NASA CONTRACTOR REPORT

NASA CR-885

VIBRATION STUDY OF A PRESSURIZED TORUS SHELL

Part II - Development and Applications of Analysis

by Atis A. Liepins

Prepared by
DYNATECH CORPORATION
Cambridge, Mass.
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • OCTOBER 1967

NASA CR-885

VIBRATION STUDY OF A PRESSURIZED TORUS SHELL

Part II - Development and Applications of Analysis

By Atis A. Liepins

Prepared under Contract No. NAS 1-6632 by
DYNATECH CORPORATION
Cambridge, Mass.

for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

FOREWORD

This report comprises the analytical portion of a two-part vibration study of a pressurized torus shell. Part I of this study, which is contained in NASA CR-884, reports the experimental investigation and was prepared by Peter F. Jordan of the Martin Company, Baltimore, Maryland, under Contract No. NAS 1-6088.

CONTENTS

ILLUSTRATIONS

TABLES

VIBRATION STUDY OF A PRESSURIZED TORUS SHELL

Part II – Development and Applications of Analysis

by Atis A. Liepins
Dynatech Corporation

SUMMARY

Previous work on the free vibrations of prèssure prestressed toroidal shells is extended. Shells constructed of a number of toroidal segments, each with a constant thickness, and various boundary conditions are included. The analysis is then applied to simple models of a toroidal shell structure for which experimental results are published. The calculated fundamental frequency and mode shape of this shell agree well with the experimental results for three levels of internal pressure.

INTRODUCTION

Several numerical analyses of the free vibrations of shells of revolution are reported in the literature. The most recent of these [1] includes the effects of prestress. Finite difference techniques are used in the numerical analysis. The formulation of the vibration problem in Reference [1], however, is such that for thin toroidal shells the finite difference mesh converges slowly [2] , and for toroidal membranes it does not give meaningful results [3]. A different formulation of the problem for toroidal shells avoids these difficulties [2, 3].

Previous analyses of the free vibrations of pressure prestressed toroidal shells [2, 3] have dealt with unsupported shells with constant wall thickness. This report extends these analyses to toroidal shells with various support conditions and varying wall thickness. The supports are located at the inner and outer circumferences. The support conditions may be completely general. The shell may consist of a number of toroidal segments forming a complete torus. Each segment has a constant thickness. (Refer to Figure 1). All segments are made of the same material.

The present analysis is applied to a toroidal shell structure for which experimental frequencies and mode shapes are published [4]. The structure consists of a toroidal shell stiffened at the outer circumference by a ring and at the inner circumference by a hub plate. Since the present analysis does not consider stiffening elements or additional masses, it is applicable to vibrations for which the stiffening elements are ineffective. Three simple models are investigated and the frequencies and mode shapes of the fundamental and first few overtones of each model are presented.

NOMENCLATURE

c	scale factor for mode shapes
h_k	thickness of k-th toroidal segment
h_o	reference thickness
k_k	$(1 - \nu^2)\, pR / Eh_k$
p	pressure
r	$(1 - \epsilon \cos \alpha)/\epsilon$
u	meridional displacement (Figure 1)
v	circumferential displacement (Figure 1)
w	normal displacement (Figure 1)
A_o	$Eh_o/(1 - \nu^2)$
C, N, Q, S $T_0, T_1, T_2 \ldots T_{12}$	defined functions
D_o	$Eh_o^3/12\,(1 - \nu^2)$
E	Young's modulus
$E_\alpha, E_\theta, E_{\alpha\theta}$	membrane strains
I_k	difference spaces in k-th segment
$M_\alpha, M_\theta, M_{\alpha\theta}$	stress couples
$N_\alpha, N_\theta, N_{\alpha\theta}$	stress resultants
Q_α, Q_θ	transverse shear stress resultants
R	radius of the generating circle of torus (Figure 1)
S_α, S_θ	membrane prestress forces
α	meridional position angle (Figure 1)
β_k	angle of k-th juncture

ϵ	ratio of the two radii of the torus (Figure 1)
η_k	h_o/h_k
θ	circumferential position angle (Figure 1)
$\kappa_\alpha, \kappa_\theta, \kappa_{\alpha\theta}$	bending strains
λ	$(\rho R^2/E\,\epsilon^2)\,\omega^2$, frequency parameter
ν	Poisson's ratio
ρ	material density
$\phi_\alpha, \phi_\theta, \phi_{\alpha\theta}$	rotations of the normal to the shell
ω	circular frequency
Γ_k	$(h_k/R)^2/12$
Δ_k	spacing between finite difference stations

Matrices

y_1, y_2	1 x 4 column matrices
A, B, C, D, E, F, H P, X_1, X_2, Y_1, Y_2	4 x 4 matrices
I	Identity matrix
Z	1 x 4 column matrix
$\phi_1, \phi_2, \phi_3, \phi_4$	4 x 4 matrices

Indices

i	station in a segment
j	station
k	segment
J	juncture station
K	number of segments
L	last station
n	Fourier index of shell vibrations (corresponds to $\frac{m}{2}$ in Reference [4])

FUNDAMENTAL EQUATIONS

The present analysis of the free vibrations of pressure prestressed toroidal shells is based upon a system of equations presented in Reference [5]. For the k-th toroidal segment these are:

Equilibrium

$$\frac{\partial}{\partial \alpha}(rN_\alpha) + \frac{\partial N_{\alpha\theta}}{\partial \theta} - N_\theta \sin\alpha + rQ_\alpha + \frac{1}{2R}\left(1 + \frac{\cos\alpha}{r}\right)\frac{\partial}{\partial \theta}(M_{\alpha\theta})$$
$$+ S_\theta\left[\frac{\partial}{\partial \theta}(E_{\alpha\theta} - \phi_{\alpha\theta}) + (E_\alpha - E_\theta)\sin\alpha\right] + rS_\alpha\left(\frac{\partial E_\alpha}{\partial \alpha} - \phi_\alpha\right)$$
$$+ pRr\phi_\alpha + \rho h_k R\omega^2 ru = 0 \qquad (1)$$

$$\frac{\partial N_\theta}{\partial \theta} + \frac{\partial}{\partial \alpha}(rN_{\alpha\theta}) + N_{\alpha\theta}\sin\alpha - Q_\theta\cos\alpha - \frac{r}{2R}\frac{\partial}{\partial \alpha}\left[\left(1 + \frac{\cos\alpha}{r}\right)M_{\alpha\theta}\right]$$
$$+ S_\theta\left[\frac{\partial E_\theta}{\partial \theta} + 2E_{\alpha\theta}\sin\alpha + \phi_\theta\cos\alpha\right] + rS_\alpha\frac{\partial}{\partial \alpha}(E_{\alpha\theta} + \phi_{\alpha\theta})$$
$$+ pRr\phi_\theta + \rho h_k R\omega^2 rv = 0 \qquad (2)$$

$$rN_\alpha - N_\theta\cos\alpha - \frac{\partial}{\partial \alpha}(rQ_\alpha) - \frac{\partial Q_\theta}{\partial \theta} + S_\theta\left[\frac{\partial \phi_\theta}{\partial \theta} + \phi_\alpha\sin\alpha - E_\theta\cos\alpha\right]$$
$$+ rS_\alpha\left(\frac{\partial \phi_\alpha}{\partial \alpha} + E_\alpha\right) - pRr(E_\alpha + E_\theta) - \rho h_k R\omega^2 rw = 0 \qquad (3)$$

$$\frac{\partial}{\partial \alpha}(rM_\alpha) + \frac{\partial M_{\alpha\theta}}{\partial \theta} - M_\theta\sin\alpha - RrQ_\alpha = 0 \qquad (4)$$

$$\frac{\partial M_\theta}{\partial \theta} + \frac{\partial}{\partial \alpha}(rM_{\alpha\theta}) + N_{\alpha\theta}\sin\alpha - RrQ_\theta = 0 \qquad (5)$$

Strain-Displacement

$$RE_\alpha = \frac{\partial u}{\partial \alpha} + w \tag{6}$$

$$rRE_\theta = \frac{\partial v}{\partial \theta} + u \sin \alpha - w \cos \alpha \tag{7}$$

$$2rRE_{\alpha\theta} = r \frac{\partial v}{\partial \alpha} + \frac{\partial u}{\partial \theta} - v \sin \alpha \tag{8}$$

$$R\kappa_\alpha = \frac{\partial \phi_\alpha}{\partial \alpha} \tag{9}$$

$$rR\kappa_\theta = \frac{\partial \phi_\theta}{\partial \theta} + \phi_\alpha \sin \alpha \tag{10}$$

$$2rR\kappa_{\alpha\theta} = r \frac{\partial \phi_\theta}{\partial \alpha} + \frac{\partial \phi_\alpha}{\partial \theta} - \phi_\theta \sin \alpha - (r + \cos \alpha)\phi_{\alpha\theta} \tag{11}$$

$$R\phi_\alpha = - \frac{\partial w}{\partial \alpha} + u \tag{12}$$

$$-rR\phi_\theta = \frac{\partial w}{\partial \theta} + v \cos \alpha \tag{13}$$

$$-2rR\phi_{\alpha\theta} = \frac{\partial u}{\partial \theta} - \frac{\partial}{\partial \alpha}(rv) \tag{14}$$

Constitutive Relations

$$Eh_k E_\alpha = N_\alpha - \nu N_\theta \tag{15}$$

$$Eh_k E_\theta = N_\theta - \nu N_\alpha \tag{16}$$

$$Eh_k E_{\alpha\theta} = (1 + \nu) N_{\alpha\theta} \tag{17}$$

$$\frac{Eh_k^3}{12} \kappa_\alpha = M_\alpha - \nu M_\theta \tag{18}$$

$$\frac{Eh_k^3}{12} \kappa_\theta = M_\theta - \nu M_\alpha \tag{19}$$

$$\frac{Eh_k^3}{12} \kappa_{\alpha\theta} = (1 + \nu) M_{\alpha\theta} \tag{20}$$

The stress resultants S_α and S_θ are known functions of pressure. They are determined from a separate analysis of the toroidal shell subjected to static internal pressure. An analysis based upon the linear membrane theory [6] gives:

$$
\begin{aligned}
S_\alpha &= pR \, \frac{1 - \frac{1}{2} \epsilon \cos \alpha}{1 - \epsilon \cos \alpha} \\
S_\theta &= \frac{1}{2} pR
\end{aligned}
\tag{21}
$$

REDUCTION TO SECOND ORDER DIFFERENTIAL EQUATIONS

The solution of the fundamental equations (1 - 20) is started by separating the variables. Set

$$\begin{bmatrix} N_\alpha, E_\alpha, M_\alpha, \kappa_\alpha, \phi_\alpha, u \\ N_\theta, E_\theta, M_\theta, \kappa_\theta, Q_\alpha, w \end{bmatrix} = \begin{bmatrix} A_o N_{\alpha n}, E_{\alpha n}, \frac{D_o}{R} M_{\alpha n}, \frac{1}{R}\kappa_{\alpha n}, \phi_{\alpha n}, Ru_n \\ A_o N_{\theta n}, E_{\theta n}, \frac{D_o}{R} M_{\theta n}, \frac{1}{R}\kappa_{\theta n}, \frac{D_o}{R^2} Q_{\alpha n}, Rw_n \end{bmatrix} \cos n\theta$$

$$\begin{bmatrix} N_{\alpha\theta}, E_{\alpha\theta}, \phi_\theta, \phi_{\alpha\theta} \\ M_{\alpha\theta}, \kappa_{\alpha\theta}, Q_\theta, v \end{bmatrix} = \begin{bmatrix} A_o N_{\alpha\theta n}, E_{\alpha\theta n}, \phi_{\theta n}, \phi_{\alpha\theta n} \\ \frac{D_o}{R} M_{\alpha\theta n}, \frac{1}{R}\kappa_{\alpha\theta n}, \frac{D_o}{R^2} Q_{\theta n}, Rv_n \end{bmatrix} \sin n\theta \qquad (22)$$

Next, define

$$\begin{aligned} S &= (\sin\alpha)/r \\ C &= (\cos\alpha)/r \\ N &= n/r \end{aligned} \qquad (23)$$

Then, use of equations (21), (22) and (23) in equations (1 - 20) yields (in nondimensional form)

$$\begin{aligned} &N'_\alpha + S(N_\alpha - N_\theta) + NN_{\alpha\theta} + \Gamma_k \eta_k^3 \left[Q_\alpha + \frac{N}{2}(1 + C) M_{\alpha\theta}\right] \\ &+ k_k \left[\frac{1}{2}N(E_{\alpha\theta} - \phi_{\alpha\theta}) + \frac{1}{2}S(E_\alpha - E_\theta) + (1 + \frac{1}{2}C)(E'_\alpha - \phi_\alpha) + \phi_\alpha\right] \\ &+ (1 - \nu^2)\epsilon^2 \lambda u = 0 \end{aligned} \qquad (24)$$

$$\begin{aligned} &N'_{\alpha\theta} + 2SN_{\alpha\theta} - NN_\theta - \Gamma_k \eta_k^3 \left[CQ_\theta + \frac{1}{2}(1 + C)(M'_{\alpha\theta} - SM_{\alpha\theta})\right] \\ &+ k_k \left[\frac{1}{2}(-NE_\theta + 2SE_{\alpha\theta} + C\phi_\theta) + (1 + \frac{1}{2}C)(E'_{\alpha\theta} + \phi'_{\alpha\theta}) + \phi_\theta\right] \\ &+ (1 - \nu^2)\epsilon^2 \lambda v = 0 \end{aligned} \qquad (25)$$

$$N_\alpha - CN_\theta - \Gamma_k\eta_k^3 \left[(Q'_\alpha + SQ_\alpha) + NQ_\theta \right] + k_k \left[\frac{1}{2}(N\phi_\theta + S\phi_\alpha - CE_\theta) + (1 + \frac{1}{2}C)(\phi'_\alpha + E_\alpha) - (E_\alpha + E_\theta) \right] - (1 - \nu^2)\,\epsilon^2 \lambda\, w = 0 \tag{26}$$

$$M'_\alpha + S(M_\alpha - M_\theta) + NM_{\alpha\theta} - Q_\alpha = 0 \tag{27}$$

$$M'_{\alpha\theta} + 2SM_{\alpha\theta} - NM_\theta - Q_\theta = 0 \tag{28}$$

$$E_\alpha = u' + w \tag{29}$$

$$E_\theta = Nv + Su - Cw \tag{30}$$

$$2E_{\alpha\theta} = v' - Sv - Nu \tag{31}$$

$$\kappa_\alpha = \phi'_\alpha \tag{32}$$

$$\kappa_\theta = S\phi_\alpha + N\phi_\theta \tag{33}$$

$$2\kappa_{\alpha\theta} = \phi'_\theta - S\phi_\theta - N\phi_\alpha - (1 + C)\,\phi_{\alpha\theta} \tag{34}$$

$$\phi_\alpha = -w' + u \tag{35}$$

$$\phi_\theta = Nw - Cv \tag{36}$$

$$2\phi_{\alpha\theta} = Nu + v' + Sv \tag{37}$$

$$\eta_k N_\alpha = E_\alpha + \nu E_\theta \tag{38}$$

$$\eta_k N_\theta = E_\theta + \nu E_\alpha \tag{39}$$

$$\eta_k N_{\alpha\theta} = (1 - \nu) E_{\alpha\theta} \tag{40}$$

$$\eta_k^3 M_\alpha = \kappa_\alpha + \nu\kappa_\theta \tag{41}$$

$$\eta_k^3 M_\theta = \kappa_\theta + \nu\kappa_\alpha \tag{42}$$

$$\eta_k^3 M_{\alpha\theta} = (1 - \nu)\,\kappa_{\alpha\theta} \tag{43}$$

In equations (24 - 43) and subsequent expressions, the subscript n has been dropped. Prime indicates differentiation with respect to α.

At this point the problem has been reduced to the simultaneous solution of equations (24 - 43). Next, we derive four simultaneous second order differential equations for u, v, ϕ_α, and Q_α. The details of this derivation may be found in Reference [2]. The result is

$$AZ'' + BZ' + CZ = 0 \tag{44}$$

where

$$Z = \begin{Bmatrix} u \\ v \\ \phi_\alpha \\ Q_\alpha \end{Bmatrix}$$

and the elements of the A, B, and C matrices are given in the Appendix. Equations (44) are the governing equations for the k-th toroidal segment. The elements of the Z matrix are the basic variables of this formulation.

The normal displacement is related to the basic variables by

$$Qw = T_1u' + T_2u + T_3v' + T_4v + T_5\phi'_\alpha + T_6\phi_\alpha - \Gamma_k\eta_k^3(Q'_\alpha + SQ_\alpha) \tag{45}$$

The Q and T's are given in the Appendix.

BOUNDARY AND JUNCTURE CONDITIONS

An application of the principle of virtual work to equations (1-5) yields the boundary conditions. On an edge where α is constant prescribe:

$$
\begin{aligned}
& N_\alpha + S_\alpha E_\alpha && \text{or} \quad u = 0 \\
& N_{\alpha\theta} - \frac{1}{2R}\left(1 + \frac{3\cos\alpha}{r}\right) M_{\alpha\theta} + S_\alpha (E_{\alpha\theta} + \phi_{\alpha\theta}) && \text{or} \quad v = 0 \\
& Q_\alpha + \frac{1}{Rr}\frac{\partial M_{\alpha\theta}}{\partial\theta} - S_\alpha \phi_\alpha && \text{or} \quad w = 0 \\
& M_\alpha && \text{or} \quad \phi_\alpha = 0
\end{aligned}
\tag{46}
$$

Applying equations (21), (22), and (23) to the boundary conditions (46) yields:

$$
\begin{aligned}
& N_\alpha + \frac{1}{\eta_k} k_k \left(1 + \frac{1}{2}C\right) E_\alpha && \text{or} \quad u = 0 \\
& N_{\alpha\theta} - \frac{1}{2}\Gamma_k \eta_k^2 \,(1 + 3C) M_{\alpha\theta} + \frac{1}{\eta_k} k_k \left(1 + \frac{1}{2}C\right)(E_{\alpha\theta} + \phi_{\alpha\theta}) && \\
& && \text{or} \quad v = 0 \\
& Q_\alpha + N M_{\alpha\theta} - \frac{k_k}{\Gamma_k \eta_k^3}\left(1 + \frac{1}{2}C\right)\phi_\alpha && \text{or} \quad w = 0 \\
& M_\alpha && \text{or} \quad \phi_\alpha = 0
\end{aligned}
\tag{47}
$$

where the subscript n has been dropped.

The boundary conditions (47) may be written in a single matrix equation as:

$$
Y_1 y_1 + Y_2 y_2 = 0 \tag{48}
$$

where y_1 and y_2 are column matrices consisting of the nondimensional force and displacement conditions respectively in equations (47); Y_1 and Y_2 are diagonal matrices that characterize the type of support condition. The elements of Y_1 and Y_2 matrices for two types of simple support, clamped support, and no support are given in the Appendix.

It is of advantage to have the boundary equation (48) expressed entirely in terms of the basic variables Z. This is accomplished by direct substitution into (47). The result is

$$\begin{aligned} y_1 &= \phi_1 Z' + \phi_2 Z \\ y_2 &= \phi_3 Z' + \phi_4 Z \end{aligned} \tag{49}$$

where the elements of the ϕ matrices are given in the Appendix.

With (49) the boundary equation (48) becomes

$$[\, Y_1\phi_1 + Y_2\phi_3 \,] Z' + [\, Y_1\phi_2 + Y_2\phi_4 \,] Z = 0 \tag{50}$$

At the juncture of two toroidal segments we require:

$$\begin{aligned} &\text{for equilibrium of forces} && y_1^- = y_1^+ \\ &\text{for continuity of deformations} && y_2^- = y_2^+ \end{aligned} \tag{51}$$

where the plus and minus superscripts refer to the quantities just before and immediately after the juncture (see Reference [7]).

With (49) the juncture equations (51) become

$$\begin{aligned} \phi_1^- (Z^-)' + \phi_2^- Z^- &= \phi_1^+ (Z^+)' + \phi_2^+ Z^+ \\ \phi_3^- (Z^-)' + \phi_4^- Z^- &= \phi_3^+ (Z^+)' + \phi_4^+ Z_4^+ \end{aligned} \tag{52}$$

NUMERICAL ANALYSIS

Since the geometry and prestress are symmetrical about $\alpha = 0$ and $\alpha = \pi$, we need to consider only one-half of the torus corresponding to the range $0 \leq \alpha \leq \pi$. Assume that one half of the torus is constructed of K segments, the k-th segment terminating at $\alpha = \beta_k$. Let each segment be subdivided by $I_k + 1$ equally spaced stations. Then the spacing between stations is

$$\Delta_1 = \beta_1 / I_1$$

$$\Delta_k = \frac{\beta_k - \beta_{k-1}}{I_k} \qquad k = 2, 3 \ldots K \tag{53}$$

and the position angle for the j-th station is

$$\alpha_j = i\Delta_1$$

$$\alpha_j = \beta_{k-1} + i\Delta_k \qquad \begin{array}{l} k = 2, 3 \ldots K \\ i = 0, 1, 2 \ldots I_k \end{array} \tag{54}$$

The last station, α_L, is at $\alpha = \pi$, where

$$L = 1 + \sum_{k=1}^{k} I_k$$

The derivatives of Z at all stations except those at junctures, α_J, are approximated by the central difference formulas

$$Z_j' = \frac{1}{2\Delta_k} (Z_{j+1} - Z_{j-1})$$

$$Z_j'' = \frac{1}{\Delta_k^2} (Z_{j+1} - 2Z_j + Z_{j-1}) \tag{55}$$

With these formulas we obtain from equation (44) the set of difference equations

$$D_j Z_{j+1} + E_j Z_j + F_j Z_{j-1} = 0 \tag{56}$$

where

$$D_j = \frac{2}{\Delta_k} A_j + B_j$$

$$E_j = -\frac{4}{\Delta_k} A_j + 2\Delta_k C_j$$

$$F_j = \frac{2}{\Delta_k} A_j - B_j \tag{57}$$

Equations (56) apply at all stations except those at junctures.

In the juncture equations (52) the first derivative is approximated by a forward or backward difference formula

$$(Z^-)'_J = \frac{1}{2\Delta^-} (3Z_J - 4Z_{J-1} + Z_{J-2})$$

$$(Z^+)'_J = \frac{-1}{2\Delta^+} (3Z_J - 4Z_{J+1} + Z_{J+2}) \tag{58}$$

Then for the J-th juncture, the juncture conditions in finite difference form are

$$\frac{\phi_1^-}{2\Delta^-} (3Z_J^- - 4Z_{J-1} + Z_{J-2}) + \phi_2^- Z_J^- = -\frac{\phi_1^-}{2\Delta^+} (3Z_J^+ - 4Z_{J+1} + Z_{J+2}) + \phi_2^+ Z_J^+ \tag{59}$$

$$\frac{\phi_3^-}{2\Delta^-} (3Z_J^- - 4Z_{J-1} + Z_{J-2}) + \phi_4^- Z_J^- = -\frac{\phi_2^+}{2\Delta^+} (3Z_J^+ - 4Z_{J+1} + Z_{J+2}) + \phi_4^+ Z_J^+ \tag{60}$$

At $\alpha = 0$ the boundary equation in finite difference form is

$$\frac{1}{2\Delta_1} (Y_1^o \phi_1 + Y_2^o \phi_2)(Z_1 - Z_{-1}) + (Y_1^o \phi_3 + Y_2^o \phi_4) Z_o = 0 \tag{61}$$

and at $\alpha = \pi$

$$\frac{1}{2\Delta_K} (Y_1^\pi \phi_1 + Y_2^\pi \phi_2)(Z_{L+1} - Z_{L-1}) + (Y_1^\pi \phi_3 + Y_2^\pi \phi_4) Z_L = 0 \tag{62}$$

The superscripts o and π denote the boundary matrices at $\alpha = 0$ and $\alpha = \pi$, respectively.

The eigenvalues of the homogeneous equations (56) and (59) through (62) are found by trial and error using Potters method [8], [2].

$$Z_j = -P_j Z_{j+1} \tag{63}$$

where

$$P_j = [E_j - F_j P_{j-1}]^{-1} D_j \tag{64}$$

applies to all j except j = 0, j = J, and j = J+1.

Write (56) for j = 0, and eliminate Z_{-1} from (61). Then by comparison with (63)

$$P_o = [\, 2\Delta_1 (Y_1^o \phi_3 + Y_2^o \phi_4) + (Y_1^o \phi_1 + Y_2^o \phi_2) F_o^{-1} E_o \,]^{-1} (Y_1^o \phi_1 + Y_2^o \phi_2)(I + F_o^{-1} D_o) \tag{65}$$

Now, write equation (56) at j = J+1, equation (63) at j = J-2, and solve equation (60) for Z_J^+. Then equation (59) becomes

$$\bar{D}_J Z_{J+1} + \bar{E}_J Z_J^- + \bar{F}_J Z_{J-1} = 0 \tag{66}$$

where

$$\begin{aligned}
\bar{D}_J &= \frac{-1}{2\Delta^+} (\phi_1^+ - X_1 X_2 \phi_3^+)(4I + D_{J+1}^{-1} E_{J+1}) \\
\bar{E}_J &= (\phi_2^- - X_1 X_2) + \frac{3}{2\Delta^-} (\phi_1^- - X_1 X_2 \phi_3^-) \\
\bar{F}_J &= -\frac{1}{2\Delta^-} (\phi_1^- - X_1 X_2 \phi_3^-)(4I + P_{J-2})
\end{aligned} \tag{67}$$

and

$$X_1 = [\, \phi_4^+ - \frac{\phi_3^+}{2\Delta^+} (3I - D_{J+1}^{-1} F_{J+1})]^{-1}$$

$$X_2 = \phi_2^+ - \frac{\phi_1^+}{2\Delta^+} (3I - D_{J+1}^{-1} F_{J+1})$$

The governing equation for the station after the juncture, where Z_J^+ is eliminated, is

$$\bar{D}_{J+1} Z_{J+2} + \bar{E}_J Z_{J+1} + \bar{F}_J Z_J^- = 0 \tag{68}$$

where

$$\bar{D}_{J+1} = D_{J+1}$$

$$\bar{E}_{J+1} = E_{J+1} - F_{J+1} X_1 \frac{\phi_3^+}{2\Delta^+} (4I + D_{J+1}^{-1} E_{J+1})$$

$$\bar{F}_{J+1} = F_{J+1} X_1 \left[\phi_4^- + \frac{3\phi_3^-}{2\Delta^-} + \frac{\phi_3^-}{2\Delta^-} (4I + P_{J-2}) P_{J-1} \right] \tag{69}$$

Then P_J and P_{J+1} are found from equation (64) using (67) and (69).

Equations (64) and (65) provide all the P's up to P_{L-1}. Then write equation (56) at $j = L$, equation (63) at $j = L-1$, and eliminate Z_{L+1} from (62). The result is

$$HZ_L = 0 \tag{70}$$

where

$$H = 2\Delta_K (Y_1^\pi \phi_3 + Y_2^\pi \phi_4) - (Y_1^\pi \phi_1 + Y_2^\pi \phi_2) D_L^{-1} E_L$$

$$+ (Y_1^\pi \phi_1 + Y_2^\pi \phi_2) (I + D_L^{-1} F_L) P_{L-1}$$

Since, in general, $Z_L \neq 0$, we must require that the determinant

$$\nabla = |H| \tag{71}$$

vanish. Equation (71) is a frequency equation.

When the vibration is axisymmetric ($n = 0$) and such that w is symmetric and u is antisymmetric about $\alpha = \pi$, $Z_L = 0$. The frequency equation for this case is obtained from equation (56) written at $i = L-1$ with the help of equation (64). It is

$$\nabla = |E_{L-1} - F_{L-1} P_{L-2}| \tag{72}$$

For a natural frequency, Z_L is found from

$$Z_{L,m} = cH_m \qquad m = 1, 2, 3, 4 \tag{73}$$

where c is an arbitrary scale factor and the H_m are the cofactors of the first row of H. The remaining Z's are calculated in the reverse order with (64). The w component of the mode shape is calculated from equation (45) where the first derivative is approximated by the first of equations (55) at all stations except at the juncture and the station after the juncture. At these stations linear approximations are used.

The mode shapes are such that either v and w are symmetric, u is antisymmetric, or v and w are antisymmetric, and u is symmetric with respect to $\alpha = 0$ and π. These two groups of modes are denoted as symmetric and antisymmetric respectively.

The foregoing computing procedure may be summarized as follows:

1. Assume a value of λ;
2. Calculate the elements of the A, B, C, D, E, F, and P matrices at all stations;
3. Calculate the determinant;
4. Repeat steps 1 - 3. From a plot of the determinant versus λ determine a λ (natural frequency) for which the determinant is zero;
5. Calculate the mode shape for a natural frequency.

The equations of this analysis are programmed in Fortran IV. The IBM 7094 II produces a determinant in approximately 1.5 seconds and a mode shape in 3 seconds when seventy-five finite difference stations are used. Computation time increases linearly with the number of finite difference stations.

RESULTS

The analysis developed in the preceding sections was applied to three simple models of a toroidal shell structure for which experimental frequencies and mode shapes are published [4]. The structure consists of a complete toroidal shell, made up of segments, each with a constant thickness; a plate, closing the hub; and a sheet metal ring at the outer circumference. A more detailed description of the model is given in Reference [4]. The average values of the structure parameters, taken from Reference [4], are given in Table 1.

Model I

The weight of Model I equals the total weight of the toroidal structure including fittings. The weights of the hub plate with fittings and the outer ring are modeled as material added to the skin in the regions $-0.2 \leq \alpha \leq 0.2$ and $\pi - 0.05 \leq \alpha \leq \pi + 0.05$, respectively. The width of these regions was chosen to keep the skin thicknesses reasonable. The wall thickness variation of Model I is shown in Figure 2. No attempt is made to model the elastic supports provided by the hub plate or the ring at the outer circumference. The model is supported in a free-free manner.

The fundamental frequency and mode shape for 0, 6, 15, and 30 psi internal pressure was calculated using double precision arithmetic (16 digits). Convergence of the finite difference mesh was established for the frequency at 15 psi pressure with 181 difference spacings distributed as follows: $I_1 = 11$, $I_2 = 18$, $I_3 = 28$, $I_4 = 119$, $I_5 = 5$. It was assumed that convergence for the frequency and mode shape at other pressures was achieved with 181 spacings. Interestingly, the frequencies converged from below the final frequency.

The overtones were calculated using single precision arithmetic (8 digits). A total of 75 difference spacings, $I_1 = I_2 = I_5 = 5$, $I_3 = 10$, $I_4 = 50$, were used. The first overtone at 0 psi pressure was also calculated with 150 spacings in single precision and with 75 spacings in double precision. The frequency calculated with 150 spacings was less than 1% lower than that calculated with 75 spacings in single precision. The frequencies calculated in single and double precision were identical to 3 significant digits.

The calculated fundamental frequencies and mode shapes are compared with experimental results in Figures 3 and 4. At 15 psi pressure the calculated and experimental frequencies and mode shapes are in very good agreement. However, the agreement deteriorates with decreasing pressure. The discrepancy between the calculated and measured results at 0 and 6 psi pressure may be due to an insufficient number of stations in the finite difference mesh.

The calculated frequencies of the first five overtones of Model I are compared to the measured overtone frequencies in Table 2. There is wide disagreement between the calculated and measured frequencies.

The mode shapes of the first five overtones of Model I for $p = 0$ are shown in Figure 5. The mode shapes for $p = 6$, 15, and 30 psi do not depart significantly from those for $p = 0$.

Models II and III

Model II disregards the weight of the hub plate with fittings and the outer ring, as well as the elastic support provided by them. The wall thickness variation of this model is shown in Figure 6. Model II is supported in a free-free manner.

Model III is an unsupported torus with a constant thickness of 0.0627 inches ($h/R = 0.00545$).

The fundamental frequencies and mode shapes of Model II were calculated with 150 finite difference spacings ($I_1 = I_2 = I_5 = 10$, $I_3 = 20$, $I_4 = 100$). Again, the fundamental frequency converged from below the final frequency. All other frequencies and mode shapes of Models II and III were calculated with 75 spacings.

The first five frequencies of Models II and III for $p = 0$, 6, 15, and 30 psi and the measured results are compared in Table 3. The calculated mode shapes are displayed in Figure 7. Although the overtone frequencies of Models II and III are closer to the experimental results than those of Model I, wide differences still exist. Furthermore, the large normal displacements at the hub of the computed symmetric modes are unrealistic because of the presence of the hub plate.

The effect of various boundary conditions on the first overtone of Model II was investigated. The following boundary conditions were considered:

Hub ($\alpha = 0$)	Outer Circumference ($\alpha = \pi$)
simple support I	free
simple support II	free
clamped support	free
elastic support	free
free	simple support I
free	simple support II
free	clamped support
free	elastic support

In the elastic support the normal displacement and the transverse shear force were elastically related. The stiffness was taken to be that of the center plate under uniform in-plane load. The remaining three conditions in the elastic support were the appropriate ones of simple support I.

The difference in frequency and mode shape for the elastic support and simple support I was found to be insignificant. This was expected as the in-plane stiffness of the center plate is high.

The torus, in this mode, has identical frequencies and mode shapes for clamped support and simple support I. Computations verified this. The frequencies and mode shapes for the remaining four combinations of supports are shown in Figure 8. There is a small difference in frequency and mode shape when the torus has a simple support I, II and no support (see Table 3 and Figure 7 for frequencies and mode shapes of the free-free torus) at the outer circumference and the hub is not supported. However, there is a significant difference in frequency and mode shape when the hub is supported and the outer circumference is free. The importance of the conditions at the hub for this mode are further demonstrated by Figure 9. Here the normal displacements for Models I and II and the measured results are displayed. Model I and the measured results are in good agreement in the region outside the crown ($90° \leq \alpha \leq 180°$). The two measured displacements inside the crown ($0 \leq \alpha \leq 90°$) fall between Model I and II results.

CONCLUSIONS

Analysis and computer program are developed for free vibrations of complete toroidal shells with pressure. These shells may be constructed of several toroidal segments, each with a constant thickness. Supports may be specified at inner and outer circumferences.

This analysis is applied to three simple models of a toroidal shell structure for which experimental results are published. Model I of this structure gives fundamental frequencies and mode shapes for three levels of internal pressure that are in good agreement with experimental results. Models more accurate than those analyzed here are necessary for the calculation of overtones.

The effects of support conditions on the frequency and mode shape of the first overtone of Model II are briefly investigated. These calculations show that the support conditions at the outer circumference have a relatively small effect on the frequency and mode shape of this mode. However, careful attention should be given to the support conditions at the hub.

APPENDIX

The elements of the A, B, and C matrices are:

$$A_{11} = Q^2 [\ 1 + k (1 + \frac{1}{2}\ C)] + Q T_1 T_7$$

$$A_{12} = Q T_3 T_7$$

$$A_{13} = Q T_5 T_7$$

$$A_{14} = - \eta^3 \Gamma Q T_7$$

$$A_{21} = Q T_1 T_9$$

$$A_{22} = Q^2 [\ \frac{1}{2}\ (1 - \nu) + k (1 + \frac{1}{2} C) + \frac{1}{8} (1 - \nu)\ \Gamma\ (1 + 3 C)^2] + Q T_3 T_9$$

$$A_{23} = Q T_5 T_9$$

$$A_{24} = - \eta^3 \Gamma Q T_9$$

$$A_{31} = Q T_1$$

$$A_{32} = Q T_3$$

$$A_{33} = Q T_5$$

$$A_{34} = -\eta^3 \Gamma Q$$

$$A_{41} = Q T_1 T_{11}$$

$$A_{42} = Q T_3 T_{11}$$

$$A_{43} = Q^2 + Q T_5 T_{11}$$

$$A_{44} = - \eta^3 \Gamma Q T_{11}$$

$$B_{11} = Q^2 S \left(1 + \frac{1}{2} k\right) + Q\, T_7 \,(T_1' + T_2) + T_1 T_8$$

$$B_{12} = \frac{1}{2} Q^2 N \left[(1+\nu) - \frac{1}{4}(1-\nu)\,\Gamma\,(1+C)(1+3C)\right] + Q\, T_7 (T_3' + T_4) + T_3 T_8$$

$$B_{13} = Q\, T_7 \,(T_5' + T_6) + T_5\, T_8$$

$$B_{14} = -\eta^3\, \Gamma\, (Q S T_7 + T_8)$$

$$B_{21} = -\frac{1}{2} Q^2 N \left[1 + \nu + \frac{1}{4}(1-\nu)\,\Gamma(1+3C)(1-C)\right] + Q\, T_9 (T_1' + T_2) + T_1 T_{10}$$

$$B_{22} = \frac{1}{2} Q^2 S \left[1 - \nu + k - \frac{1}{4}(1-\nu)\,\Gamma(1+3C)(5+3C)\right] + Q T_9 (T_3' + T_4) + T_3 T_{10}$$

$$B_{23} = \Gamma Q^2 N \left[\nu C + \frac{1}{2}(1-\nu)(1+3C)\right] + Q\, T_9 (T_5' + T_6) + T_5 T_{10}$$

$$B_{24} = -\eta^3\, \Gamma\, (Q S T_9 + T_{10})$$

$$B_{31} = Q\,(T_1' + T_2) - Q'\, T_1$$

$$B_{32} = Q\,(T_3' + T_4) - Q'\, T_3$$

$$B_{33} = Q\,(T_5' + T_6) - Q'\, T_5$$

$$B_{34} = -\eta^3\, \Gamma\, (Q S - Q')$$

$$B_{41} = Q\, T_{11} (T_1' + T_2) + T_1 T_{12}$$

$$B_{42} = -Q^2 N \left[\nu C + \frac{1}{4}(1-\nu)(1+3C)\right] + Q\, T_{11} (T_3' + T_4) + T_3 T_{12}$$

$$B_{43} = Q^2 S + Q\, T_{11} (T_5' + T_6) + T_5 T_{12}$$

$$B_{44} = -\eta^3\, \Gamma\, (Q S T_{11} + T_{12})$$

$$C_{11} = Q^2\left[(1-\nu^2)\epsilon^2\lambda + \nu C - S^2 - \frac{1}{2}(1-\nu)N^2 - \frac{1}{2}k(N^2+S^2) - \frac{1}{8}(1-\nu)\Gamma(1+C)^2 N^2\right]$$

$$+ Q T_7 T_2' + T_2 T_8$$

$$C_{12} = Q^2 N S\left[-\frac{1}{2}(3-\nu) - k + \frac{1}{8}(1-\nu)\Gamma(1+C)(1+3C)\right] + Q T_7 T_4' + T_4 T_8$$

$$C_{13} = -\frac{1}{2} Q^2 \left[k C + \frac{1}{2}(1-\nu)\Gamma(1+C)N^2\right] + Q T_7 T_6' + T_6 T_8$$

$$C_{14} = \eta^3 \Gamma\left[Q^2 - Q T_7 (C - S^2) - S T_8\right]$$

$$C_{21} = -Q^2 N S\left[\frac{1}{2}(3-\nu) + k + \frac{3}{8}(1-\nu)\Gamma(1+C)^2\right] + Q T_9 T_2' + T_2 T_{10}$$

$$C_{22} = Q^2 \Big\{(1-\nu^2)\epsilon^2\lambda - \frac{1}{2}(1-\nu)(C+S^2) - N^2 - \frac{1}{2}k(N^2+S^2+C^2) - kC - \Gamma N^2 C^2$$

$$+ \frac{1}{8}(1-\nu)\Gamma(1+3C)\left[S^2(5+3C) - C(1+3C)\right]\Big\} + Q T_9 T_4' + T_4 T_{10}$$

$$C_{23} = \Gamma Q^2 N S\left[C - \frac{1}{2}(1-\nu)\right] + Q T_9 T_6' + T_6 T_{10}$$

$$C_{24} = -\eta^3\Gamma\left[Q T_9 (C - S^2) + S T_{10}\right]$$

$$C_{31} = -Q^2 + Q T_2' - Q' T_2$$

$$C_{32} = Q T_4' - Q' T_4$$

$$C_{33} = Q^2 + Q T_6' - Q' T_6$$

$$C_{34} = -\eta^3 \Gamma\left[Q(C - S^2) - Q' S\right]$$

$$C_{41} = -\frac{1}{4}(1-\nu)(1+C)N^2 Q^2 + Q T_{11} T_2' + T_2 T_{12}$$

$$C_{42} = Q^2 N S\left[C + \nu(1+C) + \frac{1}{4}(1-\nu)(1+3C)\right] + Q T_{11} T_4' + T_4 T_{12}$$

$$C_{43} = Q^2 \left[\nu C - S^2 - \frac{1}{2} (1-\nu) N^2 \right] + Q T_{11} T_6' + T_6 T_{12}$$

$$C_{44} = -\eta^3 \left\{ Q^2 + \Gamma \left[Q T_{11} (C - S^2) + S T_{12} \right] \right\}$$

where $$T_1 = 1 - \nu C + \frac{1}{2} k C - \frac{1}{4} (1-\nu) \Gamma N^2 (1 - C) + \frac{1}{2} N T_0 \left[1 + \nu + \frac{1}{4} (1-\nu) \Gamma (1 + 3C)(1 - C) \right]$$

$$T_2 = S \left[\nu - C - k (1 + \frac{1}{2} C) + \frac{1}{2} (1-\nu) \Gamma N^2 \right] + N S T_0 \left[\frac{1}{2} (3 - \nu) + k + \frac{3}{8} (1 - \nu) \Gamma (1 + C)^2 - \frac{1}{2} (1-\nu) \Gamma (2 + 3C) \right]$$

$$T_3 = -\frac{1}{2} (1 - \nu) \Gamma N S - \frac{1}{2} S T_0 \left[1 - \nu + k + \frac{1}{4} (1-\nu) \Gamma (1 + 3C)(5 + 3C) \right]$$

$$T_4 = N \left\{ \nu - C - k (1 + C) + \Gamma \left[-N^2 C + \frac{1}{2} (1-\nu) S^2 - \frac{1}{4} (1-\nu)(1 + 3C) C \right] \right\} - T_0 \left\{ (1 - \nu^2) \epsilon^2 \lambda - \frac{1}{2} (1-\nu)(C + S^2) - N^2 - \frac{1}{2} k (N^2 + S^2 + C^2) - kC - \Gamma N^2 C^2 + \frac{1}{8} (1-\nu) \Gamma (1 + 3C) \left[S^2 (5 + 3C) - C (1 + 3C) \right] \right\}$$

$$T_5 = k (1 + \frac{1}{2} C) + \Gamma N^2 - \Gamma N T_0 \left[\nu C + \frac{1}{2} (1 - \nu)(1 + 3C) \right]$$

$$T_6 = S \left\{ \frac{1}{2} k + \Gamma N^2 + \frac{1}{2} \Gamma N T_0 \left[3 (1-\nu) + (1 - 3\nu) C \right] \right\}$$

$$T_7 = 1 - \nu C + k (1 + \frac{1}{2} C) + \frac{1}{4} (1-\nu) \Gamma (1 + C) N^2$$

$$T_8 = Q S (1 + C) \left[1 + \frac{1}{2} k - \frac{1}{2} (1 - \nu) \Gamma N^2 \right] - Q' T_7$$

$$T_9 = \frac{1}{2} (1 - \nu) \Gamma N S (2 + 3C)$$

$$T_{10} = Q N \left[-\nu + C + k (1 + C) + \Gamma N^2 C + \frac{1}{2} (1-\nu) \Gamma (1 + 3C)(C - 2S^2) - \frac{1}{2} (1 - \nu) \Gamma (1 - 3C) S^2 \right] - Q' T_9$$

$$T_{11} = \frac{1}{2}(1 + \nu) N^2$$

$$T_{12} = -(2N^2 SQ + Q' T_{11})$$

$$T_0 = \frac{\frac{1}{4}(1 - \nu)\,\Gamma\,(1 + 3C)\,N}{\frac{1}{2}(1 - \nu) + k\,(1 + \frac{1}{2}C) + \frac{1}{8}(1 - \nu)\,\Gamma\,(1 + 3C)^2}$$

$$Q = (1 - \nu^2)\,\epsilon^2 \lambda - 1 - C^2 + 2\nu C - \frac{1}{2}k\,(3C + N^2 + C^2) - \Gamma N^2 [N^2 + (1-\nu)C]$$

$$+ T_0 N \left\{ -\nu + C + k\,(1 + C) + \Gamma N^2 C + \frac{1}{2}(1 - \nu)\Gamma[(1 + 3C)C - 3(1 + C)S^2] \right\}$$

prime indicates differentiation with respect to α, and the subscript k has been dropped on η, k, and Γ. The derivatives of N, S, and C are:

$$N' = -NS$$
$$S' = C - S^2$$
$$C' = -S(1 + C)$$

Elements of ϕ_1 matrix:

$$\phi_{11} = \frac{1}{\eta}\left\{1 + k\,(1 + \frac{1}{2}C) + [1 - \nu C + k\,(1 + \frac{1}{2}C)]\,\frac{T_1}{Q}\right\}$$

$$\phi_{12} = \frac{1}{\eta}[1 - \nu C + k\,(1 + \frac{1}{2}C)]\,\frac{T_3}{Q}$$

$$\phi_{13} = \frac{1}{\eta}[1 - \nu C + k\,(1 + \frac{1}{2}C)]\,\frac{T_5}{Q}$$

$$\phi_{14} = -\eta^2 \Gamma [1 - \nu C + k\,(1 + \frac{1}{2}C)]\,\frac{1}{Q}$$

$$\phi_{21} = \frac{1-\nu}{2\eta} \Gamma (1 + 3C) NS \frac{T_1}{Q}$$

$$\phi_{22} = \frac{1-\nu}{2\eta} [1 + \frac{1}{4} \Gamma (1+3C)(1+3C+4NS \frac{T_3}{Q})] + \frac{k}{\eta} (1 + \frac{1}{2} C)$$

$$\phi_{23} = \frac{1-\nu}{2\eta} \Gamma (1 + 3C) NS \frac{T_5}{Q}$$

$$\phi_{24} = -\frac{1}{2} (1-\nu) \eta^2 \Gamma^2 (1 + 3C) \frac{NS}{Q}$$

$$\phi_{31} = -\frac{1}{\eta^3} (1 - \nu) N^2 S \frac{T_1}{Q}$$

$$\phi_{32} = \frac{1}{4\eta^3} (1 - \nu) N (1 + 3C + 4NS \frac{T_3}{Q})$$

$$\phi_{33} = -\frac{1}{\eta^3} (1 - \nu) N^2 S \frac{T_5}{Q}$$

$$\phi_{34} = (1 - \nu) \Gamma \frac{N^2 S}{Q}$$

$$\phi_{41} = \frac{1}{\eta^3} \nu N^2 \frac{T_1}{Q}$$

$$\phi_{42} = \frac{1}{\eta^3} \nu N^2 \frac{T_3}{Q}$$

$$\phi_{43} = \frac{1}{\eta^3} (1 + \nu N^2 \frac{T_5}{Q})$$

$$\phi_{44} = - \nu N^2 \Gamma / Q$$

Elements of ϕ_2 matrix:

$$\phi_{11} = \frac{1}{\eta} \left\{ \nu S + [1 - \nu C + k (1 + \frac{1}{2} C)] \frac{T_2}{Q} \right\}$$

$$\phi_{12} = \frac{1}{\eta}\left\{\nu N + \left[1 - \nu C + k\left(1 + \frac{1}{2}C\right)\right]\frac{T_4}{Q}\right\}$$

$$\phi_{13} = \frac{1}{\eta}\left[1 - \nu C + k\left(1 + \frac{1}{2}C\right)\right]\frac{T_6}{Q}$$

$$\phi_{14} = -\eta^2 \Gamma\left[1 - \nu C + k\left(1 + \frac{1}{2}C\right)\right]\frac{S}{Q}$$

$$\phi_{21} = -\frac{1-\nu}{2\eta} N\left[1 + \frac{1}{4}\Gamma(1 + 3C)\left(1 - C - 4S\frac{T_2}{Q}\right)\right]$$

$$\phi_{22} = -\frac{1-\nu}{2\eta} S\left[1 + \frac{1}{4}\Gamma(1 + 3C)\left(1 + 3C - 4N\frac{T_4}{Q}\right)\right]$$

$$\phi_{23} = \frac{1-\nu}{2\eta}\Gamma(1 + 3C)N\left(1 + S\frac{T_6}{Q}\right)$$

$$\phi_{24} = -\frac{1}{2}(1-\nu)\eta^2\Gamma^2(1 + 3C)\frac{NS^2}{Q}$$

$$\phi_{31} = \frac{1}{4\eta^3}(1-\nu)N^2\left(1 - C - 4S\frac{T_2}{Q}\right)$$

$$\phi_{32} = \frac{1}{4\eta^3}(1-\nu)NS\left(1 + 3C - 4N\frac{T_4}{Q}\right)$$

$$\phi_{33} = -\frac{1}{\eta^3}\left[(1-\nu)N^2\left(1 + S\frac{T_6}{Q}\right) + \frac{k}{\Gamma}\left(1 + \frac{1}{2}C\right)\right]$$

$$\phi_{34} = 1 + \Gamma(1-\nu)\frac{N^2 S^2}{Q}$$

$$\phi_{41} = \frac{1}{\eta^3}\nu N^2\frac{T_2}{Q}$$

$$\phi_{42} = -\frac{1}{\eta^3} \nu N \left(C - N \frac{T_4}{Q}\right)$$

$$\phi_{43} = \frac{1}{\eta^3} \nu \left(S + N^2 \frac{T_6}{Q}\right)$$

$$\phi_{44} = - \nu \Gamma \frac{N^2 S}{Q}$$

Nonzero Elements of ϕ_3 Matrix:

$$\phi_{31} = T_1/Q$$

$$\phi_{32} = T_3/Q$$

$$\phi_{33} = T_5/Q$$

$$\phi_{34} = - \eta^3 \Gamma/Q$$

Nonzero Elements of ϕ_4 Matrix:

$$\phi_{11} = \phi_{22} = \phi_{43} = 1$$

$$\phi_{31} = T_2/Q$$

$$\phi_{32} = T_4/Q$$

$$\phi_{33} = T_6/Q$$

$$\phi_{34} = -\eta^3 \Gamma S/Q$$

The subscript k has been dropped on η, k, and Γ.

Support condition matrices Y_1 and Y_2. Simple support I ($v = 0$):

$$Y_1 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & A \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & |A-1| \end{bmatrix}$$

Simple support II ($N_{\alpha\theta} = 0$):

$$Y_1 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & A \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & |A-1| \end{bmatrix}$$

Clamped support:

$$Y_1 = 0 \qquad Y_2 = I$$

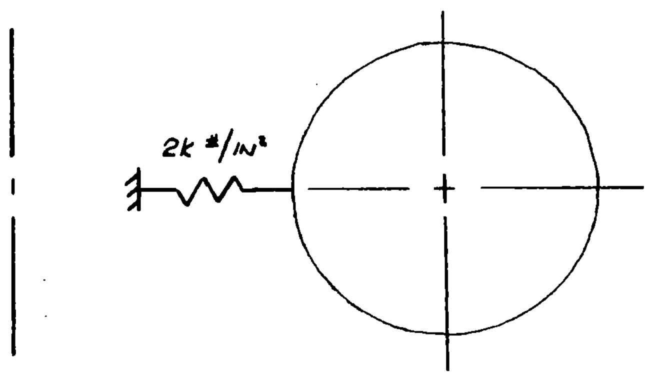
No support:

$$Y_1 = \begin{bmatrix} A & & & \\ & |A-1| & & \\ & & |A-1| & \\ & & & A \end{bmatrix} \qquad Y_2 = \begin{bmatrix} |A-1| & & & \\ & A & & \\ & & A & \\ & & & |A-1| \end{bmatrix}$$

where $A = 0$ for symmetric modes

$A = 1$ for antisymmetric modes

Example of elastic support: simple support I resting on an elastic foundation at the inner circumference



k is spring constant

The boundary matrices at $\alpha = 0$ are:

$$Y_1 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & -\frac{D_o}{kR^3}|A-1| & \\ & & & A \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & |A-1| \end{bmatrix}$$

REFERENCES

1. Cooper, P. A., "Vibration and Buckling of Prestressed Shells of Revolution," NASA TN D-3831, March 1967.

2. Liepins, A. A., "Flexural Vibrations of the Prestressed Toroidal Shell," NASA CR-296, (1965).

3. Liepins, A. A., "Free Vibrations of the Prestressed Toroidal Membrane," AIAA Journal, 3 (10), pp. 1924-1933, (1965).

4. Jordan, P. F., "Vibration Test of a Pressurized Torus Shell," AIAA Paper No. 67-73, (1967). (Reissued as NASA CR-884, 1967.)

5. Budiansky, B., and Liepins, A. A., "Vibration and Stability of Prestressed Shells: General Theory and the Membrane Torus," Presented at the Conference on Shell Theory and Analysis, Palo Alto, California, (1963).

6. Timoshenko, S., Theory of Plates and Shells, McGraw-Hill Book Company, Inc., New York, 1940, (Chapter X).

7. Budiansky, B., and Radkowski, P. P., "Numerical Analysis of Unsymmetrical Bending of Shells of Revolution," AIAA Journal, 1 (8), 1833-1842, (1963).

8. Potters, M. L., "A Matrix Method for the Solution of a Second Order Difference Equation in Two Variables," Math. Centrum Report MR 19 (1955).

Table 1

STRUCTURE PARAMETERS

R	11.5 in.
ϵ	0.7326
E	10.7×10^6 psi
ν	0.32
ρg	0.103 lb/in^3
Weight of outer ring	4.17 lbs
Total weight of structure and fittings	76.84 lbs

Table 2

CALCULATED MODEL I AND MEASURED OVERTONE FREQUENCIES

Calculated Mode	Frequency, cps			
	0 psi Pressure	6 psi	15 psi	30 psi
n = 1 Symmetric	471 (178-190) 1	476 (212-224) 1	482 (185-201) 1	490
n = 2 Antisymmetric	550 (252) 2S*	577 (301) 2S	610 (431) 2	653
n = 2 Symmetric	558 (276) 2A*	583 (441) 2 or 3A	615 (444) 2 or 3	653
n = 2 Symmetric	562 (391) 3S	590 (523) (3)	625 (521) 2	671
n = 2 Antisymmetric	564 (438) 2	592 (557) ?	627 (523) 3	671

The experimental frequencies (shown in parenthesis) are scaled from Figures 11a & 13 of Reference [4].

* 2S - n = 2, Symmetric Mode ; 2A - n = 2, Antisymmetric Mode

Table 3

CALCULATED MODEL II AND III AND MEASURED FREQUENCIES

Calculated Mode	Frequency, cps										
	0 psi pressure			6 psi pressure			15 psi pressure			30 psi pressure	
	II	III	Measured	II	III	Measured	II	III	Measured	II	III
n = 0 Antisymmetric	61.3	73.0	48.8	79.5	93.1	61.2	99.0	115	75.8	123	143
n = 1	**	**	178-190 (1)	**	**	212-224 (1)	**	**	185-201 (1)	**	**
n = 2 Symmetric	246	186	252 (2S)*	268	219	301 (2S)	296	259	431 (2)	333	311
n = 2 Antisymmetric	252	199	276 (2A)*	274	234	441 (2 or 3A)	303	276	444 (2 or 3)	343	331
n = 3 Symmetric	451	326	391 (3S)	484	372	523 (3)	526	430	521 (2)	583	509
n = 3 Antisymmetric	458	325	438 (2)	490	371	557 (?)	531	429	523 (3)	587	507

* 2S - n = 2, Symmetric Mode; 2A - n = 2, Antisymmetric Mode

** Calculated frequencies are higher than n = 3, Antisymmetric

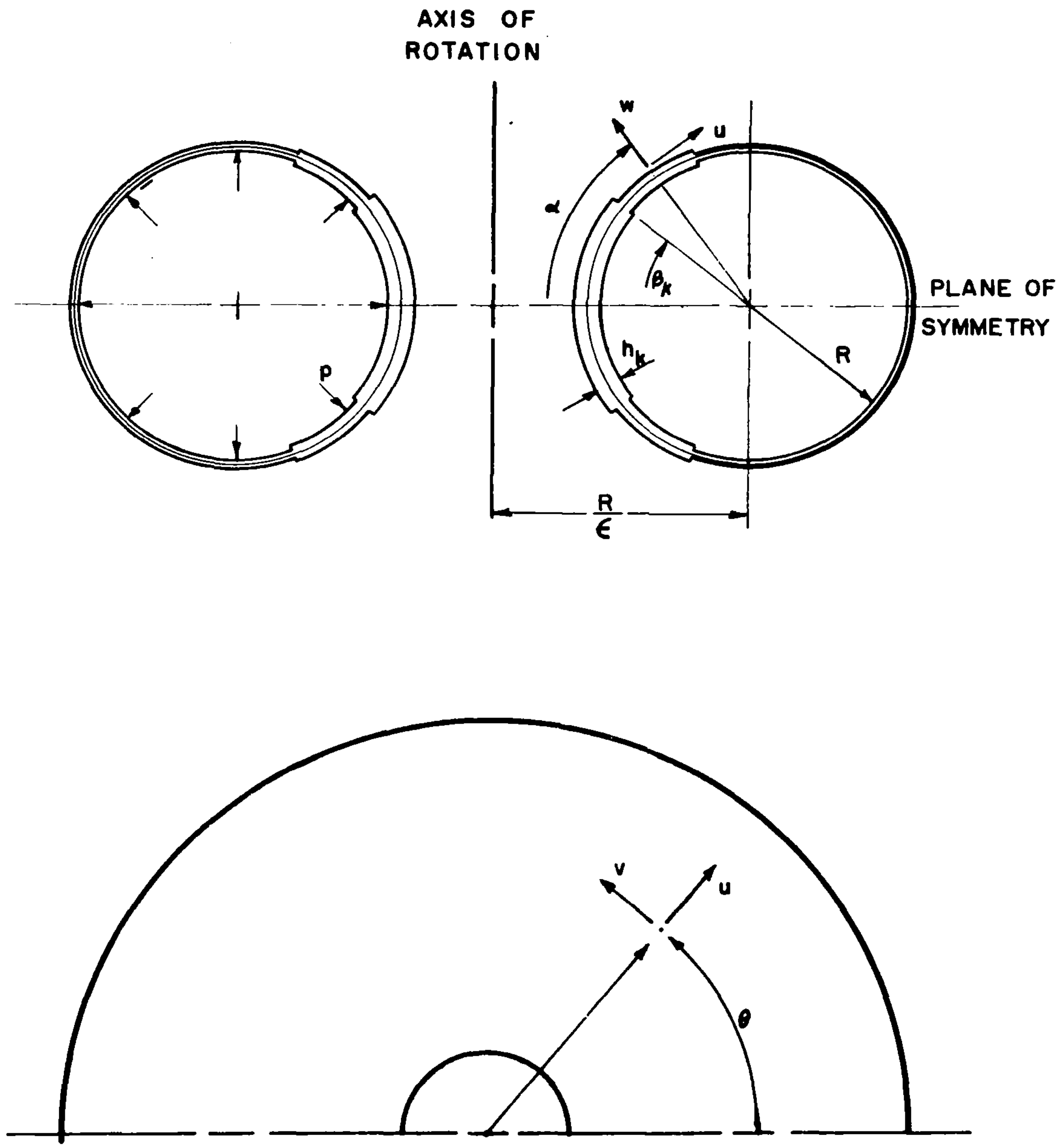
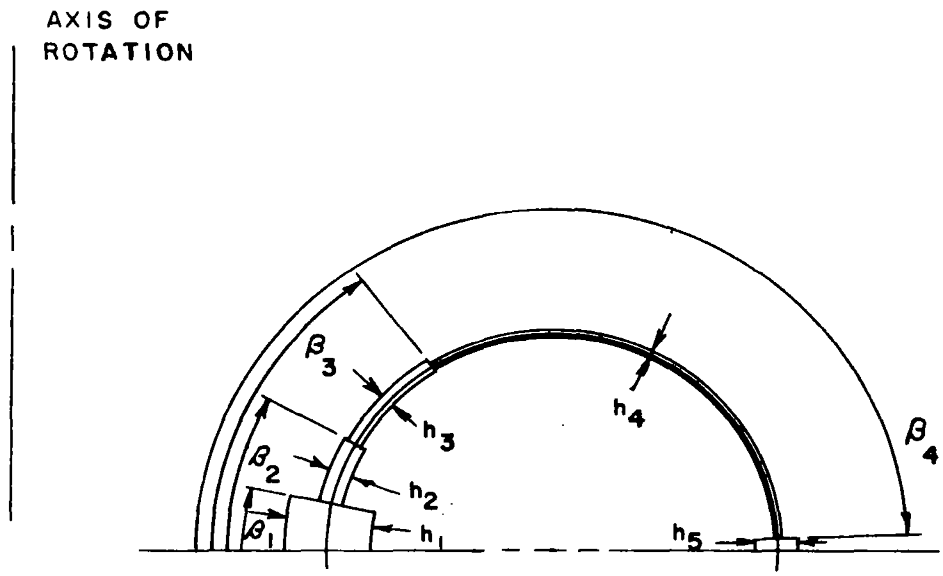
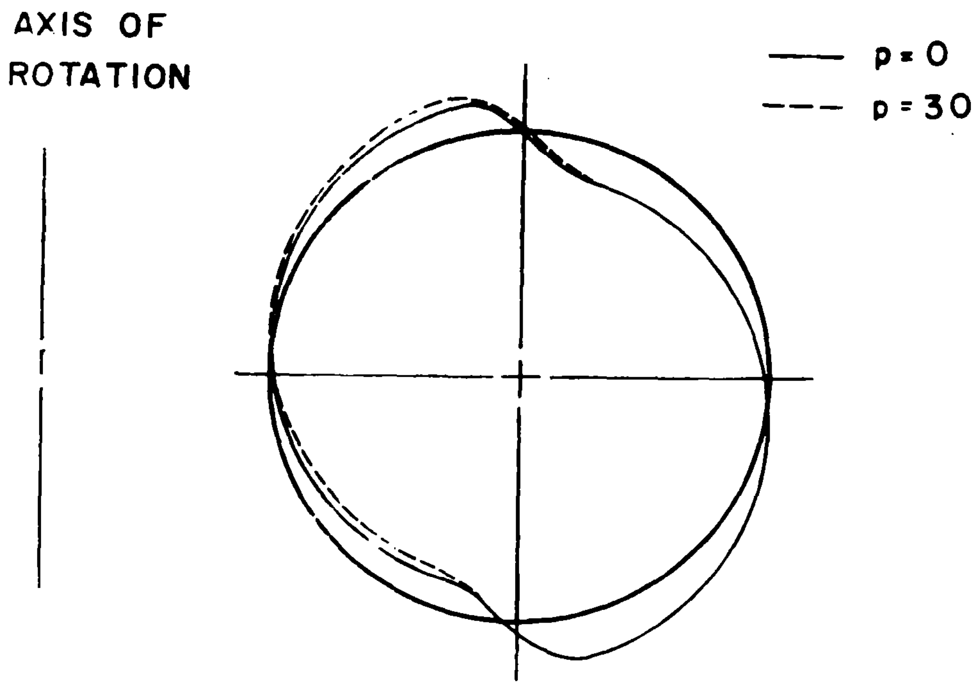


FIGURE 1 GEOMETRY AND NOTATION



k	β_k	h_k
1	0.2 RAD	1.800
2	30°	0.160
3	58°	0.100
4	π - 0.05 RAD	0.0627
5	π	0.2687

FIGURE 2 WALL THICKNESS VARIATION OF MODEL I



INTERNAL PRESSURE, PSI	CALCULATED FREQUENCY, CPS	MEASURED FREQUENCY, CPS
0	45.4	48.8*
6	59.3	61.2 *
15	74.4	75.8**
30	92.8	—

*AVERAGE OF TWO VALUES GIVEN IN FIGURE 9a OF REFERENCE [4]

** SEE DISCUSSION ON PAGE 10, REFERENCE [4]

FIGURE 3 FUNDAMENTAL MODE OF MODEL I FOR n=0

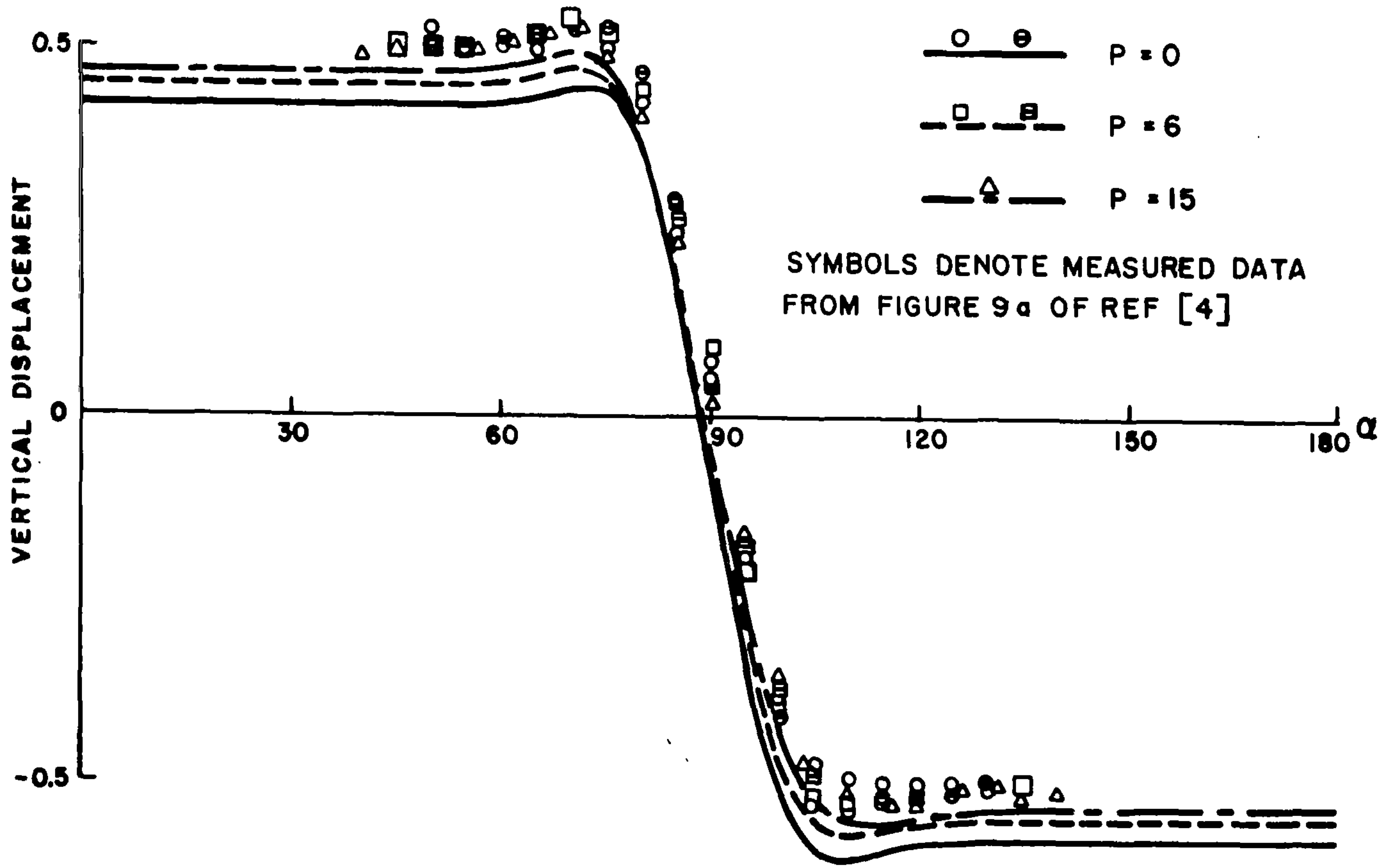


FIGURE 4: FUNDAMENTAL MODE OF MODEL I - CALCULATED AND MEASURED

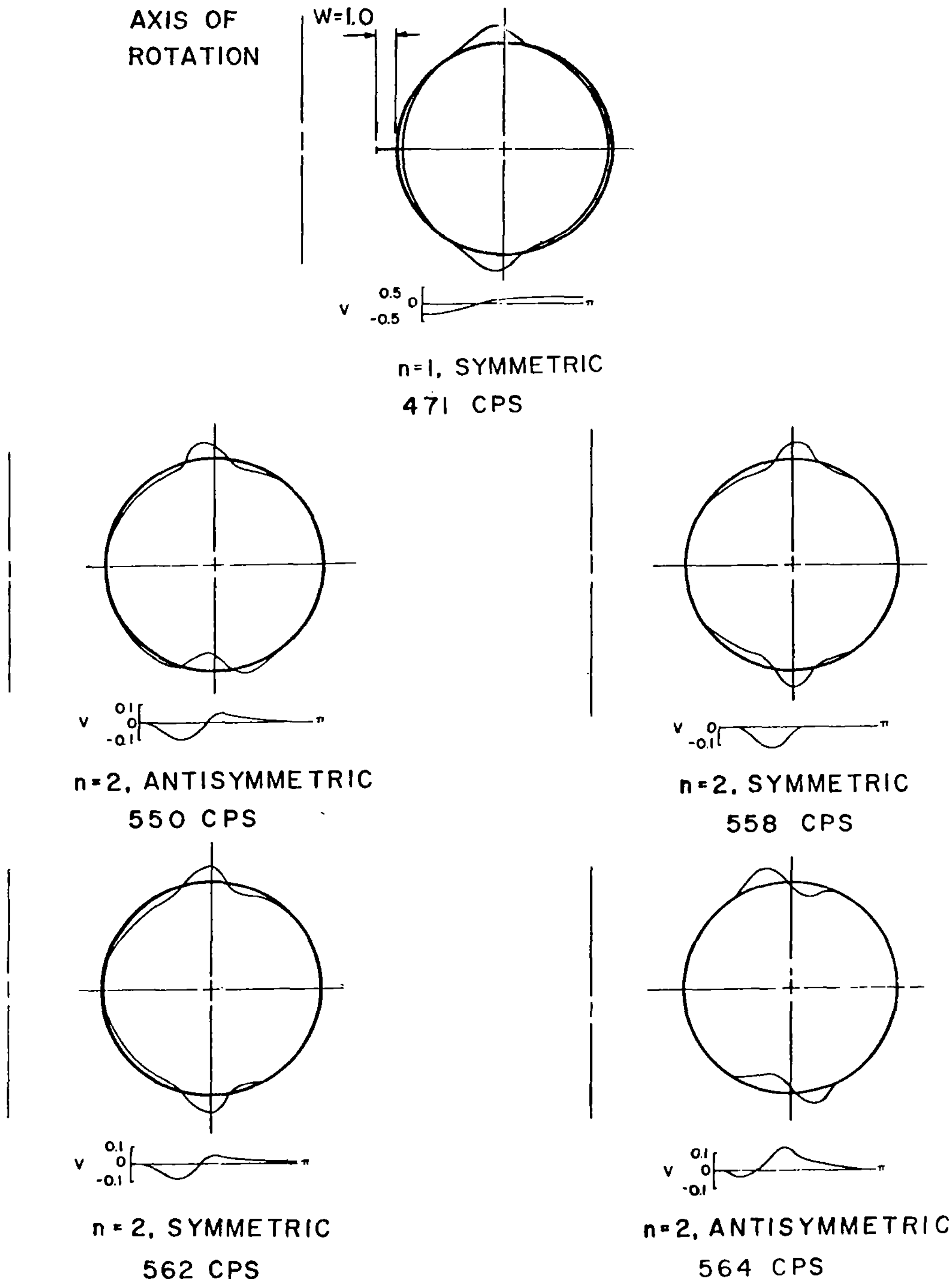
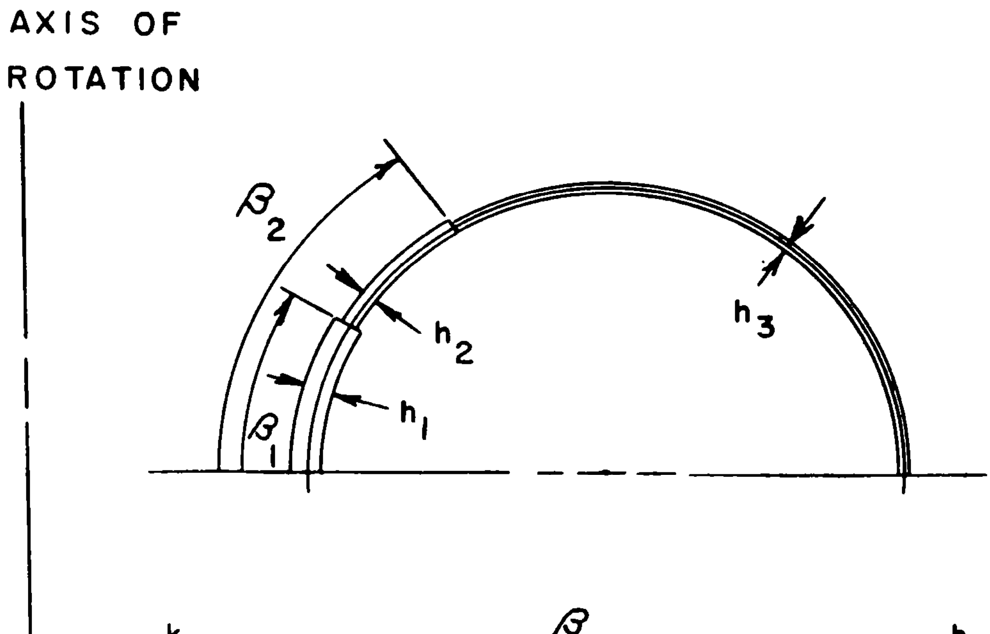


FIGURE 5 OVERTONE MODE SHAPES OF MODEL I, p=0



k	β_k	h_k
1	30°	0.160
2	58°	0.100
3	180°	0.0627

FIGURE 6 WALL THICKNESS VARIATION OF MODEL II

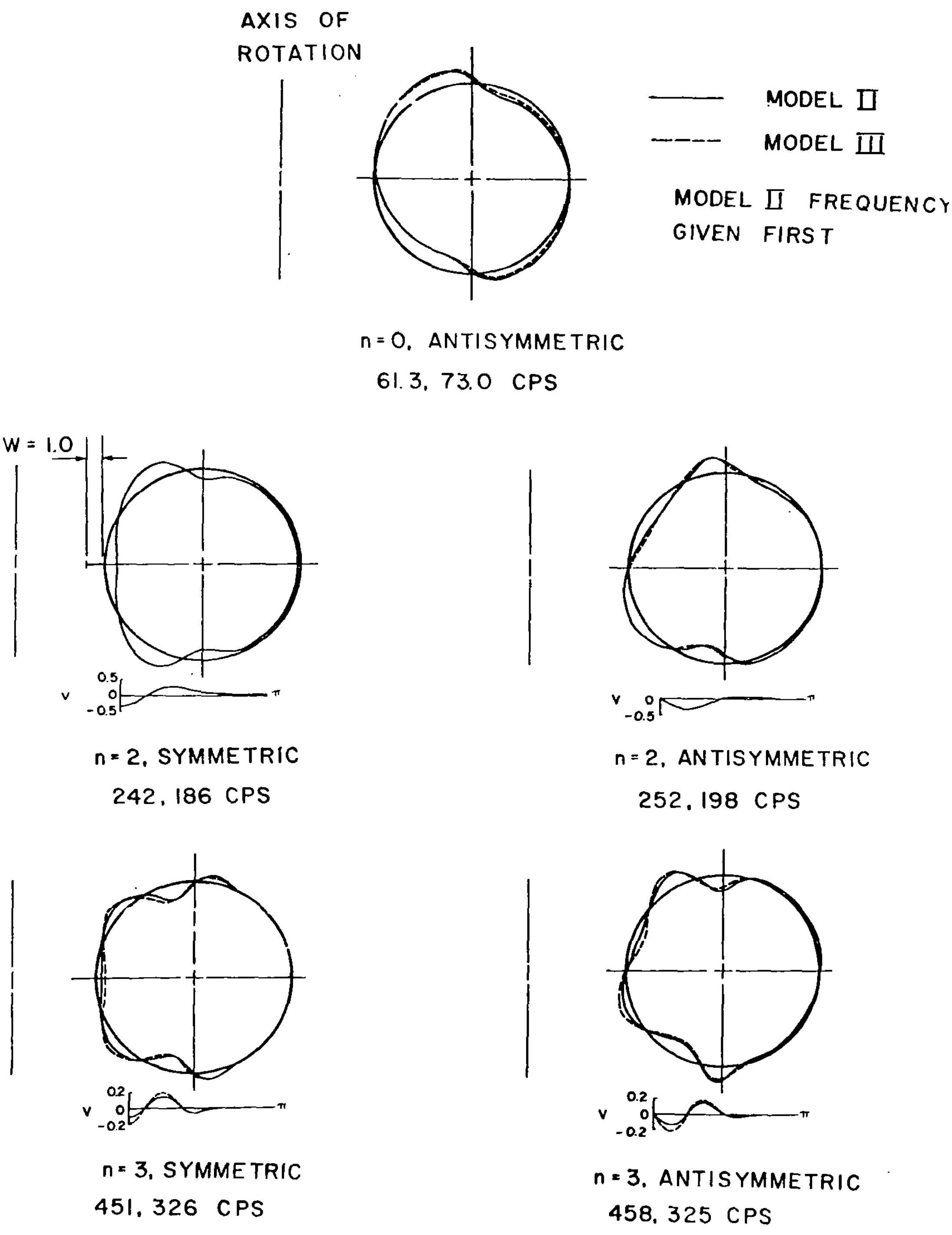


FIGURE 7 MODE SHAPES OF MODELS II AND III, p = 0

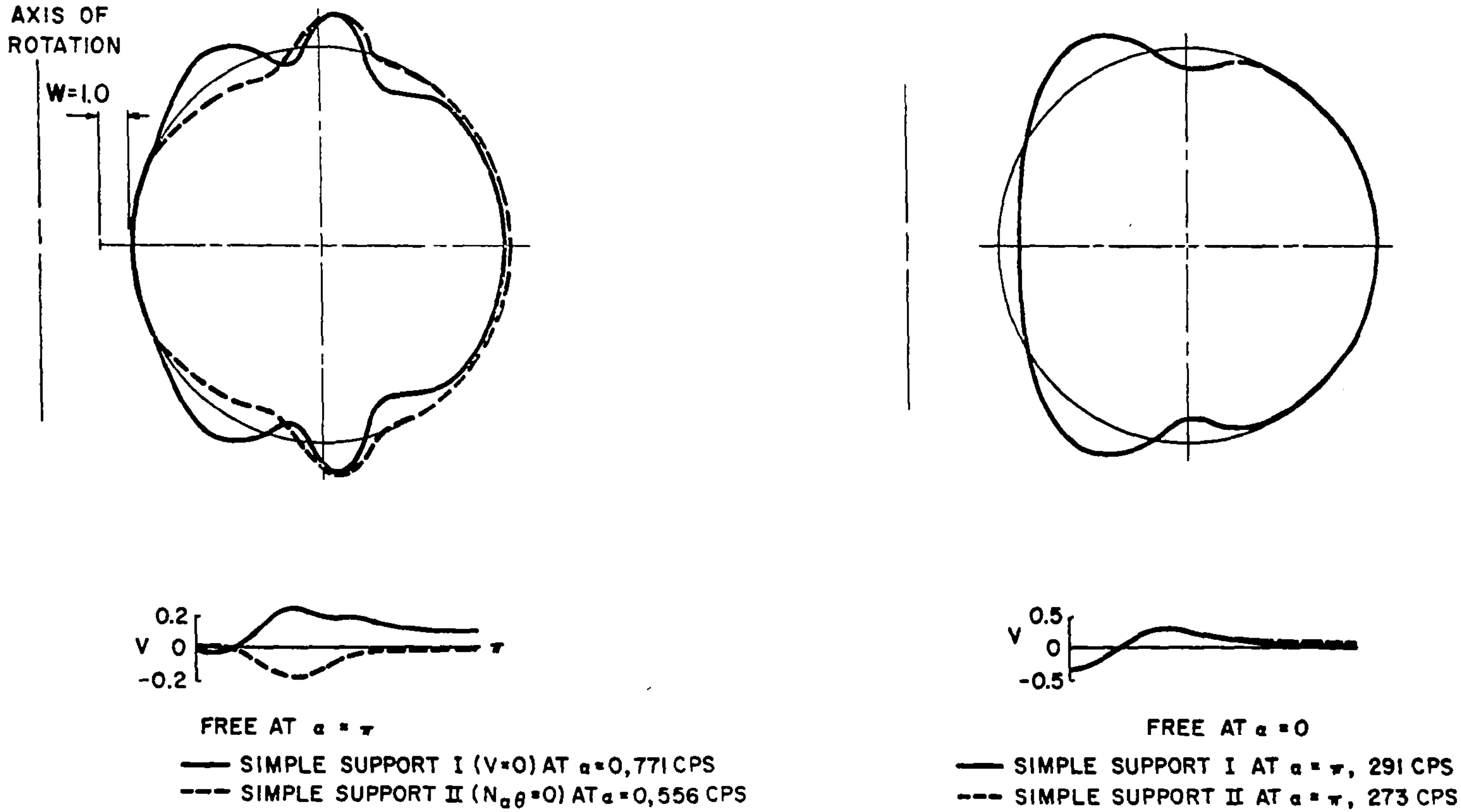


FIGURE 8: n = 2, SYMMETRIC MODE SHAPES OF MODEL II WITH SUPPORTS, p = 6 PSI

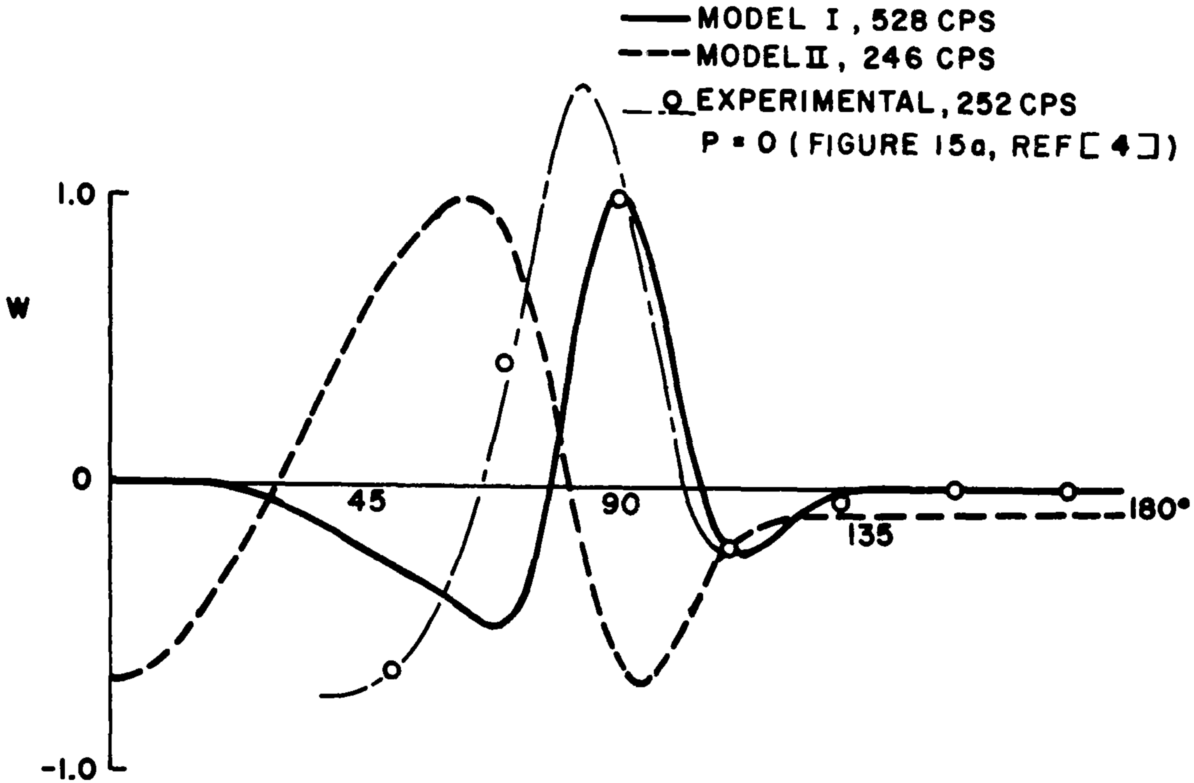


FIGURE 9: NORMAL DISPLACEMENT OF $n = 2$ SYMMETRIC MODE OF MODELS I & II , AND EXPERIMENTAL RESULTS

NASA-Langley, 1967 — 32 CR-885

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546