}(u,\underset{\sim}{k},\underset{\sim}{k}') + k\bar{\mathcal{E}}^{(-)}(\frac{-\omega-ku}{k'},\underset{\sim}{k}',\underset{\sim}{k})\right]}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*(\frac{-\omega-ku}{k'},k')\Delta^*(u,k)}$$

(I-71)

Equation (I-71) completely determines G (and hence the perturbation $\mathcal{G}$ to the correlation function) in terms of the initial perturbation, $\mathcal{G}_I$, the unperturbed distribution functions, $f_0$ and $\mathcal{G}_0$, and $\mathcal{F}$ (the perturbation to the one-particle distribution function). Substitution of Eq. (I-71) into Eq. (I-10) gives a kinetic equation for $\mathcal{F}$, (i.e., the transform of the kinetic equation for f) which is the plasma analog of the linearized Boltzmann equation. However, it is more desirable to eliminate $\mathcal{G}_0$ from Eq. (I-71). This can be done by solving the linearized equation for $\mathcal{G}_0$, Eq. (I-5). We shall carry out this task in the next chapter.

## Chapter II. Solution to the Equation for the Unperturbed Pair Correlation Function

As suggested in the preceding chapter, our procedure is to express the unperturbed pair correlation function, $\mathcal{G}_0$, as a "function" of $f_0$, the unperturbed one-particle distribution function. Then, in the third chapter, we shall use this result to simplify the quantity G, which represents the "collisional" effect in the kinetic equation as derived in the first chapter.

### 1. The "Connecting" Equation for the Unperturbed State and its Reduction to an Integral Equation

We recall that the equation which links the unperturbed functions, $g_0$ and $f_0$ is Eq. (I-5)

$$\left(\underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}} + \underset{\sim}{v}'\cdot\frac{\partial}{\partial \underset{\sim}{r}'}\right) g_0\,(\eta,\eta',\underset{\sim}{r}-\underset{\sim}{r}') = \left\{ \frac{e_\sigma}{m_\sigma}\frac{\partial f_0\,(\eta)}{\partial \underset{\sim}{v}}\cdot\left[ e_\sigma{}'\, f_0\,(\eta')\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\right.\right.$$

$$\left.\left. + \int dy''\, e_\sigma{}''\left(\frac{\partial}{\partial \underset{\sim}{r}}\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}''|}\right) g_0\,(\eta',\eta'',\underset{\sim}{r}'-\underset{\sim}{r}'')\right] + \left\{y \leftrightarrow y'\right\}\right\}. \tag{II-1}$$

Our immediate objective now is to solve for $g_0$ in terms of $f_0$. For this purpose, we take the Fourier-Transform of Eq. (I-5) w, r, t, $(\underset{\sim}{r}-\underset{\sim}{r}')$, and obtain

$$(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}')\,\mathcal{G}_0\,(\eta,\eta',\underset{\sim}{k}) = \left\{ 4\pi\frac{e_\sigma}{m_\sigma}\frac{\underset{\sim}{k}}{k^2}\cdot\frac{\partial f_0\,(\eta)}{\partial \underset{\sim}{v}}\left[ e_\sigma{}'\, f_0\,(\eta')\right.\right.$$

$$\left.\left. + \int d\eta''\, e_\sigma{}''\,\mathcal{G}_0\,(\eta',\eta'',-\underset{\sim}{k})\right]\right\} + \left\{\begin{matrix}\eta\leftrightarrow\eta' \\ \underset{\sim}{k}\leftrightarrow(-\underset{\sim}{k})\end{matrix}\right\}, \tag{II-2}$$

Since

$$\mathcal{G}_0(\eta', \eta'' - \underset{\sim}{k}) \equiv \int d\underset{\sim}{x}\, e^{i(-\underset{\sim}{k})\cdot \underset{\sim}{x}}\, \mathcal{g}_0(\eta', \eta'', \underset{\sim}{x}) \equiv \mathcal{G}_0^*(\eta', \eta'', \underset{\sim}{k}) \qquad \text{(II-3)}$$

Eq. (II-2) may be written in a somewhat compact form by employing notations defined in Chapter I; i. e.

$$\mathcal{G}_0(\eta, \eta', \underset{\sim}{k}) = \frac{1/\pi\, k^2}{(\hat{\underset{\sim}{k}} \cdot v - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)} \Big\{ [D_0(\eta', \underset{\sim}{k})\, F_0(\eta) - D_0(\eta, \underset{\sim}{k})\, F_0(\eta')]$$

$$+ D_0(\eta', \underset{\sim}{k})\, G_0(\eta, \underset{\sim}{k}) - D_0(\eta, \underset{\sim}{k})\, G_0^*(\eta', \underset{\sim}{k}) \Big\} \qquad \text{(II-4)}$$

where, we take $\epsilon$ to be a small positive real number, which decides the contour of integration in later calculations and then is put to zero finally; moreover we let

$$F_0(\eta) \equiv e_\sigma\, f_0(\eta) \quad . \qquad \text{(II-5)}$$

From the form of Eq. (II-4), it is clear that $\mathcal{G}_0$ is completely determined by $G_0$. In order to obtain an equation for $G_0$ alone, one multiplies Eq. (II-4) by $e_\sigma{}'$ first, then integrates over $\eta'$, and obtains an integral equation for $G_0$

$$\left[ 1 - \frac{1}{\pi k^2} \int d\eta'\, e_\sigma{}' \frac{D_0(\eta', \underset{\sim}{k})}{\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon} \right] G_0(\eta, \underset{\sim}{k})$$

$$= \frac{1}{\pi k^2} \int d\eta'\, e_\sigma{}' \frac{1}{(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)} \Big\{ [D_0(\eta', k)\, F_0(\eta) - D_0(\eta, k)\, F_0(\eta')]$$

$$- D_0(\eta, \underset{\sim}{k})\, G_0^*(\eta', \underset{\sim}{k}) \Big\} . \qquad \text{(II-6)}$$

Defining

$$\zeta(\eta,\underset{\sim}{k}) \equiv \frac{1}{\pi k^2}\int d\eta'\, e_\sigma{}' \frac{[D_0(\eta',\underset{\sim}{k})\, F_0(\eta) - D_0(\eta,\underset{\sim}{k})\, F_0(\eta')]}{(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' + i\epsilon)}\,, \tag{II-7}$$

and using the notation $\Delta$ defined previously (Eq. I-20), Eq. (II-6) becomes

$$\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}, k)\, G_0(\eta,\underset{\sim}{k}) = \zeta(\eta,\underset{\sim}{k}) + \frac{D_0(\eta,\underset{\sim}{k})}{\pi k^2}\int d\eta'\, e_\sigma{}' \frac{G_0{}^*(\eta',\underset{\sim}{k})}{(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' + i\epsilon)} \tag{II-8}$$

Since the kernel of this integral equation depends on $\underset{\sim}{v}'$ only in the combination of $\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'$, thus it is clear that $G_0$ is completely determined by

$$\overline{G}_0(u,k) \equiv \int d\eta\, e_\sigma\, \delta(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\, G_0(\eta,\underset{\sim}{k})\ ; \tag{II-9}$$

in fact Eq. (II-8) may be converted into (suppressing the argument $\underset{\sim}{k}$)

$$\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\, G_0(\eta) = \zeta(\eta) + \frac{D_0(\eta)}{\pi k^2}\int du'\, \frac{\overline{G}_0{}^*(u')}{(u' - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - i\epsilon)}\ . \tag{II-10}$$

Performing the barring operation on Eq. (II-10), one finds a one-dimensional integral equation relating $\overline{G}_0$ and $\overline{G}_0{}^*$

$$\Delta(u)\,\overline{G}_0(u) = \overline{\zeta}(u) + \frac{\overline{D}_0(u)}{\pi k^2}\int du' \,\frac{\overline{G}_0{}^*(u')}{(u' - u - i\epsilon)} \tag{II-11}$$

In order to solve Eq. (II-11), one writes $\Delta$, $\overline{\zeta}$, $\overline{G}_0$ in terms of their real and imaginary parts:

$$\Delta = \Delta_1 + i\Delta_2, \qquad \overline{\zeta} = \overline{\zeta}_1 + i\overline{\zeta}_2, \qquad \overline{G}_0 = \overline{G}_{01} + i\overline{G}_{02}\,. \tag{II-12}$$

One observes that through the definitions of $\zeta$ and the barring operation

$$\overline{\zeta}_2(u) = \frac{1}{k^2}\iint d\eta\, e_\sigma\, d\eta'\, e_\sigma{}'\,[\delta\,(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \hat{k}\cdot\underset{\sim}{v}')] \cdot [D_0(\eta')\,F_0(\eta) - D_0(\eta)\,F_0(\eta')] \tag{II-13}$$

or

$$\overline{\zeta}_2(u) = \frac{1}{k^2}\iint d\eta\, e_\sigma\, d\eta'\, e_\sigma{}'\,[\delta\,(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}')\,\delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}')] \cdot [D_0(\eta')\,F_0(\eta) - D_0(\eta)\,F_0(\eta')]\,. \tag{II-13a}$$

By interchanging $\eta$ and $\eta'$ in Eq. (II-13a) then

$$\overline{\zeta}_2(u) = \frac{1}{k^2}\iint d\eta'\, e_\sigma{}'\, d\eta\, e_\sigma\,[\delta\,(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})] \cdot [D_0(\eta)\,F_0(\eta') - D_0(\eta')\,F_0(\eta)] \tag{II-13b}$$

$$= -\,\overline{\zeta}_2(u)\,.$$

So that

$$\overline{\zeta}_2(u) = 0 \quad ; \tag{II-14}$$

in other words $\overline{\zeta}$ is a real function. Using this result and the previously derived relation (Eq. I-25)

$$\Delta_2 = \frac{\overline{D}_0}{k^2}$$

one may express Eq. (II-11) in real and imaginary parts:

$$[\Delta_1 (u) + i\Delta_2 (u)] [\overline{G}_{01} (u) + i \overline{G}_{02} (u)] = \overline{\zeta} (u)$$

$$+ \Delta_2 (u) \left\{ \int du' \left[\frac{1}{\pi} P\left(\frac{1}{u' - u}\right) + i\delta (u' - u)\right] \left[\overline{G}_{01} (u') - i \overline{G}_{02} (u')\right] \right\}$$

or in terms of the Hilbert transform operator, then (suppressing the argument, u)

$$\begin{aligned} &(\Delta_1 \overline{G}_{01} - \Delta_2 G_{02}) + i (\Delta_2 \overline{G}_{01} + \overline{\Delta}_1 \overline{G}_{02}) \\ &= \overline{\zeta} \Delta_2 \left\{ H \left[\overline{G}_{01}\right] + \overline{G}_{02} + i \left[\overline{G}_{01} - H \left[\overline{G}_{02}\right]\right] \right\} . \end{aligned} \tag{II-15}$$

By equating the real and the imaginary parts of Eq. (II-15), one obtains two integral equations

$$\Delta_1 \overline{G}_{01} - \Delta_2 H\left[\overline{G}_{01}\right] = \overline{\zeta} + \Delta_2 \overline{G}_{02} \tag{II-16}$$

and

$$\Delta_1 \overline{G}_{02} + \Delta_2 H\left[\overline{G}_{02}\right] = 0 \quad . \tag{II-17}$$

These singular integral equations have the general form as the one we have encountered in the preceding chapter; therefore, they may be handled with the same techniques and treat them by a similar method used previously. We shall solve them in the next section.

2. Solution to the Integral Equation

In order to solve Eq. (II-17), one introduces the complex function

$$\overline{G}_{02}(z) \equiv \frac{1}{2\pi i} \int du' \, \frac{\overline{G}_{02}(u')}{u' - z} \quad ; \tag{II-18}$$

then the corresponding Plemelj formulas are

$$\overline{G}_{02}^{(+)}(u) - \overline{G}_{02}^{(-)}(u) = \overline{G}_{02}(u) \quad , \tag{II-19}$$

$$\overline{G}_{02}^{(+)}(u) + \overline{G}_{02}^{(-)}(u) = -\, i\, H\,[\overline{G}_{02}(u)] \; . \tag{II-20}$$

Through these relations, Eq. (II-17) may be re-expressed as

$$[\Delta_1(u) + i\Delta_2(u)]\,\overline{G}_{02}^{(+)}(u) = [\Delta_1(u) - i\Delta_2(u)]\,\overline{G}_{02}^{(-)}(u) \tag{II-21}$$

One, now, needs the relations

$$2\mathfrak{D}^{(+)}(u) = \Delta_1(u) + i\Delta_2(u) \tag{II-22}$$

and

$$2\mathfrak{D}^{(-)}(u) = \Delta_1(u) - i\Delta_2(u) \; , \tag{II-23}$$

which have been derived in the preceding chapter (Eqs. (I-43) and

(I-44)). Substitution of Eqs. (II-22) and (II-23) into Eq. (II-21), yields

$$\mathfrak{D}^{(+)}(u)\,\overline{G}_{02}^{(+)}(u) - \mathfrak{D}^{(-)}(u)\,\overline{G}_{02}^{(-)}(u) = 0 \qquad \text{(II-24)}$$

According to the theory of Cauchy integrals, the complex function

$$\mathcal{E}_3(z) \equiv \mathfrak{D}(z)\,\overline{G}_{02}(z) \qquad \text{(II-25)}$$

is analytic everywhere except on the real axis and vanishes as $|z| \to \infty$. But Eq. (II-24) indicates that the jump across the real axis is zero; therefore $\mathcal{E}_3(z)$ is analytic everywhere. It follows that $\mathcal{E}_3(z)$ must be identically zero. Thus

$$\mathfrak{D}(z)\,\overline{G}_{02}(z) = 0$$

However,

$$\lim_{|z| \to \infty} \mathfrak{D}(z) = \frac{1}{2}$$

according to the definition of $\mathfrak{D}(z)$ (Eq. I-37); therefore

$$\overline{G}_{02}(z) = 0 \; . \qquad \text{(II-26)}$$

Consequently, through Eq. (II-19)

$$\overline{G}_{02}^{(+)}(u) - \overline{G}_{02}^{(-)}(u) = \overline{G}_{02}(u) \equiv 0 \;; \qquad \text{(II-27)}$$

in other words,

$$\overline{G}_0(u) = \overline{G}_{01}(u) = \overline{G}_0^{*}(u)$$

is a real function.

As a result of the fact that $\overline{G}_0$ (u) is real, Eq. (II-16) reduces to

$$\Delta_1 (u) \overline{G}_0 (u) - \Delta_2 (u) H [\overline{G}_0 (u)] = \overline{\zeta} (u) . \qquad \text{(II-16a)}$$

Precisely as in the proof of Eq. (II-27), it may be easily shown that the corresponding homogeneous equation of Eq. (II-16a); i. e.,

$$\Delta_1 (u) \overline{G}_0 (u) - \Delta_2 (u) H [\overline{G}_0 (u)] = 0 \qquad \text{(II-16b)}$$

has only a trivial solution. This implies that the solution to Eq. (II-16a) is unique.

Now, we consider the solution to Eq. (II-16a). Again, we introduce a complex function

$$\overline{G}_0 (z) \equiv \frac{1}{2\pi i} \int du' \frac{\overline{G}_0 (u')}{u' - z} . \qquad \text{(II-28)}$$

Consequently the Plemelj formulas are

$$\begin{aligned} \overline{G}_0^{(+)} (u) - \overline{G}_0^{(-)} (u) &= \overline{G}_0 (u) , \\ \overline{G}_0^{(+)} (u) + \overline{G}_0^{(-)} (u) &= - i H [\overline{G}_0 (u)] . \end{aligned} \qquad \text{(II-29)}$$

Through these relations as well as those given by Eqs. (II-22) and (II-23), Eq. (II-16a) reduces to

$$2 [\mathfrak{D}^{(-)} (u) \overline{G}_0^{(+)} (u) - \mathfrak{D}^{(+)} (u) \overline{G}_0^{(-)} (u)] = \overline{\zeta} ; \qquad \text{(II-30)}$$

or dividing Eq. (II-30) by

$$2\mathfrak{D}^{(+)}(u)\mathfrak{D}^{(-)}(u) = \frac{1}{2}\left[\Delta_1^{\,2}(u) + \Delta_2^{\,2}(u)\right] = \frac{1}{2}\,|\Delta(u)|^2 \neq 0$$

then

$$\frac{\overline{G}_0^{(+)}(u)}{\mathfrak{D}^{(+)}(u)} - \frac{\overline{G}_0^{(-)}(u)}{\mathfrak{D}^{(+)}(u)} = \frac{2\overline{\zeta}(u)}{|\Delta(u)|^2} \;. \qquad \text{(II-31)}$$

Defining

$$\tau(z) \equiv \frac{1}{2\pi i}\int du'\,\frac{1}{(u'-z)}\left[\frac{2\overline{\zeta}(u')}{|\Delta(u')|^2}\right] , \qquad \text{(II-32)}$$

and observing that

$$\begin{aligned} \tau^{(+)}(u) - \tau^{(-)}(u) &= \frac{2\zeta(u)}{|\Delta(u)|^2} \\ \tau^{(+)}(u) + \tau^{(-)}(u) &= -\,i\,H\left[\frac{2\zeta(u)}{|\Delta(u)|^2}\right] ; \end{aligned} \qquad \text{(II-33)}$$

then Eq. (II-31) can be re-written as

$$\left[\frac{\overline{G}_0(u)}{\mathfrak{D}(u)} - \tau(u)\right]^{(+)} - \left[\frac{\overline{G}_0(u)}{\mathfrak{D}(u)} - \tau(u)\right]^{(-)} = 0 \;. \qquad \text{(II-34)}$$

Therefore, following precisely the same argument stated for the function $\xi_3(z)$ (Eq. II-25), we conclude from Eq. (II-34) that the function

$$\xi_4(z) \equiv \left[\frac{\overline{G}_0(z)}{\mathfrak{D}(z)} - \tau(z)\right]$$

is identically zero, or

$$\overline{G}_0 (z) = \mathfrak{D} (z) \tau (z) \quad . \tag{II-35}$$

Substituting Eq. (II-35) into Eq. (II-28) and utilizing the Plemelj formula, we finally gain the solution to the Eq. (II-16a); i. e.

$$\overline{G}_0 (u) = [\mathfrak{D} (u) \tau (u)]^{(+)} - [\mathfrak{D} (u) \tau (u)]^{(-)} \quad . \tag{II-36}$$

In the next section, the results obtained above will be used to find the expressions for $\mathcal{G}_0$ and $G_0$.

3. Expressions for $G_0$ and $\mathcal{G}_0$

As a first step to determine $G_0$ and $\mathcal{G}_0$, we re-express $\zeta$, which is defined by Eq. (II-7), in a more compact and convenient form; i. e.

$$\zeta (\eta) = \frac{2i}{k^2} [D_0 (\eta) \overline{F}_0^{(+)} (\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}) - \overline{D}_0^{(+)} (\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}) F_0 (\eta)] \, . \tag{II-37}$$

By performing the barring operation on Eq. (II-37) then

$$\overline{\zeta} (u) = \frac{2i}{k^2} [\overline{D}_0 (u) \overline{F}_0^{(+)} (u) - \overline{D}_0^{(+)} (u) \overline{F}_0 (u)] \, ;$$

or by using the related Plemelj formulas

$$\overline{D}_0 (u) = \overline{D}_0^{(+)} (u) - \overline{D}_0^{(-)} (u) \quad ,$$

and

$$\overline{F}_0 (u) = \overline{F}_0^{(+)} (u) - \overline{F}_0^{(-)} (u) \quad ;$$

then

$$\bar{\zeta}\,(u) = \frac{2i}{k^2}\left[\bar{D}_0^{(+)}\,(u)\,\bar{F}_0^{(-)}\,(u) - \bar{D}_0^{(-)}\,(u)\,\bar{F}_0^{(+)}\,(u)\right] . \tag{II-38}$$

Presently in Eqs. (II-37) and (II-38) we utilize the relation

$$\frac{2i}{k^2}\,\bar{D}_0^{(+)} = \Delta - 1 \quad ,$$

which was derived in the first chapter. As a result, we obtain

$$\zeta\,(\eta) = \frac{2i}{k^2}\,D_0\,(\eta)\,\bar{F}_0^{(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) - \Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,F_0\,(\eta) + F_0\,(\eta)\ , \tag{II-37a}$$

and

$$\bar{\zeta}\,(u) = \bar{F}_0\,(u) + \Delta\,(u)\,\bar{F}_0^{(-)}\,(u) - \Delta^*(u)\,\bar{F}_0^{(+)}\,(u) \quad . \tag{II-38a}$$

By introducing the notations

$$\rho_0\,(\eta, \underset{\sim}{k}) \equiv \frac{F_0\,(\eta)}{|\Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})|^2}$$

and (II-39)

$$\bar{\rho}_0\,(u) \equiv \frac{\bar{F}_0\,(u)}{|\Delta\,(u)|^2} \quad ,$$

we may cast Eqs. (II-37a) and (II-38a) into different but convenient forms; i. e.

$$\frac{\zeta(\eta)}{\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})} = \frac{2i}{k^2}\,\frac{D_0(\eta)\,\overline{F}_0^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}{\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})} - F_0(\eta) + \rho_0(\eta)\,\Delta^*(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) \tag{II-37b}$$

and

$$\frac{\overline{\zeta}(u)}{|\Delta(u)|^2} = \overline{\rho}_0(u) + \frac{\overline{F}_0^{(-)}(u)}{\Delta^*(u)} - \frac{\overline{F}_0^{(+)}(u)}{\Delta(u)} \quad . \tag{II-38b}$$

Having completed the initial steps above, we proceed to find an explicit expression for $\tau^{(+)}$. Through Eqs. (II-32) and (II-38b), we find

$$\tau^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) = \frac{1}{2\pi i}\int du'\,\frac{2}{(u' - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - i\epsilon)}\left[\frac{\overline{F}_0^{(-)}(u')}{\Delta^*(u')} - \frac{\overline{F}_0^{(+)}(u')}{\Delta(u')} + \overline{\rho}_0(u')\right] . \tag{II-40}$$

The first term in the square bracket has no singularity in the lower-half plane, and in addition it vanishes at infinite in that half plane (this also applies to the second term in the upper half plane). Therefore, we can employ Cauchy's theorems to evaluate the integrals by closing the contour with a large semi-circle in the lower (and/or upper) half plane, and find

$$\tau^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) = 2\left[\overline{\rho}_0^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) - \frac{\overline{F}_0^{(+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}{\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}\right] . \tag{II-41}$$

Now the expressions $G_0$ and $\mathcal{G}_0$ can be obtained as follows:

As a consequence of the fact that $\overline{G}_0(u)$ is real, Eq. (II-10) reduces to

$$\Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\; G_0\,(\eta) = \zeta\,(\eta) + \frac{2i\,D_0\,(\eta)}{k^2}\,\overline{G}_0^{\,(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\;. \tag{II-10a}$$

Through Eq. (II-35) then

$$G_0\,(\eta) = \frac{\zeta\,(\eta)}{\Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})} + \frac{2i}{k^2}\,\frac{D_0\,(\eta)}{\Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})}\,\mathfrak{D}^{(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\;\tau^{(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\;,$$

but from Eq. (II-23)

$$\frac{2\mathfrak{D}^{(+)}}{\Delta} = 1\;.$$

Therefore

$$G_0\,(\eta) = \frac{\zeta\,(\eta)}{\Delta\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})} + \frac{i}{k^2}\,D_0\,(\eta)\;\tau^{(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\;. \tag{II-42}$$

Substitution of Eqs. (II-37b) and (II-41) into Eq. (II-42), gives

$$G_0\,(\eta) = \rho_0\,(\eta)\;\Delta^*\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) + \frac{2i}{k^2}\,D_0\,(\eta)\,\overline{\rho}_0^{\,(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) - F_0\,(\eta)\,. \tag{II-43}$$

In Eq. (II-43) returning the suppressed argument $\underset{\sim}{k}$, and defining

$$Z_0\,(\eta,\underset{\sim}{k}) \equiv \rho_0\,(\eta,\underset{\sim}{k})\;\Delta^*\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v},k) + \frac{2i}{k^2}\,D_0\,(\eta,\underset{\sim}{k})\,\overline{\rho}_0^{\,(+)}\,(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v},k) \tag{II-44}$$

We finally obtain

$$G_0\,(\eta,\underset{\sim}{k}) + F_0\,(\eta) = Z_0\,(\eta,\underset{\sim}{k})\;. \tag{II-45}$$

By taking the complex conjugate of Eq. (II-45) and changing $\eta$ to $\eta'$, we find

$$G_0{}^*(\eta', \underset{\sim}{k}) + F_0(\eta') = Z_0{}^*(\eta', \underset{\sim}{k}) \; . \tag{II-46}$$

Finally, the expression for $\mathcal{G}_0$ can be obtained by substituting Eqs. (II-45) and (II-46) into Eq. (II-4); the result is

$$\mathcal{G}_0(\eta, \eta', \underset{\sim}{k}) = \frac{1}{\pi k^2} \frac{D_0(\eta', \underset{\sim}{k})\, Z_0(\eta, \underset{\sim}{k}) - D_0(\eta, \underset{\sim}{k})\, Z_0{}^*(\eta', \underset{\sim}{k})}{(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)} \tag{II-47}$$

Now, Eqs. (II-45) - (II-47) show that we have expressed the unperturbed pair correlation function in terms of unperturbed one-particle distribution function. In the third chapter, we shall use these results to simplify the quantity G which is needed in the kinetic equation.

# Chapter III. Explicit Evaluation and Further Reduction of the Expression G

In the last chapter, we obtained the expressions for $G_0$ and $\mathcal{G}_0$ as functions of $f_0$ (the unperturbed one-particle distribution function). As suggested there, we shall utilize the results obtained to eliminate the quantities $G_0$ and $\mathcal{G}_0$ from the expression G which is given by Eq. (I-71). As a preliminary procedure, we shall introduce some short hand notations, and collect all the necessary equations (which were derived previously) as well as rewrite them in somewhat more suitable forms.

List of notations:

$$\mathcal{U}(\eta', \underset{\sim}{k}', \underset{\sim}{k}) \equiv 4\pi^2 i \frac{e_\sigma'}{m_\sigma'} \underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}'} \mathcal{F}(\eta', \underset{\sim}{k} + \underset{\sim}{k}') \quad , \tag{III-1}$$

$$D_1(\eta', k) \equiv \frac{e_\sigma'}{m_\sigma'} D_0(\eta', \underset{\sim}{k}) \quad ; \quad \rho_1(\eta', \underset{\sim}{k}) \equiv \frac{e_\sigma'}{m_\sigma'} \rho_0(\eta', \underset{\sim}{k}) \, , \tag{III-2}$$

$$Z_0(\eta, \underset{\sim}{k}) \equiv \rho_0(\eta, \underset{\sim}{k}')\Delta^*(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, k) + \frac{2i}{k^2} D_0(\eta, \underset{\sim}{k}) \bar{\rho}_0^{(+)}(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, k) \, , \tag{III-3}$$

$$Y_1(\eta', \underset{\sim}{k}) \equiv \frac{e_\sigma'}{m_\sigma'} Z_0^*(\eta', \underset{\sim}{k}) \quad , \tag{III-4}$$

$$B\left(u, \frac{-\omega - ku}{k'}\right) \equiv \frac{1}{2\pi}\left[k' \bar{\mathcal{E}}^{(-)}(u, \underset{\sim}{k}, \underset{\sim}{k}') + k \bar{\mathcal{E}}^{(-)}\left(\frac{-\omega - ku}{k'}, \underset{\sim}{k}', \underset{\sim}{k}\right)\right] . \tag{III-5}$$

We had the following equations:

$$Z_0(\eta, \underset{\sim}{k}) = F_0(\eta) + G_0(\eta, \underset{\sim}{k}) \quad , \tag{III-6}$$

$$\mathcal{G}_0(\eta, \eta', \underset{\sim}{k}) = -\frac{D_0(\eta', \underset{\sim}{k}) Z_0(\eta, \underset{\sim}{k}) - D_0(\eta, \underset{\sim}{k}) Z_0^*(\eta', \underset{\sim}{k})}{\pi k^2(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' + i\epsilon)} \quad , \tag{III-7}$$

$$P(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') = \mathcal{G}_I(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')$$
$$+ \left\{ \frac{1}{\pi k^2}\left[F_0(\eta) + G_0(\eta, \underset{\sim}{k})\right] U(\eta', \underset{\sim}{k}', \underset{\sim}{k}) - \frac{e_\sigma}{m_\sigma} \underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'} \mathcal{G}_0(\eta, \eta', \underset{\sim}{k}) \right\}$$
$$+ \left\{ \begin{matrix} \eta \leftrightarrow \eta' \\ \underset{\sim}{k} \leftrightarrow \underset{\sim}{k}' \end{matrix} \right\} \quad , \tag{III-8}$$

$$G(\eta, \underset{\sim}{k}, \underset{\sim}{k}') = -\mathcal{E}(\eta, \underset{\sim}{k}, \underset{\sim}{k}')$$
$$- \frac{2\pi}{k}\left[\overline{P}^{(-)}\left(\eta, \frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) - \frac{k'}{2\pi}\mathcal{E}(\eta, \underset{\sim}{k}, \underset{\sim}{k}')\right] \Big/ \Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}'\right)$$
$$+ \frac{2D_0(\eta, \underset{\sim}{k})}{k' k^2} \int_C du \, \frac{\overline{\overline{P}}^{(-,-)}\left(u, \frac{-\omega - ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) - B\left(u, \frac{-\omega - ku}{k'}\right)}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\, \Delta^*\left(\frac{-\omega - ku}{k'}, \underset{\sim}{k}'\right) \Delta^*(u, \underset{\sim}{k})} \quad , \tag{III-9}$$

where we have used the notations just introduced above, and have suppressed the argument $\underset{\sim}{k} + \underset{\sim}{k}'$ in $\underset{\sim}{E}$.

1. Elimination of $G_0$ and $\mathcal{G}_0$

In Eq. (III-8), we replace $G_0$ and $\mathcal{G}_0$ by the corresponding expressions given by Eqs. (III-6) and (III-7) respectively. Then Eq. (III-8) reads

$$P(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') = \mathcal{G}_I(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') + S(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') + \left\{ \begin{matrix} \eta \leftrightarrow \eta' \\ \underset{\sim}{k} \leftrightarrow \underset{\sim}{k}' \end{matrix} \right\} , \tag{III-10}$$

where

$$S(\eta,\eta',\underset{\sim}{k},\underset{\sim}{k}') \equiv \frac{1}{\pi k^2}\Bigg\{Z_0(\eta,\underset{\sim}{k})U(\eta',\underset{\sim}{k}',\underset{\sim}{k})$$

$$-\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[\frac{D_1(\eta',\underset{\sim}{k})Z_0(\eta,\underset{\sim}{k})-D_0(\eta,\underset{\sim}{k})Y_1(\eta',\underset{\sim}{k})}{(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)}\right]\Bigg\} \qquad \text{(III-11)}$$

By defining

$$S_r(\eta,\ \eta',\ \underset{\sim}{k},\ \underset{\sim}{k}') \equiv S(\eta',\ \eta,\ \underset{\sim}{k}',\ \underset{\sim}{k}) \qquad \text{(III-12)}$$

then by means of the appropriate definitions, we have:

$$\bar{S}_r(\eta,\ u'',\ \underset{\sim}{k},\ \underset{\sim}{k}') = \bar{S}(u'',\ \eta,\ \underset{\sim}{k}',\ \underset{\sim}{k}) \quad , \qquad \text{(III-13)}$$

$$\bar{\bar{S}}_r(u',\ u'',\ \underset{\sim}{k},\ \underset{\sim}{k}') = \bar{\bar{S}}(u'',\ u',\ \underset{\sim}{k}',\ \underset{\sim}{k}) \quad , \qquad \text{(III-14)}$$

and

$$\bar{\bar{S}}_r^{(-,-)}(u,\ \frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}') = \bar{\bar{S}}^{(-,-)}(\frac{-\omega-ku}{k'},\ u,\ \underset{\sim}{k}',\ \underset{\sim}{k}) \quad . \qquad \text{(III-15)}$$

The new form of P given by Eq. (III-10), with the aid of Eq. (III-15), enables us to cast Eq. (III-9) in the following form:

$$G(\eta,\underset{\sim}{k},\underset{\sim}{k}') = \mathcal{D}(\eta,\underset{\sim}{k},\underset{\sim}{k}') - \mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')$$

$$-\frac{\frac{2\pi}{k'}}{\Delta^*(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},k')}\left[\bar{S}^{(-)}(\eta,\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},\underset{\sim}{k},\underset{\sim}{k}')+\bar{S}_r^{(-)}(\eta,\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},\underset{\sim}{k},\underset{\sim}{k}')-\frac{k'}{2\pi}\mathcal{E}(\eta,\underset{\sim}{k},\underset{\sim}{k}')\right]$$

$$+\frac{2D_0(\eta,\underset{\sim}{k})}{k'k^2}\int_C du\,\frac{\bar{\bar{S}}^{(-,-)}(u,\frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}')+\bar{\bar{S}}_r^{(-,-)}(u,\frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}')-B(u,\frac{-\omega-ku}{k'})}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*(\frac{-\omega-ku}{k'},k')\Delta^*(u,k)} \quad , \qquad \text{(III-16)}$$

where

$$\mathcal{J}(\eta, \underset{\sim}{k}, \underset{\sim}{k}') \equiv -\frac{2\pi}{k'} \overline{\mathcal{G}}_I^{(-)}(\eta, \frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}')/\Delta^*(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k')$$

$$+ \frac{2D_0(\eta, \underset{\sim}{k})}{k' k^2} \int du \frac{\overline{\overline{\mathcal{G}}}_I^{(-,-)}(u\ \frac{-\omega - ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}')}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) \Delta^*(\frac{-\omega - ku}{k'}, k') \Delta^*(u, k)} \quad . \tag{III-17}$$

We recall that $\mathcal{G}_I$ is the initial perturbation to the pair correlation function.

Having eliminated the $G_0$ and $\mathcal{G}_0$ from the expression G, our next task is to simplify the obtained result. To achieve this, we need the explicit forms of $\overline{S}^{(-)}$, $\overline{S}_r^{(-)}$, and $\overline{\overline{S}}^{(-,-)}$. In the following sections, we shall derive them from the appropriate definitions with the aid of the symmetry relations, Eqs.(III-13) — (III-15).

2. Expression for $\overline{S}$ and $\overline{S}_r$

To obtain $\overline{S}$, we perform the barring operation w.r.t $\eta$ and $\underset{\sim}{k}$ on Eq. (III-11), and find

$$\overline{S}(u, \eta', \underset{\sim}{k}, \underset{\sim}{k}') = (\pi k^2)^{-1} \overline{Z}_0(u, k)\left\{U(\eta', \underset{\sim}{k}', \underset{\sim}{k}) - \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}'}\left[D_1(\eta', \underset{\sim}{k})(u - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} + i\epsilon)^{-1}\right]\right\}$$

$$+ (\pi k^2)^{-1} \overline{D}_0(u) \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}'}\left[Y_1(\eta', \underset{\sim}{k})(u - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)^{-1}\right] \quad . \tag{III-18}$$

To advance further, it is necessary to introduce the explicit form of $\overline{Z}_0$, which may be obtained from Eq. (III-3) by the barring operation,

$$\overline{Z}_0(u, k) = \overline{\rho}_0(u, k) \Delta^*(u, k) + \frac{2i}{k^2} \overline{D}_0(u) \overline{\rho}_0^{(+)}(u, k) \, . \tag{III-19}$$

By utilizing the previously derived relations,

$$\frac{2i}{k^2}\overline{D}_0(u) = \Delta(u,\ k) - \Delta^*(u,\ k) \quad , \qquad \text{(III-20)}$$

and the Plemelj formula

$$\overline{\rho}_0(u) = \overline{\rho}_0^{(+)}(u) - \overline{\rho}_0^{(-)}(u) \quad ,$$

we express $\overline{Z}_0$ in such a form that will be suitable for later calculations; i.e.,

$$\overline{Z}_0(u,\ k) = \overline{\rho}_0^{(+)}(u,\ k)\Delta(u,\ k) - \overline{\rho}_0^{(-)}(u,\ k)\Delta^*(u,\ k) \quad . \text{(III-21)}$$

With the aid of Eqs. (III-20) and (III-21), we may cast Eq. (III-18) into the following form:

$$\begin{aligned}
&\overline{S}(u, \eta', \underset{\sim}{k}, \underset{\sim}{k}') \\
&= \frac{1}{\pi k^2}\left[\overline{\rho}_0^{(+)}(u,k)\Delta(u,k) - \overline{\rho}_0^{(-)}(u,k)\Delta^*(u,k)\right]\left\{U(\eta', \underset{\sim}{k}', \underset{\sim}{k}) - \underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[\frac{D_1(\eta', \underset{\sim}{k})}{u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' + i\epsilon}\right]\right\} \\
&\quad + \frac{1}{2\pi i}\left[\Delta(u,k) - \Delta^*(u,k)\right]\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[\frac{Y_1(\eta', \underset{\sim}{k})}{u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}' + i\epsilon}\right] \quad .
\end{aligned} \qquad \text{(III-22)}$$

In Eq. (III-22) interchanging $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ and changing $\eta'$ to $\eta$; then using the symmetry relation, Eq. (III-13), we get

$$\begin{aligned}
&\overline{S}_r(\eta, u, \underset{\sim}{k}, \underset{\sim}{k}') \\
&= \frac{1}{\pi k'^2}\left[\overline{\rho}_0^{(+)}(u,k')\Delta(u,k') - \overline{\rho}_0^{(-)}(u,k')\Delta^*(u,k')\right]\left\{U(\eta, \underset{\sim}{k}, \underset{\sim}{k}') - \underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}}\left[\frac{D_1(\eta, \underset{\sim}{k}')}{u - \hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v} + i\epsilon}\right]\right\} \\
&\quad + \frac{1}{2\pi i}\left[\Delta(u,k') - \Delta^*(u,k')\right]\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}}\left[\frac{Y_1(\eta, \underset{\sim}{k}')}{u - \hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v} + i\epsilon}\right] \quad .
\end{aligned} \qquad \text{(III-23)}$$

3. The Expression for $\bar{\bar{S}}^{(-,-)}$

With the results of the last section, we shall derive $\bar{S}^{(-)}$ and then $\bar{\bar{S}}^{(-,-)}$. By definition

$$\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') \equiv \frac{1}{2\pi i}\int du' \,\frac{\bar{S}(u',\eta',\underset{\sim}{k},\underset{\sim}{k}')}{u'-u+i\epsilon} \,. \tag{III-24}$$

Using Eq. (III-22) for $\bar{S}$,

$$\begin{aligned}
&\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') \\
&= \frac{1}{2\pi i}\int du' \,\frac{[\bar{\rho}_0^{(+)}(u',k)\Delta(u',k)-\bar{\rho}_0^{(-)}(u',k)\Delta^*(u',k)]}{\pi k^2(u'-u+i\epsilon)}\left\{U(\eta',\underset{\sim}{k}',\underset{\sim}{k})-\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[\frac{D_1(\eta',k)}{u'-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon}\right]\right\} \\
&+\left(\frac{1}{2\pi i}\right)^2\int du' \,\frac{[\Delta(u',k)-\Delta^*(u',k)]}{(u'-u+i\epsilon)}\,\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[\frac{Y_1(\eta',k)}{u'-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon}\right] \,.
\end{aligned} \tag{III-25}$$

The "initial" terms in the first and second integrals are analytic in the upper-half plane, and in addition they vanish at infinity in that half plane; therefore, they contribute nothing to the integrals according to Cauchy's theorems. Consequently, Eq. (III-25) reduces to

$$\begin{aligned}
\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') &= -\frac{1}{2\pi i}\int du' \,\frac{\bar{\rho}_0^{(-)}(u',k)\Delta^*(u',k)}{\pi k^2(u'-u+i\epsilon)}\,U(\eta',\underset{\sim}{k}',\underset{\sim}{k}) \\
&+\frac{1}{2\pi i}\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left\{\int du' \,\frac{\Delta^*(u',k)\left[(\pi k^2)^{-1}\bar{\rho}_0^{(-)}(u',k)D_1(\eta',k)-(2\pi i)^{-1}Y_1(\eta',k)\right]}{(u'-u+i\epsilon)\,(u'-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)}\right\} .
\end{aligned} \tag{III-26}$$

Again, the first integral can be evaluated by the residue theorem, following a similar argument as above; while in the second one, we substitute Eq. (III-4) for $Y_1$ and then add a null integral

$$\frac{F_1(\eta')}{2\pi i}\int du' \frac{1}{(u' - u + i\epsilon)(u' - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} + i\epsilon)}$$

in the curved bracket. As a result, we find

$$\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') = (\pi k^2)^{-1}\bar{\rho}_0^{(-)}(u,k)\Delta^*(u,k)\mathcal{U}(\eta',\underset{\sim}{k}',\underset{\sim}{k})$$

$$+\frac{\underset{\sim}{E}}{2\pi i}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left\{\int du' \frac{\frac{\Delta^*(u',k)D_1(\eta',\underset{\sim}{k})}{\pi k^2}\left[\bar{\rho}_0^{(-)}(u',k)-\bar{\rho}_0^{(-)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}',k)\right]-\frac{F_1(\eta')}{2\pi i}\left[\frac{\Delta^*(u',k)}{\Delta^*(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}',k)}-1\right]}{(u'-u+i\epsilon)(u'-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)}\right\} \quad \text{(III-27)}$$

Since there is no singularity at $u' = \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}'$, the only singularity possessed by the integrand is at $u' = u$. Therefore, by closing the contour in the lower-half plane and applying Cauchy's residue theorem, we obtain

$$\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') = (\pi k^2)^{-1}\bar{\rho}_0^{(-)}(u,k)\Delta^*(u,k)\mathcal{U}(\eta',\underset{\sim}{k}',\underset{\sim}{k})$$

$$-\underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left\{\frac{\frac{\Delta^*(u,k)D_1(\eta',\underset{\sim}{k})}{\pi k^2}\left[\bar{\rho}_0^{(-)}(u,k)-\bar{\rho}_0^{(-)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}',k)\right]-\frac{F_1(\eta')}{2\pi i}\left[\frac{\Delta^*(u,k)}{\Delta^*(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}',k)}-1\right]}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)}\right\} \quad \text{(III-28)}$$

In order to utilize $\bar{S}^{(-)}$ to calculate $\bar{\bar{S}}^{(-,-)}$, we re-write Eq. (III-28) in a more compact form by defining

$$L(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') \equiv (\pi k^2)^{-1}\left\{\mathcal{U}(\eta',\underset{\sim}{k}',\underset{\sim}{k}) - \underset{\sim}{E}\cdot\frac{\partial}{\partial \underset{\sim}{v}'}\left[D_1(\eta',\underset{\sim}{k})(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)^{-1}\right]\right\}, \quad \text{(III-29)}$$

and by using the definition for $Y_1$ [cf. Eq. (III-4)]. Then Eq. (III-28) becomes

$$\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}') = \overset{*}{\Delta}(u,k)\Big\{\bar{\rho}_0^{(-)}(u,k)L(u,\eta',\underset{\sim}{k},\underset{\sim}{k}')$$

$$+\frac{\underset{\sim}{E}}{2\pi i}\cdot\frac{\partial}{\partial\underset{\sim}{v}'}\left[\frac{Y_1(\eta',\underset{\sim}{k}) - F_1(\eta')/\overset{*}{\Delta}(u,k)}{u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon}\right]\Big\}. \tag{III-30}$$

By definition

$$\bar{\bar{S}}^{(-,-)}\!\left(u,\frac{\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}'\right) = \frac{k'}{2\pi i}\int d\eta'\, e_{\sigma}{}'\,\bar{S}^{(-)}(u,\eta',\underset{\sim}{k},\underset{\sim}{k}')/(\underset{\sim}{k}'\cdot\underset{\sim}{v}'+\omega+ku+i\epsilon) \tag{III-31}$$

Hence, by substituting Eq. (III-30) for $\bar{S}^{(-)}$, we have:

$$\bar{\bar{S}}^{(-,-)}\!\left(u,\frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}'\right) = \frac{1}{2\pi i}\overset{*}{\Delta}(u,k)\int d\eta' e_{\sigma'}\left(\frac{k'}{\underset{\sim}{k}'\cdot\underset{\sim}{v}'+\omega+ku+i\epsilon}\right)$$

$$\cdot\Big\{\Big[\bar{\rho}_0^{(-)}(u,k)L(u,\eta',\underset{\sim}{k},\underset{\sim}{k}')+\frac{\underset{\sim}{E}}{2\pi i}\cdot\frac{\partial}{\partial\underset{\sim}{v}'}\left[\frac{Y_1(\eta',\underset{\sim}{k})-F_1(\eta')/\Delta^*(u,k)}{u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon}\right]\Big]\Big\}. \tag{III-32}$$

For convenience, we use the Plemelj formula to write

$$\bar{\rho}_0^{(-)} = \bar{\rho}_0^{(+)} - \bar{\rho}_0 \quad ,$$

and define

$$\mathcal{H}^{(+,+)}\!\left(u,\frac{\omega+ku}{k'}\right) \equiv \frac{1}{2\pi i}\int d\eta' e_{\sigma'}\left(\frac{k'}{\underset{\sim}{k}'\cdot\underset{\sim}{v}'+\omega+ku+i\epsilon}\right)\Big\{\bar{\rho}_0^{(+)}(u,k)L(u,\eta',\underset{\sim}{k},\underset{\sim}{k}')$$

$$+\frac{\underset{\sim}{E}}{2\pi i}\cdot\frac{\partial}{\partial\underset{\sim}{v}'}\left[Y_1(\eta',\underset{\sim}{k})(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}'+i\epsilon)^{-1}\right]\Big\}, \tag{III-33}$$

$$\mathcal{L}(u, \frac{\omega + ku}{k'}) \equiv \frac{1}{2\pi i}\int d\eta' e_{\sigma}' \left(\frac{k'}{\underset{\sim}{k}'\cdot\underset{\sim}{v}' + \omega + ku + i\epsilon}\right) L(u, \eta', \underset{\sim}{k}, \underset{\sim}{k}') \quad , \qquad \text{(III-34)}$$

$$A(u, \frac{\omega+ku}{k'}) \equiv \left(\frac{1}{2\pi i}\right)^2 \int d\eta' e_{\sigma} \; F_1(\eta')(u - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)^{-1}$$

$$\cdot \left[E \cdot \frac{\partial}{\partial \underset{\sim}{v}'}\left(\hat{\underset{\sim}{k}}' \cdot \underset{\sim}{v}' + \frac{\omega + ku}{k'} + i\epsilon\right)^{-1}\right] \quad , \qquad \text{(III-35)}$$

where in defining $\mathcal{H}^{(+,+)}$, we used the fact that the function L[defined in Eq. (III-29)] is analytic in the upper-half of the u-plane. In terms of these new notations, Eq. (III-32) reads

$$\bar{\bar{S}}^{(-,-)}(u, \frac{-\omega - ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}')$$

$$= \overset{*}{\Delta}(u,k)\left[\mathcal{H}^{(+,+)}(u, \frac{\omega + ku}{k'}) - \bar{\rho}_0(u,k)\mathcal{L}(u, \frac{\omega + ku}{k'})\right] \qquad \text{(III-36)}$$

$$+ A\left(u, \frac{\omega + ku}{k'}\right) \quad .$$

In Eq. (III-36), by interchanging $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ and then replacing u by $-(\omega + ku)/k'$ (in this order), we immediately get

$$\bar{\bar{S}}^{(-,-)}(\frac{-\omega - ku}{k'}, -u, \underset{\sim}{k}', \underset{\sim}{k})$$

$$= \overset{*}{\Delta}(\frac{-\omega - ku}{k'}, k')\left[\mathcal{H}^{(+,+)}(\frac{-\omega - ku}{k'}, -u) - \bar{\rho}_0(\frac{-\omega - ku}{k'}, k') \right. \qquad \text{(III-36-a)}$$

$$\left. \mathcal{L}(\frac{-\omega - ku}{k'}, -u)\right] + A(\frac{-\omega - ku}{k'}, -u) \quad .$$

With $\overline{\overline{S}}^{(-,-)}$ in these forms, some of the u-integration in (III-16) can be done by contour integration. As shown in the Appendix, we have the following result:

$$-\frac{2D_0(\eta,\underset{\sim}{k})}{k'k^2}\int_C du \frac{B(u,\frac{-\omega-ku}{k'})-\overline{\overline{S}}^{(-,-)}(u,\frac{-\omega-ku}{k'},\underset{\sim}{k},\underset{\sim}{k}')-\begin{bmatrix} \underset{\sim}{k}\leftrightarrow\underset{\sim}{k}'; \\ u\rightarrow-(\omega+ku)/k' \end{bmatrix}}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^{*}(\frac{-\omega-ku}{k'},k')\Delta^{*}(u,k)}$$

$$=\frac{4\pi i D_0(\eta,\underset{\sim}{k})H^{(+,+)}(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v},\frac{\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'})}{k'k^2\Delta^{*}(\frac{-\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}}{k'},k')} \tag{III-37}$$

$$-\frac{2D_0(\eta,\underset{\sim}{k})}{k'k^2}\int_C du\frac{C(u,\frac{\omega+ku}{k'})+\left[\hat{\Delta}^{*}(u,k)\overline{\rho}_0(u,k)\mathcal{I}(u,\frac{\omega+ku}{k'})\right]+\begin{bmatrix} \underset{\sim}{k}\leftrightarrow\underset{\sim}{k}'; \\ u\rightarrow-(\omega+ku)/k' \end{bmatrix}}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^{*}(\frac{-\omega-ku}{k'},k')\Delta^{*}(u,k)},$$

where we used the symbol $\begin{bmatrix} \underset{\sim}{k}\leftrightarrow\underset{\sim}{k}'; \\ u\rightarrow-(\omega+ku)/k' \end{bmatrix}$ to mean the following: in the immediately preceding square bracket, the arguments $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ are interchanged and then the argument u is replaced by $-(\omega+ku)/k'$ (in that order), and

$$C\left(u,\frac{\omega+ku}{k'}\right)\equiv B\left(u,\frac{-\omega-ku}{k'}\right)-A\left(u,\frac{\omega+ku}{k'}\right)-A\left(\frac{-\omega-ku}{k'},-u\right). \tag{III-38}$$

4. Expressions for $\overline{S}^{(-)}[\eta,-(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v})/k',\underset{\sim}{k},\underset{\sim}{k}']$ and $\overline{S}_r^{(-)}[\eta,-(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v})/k',\underset{\sim}{k},\underset{\sim}{k}']$

As an introductory step in deriving the expression of

$\bar{S}^{(-)}[\eta, -(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$, we write $S(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')$ defined in Eq. (III-11) in terms of L. This is done by using the definitions of the functions $Z_0$ and L. The result is

$$S(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}') = \left\{ [\Delta(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, k)]^{-1} F_0(\eta) + \frac{2i}{k^2} \bar{\rho}^{(+)}(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, k\, D_0(\eta, \underset{\sim}{k}) \right\} \tag{III-39}$$

$$L(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, \eta', \underset{\sim}{k}, \underset{\sim}{k}') + (\pi k^2)^{-1} D_0(\eta, \underset{\sim}{k}) \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}'} \left[ \frac{Y_1(\eta', \underset{\sim}{k})}{(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v} - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}' + i\epsilon)} \right] .$$

By definition

$$\bar{S}^{(-)}\left[\eta, \frac{-(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})}{k'} \underset{\sim}{k}, \underset{\sim}{k}'\right] \equiv \frac{k'}{2\pi i} \int d\eta' e_\sigma' \frac{S(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')}{(\underset{\sim}{k}' \cdot \underset{\sim}{v}' + \omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon)} . \tag{III-40}$$

Substituting Eq. (III-39) S, and then utilizing the definitions of Eqs. (III-33) and (III-34), we find

$$\bar{S}^{(-)}[\eta, -(\omega + \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$$

$$= [\Delta(\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, k)]^{-1} F_0(\eta) \mathcal{L}[\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, (\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k'] \tag{III-41}$$

$$+ \frac{2i}{k^2} D_0(\eta, \underset{\sim}{k}) \mathcal{H}^{(+,+)}[\hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}, (\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k'] .$$

Now the final expression which has to be derived is $\bar{S}_r^{(-)}[\eta, -(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$. By definition

$$\bar{S}_r^{(-)}[\eta, -(\omega + \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$$
$$\equiv \frac{k'}{2\pi i} \int du' \bar{S}_r(\eta, u', \underset{\sim}{k}, \underset{\sim}{k}') / (k'u' + \omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon) . \tag{III-42}$$

Substitution of Eq. (III-23) for $\bar{S}_r(\eta, u', \underset{\sim}{k}, \underset{\sim}{k}')$ in above equation, yields

$$\bar{S}_r^{(-)}[\eta, -(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$$

$$= \frac{1}{2\pi i}\int du' \frac{\left[\bar{\rho}_0^{(+)}(u',k')\Delta(u',k') - \bar{\rho}_0^{(-)}(u',k')\overset{*}{\Delta}(u',k')\right]}{\pi k'(k'u'+\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\left\{U(\eta,\underset{\sim}{k},\underset{\sim}{k}') - \underset{\sim}{E}\cdot\frac{\partial}{\partial\underset{\sim}{v}}\left[\frac{D_1(\eta,\underset{\sim}{k}')}{u'-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}+i\epsilon}\right]\right\}$$

$$+\left(\frac{1}{2\pi i}\right)^2\int du' \frac{k'[\Delta(u',k')-\overset{*}{\Delta}(u',k')]}{(k'u'+\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\underset{\sim}{E}\cdot\frac{\partial}{\partial\underset{\sim}{v}}\left[\frac{Y_1(\eta,\underset{\sim}{k}')}{u'-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}+i\epsilon}\right] \quad . \tag{III-43}$$

Following precisely the same argument and method in the derivation of Eq. (III-26), we find that the initial terms in both integrals have zero contribution; therefore Eq. (III-43) reduces to

$$\bar{S}_r^{(-)}[\eta, -(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}']$$

$$= -\frac{1}{2\pi i}\int du' \frac{\overset{*}{\Delta}(u',k')\,\bar{\rho}_0^{(-)}(u',k')}{\pi k'(k'u'+\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}U(\eta,\underset{\sim}{k},\underset{\sim}{k}') \tag{III-44}$$

$$+\left(\frac{1}{2\pi i}\right)^2\int du' \frac{\overset{*}{\Delta}(u',k')}{k'(k'u'+\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\underset{\sim}{E}\cdot\frac{\partial}{\partial\underset{\sim}{v}}\left[\frac{2i\bar{\rho}_0^{(-)}(u',k')D_1(\eta,\underset{\sim}{k}') - k'^2 Y_1(\eta,\underset{\sim}{k}')}{u'-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}+i\epsilon}\right].$$

Following a similar procedure as in the derivation of Eq. (III-27) the first integral can be treated by the residue theorem; as for the second integral, we introduce the explicit form of $Y_1$ [given by Eq. (III-4)], and add a null integral

$$\left(\frac{1}{2\pi i}\right)^2 \int du' \left(\frac{k'}{k'u + \omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon}\right) \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \left[\frac{F_1(\eta)}{u - \hat{\underset{\sim}{k}}' \cdot \underset{\sim}{v} + i\epsilon}\right]$$

to it; consequently Eq. (III-45) changes to the following form:

$$\begin{aligned} \bar{S}_r^{(-)}\left(\eta, \frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) &= \frac{1}{\pi k'^2} \Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) \bar{\rho}_0^{(-)}\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) U(\eta, \underset{\sim}{k}, \underset{\sim}{k}') \\ &+ \frac{1}{2\pi i} \int du' \left(\frac{k'}{k'u' + \omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon}\right) \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \left\{ \frac{\Delta^*(u', k') D_1(\eta, \underset{\sim}{k}') \left[\bar{\rho}_0^{(-)}(u', k') - \bar{\rho}_0^{(-)}(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}, k')\right]}{\pi k'^2 (u' - \hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v} + i\epsilon)} \right. \\ &\left. - \frac{F_1(\eta)\left[\Delta^*(u', k') - \Delta^*(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}, k')\right]}{2\pi i \Delta^*(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}, k')(u' - \hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v} + i\epsilon)} \right\} . \end{aligned} \qquad \text{(III-46)}$$

Now it is clear that there is no singularity at $u' = \hat{\underset{\sim}{k}}' \cdot \underset{\sim}{v}$ in the integrand. Consequently, Eq. (III-46) may be re-written, for the sake of convenience, in terms of $Y_1$ as the following:

$$\begin{aligned} \bar{S}_r^{(-)}\left(\eta, \frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) &= \frac{1}{\pi k'^2} \Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) \bar{\rho}_0^{(-)}\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right) U(\eta, \underset{\sim}{k}, \underset{\sim}{k}') \\ &+ \left(\frac{1}{2\pi i}\right)^2 \int du' \left(\frac{k' \Delta^*(u', k')}{k'u' + \omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon}\right) \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \left\{ \frac{\frac{2i}{k'^2} \bar{\rho}_0^{(-)}(u', k') D_1(\eta, \underset{\sim}{k}') - Y_1(\eta, \underset{\sim}{k}') + \frac{F_1(\eta)}{\Delta^*(u', k')}}{(u' - \hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v} - i\epsilon)} \right\} . \end{aligned} \qquad \text{(III-47)}$$

the integral can now be evaluated by the residue theorem, following the same argument which we have used so many times previously. Then Eq. (III-47) reduces to

$$- \frac{2\pi}{k'} \overline{S}_r^{(-)}\left(\eta, \frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) / \Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)$$

$$= -\frac{2}{k'} \overline{\rho}_0^{(-)}\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)\left\{\frac{1}{k'} U(\eta, \underset{\sim}{k}, \underset{\sim}{k}') + \underset{\sim}{E}\cdot\underset{\sim}{Q}\left[D_1(\eta, \underset{\sim}{k}')\right]\right\} \tag{III-48}$$

$$- i\underset{\sim}{E}\cdot\left\{\underset{\sim}{Q}\left[Y_1(\eta, \underset{\sim}{k}')\right] - \underset{\sim}{Q}\left[F_1(\eta)\right] / \Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)\right\},$$

where the "operator" $\underset{\sim}{Q}$ is defined as

$$\underset{\sim}{Q}[X] \equiv \left\{\frac{\frac{\partial}{\partial \underset{\sim}{v}}}{[\omega + (\underset{\sim}{k} + \underset{\sim}{k}')\cdot\underset{\sim}{v} + i\epsilon]} - \frac{\underset{\sim}{k}'}{[\omega + (\underset{\sim}{k} + \underset{\sim}{k}')\cdot\underset{\sim}{v} + i\epsilon]^2}\right\} X. \tag{III-49}$$

5. The Kinetic Equation

The results just obtained in previous sections permit us to express G as a function of $f_0$, $\mathcal{F}$, $\underset{\sim}{E}$, and $\mathcal{G}_I$. By substituting Eqs. (III-37), (III-41), and (III-48) in Eq. (III-16), we get (returning the suppressed argument, $\omega$)

$$G(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) = \mathcal{D}(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) - \mathcal{E}(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) - i\underset{\sim}{E}\cdot\underset{\sim}{Q}\left[Y_1(\eta, \underset{\sim}{k}')\right]$$

$$- \frac{2}{k'^2} \overline{\rho}_0^{(-)}\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)\left\{\frac{1}{k'} U(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) + \underset{\sim}{E}\cdot\underset{\sim}{Q}\left[D_1(\eta, \underset{\sim}{k}')\right]\right\}$$

$$+ \frac{1}{\Delta^*\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)}\left\{i\underset{\sim}{E}\cdot\underset{\sim}{Q}\left[F_1(\eta)\right] + \mathcal{E}(\eta, \underset{\sim}{k}, \underset{\sim}{k}', \omega) - \frac{2\pi F_0(\eta)}{k'\Delta(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}, k)} \mathcal{L}\left(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}, \frac{\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}\right)\right\} \tag{III-50}$$

$$- \frac{2D_0(\eta, \underset{\sim}{k})}{k' k^2} \int_C du \, \frac{C\left(u, \frac{\omega + ku}{k'}\right) + \left[\Delta^*(u, k)\overline{\rho}_0(u, k)\mathcal{L}\left(u, \frac{\omega + ku}{k'}\right)\right] + \left[\begin{matrix} \underset{\sim}{k} \leftrightarrow \underset{\sim}{k}' \\ u \rightarrow -(\omega + ku)/k' \end{matrix}\right]}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\Delta^*\left(\frac{-\omega - ku}{k'}, k'\right)\,\Delta^*(u, k)},$$

with

$$\mathbf{E} \equiv \mathbf{E}(\mathbf{k} + \mathbf{k}', \omega) \quad .$$

At this point, we are reaching the final step in the derivation of the kinetic equation. We now return to the Eq. (I-5). The form of this equation prompts us to change the arguments of G from $\mathbf{k}$ and $\mathbf{k}'$ to $\mathbf{K} \equiv \mathbf{k} + \mathbf{k}'$ and $-\mathbf{k}'$ respectively. By doing so, the G takes the following form:

$$\begin{aligned}
G(\eta, \mathbf{K}, -\mathbf{k}', \omega) &= \vartheta(\eta, \mathbf{K}, -\mathbf{k}', \omega) - \mathcal{E}(\eta, \mathbf{K}, -\mathbf{k}', \omega) - i\mathbf{E}\cdot\mathbf{Q}_1[Y_1(\eta, -\mathbf{k}')] \\
&- \frac{2}{k'^2}\overline{\rho}_0^{(-)}\left(\frac{-\omega-\mathbf{K}\cdot\mathbf{v}}{k'}, k'\right)\left\{\frac{1}{k'}U(\eta, \mathbf{K}, -\mathbf{k}', \omega) + \mathbf{E}\cdot\mathbf{Q}[D_1(\eta, -\mathbf{k}')]\right\} \\
&+ \frac{1}{\Delta^*\left(\frac{-\omega-\mathbf{K}\cdot\mathbf{v}}{k'}, k'\right)}\left\{i\mathbf{E}\cdot\mathbf{Q}_1[F_1(\eta)] + \mathcal{E}(\eta, \mathbf{K}, -\mathbf{k}', \omega) - \frac{2\pi}{k'}\frac{F_0(\eta)}{\Delta(\hat{\mathbf{K}}\cdot\mathbf{v}, K)}\mathcal{L}\left(\hat{\mathbf{K}}\cdot\mathbf{v}, \frac{\omega+\mathbf{K}\cdot\mathbf{v}}{k'}\right)\right\} \qquad \text{(III-51)} \\
&- \frac{2D_0(\eta, \mathbf{K})}{k'k^2}\int_C du \frac{C\left(u, \frac{\omega+Ku}{k'}\right) + \left[\Delta^*(u,K)\overline{\rho}_0(u,K)\mathcal{L}\left(u, \frac{\omega+Ku}{k'}\right)\right] + \begin{bmatrix} \mathbf{K} \leftrightarrow -\mathbf{k}' \,; \\ u \rightarrow -(\omega+Ku)/k' \end{bmatrix}}{(u - \hat{\mathbf{K}}\cdot\mathbf{v})\,\Delta^*\left(\frac{-\omega-Ku}{k'}, k'\right)\,\Delta^*(u,K)} ,
\end{aligned}$$

where the operator $\mathbf{Q}_1$ is obtained from Eq. (III-49) by replacing $\mathbf{k}$ by $\mathbf{K}$, and $\mathbf{k}'$ by $-\mathbf{k}'$; while $\mathbf{E}$ now is $\mathbf{E}(\mathbf{k}, \omega)$. Using this expression for G in Eq. (I-5), we obtain a linear integral equation for $\mathcal{F}$ ; i.e.,

$$\begin{aligned}
(\omega + \mathbf{k}\cdot\mathbf{v})\mathcal{F}(\eta, \mathbf{k}, \omega) &= i\,\mathcal{F}_I(\eta, \mathbf{k}) - i\frac{e_\sigma}{m_\sigma}\mathbf{E}(\mathbf{k},\omega)\cdot\frac{\partial}{\partial\mathbf{v}}f_0(\eta) \\
&- \frac{1}{2\pi^2}\frac{e_\sigma}{m_\sigma}\int d\mathbf{k}'\,\frac{\mathbf{k}'}{k'^2}\cdot\frac{\partial}{\partial\mathbf{v}}G(\eta, \mathbf{K}, -\mathbf{k}', \omega) .
\end{aligned} \qquad \text{(III-52)}$$

This resulting equation for $\mathcal{J}$ is the transformed version of the desired kinetic equation. In general, this equation is in nature similar to those equations which usually appear in transport theory. However, this equation is valid for inhomogeneous and/or arbitrary frequency systems. Furthermore, it includes the two-particle correlation effect. Therefore, it is suitable especially in the determination of the high-frequency transport properties of a plasma, as well as in handling the correlation effect on the damping of plasma oscillations. We shall apply it to derive a generalized expression of the high-frequency electrical conductivity in the second part of this thesis. As for its application to the problem of correlation damping effect, this is reserved for future work.

# PART II. APPLICATION OF THE KINETIC EQUATION.

## Chapter IV. High-Frequency Conductivity of an Inhomogeneous Non-isotropic Plasma

### 1. Introduction

The classical electrical conductivity of a plasma has been derived by many authors[10-15]. Their treatments usually started with either the Fokker-Planck equation without collective effects, or the generalized Fokker-Planck equation (including the collective effects). The derivation of the former equation is based on an analogy between the motion of charged particles and the Brownian motion[14]. This approach assumes that the characteristic macroscopic time scale is much longer than that for microscopic fluctuations (which are of order of the reciprocal of plasma frequency). The derivation of the latter equation is based on the Bogolyubov's adiabatic hypothesis. However, in a high-frequency field, both the one-particle distribution function as well as the joint two-particle correlation function are oscillating at the driving frequency. Moreover, a plasma is capable of sustaining macroscopic oscillation at or above the plasma frequency. Therefore, the electrical conductivity calculated by the methods mentioned above is not valid at high-frequency. This fact initiated new approaches to the calculation of the high-frequency conductivity of a plasma.

The classical approach to the rigorous derivation of the high-frequency electrical conductivity has been treated by using Guernsey's equation[16], and by employing C.S. Wu's operator[18,17]. Nevertheless, all these results have been established under the assumptions: (a) the system is spatially homogeneous, and (b) the unperturbed one-particle distribution function is isotropic in velocity-space. Furthermore, the approach based on the application of Guernsey's equation implicitly assumed that the unperturbed state is in thermodynamic equilibrium, (i.e., electrons and ions have Maxwellian distributions with equal temperature). In contrast, we shall derive a general expression for high-frequency conductivity which is not subject to those restrictions through the application of the kinetic equation derived in Part I.

We shall show that our result reduces to the results obtained by the previous treatments, if the proper assumptions mentioned above are made.

## 2. Scheme of Approach

As a preliminary procedure for deriving the high-frequency conductivity for an inhomogeneous and non-isotropic plasma, we look for the high-frequency limit solutions to the kinetic equation derived previously. By high-frequency limit, we mean that the driving frequency, $\omega$, is much greater than the collision frequency, $(1/t_D)$, (where $t_D$ is the cumulative $90^{\circ}$ deflection time defined by Spitzer[14]); i.e., $\omega t_D >> 1$. For simplicity we shall neglect the contributions from the initial perturbations by setting $\mathcal{G}_I$ and $\mathcal{F}_I$ to zero. Then our kinetic equation reads

$$\mathcal{F}(\eta, \underset{\sim}{k}, \omega) = \frac{-i\frac{e_\sigma}{m_\sigma}}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)}\left[\underset{\sim}{E}\cdot\frac{\partial f_0(\eta)}{\partial\underset{\sim}{v}} + \frac{1}{2\pi^2 i}\int d\underset{\sim}{k}'\,\frac{\underset{\sim}{k}'}{k'^2}\cdot\frac{\partial}{\partial\underset{\sim}{v}}G(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega)\right], \tag{IV-1}$$

with

$$\begin{aligned} G(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) = &-\mathcal{E}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) - i\underset{\sim}{E}\cdot\underset{\sim}{Q}_1\left[Y_1(\eta, -\underset{\sim}{k}')\right] \\ &- \frac{2}{k'^2}\overline{\rho}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{v}}{k'}, k'\right)\left\{\frac{1}{k'}U(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) + \underset{\sim}{E}\cdot\underset{\sim}{Q}_1\left[D_1(\eta, -\underset{\sim}{k}')\right]\right\} \\ &+ \frac{1}{\Delta^*\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{v}}{k'}, k'\right)}\left\{i\underset{\sim}{E}\cdot\underset{\sim}{Q}_1\left[F_1(\eta)\right] + \mathcal{E}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) - \frac{2\pi}{k'}\frac{F_0(\eta)}{\Delta(\hat{\underset{\sim}{K}}\cdot\underset{\sim}{v}, K)}\mathcal{L}\left(\hat{\underset{\sim}{K}}\cdot\underset{\sim}{v}, \frac{\omega + \underset{\sim}{K}\cdot\underset{\sim}{v}}{k'}\right)\right\} \\ &- \frac{2D_0(\eta, \underset{\sim}{K})}{k'K^2}\int_C du\,\frac{C\left(u, \frac{\omega+Ku}{k'}\right) + \left[\Delta^*(u,K)\overline{\rho}_0(u,K)\mathcal{L}\left(u, \frac{\omega+Ku}{k'}\right)\right] + \left[\begin{matrix}\underset{\sim}{K} \longleftrightarrow -\underset{\sim}{k}' \\ u \rightarrow -(\omega+Ku)/k'\end{matrix}\right]}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*\left(\frac{-\omega-Ku}{k'}, k'\right)\Delta^*(u,K)}, \end{aligned} \tag{IV-2}$$

and

$$\underset{\sim}{E} \equiv \underset{\sim}{E}(\underset{\sim}{k}, \omega)$$

An exact solution to Eq. (IV-1) is too difficult to obtain because of the complexity of the equation. However, in the high-frequency limit*, approximate solutions may be obtained by an iteration procedure. This is possible due to the fact that, in this limit, the collision term (the terms containing G) is of order $(1/\omega t_D)$ compared to the other terms. Thus, to zeroth order in $(1/\omega t_D)$, $\mathcal{F}$ is simply given by

$$\mathcal{F}^{(0)}(\eta, \underset{\sim}{k}, \omega) = \frac{-i \frac{e_\sigma}{m_\sigma} \underset{\sim}{E}(\underset{\sim}{k}, \omega) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_0(\eta)}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon)} \quad . \tag{IV-3}$$

In this order of approximation there is no collisional effect, hence we must proceed to next order in $1/\omega t_D$. To this end, the first iteration is obtained by substituting Eq. (IV-3) into Eq. (IV-2). We thus have an explicit solution for G (which will be denoted by $G^{(0)}$) as a function of $f_0$. Then this $G^{(0)}$ is used in Eq. (IV-1) to produce the first order (in $1/\omega t_D$) correction term (which will be denoted by $\mathcal{F}^{(1)}$) to $\mathcal{F}$ ; i.e.,

$$\mathcal{F}^{(1)}(\eta, \underset{\sim}{k}, \omega) \equiv \frac{-1}{2\pi^2} \frac{\frac{e_\sigma}{m_\sigma}}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon)} \int d\underset{\sim}{k}' \, \frac{\underset{\sim}{k}'}{k'^2} \cdot \frac{\partial}{\partial \underset{\sim}{v}} G^{(0)}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) \, . \tag{IV-4}$$

Then these $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ will be used to derive the desired expression for the conductivity.

---

*A similar treatment for the low frequency limit is also possible.

3. Explicit Expressions for $G^{(0)}$ and $\mathcal{F}^{(1)}$.

Substitution of Eq. (IV-3) into Eq. (IV-2), yields (suppressing the argument, $\omega$)

$$G^{(0)}(\eta, \mathbf{K}, -\mathbf{k}') = - \mathcal{E}^{(0)}(\eta, \mathbf{K}, -\mathbf{k}') - i\mathbf{E}^{(0)} \cdot \mathbf{Q}_1 \left[ Y_1(\eta_1 - \mathbf{k}') \right]$$

$$- \frac{2}{k'^2} \bar{\rho}_0^{(-)}\left( \frac{-\omega - \mathbf{K}\cdot\mathbf{v}}{k'}, k' \right) \left\{ \frac{1}{k'} U^{(0)}(\eta, \mathbf{K}, -\mathbf{k}') + \mathbf{E}^{(0)} \cdot \mathbf{Q}_1 \left[ D_1(\eta, -\mathbf{k}') \right] \right\}$$

$$+ \frac{1}{\Delta^*\left( \frac{-\omega - \mathbf{K}\cdot\mathbf{v}}{k'}, k' \right)} \left\{ i\mathbf{E}^{(0)} \cdot \mathbf{Q}_1 \left[ F_1(\eta) \right] + \mathcal{E}^{(0)}(\eta, \mathbf{K}, -\mathbf{k}') \right.$$

$$\left. - \frac{2\pi}{k'} \frac{F_0(\eta)}{\Delta(\hat{\mathbf{K}}\cdot\mathbf{v}, K)} \mathcal{L}^{(0)}\left( \hat{\mathbf{K}}\cdot\mathbf{v}, \frac{\omega + \mathbf{K}\cdot\mathbf{v}}{k'} \right) \right\}$$

$$\frac{2\,\nu_0(\eta, \mathbf{K})}{k' K^2} \int_C du \frac{C^{(0)}\left(u, \frac{\omega + Ku}{k'}\right) + \left[ \Delta^*(u,K)\, \bar{\rho}_0(u,K)\, \mathcal{L}^{(0)}\left(u, \frac{\omega + Ku}{k'}\right) \right] + \left[ \begin{matrix} \mathbf{K} \longleftrightarrow -\mathbf{k}' \\ u \rightarrow -(\omega + Ku)/k' \end{matrix} \right]}{(u - \hat{\mathbf{k}}\cdot\mathbf{v})\, \Delta^*\left( \frac{-\omega - Ku}{k'}, k' \right) \Delta^*(u, K)} ,$$

(IV-5)

where the superscript, (0), indicates that in those terms $\mathcal{F}$ is being replaced by $\mathcal{F}^{(0)}$. Having made this substitution, we immediately obtain the following results through the appropriate definitions and some algebra [detailed steps are shown in Appendix B]:

(a)

$$\int_C du \frac{C^{(0)}\left(u, \frac{\omega+Ku}{k'}\right)}{(u-\hat{\mathbf{k}}\cdot\mathbf{v})\,\Delta^*\left(\frac{-\omega-Ku}{k'}, k'\right)\Delta^*(u,K)}$$

$$= \frac{k'K}{2\pi i}\,\mathbf{E}^{(0)}\cdot\int d\eta' e_{\sigma'}\, h_1(\eta',\mathbf{k})\left[\frac{\dfrac{k'}{\Delta^*\left(\frac{-\omega-\mathbf{k}\cdot\mathbf{v}'}{k'}, k'\right)\Delta(\hat{\mathbf{K}}\cdot\mathbf{v}', K)}}{(\mathbf{K}\cdot\mathbf{v}'-\mathbf{K}\cdot\mathbf{v}-i\epsilon)}\right. \tag{IV-6}$$

$$\left. + \frac{\dfrac{K}{\Delta^*(-\hat{\mathbf{k}}'\cdot\mathbf{v}', k')\,\Delta^*\left(\frac{-\omega+\mathbf{k}'\cdot\mathbf{v}'}{K}, K\right)}}{(-\mathbf{k}'\cdot\mathbf{v}'+\omega+\mathbf{K}\cdot\mathbf{v}+i\epsilon)}\right],$$

$$h_1(\eta',\mathbf{k}) \equiv \frac{F_1(\eta')}{(\omega+\mathbf{k}\cdot\mathbf{v}'+i\epsilon)^2} \tag{IV-7}$$

(b)

$$\mathcal{L}^{(0)}\left(u, \frac{\omega+Ku}{k'}\right)$$

$$= -\frac{k'}{\pi K}\frac{\mathbf{E}^{(0)}}{2\pi i}\cdot\int d\eta' e_{\sigma'}\left[\frac{\mathbf{K}\, d_1(\eta',\mathbf{K};\mathbf{k}) + (\mathbf{k}\cdot\hat{\mathbf{K}})\,\mathbf{b}_1(\eta',\mathbf{k})}{(-\mathbf{k}'\cdot\mathbf{v}'+\omega+Ku+i\epsilon)}\right. \tag{IV-8}$$

$$\left. - \frac{\mathbf{k}'\, d_1(\eta',\mathbf{K};\mathbf{k})}{(Ku-\mathbf{K}\cdot\mathbf{v}'+i\epsilon)}\right];$$

where

$$d_1(\eta',\mathbf{K};\mathbf{k}) \equiv \frac{D_1(\eta',\mathbf{K})}{(\omega+\mathbf{k}\cdot\mathbf{v}'+i\epsilon)^2},$$

(IV-9)

$$\mathbf{b}_1(\eta',\mathbf{k}) = \frac{4\pi^2\left(\frac{e_\sigma}{m_\sigma}\right)^2 \frac{\partial}{\partial\mathbf{v}'} f_0(\eta')}{(\omega+\mathbf{k}\cdot\mathbf{v}'+i\epsilon)^2}.$$

Or by defining

$$\underset{\sim}{\ell}_1\left(u,\ \frac{\omega+Ku}{k'}\right)$$

$$\equiv \frac{k'}{K}\frac{1}{2\pi_i}\int d\eta' e_{\sigma}'\left[\frac{\underset{\sim}{K}d_1(\eta',\underset{\sim}{K};\underset{\sim}{k}) + (\underset{\sim}{k}\cdot\hat{\underset{\sim}{K}})\, b_1(\eta',\underset{\sim}{k})}{(-\underset{\sim}{k}'\cdot\underset{\sim}{v}' + \omega + Ku + i\epsilon)} - \frac{\underset{\sim}{k}'d_1(\eta',\underset{\sim}{K};\underset{\sim}{k})}{(Ku - \underset{\sim}{K}\cdot\underset{\sim}{v}' + i\epsilon)}\right] \tag{IV-10}$$

then

$$\mathcal{L}^{(0)}\left(u,\ \frac{\omega+ku}{k'}\right) = -\frac{1}{\pi}\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{\ell}_1\left(u,\ \frac{\omega+ku}{k'}\right)\ . \tag{IV-8-a}$$

In Eq. (IV-10), interchanging $\underset{\sim}{K}$ and $-\underset{\sim}{k}'$ then replacing u by $-(\omega+Ku)/k'$, we find

$$\underset{\sim}{\ell}_1\left(\frac{-\omega-Ku}{k'},\ -u\right)$$

$$= \frac{K}{k'}\frac{1}{2\pi i}\int d\eta' e_{\sigma'}\left[\frac{-\underset{\sim}{k}'d_1(\eta',-\underset{\sim}{k}';\underset{\sim}{k})-(\underset{\sim}{k}\cdot\hat{\underset{\sim}{k}}')\,\underset{\sim}{b}_1(\eta',\underset{\sim}{k})}{(\underset{\sim}{K}\cdot\underset{\sim}{V}'-Ku+i\epsilon)} + \frac{\underset{\sim}{K}\,d_1(\eta',-\underset{\sim}{k}';\underset{\sim}{k})}{(-\omega-Ku+\underset{\sim}{k}'\cdot\underset{\sim}{V}'+i\epsilon)}\right] \tag{IV-10-a}$$

(c)

$$-\mathcal{E}^{(0)}(\eta,\underset{\sim}{K},-\underset{\sim}{k}') - i\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1[Y_1(\eta,-\underset{\sim}{k}')] - \frac{2}{k'^2}\bar{\rho}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{V}}{k'},\underset{\sim}{k}'\right)\left\{\frac{1}{k'}\mathcal{U}^{(0)}(\eta,\underset{\sim}{K},-\underset{\sim}{k}') + \underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1[D_1(\eta,-\underset{\sim}{k}')]\right\} + \frac{1}{\Delta^*\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{V}}{k'},\underset{\sim}{k}'\right)}\left\{i\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1[F_1(\eta)] + \mathcal{E}^{(0)}(\eta,\underset{\sim}{K},-\underset{\sim}{k}')\right\}$$

$$= \underset{\sim}{E}^{(0)}\cdot\left\{\int i\frac{\partial}{\partial \underset{\sim}{V}}[F_1(\eta)-Y_1(\eta,-\underset{\sim}{k}')]\Big/(\omega+\underset{\sim}{k}\cdot\underset{\sim}{V}+i\epsilon) + \frac{2}{k'^2}\left\{\underset{\sim}{k}'\bar{\rho}_0^{(-)}(-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{V},\underset{\sim}{k}')d_1(\eta,-\underset{\sim}{k}';\underset{\sim}{k}) - \bar{\rho}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{V}}{k'},\underset{\sim}{k}'\right)\left[\underset{\sim}{k}'d_1(\eta,-\underset{\sim}{k}';\underset{\sim}{k})+(\underset{\sim}{k}\cdot\hat{\underset{\sim}{k}}')\underset{\sim}{b}_1(\eta,\underset{\sim}{k})\right]\right\} + i\underset{\sim}{k}'h_1(\eta,\underset{\sim}{k})\left\{\left[\Delta^*\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{V}}{k'},\underset{\sim}{k}'\right)\right]^{-1} - \left[\Delta^*(-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{V},\underset{\sim}{k}')\right]^{-1}\right\}\right\}. \tag{IV-11}$$

At this point, it is advantageous to condense the long tedious formulas. After examining the form of $G^{(0)}$ in Eq. (IV-5) as well as the results which appeared in (a), (b), and (c), we introduce the following notations:

$$
\begin{aligned}
\mathbf{R}_1(\eta,\mathbf{K},-\mathbf{k}') \equiv {} & \left\{ i\frac{\partial}{\partial \mathbf{v}}\left[F_1(\eta) - Y_1(\eta,-\mathbf{k}')\right]\right\} \Big/ (\omega + \mathbf{k}\cdot\mathbf{v} + i\epsilon) \\
& + \frac{2}{k'^2}\left\{ \mathbf{k}'\,\overline{\rho}_0^{(-)}(-\hat{\mathbf{k}}'\cdot\mathbf{v},k')\,d_1(\eta,-\mathbf{k}';\mathbf{k}) - \overline{\rho}_0^{(-)}\left(\frac{-\omega-\mathbf{k}\cdot\mathbf{v}}{k'},k'\right)\Big[\mathbf{k}'\,d_1(\eta,-\mathbf{k}',\mathbf{k}) \right. \\
& \left. \qquad + (\mathbf{k}\cdot\hat{\mathbf{k}}')\,\mathbf{b}_1(\eta,\mathbf{k})\Big]\right\} ,
\end{aligned}
\qquad \text{(IV-12)}
$$

$$
\mathbf{R}_2(\eta,\mathbf{K},-\mathbf{k}') \equiv i\mathbf{k}'\,h_1(\eta,\mathbf{k})\left[\frac{1}{\Delta^*\left(\frac{-\omega-\mathbf{K}\cdot\mathbf{v}}{k'},k'\right)} - \frac{1}{\Delta^*(-\hat{\mathbf{k}}'\cdot\mathbf{v},k')}\right] , \qquad \text{(IV-13)}
$$

$$
\mathbf{R}_3(\eta,\mathbf{K},-\mathbf{k}') \equiv \frac{2}{k'}\,F_0(\eta)\,\boldsymbol{\ell}_1\left(\hat{\mathbf{K}}\cdot\mathbf{v},\frac{\omega+\mathbf{K}\cdot\mathbf{v}}{k'}\right)\Big/\Delta^*\left(\frac{-\omega-\mathbf{K}\cdot\mathbf{v}}{k'},k'\right)\Delta(\hat{\mathbf{K}}\cdot\mathbf{v},K) , \qquad \text{(IV-14)}
$$

$$
\begin{aligned}
\mathbf{R}_4(\eta,\mathbf{K},-\mathbf{k}') \equiv {} & \frac{-2D_0(\eta,\mathbf{K})}{K\,2\pi i}\int d\eta' e_{\sigma'}\,h_1(\eta',\mathbf{k})\left\{\frac{\mathbf{k}'\left[\Delta^*\left(\frac{-\omega-\mathbf{K}\cdot\mathbf{v}'}{k'},k'\right)\Delta^*(\hat{\mathbf{K}}\cdot\mathbf{v}',K)\right]^{-1}}{(\mathbf{K}\cdot\mathbf{v}'-\mathbf{K}\cdot\mathbf{v}-i\epsilon)}\right. \\
& \left. + \frac{\mathbf{K}\left[\Delta^*(-\hat{\mathbf{k}}'\cdot\mathbf{v}',k')\,\Delta^*\left(\frac{-\omega+\mathbf{k}'\cdot\mathbf{v}'}{K},K\right)\right]^{-1}}{(-\mathbf{k}'\cdot\mathbf{v}'+\omega+\mathbf{K}\cdot\mathbf{v}+i\epsilon)}\right\} ,
\end{aligned}
\qquad \text{(IV-15)}
$$

$$
\begin{aligned}
\mathbf{R}_5(\eta,\mathbf{K},-\mathbf{k}') \equiv {} & \frac{2D_0(\eta,\mathbf{K})}{\pi k' K^2}\int_C du\,\frac{1}{(u-\hat{\mathbf{K}}\cdot\mathbf{v})}\left[\frac{\overline{\rho}_0(u,K)\,\boldsymbol{\ell}_1\left(u,\frac{\omega+Ku}{k'}\right)}{\Delta^*\left(\frac{-\omega-Ku}{k'},k'\right)}\right. \\
& \left. + \frac{\overline{\rho}_0\left(\frac{-\omega-Ku}{k'},k'\right)\boldsymbol{\ell}_1\left(\frac{-\omega-Ku}{k'},-u\right)}{\Delta^*(u,K)}\right] .
\end{aligned}
\qquad \text{(IV-16)}
$$

In terms of these notations, Eq. (IV-5) reads (after returning the suppressed argument, $\omega$)

$$G^{(0)}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) = \underset{\sim}{E}^{(0)}(\underset{\sim}{k}, \omega) \cdot \sum_{i=1}^{5} R_i(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) \ . \quad \text{(IV-17)}$$

By the substitution of Eq. (IV-17) into Eq. (IV-4), the first order in $1/\omega t_D$ correction to $\mathcal{F}$ becomes

$$\mathcal{F}^{(1)}(\eta, \underset{\sim}{k}, \omega)$$

$$= -\frac{1}{2\pi^2} \frac{e_\sigma/m_\sigma}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)} \int d\underset{\sim}{k}' \frac{\underset{\sim}{k}'}{k'^2} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \left\{ \underset{\sim}{E}^{(0)}(\underset{\sim}{k}, \omega) \cdot \sum_{i=1}^{5} \underset{\sim}{R}_i(\eta, \underset{\sim}{K}, -\underset{\sim}{k}', \omega) \right\} . \quad \text{(IV-18)}$$

4. High-Frequency Conductivity

Having obtained the zeroth and the first order approximations to the one-particle perturbation, the next goal is to find the corresponding order approximations for the electrical conductivities. By definition

$$\underset{\sim}{j}(\underset{\sim}{k}, \omega) \equiv \int d\eta \, e_\sigma \underset{\sim}{v} \, \mathcal{F}(\eta, \underset{\sim}{k}, \omega) \quad ,$$

Utilizing Eqs. (IV-3) and (IV-17) then

$$\underset{\sim}{j}^{(0)}(\underset{\sim}{k}, \omega) = E_\beta^{(0)}(\underset{\sim}{k}, \omega) \int d\eta \, e_\sigma \left[ \frac{-i \frac{e_\sigma}{m_\sigma} \underset{\sim}{v} \frac{\partial}{\partial v_\beta} f_0(\eta)}{\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon} \right] , \quad \text{(IV-19)}$$

$$\dot{\underset{\sim}{j}}^{(1)}(\underset{\sim}{k},\omega)=E_\beta^{(0)}(\underset{\sim}{k},\omega)\left\{-\frac{1}{2\pi^2}\int d\underset{\sim}{k}'\,\frac{k'_\alpha}{k'^2}\int d\eta\, e_\sigma\,\frac{\frac{e_\sigma}{m_\sigma}\underset{\sim}{v}\frac{\partial}{\partial v_\alpha}R_\beta(\eta,\underset{\sim}{k},-\underset{\sim}{k}',\omega)}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\right\}, \tag{IV-20}$$

where

$$\underset{\sim}{R} \equiv \sum_{i=1}^{5} \underset{\sim}{R}_i \quad . \tag{IV-21}$$

Now, by comparing Eqs. (IV-19) and (IV-20) with the Ohm's Law,

$$\underset{\sim}{j} = \underset{\sim}{E} \cdot \underset{\approx}{\sigma} \quad ,$$

We finally obtain the zeroth and the first order approximations for the desired conductivity:

$$\underset{\approx}{\sigma}^{(0)}(\underset{\sim}{k},\omega)=-i\int d\eta\, e_\sigma\,\frac{\frac{e_\sigma}{m_\sigma}\underset{\sim}{v}\frac{\partial}{\partial \underset{\sim}{v}}f_0(\eta)}{\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon} \quad , \tag{IV-22}$$

$$\underset{\approx}{\sigma}^{(1)}(\underset{\sim}{k},\omega)=-\frac{1}{2\pi^2}\int d\underset{\sim}{k}'\,\frac{k'_\alpha}{k'^2}\int d\eta\, e_\sigma\,\frac{\frac{e_\sigma}{m_\sigma}\underset{\sim}{v}\frac{\partial}{\partial v_\alpha}\underset{\sim}{R}(\eta,\underset{\sim}{k},-\underset{\sim}{k}',\omega)}{\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon} \quad , \tag{IV-23}$$

and

$$\underset{\approx}{\sigma}(k,\omega) = \underset{\approx}{\sigma}^{(0)}(k,\omega) + \underset{\approx}{\sigma}^{(1)}(k,\omega) \quad . \tag{IV-24}$$

From the definitions of the $\underset{\sim}{R}$'s [cf. Eqs. (IV-12)-(IV-16)], we observe that they are only functions of $f_0$, $\underset{\sim}{k}$, $\underset{\sim}{k}'$, and $\omega$. Therefore, for a given form of $f_0$, the electrical conductivity tensor can be calculated.

Finally, by comparing the present analysis with those previous ones which were treated by Oberman, Ron, and Dawson[16] (hereafter denoted by ORD), as well as by C.S. Wu, and Klevans[17, 18] (hereafter denoted by KW), we may list the distinguished features of the present work:

(1) The spatial inhomogeneous effect is taken into account here.

(2) The isotropic assumption for the unperturbed one particle distribution, $f_0$, is not used in our derivation.

(3) We obtain an electrical conductivity tensor instead of a scalar conductivity. Furthermore the form of the applied electric field, $\underset{\sim}{E}(\underset{\sim}{r}, t)$, as well as its orientation is not specified in our analysis.

(4) Our result applies to systems with an arbitrary number of species. In addition, it also applies to non-isothermal systems with arbitrary unperturbed distribution functions.

It should be mentioned that the last statement does not apply to KW's[18] result. In the next section we shall show how our result will reduce to those previous ones, if the appropriate assumptions are made in our formulas.

5. Spatial Homogeneous Case with Non-isotropic $f_0$

The spatial homogeneous case corresponds to $\underset{\sim}{k} \to 0$ limit in our result. By taking this limit our result reduces considerably. Then Eq. (IV-23) reads

$$\underset{\approx}{\sigma}^{(1)}(\omega) = -\frac{1}{2\pi^2\omega}\int d\underset{\sim}{k}' \frac{k'_\alpha}{k'^2}\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{v} \frac{\partial}{\partial v_\alpha} \underset{\sim}{R}(\eta, \underset{\sim}{k}', -\underset{\sim}{k}', \omega) \; ;$$

or integrating by parts,

$$\underset{\approx}{\sigma}^{(1)}(\omega) = \frac{1}{2\pi^2\omega}\int d\underset{\sim}{k}' \frac{\underset{\sim}{k}'}{k'^2}\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}(\eta, \underset{\sim}{k}', -\underset{\sim}{k}', \omega) \,. \tag{IV-25}$$

In Eqs. (IV-7) and (IV-9) taking $\underset{\sim}{k} \to 0$ limit, we find

$$h_1(\eta', \underset{\sim}{k}) \to h_1(\eta') = F_1(\eta')/\omega^2 \quad ,$$

$$d_1(\eta', \underset{\sim}{k}; \underset{\sim}{k}) \to d_1(\eta', \underset{\sim}{k}') = D_1(\eta', \underset{\sim}{k}')/\omega^2 \quad ,$$

$$\underset{\sim}{b}_1(\eta', \underset{\sim}{k}) \to \underset{\sim}{b}_1(\eta') = \frac{1}{\omega^2} 4\pi^2 \left(\frac{e_{\sigma'}}{m_{\sigma'}}\right)^2 \frac{\partial}{\partial \underset{\sim}{v}'} f_0(\eta') . \tag{IV-26}$$

Then Eq. (IV-10) reduces to

$$\underset{\sim}{l}_1\left(u, \frac{\omega + k'u}{k'}\right) = \frac{\underset{\sim}{k}'}{\omega^2}\frac{1}{2\pi i}\int d\eta' e_{\sigma'}\left[\frac{D_1(\eta', \underset{\sim}{k}')}{-\underset{\sim}{k}' \cdot \underset{\sim}{v}' + \omega + k'u + i\epsilon} - \frac{D_1(\eta', \underset{\sim}{k}')}{k'u - \underset{\sim}{k}' \cdot \underset{\sim}{v}' + i\epsilon}\right] ;$$

thus the barring operation can be done with the result

$$\underset{\sim}{\ell}_1\left(u, \frac{\omega + k' u}{k'}\right) = \frac{\hat{k}'}{\omega^2}\left[\overline{D}_1^{(+)}(u) - \overline{D}_1^{(+)}\left(\frac{\omega + k' u}{k'}\right)\right]. \qquad \text{(IV-27)}$$

Similarly, Eq. (IV-10-a) reduces to

$$\underset{\sim}{\ell}_1\left(\frac{-\omega - k' u}{k'}, -u\right) = \frac{\hat{\underset{\sim}{k}}'}{\omega^2}\left[\overline{D}_1^{(-)}(u) - \overline{D}_1^{(-)}\left(\frac{\omega + k' u}{k'}\right)\right], \qquad \text{(IV-27-a)}$$

where the identity $D_1(\eta, -\underset{\sim}{k}') \equiv - D_1(\eta, \underset{\sim}{k}')$ has been used. Consequently, in $\underset{\sim}{k} \to 0$ limit, the $\underset{\sim}{R}$'s defined in Eqs. (IV-12)-(IV-16) reduce to the following forms:

$$\begin{aligned} \underset{\sim}{R}_1(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') &= \frac{i}{\omega}\frac{\partial}{\partial \underset{\sim}{v}}\left[F_1(\eta) - Y_1(\eta, -\underset{\sim}{k}')\right] \\ &- \frac{2\underset{\sim}{k}'}{(\omega k')^2}\left[\overline{\rho}_0^{(-)}(-\hat{k}'\cdot \underset{\sim}{v}, k') - \overline{\rho}_0^{(-)}\left(\frac{-\omega - \underset{\sim}{k}'\cdot \underset{\sim}{v}}{k'}, k'\right)\right] D_1(\eta, \underset{\sim}{k}') . \end{aligned} \qquad \text{(IV-28)}$$

$$\underset{\sim}{R}_2(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = \frac{i\underset{\sim}{k}'}{\omega^2} F_1(\eta)\left[\frac{1}{\Delta^*\left(\frac{-\omega - \underset{\sim}{k}'\cdot \underset{\sim}{v}}{k'}, k'\right)} - \frac{1}{\Delta^*(-\hat{\underset{\sim}{k}}'\cdot \underset{\sim}{v}, k')}\right]; \qquad \text{(IV-29)}$$

or by using the previously defined relation

$$\Delta^* \equiv 1 + \frac{2i}{k'^2}\overline{D}_0^{(-)}$$

then

$$\underset{\sim}{R}_2(\eta,\underset{\sim}{k}',-\underset{\sim}{k}')=\frac{2\underset{\sim}{k}'}{(\omega k')^2}F_1(\eta)\,\frac{\left[\overline{D}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{k}'\cdot\underset{\sim}{v}}{k'}\right)-\overline{D}_0^{(-)}(-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v})\right]}{\Delta^*\left(\frac{-\omega-\underset{\sim}{k}'\cdot\underset{\sim}{v}}{k'},k'\right)\Delta^*(-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v},k')}\,.$$

(IV-29-a)

$$\underset{\sim}{R}_3(\eta,\underset{\sim}{k}',-\underset{\sim}{k}')=-\frac{2\underset{\sim}{k}'}{(\omega k')^2}F_0(\eta)\,\frac{\left[\overline{D}_1^{(+)}\left(\frac{\omega+\underset{\sim}{k}'\cdot\underset{\sim}{v}}{k'}\right)-\overline{D}_1^{(+)}(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v})\right]}{\Delta^*\left(\frac{-\omega-\underset{\sim}{k}'\cdot\underset{\sim}{v}}{k'},k'\right)\Delta(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v},k')}\,.$$

(IV-30)

$$\underset{\sim}{R}_4(\eta,\underset{\sim}{k}',-\underset{\sim}{k}')=\frac{-2\underset{\sim}{k}'k'}{(\omega k')^2}\,\frac{D_0(\eta,\underset{\sim}{k}')}{2\pi i}\left\{\int d\eta' e_{\sigma'}\,\frac{F_1(\eta')/\Delta^*\left(\frac{-\omega-\underset{\sim}{k}'\cdot\underset{\sim}{v}'}{k'},k'\right)\Delta^*(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}',k')}{(\underset{\sim}{k}'\cdot\underset{\sim}{v}'-\underset{\sim}{k}'\cdot\underset{\sim}{v}-i\epsilon)}\right.$$

(IV-31)

$$\left.+\int d\eta' e_{\sigma'}\,\frac{F_1(\eta')/\Delta^*(-\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}',k')\Delta^*\left(\frac{-\omega+\underset{\sim}{k}'\cdot\underset{\sim}{v}'}{k'},k'\right)}{(-\underset{\sim}{k}'\cdot\underset{\sim}{v}'+\omega+\underset{\sim}{k}'\cdot\underset{\sim}{v}+i\epsilon)}\right\}.$$

In the second integral, we change the dummy variable of integration from $\eta'$ to $-\eta'$; then we carry out the barring operation in Eq. (IV-31), and find

$$\underset{\sim}{R}_4(\eta,\underset{\sim}{k}',-\underset{\sim}{k}')=\frac{2\underset{\sim}{k}'k'}{(\omega k')^2}\,\frac{D_0(\eta,\underset{\sim}{k}')}{2\pi i}\int du\,\frac{\left[\Delta^*(u,k')\right]^{-1}}{\Delta^*\left(\frac{-\omega-k'u}{k'},k'\right)}\left[\frac{\overline{F}_1(u)}{\underset{\sim}{k}'\cdot\underset{\sim}{v}-k'u+i\epsilon}-\frac{\overline{F}_1(-u)}{\underset{\sim}{k}'\cdot\underset{\sim}{v}+\omega+k'u+i\epsilon}\right].$$

(IV-31-a)

$$\underset{\sim}{R}_5(\eta,\underset{\sim}{k}',-\underset{\sim}{k}')=-\frac{2\underset{\sim}{k}'D_0(\eta,\underset{\sim}{k}')}{\pi(\omega k'^2)^2}\left\{\int du\,\frac{\overline{\rho}_0(u,k')\left[\overline{D}_1^{(+)}(u)-\overline{D}_1^{(+)}\left(\frac{\omega+k'u}{k'}\right)\right]}{(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}-u+i\epsilon)\Delta^*\left(\frac{-\omega-k'u}{k'},k'\right)}\right.$$

$$\left.+\int du\,\frac{\overline{\rho}_0\left(\frac{-\omega-k'u}{k'},k'\right)\left[\overline{D}_1^{(-)}(u)-\overline{D}_1^{(-)}\left(\frac{\omega+k'u}{k'}\right)\right]}{(\hat{\underset{\sim}{k}}'\cdot\underset{\sim}{v}-u+i\epsilon)\,\Delta^*(u,k')}\right\}.$$

(IV-32)

By changing the dummy variable of integration of the second integral from u to $-(\omega + k'u)/k'$; then Eq. (IV-32) reads

$$\underset{\sim}{R}_5(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = -\frac{2\underset{\sim}{k}' D_0(\eta, \underset{\sim}{k}')}{\pi(\omega k'^2)^2} \int du \frac{k' \bar{\rho}_0(u, k')}{\Delta^*(\frac{-\omega - ku}{k'}, k')}$$
$$\cdot \left\{ \frac{[\bar{D}_1^{(+)}(u) - \bar{D}_1^{(+)}(\frac{\omega + k'u}{k'})]}{(\underset{\sim}{k}'\cdot\underset{\sim}{v} - ku + i\epsilon)} + \frac{[\bar{D}_1^{(-)}(\frac{-\omega - k'u}{k'}) - \bar{D}_1^{(-)}(-u)]}{(\underset{\sim}{k}'\cdot\underset{\sim}{v} + \omega + k'u + i\epsilon)} \right\}. \quad \text{(IV-32-a)}$$

By defining

$$D_2(\eta, \underset{\sim}{k}') \equiv \frac{e_\sigma}{m_\sigma} D_1(\eta, \underset{\sim}{k}') \quad ,$$

$$F_2(\eta) \equiv \frac{e_\sigma}{m_\sigma} F_1(\eta) \quad , \quad \text{(IV-33)}$$

and utilizing the reduced forms of $\underset{\sim}{R}$'s just obtained; then we can perform the $\eta$-integration in Eq. (IV-25). The results are:

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}_1(\eta, \underset{\sim}{k}', -\underset{\sim}{k}')$$
$$= \frac{2\underset{\sim}{k}'}{(\omega k'^2)^2} \int du\, \bar{D}_2(u) \left[ \bar{\rho}_0^{(-)}(\frac{-\omega - k'u}{k'}, k') - \bar{\rho}_0^{(-)}(-u, k') \right] ; \quad \text{(IV-34)}$$

or since by definition

$$\int du\, \bar{D}_2(u) \bar{\rho}_0^{(-)}(\frac{-\omega - k'u}{k'}, k') = \frac{1}{2\pi i} \int du\, \bar{D}_2(u) \int du' \frac{k' \bar{\rho}_0(u', k')}{k'u + \omega + k'u + i\epsilon}$$
$$= \int du'\, \bar{\rho}_0(u', k') \bar{D}_2^{(-)}(\frac{-\omega - k'u}{k'}) \quad ,$$

and

$$\int du\, \overline{D}_2(u)\, \overline{\rho}_0^{(-)}(-u,k') = \frac{1}{2\pi i}\int du\, \overline{D}_2(u)\int du'\, \frac{\overline{\rho}_0(u',k')}{u'+u+i\epsilon}$$

$$= \int du'\, \overline{\rho}_0(u',k')\, \overline{D}_2^{(-)}(-u') \quad ;$$

hence Eq. (IV-34) becomes

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} R_1(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = \frac{2\underset{\sim}{k}'}{(\omega k')^2}\int du\, \overline{\rho}_0(u,k')\left[\overline{D}_2^{(-)}\left(\frac{-\omega-k'u}{k'}\right) - \overline{D}_2^{(-)}(-u)\right], \quad \text{(IV-34-a)}$$

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}_2(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = \frac{2\underset{\sim}{k}'}{(\omega k')^2}\int du\, \frac{\overline{F}_2(u)\left[\overline{D}_0^{(-)}\left(\frac{-\omega-k'u}{k'}\right) - \overline{D}_0^{(-)}(-u)\right]}{\Delta^*\left(\frac{-\omega-k'u}{k'}, k'\right)\Delta^*(-u,k')}, \quad \text{(IV-35)}$$

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}_3(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = -\frac{2\underset{\sim}{k}'}{(\omega k')^2}\int du\, \frac{\overline{F}_1(u)\left[\overline{D}_1^{(+)}\left(\frac{\omega+k'u}{k'}\right) - \overline{D}_1^{(+)}(u)\right]}{\Delta^*\left(\frac{-\omega-k'u}{k'}, k'\right)\Delta(u,k')}, \quad \text{(IV-36)}$$

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}_4(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = \frac{2\underset{\sim}{k}'}{(\omega k')^2}\int du\, \frac{\overline{F}_1(u)\overline{D}_1^{(-)}(u) - \overline{F}_1(-u)\overline{D}_1^{(-)}\left(\frac{-\omega-k'u}{k'}\right)}{\Delta^*\left(\frac{-\omega-k'u}{k'}, k'\right)\Delta^*(u,k')}, \quad \text{(IV-37)}$$

$$\int d\eta\, e_\sigma \frac{e_\sigma}{m_\sigma} \underset{\sim}{R}_5(\eta, \underset{\sim}{k}', -\underset{\sim}{k}') = -\frac{4i\underset{\sim}{k}'}{(\omega k'^2)^2}\int du\, \frac{\overline{\rho}_0(u,k')}{\Delta^*\left(\frac{-\omega-k'u}{k'}, k'\right)}$$

$$\cdot\left\{\overline{D}_1^{(-)}(u)\left[\overline{D}_1^{(+)}(u) - \overline{D}_1^{(+)}\left(\frac{\omega+k'u}{k'}\right)\right] + \overline{D}_1^{(-)}\left(\frac{-\omega-k'u}{k'}\right)\left[\overline{D}_1^{(-)}\left(\frac{-\omega-k'u}{k'}\right) - \overline{D}_1^{(-)}(-u)\right]\right\}. \quad \text{(IV-38)}$$

The appearance of Eqs. (IV-34-a)-(IV-38) shows that the $\underset{\sim}{k}'$ is the only factor which has the angular dependence in $\underset{\sim}{k}'$-space. Therefore, by substituting Eqs. (IV-34-a)-(IV-38) into Eqs. (IV-25), we may perform the angular integration in $\underset{\sim}{k}'$-space. Moreover, we write

$$\frac{\omega}{k'} \equiv w \quad , \tag{IV-39}$$

and suppress the argument of k' in $\Delta$, $\Delta^*$, and $\bar{\rho}_0$. Then we find that the first order conductivity reduces to a form shown below

$$\begin{aligned}\sigma^{(1)}(\omega) = \frac{4}{3\pi\omega^3}\int dk'\int du \Bigg\{ & \bar{f}_0(u)\left[\bar{D}_2^{(-)}(-w-u)-\bar{D}_2^{(-)}(-u)\right] + \frac{\bar{F}_2(u)\left[\bar{D}_0^{(-)}(-w-u)-\bar{D}_0^{(-)}(-u)\right]}{\Delta^*(-w-u)\,\Delta^*(-u)} \\ & - \frac{\bar{F}_1(u)\left[\bar{D}_1^{(+)}(w+u)-\bar{D}_1^{(+)}(u)\right]}{\Delta^*(-w-u)\,\Delta(u)} + \frac{\bar{F}_1(u)\,\bar{D}_1^{(-)}(u)-\bar{F}_1(-u)\,\bar{D}_1^{(-)}(-w-u)}{\Delta^*(-w-u)\,\Delta^*(u)} \\ & - \frac{2i}{k'^2}\,\bar{\rho}_0(u)\,\frac{\bar{D}_1^{(-)}(u)\left[\bar{D}_1^{(+)}(u)-\bar{D}_1^{(+)}(w+u)\right]+\bar{D}_1^{(-)}(-w-u)\left[\bar{D}_1^{(-)}(-w-u)-\bar{D}_1^{(-)}(-u)\right]}{\Delta^*(-w-u)} \Bigg\}.\end{aligned} \tag{IV-40}$$

In the next section, we shall show how our result can be reduced to the one derived by Klevans and Wu[19] by assigning isotropic assumption for $f_0(\eta)$ in Eq. (IV-40).

6. Spatial Homogeneous Case with Isotropic $f_0$

The isotropic assumption for $f_0(\eta)$ produces some useful identities:

$$\overline{F}_i(-u) = \overline{F}_i(u) \quad ,$$

$$\overline{D}_i^{(\pm)}(-u) = \overline{D}_i^{(\mp)}(u) \quad , \quad ; \quad i = 0, 1, 2 \tag{IV-41}$$

$$\Delta(-u) = \Delta^*(u) \quad .$$

Utilizing these identities, and recalling that

$$\overline{\rho}_0(u) \equiv \frac{\overline{F}_0(u)}{|\Delta(u)|^2} \quad ;$$

then Eq. (IV-40) reads

$$\begin{aligned}
\sigma^{(1)}(\omega) = \frac{4}{3\pi\omega^3}\int dk' \int du \Bigg\{ & \frac{\overline{F}_0(u)\left[\overline{D}_2^{(+)}(w+u) - \overline{D}_2^{(+)}(u)\right]}{\Delta(u)\,\Delta^*(u)} + \frac{\overline{F}_1(u)\left[\overline{D}_1^{(-)}(u) - \overline{D}_1^{(+)}(w+u)\right]}{\Delta(w+u)\,\Delta^*(u)} \\
& + \frac{\overline{F}_2(u)\left[\overline{D}_0^{(+)}(w+u) - \overline{D}_0^{(+)}(u)\right] - \overline{F}_1(u)\left[\overline{D}_1^{(+)}(w+u) - \overline{D}_1^{(+)}(u)\right]}{\Delta(w+u)\,\Delta(u)} \\
& - \frac{2i}{k'^2}\,\frac{\overline{F}_0(u)\left[\overline{D}_1^{(+)}(u) - \overline{D}_1^{(+)}(w+u)\right]\left[\overline{D}_1^{(-)}(u) - \overline{D}_1^{(+)}(w+u)\right]}{\Delta(w+u)\,\Delta(u)\,\Delta^*(u)} \Bigg\} .
\end{aligned} \tag{IV-42}$$

Since $F_0(u)$ and $|\Delta(u)|^2$ are even functions of u; while $\overline{D}_2(u)$ is an odd function of u; thus

$$\int_{-\infty}^{\infty} du\, \frac{\overline{F}_0(u)\,\overline{D}_2(u)}{|\Delta(u)|^2} = 0 \quad . \tag{IV-43}$$

Therefore, by using the Plemelj formula

$$\overline{D}_2(u) = \overline{D}_2^{(+)}(u) - \overline{D}_2^{(-)}(u)$$

in the first term of Eq. (IV-42), and then utilizing Eq. (IV-43), we may cast Eq. (IV-42) in the following form

$$\sigma^{(1)}(\omega) = -\frac{4}{3\pi\omega^3}\int dk'\int du\left\{\frac{\bar{F}_0(u)\phi_2(w,u)}{\Delta(u)\Delta^*(u)} - \frac{\bar{F}_1(u)\phi_1(w,u)}{\Delta(w+u)\Delta^*(u)}\right.$$
$$\left. + \frac{\bar{F}_2(u)\psi_0(w,u) - \bar{F}_1(u)\psi_1(w,u)}{\Delta(w+u)\Delta(u)} + \frac{2i}{k'^2}\,\frac{F_0(u)\psi_1(w,u)\phi_1(w,u)}{\Delta(w+u)\Delta(u)\Delta^*(u)}\right\}, \qquad \text{(IV-44)}$$

where

$$\phi_j(w,u) = \bar{D}_j^{(-)}(u) - \bar{D}_j^{(+)}(w+u) \quad ,$$
$$\psi_j(w,u) = \bar{D}_j^{(+)}(u) - \bar{D}_j^{(+)}(w+u) \quad . \qquad j = 0,1,2 \quad \text{(IV-45)}$$

In Eq. (IV-44), we add a null term,

$$\frac{\bar{F}_0(u)\ \phi_2(w,u) - \bar{F}_0\ \phi_2(w,u)}{\Delta(w+u)\ \Delta^*(u)} \quad ,$$

in the curved bracket. Then we use the relation

$$\Delta(w+u) - \Delta(u) = -\frac{2i}{k'^2}\psi_0(w+u)$$

and find

$$\sigma^{(1)}(\omega) = -\frac{4}{3\pi\omega^3}\int dk'\int du\left\{\frac{\bar{F}_0(u)\phi_2(w,u) - \bar{F}_1(u)\phi_1(w,u)}{\Delta(w+u)\ \Delta^*(u)} + \frac{\bar{F}_2(u)\psi_0(w,u) - \bar{F}_1(u)\psi_1(w,u)}{\Delta(w+u)\Delta(u)}\right.$$
$$\left. + \frac{2i}{k'^2}\,\frac{\bar{F}_0(u)[\phi_1(w,u)\psi_1(w,u) - \phi_2(w,u)\psi_0(w,u)]}{\Delta(w+u)\,|\Delta(u)|^2}\right\}. \qquad \text{(IV-46)}$$

Further simplification can be accomplished if there are only two species, electrons and singly charged ions. Under these two conditions then

$$F_0(\eta) = e_\sigma f_0(\eta) = e_\sigma n_0 f_\sigma(\underset{\sim}{v}) \quad .$$

Through the definition of barring operating

$$\begin{aligned} \overline{F}_0(u) &= \sum_\sigma e_\sigma^2 n_0 \int d\underset{\sim}{v}\, \delta(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}) f_\sigma(\underset{\sim}{v}) \\ &\equiv \sum_\sigma e_\sigma^2 n_0 \widetilde{f}_\sigma(u) \quad . \end{aligned} \tag{IV-47}$$

$\sigma = e$, i for electrons and ions respectively. Similarly, through the appropriate definitions, we obtain

$$\overline{F}_1(u) = \sum_\sigma \frac{e_\sigma^3}{m_\sigma} n_0 \widetilde{f}_\sigma(u) = \sum_\sigma \frac{\omega_\sigma^2 e_\sigma}{4\pi} \widetilde{f}_\sigma(u) \quad , \tag{IV-48}$$

where $\omega_\sigma$ is the plasma frequency of the $\sigma$-type species by conventional definition;

$$\overline{F}_2(u) = \sum_\sigma \frac{\omega_\sigma^2 e_\sigma^2}{4\pi m_\sigma} \widetilde{f}_\sigma(u) \quad ; \tag{IV-49}$$

$$\overline{D}_0(u) = \sum_\sigma e_\sigma n_0 \widetilde{D}_\sigma(u) \quad ; \tag{IV-50}$$

$$\overline{D}_1(u) = \sum_\sigma \frac{1}{4\pi} \omega_\sigma^2 \widetilde{D}_\sigma(u) \quad ; \tag{IV-51}$$

$$\bar{D}_2(u) = \frac{1}{4\pi}\sum_{\sigma}\frac{e_\sigma \omega_\sigma^2}{m_\sigma}\tilde{D}_\sigma(u) \quad . \tag{IV-52}$$

Then from Eq. (IV-45), we have

$$\psi_0(w,u) = \sum_\sigma e_\sigma n_0\left[\tilde{D}_\sigma^{(+)}(u) - \tilde{D}_\sigma^{(+)}(w+u)\right] \equiv \sum_\sigma e_\sigma n_0\, \psi_\sigma(w,u) \ ,$$

$$\psi_1(w,u) = \frac{1}{4\pi}\sum_\sigma \omega_\sigma^2\left[\tilde{D}_\sigma^{(+)}(u) - \tilde{D}_\sigma^{(+)}(w+u)\right] \equiv \sum_\sigma \frac{1}{4\pi}\omega_\sigma^2\, \psi_\sigma(w,u) \ ,$$

$$\phi_1(w,u) = \frac{1}{4\pi}\sum_\sigma \omega_\sigma^2\left[\tilde{D}_\sigma^{(-)}(u) - \tilde{D}_\sigma^{(+)}(w+u)\right] \equiv \sum_\sigma \frac{1}{4\pi}\omega_\sigma^2\, \phi_\sigma(w,u) \ , \tag{IV-53}$$

$$\phi_2(w,u) = \frac{1}{4\pi}\sum_\sigma \frac{e_\sigma}{m_\sigma}\omega_\sigma^2\left[\tilde{D}_\sigma^{(-)}(u) - \tilde{D}_\sigma^{(+)}(w+u)\right] \equiv \sum_\sigma \frac{1}{4\pi}\frac{e_\sigma}{m_\sigma}\omega_\sigma^2\, \phi_\sigma(w,u) .$$

By using Eqs. (IV-47), (IV-48), (IV-49), and (IV-53), we obtain, after some straightforward calculations, and suppressing all the arguments [detailed steps shown in Appendix B]

$$\bar{F}_0\phi_2 - \bar{F}_1\phi_1 = \frac{e\tilde{\omega}^2}{(4\pi)^2}\left[\omega_i^2 \tilde{f}_e \phi_i - \omega_e^2 \tilde{f}_i \phi_e\right] \ ,$$

$$\bar{F}_2\psi_0 - \bar{F}_1\psi_1 = \frac{e\tilde{\omega}^2}{(4\pi)^2}\left[\omega_e^2 \tilde{f}_e \psi_i - \omega_i^2 \tilde{f}_i \psi_e\right] \ , \tag{IV-54}$$

$$\phi_1\psi_1 - \phi_2\psi_0 = \frac{\tilde{\omega}^2}{(4\pi)^2}\left[\omega_i^2 \psi_e \phi_i + \omega_e^2 \psi_i \phi_e\right] \ ,$$

where

$$\tilde{\omega}^2 = \omega_i^2 + \omega_e^2$$

Substitution of Eq. (IV-54) into Eq. (IV-46), yields

$$\sigma^{(1)}(\omega) = -\frac{2e\tilde{\omega}^2}{3(2\pi\omega)^3}\int dk'\int du\left\{\frac{[\omega_i^2\tilde{f}_e\phi_i - \omega_e^2\tilde{f}_i\phi_e]}{\Delta(w+u)\Delta^*(u)} + \frac{[\omega_e^2\tilde{f}_e\psi_i - \omega_i^2\tilde{f}_i\psi_e]}{\Delta(w+u)\,\Delta(u)}\right.$$

$$\left. + \frac{2ien_0}{k'^2}\frac{(\tilde{f}_e+\tilde{f}_i)[\omega_i^2\psi_e\phi_i + \omega_e^2\psi_i\phi_e]}{\Delta(w+u)|\Delta(u)|^2}\right\} . \qquad \text{(IV-55)}$$

Utilizing the relations

$$\omega_i^2\tilde{f}_e\phi_i - \omega_e^2\tilde{f}_i\phi_e = \tilde{\omega}^2\tilde{f}_e\phi_i - \omega_e^2(\tilde{f}_e\phi_i + \tilde{f}_i\phi_e)$$

$$\omega_e^2\tilde{f}_e\psi_i - \omega_i^2\tilde{f}_i\psi_e = -\tilde{\omega}^2\tilde{f}_i\psi_e + \omega_e^2(\tilde{f}_i\psi_e + \tilde{f}_e\psi_i)$$

$$\omega_i^2\psi_e\phi_i + \omega_e^2\psi_i\phi_e = \tilde{\omega}^2\psi_e\phi_i - \omega_e^2(\psi_e\phi_i - \psi_i\phi_e) ,$$

then $\sigma^{(1)}$ becomes

$$\sigma^{(1)}(\omega) = -\frac{2e\tilde{\omega}^4}{3(2\pi\omega)^3}\int dk'\int du\left[\frac{\tilde{f}_e\phi_i}{\Delta(w+u)\Delta^*(u)} - \frac{\tilde{f}_i\psi_e}{\Delta(w+u)\,\Delta(u)}\right.$$

$$\left. + \frac{2ien_0}{k'^2}\,\frac{(\tilde{f}_e+\tilde{f}_i)\,\psi_e\,\phi_i}{\Delta(w+u)|\Delta(u)|^2}\right]$$

$$+ \frac{2e\tilde{\omega}^2\omega_e^2}{3(2\pi\omega)^3}\int dk'\left\{\int du\left[\frac{(\tilde{f}_e\phi_i + \tilde{f}_i\phi_e)}{\Delta(w+u)\Delta^*(u)} - \frac{(\tilde{f}_i\psi_e + \tilde{f}_e\psi_i)}{\Delta(w+u)\,\Delta(u)}\right.\right. \qquad \text{(IV-56)}$$

$$\left.\left. + \frac{2ien_0(\tilde{f}_i+\tilde{f}_e)(\psi_e\phi_i - \psi_i\phi_e)}{k'^2\,\Delta(w+u)\,|\Delta(u)|^2}\right]\right\} .$$

As shown in Appendix that by utilizing the relations

$$\Delta(u)-\Delta(w+u)=\frac{2i}{k'^2}en_0[\psi_i-\psi_e] ,$$

$$\Delta^*(u)-\Delta(w+u)=\frac{2i}{k'^2}en_0[\phi_i-\phi_e] ,$$

$$\int du\,\frac{\tilde{f}_i(\phi_e-\psi_e)}{|\Delta(u)|^2}=\int du\,\frac{\tilde{f}_e(\phi_i-\psi_i)}{|\Delta(u)|^2}=0 ,$$

the integral in the curved brackets vanishes. Therefore,

$$\sigma^{(1)}(\omega)=-\frac{2e\tilde{\omega}^4}{3(2\pi\omega)^3}\int dk'\int du\left\{\frac{2ien_0[\tilde{f}_e(u)+\tilde{f}_i(u)]\psi_i(w,u)\phi_i(w,u)}{k'^2\Delta(w+u)|\Delta(u)|^2}\right.$$

$$\left.+\frac{\tilde{f}_e(u)\phi_i(w,u)}{\Delta(w+u)\Delta^*(u)}-\frac{\tilde{f}_i(u)\,\psi_e(w,u)}{\Delta(w+u)\Delta(u)}\right\} . \qquad \text{(IV-57)}$$

Eq. (IV-57) is equivalent to Eq. (IV-14) of Ref. (18) which was obtained by using an operator method. The different appearance is due to the different definitions:

(1) Klevans and Wu's[18] definitions

$$D_\sigma(u)\equiv-4\pi\frac{e_\sigma}{m_\sigma}\frac{1}{k'^2}\frac{\partial}{\partial u}\tilde{f}_\sigma(u) ,$$

$$D^{(\pm)}_\sigma(u)\equiv\int du'\,\frac{D_\sigma(u')}{u'-u\mp i\epsilon} ,$$

and

$$\underset{\sim}{E}(t) = \underset{\sim}{E}_0 \, e^{i\omega t} \quad .$$

(2) Our definitions

$$\tilde{D}_\sigma(u) \equiv -4\pi^2 \frac{e_\sigma}{m_\sigma} \frac{\partial}{\partial u} \tilde{f}_\sigma(u) \quad ,$$

$$\tilde{D}_\sigma^{(\pm)}(u) \equiv \frac{1}{2\pi i} \int du' \frac{\tilde{D}_\sigma(u')}{u' - u \mp i\epsilon} \quad ,$$

and

$$\underset{\sim}{E}(\underset{\sim}{k}, \omega) = \int_0^\infty dt \; e^{i\omega t} \underset{\sim}{E}(\underset{\sim}{k}, t) \; .$$

In Ref. (18), Klevans and Wu have shown that Eq. (IV-57) reduces to ORD's[16] result if the thermodynamic equilibrium assumption is used.

## APPENDIX A

I. Derivation of Eq. (I-52)

(1) From Eqs. (I-47), we have

$$\frac{\Gamma_1^{(-)}(u)}{D^{(-)}(u,k)} - \frac{\Gamma_2^{(-)}(u)}{D^{(-)*}[-(\omega+ku)/k',k']} \tag{A-1}$$

$$= [\Phi_1^{(-)}(u) - \Phi_2^{(-)}(u)]/2\mathfrak{D}^{(-)}(u,k)\mathfrak{D}^{(-)*}[-(\omega+k\,u)/k',k']$$

(2) From Eq. (I-51), we have

$$[\Gamma_2^{(+)}(u) - \Gamma_2^{(-)}(u)] + i\Delta_2[-(\omega-ku)/k',k']\ \Gamma_1^{(-)}(u)/\mathfrak{D}^{(-)}(u,k) \tag{A-2}$$

$$= \frac{k}{2k'}\,q\,[-(\omega-ku)/k',\underset{\sim}{k}',\underset{\sim}{k}]/\mathfrak{D}^{(-)}(u,k)$$

(3) Using Eq. (A-1), we may eliminate the factor $[\Gamma_1^{(-)}(u)/\mathfrak{D}^{(-)}(u,k)]$ from Eq. (A-2), and find

$$[\Gamma_2^{(+)}(u) - \Gamma_2^{(-)}(u)]\left\{1 - i\Delta_2[-(\omega+ku)/k',k']/\mathfrak{D}^{(-)*}[-(\omega+ku)/k',k']\right\} \tag{A-3}$$

$$= \frac{1}{2\mathfrak{D}^{(-)}(u,k)}\left\{\frac{k}{k'}\bar{q}\left(\frac{-\omega-ku}{k},\underset{\sim}{k}',\underset{\sim}{k}\right) - \frac{i\Delta_2\left(\frac{-\omega-ku}{k'},k'\right)\left[\Phi_1^{(-)}(u)-\Phi_2^{(-)}(u)\right]}{\mathfrak{D}^{(-)*}[-(\omega+ku)/k',k']}\right\}.$$

(4) By using Eq. (I-43-a) in the l.h.s. of Eq. (A-3), and Eqs. (I-43)-(II-44-a) in the r.h.s. of Eq. (A-3), we immediately obtain Eq. (I-52).

II. Derivations of Eq. (I-61) and (I-62)

(1) From Eq. (I-21), we have

$$q(\eta, \underset{\sim}{k}, \underset{\sim}{k}') \equiv i \int d\eta' e_\sigma' \frac{p(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + \underset{\sim}{k}' \cdot \underset{\sim}{v}' + i\epsilon)} \tag{I-21}$$

$$= \frac{-2\pi}{k'} \frac{1}{2\pi i} \int du\, \bar{p}(\eta, u, \underset{\sim}{k}, \underset{\sim}{k}') / [u + (\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k' + i\epsilon]$$

$$= \frac{-2\pi}{k'} \bar{p}^{(-)}[\eta, -(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v})/k', \underset{\sim}{k}, \underset{\sim}{k}'] \tag{I-61}$$

(2) From Eq. (I-35), we have

$$\Phi_1^{(-)}(u) \equiv \frac{1}{2\pi i} \int du' \left( \frac{1}{u' - u + i\epsilon} \right) \int d\eta e_\sigma \delta(u' - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}) q(\eta, \underset{\sim}{k}, \underset{\sim}{k}') . \tag{A-4}$$

Using Eq. (I-21) for q then

$$\Phi_1^{(-)}(u) = \frac{1}{2\pi i} \int du' \left( \frac{1}{u' - u + i\epsilon} \right) \int d\eta e_\sigma \delta(u' - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}) \left[ i \int d\eta' e_\sigma' \frac{p(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + \underset{\sim}{k}' \cdot \underset{\sim}{v}' + i\epsilon)} \right]$$

$$= \frac{1}{2\pi} \int du' \frac{1}{(u' - u + i\epsilon)} \int d\eta e_\sigma \delta(u' - \hat{\underset{\sim}{k}} \cdot \underset{\sim}{v}) \int d\eta' e_\sigma' \left[ \frac{p(\eta, \eta', \underset{\sim}{k}, \underset{\sim}{k}')}{\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + \underset{\sim}{k}' \cdot \underset{\sim}{v}' + i\epsilon} \right] \left[ \int_{-\infty}^{\infty} du'' \delta(u'' - \hat{\underset{\sim}{k}}' \cdot \underset{\sim}{v}') \right]$$

$$= \frac{1}{2\pi} \iint du' du'' \frac{\bar{\bar{p}}^{(-,-)}(u', u'', \underset{\sim}{k}, \underset{\sim}{k}')}{(u' - u + i\epsilon)(\omega + k u' + k' u'' + i\epsilon)} \quad . \tag{A-5}$$

(3) Similarly from Eq. (I-36), we have

$$\Phi_2^{(+)}(u) = \frac{1}{2\pi i}\int du'\left[\frac{k/k'}{u'-u-i\epsilon}\right]\int d\eta e_\sigma \delta(\hat{k}'\cdot v + \frac{\omega+ku'}{k'})\, q(\eta, k', k)$$

$$= \frac{1}{2\pi i}\int du'\left[\frac{k/k'}{u'-u-i\epsilon}\right]\int d\eta e_\sigma \delta(\hat{k}'\cdot v + \frac{\omega+ku'}{k'})\left[i\int d\eta' e_{\sigma'} \frac{p(\eta,\eta',k',k)}{(\omega+k'\cdot v+k\cdot v'+i\epsilon)}\right]$$

$$= \frac{1}{2\pi}\int du'\left[\frac{k/k'}{u'-u-i\epsilon}\right]\int d\eta e_\sigma \delta(\hat{k}'\cdot v + \frac{\omega+ku'}{k'})\left\{\int d\eta' e_{\sigma'}\left[\frac{p(\eta,\eta',k',k)}{\omega+k'\cdot v+k\cdot v'+i\epsilon}\right]\right.$$

$$\left.\cdot\left[\int_{-\infty}^{\infty} du''\,\delta(u''-\hat{k}\cdot v')\right]\right\}. \qquad \text{(A-6)}$$

In Eq. (A-6), we interchange $\eta$, and $\eta'$, and use the symmetrical relation,

$$p(\eta', \eta, k', k) = p(\eta, \eta', k, k') \quad ;$$

then we find

$$\Phi_2^{(+)}(u) = \frac{k}{2\pi k'}\int du'\left[\frac{1}{u'-u-i\epsilon}\right]\int d\eta' e_{\sigma'} \delta(\hat{k}'\cdot v' + \frac{\omega+ku'}{k'}) \qquad \text{(A-7)}$$
$$\cdot\left\{\int d\eta e_\sigma \frac{p(\eta,\eta',k,k')}{(\omega+k'\cdot v'+k\cdot v+i\epsilon)}\int du''\,\delta(u''-\hat{k}\cdot v)\right\}.$$

In Eq. (A-7), we replace u' by $-(\omega + k'u'')/k$, and u'' by u', then we obtain (after doing the barring operations)

$$\Phi_2^{(+)}(u) = -\frac{1}{2\pi}\iint du'du''\,\frac{k\,\overline{\overline{p}}(u', u'', k, k')}{(ku+\omega+k'u''+i\epsilon)(\omega+ku'+k'u''+i\epsilon)} \quad . \qquad \text{(A-8)}$$

(4) Subtracting Eq. (A-5) from Eq. (A-8), we obtain

$$\Phi_2^{(+)}(u) - \Phi_1^{(-)}(u)$$

$$= -\frac{1}{2\pi}\iint du' du'' \frac{\overline{\overline{p}}(u', u'', \underset{\sim}{k}, \underset{\sim}{k}')}{(ku+\omega+k'u''+i\epsilon)(u'-u+i\epsilon)} \tag{I-62}$$

$$= \frac{2\pi}{k'} \overline{\overline{p}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right).$$

III. Derivation of Eq. (III-37)

(1) Utilizing Eqs. (III-36) and (III-36-a) for the first and second terms in the u-integration of Eq. (III-16), we then end up immediately with

$$\int_C du \frac{\overline{\overline{S}}^{(-,-)}\left(u, \frac{-\omega-ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) + \overline{\overline{S}}^{(-,-)}\left(\frac{-\omega-ku}{k'}, u, \underset{\sim}{k}', \underset{\sim}{k}\right)}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)\Delta^*(u,k)}$$

$$= \int_C du \frac{\mathcal{H}^{(+,+)}\left(u, \frac{\omega+ku}{k'}\right)}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)} + \int_C du \frac{\mathcal{H}^{(+,+)}\left(\frac{-\omega-ku}{k'}, -u\right)}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*(u,k)}$$

$$-\int_C du \frac{\left[\Delta^*(u,k)\overline{f}_0(u,k)\mathcal{L}\left(u, \frac{\omega+ku}{k'}\right) - A\left(u, \frac{\omega+ku}{k'}\right)\right] + \left[\begin{matrix} \underset{\sim}{k} \longleftrightarrow \underset{\sim}{k}' ; \\ u \rightarrow -(\omega+ku)/k' \end{matrix}\right]}{(u-\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\Delta^*\left(\frac{-\omega-ku}{k'}, k'\right)\Delta^*(u,k)} .$$

The function $\overset{*}{\Delta}\left(\frac{-\omega - ku}{k'}, k'\right)$ is analytic in the upper-half u-plane, and likewise $\overset{*}{\Delta}(u, k)$ is analytic in the lower-half u-plane. The functions $\mathcal{H}^{(+,+)}\left(u, \frac{\omega + ku}{k'}\right)$ and $\mathcal{H}^{(+,+)}\left(\frac{-\omega - ku}{k'}, -u\right)$ enjoy similar properties respectively and in addition vanish at infinite in their half-planes of definition. Thus we can employ the Cauchy's theorems to the first two integrals by closing the contour of the first (second) integral by a large semicircle in the upper (lower) half plane. We then obtain

$$\int_C du \frac{\overline{\overline{S}}^{(-,-)}\left(u, \frac{-\omega - ku}{k'}, \underset{\sim}{k}, \underset{\sim}{k}'\right) + \overline{\overline{S}}^{(-,-)}\left(\frac{-\omega - ku}{k'}, u, \underset{\sim}{k}, \underset{\sim}{k}'\right)}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\overset{*}{\Delta}\left(\frac{-\omega - ku}{k'}, k'\right)\overset{*}{\Delta}(u, k)}$$

$$= 2\pi i \frac{\mathcal{H}^{(+,+)}\left(\hat{\underset{\sim}{k}}\cdot\underset{\sim}{v}, \frac{\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}\right)}{\overset{*}{\Delta}\left(\frac{-\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}{k'}, k'\right)} \tag{A-9}$$

$$- \int_C du \frac{\left[\overset{*}{\Delta}(u,k)\,\overline{\rho}_0(u,k)\,\mathcal{L}\left(u, \frac{\omega + ku}{k'}\right) - A\left(u, \frac{\omega + ku}{k'}\right)\right] + \left[\begin{matrix}\underset{\sim}{k} \longleftrightarrow \underset{\sim}{k}' ; \\ u \longrightarrow -(\omega + ku)/k'\end{matrix}\right]}{(u - \hat{\underset{\sim}{k}}\cdot\underset{\sim}{v})\,\overset{*}{\Delta}\left(\frac{-\omega - ku}{k'}, k'\right)\overset{*}{\Delta}(u, k)}$$

(2) Substituting Eq. (A-9) into Eq. (III-16), we immediately obtain Eq. (III-37).

APPENDIX B

In the following derivations, we presume that the procedure of changing k to K and k' to - k' has been done.

I. Derivation of Eq. (IV-6)

(1) By using $f^{(0)}$ in Eq. (III-5), we find

$$B^{(0)}\left(u, \frac{\omega+Ku}{k'}\right)$$

$$= \frac{K k'}{(2\pi i)^2}\int d\eta' e_{\sigma'} \frac{\mathbf{E}^{(0)}\cdot \frac{\partial}{\partial \mathbf{v}'} F_1(\eta')}{(\mathbf{K}\cdot \mathbf{v}' - Ku + i\epsilon)(-\mathbf{k}'\cdot \mathbf{v}' + \omega + Ku + i\epsilon)} \quad . \tag{B-1}$$

(2) With the aid of the Plemelj formula

$$\frac{1}{\mathbf{K}\cdot\mathbf{v}' - Ku - i\epsilon} = \frac{1}{\mathbf{K}\cdot\mathbf{v}' - Ku + i\epsilon} + \frac{2\pi i}{K}\,\delta(u - \hat{\mathbf{K}}\cdot\mathbf{v}') \quad ,$$

we may write Eq. (III-35) as

$$A^{(0)}\left(u, \frac{\omega+Ku}{k'}\right) = -\frac{K k'}{(2\pi i)^2}\int d\eta' e_{\sigma'} \frac{F_1(\eta')}{(\mathbf{K}\cdot\mathbf{v}' - Ku + i\epsilon)}\,\mathbf{E}^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}'}\left[\frac{1}{-\mathbf{k}'\cdot\mathbf{v}' + \omega + Ku + i\epsilon}\right]$$

$$- \frac{1}{2\pi i}\,\mathbf{E}^{(0)}\cdot\int d\eta' e_{\sigma'} \frac{k'\,\mathbf{k}\,\delta(u - \hat{\mathbf{K}}\cdot\mathbf{v}')\,F_1(\eta')}{(-\mathbf{k}'\cdot\mathbf{v}' + \omega + Ku + i\epsilon)^2} \quad . \tag{B-2}$$

(3) In Eq. (B-2), by exchanging $\mathbf{K}$ and - $\mathbf{k}'$, and then replacing u by $-(\omega + Ku)/k'$ (in this order), we find $A^{(0)}[-(\omega + Ku)/k', -u]$. By substituting these $B^{(0)}$, and $A^{(0)}$'s into Eq. (III-38), we find

$$C^{(0)}\left(u, \frac{\omega+Ku}{-k'}\right)$$

$$= \frac{k'}{2\pi i} \underset{\sim}{E}^{(0)} \cdot \int d\eta' e_{\sigma'} F_1(\eta') \left\{ \frac{\underset{\sim}{k}' \delta(u - \hat{\underset{\sim}{K}} \cdot \underset{\sim}{v}')}{(-\underset{\sim}{k}' \cdot \underset{\sim}{v}' + \omega + Ku + i\epsilon)^2} - \frac{\underset{\sim}{K} \delta\left(u + \frac{\omega - \underset{\sim}{k}' \cdot \underset{\sim}{v}'}{K}\right)}{(\underset{\sim}{K} \cdot \underset{\sim}{v}' - Ku + i\epsilon)^2} \right\} . \tag{B-3}$$

(4) Since

$$\int_c du \frac{1}{(u - \hat{\underset{\sim}{K}} \cdot \underset{\sim}{v})} = \int_{-\infty}^{\infty} du \frac{1}{(u - \hat{\underset{\sim}{K}} \cdot \underset{\sim}{v} - i\epsilon)} ;$$

therefore, by using the property of $\delta$-function to perform the u-integration, we obtain Eq. (IV-6) at once.

II. Derivation of Eq. (IV-7)

(1) By using $\mathcal{F}^{(0)}$ and the definition of $D_1$ in Eq. (III-1), we find

$$U^{(0)}(\eta', -\underset{\sim}{k}', \underset{\sim}{K}) = -K \underset{\sim}{E}^{(0)} \cdot \left[ \frac{\frac{\partial}{\partial \underset{\sim}{v}'} D_1(\eta', \underset{\sim}{K})}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v}' + i\epsilon)} + (\underset{\sim}{k} \cdot \hat{\underset{\sim}{K}}) \underset{\sim}{b}_1(\eta'; \underset{\sim}{k}) \right] , \tag{B-4}$$

where

$$\underset{\sim}{b}_1(\eta', \underset{\sim}{k}) \equiv \frac{4\pi^2 \left(\frac{e_{\sigma'}}{m_{\sigma'}}\right)^2 \frac{\partial}{\partial \underset{\sim}{v}'} f_0(\eta')}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v}' + i\epsilon)^2} . \tag{B-5}$$

Defining

$$d_1(\eta', \underset{\sim}{K}; \underset{\sim}{k}) \equiv D_1(\eta', \underset{\sim}{K}) / (\omega + \underset{\sim}{k} \cdot \underset{\sim}{v}' + i\epsilon)^2 \tag{B-6}$$

then

$$U^{(0)}(\eta', -\underset{\sim}{k}', \underset{\sim}{K})$$

$$= -K\underset{\sim}{E}^{(0)} \cdot \left\{ \frac{\partial}{\partial \underset{\sim}{v}'} \left[ \frac{D_1(\eta', \underset{\sim}{K})}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}' + i\epsilon)} \right] + \underset{\sim}{k}\, d_1(\eta', \underset{\sim}{K}; \underset{\sim}{k}) + (\underset{\sim}{k}\cdot\hat{\underset{\sim}{K}})\underset{\sim}{b}_1(\eta', \underset{\sim}{k}) \right\} . \tag{B-7}$$

(2) By using Eq. (B-7) in Eq. (III-29), we find

$$L^{(0)}(u, \eta', \underset{\sim}{K}, -\underset{\sim}{k}') = -\frac{1}{\pi K}\underset{\sim}{E}^{(0)} \cdot \left\{ \underset{\sim}{k}\, d_1(\eta', \underset{\sim}{K}; \underset{\sim}{k}) + (\underset{\sim}{k}\cdot\hat{\underset{\sim}{K}})\, \underset{\sim}{b}_1(\eta', \underset{\sim}{k}) + \frac{\partial}{\partial \underset{\sim}{v}'} \left[ \frac{(\omega + Ku - \underset{\sim}{k}'\cdot\underset{\sim}{v}')\, D_1(\eta', \underset{\sim}{K})}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}' + i\epsilon)(Ku - \underset{\sim}{K}\cdot\underset{\sim}{v}' + i\epsilon)} \right] \right\} . \tag{B-8}$$

In the first term of r.h.s., we write $\underset{\sim}{k} = (\underset{\sim}{k} + \underset{\sim}{k}') - \underset{\sim}{k}'$. And in the last term, we differentiate by parts; then we find

$$L^{(0)}(u, \eta', \underset{\sim}{K}, -\underset{\sim}{k}') = -\frac{1}{\pi K}\underset{\sim}{E}^{(0)} \cdot \left\{ \underset{\sim}{K}\, d_1(\eta', \underset{\sim}{K}; \underset{\sim}{k}) + (\underset{\sim}{k}\cdot\hat{\underset{\sim}{K}})\, \underset{\sim}{b}_1(\eta', \underset{\sim}{k}) - (\omega + Ku - \underset{\sim}{k}'\cdot\underset{\sim}{v}') \left[ \frac{\underset{\sim}{k}'\, d_1(\eta', \underset{\sim}{K}; \underset{\sim}{k})}{(Ku - \underset{\sim}{K}\cdot\underset{\sim}{v}' + i\epsilon)} - \frac{\partial}{\partial \underset{\sim}{v}'} \left( \frac{D_1(\eta', \underset{\sim}{K})}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}' + i\epsilon)(Ku - \underset{\sim}{K}\cdot\underset{\sim}{v}' + i\epsilon)} \right) \right] \right\} . \tag{B-9}$$

(3) By substituting Eq. (B-9) into Eq. (III-34), we obtain Eq. (IV-7).

III. Derivation of Eq. (IV-10)

(1) By definition

$$\underset{\sim}{Q}_1[X] \equiv \left[ \frac{1}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)} \frac{\partial}{\partial \underset{\sim}{v}} + \frac{\underset{\sim}{k}'}{(\omega + \underset{\sim}{k}\cdot\underset{\sim}{v} + i\epsilon)^2} \right] X ;$$

hence

$$-i\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1\left[Y_1(\eta,-\underset{\sim}{k}')\right]$$

$$=-i\underset{\sim}{E}^{(0)}\cdot\left[\frac{1}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\frac{\partial}{\partial\underset{\sim}{v}}Y_1(\eta,-\underset{\sim}{k}')+\frac{\underset{\sim}{k}'\,Y_1(\eta,-\underset{\sim}{k}')}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)^2}\right].$$

Using the explicit form of $Y_1$ in the second term then

$$-i\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1\left[Y_1(\eta,-\underset{\sim}{k}')\right]=\underset{\sim}{E}^{(0)}\cdot\left[\frac{-i}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\frac{\partial}{\partial\underset{\sim}{v}}Y_1(\eta,-\underset{\sim}{k}')\right.$$

$$\left.-i\underset{\sim}{k}'\frac{h_1(\eta,\underset{\sim}{k})}{\Delta^*(-\hat{k}'\cdot v',k')}-\frac{2\underset{\sim}{k}'}{k'^2}d_1(\eta,-\underset{\sim}{k}';\underset{\sim}{k})\overline{\rho}_0^{(-)}(-\hat{k}'\cdot v',k')\right]. \tag{B-10}$$

(2) In Eq. (B-7), interchanging $\underset{\sim}{K}$ and $-\underset{\sim}{k}'$ and replacing $\eta'$ by $\eta$, we find

$$\frac{1}{k'}U^{(0)}(\eta,\underset{\sim}{K},-\underset{\sim}{k}')$$

$$=-\underset{\sim}{E}^{(0)}\cdot\left[\frac{1}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\frac{\partial}{\partial\underset{\sim}{v}}D_1(\eta,-\underset{\sim}{k}')-(\underset{\sim}{k}\cdot\hat{\underset{\sim}{k}}')\underset{\sim}{b}_1(\eta,\underset{\sim}{k})\right]. \tag{B-11}$$

(3) By definition

$$\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1\left[D_1(\eta,-\underset{\sim}{k}')\right]=\underset{\sim}{E}^{(0)}\cdot\left[\frac{1}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)}\frac{\partial}{\partial\underset{\sim}{v}}+\frac{\underset{\sim}{k}'}{(\omega+\underset{\sim}{k}\cdot\underset{\sim}{v}+i\epsilon)^2}\right]D_1(\eta,-\underset{\sim}{k}'). \tag{B-12}$$

Thus

$$-\frac{2}{k'^2}\overline{\rho}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{v}}{k'},k'\right)\left\{\frac{1}{k'}U^{(0)}(\eta,\underset{\sim}{K},-\underset{\sim}{k}')+\underset{\sim}{E}^{(0)}\cdot\underset{\sim}{Q}_1\left[D_1(\eta,-\underset{\sim}{k}')\right]\right\}$$

$$=\frac{-2}{k'^2}\underset{\sim}{E}^{(0)}\cdot\left\{\overline{\rho}_0^{(-)}\left(\frac{-\omega-\underset{\sim}{K}\cdot\underset{\sim}{v}}{k'},k'\right)\left[(\underset{\sim}{k}\cdot\hat{\underset{\sim}{k}}')\underset{\sim}{b}_1(\eta,\underset{\sim}{k})+\underset{\sim}{k}'d_1(\eta,-\underset{\sim}{k}';\underset{\sim}{k})\right]\right\}. \tag{B-13}$$

(4) Using $\mathcal{F}^{(0)}$ in Eq. (I-65), we find

$$-\mathcal{E}^{(0)}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}') = \frac{i\, \underset{\sim}{E}^{(0)} \cdot \frac{\partial}{\partial \underset{\sim}{v}} F_1(\eta)}{(\omega + \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon)} \quad . \tag{B-14}$$

Thus

$$\frac{1}{\Delta^*\left(\frac{-\omega - \underset{\sim}{K}\cdot\underset{\sim}{v}}{k'}, k'\right)} \left\{ i\, \underset{\sim}{E}^{(0)} \cdot \underset{\sim}{Q}_1 [F_1(\eta)] + \mathcal{E}^{(0)}(\eta, \underset{\sim}{K}, -\underset{\sim}{k}') \right\}$$

$$= i\, \underset{\sim}{E}^{(0)} \cdot \underset{\sim}{k}'\, h_1(\eta, \underset{\sim}{k}) \Big/ \Delta^*\left(\frac{-\omega - \underset{\sim}{K}\cdot\underset{\sim}{v}}{k'}, k'\right) . \tag{B-15}$$

(5) By adding up Eqs. (B-14), (B-10), (B-13), and (B-15), we obtain Eq. (IV-10).

IV. Derivation of Eq. (IV-54)

(1)
$$\overline{F_0 \phi_2} = \left[e^2 n_0 (\tilde{f}_i + \tilde{f}_e)\right] \left[\frac{e}{4\pi}\left(\frac{\omega_i^2}{m_i}\phi_i - \frac{\omega_e^2}{m_e}\phi_e\right)\right]$$

$$= \frac{e}{(4\pi)^2}(\tilde{f}_i + \tilde{f}_e)(\omega_i^4 \phi_i - \omega_e^4 \phi_e) \quad ,$$

$$\overline{F_1 \phi_1} = \left[\frac{e}{4\pi}(\omega_i^2 \tilde{f}_i - \omega_e^2 \tilde{f}_e)\right]\left[\frac{1}{4\pi}(\omega_i^2 \phi_i + \omega_e^2 \phi_e)\right]$$

$$= \frac{e}{(4\pi)^2}\left[\tilde{f}_i(\omega_i^4 \phi_i + \omega_i^2 \omega_e^2 \phi_e) - \tilde{f}_e(\omega_i^2 \omega_e^2 \phi_i + \omega_e^4 \phi_e)\right] ,$$

$$\therefore \quad \bar{F}_0 \Phi_2 - \bar{F}_1 \Phi_1$$

$$= \frac{e}{(4\pi)^2}\left[\omega_i^2 \tilde{f}_e \Phi_i (\omega_i^2 + \omega_e^2) - \omega_e^2 \tilde{f}_i \Phi_e (\omega_e^2 + \omega_i^2)\right]$$

$$= \frac{e\tilde{\omega}^2}{(4\pi)^2}\left[\omega_i^2 \tilde{f}_e \Phi_i - \omega_e^2 \tilde{f}_i \Phi_e\right],$$

where

$$\tilde{\omega}^2 = \omega_i^2 + \omega_e^2 \quad .$$

(2)

$$\bar{F}_2 \Psi_0 = \left[\frac{e^2}{4\pi}\left(\frac{\omega_i^2}{m_i}\tilde{f}_i + \frac{\omega_e^2}{m_e}\tilde{f}_e\right)\right]\left[e n_0 (\Psi_i - \Psi_e)\right]$$

$$= \frac{e}{(4\pi)^2}(\omega_i^4 \tilde{f}_i + \omega_e^4 \tilde{f}_e)(\Psi_i - \Psi_e) \ ,$$

$$\bar{F}_1 \Psi_1 = \left[\frac{e}{4\pi}(\omega_i^2 \tilde{f}_i - \omega_e^2 \tilde{f}_e)\right]\left[\frac{1}{4\pi}(\omega_i^2 \Psi_i + \omega_e^2 \Psi_e)\right]$$

$$= \frac{e}{(4\pi)^2}(\omega_i^2 \tilde{f}_i - \omega_e^2 \tilde{f}_e)(\omega_i^2 \Psi_i + \omega_e^2 \Psi_e) \quad ,$$

$$\therefore \quad \bar{F}_2 \Psi_0 - \bar{F}_1 \Psi_1$$

$$= \frac{e}{(4\pi)^2}\left[-\omega_i^2 \tilde{f}_i \Psi_e (\omega_i^2 + \omega_e^2) + \omega_e^2 \tilde{f}_e \Psi_i (\omega_e^2 + \omega_i^2)\right]$$

$$= \frac{e\tilde{\omega}^2}{(4\pi)^2}\left[\omega_e^2 \tilde{f}_e \Psi_i - \omega_i^2 \tilde{f}_i \Psi_e\right].$$

(3) $$\psi_1 \phi_1 = \frac{1}{(4\pi)^2}(\omega_i^2 \psi_i + \omega_e^2 \psi_e)(\omega_i^2 \phi_i + \omega_e^2 \phi_e)$$

$$= \frac{1}{(4\pi)^2}\Big[\omega_i^4 \psi_i \phi_i + \omega_i^2 \omega_e^2 (\psi_e \phi_i + \psi_i \phi_e) + \omega_e^4 \psi_e \phi_e\Big] ,$$

$$\phi_2 \psi_0 = \Big[\frac{e}{4\pi}\Big(\frac{\omega_i^2}{m_i}\phi_i - \frac{\omega_e^2}{m_e}\phi_e\Big)\Big]\Big[e n_0 (\psi_i - \psi_e)\Big]$$

$$= \frac{1}{(4\pi)^2}(\omega_i^4 \phi_i - \omega_e^4 \phi_e)(\psi_i - \psi_e) ,$$

$\therefore$

$$\psi_1 \phi_1 - \phi_2 \psi_0 = \frac{\tilde{\omega}^2}{(4\pi)^2}(\omega_i^2 \psi_e \phi_i + \omega_e^2 \psi_i \phi_e).$$

V. Proof of Eq. (IV-57)

(1) From Eq. (IV-56), we define

$$I \equiv \frac{\tilde{f}_e \phi_i + \tilde{f}_i \phi_e}{\Delta(w+u)\Delta^*(u)} - \frac{\tilde{f}_i \psi_e + \tilde{f}_e \psi_i}{\Delta(w+u)\Delta(u)}$$

$$= \frac{1}{\Delta(w+u)|\Delta(u)|^2}\Big[\Delta(u)(\tilde{f}_e \phi_i + \tilde{f}_i \phi_e) - \Delta^*(u)(\tilde{f}_i \psi_e + \tilde{f}_e \psi_i)\Big].$$

since

$$\Delta(u) = \frac{2i}{k'^2} e\, n_0 (\psi_i - \psi_e) + \Delta(w+u) \quad ,$$

$$\Delta^*(u) = \frac{2i}{k'^2} e\, n_0 (\phi_i - \phi_e) + \Delta(w+u) \quad ,$$

hence

$$I = \frac{1}{|\Delta(u)|^2}\left[(\tilde{f}_e \phi_i + \tilde{f}_i \phi_e) - (\tilde{f}_i \psi_e + \tilde{f}_e \psi_i)\right]$$

$$+ \frac{2ie\,n_0}{k'^2 \Delta(w+u)|\Delta(u)|^2}\left[(\psi_i - \psi_e)(\tilde{f}_e \phi_i + \tilde{f}_i \phi_e) - (\phi_i - \phi_e)(\tilde{f}_i \psi_e + \tilde{f}_e \psi_i)\right];$$

or

$$\begin{aligned} I = {} & \frac{1}{|\Delta(u)|^2}\left[\tilde{f}_e(\phi_i - \psi_i) + \tilde{f}_i(\phi_e - \psi_e)\right] \\ & - \frac{2ie\,n_0}{k'^2 \Delta(w+u)|\Delta(u)|^2}\left[(\tilde{f}_i + \tilde{f}_e)(\psi_e \phi_i - \psi_i \phi_e)\right]. \end{aligned} \qquad \text{(B-16)}$$

(2) Using Eq. (B-16) then

$$\int du \left\{ I + \frac{2ie\,n_0}{k'^2 \Delta(w+u)|\Delta(u)|^2}\left[(\tilde{f}_i + \tilde{f}_e)(\psi_e \phi_i - \psi_i \phi_e)\right]\right\}$$

$$= \int du \frac{1}{|\Delta(u)|^2}\left[\tilde{f}_e(\phi_i - \psi_i) + \tilde{f}_i(\phi_e - \psi_e)\right]$$

$$= 0.$$

This result enables us to obtain Eq. (IV-57).

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13. ABSTRACT

A kinetic equation for an inhomogeneous and non-isotropic plasma is derived in the plasma limit. The treatment is based on the joint solution of the first two members of the BBKGY hierarchy, and on a linearization procedure about the unperturbed state. The unperturbed distribution functions. $f_0(\eta)$ and $\mathcal{G}_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$, are unspecified. Furthermore, the unperturbed pair correlation function, $\mathcal{G}_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$, is eliminated in favor of the unperturbed one-particle distribution function, $f_0(\eta)$. This elimination is accomplished by solving the equation for $\mathcal{G}_0(\eta, \eta', \underset{\sim}{r} - \underset{\sim}{r}')$. The resulting kinetic equation is free from Bogolyubov's adiabatic hypothesis; therefore, it is valid for arbitrary frequency. In the limiting case when the frequency under consideration is much higher than the collision-frequency, a general expression for the high-frequency electric conductivity tensor is derived. From this general expression the results for the homogeneous and isotropic case previously derived by Klevans and Wu, as well as the results for the thermodynamic equilibrium case derived by Oberman, Ron, and Dawson can be recovered.

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