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# AN APPROXIMATE METHOD FOR DETERMINING ABSORPTION COEFFICIENTS

MITCHELL W. FINKEL

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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List of Terms:

$F$ = Forward scattering coefficient for diffuse incidence

$F'$ = Forward scattering coefficient for parallel incidence

$B$ = Backward scattering coefficient for diffuse incidence

$B'$ = Backward scattering coefficient for parallel incidence

$\mu$ = The absorption coefficient for the diffuse flux

$\mu'$ = The absorption coefficient for the parallel flux

$t$ = Diffuse flux in the forward direction

$s$ = Diffuse flux in the backward direction

$I_x'$ = The residue of the primary beam

$J_{(\theta)}$ = Spectral radiant intensity

$A_D$ = Fraction of total absorptance assigned to the diffuse flux

$A_p$ = Fraction of total absorptance assigned to the parallel flux

$A_T$ = Total absorptance

$R$ = Body reflectance, diffuse incidence

$R'$ = Body reflectance, parallel incidence

$T$ = Body transmittance, diffuse incidence

$T'$ = Body transmittance, parallel incidence

$\tau_p'$ = Unscattered fraction of the incident beam transmitted

$\tau_D'$ = Fraction of incident beam diffusely transmitted

$\rho_p'$ = Unscattered fraction of the incident beam reflected

$\rho_D'$ = Fraction of incident beam diffusely reflected

$r_0$ = External surface reflection coefficient, parallel incidence

$r_2$ = Internal surface reflection coefficient

## Introduction

Beginning with the fact that the units of the absorption coefficient are inverse centimeters, it is apparent that some distinction should be made between the absorption coefficients $\mu'$ and $\mu$ assigned respectively to the parallel and diffuse fluxes permeating a medium. It is the purpose of this paper to show that these coefficients can be determined simply and directly by goniometric measurements. In this approximate treatment, we omit the more general case, in which the absorption coefficient may exhibit a directional dependence and consider only the mean path associated with the two coefficients. We will assume that the material is optically thin, macroscopically homogeneous and that first order scattering predominates. Further, we will consider materials in which the scattering is concentrated below the critical angle. For transmitting materials these assumptions are not excessively restrictive. It is presumed throughout that the sample is optically plane and parallel, that the measured quantity, the radiant intensity is monochromatic, and that edge effects are minimal.

## Theory

The differential equations describing the fluxes permeating a scattering medium are given by,[1,2,3]

$$\frac{dt}{dx} = F' I_x' - \mu t - Bt + Bs \tag{1}$$

$$-\frac{ds}{dx} = B' I_x' - \mu s - Bs + Bt \tag{2}$$

If as predicated, higher scattering orders are negligible, then we can simplify (1) and (2) and write,

$$\frac{dt}{dx} = F' \, I_x' - \mu t \tag{1.a}$$

$$-\frac{ds}{dx} = B' \, I_x' - \mu s \tag{2.b}$$

The attenuation of the primary beam is given by,

$$\frac{dI_x'}{dx} = -(\mu' + B' + F') \, I_x' \tag{3}$$

If we let,

$$t = e^{-\mu x} F' \, I_0' \int_0^x e^{-(\mu' - \mu + F' + B')x} \, dx$$

and impose the boundary conditions.

$$x = 0 \,, \qquad t = 0$$

$$x = X \,, \qquad t = T' - I_x'$$

the following solution is obtained,

$$T' - I_x' = \frac{e^{-\mu x} F' \, I_0'}{(\mu - \mu' - B' - F')} \left\{ e^{-(\mu' - \mu + B' + F')x} - 1 \right\}$$

which we will approximate by,

$$T' - I_x' = F' X(1 - \mu x) I_0' \tag{4}$$

Similarly, from the boundary conditions

$$x = 0, \qquad s = R'$$
$$x = X, \qquad s = 0$$

we find that,

$$R' = B' X(1 - \mu x) I_0' \tag{5}$$

It should be noted that equations (4) and (5) are, as expected, independent of $\mu'$ and that linearily with thickness as required by primary scattering is indicated for small $\mu x$.

Now in traversing a layer dx a fraction of the primary residue is scattered and the remainder transmitted, each fraction subject to a particular absorption process. In view of this partitioning of fluxes, it is clear from (4) and (5) that $(F' X + B' X) \mu x$ represents the portion of the diffuse flux absorbed. Similarly, we will let $e^{-q'x} \mu' x$, where $q' = \mu' + B' + F'$, represent the portion of the unscattered residue absorbed. It is assumed here that $I_0'$ is equal to unity. The total absorp~~tance~~ flux is then,

$$= (F' X + B' X) \mu x + e^{-q'x} \mu' x \tag{6}$$

And as conservation requires,

$$A_T = 1 - \tau_p' - \tau_D' - \rho_p' - \rho_D' \tag{7}$$

The fraction of the total absorptance we can assign to the diffuse flux is obviously,

$$A_D = \frac{(F'\ X + B'\ X)\ \mu x}{(F'\ X + B'\ X)\ \mu x + e^{-q'x}\ \mu'\ x}\ A_T \tag{8}$$

Substituting from (4) and (5) we have for small $\mu x$,

$$A_D = \frac{\left(T' - I_x{}' + R'\right)\ \mu x}{\left(T' - I_x + R'\right)\ \mu x + I_x{}'\ \mu'\ x}\ A_T \tag{9}$$

Essentially equations (4) and (5) pertain to a one dimensional medium. Consequently, the coefficient, $\mu$ is not given in terms of an actual path but rather a thickness X.[4] It suffices for our purpose if we consider a mean path. The probability that a fraction of radiation will be scattered within a solid angle

$$d\omega = \sin\theta\, d\theta\, d\varphi$$

in the direction $\theta$ is given by,[5]

$$\frac{S(\theta,\ \varphi)\ d\omega}{4\pi}$$

where $S(\theta,\ \varphi)$ is the scattering function. Therefore, for an axial symmetric case the mean direction is,

$$\langle\theta\rangle = \int \frac{\theta\ S(\theta)\ d\omega}{4\pi} \tag{10}$$

Since the scattering function is directly proportional to the radiant intensity, $J(\theta)$ for either conservative or non-conservative scattering it suffices for most cases to plot the flux,

$$2\pi\, J(\theta_i)\, \sin\theta_i\, \Delta\theta$$

as a function of $\theta$ and determine the mean by inspection. For example in figure (3) the indicated mean of about 28.5° is in good agreement with the calculated value. It follows then that if $\langle\theta\rangle$ is the mean direction, then $X/\langle\cos\theta\rangle$ is the mean path. Hence,

$$\mu X = \left(\mu'/\langle\cos\theta\rangle\right) X \tag{11}$$

Consequently, (9) can be written as,

$$A_D = \frac{\left[(T' - I_x') + R'\right]/\langle\cos\theta\rangle}{\left[(T' - I_x') + R'\right]/\langle\cos\theta\rangle + I_x'}\, A_T \tag{9.a}$$

If we now consider the boundary effects resulting from the refractive discontinuity, the parallel and diffuse fractions of radiation transmitted and reflected are,

$$\tau_p' = (1 - r_0)^2\, I'/\left(1 - r_0^2\, I'^2\right) \tag{12}$$

$$\tau_D' = (1 - r_0)\left(w + v r_0\, I'\right)/\left(1 - r_0^2\, I'^2\right) \tag{13}$$

$$\rho_p' = r_0 + r_0 (1 - r_0)^2\, I'^2/\left(1 - r_0^2\, I'^2\right) \tag{14}$$

$$\rho_D' = (1 - r_0)\left(v + w r_0\, I'\right)/\left(1 - r_0^2\, I'^2\right) \tag{15}$$

where,

$$w = (1 - r_2)\frac{(T' - I')(1 - r_2 R) + r_2 R' T}{(1 - r_2 R)^2 - r_2^2 T^2}$$

$$v = (1 - r_2)\frac{R'(1 - r_2 R) + r_2 (T' - I') T}{(1 - r_2 R)^2 - r_2^2 T^2}$$

For transmitting materials, $r_0$, $r_2$, R and R′ are often relatively small. We can therefore rewrite the previous equations as,

$$\tau_p' \sim (1 - r_0)^2 I' \tag{12.a}$$

$$\tau_D' \sim (1 - r_0)(1 - r_2)(T' - I') \tag{13.a}$$

$$\rho_p' \sim r_0 + r_0 (1 - r_0)^2 I'^2 \tag{14.a}$$

$$\rho_D' \sim (1 - r_0)(1 - r_2) R' \tag{15.a}$$

Now assuming that,

$$A_T \sim \mu x + \mu' x$$

and substituting (12.a), (13.a), and (15.a) in (9.a) we have approximately,

$$\mu x \sim \frac{(\tau_D' + \rho_D')/\langle\cos\theta\rangle}{(\tau_D' + \rho_D')/\langle\cos\theta\rangle + \tau_p'} A_T \tag{16}$$

Similarly we can show that,

$$\mu' x \sim \frac{\tau_p'}{(\tau_D' + \rho_D')/\langle\cos\theta\rangle + \tau_p'} A_T \tag{17}$$

Procedure:

Since the measured flux has been refracted and reflected at the boundary and our immediate concern is with the internal angular distribution, it is necessary that we determine $J_{(\theta)}$ in terms of the measured radiant intensity $J'_{(\theta)}$. If for the moment we ignore reflection losses and consider only the refractive effect at the boundary, then the flux incident must equal the flux emerging at the boundary. Therefore,

$$J_{(\theta)}\, d\omega = J'_{(\theta)}\, d\omega' \tag{18}$$

or

$$\frac{J'_{(\theta)}}{J_{(\theta)}} = \frac{d\omega}{d\omega'}$$

The ratio of intensities is then equal to the inverse ratio of their respective solid angles. From Snell's law

$$n \sin\theta = n' \sin\theta'$$

we find that,

$$n \cos\theta \, d\theta = n' \cos\theta' \, d\theta'$$

Hence,

$$\frac{d\omega'}{d\omega} = \frac{\sin\theta' \, d\theta' \, d\varphi}{\sin\theta \, d\theta \, d\varphi} = n^2 \frac{\cos\theta}{\cos\theta'}$$

which can also be expressed as,

$$\frac{d\omega'}{d\omega} = \frac{n^2 \sqrt{1 - \sin^2\theta'/n^2}}{\cos\theta'} \tag{19}$$

For normal incidence this reduces to the more familiar relationship,[6]

$$d\omega' = n^2\, d\omega$$

The two intensities are related by,

$$J_{(\theta)} = \left\{ \frac{n^2 \left(1 - \sin^2 \theta'/n^2\right)^{1/2}}{\cos \theta'} \right\} J'_{(\theta)}$$

If we now include reflection losses at the boundary,

$$J_{(\theta)} = \left[1 - R(\theta)\right]^{-1} \left\{ \frac{n^2 \left(1 - \sin^2 \theta'/n^2\right)^{1/2}}{\cos \theta'} \right\} J'_{(\theta)} \qquad (20)$$

where $R(\theta)$ is the general Fresnel expression,

$$R(\theta) = \frac{1}{2} \frac{\sin^2 (\theta' - \theta)}{\sin^2(\theta + \theta')} \left\{ 1 + \frac{\cos^2 (\theta + \theta')}{\cos^2 (\theta' - \theta)} \right\} \qquad (21)$$

The procedure is now rather straightforward. From the measured intensity $J'_{(\theta)}$, $J_{(\theta)}$ is calculated using equation (20). As an example, the two intensities are shown in Figure (1) for a 3.8 mm thick Irtram I sample at $1\mu$. The Fresnel reflection coefficient for a refractive index of 1.38 is plotted in Figure (2). Now as shown in Figure (3), the flux $J_{(\theta)}\, d\omega$ is plotted and a mean direction determined. Returning to equation (7) the total absorptance can be deduced by integrating the forward and back scattered flux,[7]

$$\tau_D' = 2\pi \sum_{i=1}^{\pi/2} \frac{J'_{(\theta_i)} \sin \theta_i\, \Delta\theta}{I_0'} \qquad (22)$$

and

$$\rho_D' = 2\pi \sum_{i=\pi/2}^{\pi} \frac{J_{(\theta_i')} \sin\theta_i \, \Delta\theta}{I_0'} \tag{23}$$

Alternatively, $\tau_D'$ and $\rho_D'$ can be determined directly with an ellipsoidal or integrating sphere photometer. The transmittance, $\tau_p'$ is readily determined. The reflectance, $\rho_p'$ however, is not generally susceptible to direct measurement. We can, however, solve equation (12.a) for $I'$ and then substitute in equation (14.a). So that,

$$\rho_p' \sim r_0 \left\{1 + \tau_p'^2 \Big/ (1 - r_0)^2\right\} \qquad ((24)$$

The rest is a matter of substituting in the appropriate equation.

Results:

We can in some measure validate both the assumptions and method outlined here for determining the absorption coefficients $\mu$ and $\mu'$. We begin by restating equations (13.a) and (15.a),

$$T' = \frac{\tau_D'}{(1 - r_0)(1 - r_2)} + \frac{\tau_p'}{(1 - r_0)^2} \tag{25}$$

$$R' = \rho_D' \Big/ (1 - r_0)(1 - r_2) \tag{26}$$

Again, following Ryde we note that as $Q \rightarrow 1$ and $P \rightarrow 0$, the equations for the body transmittance,

$$T' = \frac{QK + Pe^{-qx} B \sinh KX}{(\mu + B) \sinh KX + K \cosh KX} - (Q - 1)\, e^{-q'x} \tag{27}$$

and reflectance,

$$R' = \frac{Pe^{-q'x} K + QB \sinh KX}{(\mu + B) \sinh KX + K \cosh KX} - P \tag{28}$$

can be written as,

$$T' \sim T = \frac{K}{(\mu + B) \sinh KX + K \cosh KX} \tag{27.a}$$

and

$$R' \sim R = \frac{B \sinh KX}{(\mu + B) \sinh KX + K \cosh KX} \tag{28.a}$$

where,

$$KX = \sqrt{\mu X (\mu X + 2BX)} \tag{29}$$

Significantly, both R and T are a function of $\mu$ and B. Furthermore, an approximate value of the backscattering coefficient, B is obtained from the ratio,

$$\frac{R}{T} = \frac{BX \sinh KX}{KX}$$

Expanding sinh KX,

$$\frac{R}{T} = \frac{B}{K} \left\{ KX + \frac{(KX)^3}{3!} + \frac{(KX)^5}{5!} + \cdots \right\}$$

we find that,

$$\frac{R}{T} \sim BX$$

or substituting from (25) and (26),

$$BX \sim \frac{\rho_D'}{\tau_D' + \dfrac{\tau_P'\left(1 - r_2\right)}{\left(1 - r_0\right)}} \tag{30}$$

The following measured values were obtained, $\tau_P' = 0.23$, $\tau_D' = 0.45$, $\rho_D' = 0.0725$ and from (24) $\rho_P' = 0.0274$. Assuming $r_2 = 0.15$ and given that $r_0 = 0.026$, $\langle \cos\theta \rangle = 0.879$, we find from (30), (29) and (16) that $BX = 0.1114$, $KX = 0.246$ and $\mu x = 0.1586$. Inserting the above values in (25) and (27.a) we find that $T = 0.785$ and $T = 0.769$, indicating agreement within 2%. Similarly, from (26) and (28.a) we find that $R = 0.08624$) and $R = 0.087571$, indicating agreement within 1.5%. The measured values for other thickness of Irtram I further substantiates the values obtained for BX and $\mu x$ reported here.[8]

## Conclusions:

An approximate method has been outlined for determining the absorption coefficient $\mu$ and $\mu'$ which appears to be applicable to optically thin materials where primary scattering predominates. The method has at least the advantage of following directly from a fairly conventional measurement. The coefficients are given in terms of the reflected and transmitted fluxes weighted by their mean paths. Some of the assumptions made, particularly a method for determining the interval reflection coefficient, $r_2$ will be the subject of another paper now in preparation.

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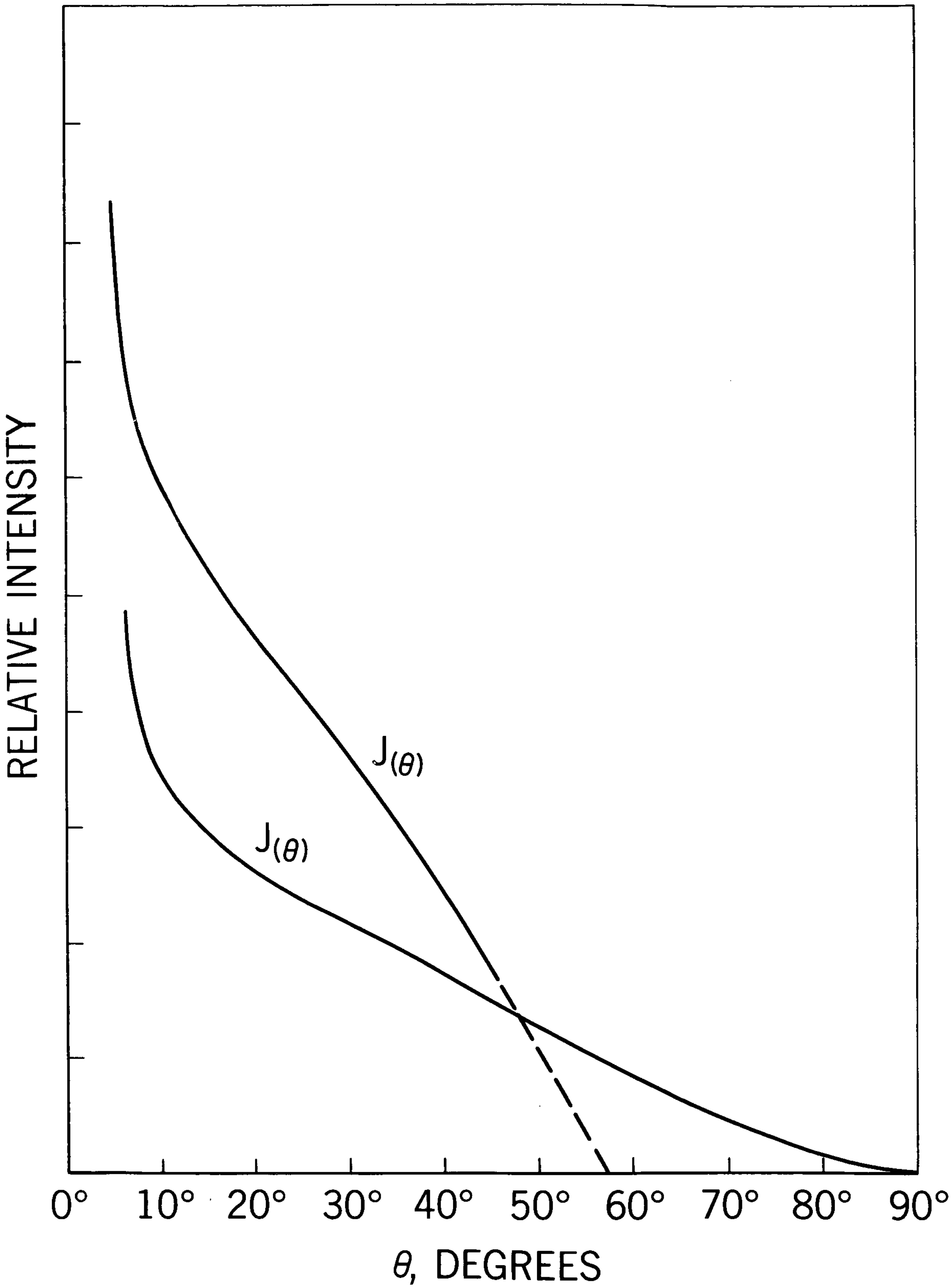
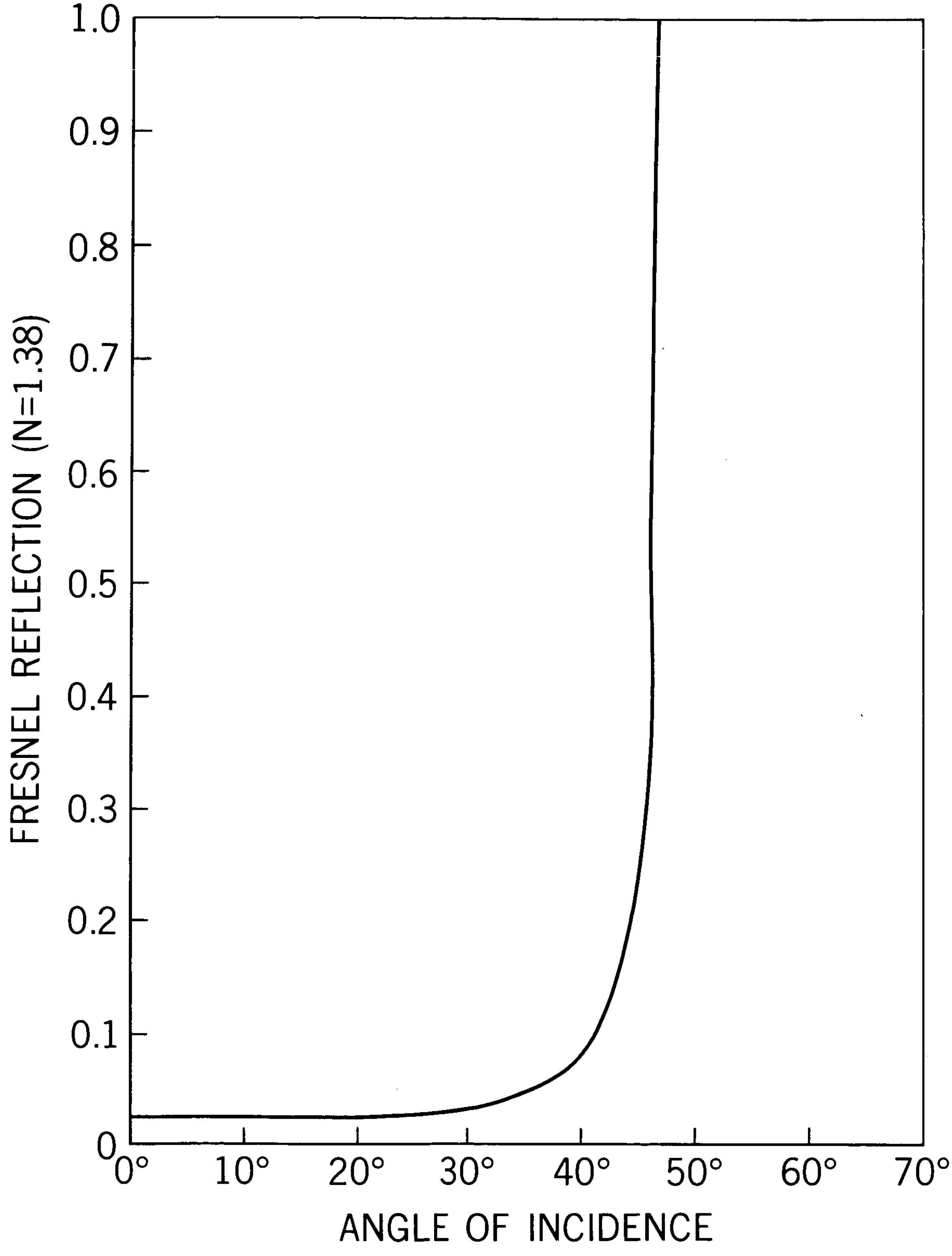



Figure 1. Angular distribution of radiant intensity for Irtran I (3.8 mm) at 1 micron



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
FRESNEL REFLECTION (N=1.38)
0°
10°
20°
30°
40°
50°
60°
70°
ANGLE OF INCIDENCE


Figure 2

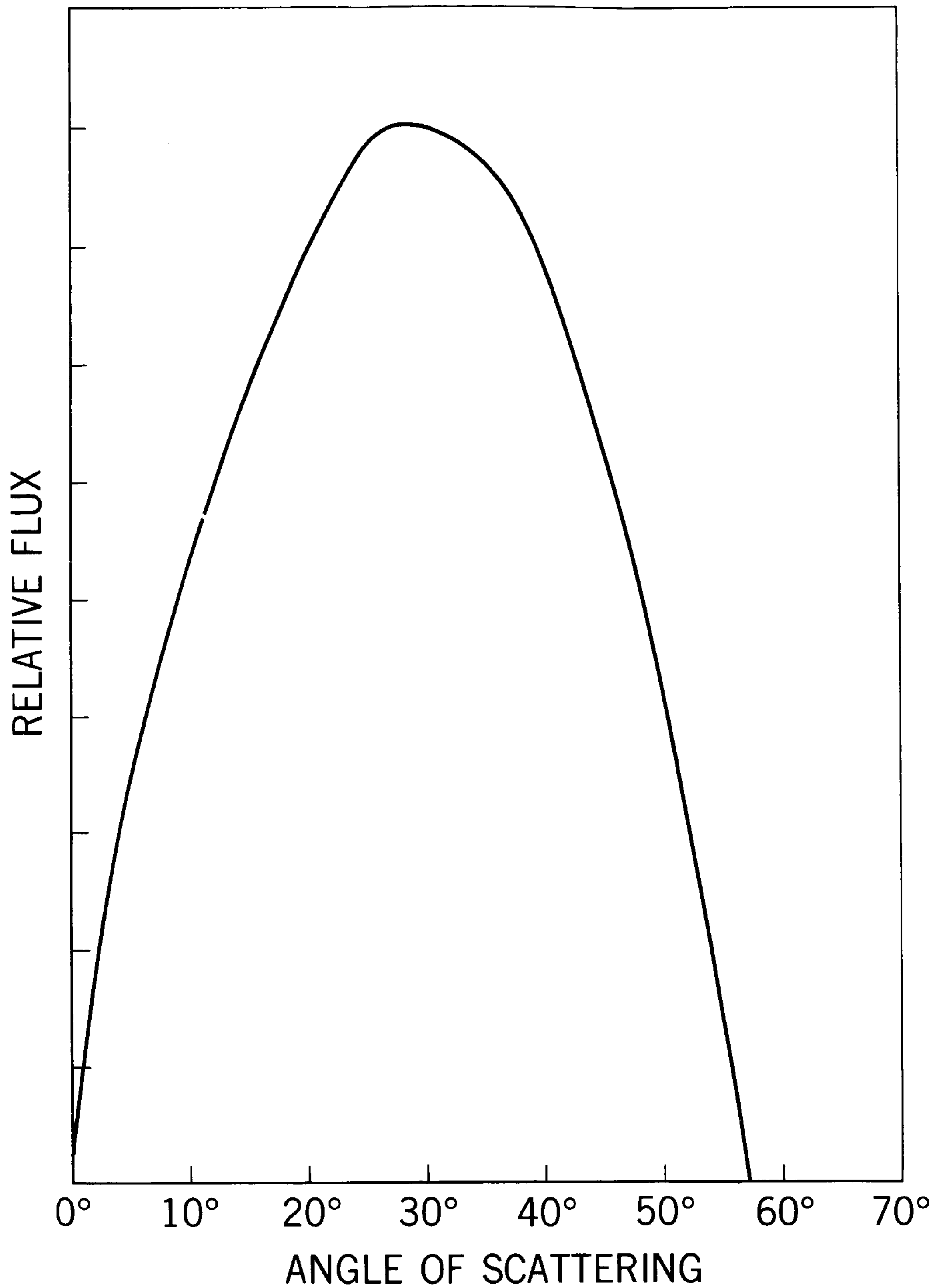



Figure 3. Flux distribution at normal incidence for Irtran I (3.8 mm) at 1 micron