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Orbital Regression of Synchronous Satellites Due to the Combined Gravitational Effects of The Sun, the Moon and the Oblate Earth

R. H. Frick

August 1967

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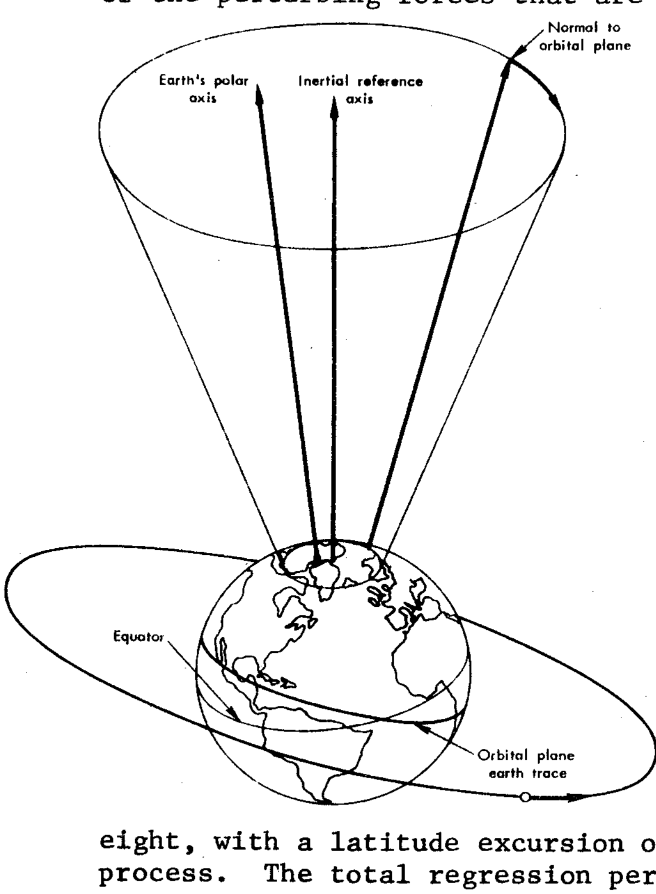
R-454-NASA, Orbital Regression of Synchronous Satellites Due to the Combined Gravitational Effects of the Sun, the Moon, and the Oblate Earth, R. H. Frick, RAND Report, August 1967, 186 pp.

PURPOSE: To analyze the long-term behavior of earth satellites due to the gravitational effects of the sun, the moon, and the oblate earth.

RELATED TO: RAND's continuing study of satellite orbit control requirements for NASA. The study extends the results obtained in R-399-NASA, Perturbations of a Synchronous Satellite, May 1962.

DISCUSSION AND METHODOLOGY: Now that long-lived earth satellites are orbited on a relatively permanent basis, it is important to know the effect of long-term orbital perturbations, and the cost of controlling them. It is also interesting to consider the effect of such perturbations on the motion of the increasing debris in orbit. While the earth's inverse-square law gravitational field is the major attraction on a satellite, perturbing forces arise from the attraction of the sun, the moon, and the oblateness of the earth. The resultant force field is neither central in direction nor inverse-square in magnitude. The components of the perturbing forces that are normal to the orbital plane perturb the plane's orientation relative to inertial space. Those lying in the orbital plane cause changes in the shape and orientation of the orbit in its plane. The in-plane motion was analyzed in R-399-NASA; in the present study emphasis is on the determination of the orbital plane itself. The analysis applies to satellites in near-circular orbits at any inclination and with orbital radii less than 10 earth radii. The perturbed motion of an uncontrolled satellite is described as seen from inertial space and as seen from the rotating earth.



PRINCIPAL FINDINGS: The motion of the orbital plane is such that its normal describes a conical surface relative to inertial space as shown in the figure. The ground trace of a synchronous-altitude orbit lying in the reference plane is a figure eight with crossing-point on the equator and a maximum latitude excursion of 7°20'; this does not vary with time. An orbit at an angle to the reference plane has a figure-eight ground trace which varies with the regression period. For a synchronous orbit that is originally equatorial and "stationary," the ground trace develops from a point to a figure eight, with a latitude excursion of 14°40' after 26.6 years, and then reverses the process. The total regression period is about 53 years. A fuel expenditure proportional to the sine of twice the inclination angle relative to the reference plane is required to stop orbital regression.

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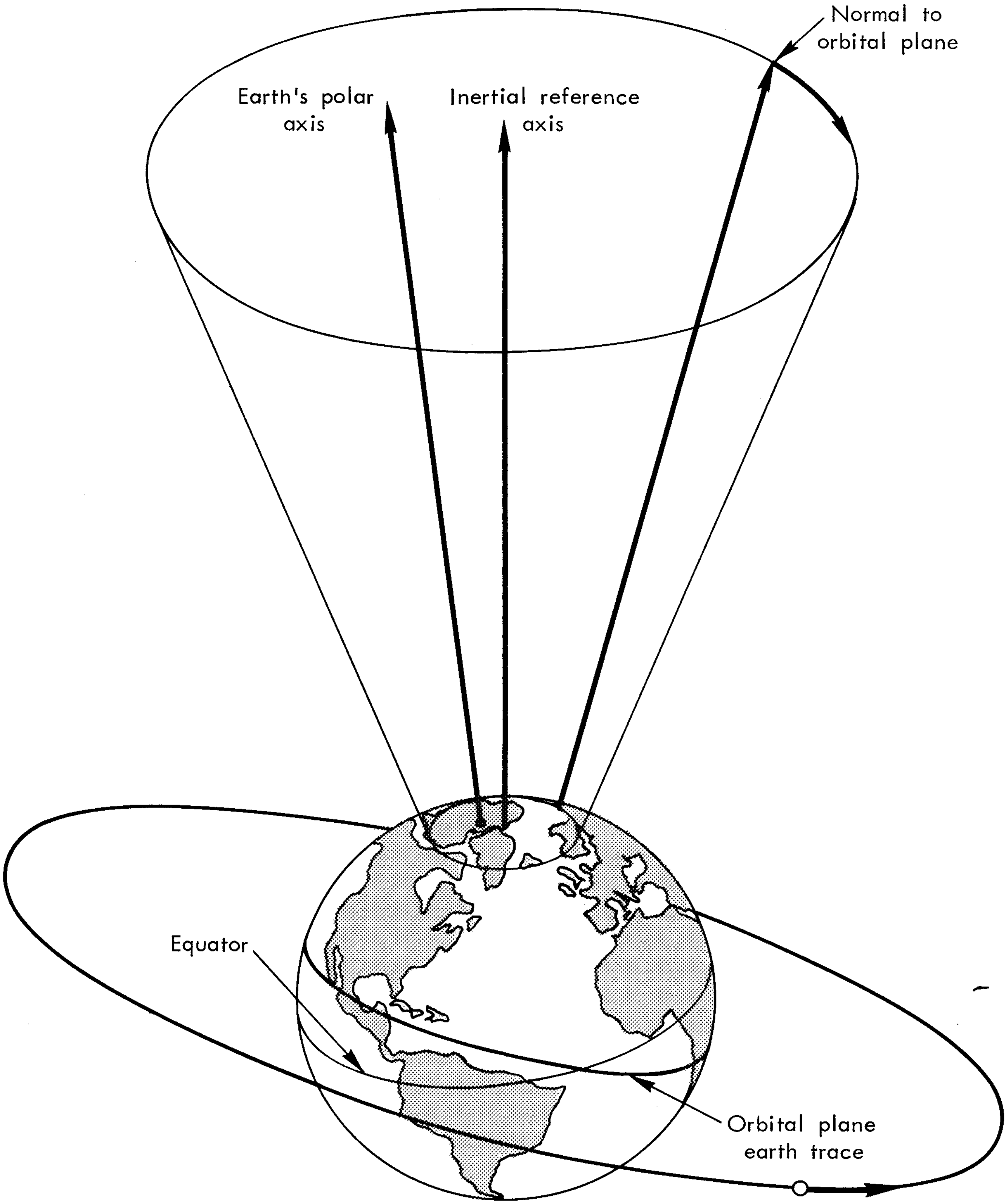
PREFACE

As part of a continuing RAND study for the National Aeronautics and Space Administration of the perturbations of earth satellites and resulting orbital control requirements, this Report investigates the gravitational effects of the sun, the moon and the oblate earth on the orbital behavior of earth satellites. An extension of the results obtained earlier in R-399-NASA, the analysis provides determination of orbital control propulsion requirements and of the orbital regression of earth satellites with no restriction on orbital altitude or inclination. The general analytical solution for the regression of a satellite obtained in the R can also be specialized to explain lunar regression and the behavior of low-altitude satellites.

The Report should be of interest to people dealing with satellite systems which require high-precision long-term orbital control.



ORBITAL REGRESSION

SUMMARY

This Report presents an analysis of the long-term behavior of earth satellites due to the gravitational effects of the sun, the moon and the oblate earth. The importance of these effects has increased in recent years because of interest in more precise orbital control and an increase in expected payload lifetime. The fuel requirement for maintaining precise long-term orbital control in the presence of these gravitational perturbations tends to dominate the overall fuel requirement for orbit and attitude control. As a result, it is of increasing interest to determine first whether the magnitude of the long-term orbital perturbations of an uncontrolled satellite is compatible with the requirements of a given satellite mission during its expected payload lifetime. If the orbital perturbations exceed acceptable values, it is necessary to investigate methods whereby these perturbations can be controlled, either actively or passively. In addition, it is of interest to consider the effect of these long-term perturbations on the motion of the increasing collection of debris in orbit. This Report deals with all of these problems.

The analysis presented here applies to satellites in near-circular orbits at any inclination and with orbital radii which are small compared to the radius of the moon's orbit (i.e., less than 10 earth radii). The formulation takes into account the rotation of the earth around the sun, the rotation of the earth-moon system about its center of mass and the regression of the moon's orbital plane about the normal to the ecliptic.

The major effect of the perturbing influences considered is to produce motion of the orbital plane relative to inertial space. The nature

of this motion can be completely described by the trace of the normal to the orbital plane on a sphere concentric with the earth. It is shown that for an orbit of a given radius an orbital orientation can be found which remains invariant relative to inertial space. This invariant plane has a common intersection with the earth's equatorial plane and the plane of the ecliptic, while its inclination to the latter is always less than that of the equatorial plane. For low-altitude orbits, the invariant plane is very nearly equatorial, with an inclination of $23^{\circ}27'$ relative to the ecliptic. As the orbital altitude increases, the value of the inclination decreases to $16^{\circ}7'$ at synchronous altitude and approaches zero for extremely high orbits.

For an orbit of a given altitude, the trace of the normal to the invariant plane on the sphere described above is a single point between the earth's polar axis and the normal to the ecliptic. For orbits of the same radius but different orientations, two types of motion are possible. If the initial inclination of the orbit relative to the corresponding invariant or reference plane is less than about 80°, the normal to the orbital plane rotates about the normal to the reference plane with an essentially constant angular rate and inclination angle. The resulting trace on the sphere is a circle with center on the normal to the reference plane. If the initial inclination is in excess of 80°, the trace of the normal to the orbital plane on the sphere may be an ellipse with its center on a line in the direction of the vernal equinox and major axis in the reference plane.

The regression period at zero inclination varies from .1 year for a surface orbit to about 53 years for a synchronous orbit and a maximum of about 75 years for an orbit of radius equal to 9 earth radii. As the inclination increases, the period varies inversely as the cosine of the inclination angle. On the other hand, the regression period corresponding to the elliptical contours has a minimum value as the ellipse approaches a point, and increases toward infinity as the major axis approaches 90°. However, since high-inclination orbits are of relatively little interest, the emphasis in this Report is on the first type of regression, which is illustrated by the frontispiece.

It should be noted that superposed on this steady-state motion are oscillatory perturbations in both regression rate and orbital inclination which cause the instantaneous position of the normal to the orbital plane to oscillate relative to its steady-state motion. However, it is shown that the displacement is less than half a degree.

It is of particular interest to observe the effect of this orbital regression on the relative motion of synchronous altitude satellites as seen from the rotating earth. Since the orbital altitude is assumed to be constant, this relative motion is completely described by the trace of the subsatellite point on the earth's surface.

The reference plane corresponding to a synchronous altitude orbit has an inclination of $16^{\circ}7'$ relative to the ecliptic, as compared with an inclination of $23^{\circ}27'$ for the earth's equatorial plane. Since the orientation of a synchronous orbit in this plane remains invariant relative to inertial space, its inclination of $7^{\circ}20'$ relative to the earth's equatorial plane is also invariant. As a result, the trace of the subsatellite point on the surface of the rotating earth is the characteristic figure-eight pattern with a maximum latitude excursion on either side of the equator equal to the inclination angle of $7^{\circ}20'$ relative to the equatorial plane. In addition, the maximum longitude excursion relative to the equatorial crossing position is of the order of ±15 min of arc. This ground trace is repeated once each orbit with no variation in size or shape. Similarly, if the orbital plane of a synchronous orbit is perpendicular to the reference plane and polar relative to the earth, it remains stationary relative to inertial space, and its ground trace on the rotating earth also repeats itself on each orbit. However, for such an orbit, the ground trace varies from -90° to $+90^{\circ}$ in latitude each day.

For any other inclination of a synchronous orbit relative to the reference plane, it is found that the maximum latitude or amplitude of the figure eight varies as a function of time. This is due to the fact that the maximum latitude is equal to the inclination of the orbit relative to the equatorial plane, and although the inclination relative to the reference plane is fixed, that relative to the equatorial plane varies as the orbit regresses. The resulting variation in the ground trace amplitude has a periodicity equal to that of the regression and a magnitude which can never exceed $14^{\circ}40'$. In addition, it is found that the longitude of the equatorial crossing also oscillates with the regression period and with an amplitude which may be as large as $7^{\circ}20'$, depending on the orbital inclination relative to the reference plane.

In regard to these variations in the size and shape of the ground trace, it is of particular interest to consider the long-term behavior of a satellite which is initially in a synchronous equatorial orbit. Such a satellite is ordinarily referred to as stationary since it appears to be fixed relative to the earth. However, its orbit is actually inclined to its reference plane at an angle of $7^{\circ}20'$ and has a regression period of about 53 years. As the orbital plane regresses, its inclination to the equatorial plane increases from 0° at an average initial rate of $.863^{\circ}$ per year. At the end of half the regression period this inclination reaches a maximum of $14^{\circ}40'$, after which it decreases symmetrically to 0° after a complete regression period. Since the maximum latitude excursion during an orbit is equal to the orbital inclination to the equator, the resultant ground trace is initially an equatorial point but develops into a figure eight which reaches a maximum amplitude of $14^{\circ}40'$ before decreasing to the original equatorial point at the end of the regression period. During this cycle, the position of the equatorial crossing oscillates with the regression period with an amplitude of $.47^{\circ}$, moving to the east of its initial position during the first half of the cycle and to the west during the second half. It should be noted that these variations in longitude are superposed on the shorter period ($\sim$ 2-year) oscillations due to the equatorial ellipticity described in Ref. 1.

It is seen that a passive satellite cannot remain truly stationary relative to the rotating earth, and that its earth trace can remain invariant only for certain orbital inclinations. Since a given satellite mission may require a fixed ground trace which is not inherently invariant, it is of interest to determine the amount of control necessary to produce the desired invariance. It is seen that such an invariant ground trace can exist only if the orientation of the orbital plane remains fixed in inertial space. By applying appropriate control impulses normal to the orbital plane, it is possible to reduce the steady-state orbital regression rate to zero. Under these conditions, the orbital orientation and the resulting ground trace have the desired invariance. The control impulse required per year to achieve this invariance is proportional to the sine of twice the inclination angle of the orbit to its reference plane. Thus, the magnitude of the control impulse per year depends as follows on the desired value of the ground trace amplitude. For an amplitude of 0°, the required control impulse has an average value of 152 ft/sec/year, which decreases to zero for an amplitude of $7^{\circ}20'$, after which it increases to a maximum of 580 ft/sec/year for an amplitude of 45°. The impulse requirement for amplitudes between 45° and 90° is a mirror image of that from 0° to 45°, decreasing to zero at $82^{\circ}40'$ and increasing again to 152 ft/sec/year at 90°. It should be noted that these values represent a long-term average control requirement, neglecting the oscillatory components of the orbital regression. In the event that it is necessary to control these oscillatory variations, the control requirement in a given year might deviate from its average value by as much as 30 ft/sec, depending on the amplitude and phase of the oscillatory terms.

If instead of an invariant ground trace, a given mission requires merely an upper limit on its latitude excursion, it may be possible to satisfy this condition passively. If the initial orbital inclination to the equator is made equal to the upper limit of the latitude excursion, then by a suitable choice of the initial regression phase the inclination to the orbital plane will decrease to 0° before it again increases to its initial value.

In this way the time during which the latitude excursion remains below its upper limit is maximized. If this time is longer than the expected payload lifetime, this passive technique can be used to satisfy the mission requirement. However, the decision regarding the use of active or passive orbit control depends on the tolerances in ground trace amplitude and the required mission lifetime.

Finally, the analysis of orbital regression developed here for artificial satellites is extended to include the regression of the moon. This requires an expansion of the basic theory to include higher order terms as shown in Appendix E.

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CONTENTS

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LIST OF FIGURES

LIST OF SYMBOLS

A_i	amplitude of i^{th} oscillatory component of $\dot{\alpha}$
a_{cz}	control acceleration
a_{io}	constant coefficient of least-squares fit to α
a_{il}	linear coefficient of least-squares fit to α
a_x, b_x, c_x	direction cosines of x axis in the x_1, y_1, z_1 system
a_y, b_y, c_y	direction cosines of y axis in the x_1, y_1, z_1 system
a_z, b_z, c_z	direction cosines of z axis in the x_1, y_1, z_1 system
B_i	amplitude of i^{th} oscillatory component of $\dot{\psi}$ sin α
b_{io}	constant coefficient of least-squares fit to ψ sin α
b_{il}	linear coefficient of least-squares fit to ψ sin α
D_o	total angular bias of orbital normal
D_1	total angular drift of orbital normal in time T
F_{cz}	magnitude of control force
$\overline{F}_{Em}$	force on the moon due to the earth
$\overline{F}_{Es}$	force on the satellite due to the earth
$\overline{F}_{mE}$	force on the earth due to the moon
$\overline{F}_{ms}$	force on the satellite due to the moon
$\overline{F}_{SE}$	force on the earth due to the sun
$\overline{F}_{Sm}$	force on the moon due to the sun
$\overline{F}_{Ss}$	force on the satellite due to the sun
G	universal gravitational constant
$\overline{i}$, $\overline{j}$, $\overline{k}$	unit vectors of the x, y, z system
$\overline{i}_o$, $\overline{j}_o$, $\overline{k}_o$	unit vectors of the x_o, y_o, z_o system

$\bar{i}_G, \bar{j}_G, \bar{k}_G$	unit vectors of the x_G, y_G, z_G system
$\bar{i}_m, \bar{j}_m, \bar{k}_m$	unit vectors of the x_m, y_m, z_m system
$\bar{i}_1, \bar{j}_1, \bar{k}_1$	unit vectors of the x_1, y_1, z_1 system
J_2	oblateness coefficient in the earth's potential function
M_E	mass of the earth
M_S	mass of the sun
M_m	mass of the moon
M_s	mass of the satellite
m	ratio of the earth's orbital rate to that of the moon
P	orbital regression rate at zero inclination
Q	oscillatory amplitude factor
R_o	mean earth radius
$\bar{R}$	vector from the center of the sun to the center of mass of the earth-moon system
$\bar{R}_E$	vector from the center of the sun to the center of the earth
$\bar{R}_m$	vector from the center of the sun to the center of the moon
δr	perturbation of r
r_o	steady-state value of r
$\bar{r}$	vector from the center of the earth to the satellite
$\bar{r}_S$	vector from the center of the sun to the satellite
$\bar{r}_m$	vector from the center of the moon to the satellite
$\bar{r}_1$	unit vector along $\bar{R}$
S	oscillatory amplitude factor
T	duration of regression
T_N	nodical period of the moon

T_R	satellite regression period
T_S	sidereal period of the moon
t	time
t_n	time for n passages of the ascending node in the reference plane
t_o	time interval between reference plane crossing and equatorial crossing
ΔV_z	velocity increment required from control system
X, Y	components of the deflection of the orbital normal
$\bar{X}$, $\bar{Y}$	linear least-squares fits to X and Y
x, y, z	satellite coordinate system
x_G, y_G, z_G	nonrotating geocentric coordinate system with z_G along the earth's polar axis
x_S, y_S, z_S	heliocentric inertial coordinate system
x_m, y_m, z_m	lunar coordinate system
x_o, y_o, z_o	nonrotating geocentric coordinate system with z_o normal to the ecliptic
x_1, y_1, z_1	geocentric coordinate system with x_1y_1 plane inclined at an angle α_1 to the ecliptic and x_1 axis along x_o
α	inclination of the satellite orbital plane to the x_1y_1 reference plane
$\delta\alpha$	perturbation of α
α_G	inclination of the satellite orbital plane to the earth's equatorial plane
α_m	inclination of the moon's orbital plane to the ecliptic
α_o	steady-state value of α
α_1	inclination of the x_1y_1 reference plane to the ecliptic
β	inertial longitude of the satellite

$\Delta\beta$	longitude difference between satellite and initial subsatellite point
β_n	inertial longitude of initial subsatellite point after n crossings of the reference plane
β_o	inertial longitude of the satellite at time of equatorial crossing
$\Delta\beta_o$	value of $\Delta\beta$ at time of equatorial crossing
β_1	current inertial longitude of initial subsatellite point
γ	satellite latitude
γ_m	maximum latitude excursion during one orbit
Θ	orbital angle of the earth around the sun
θ	orbital angle of the satellite around the earth
θ_m	orbital angle of the moon around the earth
θ_o	value of θ at time of equatorial crossing
$\Delta\theta_o$	change in θ_o after one orbit
$\delta\dot{\theta}$	perturbation of $\dot{\theta}$
$\dot{\theta}_E$	earth rotation rate
$\dot{\theta}_o$	steady-state orbital angular rate of the satellite
λ	inclination of equatorial plane to the ecliptic
$\delta\rho$	perturbation of the magnitude of $\overline{\rho}$
ρ_o	steady-state magnitude of $\overline{\rho}$
$\delta\rho_o$	initial value of $\delta\rho$
$\overline{\rho}$	vector from the center of the earth to the center of the moon
$\overline{\rho}_E$	vector from the center of the earth to the center of mass of the earth-moon system
σ_D	combined root-mean-square deviation
σ_X	root-mean-square deviation of X from $\overline{X}$

σ_Y	root-mean-square deviation of Y from $\bar{Y}$
τ	arbitrary time interval
$\Delta\tau$	difference between equatorial crossing period and nodical period
τ_o	nodical period of the satellite
ψ	satellite orbital regression angle
ψ_m	lunar regression angle
ψ_n	value of ψ after n reference plane crossings
ψ_o	value of ψ at time of equatorial crossing
$\Delta\psi_o$	increase of ψ_o after one orbit
$\delta\dot{\psi}$	perturbation of $\dot{\psi}$
$\dot{\psi}(0)$	regression rate of the satellite at zero inclination
$\dot{\psi}_c$	regression rate of the satellite in the presence of orbital control forces
$\dot{\psi}_o$	steady-state regression rate of the satellite
ω	component of the moon's angular velocity normal to its orbital plane
$\delta\omega$	perturbation of ω
ω_i	oscillatory frequency of the i^{th} component in either α or $\psi \sin \alpha$
ω_o	steady-state value of ω
$\delta\omega_o$	initial value of $\delta\omega$
$\bar{\omega}_E$	vector angular velocity of the earth around the sun
$\bar{\omega}_m$	vector angular velocity of the moon relative to inertial space
$\bar{\omega}_o$	vector angular velocity of the x, y, z system relative to inertial space

I. INTRODUCTION

In the ten years since the launching of the first Sputnik, a rather impressive amount of hardware has been placed in orbit around the earth in the fulfillment of various satellite missions. During the early years of this period, the lifetime of a given satellite mission was short and there was little interest in precise long-term orbital control. However, in recent years, as technology has developed which can take advantage of precisely controlled synchronous orbits with long-life potential, it has become important to know (1) the effect of long-term orbital perturbations on the satellite mission, and (2) the cost of controlling these perturbations.

It is well known that if the only force on a satellite is the inverse-square-law gravitational attraction of the earth, then the resulting orbit is an ellipse with one focus at the earth's center. In addition, the direction of the normal to this orbital plane remains fixed relative to inertial space. While the earth's inverse-square field is the major attraction on the satellite, it is necessary to consider the perturbing forces which might produce long-term changes in the basic orbital motion described above. Three such forces are those due to the attractions of the sun and the moon and that arising from the oblateness of the earth. Since the resultant force field when these effects are included is neither central in direction nor inverse-square in magnitude, the resulting orbital perturbations may be of two types. Those components of the perturbing forces which are normal to the orbital plane produce perturbations in the plane's orientation relative to inertial space. Those components which lie in the plane cause alterations in the shape and orientation of the orbit in its plane.

In Ref. 1, the effects of the sun and the moon on a satellite in a synchronous equatorial orbit are determined. The results show that the in-plane perturbations are of the nature of small amplitude oscillations in the satellite's position relative to its nominal unperturbed position. The maximum excursion is of the order of 45 mi. It is also found that the perturbations in the attitude of the orbital plane are of the nature of a slow change in its inclination to the equatorial plane at a rate of about $.85^{\circ}$/year. The analysis also indicates a slow sinusoidal increase to a maximum inclination of about 20° and a return to 0° after a period of about 73 years. However, these two values are only approximate, since a 20° angle exceeds the small angle assumption used in the perturbation analysis.

This Report gives a more general determination of the orbital perturbations resulting from the gravitational attractions of the sun and moon and from the oblateness of the earth, and it places no restriction on the magnitude of the orbital inclination. The emphasis is primarily on the determination of the motion of the orbital plane since it is not anticipated that the in-plane motion will differ greatly from that determined in Ref. 1. By means of the analysis presented here, the perturbed motion of an uncontrolled satellite is described both as seen from inertial space and as seen from the rotating earth. In addition, the fuel requirement to maintain a fixed orientation of the orbital plane relative to inertial space is determined.

II. METHOD OF ANALYSIS

STATEMENT OF THE PROBLEM

The problem to be solved in this Report can be stated as follows: If a satellite is in a circular orbit around the earth with a known initial orientation of its orbital plane relative to inertial space, determine the motion of this orbital plane as it is affected by the gravitational attraction of the sun, the moon and the oblate earth.

For the purposes of this analysis it is assumed that the orbital altitude is sufficiently high that forces due to residual drag can be neglected. In addition, by assuming a small area-to-mass ratio for the orbiting object, the effects of solar radiation pressure can also be neglected. The ellipticity of the earth's equatorial section is also omitted since its effect on orbital regression is negligible.

Finally, the positions of the sun and the moon relative to the earth are specified as known functions of time according to the following model. The center of mass of the earth-moon system moves on a circular orbit around the sun with a period of one sidereal year, and the plane of the motion is that of the ecliptic.* The earth and moon rotate about their common center of mass with a constant separation and a period of one sidereal month. The plane of this rotation is inclined at an angle of $5^{\circ}8'$ to the ecliptic and regresses about the normal to the ecliptic with a period of 18.6 years.

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*Strictly speaking, the plane of the ecliptic is defined by the motion of the earth's center of mass; however this differs in orientation by about 1 sec of arc from the plane defined above.

DEFINITION OF COORDINATES

Reference Systems

In the formulation of the equations of motion and the description of the resulting motion, it is convenient to define the following reference systems.

x_S, y_S, z_S. This is an inertial reference system with its origin at the center of the sun; its $x_S y_S$ plane is the ecliptic and its x_S axis is in the direction of the earth at the time of the vernal equinox.

x_o, y_o, z_o. This and all the remaining reference systems are geocentric. This particular one maintains its axes parallel to the corresponding ones in the x_S, y_S, z_S system. Thus, every point of this system is under a constant acceleration in a direction parallel to the earth-sun line. The unit vectors associated with this and the previous system are represented as $\bar{i}_o$, $\bar{j}_o$, $\bar{k}_o$.

x_1, y_1, z_1. This system is rotated relative to x_o, y_o, z_o through an angle α_1 about their common x axes as shown in Fig. 1. This system is the one relative to which the motions of the orbital plane are expressed. The appropriate value of α_1 is determined in the course of the analysis. The associated unit vectors are represented as $\bar{i}_1$, $\bar{j}_1$, $\bar{k}_1$.

x_G, y_G, z_G. This system has a common x axis with the two previous systems, while the $x_G y_G$ plane is the earth's equatorial plane, which makes an angle $\lambda = 23^o 27'$ with the ecliptic or $x_o y_o$ plane as shown in Fig. 1. The associated unit vectors are represented as $\bar{i}_G$, $\bar{j}_G$, $\bar{k}_G$.

x, y, z. This is the orbital coordinate system with the x axis passing through the satellite and the xy plane representing the instantaneous orbital plane. The orientation of this system relative to the

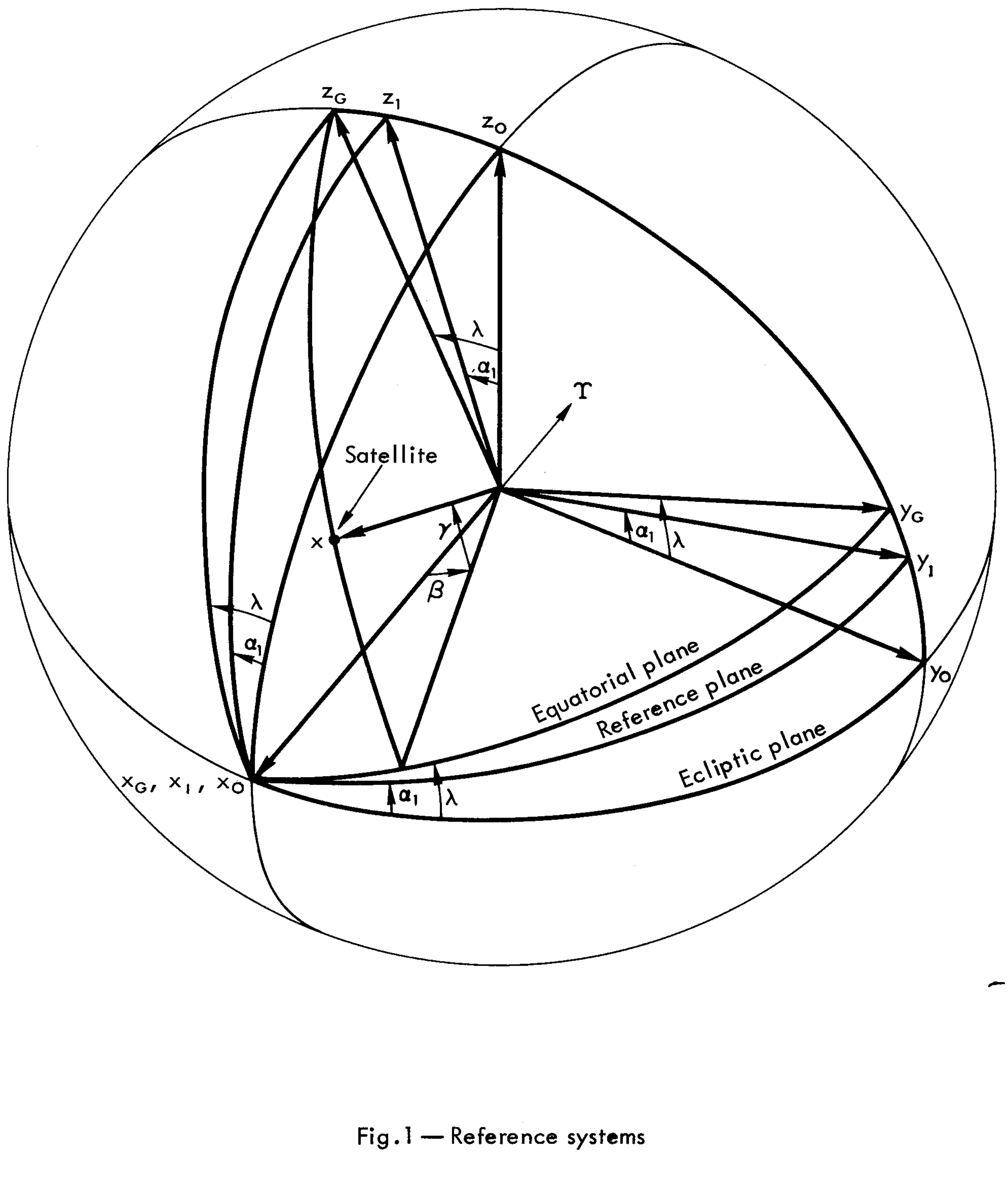

Fig. 1 — Reference systems

x_1, y_1, z_1 system is specified by the three angles α, ψ and θ as shown in Fig. 2. The angle α is the angle between the xy (orbital) plane and the x_1y_1 (reference) plane. The angle ψ is the regression angle measured from the x_1 axis to the line of nodes, ON. Finally the angle θ is the orbital angle measured from the line of nodes to the x axis. The associated unit vectors are represented as $\bar{i}$, $\bar{j}$, $\bar{k}$.

It should be noted that an additional relation between α, ψ and θ is required to insure that the xy plane is indeed the instantaneous orbital plane. This relation is determined in the derivation of the equations of motion.

$\underline{x_m, y_m, z_m}$. This is the lunar reference system in which the x_m axis passes through the center of mass of the moon and the x_my_m plane represents the moon's orbital plane. The orientation of this system relative to the x_o, y_o, z_o system is specified by the three angles α_m, ψ_m and θ_m as shown in Fig. 3, where it is seen that these angles are analogous to α, ψ and θ for the satellite orbital system. As before, the associated unit vectors are represented as $\bar{i}_m$, $\bar{j}_m$, $\bar{k}_m$.

The direction cosines relating these various systems are listed in Appendix A.

Position Vectors

The relative positions of the sun, moon, earth and satellite can be described vectorially as shown in Fig. 4, in which the various vectors are defined as follows.

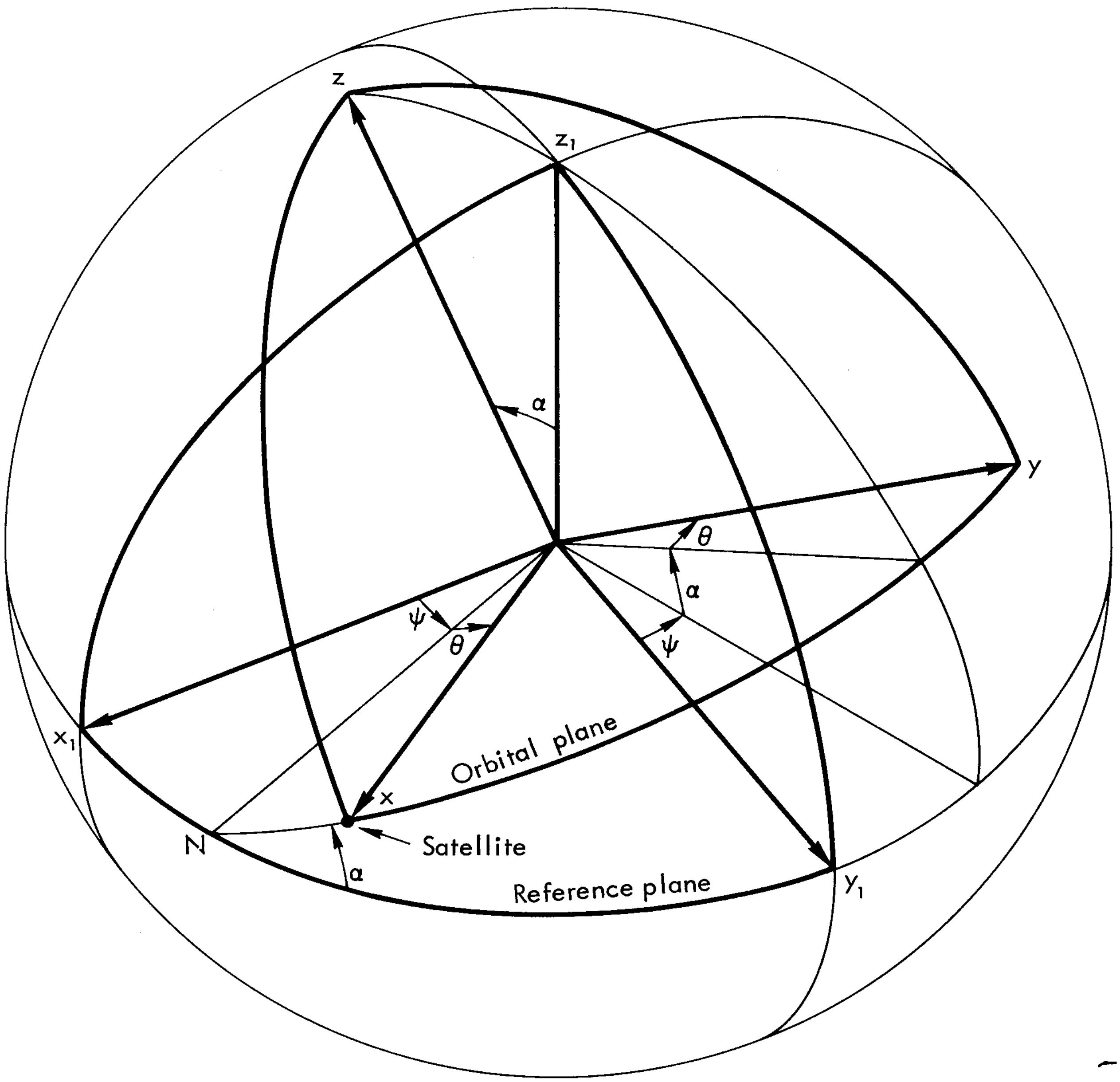

Fig. 2 — Satellite orbital reference systems

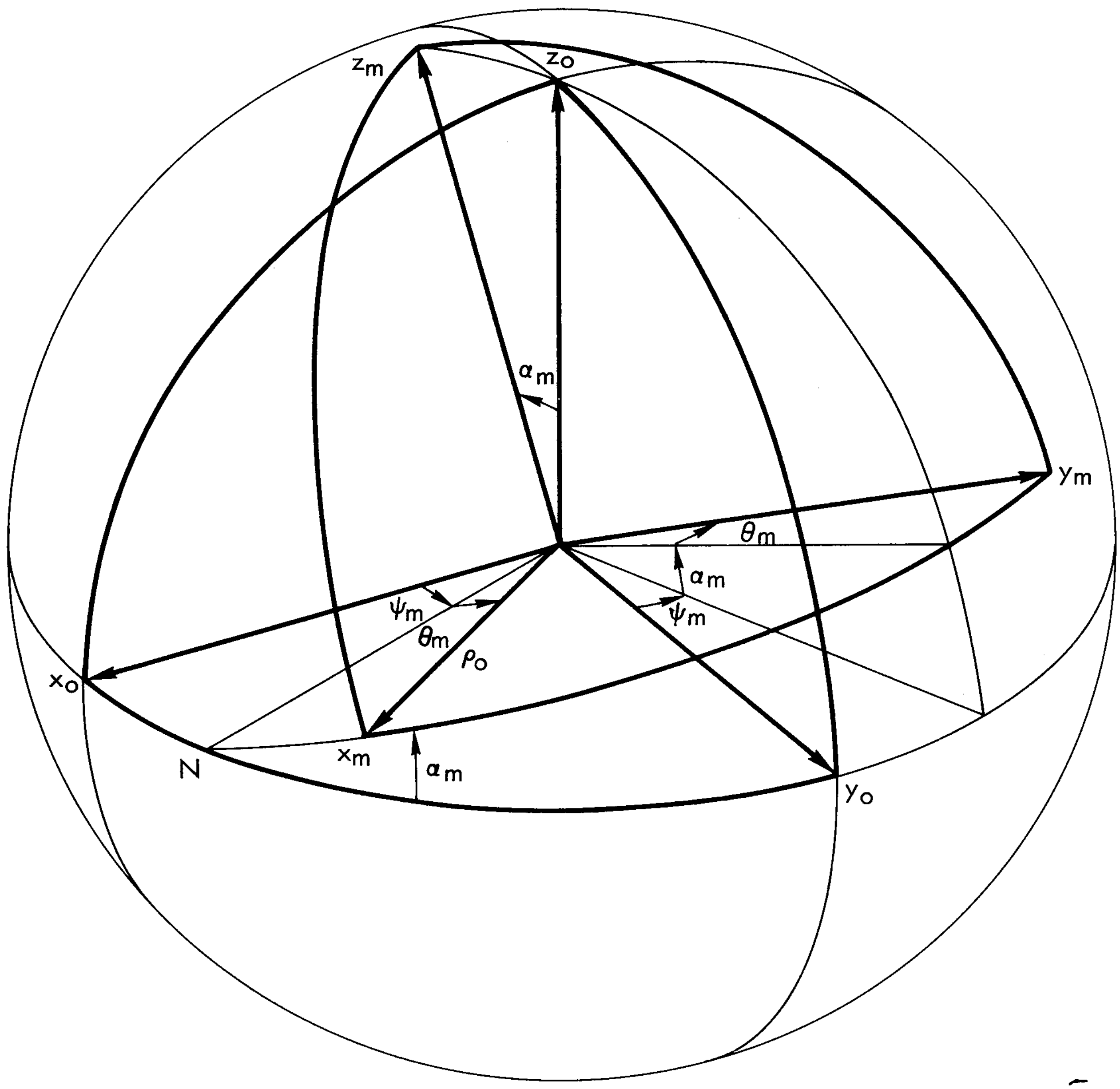

Fig. 3 — Lunar orbital reference system

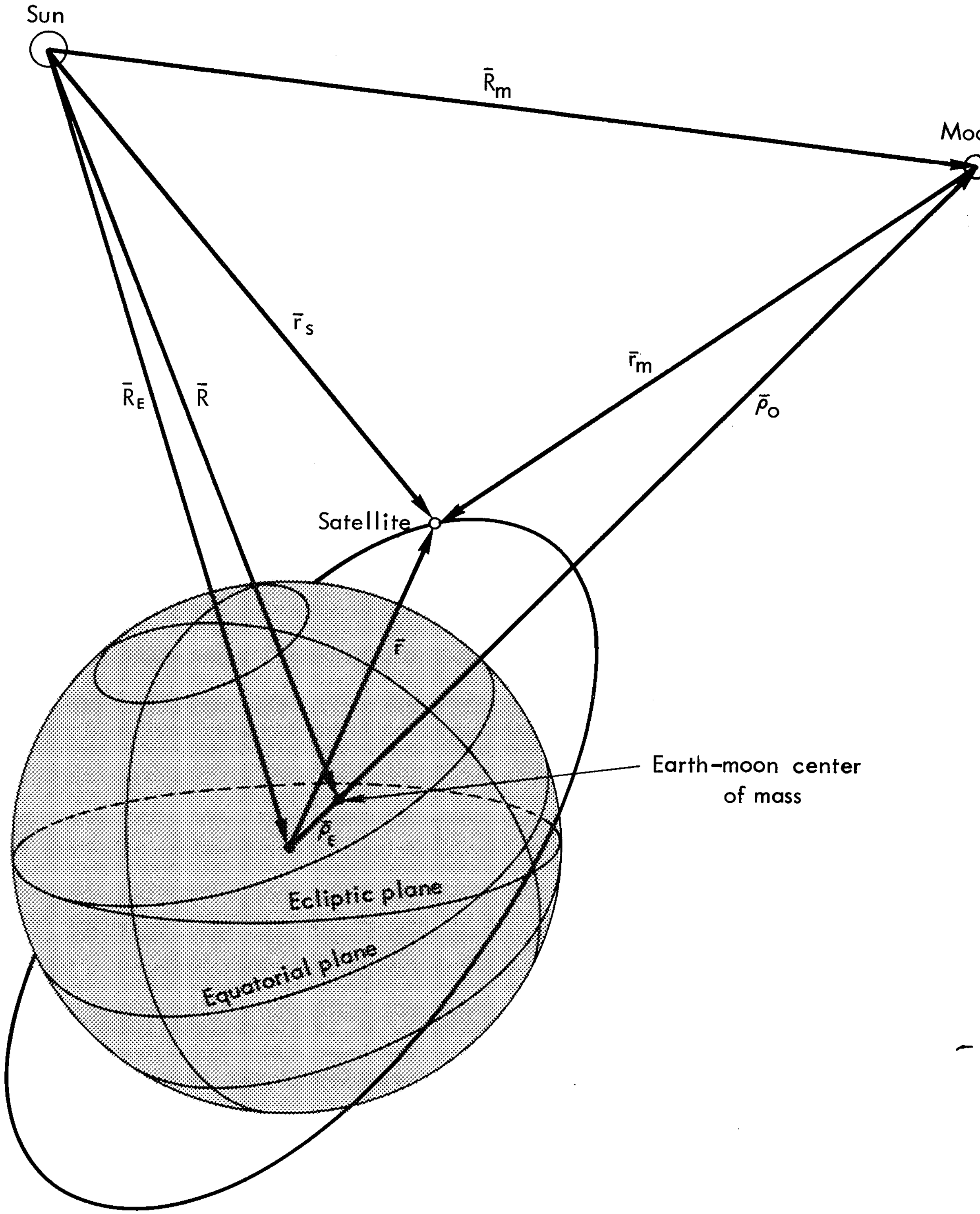

Fig. 4 — Position vectors

The vector $\overline{R}$ is from the center of the sun to the center of mass of the earth-moon system. In accordance with the assumed motion of this system around the sun,

$$\overline{R} = R\overline{r}_1 = R(\overline{i}_o \cos \Theta + \overline{j}_o \sin \Theta) \tag{1}$$

where R is constant and equal to the earth's orbital radius, while Θ varies linearly with time.

The vector $\overline{\rho}_o$ is from the center of the earth to the center of the moon and lies along the x_m axis. It can be expressed as

$$\overline{\rho}_o = \rho_o \overline{i}_m \tag{2}$$

where ρ_o is a constant and equal to the earth-moon separation.

The vector $\overline{r}$ is from the earth's center to the satellite along the x axis and can be expressed as

$$\overline{r} = r\overline{i} \tag{3}$$

where r is the magnitude of $\overline{r}$ and is one of the orbital variables to be determined.

In addition to these basic vectors, it is convenient to define certain others that are of use in the formulation of the equations of motion, although they can be expressed in terms of those already defined. The vector $\overline{\rho}_E$ from the earth's center to the center of mass of the earth-moon system can be expressed as

$$\overline{\rho}_E = \frac{1}{\mu} \overline{\rho}_o \tag{4}$$

where

$$\mu = \frac{M_E + M_m}{M_m} \tag{5}$$

and M_E and M_m are the masses of the earth and moon respectively.

The vector $\overline{R}_E$ from the sun's center to the earth's center is given by

$$\overline{R}_E = \overline{R} - \frac{1}{\mu}\,\overline{\rho}_o \tag{6}$$

The vector $\overline{R}_m$ from the sun's center to the moon's center is given by

$$\overline{R}_m = \overline{R} + \left(1 - \frac{1}{\mu}\right)\overline{\rho}_o \tag{7}$$

The vector $\overline{r}_S$ from the sun's center to the satellite is expressed as

$$\overline{r}_S = \overline{R} - \frac{1}{\mu}\,\overline{\rho}_o + \overline{r} \tag{8}$$

The vector $\overline{r}_m$ from the moon's center to the satellite is given by

$$\overline{r}_m = \overline{r} - \overline{\rho}_o \tag{9}$$

Angular Velocity Vectors

The relative motion of the various coordinate systems defined previously can be expressed in terms of the following angular velocity vectors.

The angular velocity, $\overline{\omega}_E$, of the center of mass of the earth-moon system around the sun can be expressed as

$$\overline{\omega}_E = \dot{\Theta}\,\overline{k}_o \tag{10}$$

where $\dot{\Theta}$ is a constant equal to .0172 rad/solar day. This corresponds to a period of one sidereal year.

The angular velocity, $\overline{\omega}_m$, of the earth-moon system about its center of mass and relative to inertial space is given by the expression

$$\overline{\omega}_m = \dot{\theta}_m \overline{k}_m + \dot{\psi}_m \overline{k}_o \tag{11}$$

where $\dot{\theta}_m$ is the moon's orbital rate of .22998 rad/solar day, corresponding to a period of one sidereal month, while $\dot{\psi}_m$ is the regression rate of the moon's orbital plane and has a value of -9.249×10^{-4} rad/solar day, corresponding to a period of 18.6 years.

The angular velocity $\overline{\dot{\theta}}_E$ of the earth about its axis is given by

$$\overline{\dot{\theta}}_E = \dot{\theta}_E \overline{k}_G \tag{12}$$

where $\dot{\theta}_E$ is equal to 6.3004 rad/solar day, corresponding to a period of one sidereal day.

The orbital angular velocity, $\overline{\omega}_o$, of the satellite around the earth's center relative to inertial space is expressed in terms of the three orbital angles as follows

$$\overline{\omega}_o = \dot{\psi}\overline{k}_1 + \dot{\alpha}\,(\overline{i}_1 \cos\psi + \overline{j}_1 \sin\psi) + \dot{\theta}\overline{k} \tag{13}$$

where ψ, α and θ are the orbital variables to be determined together with r.

FORMULATION OF THE EQUATIONS OF MOTION

The derivation of the equations of motion of the satellite relative to the earth is presented in this section in abbreviated form; the details are given in Appendix B.

The equations of motion of the satellite, the earth and the moon relative to inertial space can be expressed in vector form as follows

$$\text{Satellite:} \qquad \ddot{\bar{r}}_S = \ddot{\bar{R}} - \frac{1}{\mu}\ddot{\bar{\rho}}_o + \ddot{\bar{r}} = \frac{\bar{F}_{Ss} + \bar{F}_{ms} + \bar{F}_{Es}}{M_s} \tag{14}$$

$$\text{Earth:} \qquad \ddot{\bar{R}}_E = \ddot{\bar{R}} - \frac{1}{\mu}\ddot{\bar{\rho}}_o = \frac{\bar{F}_{SE} + \bar{F}_{mE}}{M_E} \tag{15}$$

$$\text{Moon:} \qquad \ddot{\bar{R}}_m = \ddot{\bar{R}} + \left(1 - \frac{1}{\mu}\right)\ddot{\bar{\rho}}_o = \frac{\bar{F}_{Sm} + \bar{F}_{Em}}{M_m} \tag{16}$$

In this derivation the dot notation for time derivatives signifies a derivative relative to inertial space, while d/dt is a derivative relative to the orbital (x, y, z) reference system. In addition, the subscripts S, E, m and s refer to the sun, the earth, the moon and the satellite, respectively. When they are used with M, the mass of the body they represent is indicated. When they are used with a vector force, $\bar{F}$, the first subscript determines the attracting body, while the second defines the body on which the force acts.

The vector equation of motion of the satellite relative to the earth is obtained as the difference between Eqs. (14) and (15) in the form

$$\ddot{\bar{r}} = \frac{\bar{F}_{Ss} + \bar{F}_{ms} + \bar{F}_{Es}}{M_s} - \frac{\bar{F}_{SE} + \bar{F}_{mE}}{M_E} \tag{17}$$

An evaluation of the forces on the right reduces Eq. (17) to the following form

$$\ddot{\bar{r}} = -\frac{GM_E}{r^3}\left[\left(1 - \frac{3J_2R_o^2}{2r^2}\left[\frac{5}{r^2}(\bar{r}\cdot\bar{k}_G)^2 - 1\right]\right)\bar{r} + \frac{3J_2R_o^2}{r^2}(\bar{r}\cdot\bar{k}_G)\,\bar{k}_G\right]$$

$$- \frac{\dot{\theta}_m^2}{\mu}\left[\bar{r} - \frac{3}{\rho_o^2}(\bar{r}\cdot\bar{\rho}_o)\,\bar{\rho}_o\right]$$

$$- \dot{\Theta}^2\left[\bar{r} - \frac{3}{R^2}(\bar{r}\cdot\bar{R})\,\bar{R}\right] \tag{18}$$

where R_o is the earth's equatorial radius, G is the universal gravitational constant and J_2 is the coefficient of the oblateness term in the earth's potential function.

The three component equations of motion corresponding to the x, y and z axes are obtained from Eq. (18) in the form

$$(\ddot{\bar{r}} \cdot \bar{i}) = -\frac{GM_E}{r^2} - \frac{3J_2GM_ER_o^2}{2r^4}\left[1 - 3(\bar{k}_G \cdot \bar{i})^2\right]$$

$$-\frac{r\dot{\theta}_m^2}{\mu}\left[1 - 3(\bar{i}_m \cdot \bar{i})^2\right]$$

$$-r\dot{\Theta}^2\left[1 - 3(\bar{r}_1 \cdot \bar{i})^2\right] \qquad (19)$$

$$(\ddot{\bar{r}} \cdot \bar{j}) = -\frac{3J_2GM_ER_o^2}{r^4}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{j})$$

$$+\frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{j})$$

$$+3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{j}) \qquad (20)$$

$$(\ddot{\bar{r}} \cdot \bar{k}) = -\frac{3J_2GM_ER_o^2}{r^4}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{k})$$

$$+\frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{k})$$

$$+3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k}) \qquad (21)$$

The quantity $\ddot{\bar{r}}$ is given by the relation

$$\ddot{\bar{r}} = \frac{d^2\bar{r}}{dt^2} + 2\left[\omega_o \times \frac{d\bar{r}}{dt}\right] + \left[\frac{d\bar{\omega}_o}{dt} \times \bar{r}\right] + \left[\bar{\omega}_o \times [\bar{\omega}_o \times \bar{r}]\right] \qquad (22)$$

from which the left sides of Eqs. (19), (20) and (21) can be expressed as

$$(\ddot{\bar{r}} \cdot i) = \frac{d^2r}{dt^2} + r\left[(\bar{\omega}_o \cdot \bar{i})^2 - (\bar{\omega}_o \cdot \bar{\omega}_o)\right] \tag{23}$$

$$(\ddot{\bar{r}} \cdot \bar{j}) = 2\,\frac{dr}{dt}\,(\bar{\omega}_o \cdot \bar{k}) + r\left(\frac{d\bar{\omega}_o}{dt} \cdot \bar{k}\right) + r(\bar{\omega}_o \cdot \bar{i})\,(\bar{\omega}_o \cdot \bar{j}) \tag{24}$$

$$(\ddot{\bar{r}} \cdot \bar{k}) = -\,2\,\frac{dr}{dt}\,(\bar{\omega}_o \cdot \bar{j}) - r\left(\frac{d\bar{\omega}_o}{dt} \cdot \bar{j}\right) + r(\bar{\omega}_o \cdot \bar{i})\,(\bar{\omega}_o \cdot \bar{k}) \tag{25}$$

At this point, it is necessary to determine the constraint equation which insures that the xy plane is the instantaneous orbital plane. This condition requires that $\dot{\bar{r}}$ as well as $\bar{r}$ must lie in the xy plane. This can be satisfied if

$$(\dot{\bar{r}} \cdot \bar{k}) = 0 \tag{26}$$

from which it follows that

$$(\bar{\omega}_o \cdot \bar{j}) = 0 \tag{27}$$

and

$$\left(\frac{d\bar{\omega}_o}{dt} \cdot \bar{j}\right) = 0 \tag{28}$$

Thus, Eqs. (23), (24) and (25) can be simplified to the form

$$(\ddot{\bar{r}} \cdot \bar{i}) = \frac{d^2r}{dt} - r(\bar{\omega}_o \cdot \bar{k})^2 \tag{29}$$

$$(\ddot{\bar{r}} \cdot \bar{j}) = \frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o \cdot \bar{k})\right] \tag{30}$$

$$(\ddot{\bar{r}} \cdot \bar{k}) = r(\bar{\omega}_o \cdot \bar{i})\ (\bar{\omega}_o \cdot \bar{k}) \tag{31}$$

where

$$(\bar{\omega}_o \cdot \bar{i}) = \dot{\psi} \sin\theta \sin\alpha + \dot{\alpha} \cos\theta \tag{32}$$

$$(\bar{\omega}_o \cdot \bar{j}) = \dot{\psi} \cos\theta \sin\alpha - \dot{\alpha} \sin\theta \tag{33}$$

$$(\bar{\omega}_o \cdot \bar{k}) = \dot{\psi} \cos\alpha + \dot{\theta} \tag{34}$$

The complete equations of motion of the satellite are obtained by combining Eqs. (19)-(21), (27), (29) and (30)-(34) to give

$$\frac{d^2r}{dt^2} - r(\dot{\theta} + \dot{\psi}\cos\alpha)^2 = -\frac{GM_E}{r^2} - \frac{3J_2GM_ER_o^2}{2r^4}\left[1 - 3(\bar{k}_G \cdot \bar{i})^2\right]$$
$$- \frac{r\dot{\theta}_m^2}{\mu}\left[1 - 3(\bar{i}_m \cdot \bar{i})^2\right]$$
$$- r\dot{\Theta}^2\left[1 - 3(\bar{r}_1 \cdot \bar{i})^2\right] \tag{35}$$

$$\frac{1}{r}\frac{d}{dt}\left[r^2(\dot{\theta} + \dot{\psi}\cos\alpha)\right] = -\frac{3J_2GM_ER_o^2}{r^4}(\bar{k}_G \cdot \bar{i})\ (\bar{k}_G \cdot \bar{j})$$
$$+ \frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})\ (\bar{i}_m \cdot \bar{j})$$
$$+ 3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})\ (\bar{r}_1 \cdot \bar{j}) \tag{36}$$

$$\dot{\alpha} = - \frac{3J_2 GM_E R_o^2}{(\dot{\theta} + \dot{\psi} \cos \alpha) r^5} (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{k}) \cos \theta$$

$$+ \frac{3\dot{\theta}_m^2}{\mu(\dot{\theta} + \dot{\psi} \cos \alpha)} (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{k}) \cos \theta$$

$$+ \frac{3\dot{\Theta}^2}{(\dot{\theta} + \dot{\psi} \cos \alpha)} (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{k}) \cos \theta \tag{37}$$

$$\dot{\psi} = - \frac{3J_2 GM_E R_o^2}{(\dot{\theta} + \dot{\psi} \cos \alpha) r^5 \sin \alpha} (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{k}) \sin \theta$$

$$+ \frac{3\dot{\theta}_m^2}{\mu(\dot{\theta} + \dot{\psi} \cos \alpha) \sin \alpha} (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{k}) \sin \theta$$

$$+ \frac{3\dot{\Theta}^2}{(\dot{\theta} + \dot{\psi} \cos \alpha) \sin \alpha} (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{k}) \sin \theta \tag{38}$$

These four equations constitute the desired orbital equations of motion in terms of the variables r, θ, α and ψ.

MOTION OF THE ORBITAL PLANE

The exact solution of the equations of motion as presented above does not appear feasible since the terms resulting from earth oblateness, the sun and the moon are rather involved functions of the unknowns r, θ, α and ψ. To simplify the problem, it is assumed that the quantities r, $\dot{\theta}$, α and $\dot{\psi}$ can be represented as follows by a steady-state value plus a perturbation.

$$r = r_o + \delta r \tag{39}$$

$$\dot{\theta} = \dot{\theta}_o + \delta\dot{\theta} \tag{40}$$

$$\dot{\psi} = \dot{\psi}_o + \delta\dot{\psi} \tag{41}$$

$$\alpha = \alpha_o + \delta\alpha \tag{42}$$

Since the forcing terms due to earth oblateness, the sun and the moon in Eqs. (35) through (38) are in the nature of perturbations, it is assumed that in these terms r, $\dot{\theta}$, $\dot{\psi}$ and α can be replaced by their steady-state values, while θ and ψ are expressed as

$$\theta = \dot{\theta}_o t \tag{43}$$

$$\psi = \dot{\psi}_o t \tag{44}$$

In addition, it is assumed that $\dot{\theta}$ is much greater than $\dot{\psi}$, and that the nominal orbit is circular. Thus, the steady-state values of r and $\dot{\theta}$ are given by

$$r = r_o \tag{45}$$

$$\dot{\theta}^2 = \dot{\theta}_o^2 = \frac{GM_E}{r_o^3} \tag{46}$$

With these assumptions, Eqs. (37) and (38) can be written as

$$\dot{\alpha} = -\frac{3J_2R_o^2\dot{\theta}_o}{r_o^2}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{k})\cos\theta$$

$$+\frac{3\dot{\theta}_m^2}{\mu\dot{\theta}_o}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{k})\cos\theta$$

$$+\frac{3\dot{\Theta}^2}{\dot{\theta}_o}(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k})\cos\theta \tag{47}$$

$$\dot{\psi} = -\frac{3J_2R_o^2\dot{\theta}_o}{r_o^2\sin\alpha_o}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{k})\sin\theta$$

$$+\frac{3\dot{\theta}_m^2}{\mu\dot{\theta}_o\sin\alpha_o}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{k})\sin\theta$$

$$+\frac{3\dot{\Theta}^2}{\dot{\theta}_o\sin\alpha_o}(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k})\sin\theta \tag{48}$$

thus giving $\dot{\alpha}$ and $\dot{\psi}$ as functions of time.

If the direction cosines tabulated in Appendix A are substituted in Eqs. (47) and (48), they can be reduced to the form

$$\dot{\alpha} = \sum_{i=1}^{127} A_i \sin\omega_i t \tag{49}$$

$$\dot{\psi} = \dot{\psi}_o + \frac{1}{\sin\alpha_o}\sum_{i=1}^{127} B_i \cos\omega_i t \tag{50}$$

The details of this transformation are presented in Appendix C together with a tabulation of the resulting expressions for the ω_i, A_i and B_i values, as well as for $\dot{\psi}_o$.

An examination of this tabulation shows that the ω_i values are linear combinations of the frequencies $\dot{\theta}_o$, $\dot{\theta}_m$, $\dot{\Theta}$, $\dot{\psi}_m$ and $\dot{\psi}_o$. In the case of A_i and B_i, they may be functions of any or all of the quantities α_o, α_1, α_m, $\dot{\theta}_o$, r_o, J_2, $\dot{\Theta}$ and $\dot{\theta}_m$. The presence of J_2 in a given amplitude indicates that this term is at least partially the result of earth oblateness. Similarly, $\dot{\Theta}$ and $\dot{\theta}_m$ indicate contributions due to the sun and the moon, respectively. Thus, it is seen that the steady-state regression rate, $\dot{\psi}_o$, as well as the first seven oscillatory terms are the result of a combination of all three perturbing influences. On the other hand, those terms for which $8 \le i \le 22$ are entirely due to solar influence, while the rest of the terms $(23 \le i \le 127)$ are due to the moon.

The solutions for α and ψ can now be obtained by integrating Eqs. (40) and (50) to give

$$\alpha = \alpha_o + \sum_{i=1}^{127} \frac{A_i}{\omega_i} [1 - \cos \omega_i t] \tag{51}$$

$$\psi = \dot{\psi}_o t + \frac{1}{\sin \alpha_o} \sum_{i=1}^{127} \frac{B_i}{\omega_i} \sin \omega_i t \tag{52}$$

under the assumption that at $t = 0$, $\alpha = \alpha_o$ and $\psi = 0$.

It is now necessary to define the angle α_1 between the reference $(x_1 y_1)$ plane and the plane of the ecliptic since the quantities $\dot{\psi}_o$,

A_i and B_i are all functions of α_1. The selection of α_1 is dictated by the fact that the analysis leading up to Eqs. (51) and (52) is based on the assumption that the angle α between the orbital (xy) plane and the reference (x_1y_1) plane remains essentially constant and equal to α_o.

In view of this fact, a value of α_1 should be selected which minimizes the amplitudes of the oscillatory terms in Eq. (51). Particular emphasis should be placed on the low-frequency terms because of the factor $1/\omega_i$. An examination of the A_i expressions in Appendix C shows that the only reasonable choice for α_1 is that value which makes A_1, the coefficient associated with the frequency, ψ_o, identically zero, as follows.

$$\left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2}\left(2 - 3\sin^2\alpha_m\right)\right]\sin 2\alpha_1 - \frac{2J_2\dot{\theta}_o^2R_o^2}{\dot{\Theta}^2r_o^2}\sin 2(\lambda - \alpha_1) = 0 \tag{53}$$

which can be solved for α_1 in the form

$$\tan 2\alpha_1 = \frac{\dfrac{2J_2\dot{\theta}_o^2R_o^2}{\dot{\Theta}^2r_o^2}\sin 2\lambda}{1 + \dfrac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2}\left(2 - 3\sin^2\alpha_m\right) + \dfrac{2J_2\dot{\theta}_o^2R_o^2}{\dot{\Theta}^2r_o^2}\cos 2\lambda} \tag{54}$$

This selection of α_1 also makes the amplitudes A_4, A_6, B_1, B_4 and B_6 identically zero since they all have the left side of Eq. (53) as a factor.

In addition, the amplitudes A_2, A_3, A_7, B_2, B_3 and B_7 all involve the factor f, given by

$$f = \left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m)\right] \sin^2 \alpha_1$$

$$+ \frac{2J_2\dot{\theta}_o^2 R_o^2}{\dot{\Theta}^2 r_o^2} \sin^2 (\lambda - \alpha_1) \tag{55}$$

While no choice of α_1 can make the factor f vanish, the value determined by Eq. (54) does minimize f since

$$\frac{df}{d\alpha_1} = \left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m)\right] \sin 2 \alpha_1$$

$$- \frac{2J_2\dot{\theta}_o^2 R_o^2}{\dot{\Theta}^2 r_o^2} \sin 2 (\lambda - \alpha_1) \tag{56}$$

which is equal to zero from Eq. (53) .

By the same type of reasoning it can be shown that this choice of α_1 maximizes the amplitudes A_5 and B_5 associated with the frequency, $2\dot{\theta}_o$. However, this is a high-frequency term with a very small amplitude. Thus, it causes very little change in the inclination angle, α.

With the determination of α_1 it is now possible to use Eqs. (51) and (52) to describe the resulting motion of the orbital plane. In Appendix D it is shown the remaining oscillatory terms result in less than a $.5^o$ variation in the normal to the orbital plane. Thus, to a very good approximation, the motion of the orbital plane is represented by

$$\alpha = \alpha_o \tag{57}$$

$$\psi = \dot{\psi}_o t \tag{58}$$

which means that the orbital plane maintains a constant inclination, α_o, relative to the selected reference plane but regresses about the normal to the reference plane at a steady rate of $\dot{\psi}_o$ given in Appendix C as

$$\dot{\psi}_o = -\frac{3\dot{\Theta}^2 \cos \alpha_o}{8\dot{\theta}_o}\left[\left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2}(2 - 3 \sin^2 \alpha_m)\right](2 - 3 \sin^2 \alpha_1) + \frac{2J_2\dot{\theta}_o^2 R_o^2}{\dot{\Theta}^2 r_o^2}\left[2 - 3 \sin^2 (\lambda - \alpha_1)\right]\right] \tag{59}$$

Thus, the motion of the orbital plane is completely described by Eqs. (54), (57), (58) and (59).

In obtaining the above result, it was assumed in Eq. (44) that ψ could be replaced by $\dot{\psi}_o t$ in Eqs. (37) and (38). However, an examination of Eqs. (50) and (59) shows that as the orbital inclination, α_o, approaches 90^o, the above assumption is no longer valid, since $\dot{\psi}_o$ approaches zero and Eq. (50) is dominated by the oscillatory terms.

In Appendix F, the behavior of these high-inclination orbits is investigated in more detail. It is found that the representation of the motion as described by Eqs. (57) through (59) is a good approximation of the actual motion as long as the regression takes place about the z_1 axis.

III. RESULTS AND DISCUSSION

DETERMINATION OF ORBITAL REGRESSION

On the basis of the analysis presented in the previous section, it is now possible to make a quantitative determination of the orbital regression rate and its axis of rotation.

Reference Plane

An examination of Eq. (54) shows that the inclination, α_1, of the reference plane is a function of orbital radius. The relationship is shown in Fig. 5, which gives α_1 as a function of r_o. It is seen that for low-altitude orbits, for which the earth's oblateness is the dominant perturbing influence, the reference plane is very nearly coincident with the earth's equatorial plane ($\alpha_1 = 23^\circ 27'$). This is due to the fact that the equatorial plane is one of symmetry for the oblateness effect. Similarly, for high-altitude orbits, for which the combination of the solar and lunar effects becomes dominant, the reference plane approaches the plane of the ecliptic ($\alpha_1 = 0^\circ$). Again, this is due to the symmetry of these two effects relative to this plane.

Steady-State Motion

The steady-state regression of the orbital plane as specified by Eqs. (57) and (58) is represented in Fig. 6, in which the normal to the orbital plane (the z axis) maintains a fixed angle, α_o, with respect to the normal to the reference plane (the z_1 axis). At the same time, the z axis rotates about z_1 at an angular rate $\dot{\psi}_o$, tracing a circular contour on the spherical surface as shown. The residual oscillatory terms

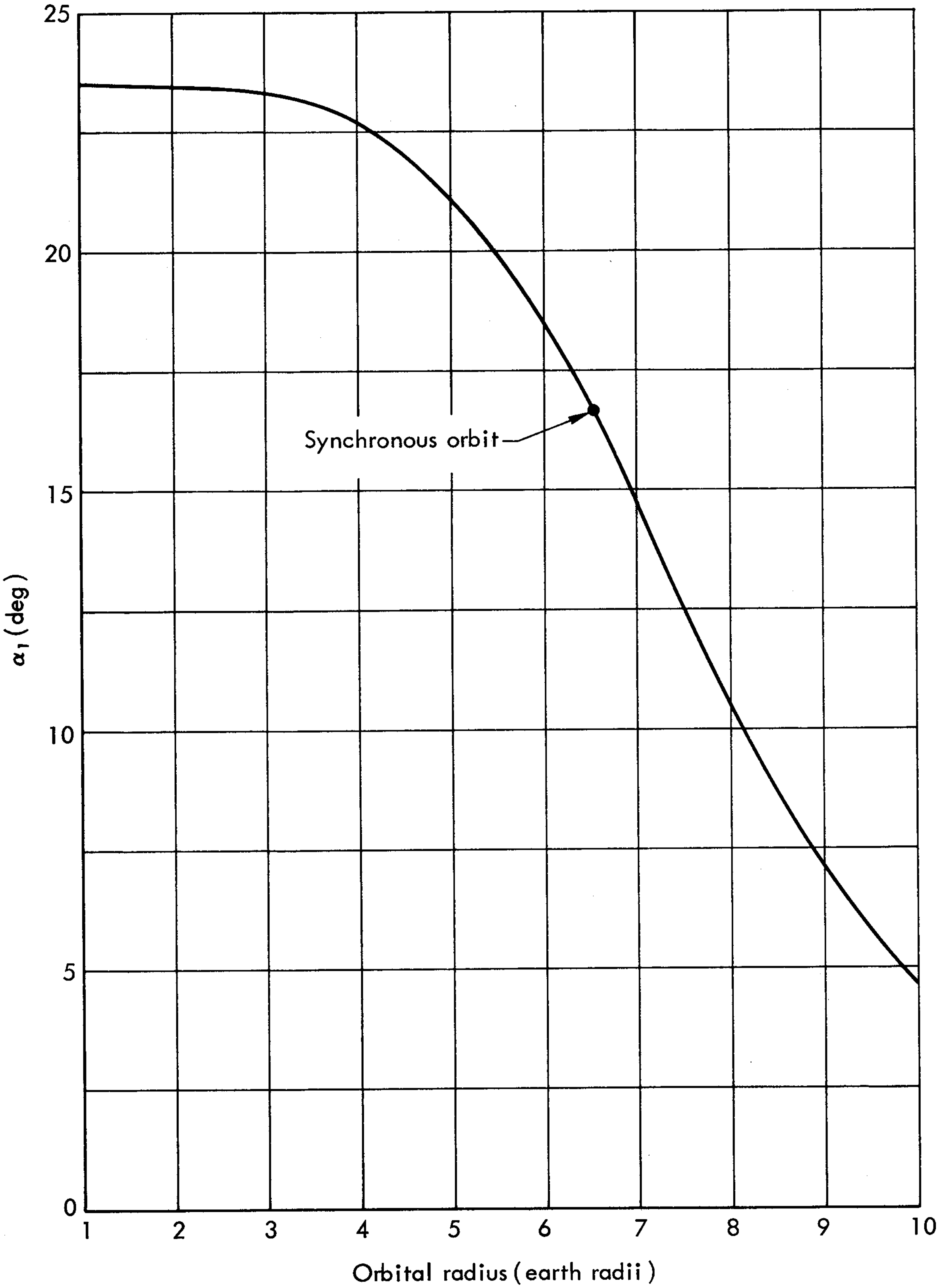

Fig. 5 — Dependence of reference plane inclination on orbital radius

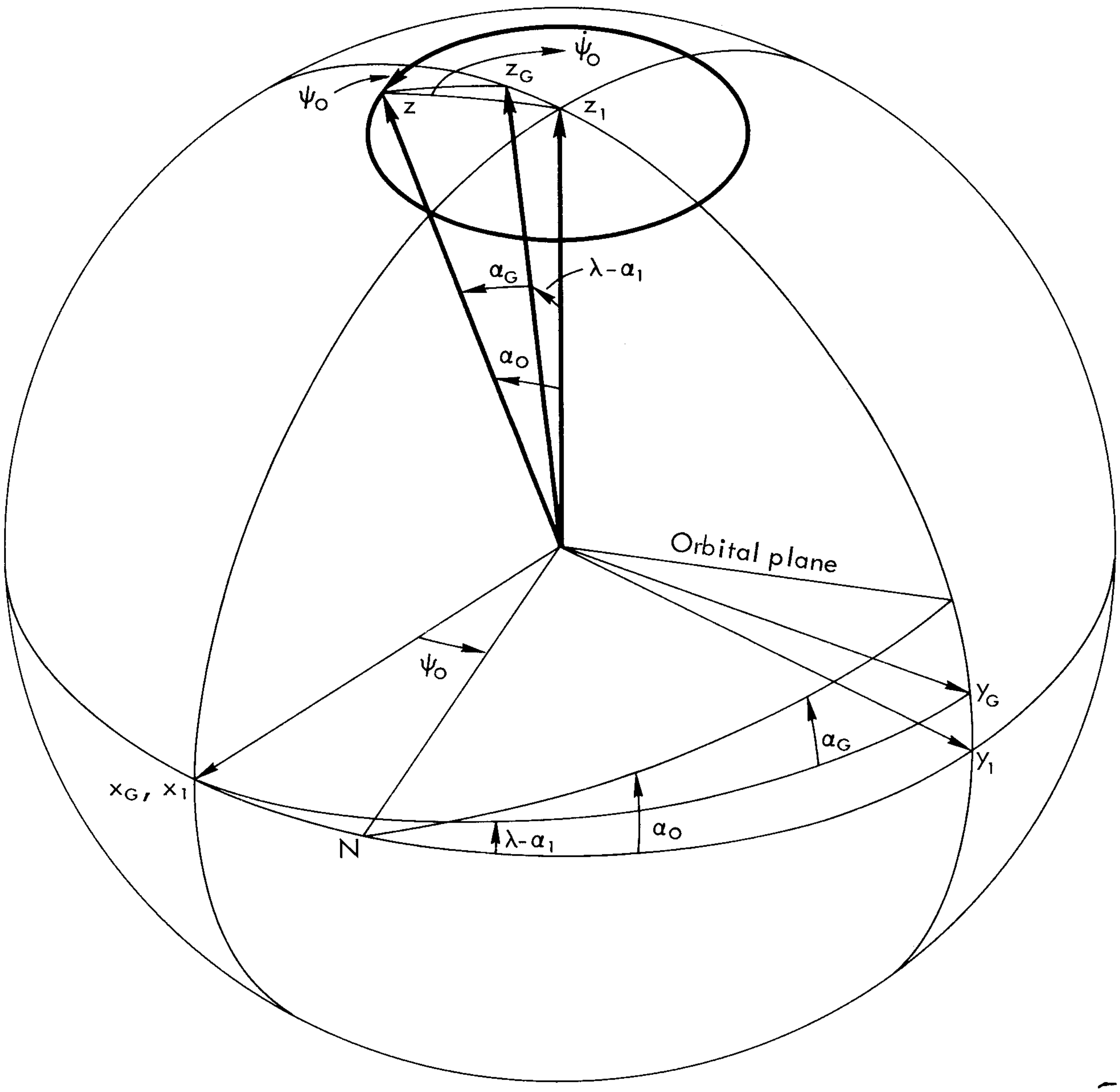

Fig. 6 — Steady-state orbital regression

in Eqs. (51) and (52) cause small variations in this trace, but, as shown in Appendix D, these residual oscillatory terms are negligible.

While the z axis maintains a fixed angle, α_o, relative to the z_1 axis during its regression, it is seen from Fig. 6 that the angle α_G between the z axis and the earth's polar axis (the z_G axis) varies between the limits

$$\alpha_{G_{min}} = |\alpha_o - (\lambda - \alpha_1)| \tag{60}$$

and

$$\alpha_{G_{max}} = \alpha_o + (\lambda - \alpha_1) \tag{61}$$

Thus, α_G, which represents the inclination of the orbital plane to the earth's equatorial plane, will vary slowly as a result of the regression about the z_1 axis.

The regression period, the time required for the z axis to make one complete rotation about z_1, is given by the relation

$$T_R = \frac{2\pi}{|\dot{\psi}_o|} \tag{62}$$

By combining Eqs. (54), (59) and (62), T_R can be determined as a function of the constant orbital inclination, α_o, relative to the reference plane and the orbital radius, r_o. This relationship is shown in Fig. 7. It is seen that in general the period increases as the orbital inclination increases, becoming infinite for α_o equal to 90^o. The period also increases with orbital radius, but appears to reach a maximum in the vicinity of 10 earth radii. Presumably, this is due to the increase in the effect

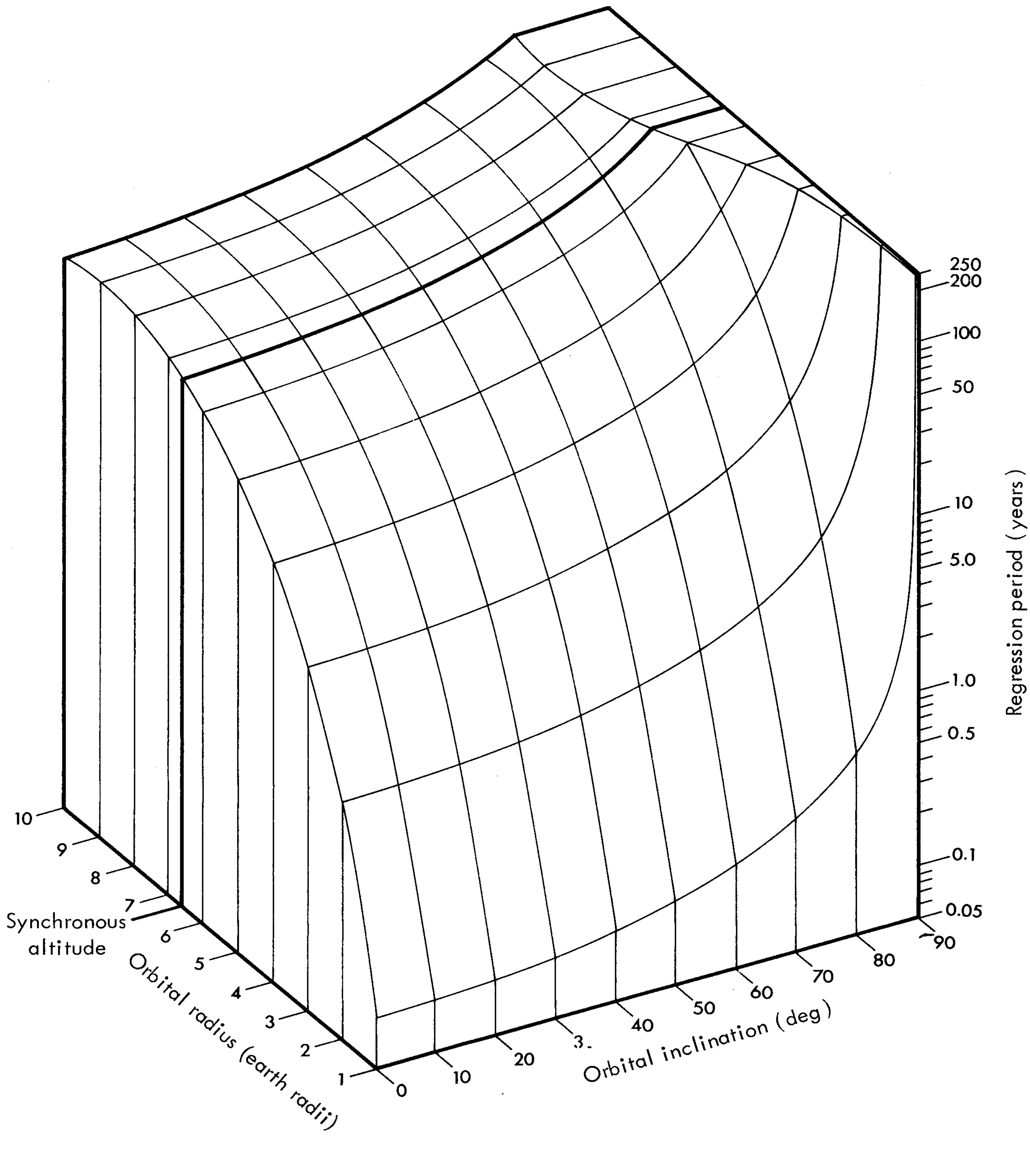

Fig.7 — Dependence of regression period on orbital inclination and orbital radius

of the sun and moon and to the corresponding decrease in the effect of earth oblateness at this altitude.

The numerical values of the regression period range from .099 year for a surface orbit at zero inclination to 52.84 years for a similar inclination at synchronous altitude. At a given altitude, the period for an inclination, α_o, is obtained by dividing the zero inclination period by cos α_o.

Special Cases

It is of interest to examine certain special cases of the expression for the orbital regression rate given by Eq. (59).

Earth Oblateness. If the oblateness of the earth is the only perturbing influence, its effect is given by setting $\dot{\Theta}$ and $\dot{\theta}_m$ equal to zero in Eqs. (54) and (59). Under these conditions, Eq. (54) gives a value of α_1 equal to λ. Thus, the reference plane is the earth's equatorial plane, and the regression takes place about the earth's polar axis. Under these same conditions, Eq. (59) reduces to

$$\dot{\psi}_o = - \frac{3J_2R_o^2\dot{\theta}_o}{2r_o^2} \cos \alpha_o \tag{63}$$

which is the usual form for the regression due to oblateness. (See Ref. 2.)

Sun and Moon. If the effect of the earth's oblateness is negligible, as in the case of large orbital radii, J_2 can be set equal to zero, with the result that the angle α_1 from Eq. (54) becomes zero. Thus, the reference plane becomes the plane of the ecliptic, and the resulting regression takes place about the normal to the ecliptic. The regression rate as given by Eq. (59) reduces to

$$\dot{\psi}_o = - \frac{3\dot{\Theta}^2 \cos \alpha_o}{4\dot{\theta}_o} - \frac{3\dot{\theta}_m^2 (2 - 3 \sin^2 \alpha_m) \cos \alpha_o}{8\mu \dot{\theta}_o} \tag{64}$$

where the first term is the solar effect and the second is that due to the moon. Numerical evaluation shows that the regression rate due to the moon is approximately twice that due to the sun.

Lunar Regression. The expressions developed in the previous section can also be used to determine the regression of the moon itself due to the influences of the earth's oblateness and the sun's gravitational attraction. As applied to the moon, Eq. (59) can be restated in the following form:

$$\dot{\psi}_m = - \frac{3\dot{\Theta}^2 \cos \alpha_m}{4\dot{\theta}_m} \left[1 + \frac{J_2 \dot{\theta}_m^2 R_o^2}{\dot{\Theta}^2 \rho_o^2} (2 - 3 \sin^2 \lambda)\right] \tag{65}$$

where the first term in the bracket is due to the solar effect and the second term is that resulting from the earth's oblateness. Actually, the oblateness term is negligible, and Eq. (65) can be rewritten as

$$\dot{\psi}_m = - \frac{3\dot{\Theta}^2 \cos \alpha_m}{4\dot{\theta}_m} \doteq - \frac{3\dot{\Theta}^2}{4\dot{\theta}_m} \tag{66}$$

By means of this relation, the regression period of the moon is found to be 17.9 years instead of the accepted 18.6 years. The reason for this discrepancy is that Eq. (66) does not include the higher order terms which are necessary in calculating the orbital regression rate of the moon. If these are included, Eq. (66) then has the form

$$\dot{\psi}_m = -\frac{3\dot{\Theta}^2}{4\dot{\theta}_m}\left[1 - \frac{3}{8}\left(\frac{\dot{\Theta}}{\dot{\theta}_m}\right) - \frac{91}{32}\left(\frac{\dot{\Theta}}{\dot{\theta}_m}\right)^2\right] \tag{67}$$

and the regression period based on this relation is the accepted 18.6 years. The details of the derivation of Eq. (67) are given in Appendix E.

While Eq. (67) is necessary in explaining the behavior of the moon, where the ratio $\dot{\Theta}/\dot{\theta}_m$ is of the order of 1/13, the higher order terms are negligible in the case of artificial satellites, for which this ratio is of the order of 1/365 or less. Thus, Eq. (66) is adequate for the purposes of this Report.

APPLICATION TO SYNCHRONOUS ORBITS

While the foregoing determination of orbital regression applies for any orbital altitude, the discussion will now be limited to synchronous altitude orbits.

Condition for Synchronism

From Eqs. (54) and (59), it is seen that for a synchronous altitude orbit (r_o = 26195.2 mi) the inclination of the reference plane, α_1, is equal to $16^{\circ}7'$, while the steady-state regression rate is given by

$$\dot{\psi}_o = \dot{\psi}(0) \cos \alpha_o \tag{68}$$

where $\dot{\psi}(0)$ has a value of -3.257×10^{-4} rad/solar day. In the case of an unperturbed equatorial satellite, synchronism is achieved by setting the orbital angular rate, $\dot{\theta}_o$, equal to the earth's spin rate relative to inertial space, $\dot{\theta}_E$. Under these conditions the satellite remains stationary above a fixed point on the equator. If this same unperturbed satellite is placed in an orbit with an inclination α_G relative to the equatorial plane, it is obvious that the subsatellite point can no longer remain fixed since it must vary in latitude between $+\alpha_G$ and $-\alpha_G$ during each orbital period. However, if the orbital angular rate is again set equal to $\dot{\theta}_E$, the resulting ground trace of the subsatellite point is a fixed curve on the rotating earth. This curve is in the form of a figure eight with its crossing point on the equator. Thus, although the satellite itself is no longer stationary relative to the earth, it's ground trace pattern is.

If perturbations are also considered with the resulting regression of the orbital plane, it is necessary to modify the orbital angular rate to compensate for this regression. The appropriate orbital angular rate can be determined from Fig. 8. This figure shows the intersections with the earth's surface of the reference, equatorial and orbital planes. It is assumed that the orbital plane is initially in the position indicated by the dotted line and that the initial subsatellite point coincides with x_1. At the end of a time t_n required for n crossings of the reference plane, the subsatellite point is at S_n at an angular distance ψ_n from x_1. In this same time the point A on the earth, which was the initial subsatellite point, is displaced from x_1 by an angular distance β_n. The quantities t_n, ψ_n and β_n can be expressed as follows

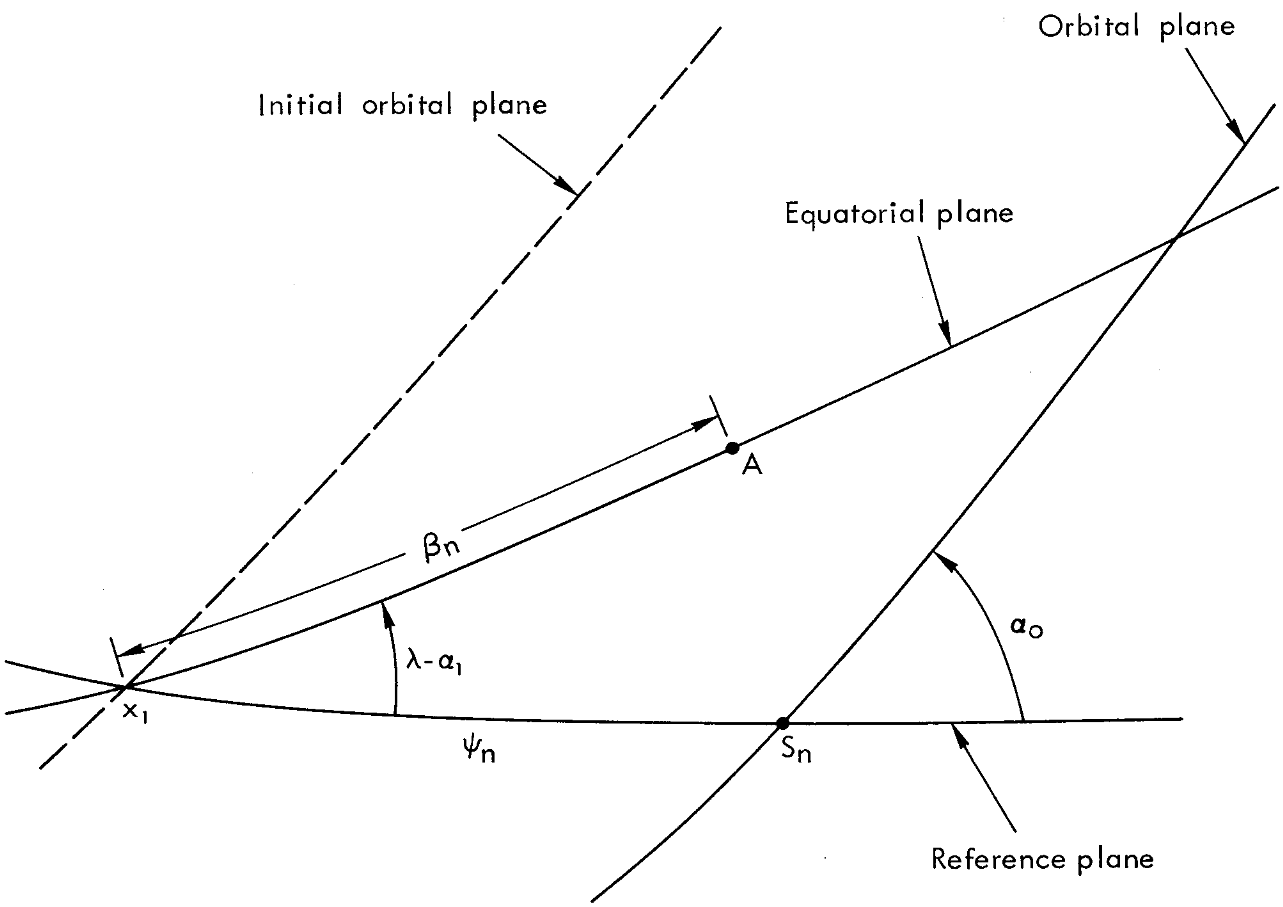

Fig. 8 — Determination of condition for synchronism

$$t_n = \frac{2n\pi}{\dot{\theta}_o} \tag{69}$$

$$\psi_n = \dot{\psi}_o t_n \tag{70}$$

$$\beta_n = \dot{\theta}_E t_n - 2n\pi \tag{71}$$

where $\dot{\theta}_o$ is the orbital angular rate to be determined. If the orbital angular rate is adjusted so that β_n is equal to ψ_n, then the subsatellite point and A will be coincident whenever the orbital plane passes through the x_1 axis (i.e., at intervals of half the regression period). The condition that β_n equals ψ_n can be expressed by combining Eqs. (69) through (71) to give

$$\dot{\theta}_o = \dot{\theta}_E - \dot{\psi}_o \tag{72}$$

With this value of orbital angular rate the resulting ground trace is again a figure eight. However, as will be seen, its size and its position relative to the rotating earth are no longer constant. Although neither the satellite nor its ground trace are stationary relative to the earth, synchronism is maintained since the motion of the ground trace pattern in longitude has no secular terms. Thus, Eq. (72) represents the condition for synchronism in this case.

From Eq. (72), it is seen that the desired synchronous orbital angular rate depends on the orbital inclination through $\dot{\psi}_o$. However, this variation in $\dot{\theta}_o$ corresponds to a change of the order of .2 mi in orbital radius. This change in r_o is negligible insofar as the determination of either α_1 or $\dot{\psi}_o$ is concerned.

Determination of the Ground Trace

The geometry for the determination of the satellite ground trace is shown in Fig. 9, where the angle β is the inertial longitude of the satellite measured from the x_1 axis and γ is the satellite latitude.

The unit vector $\bar{i}$ along the satellite radius vector can be expressed relative to the x_G, y_G, z_G coordinate system in the form

$$\bar{i} = (\cos\gamma \cos\beta)\, \bar{i}_G + (\cos\gamma \sin\beta)\, \bar{j}_G + (\sin\gamma)\, \bar{k}_G \tag{73}$$

The same unit vector can be expressed relative to the x_1, y_1, z_1 system as

$$\bar{i} = a_x \bar{i}_1 + b_x \bar{j}_1 + c_x \bar{k}_1 \tag{74}$$

where a_x, b_x and c_x are given in Appendix A.

If Eq. (74) is projected into the x_G, y_G, z_G system it becomes

$$\bar{i} = a_x \bar{i}_G + \left[b_x \cos(\lambda - \alpha_1) + c_x \sin(\lambda - \alpha_1)\right] \bar{j}_G$$

$$+ \left[- b_x \sin(\lambda - \alpha_1) + c_x \cos(\lambda - \alpha_1)\right] \bar{k}_G \tag{75}$$

By equating components of Eqs. (73) and (75), the following expressions for β and γ are obtained.

$$\tan\beta = \frac{b_x \cos(\lambda - \alpha_1) + c_x \sin(\lambda - \alpha_1)}{a_x} \tag{76}$$

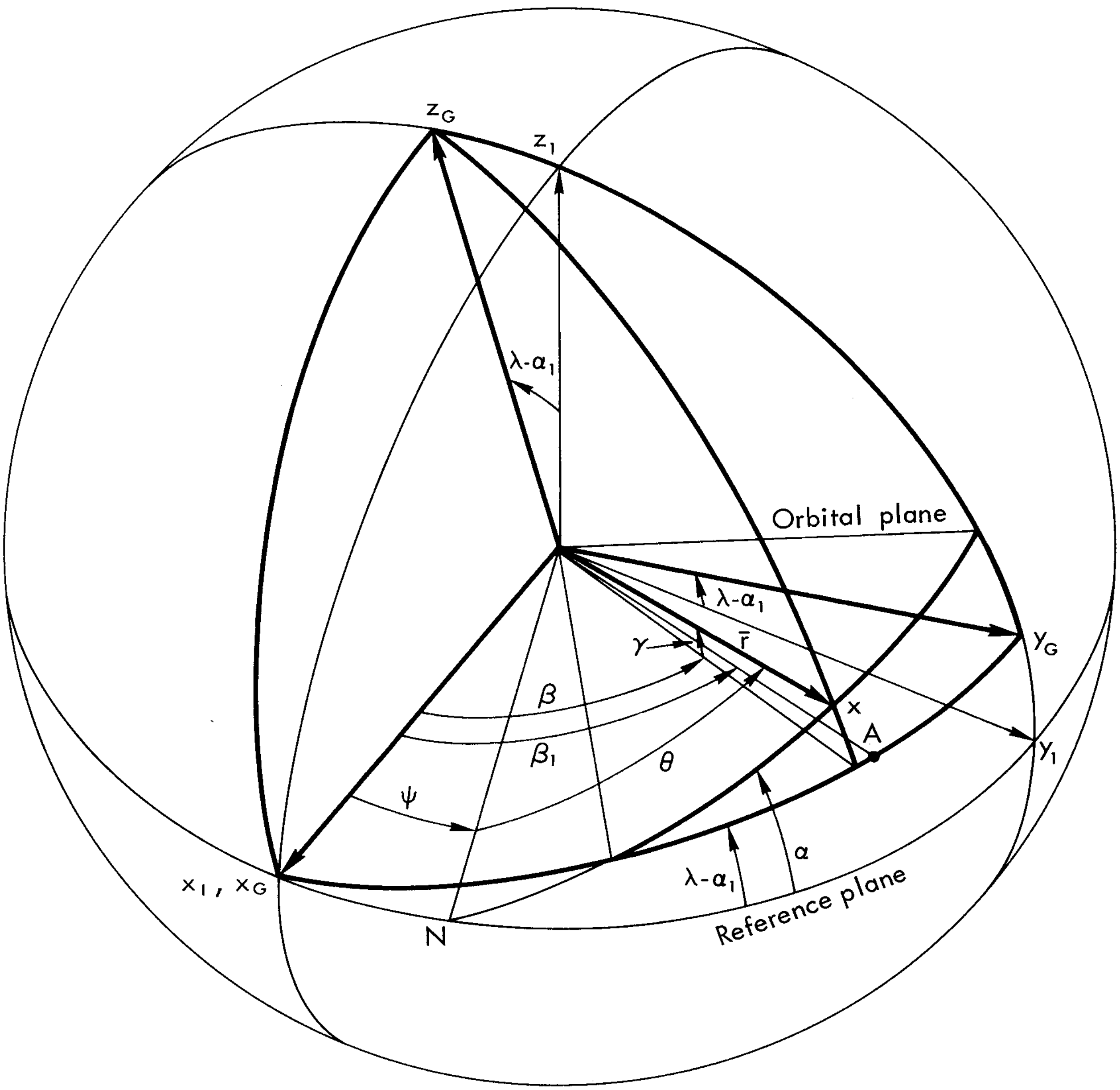

Fig.9 — Ground trace determination

$$\sin \gamma = - b_x \sin(\lambda - \alpha_1) + c_x \cos(\lambda - \alpha_1) \tag{77}$$

In Fig. 10 it is assumed that initially the orbital plane passes through x_1 and the satellite is at x_1. The figure then represents the geometry at some later time, T, corresponding to some integral number of equatorial crossings. S represents the satellite position and A is the current position of the initial subsatellite point. The value of θ_o can be expressed in terms of ψ_o by setting γ equal to zero and replacing ψ and θ by ψ_o and θ_o in Eq. (77). Solution of the resulting expression gives

$$\tan \theta_o = \frac{\sin \psi_o \sin(\lambda - \alpha_1)}{\sin \alpha_o \cos(\lambda - \alpha_1) - \cos \alpha_o \cos \psi_o \sin(\lambda - \alpha_1)} \tag{78}$$

where

$$\psi_o = \dot{\psi}_o T \tag{79}$$

From Fig. 10, it is seen that the reference plane crossing immediately prior to T occurs at time T - t_o, where

$$t_o = \frac{\theta_o}{\dot{\theta}_o} \tag{80}$$

and $\dot{\theta}_o$ is given by Eq. (72).

If t is the time elapsed since this reference plane crossing, then the general expression for ψ in Fig. 9 is given by

$$\psi = \dot{\psi}_o (T - t_o + t) \tag{81}$$

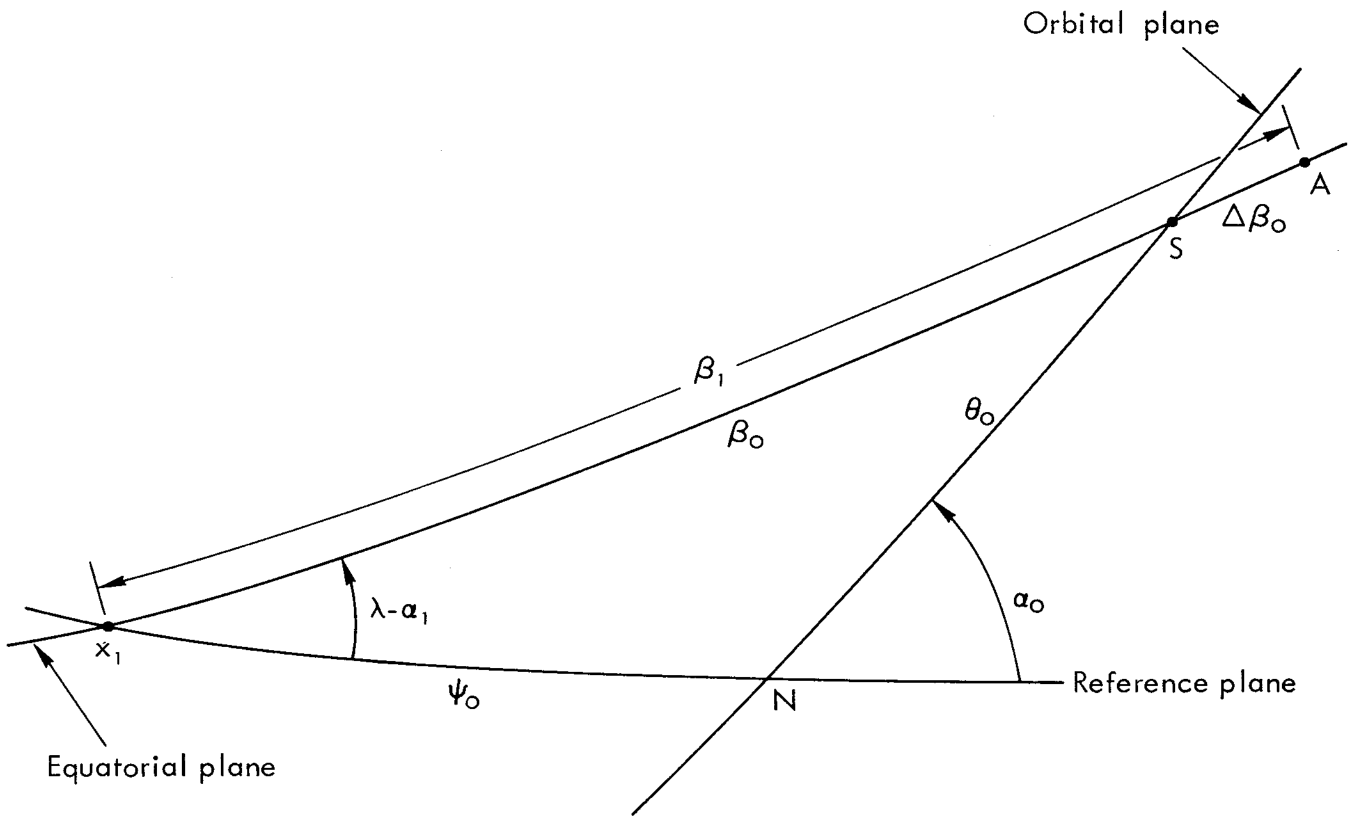

Fig.10 — Equatorial crossing geometry

The angle θ in Fig. 9 can also be expressed as

$$\theta = \dot{\theta}_o(T - t_o + t) - 2n\pi \tag{82}$$

where n is the number of orbits completed at the time of reference plane crossing. Since n is given by

$$n = \frac{\dot{\theta}_o(T - t_o)}{2\pi} \tag{83}$$

Eq. (82) can be expressed as

$$\theta = \dot{\theta}_o t \tag{84}$$

If β_1 represents the current inertial longitude of the initial sub-satellite point, A, it can be expressed as

$$\beta_1 = \dot{\theta}_E(T - t_o + t) - 2n\pi \tag{85}$$

Elimination of $\dot{\theta}_o$ and n between Eqs. (72), (83) and (85) gives

$$\beta_1 = \dot{\theta}_E t + \dot{\psi}_o(T - t_o) \tag{86}$$

If Eqs. (72), (81), (84) and (86) are combined, the following value for β_1 is obtained:

$$\beta_1 = \theta + \psi \tag{87}$$

The longitude of the satellite relative to the reference point A on the rotating earth is given by

$$\Delta\beta = \beta - \beta_1 \tag{88}$$

while its latitude is equal to γ. The relative longitude, $\Delta\beta$, and the latitude, γ, can be computed as a function of t by means of Eqs. (76)-(81), (84), (87) and (88).

For a given value of T, a determination of $\Delta\beta$ and γ versus t over one orbital period determines the ground trace on the rotating earth. By taking various values of T over one regression period, the effect of the phase of the regression can be shown for various initial orbital inclinations as follows.

Orbit in Reference Plane. In this case, the determination of $\Delta\beta$ and γ can be simplified considerably since α_o is zero and t_o is equal to T. Thus, Eqs. (76), (77), (87) and (88) become

$$\tan \beta = \cos(\lambda - \alpha_1) \tan \dot{\theta}_E t \tag{89}$$

$$\sin \gamma = - \sin(\lambda - \alpha_1) \sin \dot{\theta}_E t \tag{90}$$

$$\beta_1 = \dot{\theta}_E t \tag{91}$$

$$\Delta\beta = \beta - \beta_1 \tag{92}$$

The resulting plot of γ versus $\Delta\beta$ is shown in Fig. 11, where it is seen that the maximum value of latitude is equal to $\lambda - \alpha_1$ or $\pm 7^\circ 20'$, which is the inclination of the orbital plane to the equatorial plane. Since the time T does not appear in Eqs. (89) through (92), the ground trace as shown will contine to repeat since this particular orbit remains fixed in inertial space.

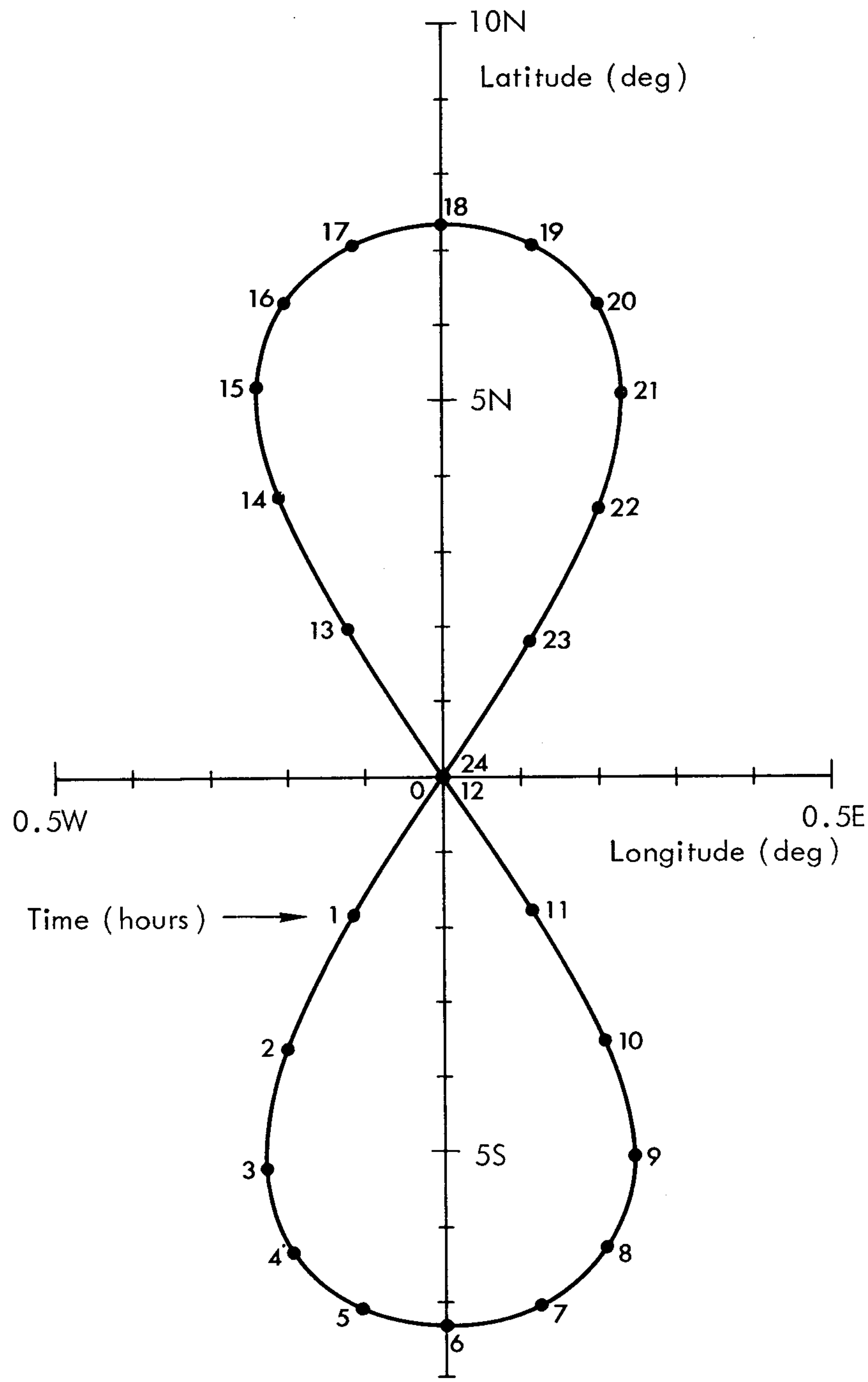

Fig. 11 — Ground trace (orbit in reference plane, $\alpha_o = 0$)

Orbital Plane Normal to Reference Plane. In this case, the equations for the determination of $\Delta\beta$ and γ are simplified by setting α_o equal to 90^o. Under these conditions, the ground trace is determined by the following relations:

$$\tan \beta = \frac{\sin(\lambda - \alpha_1) \sin \dot{\theta}_E t + \cos(\lambda - \alpha_1) \sin \psi_o \cos \dot{\theta}_E t}{\cos \psi_o \cos \dot{\theta}_E t} \tag{93}$$

$$\sin \gamma = - \sin(\lambda - \alpha_1) \sin \psi_o \cos \dot{\theta}_E t + \cos(\lambda - \alpha_1) \sin \dot{\theta}_E t \tag{94}$$

$$\beta_1 = \dot{\theta}_E t - \theta_o + \psi_o \tag{95}$$

$$\tan \theta_o = \sin \psi_o \tan(\lambda - \alpha_1) \tag{96}$$

$$\Delta\beta = \beta - \beta_1 \tag{97}$$

As shown in Appendix F, ψ_o may oscillate about a value of either 90^o or 270^o, with an amplitude less than 90^o and a period greater than 269 years.

By means of these equations, the ground traces shown in Fig. 12 have been computed for ψ_o equal to 0^o, $\pm 90^o$ and 180^o. It is seen that as $|\psi_o|$ increases from 0^o to 180^o the amplitude of the figure eight increases from $90 - (\lambda - \alpha_1)$, or $82^o 40'$, to $90 + (\lambda - \alpha_1)$, or $102^o 40'$. At the same time the maximum latitude ranges from $82^o 40'$ at $|\psi_o|$ equal to 0^o,

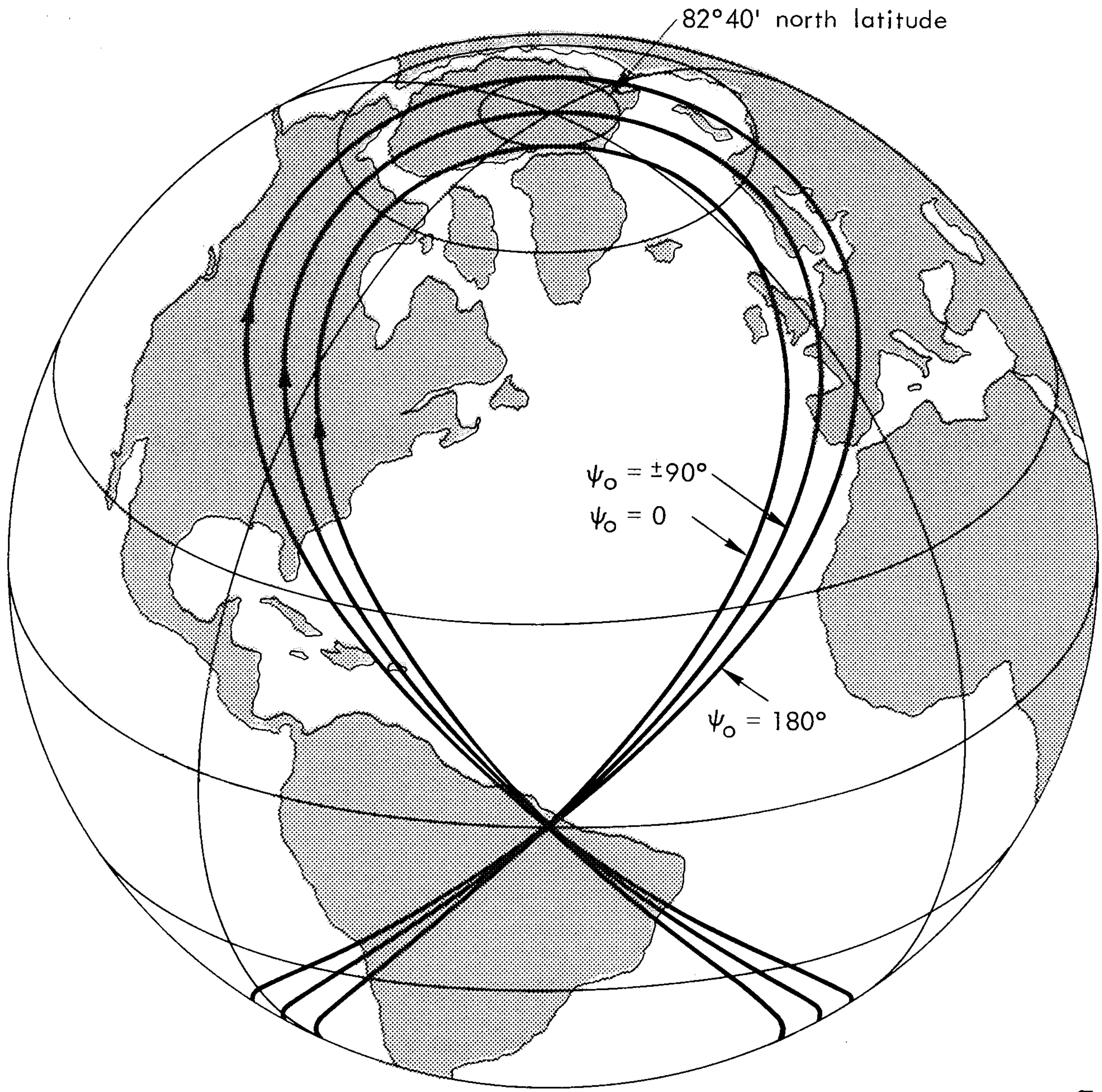

Fig. 12 — Ground trace (orbital plane normal to reference plane, $\alpha_o = 90°$)

to 90° when $|\psi_o|$ equals 90°, and back to $82^{\circ}40'$ for $|\psi_o|$ equal to 180°. The longitude variations for $|\psi_o| < 90^{\circ}$ are oscillatory in nature, but, for $|\psi_o| > 90^{\circ}$, the longitude $\Delta\beta$ increases monotonically. This is due to the fact that the ground trace encircles the earth's axis in a negative sense for these latter conditions.

It should be noted that only the ground trace for ψ_o equal to 90° is actually fixed relative to the earth. Those corresponding to other initial values of ψ_o are subject to large amplitude oscillations in longitude.

Orbit in Equatorial Plane. In this case the orbital plane is initially in the earth's equatorial plane, with α_o set equal to $\lambda - \alpha_1$, which is equal to $7^{\circ}20'$. From Eq. (59), it can be shown that the orbital regression rate is equal to -3.231×10^{-4} rad/solar day, which corresponds to a regression period of 53.249 years. The ground trace can now be determined by means of Eqs. (76) through (78). Since the trace changes as the orbit regresses, the computation is made at five-year intervals in T up to 25 years. The resulting patterns are presented in Fig. 13, where it is seen that during the first half of the regression period the ground trace grows from a single point to a figure-eight pattern which attains its maximum size after half of one regression period. At this time, the ground trace has dimensions of $\pm 14^{\circ}40'$ in latitude and $\pm 1^{\circ}$ in longitude. In addition to the variation in size of the earth trace, its equatorial crossing moves relative to the origin, which is the initial subsatellite point. During the first half of the regression period, the equatorial crossing moves to the east, reaching a maximum displacement of about $.6^{\circ}$, which then decreases to zero after half the regression period. The behavior of the ground trace during the second half of the

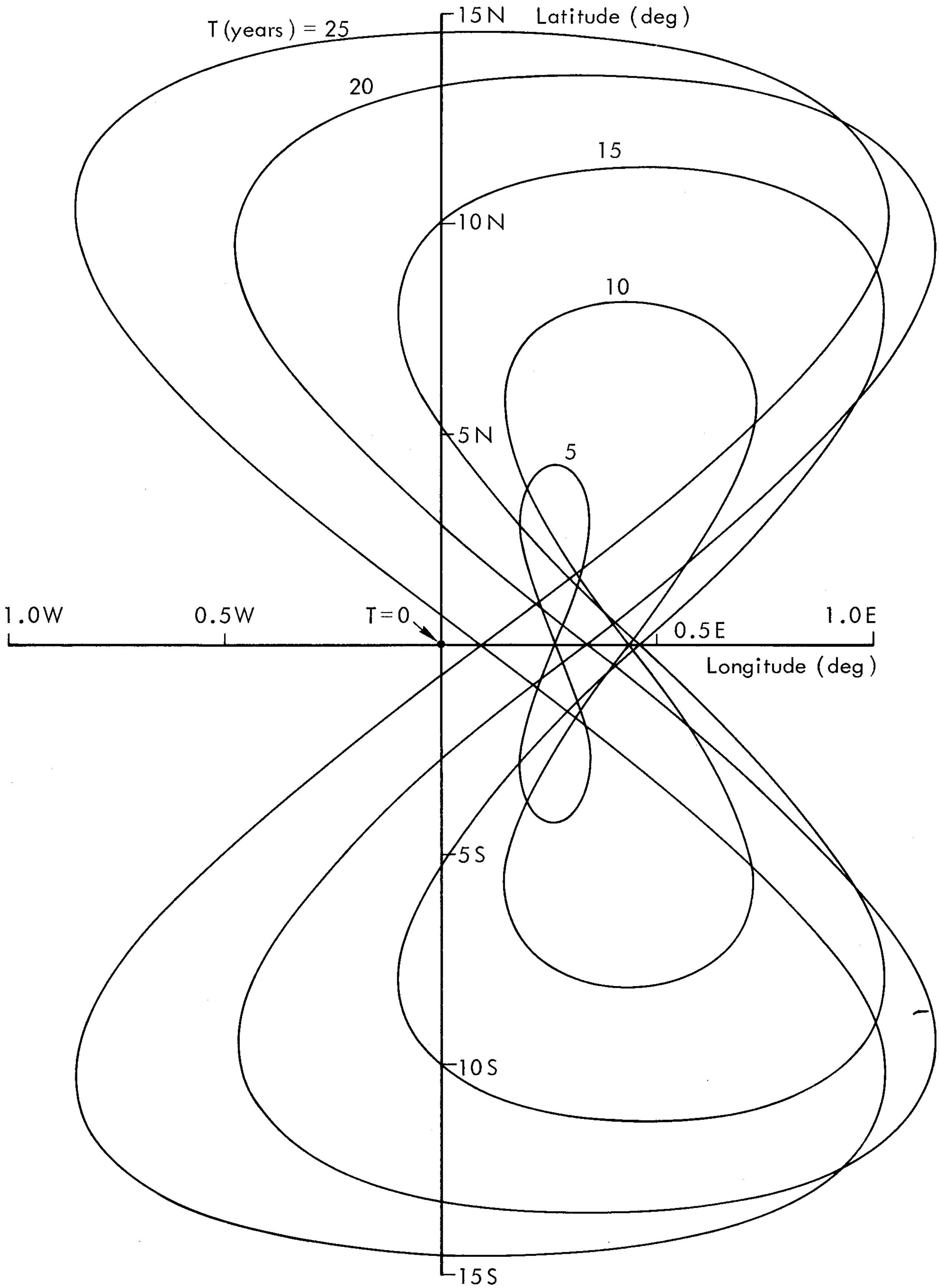

Fig. 13—Ground trace (orbital plane initially equatorial, $\alpha_o = 7° \, 20'$)

regression period is the reverse of that just described in that the pattern decreases in size and degenerates to a point at the end of one regression period. Also, the equatorial crossing drifts toward the west, reaching a maximum displacement of $.6^{\circ}$ and returning to 0° at the end of the regression period. The behavior of the ground trace during the second half of the regression cycle is a mirror image of that shown in Fig. 13.

If the geometrical representation used in Fig. 6 is adapted to this case, the relationship between the z, z_1 and z_G axes and the trace of the z axis on the reference sphere is shown in Fig. 14. Since α_o and $\lambda - \alpha_1$ are equal, the earth's polar axis intersects the z axis trace. Also, the arc of the great circle through z and z_G is a measure of the instantaneous inclination, α_G, of the orbit relative to the earth's equatorial plane. Thus, as the z axis moves around its trace, the value of α_G varies from 0° at z_G to a maximum of $2(\lambda - \alpha_1)$ or $14^{\circ}40'$ at a point diametrically opposite to z_G. This instantaneous value of α_G is equal to the maximum latitude excursion of the ground traces of Fig. 13.

Orbit at an Arbitrary Inclination. As an example of the general behavior of an inclined synchronous orbit, a value of 30° for α_o is selected. This inclination corresponds to a regression rate, $\dot{\psi}_o$, of -2.821×10^{-4} rad/solar day, which results in a regression period of 60.98 years. The resulting earth traces at five-year intervals up to 30 years are shown in Fig. 15. It is seen that initially the characteristic figure-eight pattern has an amplitude in latitude of $\alpha_o - (\lambda - \alpha_1)$ or $22^{\circ}40'$. During the first half of the regression period, this amplitude increases to a maximum equal to $\alpha_o + (\lambda - \alpha_1)$ or $37^{\circ}20'$. During this time

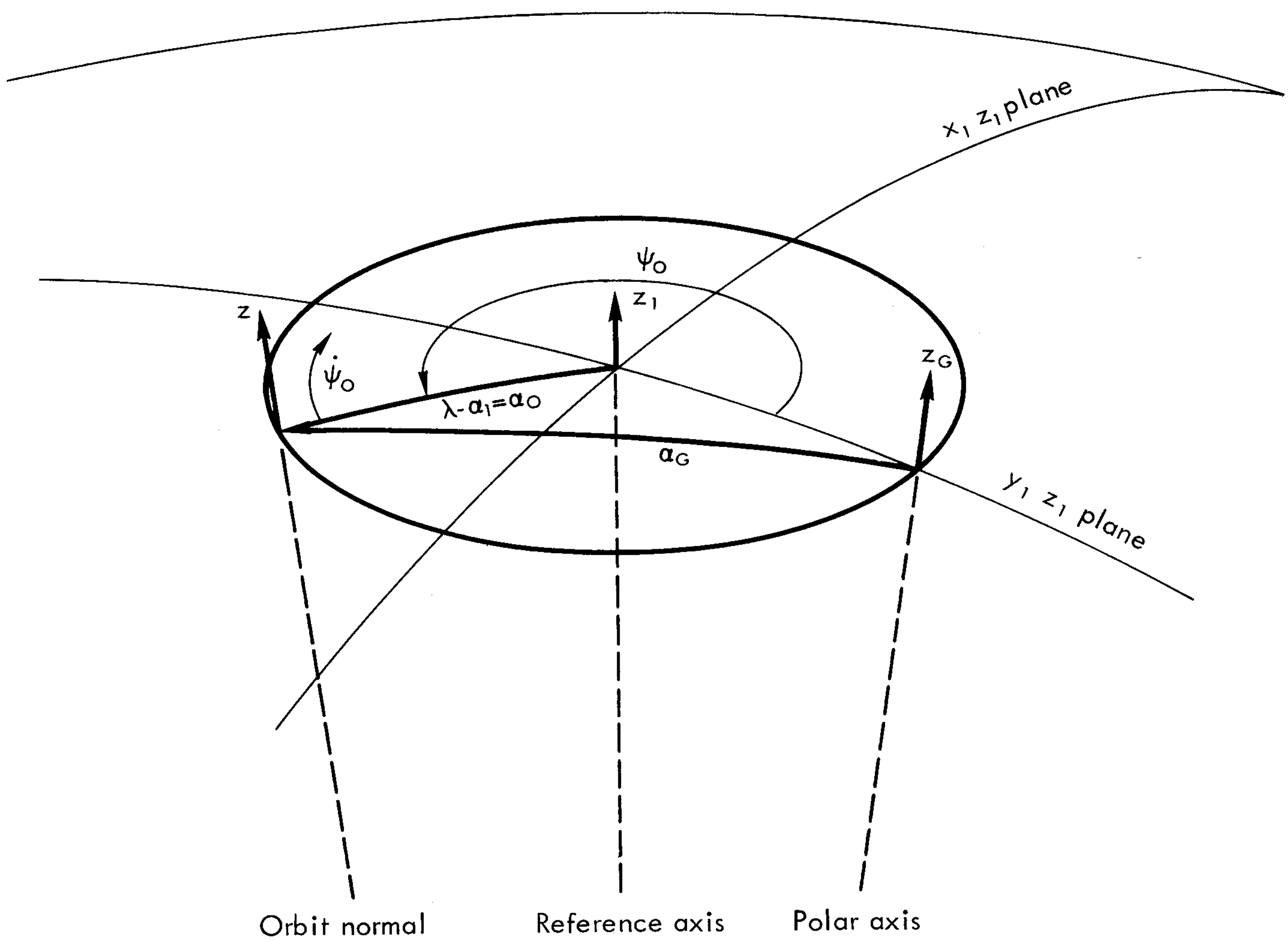

Fig. 14— Regression of a synchronous equatorial satellite

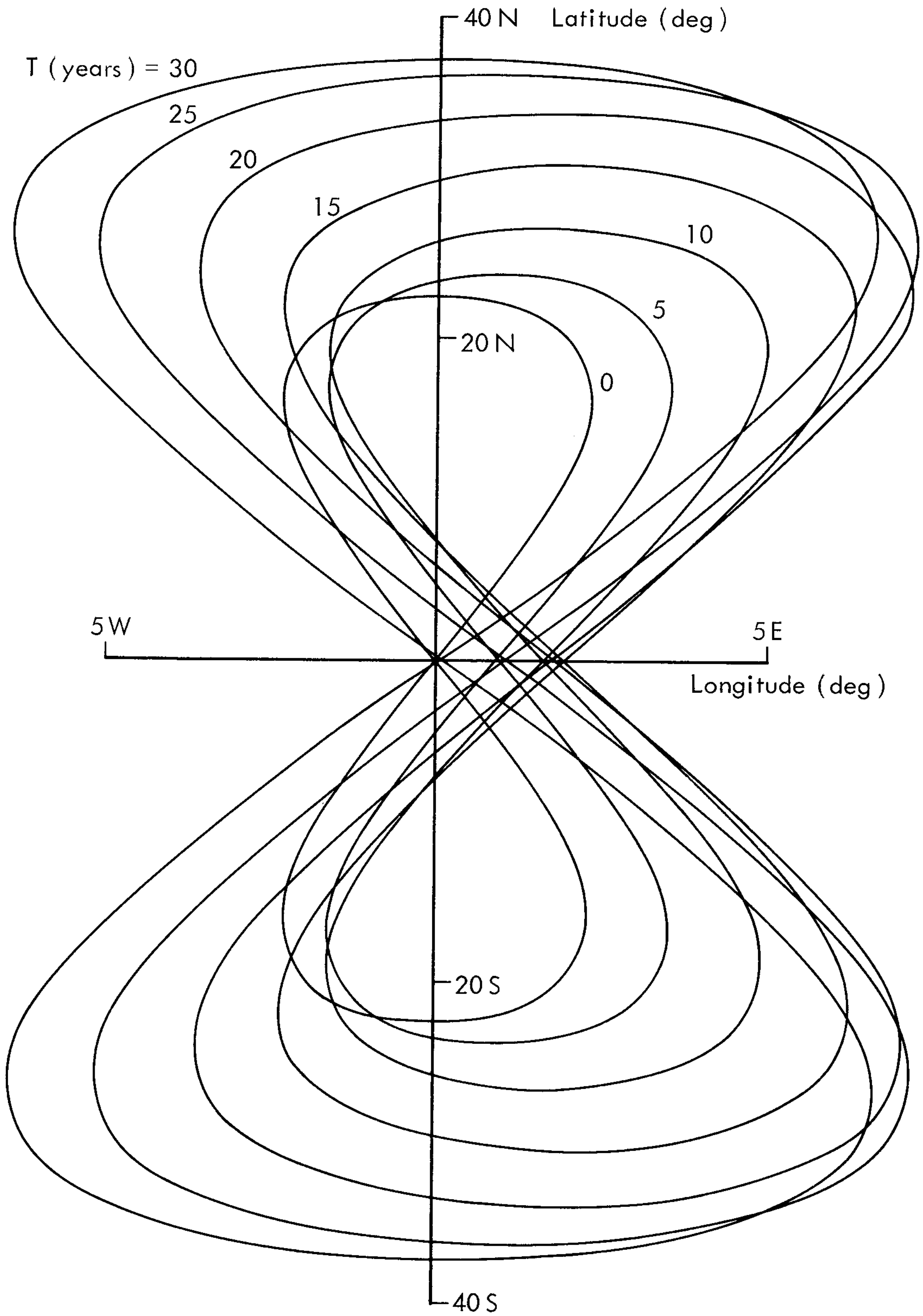

Fig. 15 — Ground trace ($\alpha_o = 30°$)

the equatorial crossing point, which is initially at the origin, drifts in an easterly direction, reaching a maximum of about 2^o, after which the drift reverses and the crossing point returns to 0^o at the end of one-half of the regression period. As in the previous case, the behavior of the ground trace during the second half of the regression period is a mirror image of that during the first half.

The relationship of the z, z_1 and z_G axes is shown in Fig. 16 for this case. It is seen that the instantaneous orbital inclination, α_G, relative to the equatorial plane lies between $\alpha_o - (\lambda - \alpha_1)$ and $\alpha_o + (\lambda - \alpha_1)$, which are also the limits of the maximum daily latitude excursions of the ground traces of Fig. 15. In general, this relationship can be stated as

$$|\alpha_o - (\lambda - \alpha_1)| \le \alpha_G \le \alpha_o + (\lambda - \alpha_1) \tag{98}$$

Size and Position of the Ground Trace

It is seen from the previous examples that the size and position of the ground trace are functions of both the orbital inclination, α_o, and the time. In order to describe these variations, it is convenient to determine both the maximum latitude of the ground trace and the position of the equatorial crossing as functions of α_o and time.

Variation of Maximum Latitude. Since the maximum latitude during a single traversal of the ground trace is equal to the inclination, α_G, of the orbit to the equatorial plane, this maximum latitude, γ_m, can be expressed as

$$\cos \gamma_m = (\bar{k} \cdot \bar{k}_G) \tag{99}$$

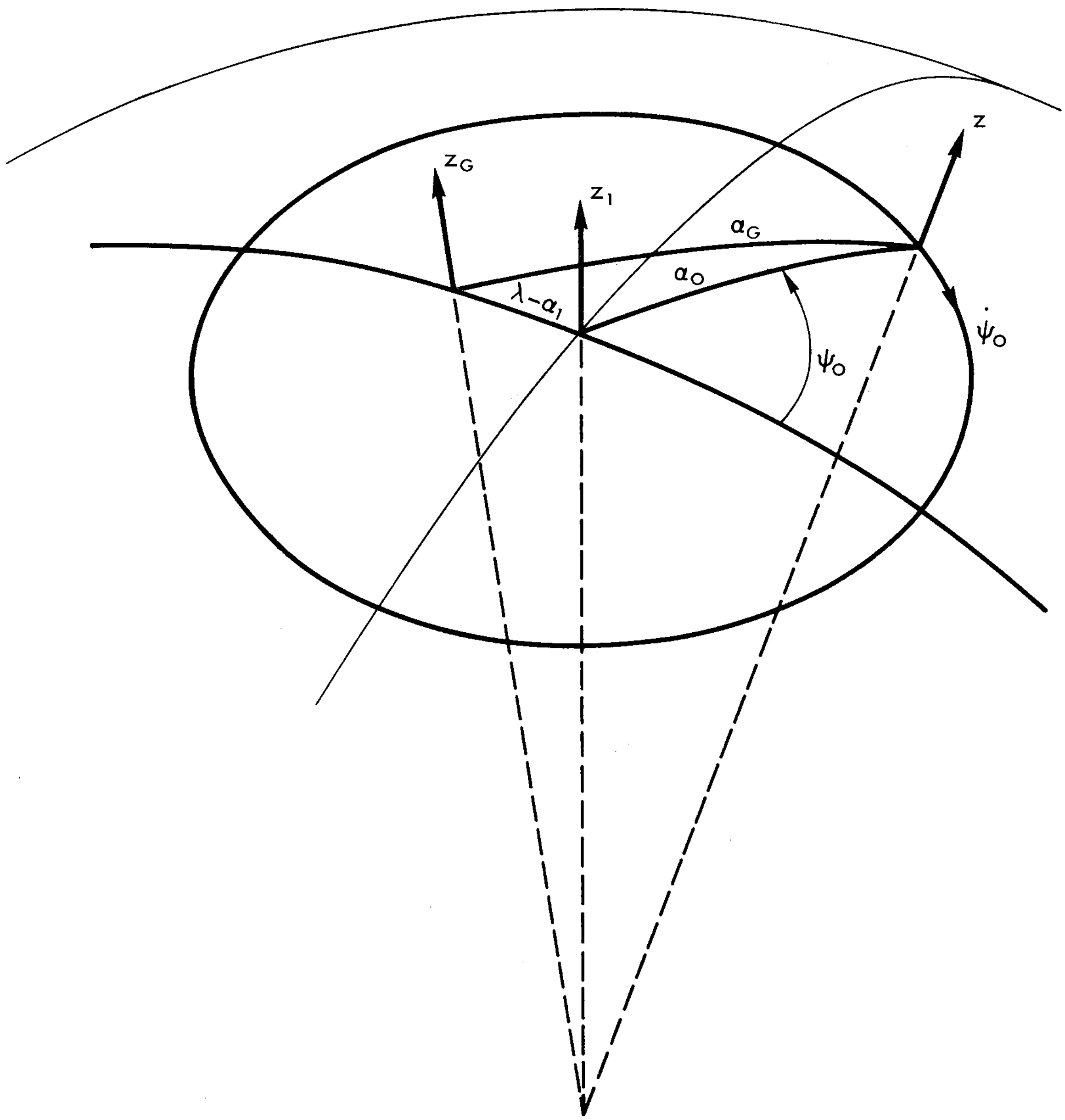

Fig. 16 — Regression of a synchronous satellite (α_o = 30°)

which can be expressed from Appendix A as

$$\cos \gamma_m = \cos \alpha_o \cos(\lambda - \alpha_1) + \sin \alpha_o \sin(\lambda - \alpha_1) \cos \psi \tag{100}$$

where

$$\psi = \dot{\psi}_o T \tag{101}$$

and $\dot{\psi}_o$ is given by Eq. (59).

The dependence of γ_m on T and α_o is shown in Fig. 17, where T has been normalized by dividing by T_R. It is seen that the maximum possible variation of γ_m for a given value of α_o is $14^o40'$. In addition, for those contours for $\alpha_o > 82^o40'$, the central portion corresponds to orbits which encircle the earth's pole in a negative direction.

A further examination of Fig. 17 shows that, with the exception of the contour for $\alpha_o = \lambda - \alpha_1$, all of the α_o contours have a zero slope at T = 0. However, for $\alpha_o = \lambda - \alpha_1$ the slope is $.8630^o$ per year, which also represents the initial rate of change of the orbital inclination relative to the equatorial plane.

Figure 18 shows both this variation of γ_m for α_o equal to $\lambda - \alpha_1$, and the equivalent curve from Fig. 10 in Ref. 1. It is seen that the present analysis gives a value of 53.249 years for the period of the latitude variation and a maximum amplitude of $14^o40'$, compared with the values of 73.6 years and 20^o obtained by the more approximate methods of Ref. 1.

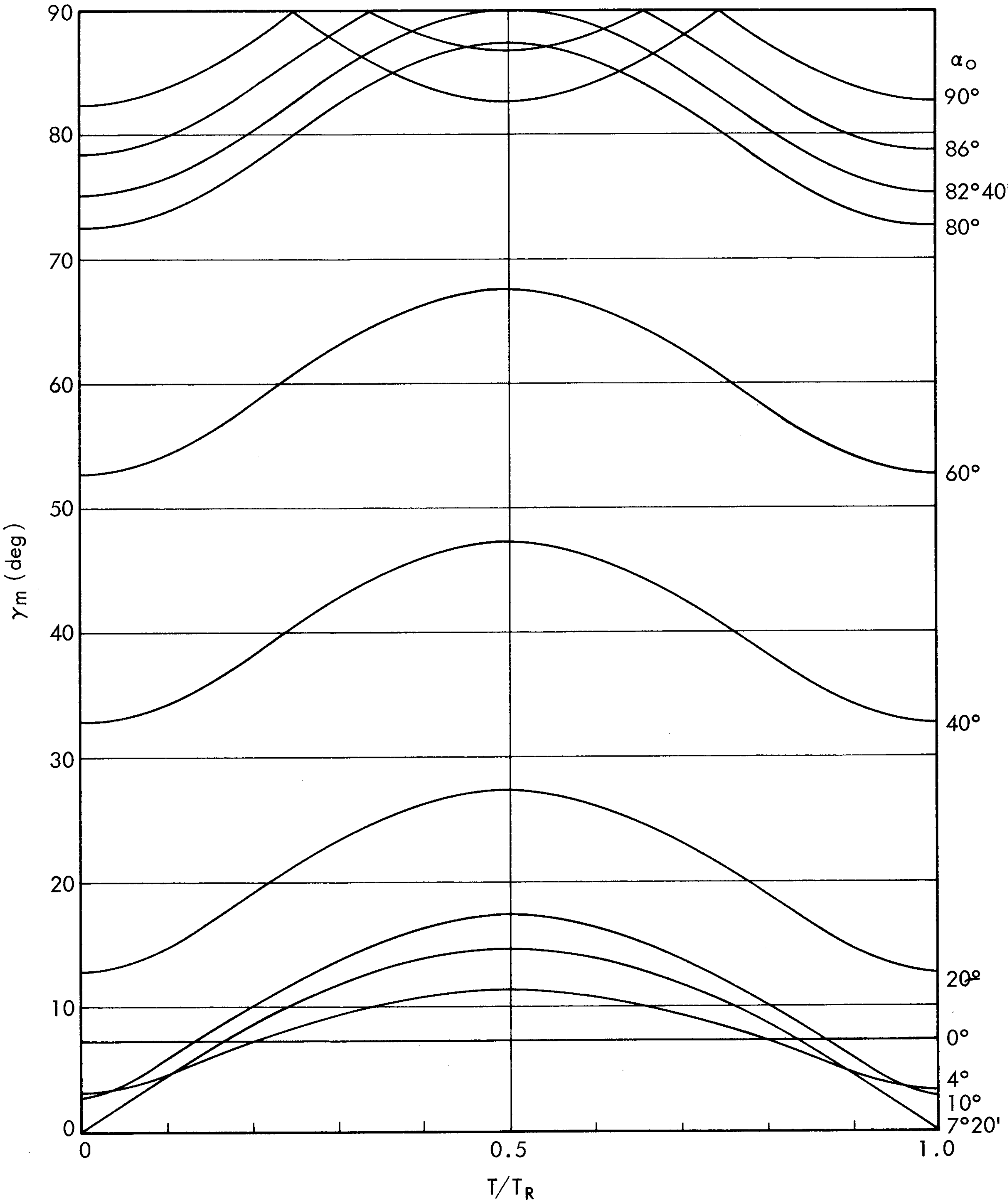

Fig.17 — Maximum latitude as a function of α_o and T

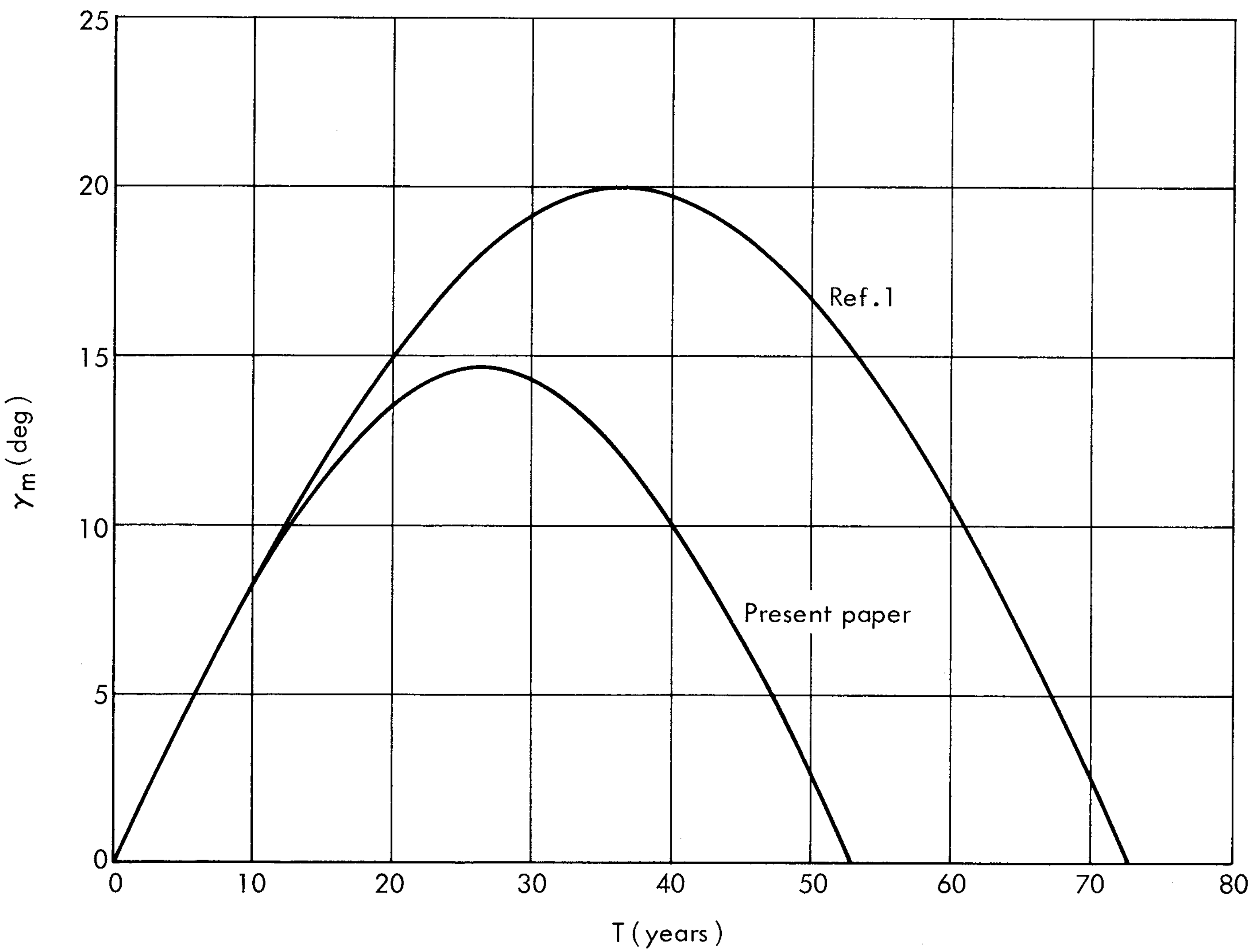

Fig. 18 — Maximum latitude as a function of time for an initially equatorial orbit

Equatorial Crossing Position. The variation in the position of the equatorial crossing point as a function of time and orbital inclination can be determined as follows. At the time of equatorial crossing, the inertial longitude of the satellite can be obtained from Eq. (76) in the form

$$\tan \beta_o = \frac{\left[\begin{array}{l}\cos \theta_o \sin \psi_o \cos(\lambda-\alpha_1) \\ + \sin \theta_o \left[\sin \alpha_o \sin(\lambda-\alpha_1) + \cos \alpha_o \cos(\lambda-\alpha_1) \cos \psi_o\right]\end{array}\right]}{\cos \theta_o \cos \psi_o - \sin \theta_o \cos \alpha_o \sin \psi_o} \tag{102}$$

Elimination of θ_o between Eqs. (78) and (102) gives

$$\tan \beta_o = \frac{\sin \psi_o \sin \alpha_o}{- \cos \alpha_o \sin(\lambda - \alpha_1) + \sin \alpha_o \cos(\lambda - \alpha_1) \cos \psi_o} \tag{103}$$

The longitude of the initial subsatellite point is given by Eq. (87) as

$$\beta_{1o} = \theta_o + \psi_o \tag{104}$$

Thus, the longitude difference between the current equatorial crossing and the initial subsatellite point is given by

$$\Delta\beta_o = \beta_o - \theta_o - \psi_o \tag{105}$$

By means of Eqs. (79), (103) and (105), $\Delta\beta_o$ can be computed as a function of α_o and the time T. The resulting relation is shown in Fig. 19, where T has been normalized as in Fig. 17. It is seen that the resulting surface is such that for a given value of α_o the quantity $\Delta\beta_o$ has a maximum at approximately one-quarter of the regression period and a minimum at

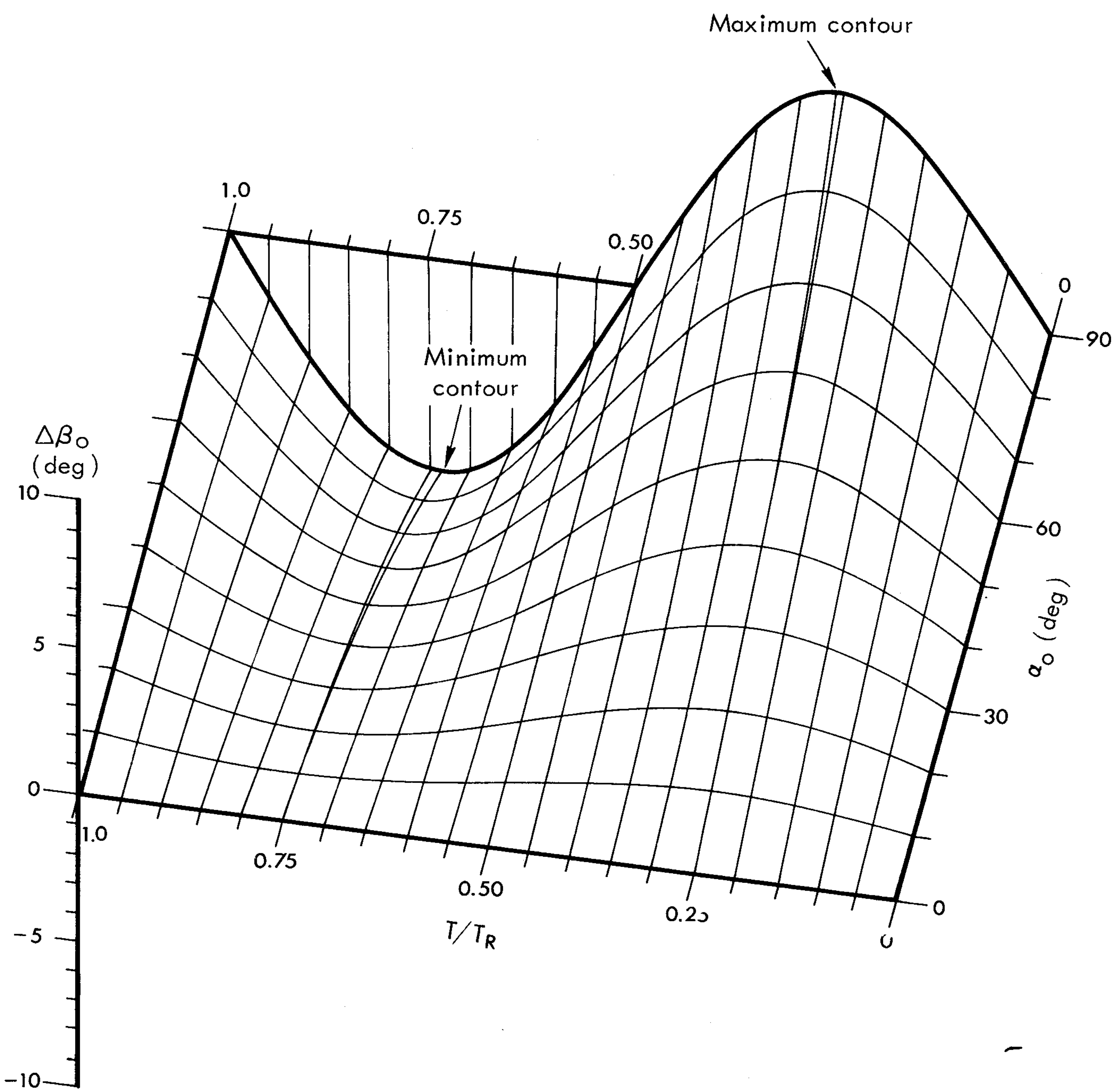

Fig. 19 — Longitude of equatorial crossing as a function of α_o and T

about three fourths of this period. The value of this maximum ranges from zero at α_o equal to 0^o to approximately $\lambda - \alpha_1$ at α_o equal to 90^o. The exact expression for this maximum and its time of occurrence can be found by differentiating Eq. (105) with respect to time and equating to zero. This results in

$$\Delta\beta_{o_{max}} = \tan^{-1}\left[\frac{8 \sin \frac{\alpha_o}{2} \sin\left(\frac{\lambda-\alpha_1}{2}\right)\sqrt{\frac{\cos \alpha_o + \cos(\lambda-\alpha_1)}{2}}}{[3 \cos(\lambda-\alpha_1) + 3 \cos \alpha_o - \cos \alpha_o \cos(\lambda-\alpha_1) - 1]}\right] \tag{106}$$

and

$$\frac{T_{max}}{T_R} = \frac{1}{2\pi} \cos^{-1}\left[- \tan \frac{\alpha_o}{2} \tan\left(\frac{\lambda - \alpha_1}{2}\right)\right] \tag{107}$$

It is seen from Eq. (107) that the maximum occurs very slightly past one-quarter of the regression period. In a similar manner it can be shown that the minimum of $\Delta\beta_o$ has a magnitude equal to the negative of Eq. (106) and a time of occurence given by Eq. (107) as slightly before three-quarters of the regression period. Contours for these maxima and minima are shown in Fig. 19.

It should be noted that Fig. 19 applies only to regression about the z_1 axis and that the effect would actually be superposed on the shorter period oscillation due to equatorial ellipticity described in Ref. 1.

Period Between Equatorial Crossings

The nodical period is defined as the time interval between crossings of the reference plane at the ascending node and is given by

$$\tau_o = \frac{2\pi}{\dot{\theta}_o} = \frac{2\pi}{\dot{\theta}_E - \dot{\psi}_o} \tag{108}$$

However, the period between successive equatorial crossings is not necessarily equal to τ_o when orbital regression is present. The amount of this deviation from τ_o can be determined as follows. Figure 20 shows the position of the orbital plane at the times of two successive crossings of the equatorial plane. If these two crossing times are T and $T + \tau_o + \Delta\tau$, then the associated values of ψ are ψ_o and $\psi_o + \Delta\psi_o$, and those for θ are θ_o and $\theta_o + \Delta\theta_o$. The value of the increment $\Delta\psi_o$ is given by

$$\Delta\psi_o = \dot{\psi}_o(\tau_o + \Delta\tau) \tag{109}$$

On the other hand, since the angle θ increases by approximately 2π between two equatorial crossings, the angle $\Delta\theta_o$ can be determined from the relation

$$\theta_o + \Delta\theta_o = \theta_o + \dot{\theta}_o(\tau_o + \Delta\tau) - 2\pi \tag{110}$$

which together with Eq. (108) gives

$$\Delta\theta_o = \dot{\theta}_o\Delta\tau \tag{111}$$

At the time of the first equatorial crossing, T, the latitude is 0^o. Thus, substition of ψ_o and θ_o for ψ and θ in Eq. (77) gives

$$- \sin(\lambda - \alpha_1) \left[\cos\theta_o \sin\psi_o + \sin\theta_o \cos\psi_o \cos\alpha_o\right]$$
$$+ \sin\theta_o \sin\alpha_o \cos(\lambda - \alpha_1) = 0 \tag{112}$$

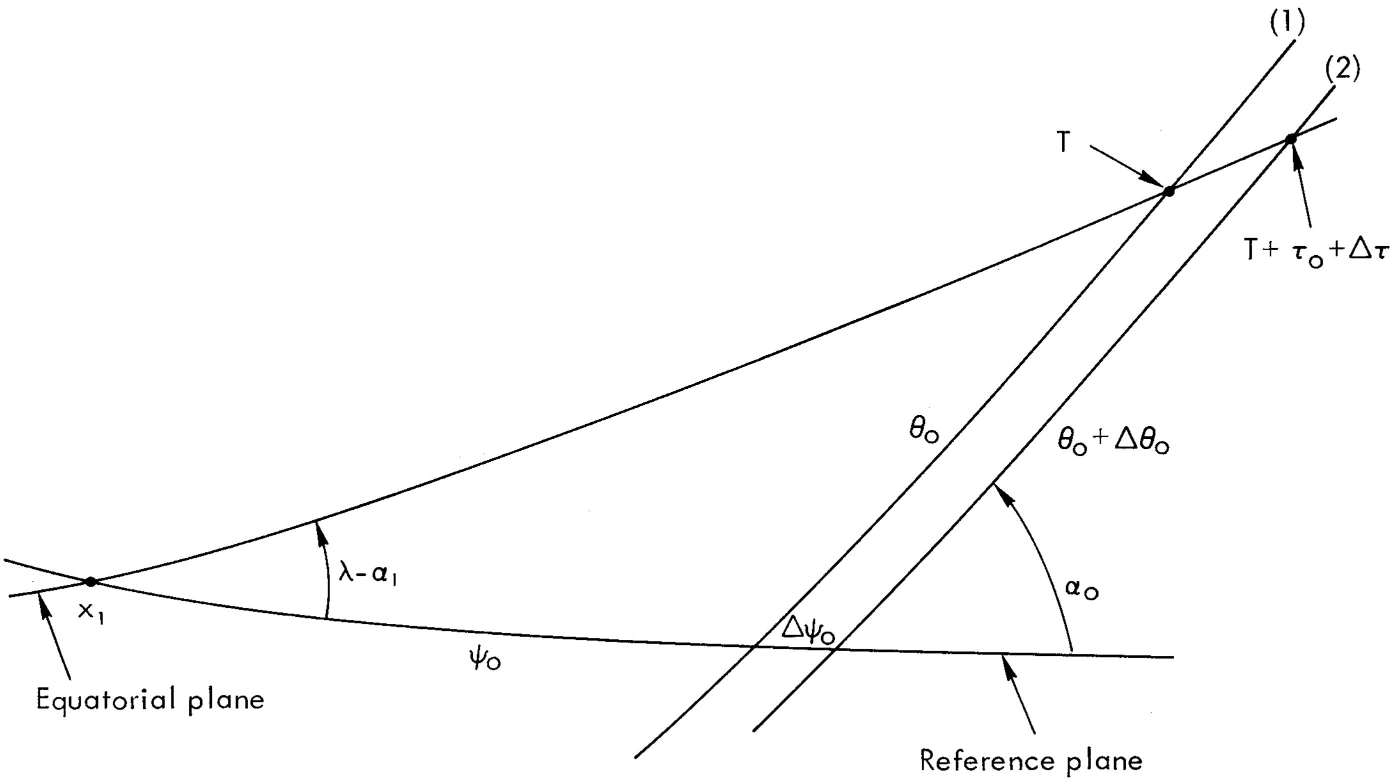

Fig.20—Determination of period between equatorial crossings

Similarly, at the time, $T + \tau_o + \Delta\tau$, of the next equatorial crossing, the latitude is again 0^o, and the substitution of $\psi_o + \Delta\psi_o$ and $\theta_o + \Delta\theta_o$ for ψ and θ in Eq. (77) gives a second relation of the form

$$- \sin(\lambda - \alpha_1) \Big[\cos(\theta_o + \Delta\theta_o) \sin(\psi_o + \Delta\psi_o) + \sin(\theta_o + \Delta\theta_o) \cos(\psi_o + \Delta\psi_o) \cos \alpha_o\Big] + \sin(\theta_o + \Delta\theta_o) \sin \alpha_o \cos(\lambda - \alpha_1) = 0 \tag{113}$$

By expanding Eq. (113) for small values of $\Delta\theta_o$ and $\Delta\psi_o$ and subtracting Eq. (112), the following relation between $\Delta\theta_o$ and $\Delta\psi_o$ is obtained.

$$\Delta\theta_o \Big[(\sin \theta_o \sin \psi_o - \cos \theta_o \cos \psi_o \cos \alpha_o) \sin(\lambda - \alpha_1) + \cos \theta_o \sin \alpha_o \cos(\lambda - \alpha_1)\Big] + \Delta\psi_o \Big[\cos \theta_o \cos \psi_o - \sin \theta_o \sin \psi_o \cos \alpha_o\Big] \sin(\lambda - \alpha_1) = 0 \tag{114}$$

Elimination of $\Delta\theta_o$ and $\Delta\psi_o$ between Eqs. (109), (111) and (114) gives the following solution for $\Delta\tau$.

$$\Delta\tau = \frac{\dot{\psi}_o \tau_o \sin(\lambda-\alpha_1) \Big[- \cos \alpha_o \sin(\lambda-\alpha_1) + \sin \alpha_o \cos(\lambda-\alpha_1) \cos \psi_o\Big]}{\left\{\dot{\theta}_o \Big[\sin^2 \psi_o \sin^2(\lambda-\alpha_1) + \sin \alpha_o \cos(\lambda-\alpha_1) - \cos \alpha_o \sin(\lambda-\alpha_1) \cos \psi_o)^2\Big] + \dot{\psi}_o \sin(\lambda-\alpha_1) \Big[\cos \alpha_o \sin(\lambda-\alpha_1) - \sin \alpha_o \cos(\lambda-\alpha_1) \cos \psi_o\Big]\right\}} \tag{115}$$

where ψ_o is given by Eq. (79).

Figure 21 shows the relationship between $\Delta\tau$, α_o and T specified by Eq. (115). It is seen that the period between equatorial crossings differs from τ_o by only a few seconds for most values of α_o and T. However, in the vicinity of the singularity at $\alpha_o = \lambda - \alpha_1$ and T = 0, $\Delta\tau$ can be appreciable since under these conditions the orbital plane and the equatorial plane are very nearly coplanar and the position of equatorial crossing can shift very rapidly.

ORBITAL INCLINATION CONTROL

In the preceding discussion it has been shown that in the absence of any control forces the characteristic figure-eight ground trace may have long period variations in size and position due to orbital regression. For many applications it may be desirable to limit these variations. This is particularly true of the excursion in latitude where, for coverage reasons, it may be necessary that the satellite latitude excursion be restricted. This sort of control can be accomplished in two ways, the most obvious being to apply control forces in such a way as to reduce the orbital regression rate to zero, with the result that the orbital plane remains fixed in inertial space and the ground trace remains fixed in size and position on the earth. The other method of control is passive and is achieved by injecting the satellite into orbit in such a way that even in the presence of regression the period during which the latitude remains less than a given value is maximized.

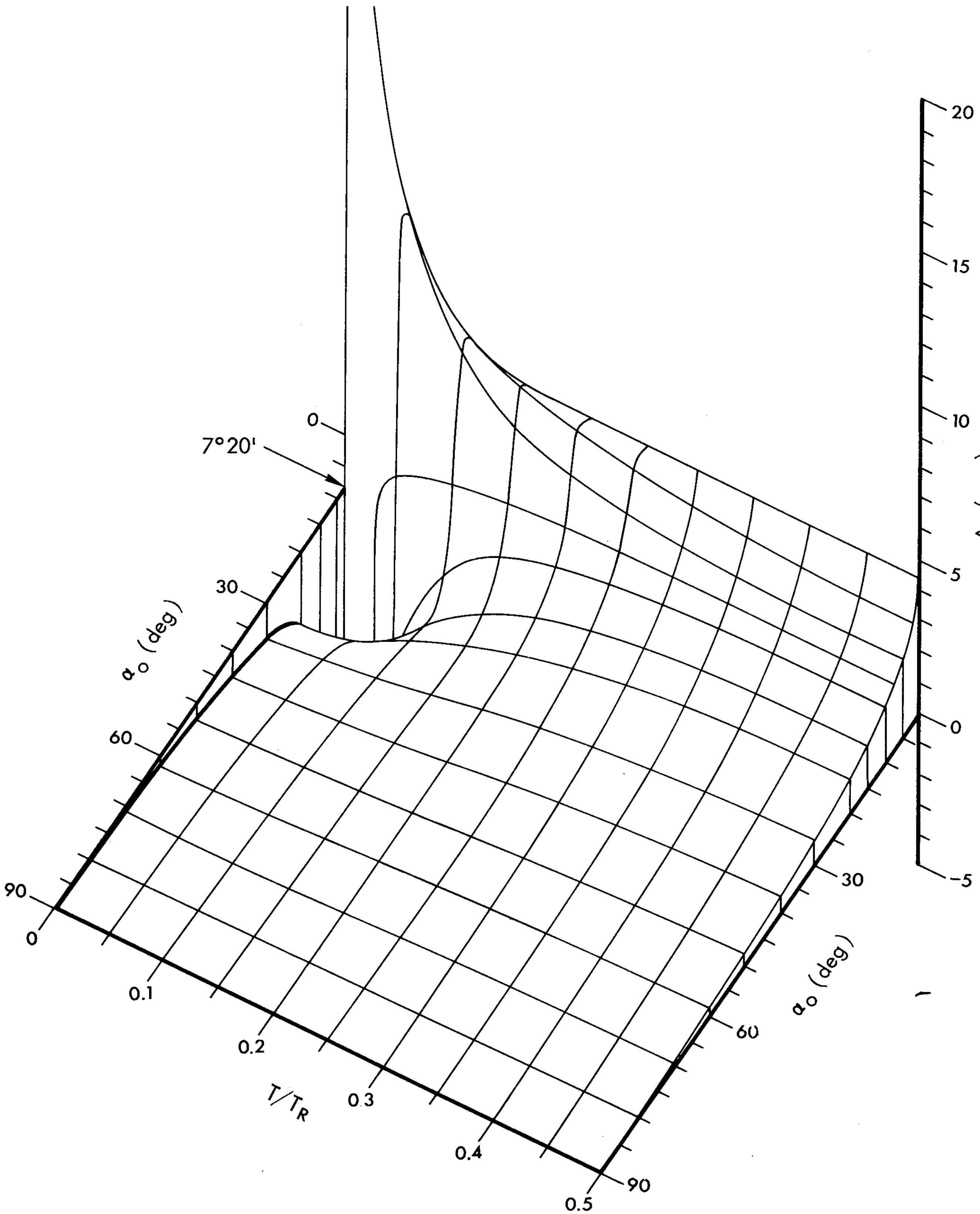

Fig. 21 — Period between equatorial crossings as a function of α_0 and T

Active Control

In connection with this method it is of interest to determine the amount of propulsion required to reduce the regression rate to zero thus fixing the ground trace relative to the earth.

If it is assumed that a control force F_{cz} is applied in the z direction perpendicular to the orbital plane, Eq. (21) has an additive term of the form F_{cz}/M_s on its right side, and the resulting expression for the regression rate corresponding to Eq. (48) becomes

$$\dot{\psi} = - \frac{3J_2R_o^2\dot{\theta}_o}{r_o^2 \sin \alpha_o} (\overline{k}_G \cdot \overline{i}) (\overline{k}_G \cdot \overline{k}) \sin \theta$$

$$+ \frac{3\dot{\theta}_m^2}{\mu\dot{\theta}_o \sin \alpha_o} (\overline{i}_m \cdot \overline{i}) (\overline{i}_m \cdot \overline{k}) \sin \theta$$

$$+ \frac{3\dot{\Theta}^2}{\dot{\theta}_o \sin \alpha_o} (\overline{r}_1 \cdot \overline{i}) (\overline{r}_1 \cdot \overline{k}) \sin \theta$$

$$+ \frac{F_{cz} \sin \theta}{M_s\dot{\theta}_o \sin \alpha_o} \tag{116}$$

The resulting steady-state regression in the presence of the force F_{cz} is given by

$$\dot{\psi}_c = \frac{1}{T}\int_o^T \dot{\psi}\, dt \tag{117}$$

where T is a time large compared to any of the oscillatory periods associated with $\dot{\psi}$. Combination of Eqs. (116) and (117) gives

$$\dot{\psi}_c = \dot{\psi}_o + \frac{1}{r_o M_s \dot{\theta}_o T \sin \alpha_o} \int_o^T F_{cz} \sin \theta \, dt \tag{118}$$

To reduce the steady-state regression rate, $\dot{\psi}_c$, to zero, it is necessary that

$$\frac{1}{T} \int_o^T a_{cz} \sin \theta \, dt = - r_o \dot{\theta}_o \dot{\psi}_o \sin \alpha_o \tag{119}$$

where a_{cz} is the control acceleration perpendicular to the orbital plane.

From the form of Eq. (119), it is seen that accelerations applied when θ is 90^o are more effective in controlling the regression rate. Thus, it is assumed that the control consists of a single impulsive force applied each time θ is equal to 90^o. Under these conditions the left side of Eq. (119) represents the required velocity change per unit time, $\Delta V_z/T$, so that

$$\frac{\Delta V_z}{T} = - r_o \dot{\theta}_o \dot{\psi}_o \sin \alpha_o \tag{120}$$

Combination of Eqs. (59) and (120) gives

$$\frac{\Delta V_z}{T} = \frac{3 r_o \dot{\Theta}^2 \sin 2\alpha_o}{16} \left\{ \left[1 + \frac{\dot{\theta}_m^2}{2\mu \dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m) \right] (2 - 3 \sin^2 \alpha_1) + \frac{2 J_2 \dot{\theta}_o^2 R_o^2}{\dot{\Theta}^2 r_o^2} \left[2 - 3 \sin^2 (\lambda - \alpha_1) \right] \right\} \tag{121}$$

Thus, the propulsion requirement as specified by Eq. (121) depends only on the inclination angle, α_o, relative to the reference plane.

Since the orbital inclination angle, α_G, relative to the equatorial plane determines the maximum latitude excursion of the satellite, α_G is the parameter which would be specified. The geometrical relationship between α_G and α_o is shown in Fig. 22, where it is seen that α_o varies depending on the position of the z axis on the constant α_G contour. However, α_o ranges between extremal values of $(\lambda - \alpha_1) + \alpha_G$ and $(\lambda - \alpha_1) - \alpha_G$. These values occur when ψ_o is equal to 0^o and 180^o respectively. Thus, the propulsion requirement is also bounded by the expressions obtained by substituting the above extremal values for α_o in Eq. (121). The propulsion requirement can then be expressed as

$$\frac{\Delta V_z}{T} = F_o \sin 2(\lambda - \alpha_1 \pm \alpha_G) \tag{122}$$

where F_o is the amplitude of Eq. (121) and the use of the plus or minus sign is determined by which of the two gives the smaller absolute value for $\Delta V_z/T$. The propulsion requirement as a function of α_G is shown in Fig. 23, where the solid lines represent the minimum condition specified.

It is seen that for α_G equal to zero (equatorial orbit) the propulsion requirement is 151.9 ft/sec/year, which is in good agreement with the value of 150 ft/sec/year obtained in Ref. 1. For α_G equal to $7^o20'$, the orbit lies in the reference plane and remains there by definition. Thus, there is no propulsion required for orbital control. As α_G increases, the propulsion requirement reaches a maximum of 580.4 ft/sec/year when α_G equals 45^o. Although Fig. 23 indicates a zero value for the

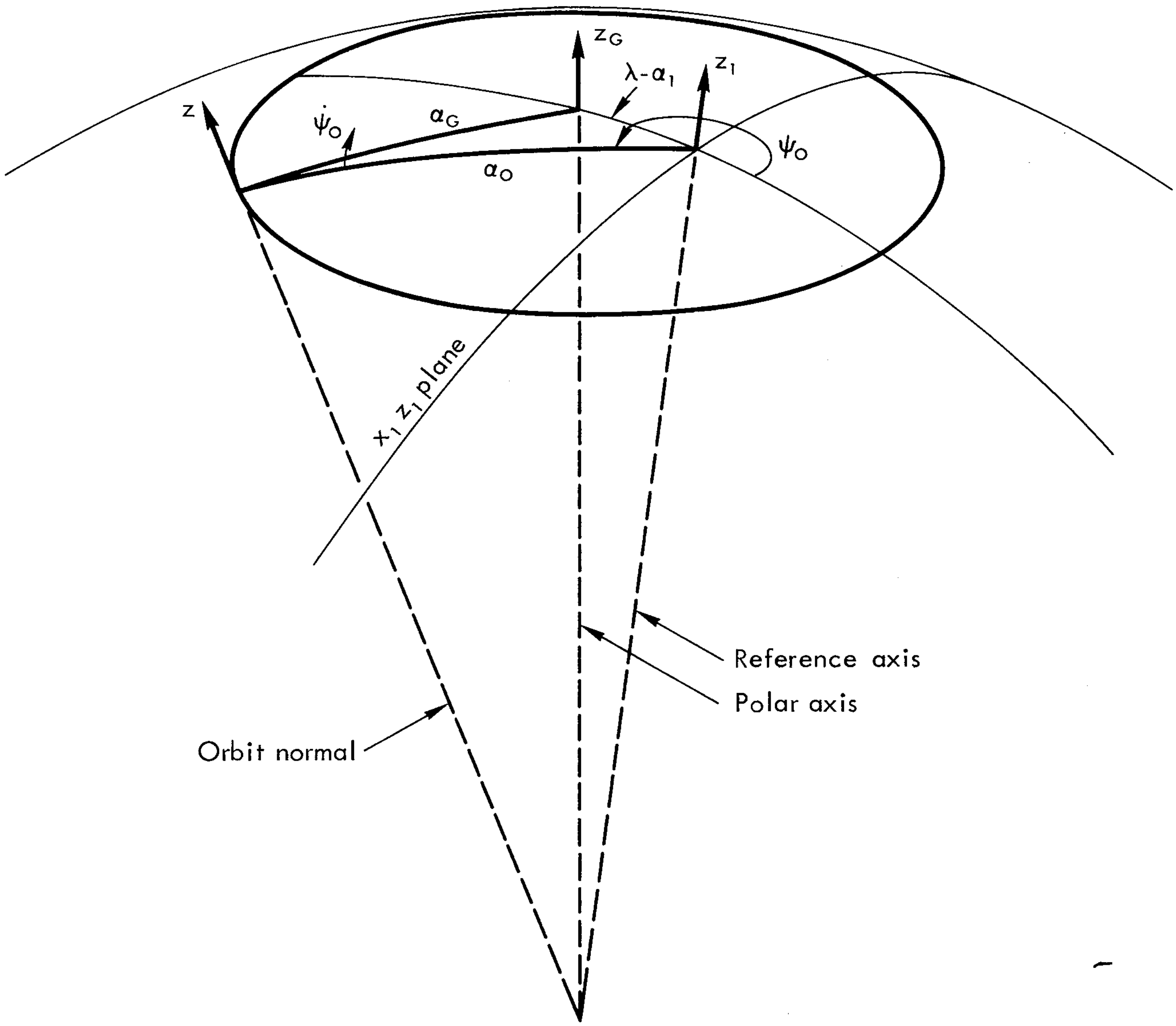

Fig. 22 — Variation of α_0 if α_G is held constant

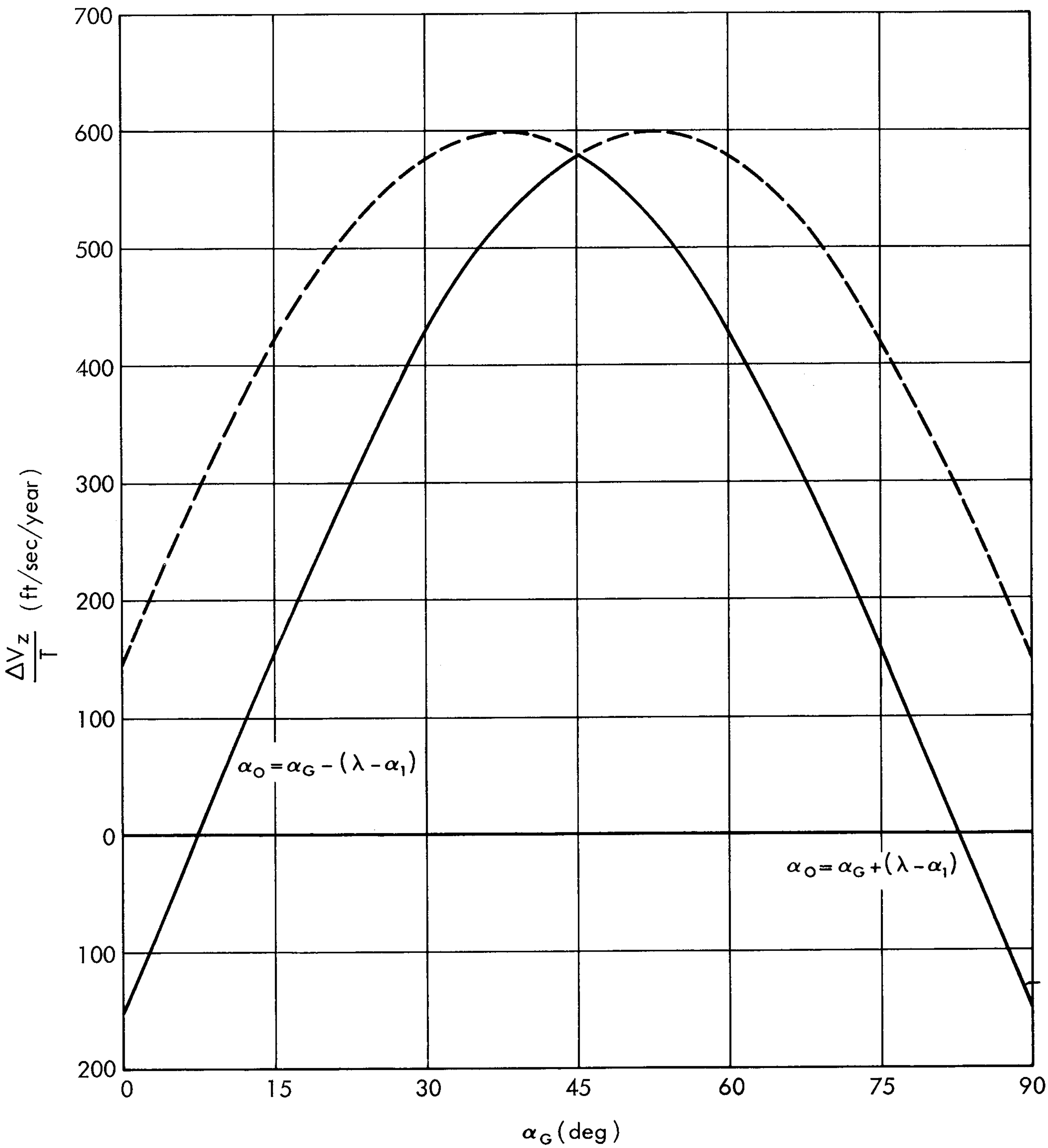

Fig. 23 — Propulsion requirement for orbit inclination control

propulsion requirement when α_G equals $82^\circ 40'$, this orientation is of little interest since the orbit is in unstable equilibrium, as shown in Appendix F. However, on the basis of Appendix F, there is a second stable orbital orientation with the orbital plane perpendicular to the direction of the vernal equinox. This orbit also has a zero propulsion requirement.

Passive Control

In this method of ground trace control it is necessary to select an initial phase of the regression cycle such that the latitude excursion is bounded for a maximum period of time. To accomplish this, use is again made of the reference sphere as shown in Fig. 24, where the circle about z_G is a locus of constant α_G and thus constant γ_m as defined in Eq. (100). The circle about z_1 is the path of the z axis as the orbit regresses. If the z axis is initially at A, then the maximum latitude excursion will remain less than its initial value, γ_{mo}, during the time it takes the z axis to regress from A to B. This time can be obtained by solving Eq. (100) for T to give

$$T = \frac{2}{\dot{\psi}_o} \cos^{-1} \left[\frac{\cos \gamma_{mo} - \cos \alpha_o \cos(\lambda - \alpha_1)}{\sin \alpha_o \sin(\lambda - \alpha_1)} \right] \tag{123}$$

By differentiation of Eq. (123) it can be shown that for a given value of γ_{mo} the maximum value of T occurs when α_o is given by

$$\cos \alpha_o = \frac{\cos(\lambda - \alpha_1)}{\cos \gamma_{mo}} \tag{124}$$

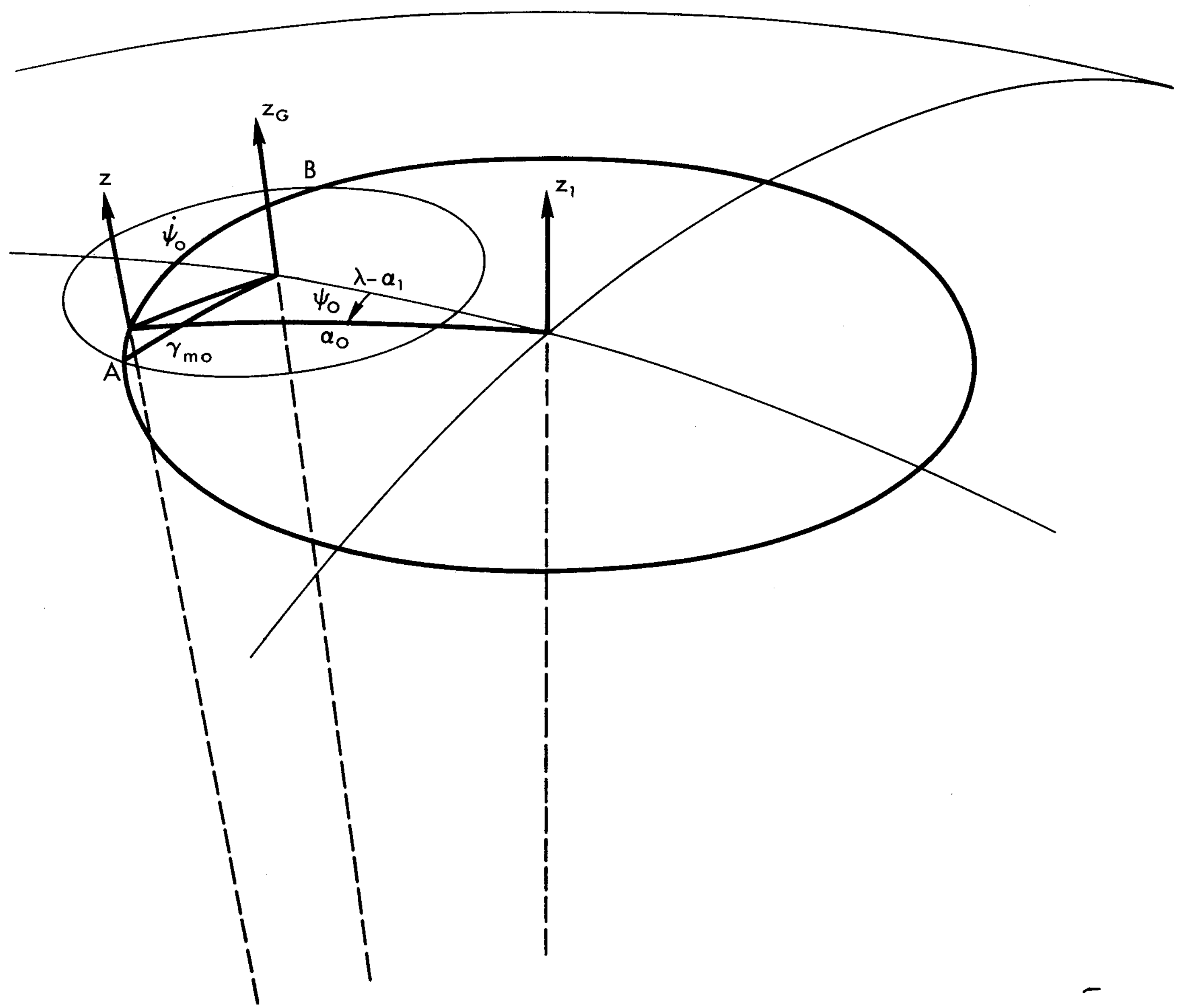

Fig. 24 — Passive orbital control

In obtaining Eq. (124) the dependence of $\dot{\psi}_o$ on α_o is neglected since only values of α_o less than $\lambda - \alpha_1$ are considered. The resulting expression for the maximum value of T becomes

$$T_m = \frac{2}{\dot{\psi}_o} \cos^{-1}\left[\frac{\sqrt{2 \sin(\lambda - \alpha_1 + \gamma_{mo}) \sin(\lambda - \alpha_1 - \gamma_{mo})}}{\sin(\lambda - \alpha_1)}\right] \tag{125}$$

In Fig. 25, T_m is plotted as a function of the maximum permissible latitude, γ_{mo}. It is seen that for α_o equal to $\lambda - \alpha_1$, T_m reaches a value of 26.4 years.

For values of γ_{mo} greater than $\lambda - \alpha_1$, T_m is infinite since the γ_m contour of Fig. 24 encloses the z_1 axis, which is one of the naturally stable positions for the z axis.

Thus, for any given satellite application, if the specifications of maximum permissible latitude variation and minimum acceptable lifetime are represented by a point to the left of the curve in Fig. 25, then it is necessary to use an active control system with its associated propulsion requirements as shown in Fig. 23.

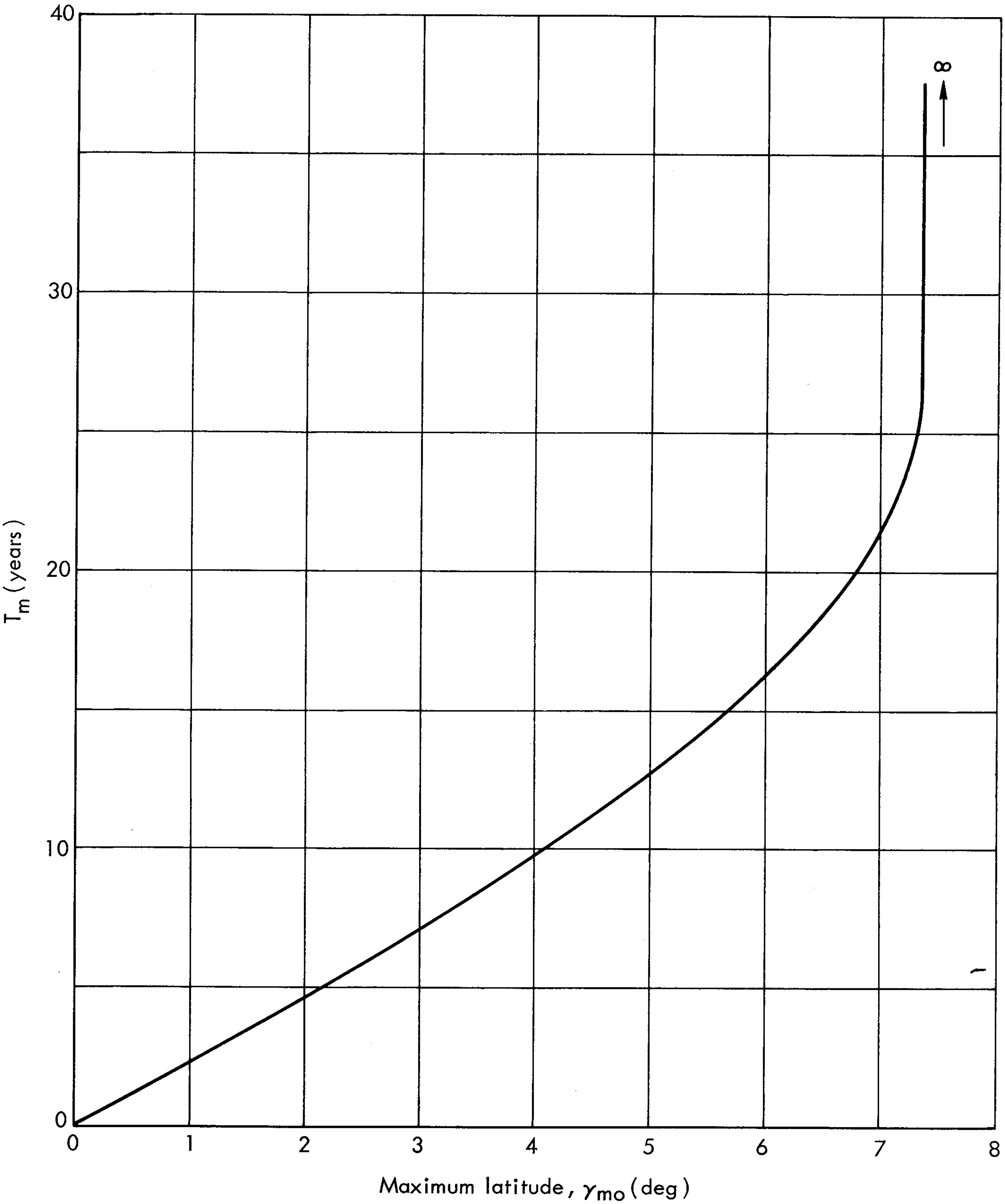

Fig. 25 — Time interval during which latitude is less than γ_{mo}

IV. CONCLUSIONS

As a result of the analysis presented in this Report, the following general conclusions can be stated regarding the effects of the sun, the moon and the oblateness of the earth in perturbing satellites in nominally circular orbits.

- For a given orbital radius, an orientation of the orbital plane can be found which remains essentially constant relative to inertial space under the influence of the assumed perturbing forces.
- The plane of the ecliptic, the earth's equatorial plane and the constant reference plane defined above have a common line of nodes, and the inclination of the reference plane relative to the ecliptic is less than that of the equatorial plane, the exact value being a function of orbital radius.
- If an orbit is inclined relative to its reference plane at an angle less than 79°, its orbital plane will move in such a manner that its inclination relative to the reference plane remains essentially constant while its line of nodes in the reference plane regresses at a steady-state angular rate proportional to the cosine of the inclination angle.
- A highly elliptical type of regression can take place about the direction of the vernal equinox if the inclination of the orbit exceeds 79°.
- In addition to the steady-state motion, there are a large number of oscillatory terms in both inclination angle and regression angle. These result in an oscillation in the direction of the normal to the orbital plane of less than $.5^{\circ}$ relative to its steady-state motion.

In addition to these general conclusions, which apply to orbits of any radius, certain others are reached which apply only to synchronous altitude orbits regressing about the normal to the reference plane

- The reference plane for a synchronous altitude orbit has an inclination of $16^{\circ}7'$ relative to the ecliptic and $-7^{\circ}20'$ relative to the equatorial plane.
- The regression period of a synchronous orbit is 52.81 years at an inclination of 0° and varies inversely as the cosine of the inclination angle, α_o, relative to the reference plane.

- The ground trace of an orbit lying in the reference plane is a figure eight with crossing point on the equator and a maximum latitude excursion of $7^{\circ}20'$. This trace is invariant in size and location as a function of time.

- An orbit inclined to the reference plane has a ground trace which is also a figure eight, but both its maximum latitude excursion and its equatorial crossing point undergo oscillatory variations with a period equal to the regression period.

- In the case of a synchronous orbit which is initially equatorial, it is found that the ground trace develops from a point to a figure eight with a latitude excursion of $14^{\circ}40'$ after 26.6 years and then shrinks back to a point after another 26.6 years.

- To stop the orbital regression and thus fix the satellite ground trace requires a fuel expenditure proportional to the sine of twice the inclination angle relative to the reference plane.

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Appendix A

DIRECTION COSINES

The following tables present the direction cosines relating the unit vectors of the various coordinate systems defined in the body of the report.

REFERENCE COORDINATE SYSTEM

	$\bar{i}_o$	$\bar{j}_o$	$\bar{k}_o$
$\bar{i}_1$	1	0	0
$\bar{j}_1$	0	$\cos \alpha_1$	$\sin \alpha_1$
$\bar{k}_1$	0	$-\sin \alpha_1$	$\cos \alpha_1$

EQUATORIAL COORDINATE SYSTEM

	$\bar{i}_1$	$\bar{j}_1$	$\bar{k}_1$
$\bar{i}_G$	1	0	0
$\bar{j}_G$	0	$\cos(\lambda - \alpha_1)$	$\sin(\lambda - \alpha_1)$
$\bar{k}_G$	0	$-\sin(\lambda - \alpha_1)$	$\cos(\lambda - \alpha_1)$

SATELLITE ORBITAL SYSTEM

	$\bar{i}_1$	$\bar{j}_1$	$\bar{k}_1$
$\bar{i}$	$a_x = \cos\theta\cos\psi - \sin\theta\cos\alpha\sin\psi$	$b_x = \cos\theta\sin\psi + \sin\theta\cos\alpha\cos\psi$	$c_x = \sin\theta\sin\alpha$
$\bar{j}$	$a_y = -\sin\theta\cos\psi - \cos\theta\cos\alpha\sin\psi$	$b_y = -\sin\theta\sin\psi + \cos\theta\cos\alpha\cos\psi$	$c_y = \cos\theta\sin\alpha$
$\bar{k}$	$a_z = \sin\alpha\sin\psi$	$b_z = -\sin\alpha\cos\psi$	$c_z = \cos\alpha$

LUNAR ORBITAL SYSTEM

	$\bar{i}_o$	$\bar{j}_o$	$\bar{k}_o$
$\bar{i}_m$	$a_{xm} = \cos\theta_m\cos\psi_m - \sin\theta_m\cos\alpha_m\sin\psi_m$	$b_{xm} = \cos\theta_m\sin\psi_m + \sin\theta_m\cos\alpha_m\cos\psi_m$	$c_{xm} = \sin\theta_m\sin\alpha_m$
$\bar{j}_m$	$a_{ym} = -\sin\theta_m\cos\psi_m - \cos\theta_m\cos\alpha_m\sin\psi_m$	$b_{ym} = -\sin\theta_m\sin\psi_m + \cos\theta_m\cos\alpha_m\cos\psi_m$	$c_{ym} = \cos\theta_m\sin\alpha_m$
$\bar{k}_m$	$a_{zm} = \sin\alpha_m\sin\psi_m$	$b_{zm} = -\sin\alpha_m\cos\psi_m$	$c_{zm} = \cos\alpha_m$

Appendix B

EQUATIONS OF MOTION

In the body of this Report, the equations of motion of the satellite, the earth and the moon relative to inertial space are given in Eqs. (14), (15) and (16) in the form

$$\text{Satellite:} \qquad \ddot{\bar{r}}_s = \frac{\bar{F}_{Ss}}{M_s} + \frac{\bar{F}_{ms}}{M_s} + \frac{\bar{F}_{Es}}{M_s} \tag{B-1}$$

$$\text{Earth:} \qquad \ddot{\bar{R}}_E = \frac{\bar{F}_{SE}}{M_E} + \frac{\bar{F}_{mE}}{M_E} \tag{B-2}$$

$$\text{Moon:} \qquad \ddot{\bar{R}}_m = \frac{\bar{F}_{Sm}}{M_m} + \frac{\bar{F}_{Em}}{M_m} \tag{B-3}$$

The vector equation of motion of the satellite relative to the earth is obtained as the difference between Eqs. (B-1) and (B-2) as given in Eq. (17) and below

$$\ddot{\bar{r}} = \frac{\bar{F}_{Ss} + \bar{F}_{ms} + \bar{F}_{Es}}{M_s} - \frac{\bar{F}_{SE} + \bar{F}_{mE}}{M_E} \tag{B-4}$$

Similarly, the vector equation of motion of the moon relative to the earth is given by the difference between Eqs. (B-2) and (B-3) as

$$\ddot{\bar{\rho}}_o = \frac{\bar{F}_{Sm} + \bar{F}_{Em}}{M_m} - \frac{\bar{F}_{SE} + \bar{F}_{mE}}{M_E} \tag{B-5}$$

Finally, the vector equation of motion of the center of mass of the earth-moon system relative to inertial space is obtained as follows. The position of the center of mass is located by the vector $\overline{R}$, which is defined by the relation

$$(M_E + M_m)\,\overline{R} = M_E\overline{R}_E + M_m\overline{R}_m \tag{B-6}$$

Elimination of R_E and R_m between Eqs. (B-2), (B-3) and (B-6) gives the desired equation for center of mass motion as

$$\ddot{\overline{R}} = \frac{\overline{F}_{SE} + \overline{F}_{Sm} + \overline{F}_{Em} + \overline{F}_{mE}}{M_E + M_m} \tag{B-7}$$

GRAVITATIONAL ATTRACTIONS

The various gravitational forces involved in the foregoing equations are evaluated as follows.

Force of the Sun on the Satellite

The force $\overline{F}_{Ss}$ can be expressed as

$$\overline{F}_{Ss} = \frac{GM_SM_s}{r_S^3}\,\overline{r}_S \tag{B-8}$$

where

$$\overline{r}_S = \overline{R} - \frac{\overline{\rho}_o}{\mu} + \overline{r} \tag{B-9}$$

The magnitude of $\overline{r}_S$ is then given to first order as

$$r_S^2 = R^2 + 2(\overline{r}\cdot\overline{R}) - 2\left(\frac{\overline{\rho}_o}{\mu}\cdot\overline{R}\right) \tag{B-10}$$

from which the inverse cube term can be expressed by a binomial expansion as

$$\frac{1}{r_S^3} = \frac{1}{R^3}\left[1 - \frac{3}{R^2}\,(\bar{r} \cdot \bar{R}) + \frac{3}{R^2}\left(\frac{\bar{\rho}_o}{\mu} \cdot \bar{R}\right)\right] \tag{B-11}$$

Substitution of Eqs. (B-9) and (B-11) in Eq. (B-8) gives

$$\bar{F}_{Ss} = -\frac{GM_S M_s}{R^3}\left[\bar{R} - \frac{\bar{\rho}_o}{\mu} + \bar{r} - \frac{3}{R^2}\,(\bar{r} \cdot \bar{R})\,\bar{R} + \frac{3}{R^2}\left(\frac{\bar{\rho}_o}{\mu} \cdot \bar{R}\right)\bar{R}\right] \tag{B-12}$$

neglecting terms of the order of r^2/R^2.

<u>Force of the Moon on the Satellite</u>

The force $\bar{F}_{ms}$ is given by

$$\bar{F}_{ms} = -\frac{GM_m M_s}{r_m^3} \tag{B-13}$$

where

$$\bar{r}_m = -\bar{\rho}_o + \bar{r} \tag{B-14}$$

From these relations the first-order expansion for $\bar{F}_{ms}$ is obtained in the same manner as above in the form

$$\bar{F}_{ms} = -\frac{GM_m M_s}{\rho_o^3}\left[-\bar{\rho}_o + \bar{r} - \frac{3(\bar{\rho}_o \cdot \bar{r})}{\rho_o^2}\,\bar{\rho}_o\right] \tag{B-14}$$

neglecting terms of the order of r^2/ρ_o^2.

Force of the Earth on the Satellite

The gravitational potential of the earth at a given point can be expressed in terms of the radial distance r from the earth's center and the distance z_G from the equatorial plane as follows

$$U = \frac{GM_E M_s}{r}\left[1 - \frac{J_2 R_o^2}{2r^2}\left(\frac{3z_G^2}{r^2} - 1\right)\right] \tag{B-15}$$

where G is the universal gravitational constant, J_2 is the coefficient due to earth oblateness ($J_2 = 1.08219 \times 10^{-3}$) and R_o is the mean radius of the earth. The force components in the r and z directions are found by differentiation to be

$$F_r = \frac{\partial U}{\partial r}$$

$$= -\frac{GM_E M_s}{r^2}\left[1 - \frac{3J_2 R_o^2}{2r^2}\left(\frac{5z_G^2}{r^2} - 1\right)\right] \tag{B-16}$$

and

$$F_z = \frac{\partial U}{\partial z_G}$$

$$= -\frac{3GM_E M_s J_2 R_o^2}{r^5} z_G \tag{B-17}$$

Since

$$z_G = (\bar{r} \cdot \bar{k}_G) \tag{B-18}$$

the force exerted by the earth on the satellite can be expressed in vector form by combining Eqs. (B-16) through (B-18) to give

$$\overline{F}_{Es} = - \frac{GM_E M_s}{r^3} \left[\left[1 - \frac{3J_2 R_o^2}{2r^2} \left(\frac{5(\overline{r} \cdot \overline{k}_G)}{r^2} - 1 \right) \right] \overline{r} + \frac{3J_2 R_o^2}{r^2} (\overline{r} \cdot \overline{k}_G) \overline{k}_G \right] \tag{B-19}$$

Force of the Sun on the Earth

The force $\overline{F}_{SE}$ can be expressed as

$$\overline{F}_{SE} = - \frac{GM_S M_E}{R_E^3} \overline{R}_E \tag{B-20}$$

where

$$\overline{R}_E = \overline{R} - \frac{\overline{\rho}_o}{\mu} \tag{B-21}$$

Again, by a binomial expansion, $\overline{F}_{SE}$ becomes

$$\overline{F}_{SE} = - \frac{GM_S M_E}{R^3} \left[\overline{R} - \frac{\overline{\rho}_o}{\mu} + \frac{3}{\mu R^2} (\overline{\rho}_o \cdot \overline{R}) \overline{R} \right] \tag{B-22}$$

neglecting terms of the order of $\rho_o^2/\mu R^2$.

Force of the Moon on the Earth

In this case the force $\overline{F}_{mE}$ is given by

$$\overline{F}_{mE} = \frac{GM_m M_E}{\rho_o^3} \overline{\rho}_o \tag{B-23}$$

Force of the Sun on the Moon

The force $\bar{F}_{Sm}$ is given by the relation

$$\bar{F}_{Sm} = - \frac{GM_S M_m}{R_m^3} \bar{R}_m \tag{B-24}$$

where

$$\bar{R}_m = \bar{R} + \left(1 - \frac{1}{\mu}\right) \bar{\rho}_o \tag{B-25}$$

These two relations can be expanded as before to give

$$\bar{F}_{Sm} = - \frac{GM_S M_m}{R^3} \left[\bar{R} + \left(1 - \frac{1}{\mu}\right) \bar{\rho}_o - \frac{3}{R^2} \left(1 - \frac{1}{\mu}\right) \left(\bar{\rho}_o \cdot \bar{R}\right) \bar{R} \right] \tag{B-26}$$

neglecting terms of the order of ρ_o^2/R^2.

Force of the Earth on the Moon

In this case the force $\bar{F}_{Em}$ is given by the relation

$$\bar{F}_{Em} = - \frac{GM_E M_m}{\rho_o^3} \bar{\rho}_o \tag{B-27}$$

MOTION OF THE MOON

Substitution for the forces on the right side of Eq. (B-5) gives the following equation for the motion of the moon relative to the earth:

$$\ddot{\bar{\rho}}_o = - \frac{G(M_E + M_m)}{\rho_o^3} \bar{\rho}_o$$

$$- \frac{GM_s}{R^3} \left[\bar{\rho}_o - \frac{3}{R^2} (\bar{\rho}_o \cdot \bar{R}) \bar{R} \right] \quad \text{(B-28)}$$

which can be simplified to

$$\ddot{\bar{\rho}}_o = - \frac{G(M_E + M_m)}{\rho_o^3} \bar{\rho}_o \quad \text{(B-29)}$$

since the second term in Eq. (B-28) is less than one percent of the first term.

Since it has been assumed that the moon rotates at a fixed angular rate and at a constant distance from the earth, its acceleration is given by

$$\ddot{\bar{\rho}}_o = - \dot{\theta}_m^2 \bar{\rho}_o \quad \text{(B-30)}$$

which together with Eq. (B-29) gives

$$\frac{\dot{\theta}_m^2}{\mu} = \frac{GM_m}{\rho_o^3} \quad \text{(B-31)}$$

where

$$\mu = \frac{M_E + M_m}{M_m} \quad \text{(B-32)}$$

MOTION OF THE EARTH-MOON SYSTEM

In a similar manner, the forces in Eq. (B-7) can be eliminated to give

$$\ddot{\bar{R}} = -\frac{GM_S}{R^3}\bar{R} \tag{B-33}$$

as the equation of motion for the center of mass of the earth-moon system. Since this motion is also assumed to have a uniform angular rate, $\dot{\Theta}$, and a fixed value of R, the acceleration is given by

$$\ddot{\bar{R}} = -\dot{\Theta}^2\bar{R} \tag{B-34}$$

which combined with Eq. (B-33) gives the following for the earth's orbital angular rate

$$\dot{\Theta}^2 = \frac{GM_S}{R^3} \tag{B-35}$$

MOTION OF THE SATELLITE

Vector Equation of Motion

Substitution of the appropriate force expressions in Eq. (B-4) and elimination of the quantities GM_S and GM_m by mean of Eqs. (B-31) and (B-35) give the following vector equation of motion for the satellite:

$$\begin{aligned}\ddot{\bar{r}} = &-\frac{GM_E}{r^3}\left[\left(1 - \frac{3J_2R_o^2}{2r^2}\left[\frac{5}{r^2}(\bar{r}\cdot\bar{k}_G)^2 - 1\right]\right)\bar{r} + \frac{3J_2R_o^2}{r^2}(\bar{r}\cdot\bar{k}_G)\,\bar{k}_G\right] \\ &-\dot{\Theta}^2\left[\bar{r} - \frac{3}{R^2}(\bar{r}\cdot\bar{R})\,\bar{R}\right] \\ &-\frac{\dot{\theta}_m^2}{\mu}\left[\bar{r} - \frac{3}{\rho_o^2}(\bar{r}\cdot\bar{\rho}_o)\,\bar{\rho}_o\right]\end{aligned} \tag{B-36}$$

which corresponds to Eq. (18).

Substitution of $r\bar{i}$, $R\bar{r}_1$ and $\rho_o \bar{i}_m$ for $\bar{r}$, $\bar{R}$ and $\bar{\rho}_o$ respectively in Eq. (B-36) gives

$$\ddot{\bar{r}} = -\frac{GM_E}{r^2}\left[\left(1 - \frac{3J_2R_o^2}{2r^2}\left[5(\bar{i}\cdot\bar{k}_G)^2 - 1\right]\right)\bar{i} + \frac{3J_2R_o^2}{r^2}(\bar{i}\cdot\bar{k}_G)\,\bar{k}_G\right]$$

$$- r\dot{\Theta}^2\left[\bar{i} - 3(\bar{i}\cdot\bar{r}_1)\,\bar{r}_1\right]$$

$$- \frac{r\dot{\theta}_m^2}{\mu}\left[\bar{i} - 3(\bar{i}\cdot\bar{i}_m)\,\bar{i}_m\right] \qquad \text{(B-37)}$$

The left side of Eq. (B-37) can also be expressed in the form

$$\ddot{\bar{r}} = \frac{d^2\bar{r}}{dt^2} + 2\left[\bar{\omega}_o \times \frac{d\bar{r}}{dt}\right] + \left[\frac{d\bar{\omega}_o}{dt} \times \bar{r}\right] + \left[\bar{\omega}_o \times [\bar{\omega}_o \times \bar{r}]\right] \qquad \text{(B-38)}$$

where the derivatives on the right are relative to the x, y, z coordinate system, and $\bar{\omega}_o$ is the angular velocity of the x, y, z system relative to inertial space.

Substitution of $r\bar{i}$ for r in Eq. (B-38) gives

$$\ddot{\bar{r}} = \frac{d^2r}{dt^2}\,\bar{i} + 2\,\frac{dr}{dt}\left[\bar{\omega}_o \times \bar{i}\right] + r\left[\frac{d\bar{\omega}_o}{dt} \times \bar{i}\right] + r\left[(\bar{\omega}_o\cdot\bar{i})\,\bar{\omega}_o - (\bar{\omega}_o\cdot\bar{\omega}_o)\,\bar{i}\right] \qquad \text{(B-39)}$$

Component Equation of Motion

If the $\bar{i}$, $\bar{j}$ and $\bar{k}$ components of Eqs. (37) and (39) are equated, the following equations of motion are obtained

$$\frac{d^2 r}{dt^2} + r\left[(\bar{\omega}_o \cdot \bar{i})^2 - (\bar{\omega}_o \cdot \bar{\omega}_o)\right] = -\frac{GM_E}{r^2}$$

$$- \frac{3GM_E J_2 R_o^2}{2r^4}\left[1 - 3(\bar{k}_G \cdot \bar{i})^2\right]$$

$$- r\dot{\Theta}^2\left[1 - 3(\bar{r}_1 \cdot \bar{i})^2\right]$$

$$- \frac{r\dot{\theta}_m^2}{\mu}\left[1 - 3(\bar{i}_m \cdot \bar{i})^2\right] \tag{B-40}$$

$$2\frac{dr}{dt}(\bar{\omega}_o \cdot \bar{k}) + r\left(\frac{d\bar{\omega}_o}{dt} \cdot \bar{k}\right) + r(\bar{\omega}_o \cdot \bar{i})(\bar{\omega}_o \cdot \bar{j}) =$$

$$- \frac{3GM_E J_2 R_o^2}{r^4}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{j})$$

$$+ 3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{j})$$

$$+ \frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{j}) \tag{B-41}$$

$$- 2 \frac{dr}{dt} (\bar{\omega}_o \cdot \bar{j}) - r\left(\frac{d\bar{\omega}_o}{dt} \cdot \bar{j}\right) + r(\bar{\omega}_o \cdot \bar{i}) (\bar{\omega}_o \cdot \bar{k}) =$$

$$- \frac{3GM_E J_2 R_o^2}{r^4} (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{k})$$

$$+ 3r\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{k})$$

$$+ \frac{3r\dot{\theta}_m^2}{\mu} (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{k}) \qquad \text{(B-42)}$$

These three equations can be further simplified to

$$\frac{d^2 r}{dt^2} - r\left[(\bar{\omega}_o \cdot \bar{j})^2 + (\bar{\omega}_o \cdot \bar{k})^2\right] = - \frac{GM_E}{r^2}$$

$$- \frac{3GM_E J_2 R_o^2}{2r^4} \left[\bar{1} - 3(\bar{k}_G \cdot \bar{i})^2\right]$$

$$- r\dot{\Theta}^2 \left[1 - 3(\bar{r}_1 \cdot \bar{i})^2\right]$$

$$- \frac{r\dot{\theta}_m^2}{\mu} \left[1 - 3(\bar{i}_m \cdot \bar{i})^2\right] \qquad \text{(B-43)}$$

$$\frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o \cdot \bar{k})\right] + r(\bar{\omega}_o \cdot \bar{i})(\bar{\omega}_o \cdot \bar{j}) =$$

$$- \frac{3GM_E J_2 R_o^2}{r^4}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{j})$$

$$+ 3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{j})$$

$$+ \frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{j}) \qquad \text{(B-44)}$$

$$-\frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o \cdot \bar{j})\right] + r(\bar{\omega}_o \cdot \bar{i})(\bar{\omega}_o \cdot \bar{k}) =$$

$$- \frac{3GM_E J_2 R_o^2}{r^4}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{k})$$

$$+ 3r\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k})$$

$$+ \frac{3r\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{k}) \qquad \text{(B-45)}$$

Constraint Equation

In order that the xy plane always be the instantaneous orbit plane, it is necessary that the vectors $\bar{r}$ and $\dot{\bar{r}}$ both lie in this plane. The vector $\bar{r}$ satisfies this condition since it is along the x axis by definition. In the case of $\dot{\bar{r}}$ the following relation must be satisfied

$$(\dot{\bar{r}} \cdot \bar{k}) \equiv 0 \tag{B-46}$$

which states that $\dot{\bar{r}}$ is perpendicular to the z axis and thus lies in the xy plane as desired.

Equation (B-46) can be simplified by combining it with the expression

$$\dot{\bar{r}} = \frac{d\bar{r}}{dt} + [\bar{\omega}_o \times \bar{r}] \tag{B-47}$$

which gives

$$\left(\frac{d\bar{r}}{dt} \cdot \bar{k}\right) + \left([\bar{\omega}_o \times \bar{r}] \cdot \bar{k}\right) \equiv 0 \tag{B-48}$$

Substitution of $r\bar{i}$ for $\bar{r}$ reduces Eq. (B-48) to

$$\frac{dr}{dt}\,(\bar{i} \cdot \bar{k}) + r(\bar{\omega}_o \cdot [\bar{i} \times \bar{k}]) \equiv 0 \tag{B-49}$$

or

$$(\bar{\omega}_o \cdot \bar{j}) \equiv 0 \tag{B-50}$$

Complete Equations of Motion

By means of Eq. (B-50), the three equations, Eqs. (B-43), (B-44) and (B-45), can be simplified to

$$\frac{d^2r}{dt^2} - r(\bar{\omega}_o \cdot \bar{k})^2 = -\frac{GM_E}{r^2}$$

$$- \frac{3GM_E J_2 R_o^2}{2r^4}[1 - 3(\bar{k}_G \cdot \bar{i})^2]$$

$$- r\dot{\Theta}^2[1 - 3(\bar{r}_1 \cdot \bar{i})^2]$$

$$- \frac{r\dot{\theta}_m^2}{2}[1 - 3(\bar{i}_m \cdot \bar{i})^2] \qquad \text{(B-51)}$$

$$\frac{d}{dt}[r^2(\bar{\omega}_o \cdot \bar{k})] = -\frac{3GM_E J_2 R_o^2}{r^3}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{j})$$

$$+ 3r^2\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{j})$$

$$+ \frac{3r^2\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{j}) \qquad \text{(B-52)}$$

$$(\bar{\omega}_o \cdot \bar{i})(\bar{\omega}_o \cdot \bar{k}) = -\frac{3GM_E J_2 R_o^2}{r^5}(\bar{k}_G \cdot \bar{i})(\bar{k}_G \cdot \bar{k})$$

$$+ 3\dot{\Theta}^2(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k})$$

$$+ \frac{3\dot{\theta}_m^2}{\mu}(\bar{i}_m \cdot \bar{i})(\bar{i}_m \cdot \bar{k}) \qquad \text{(B-53)}$$

The three angular velocity components can be expressed by means of Eq. (13) so that

$$(\bar{\omega}_o \cdot \bar{i}) = \dot{\psi} \sin \theta \sin \alpha + \dot{\alpha} \cos \theta \qquad \text{(B-54)}$$

$$(\bar{\omega}_o \cdot \bar{j}) = \dot{\psi} \cos \theta \sin \alpha - \dot{\alpha} \sin \theta \qquad \text{(B-55)}$$

$$(\bar{\omega}_o \cdot \bar{k}) = \dot{\theta} + \dot{\psi} \cos \alpha \qquad \text{(B-56)}$$

Substitution of Eqs. (B-54) through (B-56) into Eqs. (B-50) through (B-53) gives

$$\frac{d^2 r}{dt^2} - r(\dot{\theta} + \dot{\psi} \cos \alpha)^2 = - \frac{GM_E}{r^2} - \frac{3GM_E J_2 R_o^2}{2r^4} [1 - 3(\bar{k}_G \cdot \bar{i})^2] - r\dot{\Theta}^2 [1 - 3(\bar{r}_1 \cdot \bar{i})^2] - \frac{r\dot{\theta}_m^2}{\mu} [1 - 3(\bar{i}_m \cdot \bar{i})^2] \qquad \text{(B-57)}$$

$$\frac{d}{dt} [r^2(\dot{\theta} + \dot{\psi} \cos \alpha)] = - \frac{3GM_E J_2 R_o^2}{r^3} (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{j}) + 3r^2\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{j}) + \frac{3r^2\dot{\theta}_m^2}{\mu} (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{j}) \qquad \text{(B-58)}$$

$$\dot{\psi} = - \frac{3GM_E J_2 R_o^2 (\bar{k}_G \cdot \bar{i})\ (\bar{k}_G \cdot \bar{k}) \sin\theta}{r^5 (\dot{\theta} + \dot{\psi}\cos\alpha) \sin\alpha}$$

$$+ \frac{3\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i})\ (\bar{r}_1 \cdot \bar{k}) \sin\theta}{(\dot{\theta} + \dot{\psi}\cos\alpha)\sin\alpha}$$

$$+ \frac{3\dot{\theta}_m^2 (\bar{i}_m \cdot \bar{i})\ (\bar{i}_m \cdot \bar{k}) \sin\theta}{\mu(\dot{\theta} + \dot{\psi}\cos\alpha)\sin\alpha} \qquad \text{(B-59)}$$

$$\dot{\alpha} = - \frac{3GM_E J_2 R_o^2 (\bar{k}_G \cdot \bar{i})\ (\bar{k}_G \cdot \bar{k}) \cos\theta}{r^5 (\theta + \psi\cos\alpha)}$$

$$+ \frac{3\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i})\ (\bar{r}_1 \cdot \bar{k}) \cos\theta}{(\dot{\theta} + \dot{\psi}\cos\alpha)}$$

$$+ \frac{3\dot{\theta}_m^2 (\bar{i}_m \cdot \bar{i})\ (\bar{i}_m \cdot \bar{k}) \cos\theta}{\mu(\dot{\theta} + \dot{\psi}\cos\alpha)} \qquad \text{(B-60)}$$

These four equations represent the desired equations of motion, and the scalar products of the unit vectors can be evaluated by means of the direction cosines given in Appendix A.

Appendix C

OSCILLATORY AMPLITUDES

INTRODUCTION

In the main part of this Report, expressions are derived for $\dot{\alpha}$, the rate of change of orbital inclination, and $\dot{\psi}$, the orbital regression rate, in the form

$$\dot{\psi} = - \frac{3GM_E J_2 R_o^2 (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{k}) \sin \theta}{r^5 (\dot{\theta} + \dot{\psi} \cos \alpha) \sin \alpha} + \frac{3\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{k}) \sin \theta}{(\dot{\theta} + \dot{\psi} \cos \alpha) \sin \alpha} + \frac{3\dot{\theta}_m^2 (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{k}) \sin \theta}{\mu (\dot{\theta} + \dot{\psi} \cos \alpha) \sin \alpha} \tag{C-1}$$

$$\dot{\alpha} = - \frac{3GM_E J_2 R_o^2 (\bar{k}_G \cdot \bar{i}) (\bar{k}_G \cdot \bar{k}) \cos \theta}{r^5 (\dot{\theta} + \dot{\psi} \cos \alpha)} + \frac{3\dot{\Theta}^2 (\bar{r}_1 \cdot \bar{i}) (\bar{r}_1 \cdot \bar{k}) \cos \theta}{(\dot{\theta} + \dot{\psi} \cos \alpha)} + \frac{3\dot{\theta}_m^2 (\bar{i}_m \cdot \bar{i}) (\bar{i}_m \cdot \bar{k}) \cos \theta}{\mu (\dot{\theta} + \dot{\psi} \cos \alpha)} \tag{C-2}$$

corresponding to Eqs. (37) and (38). It is also indicated in Eqs. (49) and (50) that these expressions can be represented as

$$\dot{\psi} = \dot{\psi}_o + \frac{1}{\sin \alpha_o} \sum_{i=1}^{127} B_i \cos \omega_i t \tag{C-3}$$

and

$$\dot{\alpha} = \sum_{i=1}^{127} A_i \sin \omega_i t \tag{C-4}$$

In this appendix the method of determining the quantities $\dot{\psi}_o$, A_i, B_i and ω_i is indicated and the resulting expressions are tabulated.

METHOD OF COMPUTATION

If the scalar products on the right sides of Eqs. (C-1) and (C-2) are evaluated by means of the direction cosines in Appendix A, the resulting expressions can be expanded as a constant plus a summation of cosine terms in the case of Eq. (C-1) and as a summation of sine terms in Eq. (C-2).

The arguments of these sine and cosine terms are in the form of linear combinations of the angles θ, Θ, θ_m, ψ_m and ψ, while their amplitudes are functions of the angles α, α_m and α_1 as well as the coefficients of the terms in Eqs. (C-1) and (C-2). If it is assumed that the oscillatory variations of $\dot{\psi}$ are small compared to its steady-state value, $\dot{\psi}_o$, then the angle ψ on the right sides of Eqs. (C-1) and (C-2) can be replaced by $\dot{\psi}_o t$. Similarly, if it is assumed that the oscillatory variations of α are small, then α can be replaced by its steady-state value, α_o.

Finally, it is assumed that $\dot{\psi}$ is small compared to $\dot{\theta}$ so that the term $\dot{\theta} + \dot{\psi} \cos \alpha$ can be replaced by $\dot{\theta}_o$. Since the solution of Eqs. (C-1) and (C-2) for ψ and α depends upon these assumptions, it is essential that the resulting solutions verify the assumptions; if this were not the case, the method would be invalid. This aspect of the problem is investigated in Appendix D.

If, in addition, the following substitutions are made

$$\begin{aligned} \theta &= \dot{\theta}_o t \\ \Theta &= \dot{\Theta} t \\ \theta_m &= \dot{\theta}_m t \\ \psi_m &= \dot{\psi}_m t \end{aligned} \tag{C-5}$$

then Eqs. (C-1) and (C-2) take on the form indicated in Eqs. (C-3) and (C-4), where the ω_i values are linear combinations of the angular rates $\dot{\theta}_o$, $\dot{\Theta}$, $\dot{\theta}_m$, $\dot{\psi}_m$ and $\dot{\psi}_o$, and the quantities $\dot{\psi}_o$, A_i and B_i are functions of α_o, α_m and α_1.

The right sides of Eqs. (C-1) and (C-2) each consist of three terms arising from the effects of earth oblateness, the sun and the moon. Thus, in the determination of $\dot{\psi}_o$ as well as A_i and B_i, the presence of J_2 indicates an oblateness effect, while $\dot{\Theta}$ and $\dot{\theta}_m$ indicate solar and lunar effects respectively. Actually, as will be seen, $\dot{\psi}_o$ and some of the A_i's and B_i's have contributions from all three.

ANALYTICAL EXPRESSIONS

Since the details of the actual analysis are rather lengthy, only the resulting amplitudes and frequencies are given here. In presenting these expressions it is convenient to make the following definitions:

$$K = \frac{GM_E J_2 R_o^2}{\dot{\Theta}^2 r_o^5} \tag{C-6}$$

$$P = \frac{3\dot{\Theta}^2}{8\dot{\theta}_o} \left\{ \left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m) \right] (2 - 3 \sin^2 \alpha_1) + 2K \left[2 - 3 \sin^2(\lambda - \alpha_1) \right] \right\} \tag{C-7}$$

$$Q = \frac{3\dot{\Theta}^2}{8\dot{\theta}_o} \left\{ \left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m) \right] \sin^2 \alpha_1 + 2K \sin^2(\lambda - \alpha_1) \right\} \tag{C-8}$$

$$S = \frac{3\dot{\Theta}^2}{8\dot{\theta}_o} \left\{ \left[1 + \frac{\dot{\theta}_m^2}{2\mu\dot{\Theta}^2} (2 - 3 \sin^2 \alpha_m) \right] \sin 2 \alpha_1 - 2K \sin 2(\lambda - \alpha_1) \right\} \tag{C-9}$$

Steady-State Regression Rate

The summation of the constant terms in Eq. (C-1) gives

$$\dot{\psi}_o = - P \cos \alpha_o \tag{C-10}$$

and since P is inherently positive, $\dot{\psi}_o$ is negative for direct orbits ($|\alpha_o| < 90^o$).

Oscillatory Amplitudes and Frequencies

Combination Terms. Those oscillations resulting from all three of the perturbing influences have the following amplitudes and frequencies.

$$\omega_1 = \dot{\psi}_o$$

$$A_1 = - S \cos \alpha_o \qquad B_1 = - S \cos 2\alpha_o$$

$$\omega_2 = 2\dot{\psi}_o$$

$$A_2 = Q \sin \alpha_o \qquad B_2 = A_2 \cos \alpha_o$$

$$\omega_3 = 2\dot{\theta}_o - 2\dot{\psi}_o$$

$$A_3 = - Q \sin^2 \frac{\alpha_o}{2} \sin \alpha_o \qquad B_3 = - A_3$$

$$\omega_4 = 2\dot{\theta}_o - \dot{\psi}_o$$

$$A_4 = S \sin^2 \frac{\alpha_o}{2} (1 + 2 \cos \alpha_o) \qquad B_4 = - A_4$$

$$\omega_5 = 2\dot{\theta}_o$$

$$A_5 = - P \sin \alpha_o \cos \alpha_o \qquad B_5 = - A_5$$

$$\omega_6 = 2\dot{\theta}_o + \dot{\psi}_o$$

$$A_6 = S \cos^2 \frac{\alpha_o}{2} (1 - 2 \cos \alpha_o) \qquad B_6 = - A_6$$

$$\omega_7 = 2\dot{\theta}_o + 2\dot{\psi}_o$$

$$A_7 = Q \cos^2 \frac{\alpha_o}{2} \sin \alpha_o \qquad B_7 = - A_7$$

Solar Terms. Those terms arising from the solar effect only are as follows, where N represents $\dot{\Theta}^2/\dot{\theta}_o$.

$$\omega_8 = 2\dot{\Theta} - 2\dot{\psi}_o$$

$$A_8 = -\frac{3}{4} N \cos^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_8 = \frac{3}{8} N \cos^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_9 = 2\dot{\Theta} - \dot{\psi}_o$$

$$A_9 = -\frac{3}{4} N \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_9 = \frac{3}{4} N \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{10} = 2\dot{\Theta}$$

$$A_{10} = 0$$

$$B_{10} = -\frac{9}{16} N \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{11} = 2\dot{\Theta} + \dot{\psi}_o$$

$$A_{11} = -\frac{3}{4} N \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{11} = -\frac{3}{4} N \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{12} = 2\dot{\Theta} + 2\dot{\psi}_o$$

$$A_{12} = \frac{3}{4} N \sin^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{12} = \frac{3}{8} N \sin^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{13} = 2\dot{\theta}_o - 2\dot{\Theta} - 2\dot{\psi}_o$$

$$A_{13} = -\frac{3}{4} N \sin^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{13} = -A_{13}$$

$$\omega_{14} = 2\dot{\theta}_o - 2\dot{\Theta} - \dot{\psi}_o$$

$$A_{14} = \frac{3}{4} N \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (1 + \cos 2\alpha_o) \sin^2 \frac{\alpha_o}{2}$$

$$B_{14} = - A_{14}$$

$$\omega_{15} = 2\dot{\theta}_o - 2\dot{\Theta}$$

$$A_{15} = - \frac{9}{32} N \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{15} = - A_{15}$$

$$\omega_{16} = 2\dot{\theta}_o - 2\dot{\Theta} + \dot{\psi}_o$$

$$A_{16} = - \frac{3}{4} N \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (1 - 2 \cos \alpha_o) \cos^2 \frac{\alpha_o}{2}$$

$$B_{16} = - A_{16}$$

$$\omega_{17} = 2\dot{\theta}_o - 2\dot{\Theta} + 2\dot{\psi}_o$$

$$A_{17} = \frac{3}{4} N \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{17} = - A_{17}$$

$$\omega_{18} = 2\dot{\theta}_o + 2\dot{\Theta} - 2\dot{\psi}_o$$

$$A_{18} = - \frac{3}{4} N \cos^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{18} = - A_{18}$$

$$\omega_{19} = 2\dot{\theta}_o + 2\dot{\Theta} - \dot{\psi}_o$$

$$A_{19} = - \frac{3}{4} N \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (1 + 2 \cos \alpha_o) \sin^2 \frac{\alpha_o}{2}$$

$$B_{19} = - A_{19}$$

$$\omega_{20} = 2\dot{\theta}_o + 2\dot{\Theta}$$

$$A_{20} = - \frac{9}{32} N \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{20} = - A_{20}$$

$$\omega_{21} = 2\dot{\theta}_o + 2\dot{\Theta} + \dot{\psi}_o$$

$$A_{21} = \frac{3}{4} N \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (1 - 2 \cos \alpha_o) \cos^2 \frac{\alpha_o}{2}$$

$$B_{21} = - A_{21}$$

$$\omega_{22} = 2\dot{\theta}_o + 2\dot{\Theta} + 2\dot{\psi}_o$$

$$A_{22} = \frac{3}{4} N \sin^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{22} = - A_{22}$$

Lunar Terms. Those terms resulting entirely from the lunar effect are as follows, where M represents $\dot{\theta}_m^2/\mu\dot{\theta}_o$:

$$\omega_{23} = \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{23} = \frac{3}{32} M \sin 2\alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{23} = - \frac{3}{32} M \sin 2\alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{24} = \dot{\psi}_m - \dot{\psi}_o$$

$$A_{24} = - \frac{3}{16} M \sin 2\alpha_m [(\cos \alpha_1 + \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{24} = \frac{3}{16} M \sin 2\alpha_m (\cos \alpha_1 + \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{25} = \dot{\psi}_m$$

$$A_{25} = 0$$

$$B_{25} = - \frac{9}{32} M \sin 2\alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$\omega_{26} = \dot{\psi}_m + \dot{\psi}_o$$

$$A_{26} = -\frac{3}{16} M \sin 2\alpha_m [(\cos \alpha_1 - \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{26} = -\frac{3}{16} M \sin 2\alpha_m (\cos \alpha_1 - \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{27} = \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{27} = -\frac{3}{32} M \sin 2\alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{27} = \frac{3}{32} M \sin 2\alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{28} = 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{28} = -\frac{3}{8} M \sin^2 \alpha_m \cos^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{28} = \frac{3}{16} M \sin^2 \alpha_m \cos^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{29} = 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{29} = -\frac{3}{8} M \sin^2 \alpha_m \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{29} = \frac{3}{8} M \sin^2 \alpha_m \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{30} = 2\dot{\psi}_m$$

$$A_{30} = 0$$

$$B_{30} = -\frac{9}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{31} = 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{31} = -\frac{3}{8} M \sin^2 \alpha_m \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{31} = -\frac{3}{8} M \sin^2 \alpha_m \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{32} = 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{32} = \frac{3}{8} M \sin^2 \alpha_m \sin^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{32} = \frac{3}{16} M \sin^2 \alpha_m \sin^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{33} = 2\dot{\theta}_m - 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{33} = -\frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{33} = \frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{34} = 2\dot{\theta}_m - 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{34} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{34} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{35} = 2\dot{\theta}_m - 2\dot{\psi}_m$$

$$A_{35} = 0$$

$$B_{35} = -\frac{9}{16} M \sin^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{36} = 2\dot{\theta}_m - 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{36} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{36} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{37} = 2\dot{\theta}_m - 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{37} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{37} = \frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{38} = 2\dot{\theta}_m - \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{38} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{38} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{39} = 2\dot{\theta}_m - \dot{\psi}_m - \dot{\psi}_o$$

$$A_{39} = \frac{3}{8} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{39} = -\frac{3}{8} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m (\cos \alpha_1 - \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{40} = 2\dot{\theta}_m - \dot{\psi}_m$$

$$A_{40} = 0$$

$$B_{40} = -\frac{9}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$\omega_{41} = 2\dot{\theta}_m - \dot{\psi}_m + \dot{\psi}_o$$

$$A_{41} = \frac{3}{8} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [\cos \alpha_1 + \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{41} = \frac{3}{8} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m (\cos \alpha_1 + \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{42} = 2\dot{\theta}_m - \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{42} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{42} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{43} = 2\dot{\theta}_m - 2\dot{\psi}_o$$

$$A_{43} = -\frac{3}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin \alpha_o$$

$$B_{43} = \frac{3}{64} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{44} = 2\dot{\theta}_m - \dot{\psi}_o$$

$$A_{44} = \frac{9}{32} M \sin^2 \alpha_m \sin 2\alpha_1 \cos \alpha_o$$

$$B_{44} = -\frac{9}{32} M \sin^2 \alpha_m \sin 2\alpha_1 \cos 2\alpha_o$$

$$\omega_{45} = 2\dot{\theta}_m$$

$$A_{45} = 0$$

$$B_{45} = -\frac{9}{32} M \sin^2 \alpha_m (2 - 3 \sin^2 \alpha_1) \sin 2\alpha_o$$

$$\omega_{46} = 2\dot{\theta}_m + \dot{\psi}_o$$

$$A_{46} = -\frac{9}{32} M \sin^2 \alpha_m \sin 2\alpha_1 \cos \alpha_o$$

$$B_{46} = -\frac{9}{32} M \sin^2 \alpha_m \sin 2\alpha_1 \cos 2\alpha_o$$

$$\omega_{47} = 2\dot{\theta}_m + 2\dot{\psi}_o$$

$$A_{47} = \frac{3}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin \alpha_o$$

$$B_{47} = \frac{3}{64} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{48} = 2\dot{\theta}_m + \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{48} = -\frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{48} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{49} = 2\dot{\theta}_m + \dot{\psi}_m - \dot{\psi}_o$$

$$A_{49} = \frac{3}{8} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{49} = -\frac{3}{8} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m (\cos \alpha_1 + \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{50} = 2\dot{\theta}_m + \dot{\psi}_m$$

$$A_{50} = 0$$

$$B_{50} = \frac{9}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$\omega_{51} = 2\dot{\theta}_m + \dot{\psi}_m + \dot{\psi}_o$$

$$A_{51} = \frac{3}{8} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1) \cos \alpha_o - \sin \alpha_1 \sin \alpha_o]$$

$$B_{51} = \frac{3}{8} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m (\cos \alpha_1 - \cos 2\alpha_1) \cos 2\alpha_o$$

$$\omega_{52} = 2\dot{\theta}_m + \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{52} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin \alpha_o$$

$$B_{52} = - \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 \sin 2\alpha_o$$

$$\omega_{53} = 2\dot{\theta}_m + 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{53} = - \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{53} = \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{54} = 2\dot{\theta}_m + 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{54} = - \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{54} = \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{55} = 2\dot{\theta}_m + 2\dot{\psi}_m$$

$$A_{55} = 0$$

$$B_{55} = - \frac{9}{16} M \cos^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$\omega_{56} = 2\dot{\theta}_m + 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{56} = -\frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos \alpha_o$$

$$B_{56} = -\frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 \cos 2\alpha_o$$

$$\omega_{57} = 2\dot{\theta}_m + 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{57} = \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin \alpha_o$$

$$B_{57} = \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin 2\alpha_o$$

$$\omega_{58} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{58} = -\frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{58} = -A_{58}$$

$$\omega_{59} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{59} = \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{59} = -A_{59}$$

$$\omega_{60} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_m$$

$$A_{60} = -\frac{9}{32} M \cos^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{60} = -A_{60}$$

$$\omega_{61} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{61} = \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{61} = -A_{61}$$

$$\omega_{62} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{62} = \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{62} = - A_{62}$$

$$\omega_{63} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{63} = - \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{63} = - A_{63}$$

$$\omega_{64} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_m - \dot{\psi}_o$$

$$A_{64} = - \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{64} = - A_{64}$$

$$\omega_{65} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_m$$

$$A_{65} = \frac{9}{32} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{65} = - A_{65}$$

$$\omega_{66} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_m + \dot{\psi}_o$$

$$A_{66} = - \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1)(\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{66} = - A_{66}$$

$$\omega_{67} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{67} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{67} = - A_{67}$$

$$\omega_{68} = 2\dot{\theta}_o - 2\dot{\theta}_m - 2\dot{\psi}_o$$

$$A_{68} = -\frac{9}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{68} = - A_{68}$$

$$\omega_{69} = 2\dot{\theta}_o - 2\dot{\theta}_m - \dot{\psi}_o$$

$$A_{69} = \frac{9}{64} M \sin^2 \alpha_m \sin 2\alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{69} = - A_{69}$$

$$\omega_{70} = 2\dot{\theta}_o - 2\dot{\theta}_m$$

$$A_{70} = -\frac{9}{64} M \sin^2 \alpha_m (2 - 3 \sin^2 \alpha_1) \sin 2\alpha_o$$

$$B_{70} = - A_{70}$$

$$\omega_{71} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_o$$

$$A_{71} = -\frac{9}{64} M \sin^2 \alpha_m \sin 2\alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{71} = - A_{71}$$

$$\omega_{72} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_o$$

$$A_{72} = \frac{9}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{72} = - A_{72}$$

$$\omega_{73} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{73} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{73} = - A_{73}$$

$$\omega_{74} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_m - \dot{\psi}_o$$

$$A_{74} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{74} = -A_{74}$$

$$\omega_{75} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_m$$

$$A_{75} = -\frac{9}{32} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{75} = -A_{75}$$

$$\omega_{76} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_m + \dot{\psi}_o$$

$$A_{76} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{76} = -A_{76}$$

$$\omega_{77} = 2\dot{\theta}_o - 2\dot{\theta}_m + \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{77} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{77} = -A_{77}$$

$$\omega_{78} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{78} = -\frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \alpha_o$$

$$B_{78} = -A_{78}$$

$$\omega_{79} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{79} = -\frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{79} = -A_{79}$$

$$\omega_{80} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_m$$

$$A_{80} = -\frac{9}{32} M \sin^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{80} = -A_{80}$$

$$\omega_{81} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{81} = -\frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{81} = -A_{81}$$

$$\omega_{82} = 2\dot{\theta}_o - 2\dot{\theta}_m + 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{82} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{82} = -A_{82}$$

$$\omega_{83} = 2\dot{\theta}_o - 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{83} = -\frac{3}{8} M \sin^2 \alpha_m \sin^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{83} = -A_{83}$$

$$\omega_{84} = 2\dot{\theta}_o - 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{84} = \frac{3}{16} M \sin^2 \alpha_m \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{84} = -A_{84}$$

$$\omega_{85} = 2\dot{\theta}_o - 2\dot{\psi}_m$$

$$A_{85} = -\frac{9}{64} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{85} = -A_{85}$$

$$\omega_{86} = 2\dot{\theta}_o - 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{86} = \frac{3}{16} M \sin^2 \alpha_m \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{86} = - A_{86}$$

$$\omega_{87} = 2\dot{\theta}_o - 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{87} = \frac{3}{8} M \sin^2 \alpha_m \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{87} = - A_{87}$$

$$\omega_{88} = 2\dot{\theta}_o - \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{88} = \frac{3}{16} M \sin \alpha_m \cos \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{88} = - A_{88}$$

$$\omega_{89} = 2\dot{\theta}_o - \dot{\psi}_m - \dot{\psi}_o$$

$$A_{89} = \frac{3}{16} M \sin \alpha_m \cos \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{89} = - A_{89}$$

$$\omega_{90} = 2\dot{\theta}_o - \dot{\psi}_m$$

$$A_{90} = - \frac{9}{32} M \sin \alpha_m \cos \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{90} = - A_{90}$$

$$\omega_{91} = 2\dot{\theta}_o - \dot{\psi}_m + \dot{\psi}_o$$

$$A_{91} = \frac{3}{16} M \sin \alpha_m \cos \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1)(\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{91} = - A_{91}$$

$$\omega_{92} = 2\dot{\theta}_o - \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{92} = -\frac{3}{16} M \sin \alpha_m \cos \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{92} = -A_{92}$$

$$\omega_{93} = 2\dot{\theta}_o + \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{93} = \frac{3}{16} M \sin \alpha_m \cos \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{93} = -A_{93}$$

$$\omega_{94} = 2\dot{\theta}_o + \dot{\psi}_m - \dot{\psi}_o$$

$$A_{94} = -\frac{3}{16} M \sin \alpha_m \cos \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{94} = -A_{94}$$

$$\omega_{95} = 2\dot{\theta}_o + \dot{\psi}_m$$

$$A_{95} = -\frac{9}{32} M \sin \alpha_m \cos \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{95} = -A_{95}$$

$$\omega_{96} = 2\dot{\theta}_o + \dot{\psi}_m + \dot{\psi}_o$$

$$A_{96} = -\frac{3}{16} M \sin \alpha_m \cos \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1)(\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{96} = -A_{96}$$

$$\omega_{97} = 2\dot{\theta}_o + \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{97} = -\frac{3}{16} M \sin \alpha_m \cos \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{97} = -A_{97}$$

$$\omega_{98} = 2\dot{\theta}_o + 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{98} = -\frac{3}{8} M \sin^2 \alpha_m \cos^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{98} = -A_{98}$$

$$\omega_{99} = 2\dot{\theta}_o + 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{99} = -\frac{3}{16} M \sin^2 \alpha_m \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{99} = -A_{99}$$

$$\omega_{100} = 2\dot{\theta}_o + 2\dot{\psi}_m$$

$$A_{100} = -\frac{9}{64} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{100} = -A_{100}$$

$$\omega_{101} = 2\dot{\theta}_o + 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{101} = -\frac{3}{16} M \sin^2 \alpha_m \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{101} = -A_{101}$$

$$\omega_{102} = 2\dot{\theta}_o + 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{102} = \frac{3}{8} M \sin^2 \alpha_m \sin^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{102} = -A_{102}$$

$$\omega_{103} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{103} = -\frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{103} = -A_{103}$$

$$\omega_{104} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{104} = \frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{104} = - A_{104}$$

$$\omega_{105} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_m$$

$$A_{105} = - \frac{9}{32} M \sin^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{105} = - A_{105}$$

$$\omega_{106} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{106} = \frac{3}{8} M \sin^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{106} = - A_{106}$$

$$\omega_{107} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{107} = \frac{3}{4} M \sin^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{107} = - A_{107}$$

$$\omega_{108} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{108} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{108} = - A_{108}$$

$$\omega_{109} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_m - \dot{\psi}_o$$

$$A_{109} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{109} = - A_{109}$$

$$\omega_{110} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_m$$

$$A_{110} = -\frac{9}{32} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{110} = - A_{110}$$

$$\omega_{111} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_m + \dot{\psi}_o$$

$$A_{111} = \frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1) (\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{111} = - A_{111}$$

$$\omega_{112} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{112} = -\frac{3}{16} M \sin^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 + \cos \alpha_o) \sin \alpha_o$$

$$B_{112} = - A_{112}$$

$$\omega_{113} = 2\dot{\theta}_o + 2\dot{\theta}_m - 2\dot{\psi}_o$$

$$A_{113} = -\frac{9}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{113} = - A_{113}$$

$$\omega_{114} = 2\dot{\theta}_o + 2\dot{\theta}_m - \dot{\psi}_o$$

$$A_{114} = \frac{9}{64} M \sin^2 \alpha_m \sin 2\alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{114} = - A_{114}$$

$$\omega_{115} = 2\dot{\theta}_o + 2\dot{\theta}_m$$

$$A_{115} = -\frac{9}{64} M \sin^2 \alpha_m (2 - 3 \sin^2 \alpha_1) \sin 2\alpha_o$$

$$B_{115} = - A_{115}$$

$$\omega_{116} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_o$$

$$A_{116} = -\frac{9}{64} M \sin^2 \alpha_m \sin 2\alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{116} = -A_{116}$$

$$\omega_{117} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_o$$

$$A_{117} = \frac{9}{32} M \sin^2 \alpha_m \sin^2 \alpha_1 \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{117} = -A_{117}$$

$$\omega_{118} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{118} = -\frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{118} = -A_{118}$$

$$\omega_{119} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_m - \dot{\psi}_o$$

$$A_{119} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 + \cos 2\alpha_1)(\cos \alpha_o - \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{119} = -A_{119}$$

$$\omega_{120} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_m$$

$$A_{120} = \frac{9}{32} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin 2\alpha_1 \sin 2\alpha_o$$

$$B_{120} = -A_{120}$$

$$\omega_{121} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_m + \dot{\psi}_o$$

$$A_{121} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m [(\cos \alpha_1 - \cos 2\alpha_1)(\cos \alpha_o + \cos 2\alpha_o) - \sin \alpha_1 \sin \alpha_o]$$

$$B_{121} = -A_{121}$$

$$\omega_{122} = 2\dot{\theta}_o + 2\dot{\theta}_m + \dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{122} = \frac{3}{16} M \cos^2 \frac{\alpha_m}{2} \sin \alpha_m \sin \alpha_1 (\cos \alpha_1 - \cos \alpha_o) \sin \alpha_o$$

$$B_{122} = - A_{122}$$

$$\omega_{123} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$A_{123} = - \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \cos^4 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{123} = - A_{123}$$

$$\omega_{124} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_m - \dot{\psi}_o$$

$$A_{124} = - \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \cos^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o - \cos 2\alpha_o)$$

$$B_{124} = - A_{124}$$

$$\omega_{125} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_m$$

$$A_{125} = - \frac{9}{32} M \cos^4 \frac{\alpha_m}{2} \sin^2 \alpha_1 \sin 2\alpha_o$$

$$B_{125} = - A_{125}$$

$$\omega_{126} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_m + \dot{\psi}_o$$

$$A_{126} = - \frac{3}{8} M \cos^4 \frac{\alpha_m}{2} \sin^2 \frac{\alpha_1}{2} \sin \alpha_1 (\cos \alpha_o + \cos 2\alpha_o)$$

$$B_{126} = - A_{126}$$

$$\omega_{127} = 2\dot{\theta}_o + 2\dot{\theta}_m + 2\dot{\psi}_m + 2\dot{\psi}_o$$

$$A_{127} = \frac{3}{4} M \cos^4 \frac{\alpha_m}{2} \sin^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_o}{2} \sin \alpha_o$$

$$B_{127} = - A_{127}$$

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Appendix D

OSCILLATORY DEVIATIONS FROM STEADY-STATE REGRESSION

STATEMENT OF THE PROBLEM

In the body of this Report it is shown that under the influence of the assumed perturbing forces, a satellite orbit is subject to a steady-state regression of the orbital plane. This motion is represented in Fig. 6 where the normal to the orbital plane traces out a circle on the reference sphere at a constant angular rate.

Analytically, this motion is described by Eqs. (51) and (52), where it is seen that there are oscillatory terms superposed on the steady-state solutions for α and ψ. It is pointed out in Appendix C that these solutions for α and ψ are valid only if the oscillatory terms are small. Therefore, this appendix investigates the magnitude of these residual oscillations.

REFERENCE SYSTEM

To describe this oscillatory effect it is convenient to define an X, Y coordinate system as shown in Fig. 26. This sytem moves in such a way that the XY plane remains perpendicular to the steady-state orbital normal and also tangent to the reference sphere of Fig. 6. The origin thus lies at the point of tangency, and the X axis remains tangent to the steady-state Z axis trace of Fig. 6. In such a coordinate system, any oscillatory components in either α or ψ results in a departure of the instantaneous orbital normal from its steady-state position at the origin of the X, Y system.

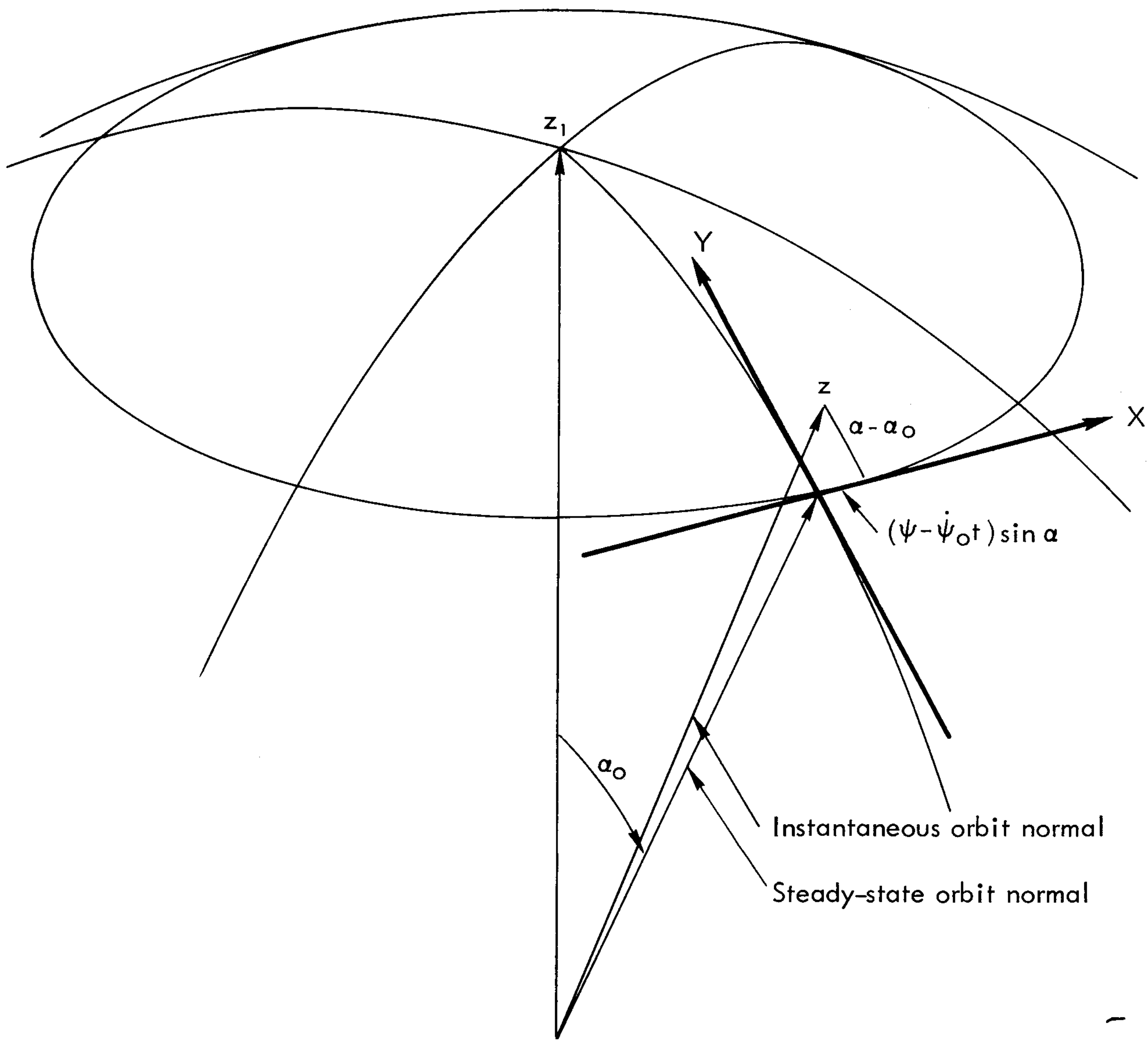

Fig. 26 — Reference system for oscillatory deviations of the orbit normal

OSCILLATORY MOTION

In the expressions for α and ψ given by Eqs. (51) and (52), it is assumed that at zero time all of the oscillatory components are in phase. This can be generalized by introducing a new time origin at a time τ and measuring time from this. Thus, Eqs. (51) and (52) are modified as follows

$$\alpha = \alpha_o + \sum_{i=1}^{127} \frac{A_i}{\omega_i} [\cos \omega_i \tau - \cos \omega_i (\tau + t)] \qquad \text{(D-1)}$$

$$\psi = \dot{\psi}_o t + \frac{1}{\sin \alpha_o} \sum_{i=1}^{127} \frac{B_i}{\omega_i} [\sin \omega_i (\tau + t) - \sin \omega_i \tau] \qquad \text{(D-2)}$$

By making τ large, the phases of the various components are essentially random since the frequencies are assumed to be incommensurable.

From Eqs. (D-1) and (D-2), the X, Y coordinates of the intersection of the instaneous orbital normal with the XY plane are given by

$$X = (\psi - \dot{\psi}_o t) \sin \alpha_o$$

$$= \sum_{i=1}^{127} \frac{B_i}{\omega_i} [\sin \omega_i (\tau + t) - \sin \omega_i \tau] \qquad \text{(D-3)}$$

$$Y = \alpha - \alpha_o$$

$$= \sum_{i=1}^{127} \frac{A_i}{\omega_i} [\cos \omega_i \tau - \cos \omega_i (\tau + t)] \qquad \text{(D-4)}$$

An examination of the expressions for the ω_i values in Appendix C shows that some of these can vanish for particular combinations of inclination angle and orbital radius. Those frequencies which can become zero are as follows

$$\omega_2 = 2\dot{\psi}_o$$

$$\omega_{11} = 2\dot{\Theta} + \dot{\psi}_o$$

$$\omega_{12} = 2\dot{\Theta} + 2\dot{\psi}_o$$

$$\omega_{23} = \dot{\psi}_m - 2\dot{\psi}_o$$

$$\omega_{24} = \dot{\psi}_m - \dot{\psi}_o$$

$$\omega_{27} = 2\dot{\psi}_m - 2\dot{\psi}_o$$

$$\omega_{29} = 2\dot{\psi}_m - \dot{\psi}_o$$

The contours in the $r_o\ \alpha_o$ plane for which these frequencies vanish are shown in Fig. 27.

When a particular ω_i does become very small, the amplitude of the corresponding low-frequency oscillatory terms in Eqs. (D-1) and (D-2) become large due to the $1/\omega_i$ factor. However, the importance of such terms depends on the period of time over which the motion is observed. For this reason, in the determination of the effective oscillatory amplitude an observation period, T, is defined, and the magnitude of the effective oscillation depends on this value of T.

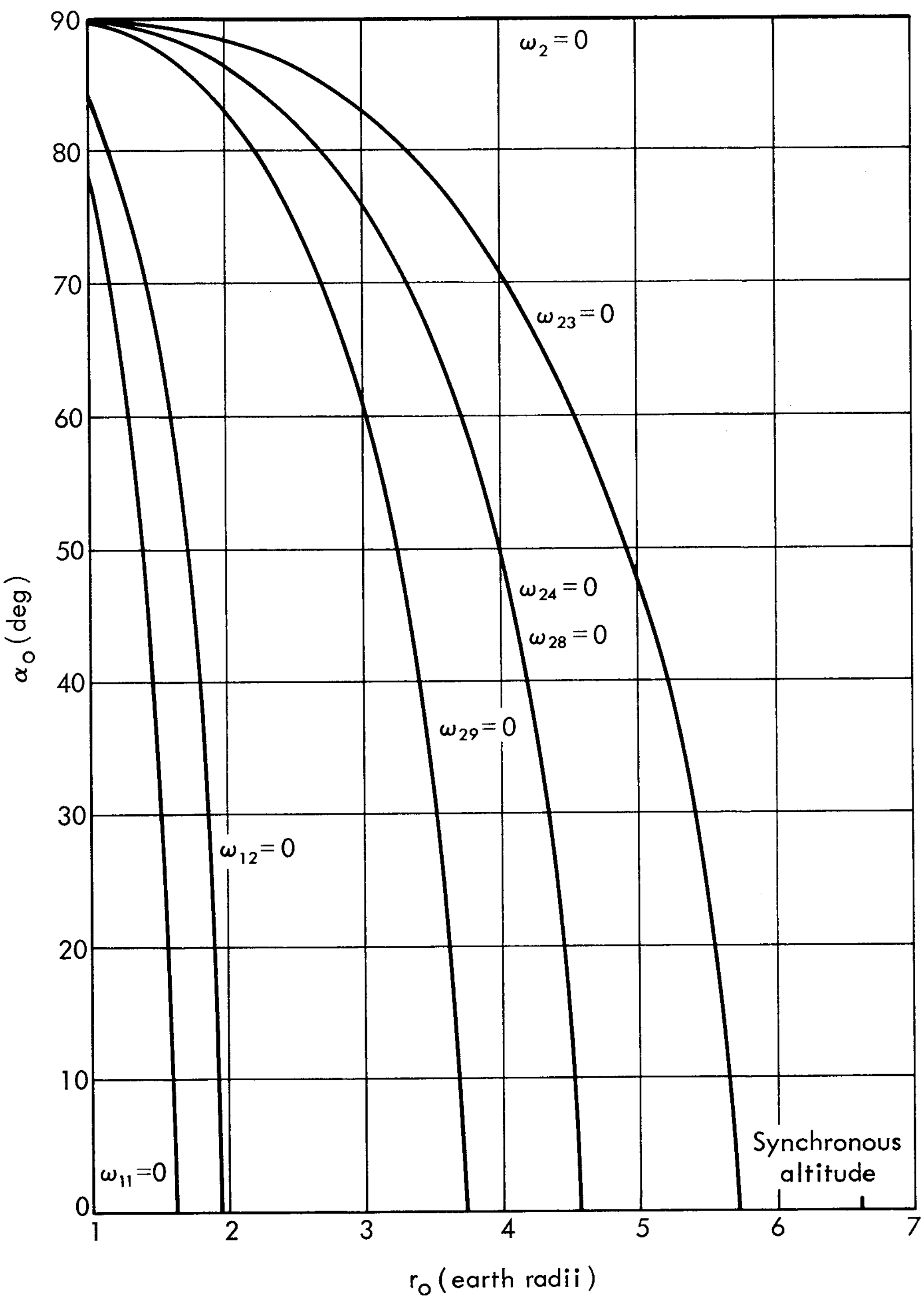

Fig. 27 — Zero frequency loci

LEAST-SQUARES FIT

After selecting a value for the interval T, a linear least-squares fit, $\bar{X}$, is determined for the function X over this interval in the form

$$\bar{X} = \sum_{i=1}^{127} b_{io} + t \sum_{i=1}^{127} b_{i1} \tag{D-5}$$

where

$$b_{io} = - \frac{B_i}{\omega_i^2 T} \left[\omega_i T - 2 \sin \frac{\omega_i T}{2}\right] \sin \omega_i \tau \tag{D-6}$$

and

$$b_{i1} = \frac{12 B_i}{\omega_i^3 T^3} \left[2 \sin \frac{\omega_i T}{2} - \omega_i T \cos \frac{\omega_i T}{2}\right] \cos \omega_i \tau \tag{D-7}$$

The amplitudes of $\sin \omega_i \tau$ and $\cos \omega_i \tau$ in Eqs. (D-6) and (D-7) are shown in Figs. 28 and 29.

The function $\bar{X}$ represents the average trend of the function X over the interval between $\tau - \frac{T}{2}$ and $\tau + \frac{T}{2}$. The mean-square deviation of X from $\bar{X}$ is then given by

$$\sigma_X^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} [X - \bar{X}]^2 \, dt$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left[\sum_{i=1}^{127} \left\{ \frac{B_i}{\omega_i} [\sin \omega_i(\tau + t) - \sin \omega_i \tau] - b_{io} - b_{i1} t \right\} \right]^2 dt$$

$$= \sum_{r=1}^{127} \sum_{s=1}^{127} X_{rs} \tag{D-8}$$

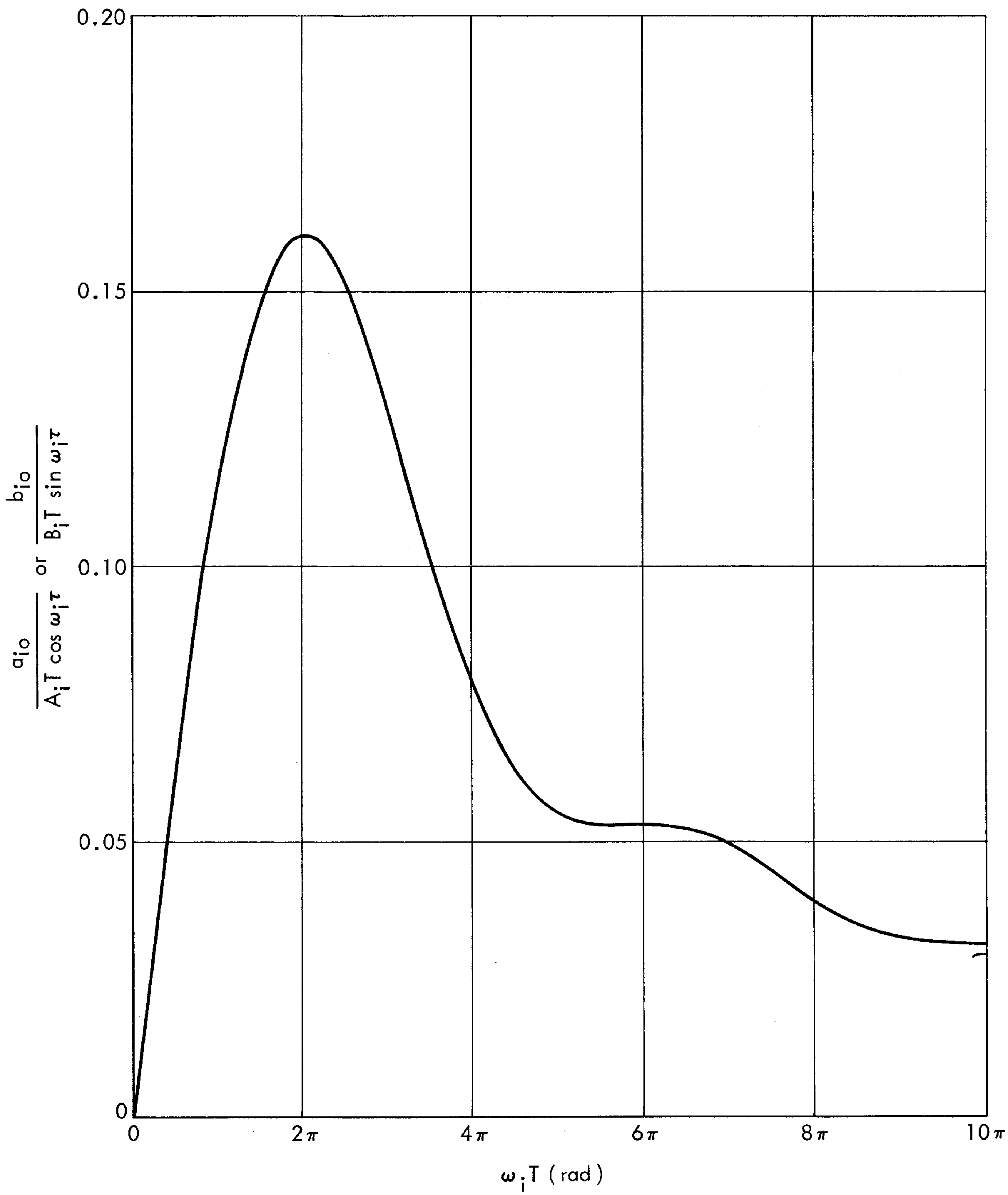

Fig. 28 — Dependence of a_{io} and b_{io} on $\omega_i T$

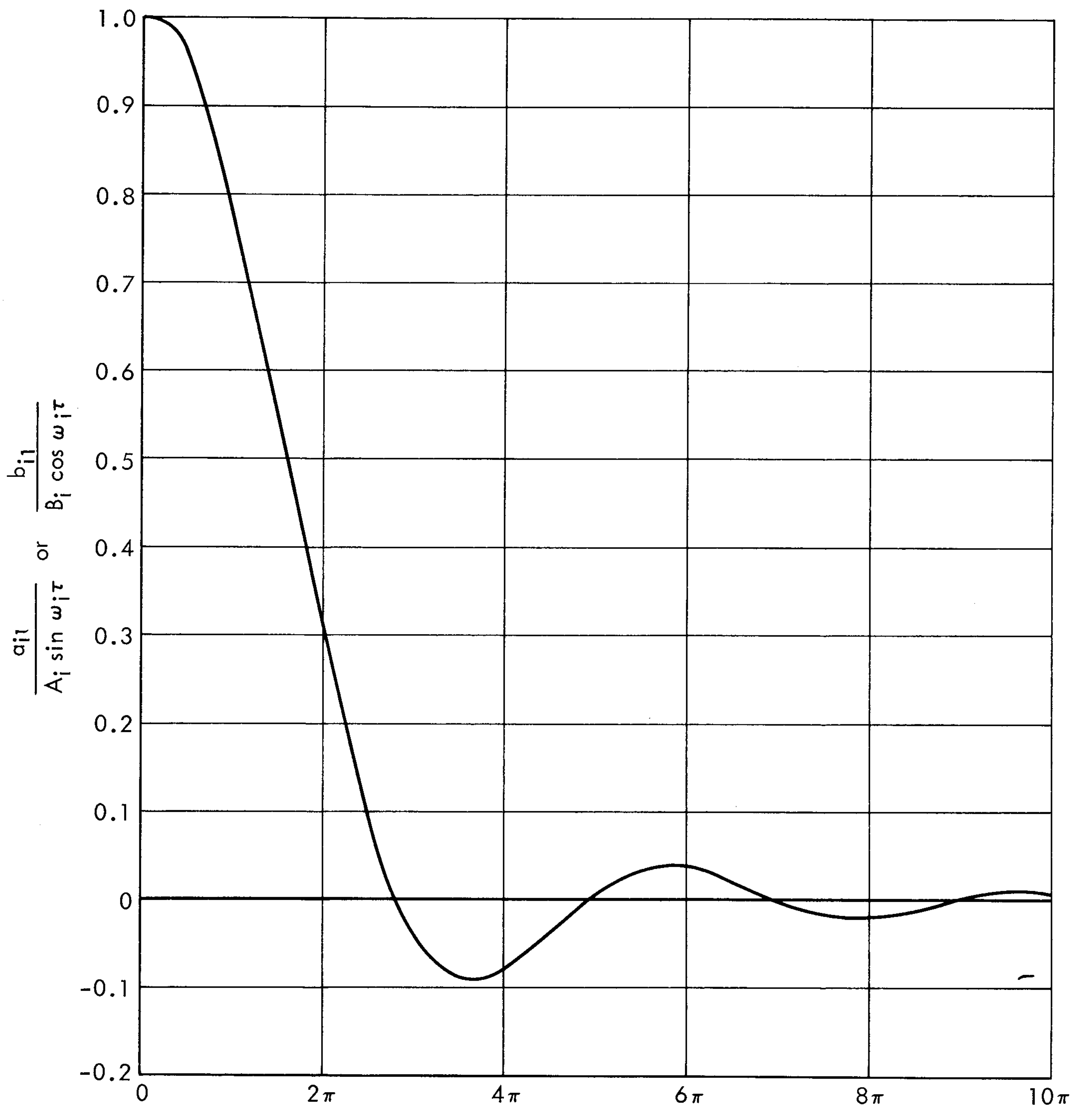

Fig. 29 — Dependence of a_{i1} and b_{i1} on $\omega_i T$

where

$$X_{rs} = B_r B_s [A \cos(\omega_r - \omega_s) \tau$$

$$- B \sin \omega_r \tau \sin \omega_s \tau - C \cos \omega_r \tau \cos \omega_s \tau] \qquad \text{(D-9)}$$

and

$$A = \frac{1}{2\omega_r \omega_s} \left[\frac{2}{(\omega_r - \omega_s)T} \sin \frac{(\omega_r - \omega_s)T}{2} \right.$$

$$\left. - \frac{2}{(\omega_r + \omega_s)T} \sin \frac{(\omega_r + \omega_s)T}{2} \right] \qquad \text{(D-10)}$$

$$B = \frac{2}{\omega_r^2 \omega_s^2 T^2} \left[2 \sin \frac{\omega_r T}{2} \sin \frac{\omega_s T}{2} \right.$$

$$\left. - \frac{\omega_r \omega_s T}{\omega_r + \omega_s} \sin \frac{(\omega_r + \omega_s)T}{2} \right] \qquad \text{(D-11)}$$

$$C = \frac{12}{\omega_r^3 \omega_s^3 T^4} \left[2 \sin \frac{\omega_r T}{2} - \omega_r T \cos \frac{\omega_r T}{2} \right] \left[2 \sin \frac{\omega_s T}{2} \right.$$

$$\left. - \omega_s T \cos \frac{\omega_s T}{2} \right] \qquad \text{(D-12)}$$

In a similar manner the trend of the function Y can be determined as

$$\bar{Y} = \sum_{i=1}^{127} a_{io} + t \sum_{i=1}^{127} a_{i1} \qquad \text{(D-13)}$$

where

$$a_{io} = \frac{A_i}{\omega_i^2 T}\left[\omega_i T - 2\sin\frac{\omega_i T}{2}\right]\cos\omega_i\tau \tag{D-14}$$

and

$$a_{i1} = \frac{12A_i}{\omega_i^3 T^3}\left[2\sin\frac{\omega_i T}{2} - \omega_i T\cos\frac{\omega_i T}{2}\right]\sin\omega_i\tau \tag{D-15}$$

As before, the mean-square deviation of Y from $\overline{Y}$ is given by

$$\sigma_Y^2 = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} [Y - \overline{Y}]^2\,dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\left[\sum_{i=1}^{127}\left[\frac{A_i}{\omega_i}[\cos\omega_i\tau - \cos\omega_i(\tau + t)] - a_{io} - a_{i1}t\right]\right]^2 dt$$

$$= \sum_{r=1}^{127}\sum_{s=1}^{127} Y_{rs} \tag{D-16}$$

where

$$Y_{rs} = A_r A_s\Big[A\cos(\omega_r - \omega_s)\,\tau$$

$$- B\cos\omega_r\tau\cos\omega_s\tau$$

$$- C\sin\omega_r\tau\sin\omega_s\tau\Big] \tag{D-17}$$

COMBINED EFFECT

As a result of the X and Y variations described above, the total steady-state displacement of the normal of the orbital plane from the origin of the X, Y system is given by

$$D_o = \sqrt{\left(\sum_{i=1}^{127} a_{io}\right)^2 + \left(\sum_{i=1}^{127} b_{io}\right)^2} \tag{D-18}$$

Since this quantity is a constant for the interval T, it can be absorbed into the initial value of α_o and the initial value of the regression angle ψ_o. This amounts to shifting the origin of the X, Y system by an amount D_o.

However, there is still a steady drift rate of the orbital normal relative to this new origin, which can be expressed as

$$D_1 = \sqrt{\left(\sum_{i=1}^{127} a_{i1}\right)^2 + \left(\sum_{i=1}^{127} b_{i1}\right)^2} \tag{D-19}$$

from which the total drift during the time interval T is given by

$$\Delta D = D_1 T \tag{D-20}$$

The mean-square deviation from this steady drift can be expressed in terms of the components as

$$\begin{aligned} \sigma_D^2 &= \sigma_X^2 + \sigma_Y^2 \\ &= \sum_{r=1}^{127} \sum_{s=1}^{127} (X_{rs} + Y_{rs}) \end{aligned} \tag{D-21}$$

From Eqs. (D-9) and (D-17) the argument of Eq. (D-21) can be expressed as

$$X_{rs} + Y_{rs} = (A_r A_s + B_r B_s)\left[\frac{A-C}{2} + \frac{A-B}{2}\right]\cos(\omega_r - \omega_s)\tau$$

$$+ (A_r A_s - B_r B_s)\left[\frac{A-C}{2} - \frac{A-B}{2}\right]\cos(\omega_r + \omega_s)\tau \qquad \text{(D-22)}$$

Thus, the deviation of the orbital regression from a steady-state motion can be described by the drift, ΔD, and the deviation from this drift, σ_D.

NUMERICAL RESULTS

Drift

In the determination of ΔD it is seen from Eqs. (D-7), (D-15), (D-19) and (D-20) that ΔD is a function not only of the averaging interval T but also the arbitrary time τ. However, the result should be independent of τ, and it appears reasonable to replace cos $\omega_i\tau$ and sin $\omega_i\tau$ by their root-mean-square value of $1/\sqrt{2}$ in the evaluation of ΔD.

By means of the relations developed above, the value of ΔD can be computed as a function of the orbital inclination angle of a synchronous altitude satellite for values of T equal to 50, 100 and 500 years. The results of these computations are shown in Fig. 30. It should be noted that the summations are taken over only the twelve lowest frequency terms, which are ω_1, ω_2 and ω_{23} through ω_{32}. It is found that the contributions of the terms whose frequencies involve $\dot{\theta}_m$, $\dot{\Theta}$ and $\dot{\theta}_o$, are negligible. In Fig. 30 it is seen that for values of T which are appreciably larger than the oscillatory periods included, the values for ΔD are independent of

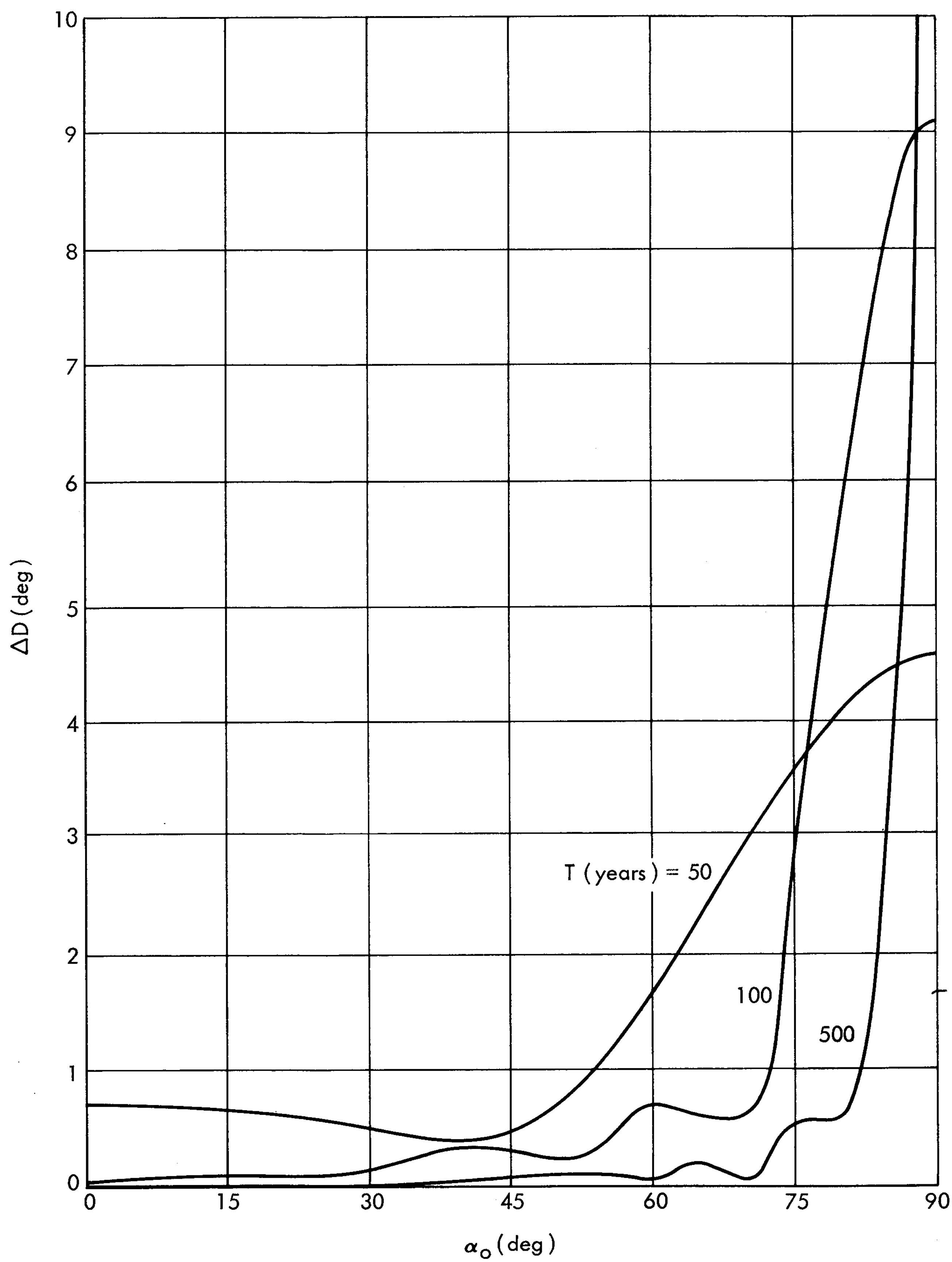

Fig. 30 — Drift angle as a function of α_o (synchronous orbit)

T and less than $.5^{\circ}$. However, as α_{o} approaches 90° the value of ω_2 or $2\dot{\psi}_{o}$ approaches zero. In this region of high inclination the summations in Eq. (D-19) are dominated by the terms a_{21} and b_{21}, which have the form shown in Fig. 29. Thus, the fluctuations in Fig. 30 are simply a reflection of Fig. 29 plotted as a function of α_{o}. The magnitude of the maximum at $\alpha_{o} = 90^{\circ}$ increases with the averaging time, T.

For comparison, Fig. 31 shows the variation of ΔD for an orbital radius of 4 earth radii and for T equal to 100 years. In this case, several maxima exist, as would be expected from Fig. 27. The maximum at $\alpha_{o} \doteq 50^{\circ}$ results from both ω_{24} and ω_{27} approaching zero while ω_{23} gives a maximum at $\alpha_{o} \doteq 70^{\circ}$; as before, ω_2 causes a maximum at $\alpha_{o} = 90^{\circ}$. The maxima again are due to the dominance of the terms associated with the near-zero frequencies, and in the case of those at 50° and 70° the form of Fig. 29 appears above and below the zero frequency position, resulting in a symmetrical peak at each of these positions. As in the previous case, these maxima are accentuated as T increases, while the general level of the rest of the curve is relatively constant for values of T greater than the oscillatory periods.

On the basis of Figs. 30 and 31 as well as other cases not shown here, it is found that the general level of ΔD decreases as the orbital altitude decreases.

Mean-Square Deviation

In the determination of σ_D^2 by means of Eqs. (D-21) and (D-22) it is again found that σ_D^2 depends on τ as well as T. However, examination of the argument of Eq. (D-21) as expressed in Eq. (D-22) shows that the maximum value of $X_{rs} + Y_{rs}$ is given by

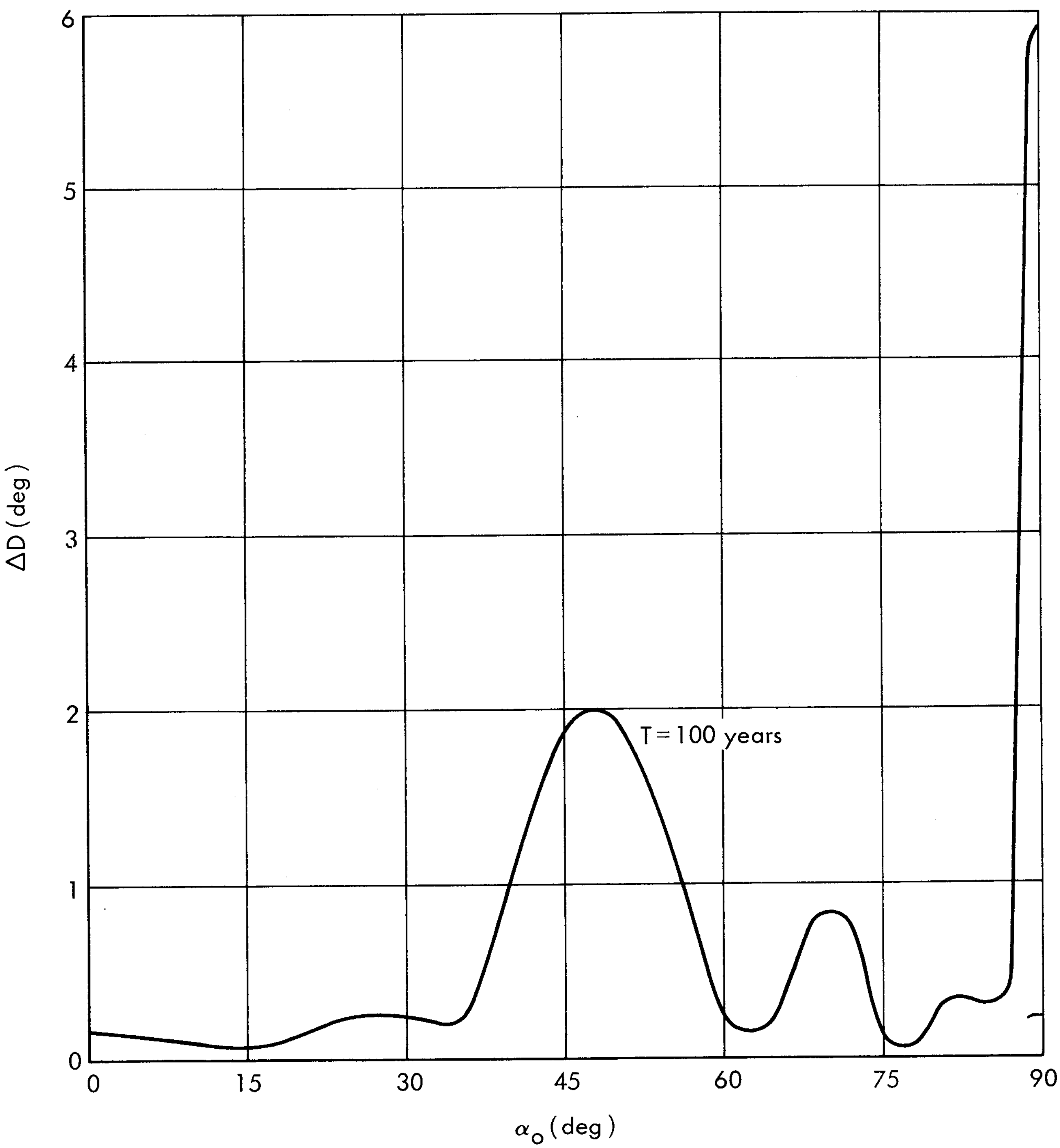

Fig.31 — Drift angle as a function of α_o (r_o = 4 earth radii)

$$(X_{rs} + Y_{rs})_{max} = \left|(A_r A_s + B_r B_s)\left(\frac{A-C}{2} + \frac{A-B}{2}\right)\right|$$

$$+ \left|(A_r A_s - B_r B_s)\left(\frac{A-C}{2} - \frac{A-B}{2}\right)\right| \qquad \text{(D-23)}$$

Thus, an upper bound on σ_D^2 can be established by the relation

$$\sigma_D^2 \leq \sum_{r=1}^{127} \sum_{s=1}^{127} (X_{rs} + Y_{rs})_{max} \qquad \text{(D-24)}$$

The mean-square deviation σ_D^2 is evaluated as a function of the orbital inclination of a synchronous altitude orbit for values of T equal to 50, 100 and 500 years. The results are shown in Fig. 32. The summations of Eq. (D-24) are taken over the twelve lowest frequency terms, as in the case of ΔD.

An examination of Fig. 32 shows that for all values of T the curves are essentially the same for the lower values of α_o. However, as α_o increases, σ_D goes through a maximum. The height of this maximum increases with T, while the corresponding value of α_o approaches 90^o. As in the case of ΔD this maximum results from the fact that ω_2 is approaching zero and the summation in Eq. (D-24) is dominated by the term $(X_{22} + Y_{22})_{max}$. Figure 33 represents the limiting curves for Eq. (D-22) when r = s. Thus, the heavy portions of these curves represent $(X_{ii} + Y_{ii})_{max}$ which appears in σ_D^2. A comparison of Fig. 33 and Fig. 32 shows that the maximum in Fig. 32 is equivalent to a plot of Fig. 33 as a function of α_o for i = 2.

In view of the long solution time required to determine the curves in Fig. 32, less detailed information is available for other orbital

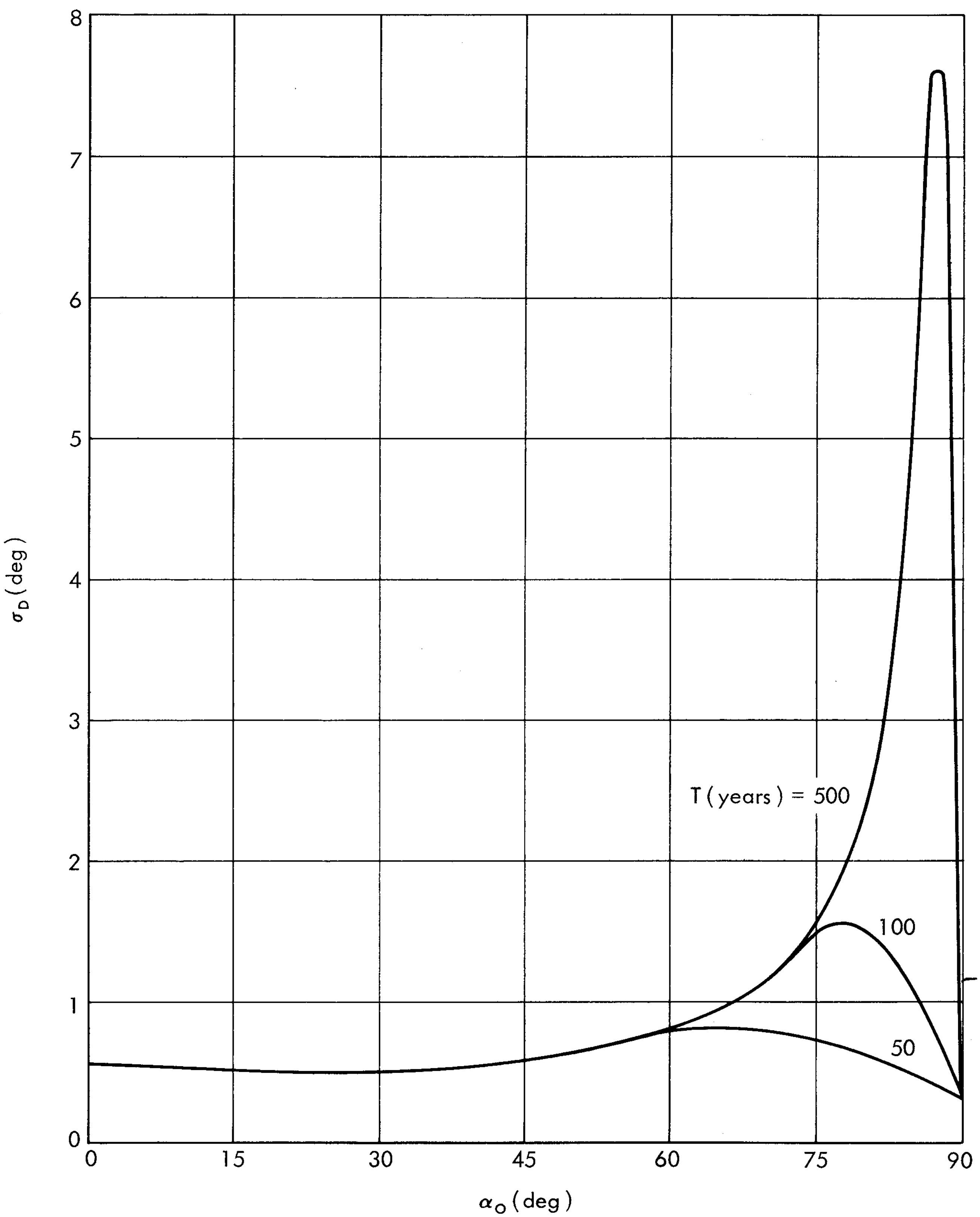

Fig. 32 — Root-mean-square deviation of the orbit normal as a function of α_o (synchronous orbit)

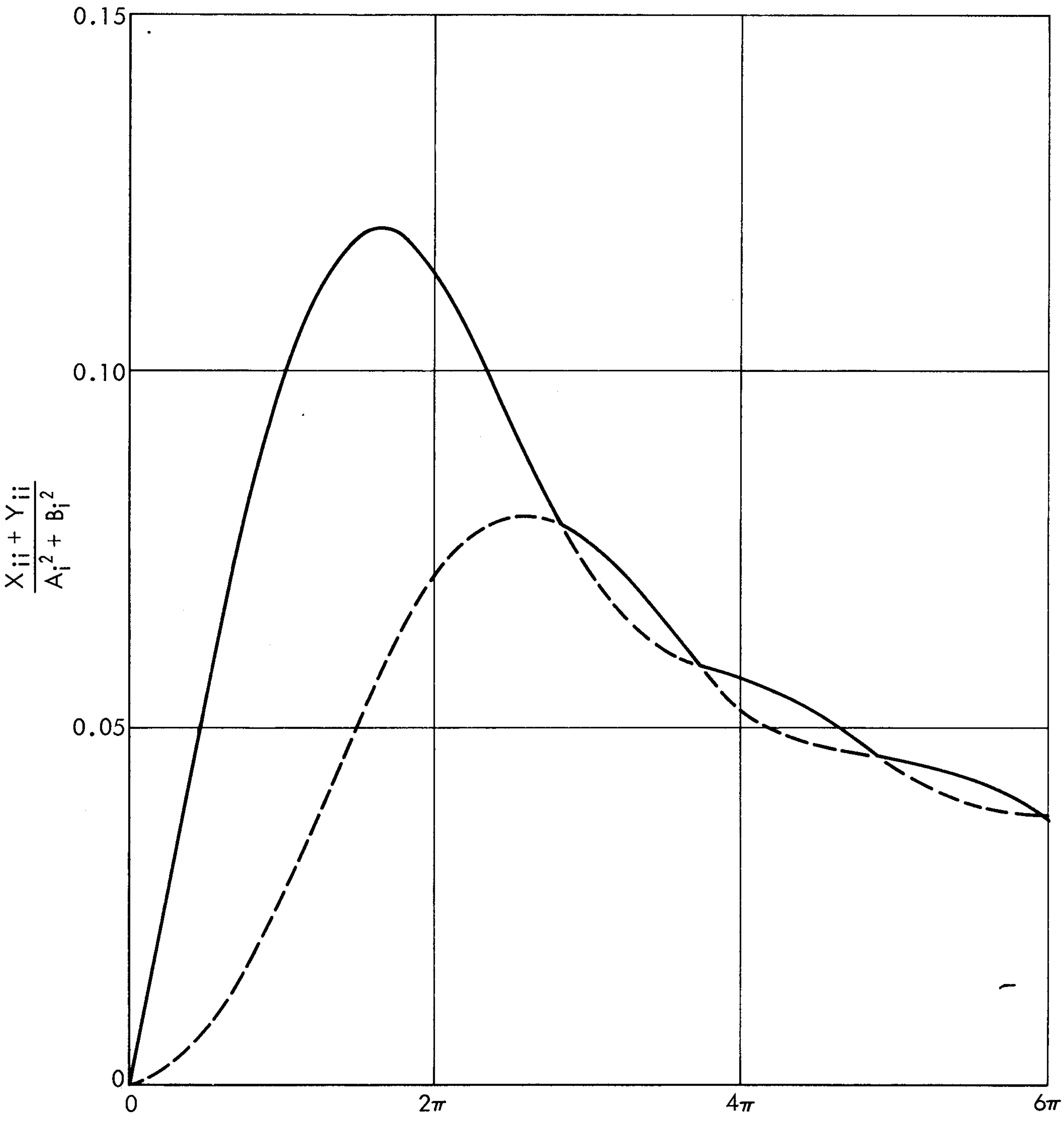

Fig.33—Dependence of $X_{ii} + Y_{ii}$ on $\omega_i T$

altitudes. However, it is found that the magnitude of σ_D^2 decreases with the orbital radius.

Figure 34 shows the details of the σ_D^2 curve for an orbital radius of 4 earth radii and an averaging time of 500 years for inclination angles in the vicinity of $\alpha_o = 50^o$. This corresponds to the region in which both ω_{24} and ω_{27} vanish. It is seen that maxima occur on either side of the value of α_o for which the two frequencies vanish. The shape of these maxima is also related to that shown in Fig. 33 for i = 24 and 27.

DISCUSSION

On the basis of the foregoing analysis, it is seen that the values of both ΔD and σ_D increase with orbital radius. Since the highest radius orbit considered here is that at synchronous altitude, Figs. 30 and 32 represent upper bounds for ΔD and σ_D respectively. An examination of Figs. 30 and 32 shows that both the total drift and the root-mean-square deviation are less than $.5^o$ as long as the averaging time is appreciably larger than the periods of the oscillations included and if the orbital inclination angle, α_o, is less than about 75^o. For orbital inclinations between 75^o and 90^o, significant peaks occur in the curves for both ΔD and σ_D. Thus, the steady-state motion described by Eqs. (57) and (58) is not valid at these high inclination angles where $\dot{\psi}$ becomes very small. As indicated previously, the behavior of these high-inclination orbits is determined in Appendix F.

Thus, it is seen that for inclination angles up to about 75^o the normal to the orbital plane deviates less than $.5^o$ from its position as described by Eqs. (57) and (58). For an inclination greater than

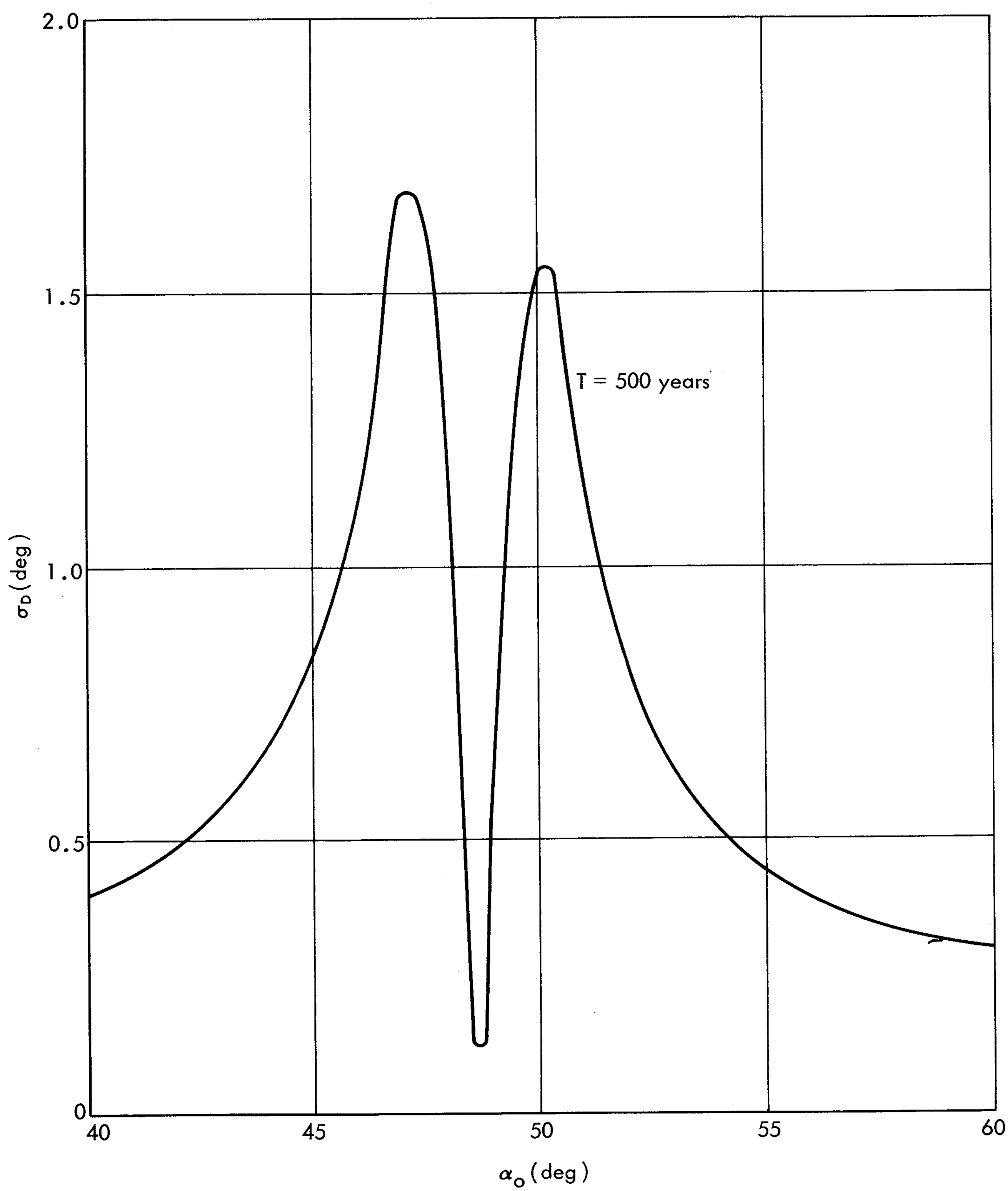

Fig. 34 — Root-mean-square deviation of the orbit normal as a function of α_o (r_o = 4 earth radii)

75^{o}, the regression may take place about the x_1 axis as described in Appendix F. Since these high-inclination orbits are of less interest, the contributions of the residual oscillatory terms for this second type of regression have not been evaluated. However, it does not appear that any large oscillatory contribution would result since many of the A_i and B_i values vanish at an inclination of 90^{o}.

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Appendix E

LUNAR REGRESSION

The satellite equations of motion developed in Appendix B should also apply to the case of the earth's most prominent satellite, the moon. However, as indicated earlier, it is necessary to go to a higher order approximation of the solution before the moon's orbital regression can be adequately described. The details of this determination are presented below.

MODIFICATION OF EQUATIONS OF MOTION

The general equations of motion of an earth satellite, including perturbations due to earth oblateness, the sun and the moon, are developed in Appendix B as Eqs. (B-57) through (B-60). In applying these equations to the case of the moon, a number of simplifcations can be made.

Of the three perturbations considered, the only significant one in the case of the moon is that due to the sun. At the moon's altitude the effect of earth oblateness is negligible, and the moon obviously cannot act as a perturbing influence on itself. Thus, the right side of Eqs. (B-57) through (B-60) should be modified by setting J_2 and $\dot{\theta}_m$ equal to zero.

In the original development, the reaction of the satellite on the earth is neglected. However, this is not possible for a satellite as large as the moon. Thus, in Eq. (B-57) it is necessary to replace GM_E by $G(M_E + M_m)$.

An examination of Eq. (54) shows that since the earth oblateness effect is negligible ($J_2 = 0$), the angle α_1 is zero. Thus, the reference plane becomes the plane of the ecliptic, and the x_o, y_o, z_o and x_1, y_1, z_1 reference systems are identical.

Finally, it can also be assumed that the moon's orbital inclination to the ecliptic is sufficiently small that cos α is unity while sin α is equal to α.

Under the above assumptions, Eqs. (B-57) through (B-60) become

$$\frac{d^2\rho}{dt^2} - \rho\omega^2 = - \frac{G(M_E + M_m)}{\rho^2} - \rho\dot{\Theta}^2\left[1 - 3(i \cdot r_1)^2\right] \tag{E-1}$$

$$\frac{d}{dt}[\rho^2\omega] = 3\rho^2\dot{\Theta}^2(\overline{r}_1 \cdot \overline{i})\ (\overline{r}_1 \cdot \overline{j}) \tag{E-2}$$

$$\omega\dot{\psi} = \frac{3\dot{\Theta}^2(\overline{r}_1 \cdot \overline{i})\ (\overline{r}_1 \cdot \overline{k}) \sin \theta}{\alpha} \tag{E-3}$$

$$\omega\dot{\alpha} = 3\dot{\Theta}^2(\overline{r}_1 \cdot \overline{i})\ (\overline{r}_1 \cdot \overline{k}) \cos \theta \tag{E-4}$$

where

$$\omega = \dot{\theta} + \dot{\psi} \tag{E-5}$$

and r has been replaced by ρ.

By means of the direction cosines listed in Appendix A, the scalar products can be expressed as

$$(\overline{r}_1 \cdot \overline{i}) = \cos(\theta - \Theta + \psi) \tag{E-6}$$

$$(\overline{r}_1 \cdot \overline{j}) = - \sin(\theta - \Theta + \psi) \tag{E-7}$$

$$(r_1 \cdot k) = - \alpha \sin(\Theta - \psi) \tag{E-8}$$

Substitution of Eqs. (E-6) through (E-8) in Eqs. (E-1) through (E-4) gives

$$\frac{d^2\rho}{dt^2} - \rho\omega^2 = - \frac{G(M_E + M_m)}{\rho^2}$$

$$+ \frac{\rho\dot{\Theta}^2}{2}\left[1 + 3 \cos 2(\theta + \psi - \Theta)\right] \tag{E-9}$$

$$\frac{d}{dt}(\rho^2\omega) = - \frac{3}{2} \rho^2\dot{\Theta}^2 \sin 2(\theta + \psi - \Theta) \tag{E-10}$$

$$\dot{\psi} = - \frac{3\dot{\Theta}^2}{\omega} \sin \theta \sin(\Theta - \psi) \cos(\theta + \psi - \Theta) \tag{E-11}$$

$$\dot{\alpha} = - \frac{3\dot{\Theta}^2}{\omega} a \cos \theta \sin(\Theta - \psi) \cos (\theta + \psi - \Theta) \tag{E-12}$$

These expressions represent the desired equations of motion of the moon relative to the earth as affected by solar perturbations.

METHOD OF SOLUTION

In the case of the moon, it is not possible to determine $\dot{\psi}$ and $\dot{\alpha}$ by substituting the unperturbed values of θ and ω in Eqs. (E-11) and

(E-12). Instead, it is necessary to take into account the effect of perturbations in ω and θ on the equation for $\dot{\psi}$ and $\dot{\alpha}$, which in turn necessitates a solution for the perturbed in-plane motion as described by Eqs. (E-9) and (E-10).

In-Plane Equations

If ω_o represents the observed sidereal rate of the moon around the earth, then the corresponding orbital radius in the absence of perturbations is defined by the relation

$$\rho_o = \sqrt[3]{\frac{G(M_E + M_m)}{\omega_o^2}} \tag{E-13}$$

Substitution of Eq. (E-13) in Eq. (E-9) gives

$$\frac{d^2\rho}{dt^2} - \rho\omega^2 = -\frac{\rho_o^3\omega_o^2}{\rho^2} + \frac{\rho\dot{\Theta}^2}{2}\left[1 + 3\cos(\theta + \psi + \Theta)\right] \tag{E-14}$$

as the form of the radial in-plane equation.

While ρ_o and ω_o are the unperturbed solutions of Eqs. (E-10) and (E-14), the quantities $\delta\rho$ and $\delta\omega$ represent the perturbations to these solutions which occur due to the sun. By making the following substitutions in Eqs. (E-10) and (E-14),

$$\rho = \rho_o + \delta\rho \tag{E-15}$$

$$\omega = \omega_o + \delta\omega \tag{E-16}$$

$$\Theta = \dot{\Theta} t \tag{E-17}$$

$$\theta + \psi = \omega_o t \tag{E-18}$$

$$\omega_o - \dot{\Theta} = \dot{\xi} \tag{E-19}$$

the following perturbation equations are obtained in terms of $\delta\rho$ and $\delta\omega$.

$$\frac{d^2\delta\rho}{dt^2} - 3\omega_o^2\delta\rho - 2\rho_o\omega_o\delta\omega = \frac{\rho_o\dot{\Theta}^2}{2}\left[1 + 3\cos 2\dot{\xi}t\right] \tag{E-20}$$

$$2\omega_o\rho_o\frac{d\delta\rho}{dt} + \rho_o^2\frac{d\delta\omega}{dt} = -\frac{3\rho_o^2\dot{\Theta}^2}{2}\sin 2\dot{\xi}t \tag{E-21}$$

The solution to these equations is given by

$$\delta\rho = \left[2\left(2\delta\rho_o + \rho_o\frac{\delta\omega_o}{\omega_o}\right) - \frac{\rho_o m^2(2 + m)}{2(1 - m)}\right]$$

$$+ \cos\omega_o t\left[-\left(3\delta\rho_o + 2\rho_o\frac{\delta\omega_o}{\omega_o}\right) + \frac{2\rho_o m^2(3 - m - m^2)}{3 - 8m + 4m^2}\right]$$

$$- \cos 2\dot{\xi}t\left[\frac{\rho_o m^2(2 - m)}{6(1 - m)(3 - 8m + 4m^2)}\right] \tag{E-22}$$

and

$$\frac{\delta\omega}{\omega_o} = \left[-\frac{3}{\rho_o}\left(2\delta\rho_o + \frac{\rho_o\delta\omega_o}{\omega_o}\right) + \frac{m^2(5 + 4m)}{4(1 - m)}\right]$$

$$+ 2\cos\omega_o t\left[\frac{1}{\rho_o}\left(3\delta\rho_o + \frac{2\rho_o\delta\omega_o}{\omega_o}\right) - \frac{2m^2(3 - m - m^2)}{3 - 8m + 4m^2}\right]$$

$$+ \cos 2\dot{\xi}t\left[\frac{3m^2(11 - 12m + 4m^2)}{4(1 - m)(3 - 8m + 4m^2)}\right] \tag{E-23}$$

where m is the ratio of the earth's angular rate around the sun to the moon's angular rate around the earth, and $\delta\rho_o$ and $\delta\omega_o$ are the initial values of $\delta\rho$ and $\delta\omega$.

Since $\delta\rho_o$ and $\delta\omega_o$ are arbitrary constants, they can be used to simplify the above solution by setting the coefficient of $\cos\omega_o t$ and the constant term in Eq. (E-23) equal to zero as follows

$$2\delta\rho_o + \frac{\rho_o\delta\omega_o}{\omega_o} = \frac{\rho_o m^2(5 + 4m)}{12(1 - m)} \tag{E-24}$$

$$3\delta\rho_o + \frac{2\rho_o\delta\omega_o}{\omega_o} = \frac{2\rho_o m^2(3 - m - m^2)}{3 - 8m + 4m^2} \tag{E-25}$$

These two conditions reduce Eqs. (E-22) and (E-23) to

$$\delta\rho = -\frac{\rho_o m^2}{6} - \frac{3\rho_o m^2(2 - m)}{2(1 - m)(3 - 8m + 4m^2)}\cos 2\dot{\xi}t \tag{E-26}$$

and

$$\delta\omega = \frac{3\omega_o(11 - 12m + 4m^2)}{4(1 - m)(3 - 8m + 4m^2)} \cos 2\dot{\xi}t \tag{E-27}$$

Substitution of these values of $\delta\rho$ and $\delta\omega$ in Eqs. (E-15) and (E-16) gives the perturbed solutions for ρ and ω as

$$\rho = \rho_o\left[1 - \frac{m^2}{6} - m^2\left(1 + \frac{19}{6} m\right) \cos 2\dot{\xi}t\right] \tag{E-28}$$

$$\omega = \omega_o\left[1 + \left(\frac{11}{4} m^2 + \frac{85}{12} m^3\right) \cos 2\dot{\xi}t\right] \tag{E-29}$$

where powers of m greater than three are neglected in the expansion of $\delta\rho$ and $\delta\omega$.

From Eqs. (E-28) and (E-29) it is seen that as a result of the choice of $\delta\rho_o$ and $\delta\omega_o$, the perturbed solution still has the observed mean orbital rate ω_o. However, to achieve this, the mean orbital radius must be less than the unperturbed radius by an amount $\rho_o m^2/6$, which is of the order of 220 mi.

Out-of-Plane Equations

Regression. It is now possible to determine the moon's regression rate, $\dot{\psi}$, by means of Eq. (E-11), which can be transformed into the following form:

$$\dot{\psi} = -\frac{3\dot{\Theta}^2}{4\omega}\Big[1 - \cos 2\theta - \cos 2(\Theta - \psi) + \cos 2(\theta + \psi - \Theta)\Big] \tag{E-30}$$

If $\dot{\psi}_o$ and $\delta\dot{\psi}$ represent the steady-state and oscillatory components, respectively, of the regression rate, then $\dot{\psi}$ can be represented as

$$\dot{\psi} = \dot{\psi}_o + \delta\dot{\psi} \qquad \text{(E-31)}$$

and the problem reduces to a determination of $\dot{\psi}_o$ from Eq. (E-30). Before this can be done, it is necessary to obtain expressions for the quantities θ and ψ as functions of time as follows.

From Eq. (E-5), $\dot{\theta}$ can be expressed as

$$\dot{\theta} = \omega - \dot{\psi}$$

$$= \omega_o - \dot{\psi}_o + \delta\omega - \delta\dot{\psi} \qquad \text{(E-32)}$$

which can be integrated to give

$$\theta = (\omega_o - \dot{\psi}_o)\ t + \int \delta\omega dt - \int \delta\dot{\psi} dt \qquad \text{(E-33)}$$

Since the quantities $\delta\omega$ and $\delta\dot{\psi}$ by definition have no steady-state value, their integrals will represent small oscillatory angles.

Similarly, the angle ψ can be expressed as

$$\psi = \dot{\psi}_o t + \int \delta\dot{\psi} dt \qquad \text{(E-34)}$$

If Eqs. (E-17), (E-29), (E-32) and (E-34) are substituted in Eq. (E-30), the following expression for $\dot{\psi}$ is obtained to the order of m^4

$$\dot{\psi} = - \frac{3\omega_o m^2}{4} \Big[(1 - \frac{11}{4} m^2) - (1 - \frac{11}{8} m^2) \cos 2(\omega_o - \dot{\psi}_o) t$$

$$- (1 - \frac{11}{4} m^2) \cos 2(\dot{\Theta} - \dot{\psi}_o) t$$

$$+ (1 - \frac{11}{4} m^2) \cos 2(\omega_o - \dot{\Theta}) t$$

$$+ \frac{11}{8} m^2 \cos 2(\omega_o - 2\dot{\Theta} + \dot{\psi}_o) t$$

$$- 2\Big[\sin 2(\omega_o - \dot{\psi}_o) t + \sin 2(\dot{\Theta} - \dot{\psi}_o) t \Big] \int \delta\dot{\psi} dt \Big] \qquad \text{(E-35)}$$

The values of $\dot{\psi}_o$ and $\delta\dot{\psi}$ can be determined by successive iterations of Eq. (E-35) as follows. As a first approximation

$$\dot{\psi}_o = - \frac{3\omega_o m^2}{4} (1 - \frac{11}{4} m^2) \qquad \text{(E-36)}$$

while $\delta\dot{\psi}$ is given by the remainder of Eq. (E-35), neglecting the integral term

$$\delta\dot{\psi} = - \frac{3\omega_o m^2}{4} \Big[- (1 - \frac{11}{8} m^2) \cos 2(\omega_o - \dot{\psi}_o) t$$

$$- (1 - \frac{11}{4} m^2 - \frac{59}{12} m^3) \cos 2(\dot{\Theta} - \dot{\psi}_o) t$$

$$+ \text{ other oscillatory terms} \Big] \qquad \text{(E-37)}$$

Integration of Eq. (E-37) gives

$$\int \delta\dot{\psi}dt = \frac{3}{8} m^2 \left[\sin 2(\omega_o - \dot{\psi}_o) t + \frac{1}{m} (1 - \frac{3}{4} m) \sin 2(\dot{\Theta} - \dot{\psi}_o) t + \text{other oscillatory terms}\right] \quad \text{(E-38)}$$

Substitution of Eq. (E-38) in Eq. (E-35) gives an improved determination of $\dot{\psi}$ in the form

$$\dot{\psi} = - \frac{3\omega_o m^2}{4} \left[(1 - \frac{3}{8} m - \frac{91}{32} m^2) + \text{oscillatory terms}\right] \quad \text{(E-39)}$$

A second iteration is not necessary since no additional terms of the order of m^4 arise in the steady-state regression rate. Thus, the regression rate of the moon is given by

$$\dot{\psi}_o = - \omega_o\left[\frac{3}{4} m^2 - \frac{9}{32} m^3 - \frac{273}{128} m^4\right] \quad \text{(E-40)}$$

which is identical with the expression determined in Ref. 3 using the method of Delaunay.

By equating the steady-state parts of Eq. (E-5), the mean angular rate $\dot{\theta}_o$ is obtained as

$$\dot{\theta}_o = \omega_o - \dot{\psi}_o \quad \text{(E-41)}$$

which when combined with Eq. (E-40) gives

$$\dot{\theta}_o = \omega_o\left[1 + \frac{3}{4} m^2 - \frac{9}{32} m^3 - \frac{273}{128} m^4\right] \tag{E-42}$$

Since the angle θ is measured from the line of nodes, the period associated with $\dot{\theta}_o$ is the nodical period or the period between passages of the ascending node, while the period associated with ω_o is the sidereal period as indicated previously. The relation of these two periods is thus obtained from Eq. (E-42) as

$$T_S = T_N\left[1 + \frac{3}{4} m^2 - \frac{9}{32} m^3 - \frac{273}{128} m^4\right] \tag{E-43}$$

Inclination. It is also possible to determine the variation in orbital inclination by expanding Eq. (E-12) as follows

$$\dot{\alpha} = -\frac{3\dot{\Theta}^2\alpha}{4\omega}\Big[\sin 2\theta - \sin 2(\theta - \Theta + \psi) + \sin 2(\Theta - \psi)\Big] \tag{E-44}$$

A comparison with Eq. (E-30) shows that the oscillatory amplitudes in Eq. (E-44) are reduced considerably by the factor α which is equal to about .1 rad. Thus, the determination of α is somewhat simpler than that of ψ described above.

Equation (E-44) can be written in the form

$$\dot{\alpha} = - \frac{3\omega_o m^2 \alpha}{4} \Big[\sin 2(\omega_o - \dot{\psi}_o)\, t$$

$$- \sin 2(\omega_o - \dot{\Theta})\, t$$

$$+ \sin 2(\dot{\Theta} - \dot{\psi}_o)\, t \Big] \qquad \text{(E-45)}$$

where ω is replaced by ω_o and the terms in $\delta\omega$ and $\delta\dot{\psi}$ are neglected. Integration of Eq. (E-45) gives

$$\log \frac{\alpha}{\alpha_o} = - \frac{3\omega_o m^2}{8} \Big[\frac{1}{\omega_o - \dot{\psi}_o} \Big(1 - \cos 2(\omega_o - \dot{\psi}_o)\, t\Big)$$

$$- \frac{1}{\omega_o - \dot{\Theta}} \Big(1 - \cos 2(\omega_o - \dot{\Theta}) t\Big)$$

$$+ \frac{1}{(\dot{\Theta} - \dot{\psi}_o)} \Big(1 - \cos 2(\dot{\Theta} - \dot{\psi}_o)\, t\Big) \Big] \qquad \text{(E-46)}$$

and since the terms on the right are small, Eq. (E-46) can be expressed to the order of m^2 as

$$\alpha = \alpha_o - \frac{3m^2 \alpha_o}{8} \Big[\frac{1}{m} \Big(1 - \cos 2(\dot{\Theta} - \dot{\psi}_o)\, t\Big)$$

$$- \cos 2(\omega_o - \dot{\psi}_o)\, t$$

$$+ \cos 2(\omega_o - \dot{\Theta})\, t \Big] \qquad \text{(E-47)}$$

The constant term inside the bracket can be absorbed in α_o so that

$$\alpha = \alpha_o + \frac{3m^2\alpha_o}{8}\left[\frac{1}{m}\cos 2(\dot{\Theta} - \dot{\psi}_o)\, t \right.$$

$$\left. + \cos 2(\omega_o - \dot{\psi}_o)\, t - \cos 2(\omega_o - \dot{\Theta})\, t\right] \qquad \text{(E-48)}$$

The first and largest of the oscillatory terms in Eq. (E-48) has an amplitude of $3m\alpha_o/8$ or about 8.6 min of arc, while the other two have amplitudes of about .6 min of arc.

Thus, it is seen that the inclination of the moon's orbit remains essentially constant.

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Appendix F

REGRESSION OF HIGH INCLINATION ORBITS

INTRODUCTION

In the body of this Report, the solutions for the orbital inclination angle, α, and orbital regression angle, ψ, are given by Eqs. (51) and (52). In order to obtain these expressions it was assumed that where ψ appears on the right side of Eqs. (37) and (38) it can be replaced by $\dot{\psi}_o t$. In view of the fact that $\dot{\psi}_o$ is proportional to the cosine of the inclination angle, it becomes very small as the inclination approaches 90^o. An examination of the resulting solution for ψ given by Eq. (52) shows that under these conditions the oscillatory terms are no longer negligible. Thus, for high-inclination orbits, the above assumption does not hold. This is indicated by the increase in the values of ΔD and σ_D in the vicinity of 90^o as shown in Figs. 30 and 32. While the assumption does appear to be valid up to inclinations of the order of 75^o, it is of interest to investigate the behavior of higher inclination orbits.

METHOD OF ANALYSIS

If the substitution of $\dot{\psi}_o t$ for ψ is not made, then Eqs. (49) and (50) can be written in the form

$$\dot{\alpha} = A_1 \sin \psi + A_2 \sin 2\psi + \sum_{i=3}^{127} A_i \sin \omega_i t \qquad \text{(F-1)}$$

$$\dot{\psi} = \dot{\psi}_o + \frac{1}{\sin \alpha} \left[B_1 \cos \psi + B_2 \cos 2\psi + \sum_{i=3}^{127} B_i \cos \omega_i t \right] \qquad \text{(F-2)}$$

While ψ is also present in the other oscillatory terms, it constitutes a slowly varying phase angle of a higher frequency oscillation and can be neglected. However, when it stands alone, as in the first two oscillatory terms, it must be considered. An examination of the numerical values of A and B shows that the summation terms in Eqs. (F-1) and (F-2) are negligible, so that

$$\dot{\alpha} = A_1 \sin \psi + A_2 \sin 2\psi \tag{F-3}$$

$$\dot{\psi} = \dot{\psi}_o + \frac{1}{\sin \alpha}\left[B_1 \cos \psi + B_2 \cos 2\psi\right] \tag{F-4}$$

If the reference plane is defined in the same manner as before to make A_1 and B_1 zero, then Eqs. (F-3) and (F-4) can be expressed as

$$\dot{\alpha} = Q \sin \alpha \sin 2\psi \tag{F-5}$$

$$\dot{\psi} = -(P - Q \cos 2\psi) \cos \alpha \tag{F-6}$$

where P and Q are defined by Eqs. (C-7) and (C-8) of Appendix C.

Combination of Eqs. (F-5) and (F-6) gives the relation

$$\frac{d\alpha}{d\psi} = -\frac{Q \sin \alpha \sin 2\psi}{(P - Q \cos 2\psi)\cos \alpha} \tag{F-7}$$

which can be integrated to give

$$\sin \alpha = \sin \alpha_o \sqrt{\frac{P - Q \cos 2\psi_o}{P - Q \cos 2\psi}} \tag{F-8}$$

as the functional relation between α and ψ. The quantities α_o and ψ_o represent the initial values of α and ψ, and, in accordance with Eqs. (51) and (52), ψ_o is equal to zero when α equals α_o. Thus, Eq. (F-8) becomes

$$\sin \alpha = \sin \alpha_o \sqrt{\frac{P - Q}{P - Q \cos 2\psi}} \tag{F-9}$$

However, for the purposes of this analysis it is more convenient to select the initial position when ψ is 90^o and α has a value of α_o'. Under these conditions, Eq. (F-8) becomes

$$\sin \alpha = \sin \alpha_o' \sqrt{\frac{P + Q}{P - Q \cos 2\psi}} \tag{F-10}$$

while α_o and α_o' are related by the expression

$$\sin \alpha_o' = \sin \alpha_o \sqrt{\frac{P - Q}{P + Q}} \tag{F-11}$$

By means of Eq. (F-10), the trace of the normal to the orbital plane on a unit sphere can be determined for any given value of α_o'. In Fig. 35, several of these traces are shown for various orbital inclinations of a synchronous altitude orbit. In this figure, the angle ψ is measured in the x_1y_1 plane from the negative y_1 axis while α is the elevation angle measured from the z_1 axis. It is seen that for values of α_o' up to about 75^o the traces encircle the z_1 axis at a relatively constant value for the inclination angle, α. This corresponds to the regression described in the body of the report and pictured in Fig. 6.

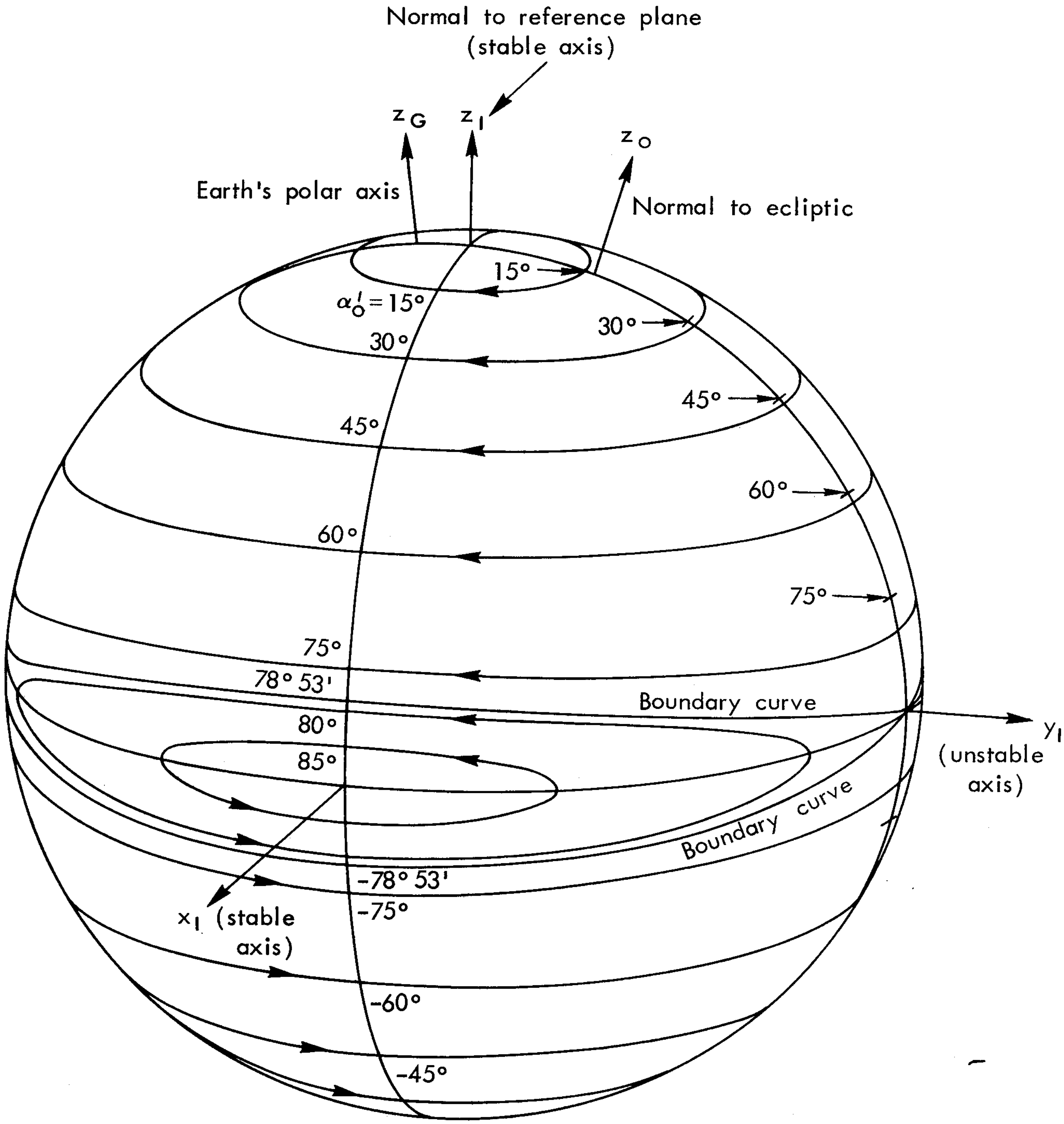

Fig. 35—Regression of a synchronous altitude orbit

For values of α_o' such that

$$\sin \alpha_o' > \sqrt{\frac{P - Q}{P + Q}} \tag{F-12}$$

the nature of the traces change and they become elongated ellipses which encircle the x_1 axis.

The boundaries between the two types of trace correspond to the conditions

$$\sin \alpha_o' = \sqrt{\frac{P - Q}{P + Q}} \tag{F-13}$$

or a value $\alpha_o' = 78^o53'$ for synchronous altitude orbits. The actual contours lie in planes including the y_1 axis and inclined at angles of $\pm 11^o 7'$ to the $x_1 y_1$ reference plane.

The periodicity of the motion corresponding to these traces can be determined if the angle α is eliminated between Eqs. (F-6) and (F-10) to give

$$\dot{\psi} = - \sqrt{(P - Q \cos 2\psi) (P \cos^2 \alpha_o' - Q \sin^2 \alpha_o' - Q \cos 2\psi)} \tag{F-14}$$

which can be solved for t in the form

$$t = - \int_{\frac{\pi}{2}}^{\psi} \frac{dx}{\sqrt{(P - Q \cos 2\psi) (P \cos^2 \alpha_o' - Q \sin^2 \alpha_o' - Q \cos 2\psi)}} \tag{F-15}$$

By means of the transformations given in Ref. 4, Eq. (F-15) can be expressed as an elliptic integral of the first kind. The resulting expression for the period of the motion is given by the following equations. If $\sin \alpha_o' \leq \sqrt{\frac{P - Q}{P + Q}}$, then

$$T = \frac{4}{\sqrt{P^2 - Q^2} \cos \alpha_o'} \int_o^{\frac{\pi}{2}} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} \tag{F-16}$$

where

$$k^2 = \frac{2Q}{P - Q} \tan^2 \alpha_o' \tag{F-17}$$

If $\sin \alpha_o' \geq \sqrt{\frac{P - Q}{P + Q}}$

$$T = \frac{4}{\sqrt{2Q(P + Q)} \sin \alpha_o'} \int_o^{\frac{\pi}{2}} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} \tag{F-18}$$

where

$$k^2 = \frac{P - Q}{2Q \tan^2 \alpha_o'} \tag{F-19}$$

Thus, it is seen that the period given by Eq. (F-16) corresponds to the more conventional regression about the z_1 axis while that given by Eq. (F-18) is associated with the elongated elliptical traces of Fig. 35. A plot of the regression period as a function of α_o' is shown in Fig. 36.

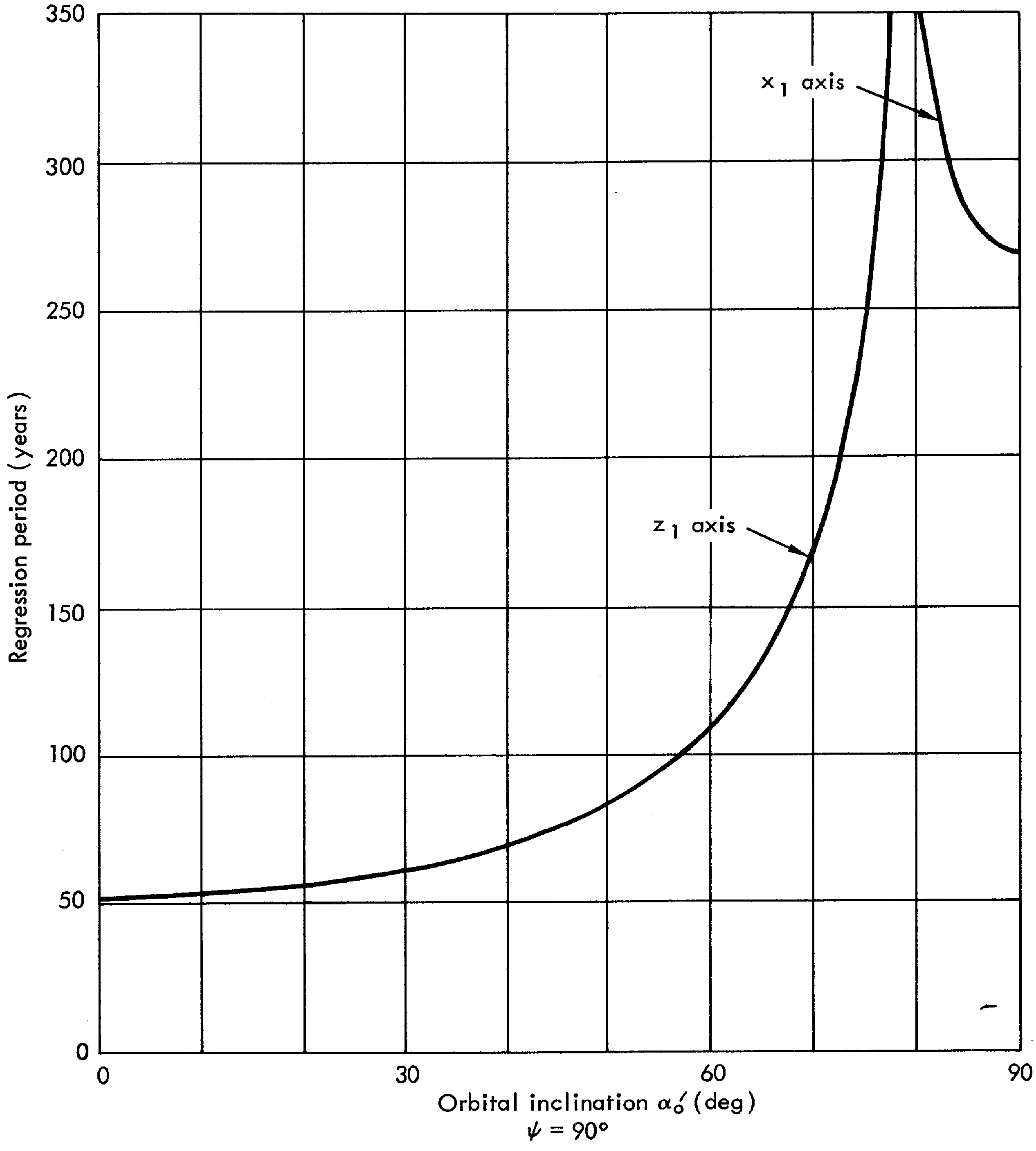

Fig. 36 — Periods of regression about x_1 and z_1 axes

RESULTS AND DISCUSSION

In the body of the Report it is shown that the x_1y_1 plane has the property that an orbit in this plane would maintain its orientation relative to inertial space. However, an examination of Fig. 35 shows that a second stable configuration exists with the orbit in the y_1z_1 plane. Since the earth's axis lies in this plane, such an orbit would not only remain fixed relative to inertial space but would remain polar relative to the earth. On the other hand, an orbit established in the x_1z_1 plane is in unstable equilibrium and may regress about either the x_1 or z_1 axis, depending on the direction of its initial disturbance.

In order to compare the regression periods determined by this method with those shown in Fig. 7, it is necessary to express Eq. (F-16) in terms of the inclination angle when ψ is equal to zero. This is done by combining Eqs. (F-11) and (F-16) to give

$$T = \frac{4}{P \cos \alpha_o} \int_0^{\frac{\pi}{2}} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} \left[\frac{1}{(1 - \frac{Q}{P}) \sqrt{1 + \frac{2Q}{P - Q} \cos^2 \alpha_o}}\right] \tag{F-20}$$

As compared with the expression

$$T = \frac{2\pi}{P \cos \alpha_o} \tag{F-21}$$

determined in the body of the Report as Eq. (62). Figure 37 is a plot of Eqs. (F-20) and (F-21) for synchronous altitude orbits. It is seen that the agreement is excellent up to large values of α_o.

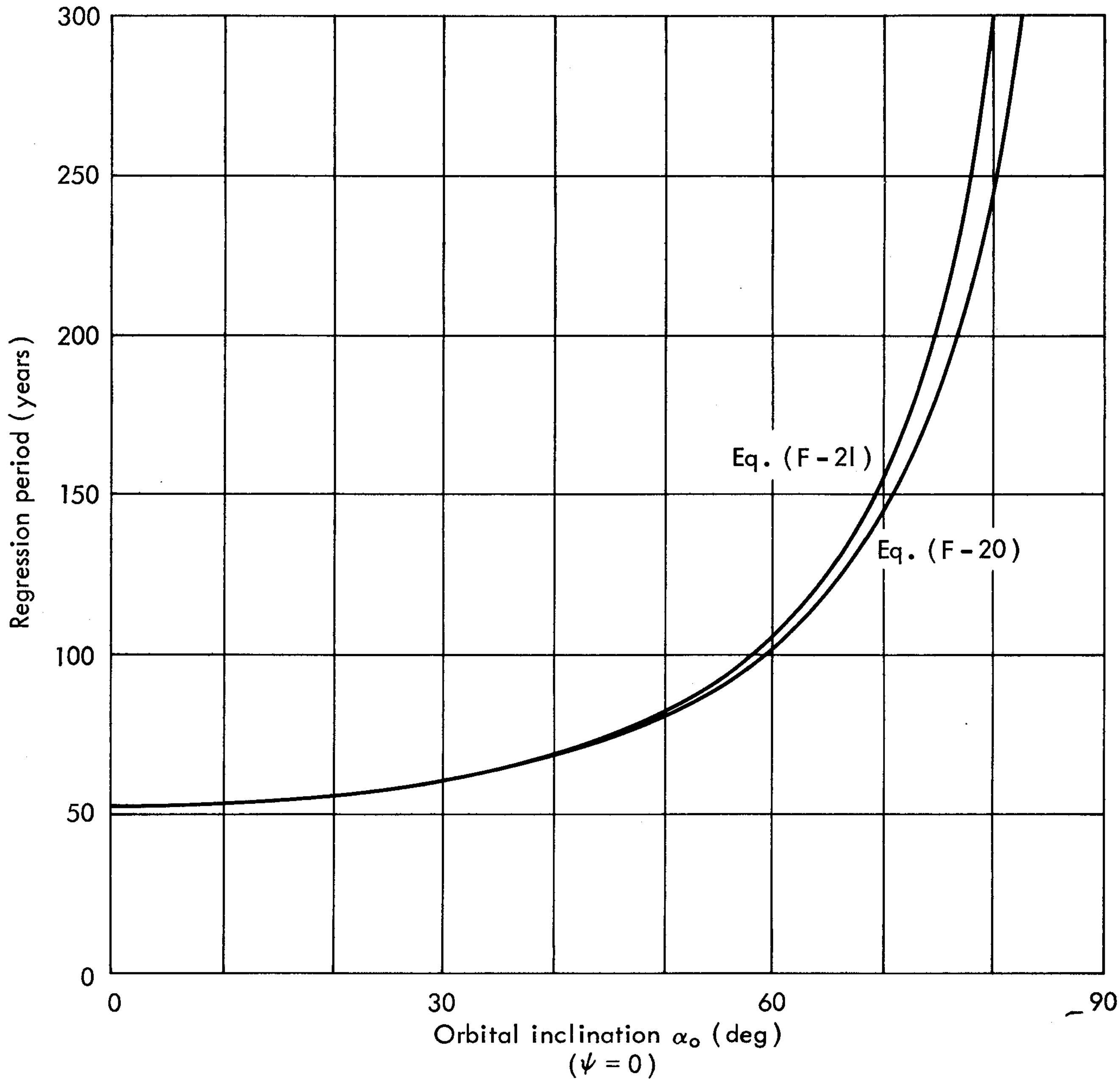

Fig. 37— Comparison of Eqs. (F-20) and (F-21) for regression period

The maximum variation of orbital inclination during regression about the z_1 axis is equal to the difference between α_o and α_o'. This can be seen in Fig. 35, where the α_o' values are indicated relative to the actual position of the trace when it crosses the y_1z_1 plane. It is seen that the variation is relatively small, having reached a value of only 5^o at an inclination of 75^o. Thus, the representation of this type of regression by a conical motion of the normal to the orbital plane as shown in Fig. 6 also appears to be valid as long as the regression is about the z_1 axis.

The results obtained in this appendix are in excellent agreement with those obtained in Ref. 5 by Allan and Cook who used eigenvalue methods to determine the nature of the regression and its periodicity.

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