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SPINNING PARABOLOIDAL TENSION NETWORKS

by William M. Robbins, Jr.

Prepared by
ASTRO RESEARCH CORPORATION
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SUMMARY

The requirements for forming a spinning paraboloid of revolution from perfectly flexible fibers are investigated. It is shown that such a surface can be made with two symmetrically placed sets of spiral fibers, either with or without a parallel-circle set. In either case there is a minimum radius at which a given design must be truncated and which, for reasonable surface geometry, limits the ratio of outer to inner radius to two or less.

INTRODUCTION

Current interest in the design of very large reflectors for orbiting radiotelescopes operating at low frequencies has led to the investigation of methods whereby the surface density of radio reflectors can be made as low as possible. This need for lightness, coupled with the requirements for reflectivity, for operation in a micrometeroid environment, and for maintaining a geometrically precise surface, has led the Astro Research Corporation to the investigation of networks of ribbons as radio reflectors.

Methods whereby a reflector, in the form of a paraboloid of revolution, can be formed by using a network of flexible fibers is the subject of report. The fibers are maintained in tension by spinning the paraboloid about its axis of symmetry and by supplying an axial tension force by means of a central compression column.

It is assumed that the network structure is fine enough that membrane theory can be applied to the surface.

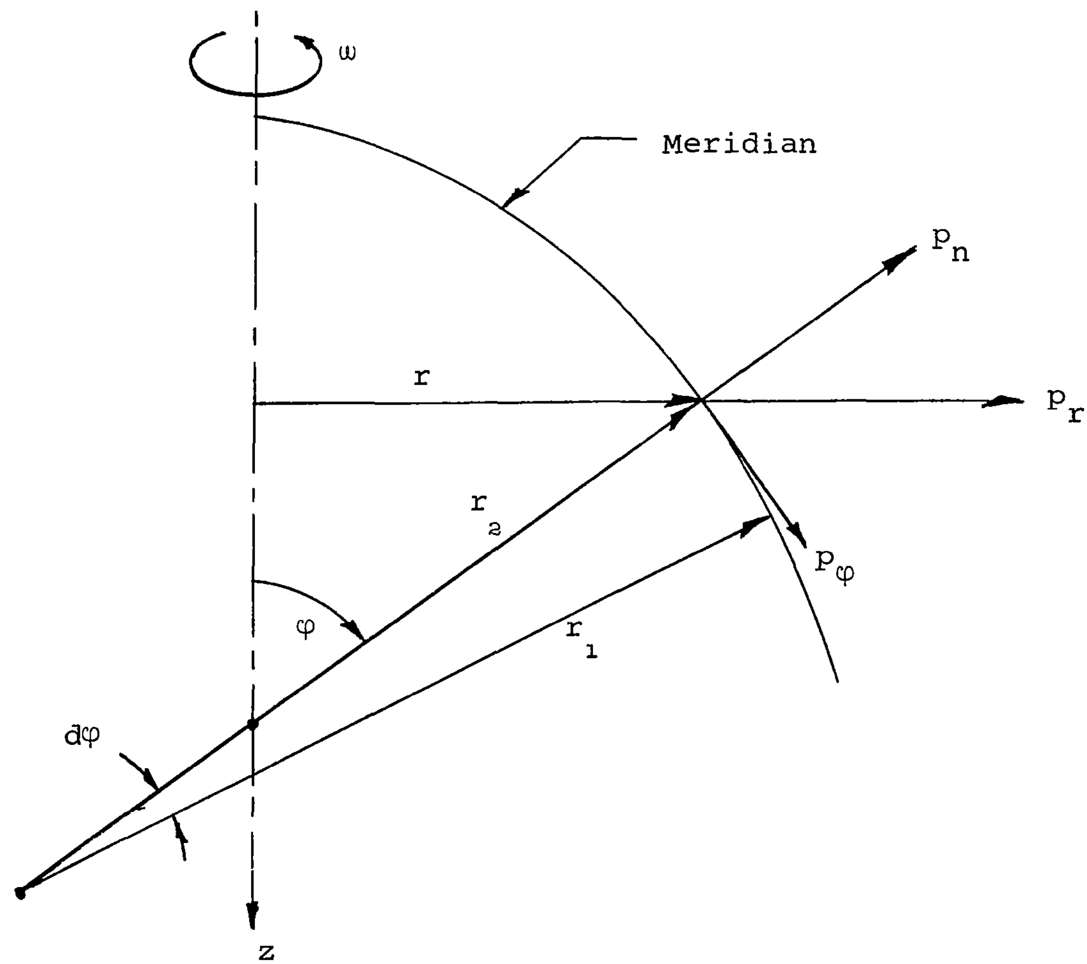
SYMBOLS

A	cross-sectional area of fiber
A_a	cross-sectional area of parallel-circle fiber
A_c	cross-sectional area of spiral fiber
C	characteristic parameter for paraboloid formed of two families of fibers
F	total axial force
f	focal distance of paraboloid
ℓ	length of fiber element
M_R	total mass on rim
M'_R	mass loading per unit of length on rim
m''	mass per unit surface area
m'	mass per unit fiber length
N_R	radial force per unit of length as supplied by rim mass
N_α	membrane force supplied by tensioning fibers
N_θ	latitudinal membrane force
N_φ	meridional membrane force
$N_{\varphi R}$	value of N_φ at rim
n	focal ratio of paraboloid
n'	number of fibers in each spiral set
p_n	normal force per unit surface area

p_r	centrifugal force per unit surface area
p_φ	meridional force per unit surface area
R	radius at rim
r	radius
r_f	r/f = normalized radius
s	spacing between fibers
s_m	spacing between parallel-circle fibers
T	tension in fiber
T_a	tension in parallel circle fiber
T_c	tension in spiral fiber
z	axial distance coordinate
α	angle between tensioning fibers and plane normal to axis of symmetry
γ	angle between fiber and local meridian
ΔC_o	circumferential fiber spacing at point where $\cos\gamma = 1$
ρ	density of fiber
σ_a	stress in parallel-circle fiber
σ_c	stress in spiral fiber
φ	meridional angle
Ω_f	characteristic parameter for paraboloid formed of two families of fibers
ω	spin rate of paraboloid

THE GENERAL MEMBRANE FORCES

Consider a thin shell of revolution, without bending stiffness, spinning with constant speed ω about its axis of symmetry. Let the only body force be the centrifugal force p_r which results from the spin, and which produces the meridional and latitudinal membrane forces N_φ and N_θ. The following sketch shows the geometry of a meridional cut of the surface, where r_1 is the radius of curvature of the meridian.



The following relations can be written by inspection.

$$r = r_2 \cdot \sin\varphi \tag{1}$$

$$\frac{dr}{d\varphi} = r_1 \cdot \cos\varphi \tag{2}$$

$$p_n = p_r \cdot \sin\varphi \tag{3}$$

$$p_\varphi = p_r \cdot \cos\varphi \tag{4}$$

Also

$$p_r = m'' \omega^2 r \tag{5}$$

where m'' is the mass per unit surface area.

The equations of equilibrium are (see ref. 1, p. 23):

$$\frac{d\left(N_\varphi \cdot r\right)}{d\varphi} - r_1 N_\theta \cdot \cos\varphi = - p_\varphi \cdot r \cdot r_1 \tag{6}$$

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} = p_n \tag{7}$$

These equations can be combined to yield

$$\frac{d\left(N_\varphi \cdot r\right)}{d\varphi} + \left(N_\varphi \cdot r\right) \cot\varphi = 0 \tag{8}$$

which has the solution

$$2\pi r N_\varphi \cdot \sin\varphi = F \tag{9}$$

where F is the total axial component of force carried by the membrane, and supplied by some external source, such as a compression column. Equation (9) can also be written as

$$N_\varphi = \frac{F}{2\pi r \cdot \sin\varphi} \tag{10}$$

Rearranging (7) and substituting (1), (2), (5), and (10) yields

$$N_\theta = m''\omega^2 r^2 - \frac{F}{2\pi}\cdot\cot\varphi\cdot\csc\varphi\cdot\frac{d\varphi}{dr} \tag{11}$$

THE PARABOLOID OF REVOLUTION

The general shell of revolution will now be restricted to a paraboloid of revolution by defining the relationship between z and r such that

$$z = \frac{1}{4f}\cdot r^2 \tag{12}$$

where f is the focal distance. For reference, the meridional shape of a paraboloid of revolution is shown in figure 1, where the "depth" of the paraboloid can be seen for various values of the focal ratio n .

Let $r_f = \frac{r}{f}$ Then by differentiation

$$\tan\varphi = \frac{dz}{dr} = \left(\frac{r_f}{2}\right) \tag{13}$$

Further manipulation leads to the following three expressions.

$$\sin\varphi = \frac{\left(\frac{r_f}{2}\right)}{\sqrt{1 + \left(\frac{r_f}{2}\right)^2}} \tag{14}$$

$$\frac{d\varphi}{dr}\cdot r\cdot \cot\varphi = \frac{1}{1+\left(\frac{r_f}{2}\right)^2} \tag{15}$$

$$\cot\varphi\cdot \csc\varphi\cdot\frac{d\varphi}{dr} = \frac{1}{r}\cdot\frac{1}{\frac{r_f}{2}\sqrt{1+\left(\frac{r_f}{2}\right)^2}} \tag{16}$$

Equations (10) and (11) can now be written as

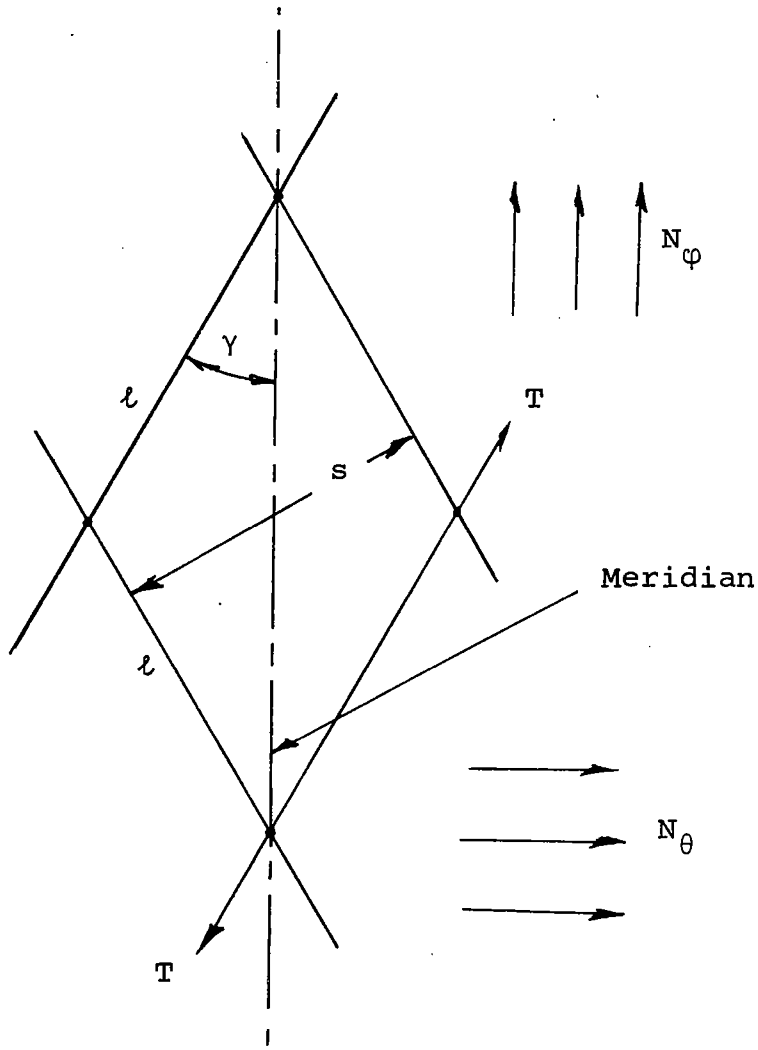
$$N_\varphi = \frac{F}{2\pi r}\,\frac{\sqrt{1+\left(\frac{r_f}{2}\right)^2}}{\left(\frac{r_f}{2}\right)} \tag{17}$$

and

$$N_\theta = m''\omega^2 r^2 - \frac{F}{2\pi r}\cdot\frac{1}{\left(\frac{r_f}{2}\right)\sqrt{1+\left(\frac{r_f}{2}\right)^2}} \tag{18}$$

NETWORK OF TWO FAMILIES OF FIBERS

Let the paraboloidal surface be formed of two symmetrically placed and perfectly flexible sets of spiral fibers connected at the intersections and forming diamond-shaped meshes over the surface as shown in figure 2. Let the dimensions of the meshes be small enough, in comparison with the total dimensions of the paraboloidal surface, that the surface can be treated approximately as a membrane. The forces on such a mesh are shown in the sketch below.



where:

ℓ = length of fiber element

s = spacing between fibers

γ = angle between fiber and local meridian

T = tension in fiber

By inspection it follows that

$$N_\varphi \cdot \ell \cdot \sin\gamma = T \cdot \cos\gamma \tag{19}$$

$$N_\theta \cdot \ell \cdot \cos\gamma = T \cdot \sin\gamma \tag{20}$$

$$\ell = \frac{\pi r}{n' \cdot \sin\gamma} \tag{21}$$

$$s = \frac{2\pi r}{n'} \cdot \cos\gamma \tag{22}$$

where n' is the number of fibers in each spiral set.

From (19) and (20)

$$\frac{N_\theta}{N_\varphi} = \tan^2\gamma \tag{23}$$

which indicates that, because of the in-plane shear compliance of the two-fiber network, there must be a specific relationship between N_θ and N_φ for each value of γ .

For such a network of two symmetrically placed families of fibers, the mass per unit of surface area is given by

$$m'' = \frac{2m'}{s} = \frac{2\rho A}{s} \tag{24}$$

where:

m'' = mass per unit of surface area

m' = mass per unit of fiber length

ρ = density of fiber

A = cross-sectional area of fiber

When (22) is substituted into (24):

$$m'' = \frac{\rho A n'}{\pi r \cdot \cos\gamma} \tag{25}$$

When (17), (18) and (25) are substituted into (23), an equation in $\cos\gamma$ is obtained.

$$\cos^2\gamma + 2C\left(\frac{r_f}{2}\right)\cdot\sqrt{1+\left(\frac{r_f}{2}\right)^2}\cdot\cos\gamma - \frac{1+\left(\frac{r_f}{2}\right)^2}{\left(\frac{r_f}{2}\right)^2} = 0 \tag{26}$$

where

$$C = \frac{4\rho A n' \omega^2 f^2}{F} \tag{27}$$

Let $\cos\gamma_{(r_f=1)}$ be the value of $\cos\gamma$ when $r_f = 1$.
Then (26) can be solved for C at $r_f = 1$ which leads to

$$C = \frac{2\sqrt{5}\left[1 - \frac{1}{5}\cdot\cos^2\gamma_{(r_f=1)}\right]}{\cos\gamma_{(r_f=1)}} \tag{28}$$

and C vs. $\cos\gamma_{(r_f=1)}$ is shown in figure 3.

Solving (26) for $\cos\gamma$ yields

$$\cos\gamma = C\cdot\frac{r_f}{2}\sqrt{1+\left(\frac{r_f}{2}\right)^2}\cdot\left[\sqrt{1+\frac{1}{C^2\left(\frac{r_f}{2}\right)^4}} - 1\right] \tag{29}$$

In figure 4, γ as a function of r_f is shown for various values of $\cos\gamma_{(r_f=1)}$. It should be noted that there is always a minimum possible value of r_f for each value of $\cos\gamma_{(r_f=1)}$ and it occurs when $\gamma = 0$, or when the fibers lie along a meridian. By setting $\cos\gamma = 1$ in (26) it is possible to find C in terms of $r_{f(min)}$.

$$C = \frac{1}{2\left(\frac{r_{f(min)}}{2}\right)^3 \sqrt{1 + \left(\frac{r_{f(min)}}{2}\right)^2}} \tag{30}$$

and C vs $r_{f(min)}$ is also plotted in figure 3. The relationship between $r_{f(min)}$ and $\cos\gamma_{(r_f=1)}$ can be found from figure 3 and is shown in figure 5.

The manner in which γ varies with radius for various $\cos\gamma_{(r_f=1)}$ is shown in figure 6 where γ is plotted as a function of $r_f/r_{f(min)}$. It can be seen that if some upper and lower limits are placed on γ, then the largest ratio of outer radius to inner radius is achieved when $\cos\gamma_{(r_f=1)}$ is small, which corresponds to an outer radius not large compared to the focal distance, or to a "flat dish".

The stress in the fibers can be found from (19) and (21) to be

$$\sigma = \frac{T}{A} = \frac{N_\varphi \pi r}{An'\cdot\cos\gamma} \tag{31}$$

Substituting (17) into (31) allows a normalized stress to be written as

$$\frac{\sigma}{\rho\omega^2 f^2} = \frac{2}{C\cdot\cos\gamma} \frac{\sqrt{1 + \left(\frac{r_f}{2}\right)^2}}{\left(\frac{r_f}{2}\right)} \tag{32}$$

where $\rho\omega^2 f^2$ is the stress that would exist in a hoop of radius f when it is spun about its axis of symmetry at the rate ω. The normalized fiber stress as a function of r_f is shown in figure 7 for various values of $\cos\gamma_{(r_f=1)}$. It is of interest to note that $\sigma/\rho\omega^2 f^2$ at any r_f never differs very much from that of a hoop at r_f, i.e.

$$\frac{\sigma}{\rho\omega^2 f^2} \simeq r_f^2 \tag{33}$$

The length of a fiber element is given by (21) and can be normalized by dividing it by the circumferential fiber spacing at the minimum radius to yield

$$\frac{\ell}{\Delta C_o} = \frac{r_f}{2r_{f(min)}\sin\gamma} \tag{34}$$

which is displayed in figure 8 for several values of $\cos\gamma_{(r_f=1)}$.

NETWORK OF THREE FAMILIES OF FIBERS

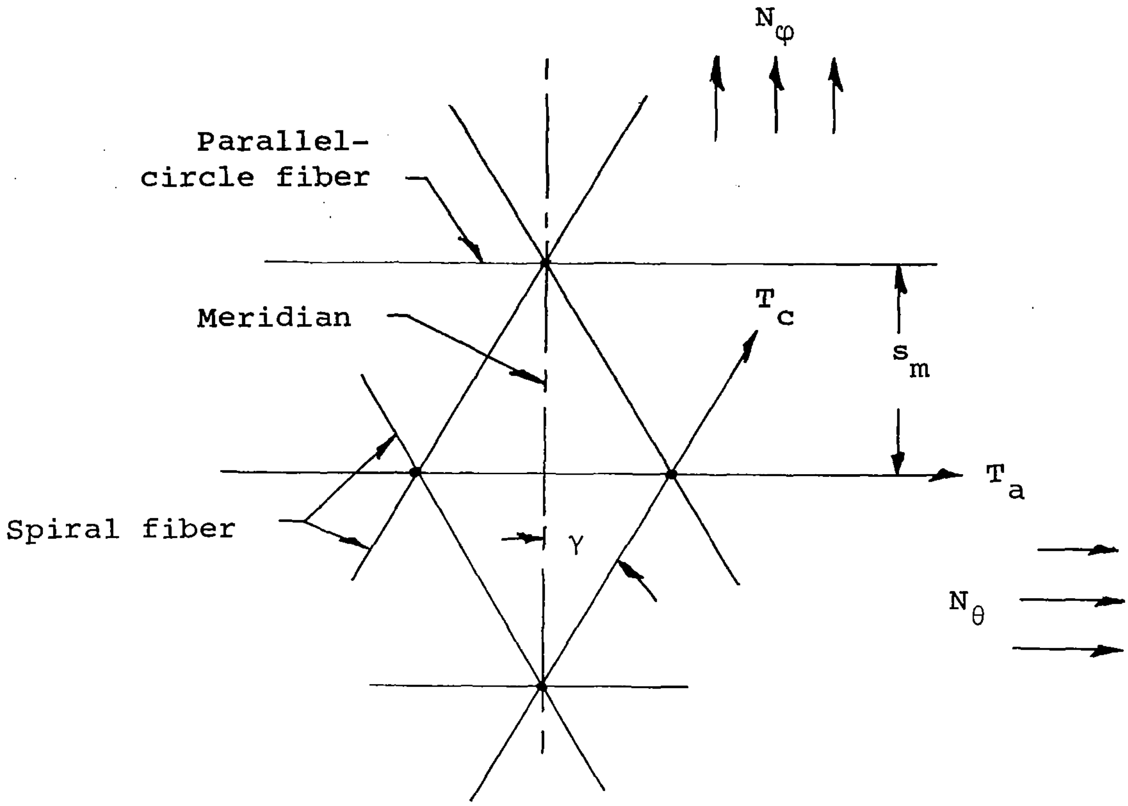
Let the paraboloid of revolution now be formed with a network of three families of perfectly flexible fibers, one family being along parallel circles and the other two being symmetrically placed spiral sets. The fibers are jointed at the intersections to form triangles over the surface as shown in figure 9 and the sketch below.

The same symbols are used as in the previous section, except that

T_a = tension in parallel-circle fiber

T_c = tension in spiral fiber

s_m = spacing between parallel-circle fibers



From the sketch it can be seen that:

$$s_m = \frac{\pi r}{n' \cdot \tan\gamma} \tag{35}$$

$$m'' = \frac{\rho n'}{\pi r}\left[\frac{A_c}{\cos\gamma} + A_a \cdot \tan\gamma\right] \tag{36}$$

$$N_\theta = \frac{T_a}{s_m} + \frac{T_c \cdot \sin\gamma}{s_m} \tag{37}$$

$$N_\varphi = \frac{T_c \cdot \cos\gamma}{s_m \cdot \tan\gamma} \tag{38}$$

where:

A_c = cross-sectional area of spiral fibers

A_a = cross-sectional area of parallel-circle fibers

At this point various possible restrictions can be chosen to more closely define the system. Two assumptions which make the analysis relatively simple are that the cross-sectional area of all the fibers are the same and that the meshes are equilateral triangles, i.e.,

$$A_c = A_a = A$$

$$\gamma = 30°$$

Then:

$$s_m = s = \frac{\sqrt{3}\pi r}{n'} \tag{39}$$

$$m'' = \frac{\sqrt{3}}{\pi} \cdot \frac{\rho n' A}{r} \tag{40}$$

$$N_\theta = \frac{T_a}{s} + \frac{1}{2} \cdot \frac{T_c}{s} = \frac{A}{s}\left(\sigma_a + \frac{\sigma_c}{2}\right) \tag{41}$$

$$N_\varphi = \frac{3}{2} \cdot \frac{T_c}{s} = \frac{3A}{2s} \cdot \sigma_c \tag{42}$$

From (9), (38), and (39)

$$F = \sqrt{3} n' T_c \cdot \sin\varphi = \sqrt{3} \cdot n' A \sigma_c \cdot \sin\varphi \tag{43}$$

From (14)

$$\sin\varphi_{\left(r_f = 1\right)} = \frac{1}{\sqrt{5}} \tag{44}$$

and (43) becomes

$$F = \sqrt{\frac{3}{5}} \cdot n' A \sigma_{c(r_f=1)} \tag{45}$$

Equations (41) and (42) can be solved for σ_c and σ_a yielding:

$$\sigma_c = \frac{2N_\varphi s}{3A} \tag{46}$$

$$\sigma_a = \frac{\left(N_\theta - \frac{1}{3} \cdot N_\varphi\right) s}{A} = \frac{s}{A} \cdot N_\theta - \frac{1}{2} \cdot \sigma_c \tag{47}$$

When (17), (18), (39), (40), and (45) are substituted into (46) and (47), σ_c and σ_a are obtained as

$$\sigma_c = \frac{\sigma_{c(r_f=1)}}{\sqrt{5}} \cdot \frac{\sqrt{1 + \left(\frac{r_f}{2}\right)^2}}{\left(\frac{r_f}{2}\right)} \tag{48}$$

$$\sigma_a = 3\rho\omega^2 f^2 r_f^2 - \frac{\sigma_{c(r_f=1)}}{2\sqrt{5}} \cdot \frac{\left[2 - \left(\frac{r_f}{2}\right)^2\right]}{\left(\frac{r_f}{2}\right)\sqrt{1 + \left(\frac{r_f}{2}\right)^2}} \tag{49}$$

Defining a dimensionless parameter Ω_f to be

$$\Omega_f = \frac{\rho\omega^2 f^2}{\sigma_{c(r_f=1)}} \tag{50}$$

allows σ_c and σ_a to be written in normalized form as

$$\frac{\sigma_c}{\rho\omega^2 f^2} = \frac{1}{\sqrt{5} \cdot \Omega_f} \cdot \frac{\sqrt{1 + \left(\frac{r_f}{2}\right)^2}}{\left(\frac{r_f}{2}\right)} \tag{51}$$

$$\frac{\sigma_a}{\rho\omega^2 f^2} = 3r_f^2 - \frac{\left[2 - \left(\frac{r_f}{2}\right)^2\right]}{2\sqrt{5}\cdot\Omega_f\left(\frac{r_f}{2}\right)\cdot\sqrt{1 + \left(\frac{r_f}{2}\right)^2}} \tag{52}$$

The normalized parallel-circle-fiber stress $\sigma_c/\rho\omega^2 f^2$ and the normalized spiral-fiber stress $\sigma_a/\rho\omega^2 f^2$ are shown in figures 10 and 11, respectively.

It is of interest to note that C of (27) is related to Ω_f by

$$\frac{C \cos\gamma_{\left(r_f=1\right)}}{2\sqrt{5}} = \Omega_f$$

RIM MASS

It is quite apparent that the meridional force N_φ must be supplied at the rim by some external system of forces. Such forces might be supplied by a system of tensioning fibers (assumed here to have negligible mass) in the form of a cone, in addition to radial forces supplied by a uniform distribution of mass along the rim. Such a system is shown in figure 12, where:

M_R = total mass on rim

M_R' = mass loading per unit of length on rim

α = angle between tensioning fibers and plane normal to axis of symmetry

R = radius at rim

$N_{\varphi R}$ = value of N_φ at rim

N_α = membrane force supplied by tensioning fibers

N_R = radial force per unit of length as supplied by rim mass

By summing the axial forces

$$N_\alpha \cdot \sin\alpha = \frac{F}{2\pi r} = N_{\varphi R} \cdot \sin\varphi_R \tag{53}$$

By summing the radial forces:

$$N_\alpha \cdot \cos\alpha + N_{\varphi R} \cdot \cos\varphi_R = N_R = M_R' \omega^2 R \tag{54}$$

By combining (53) and (54) the two following relationships are obtained:

$$N_R = N_{\varphi R} \cdot \frac{\sin\left(\varphi_R + \alpha\right)}{\sin\alpha} \tag{55}$$

$$M_R' = \frac{F}{2\pi\omega^2 R^2}\left(\cot\varphi_R + \cot\alpha\right) \tag{56}$$

Also the total mass M_R is:

$$M_R = 2\pi R M_R' = \frac{F}{\omega^2 R}\left(\cot\varphi_R + \cot\alpha\right) \tag{57}$$

It is of interest to note that the above relationships still hold if the tensioning fibers are not massless or even if they are replaced by a membrane. It is only required that α then be measured at the rim.

CONCLUDING REMARKS

It is possible to form a spinning paraboloid of revolution by a network of either two or three families of fibers. In the case of two families of fibers the placement of the fibers across the entire surface is determined by their orientation at a single point. In the case of three families of fibers, considerably more latitude of design is allowed, the only major restriction being that tension be maintained in each element.

Astro Research Corporation
Santa Barbara, California, December 16, 1966.

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1. Flügge, Wilhelm: Stresses in Shells. Second printing, Springer-Verlag (Berlin), 1962.

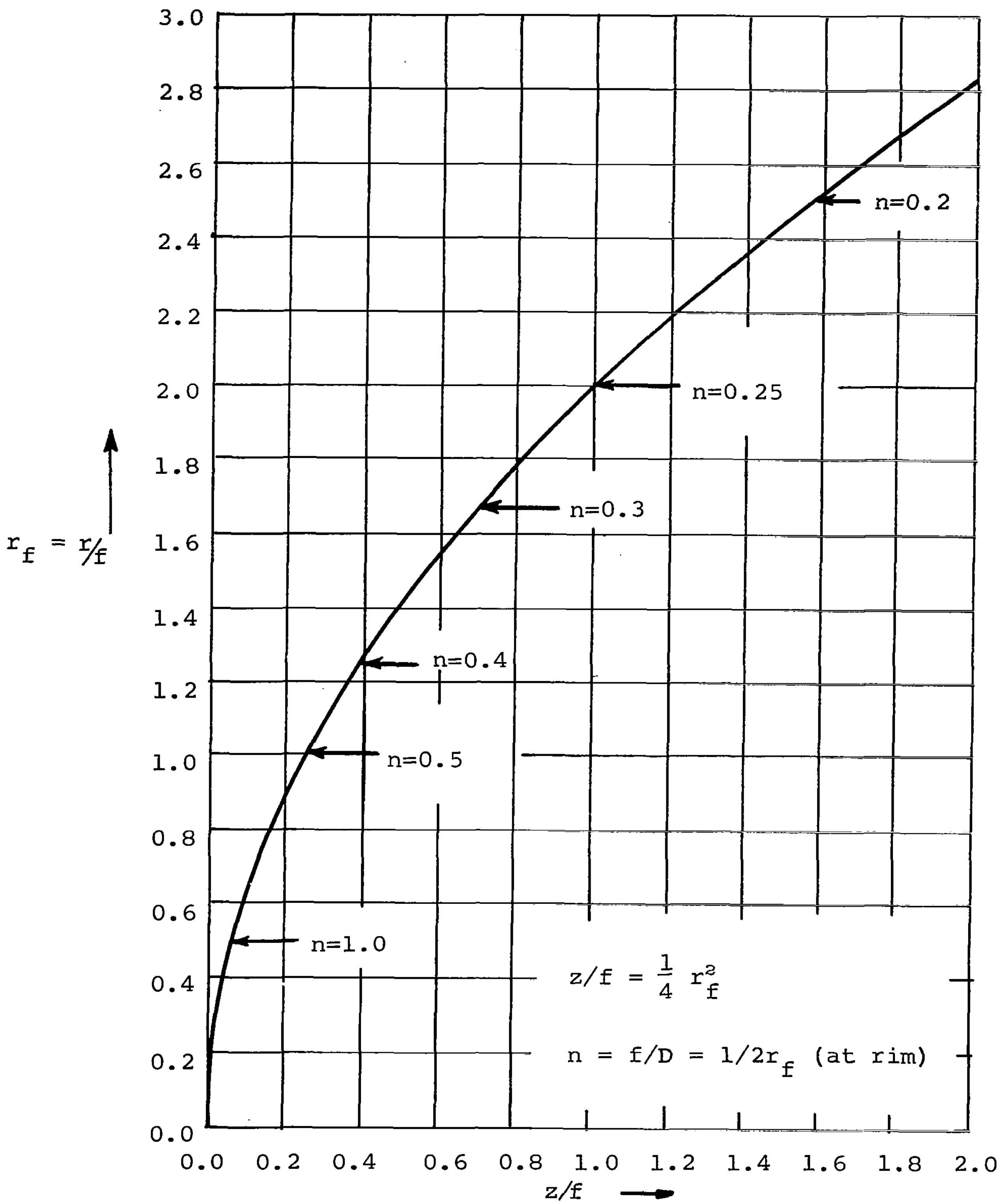

Figure 1. — Paraboloidal Shape

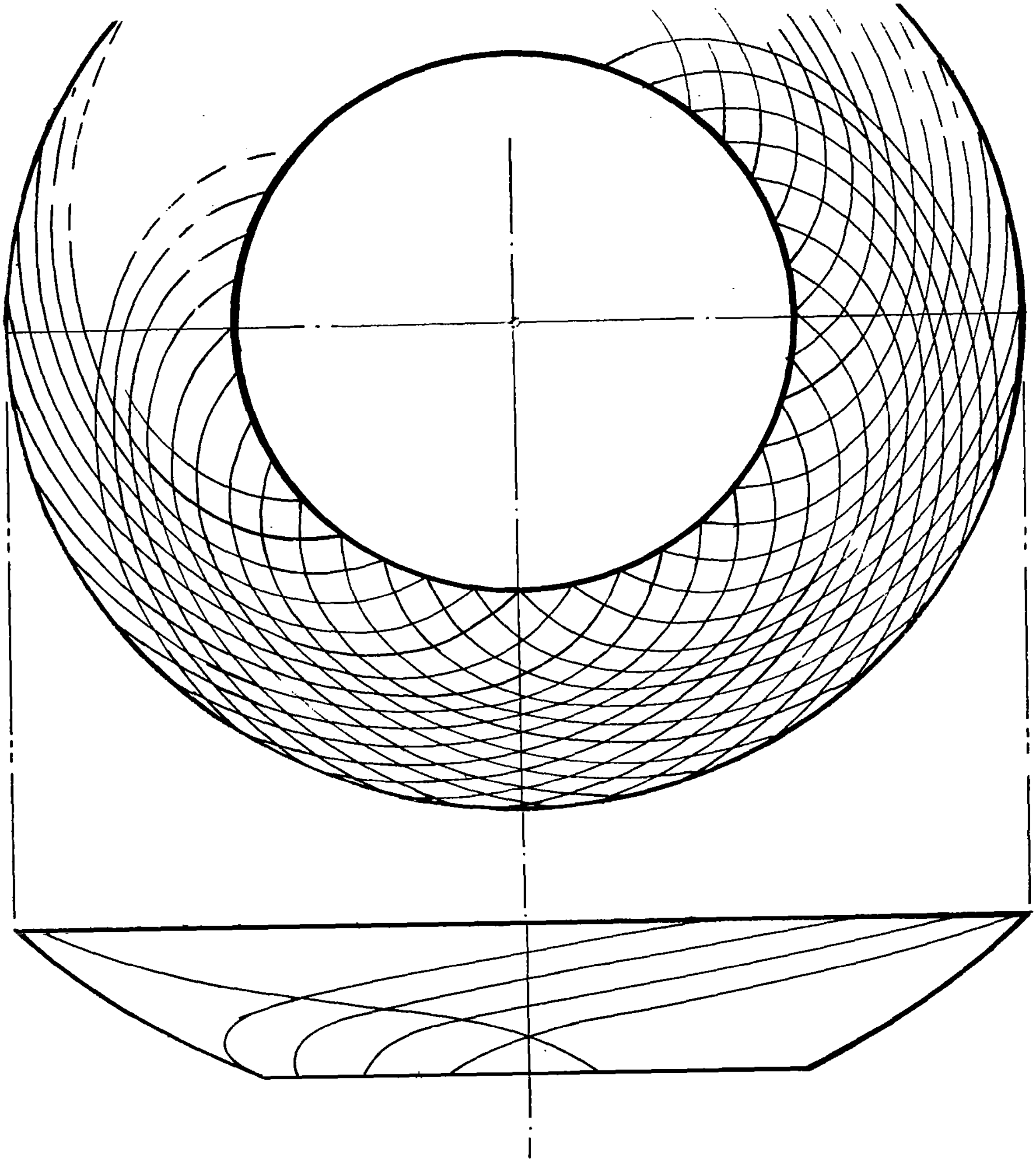

Figure 2. — Paraboloid Formed of Two Families of Fibers

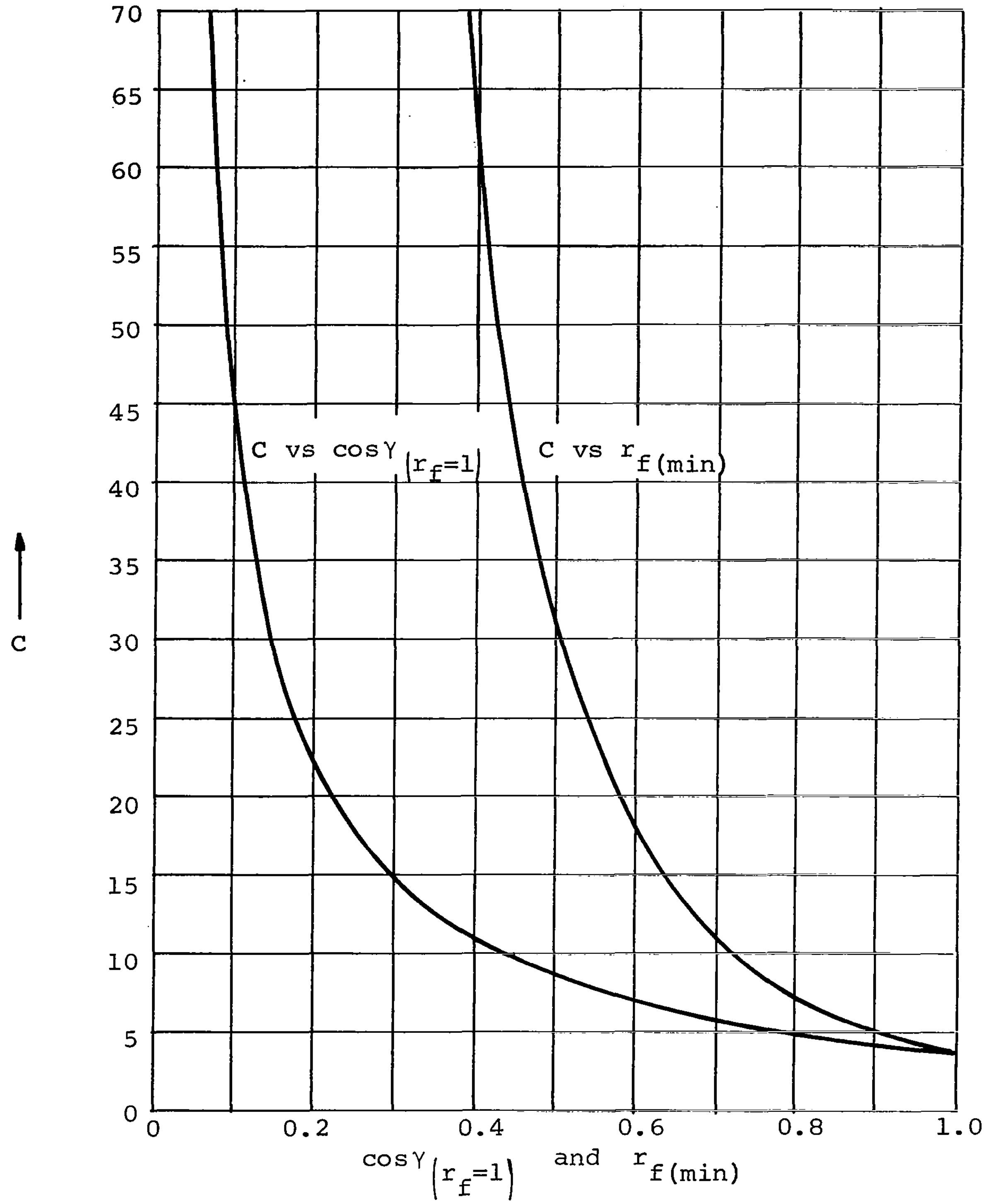

Figure 3. — Characteristic Parameter for Paraboloid of Two Families of Fibers

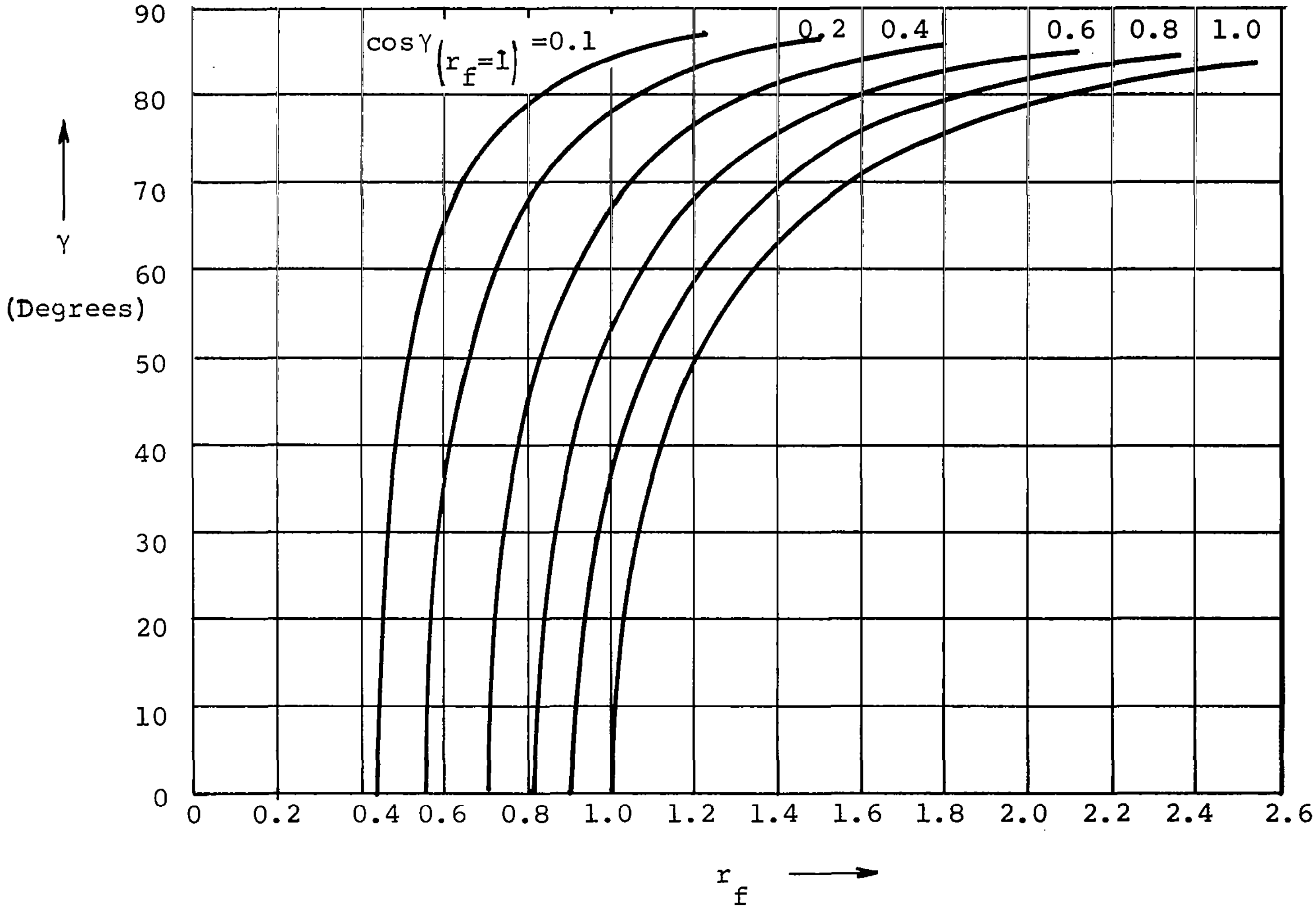

Figure 4. — Variation of Angle γ with r_f for Paraboloid of Two Families of Fibers

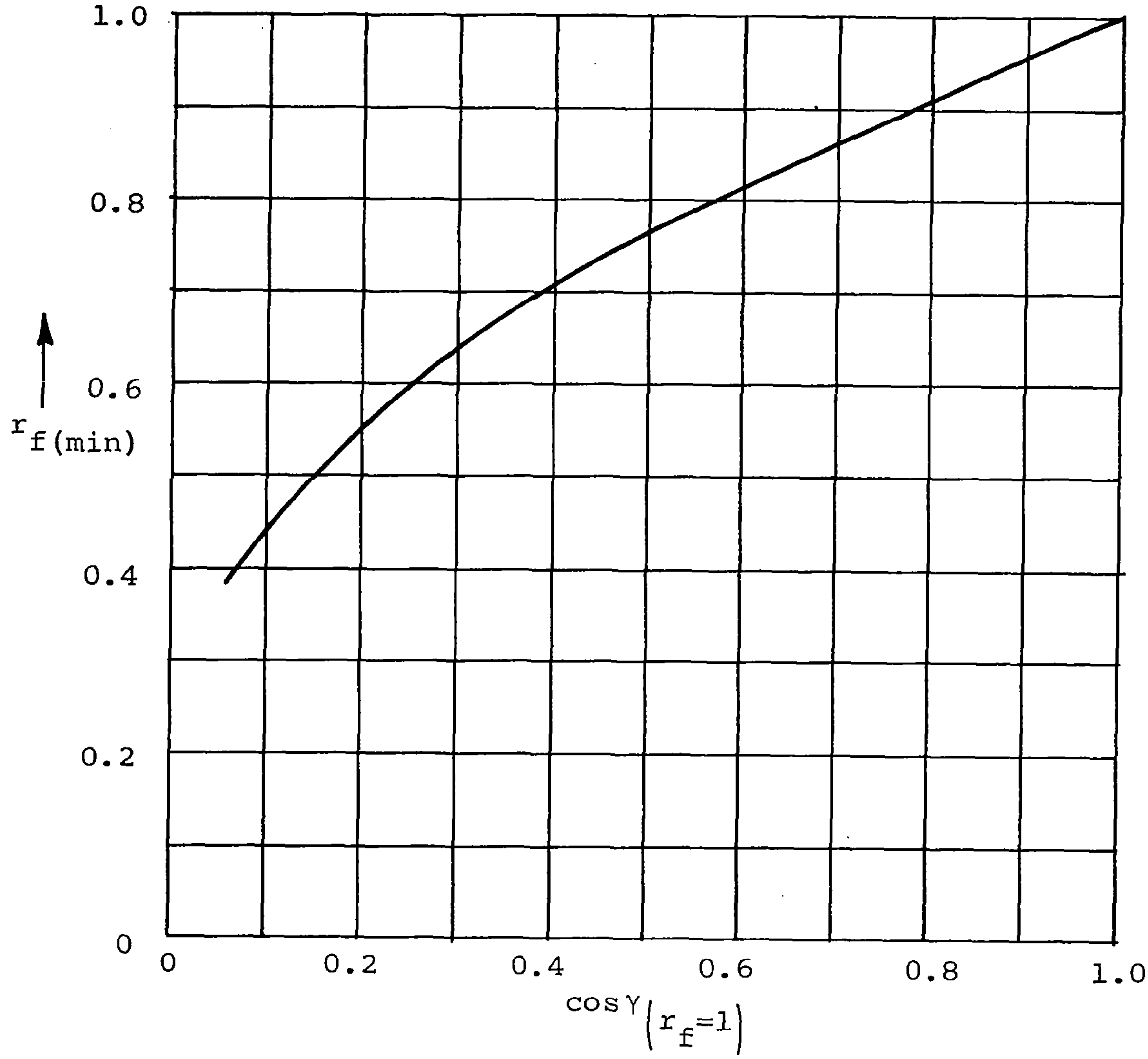

Figure 5. — Minimum Radius for Paraboloid of Two Families of Fibers

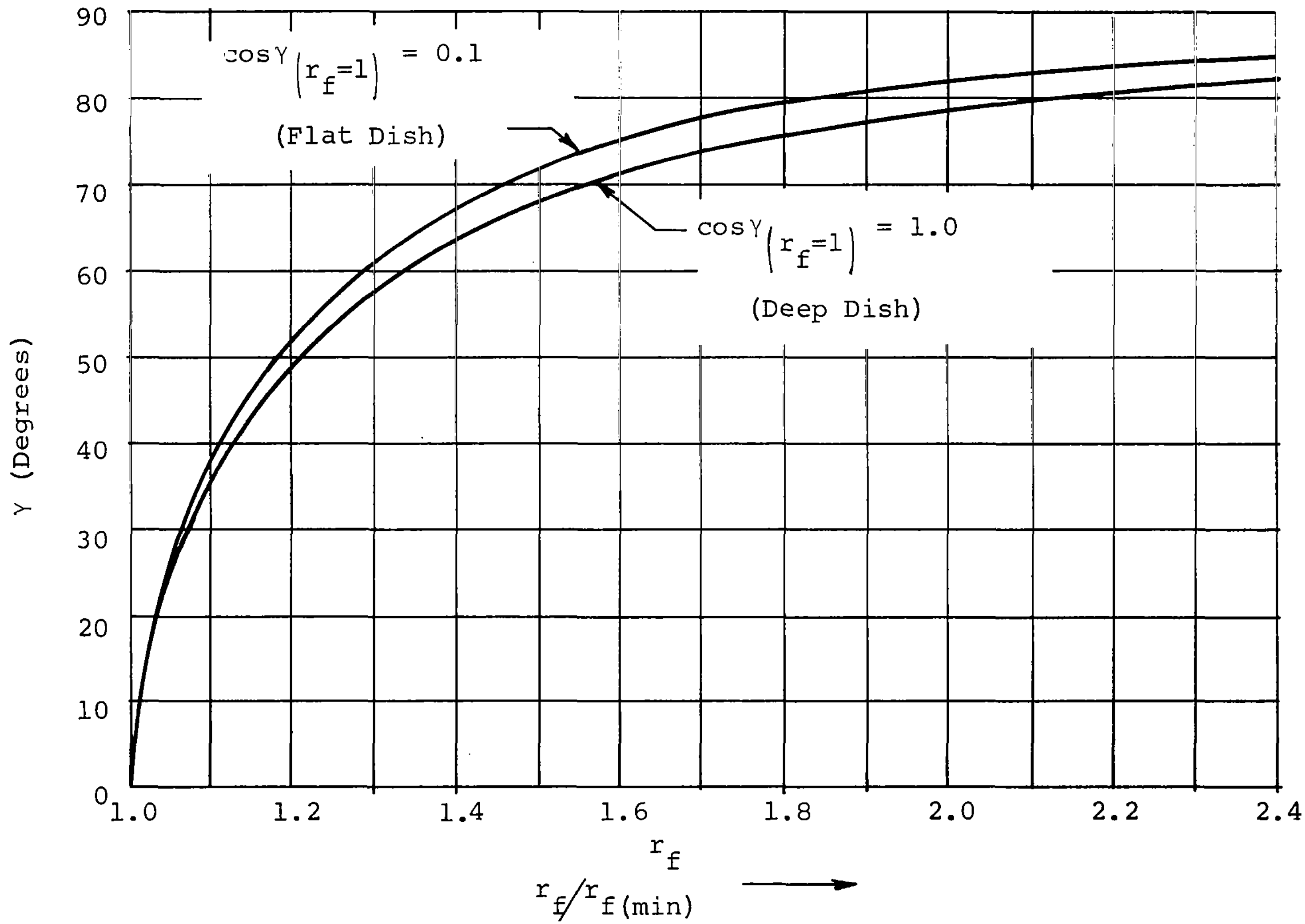

Figure 6. — Variation of the Angle γ with $r_f/r_{f(min)}$ for Paraboloid of Two Families of Fibers

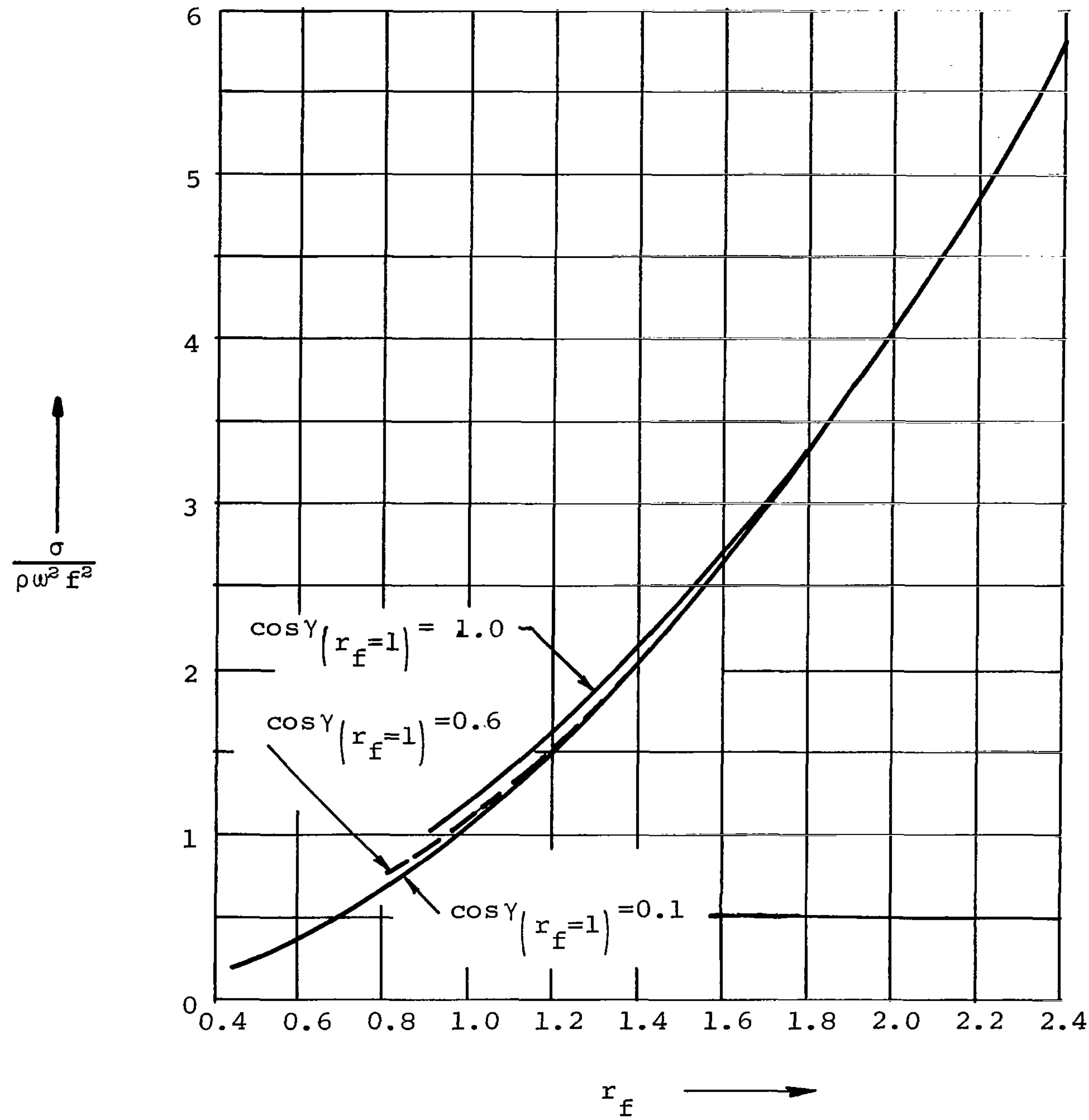

Figure 7. — Normalized Fiber Stress in Paraboloid of Two Families of Fibers

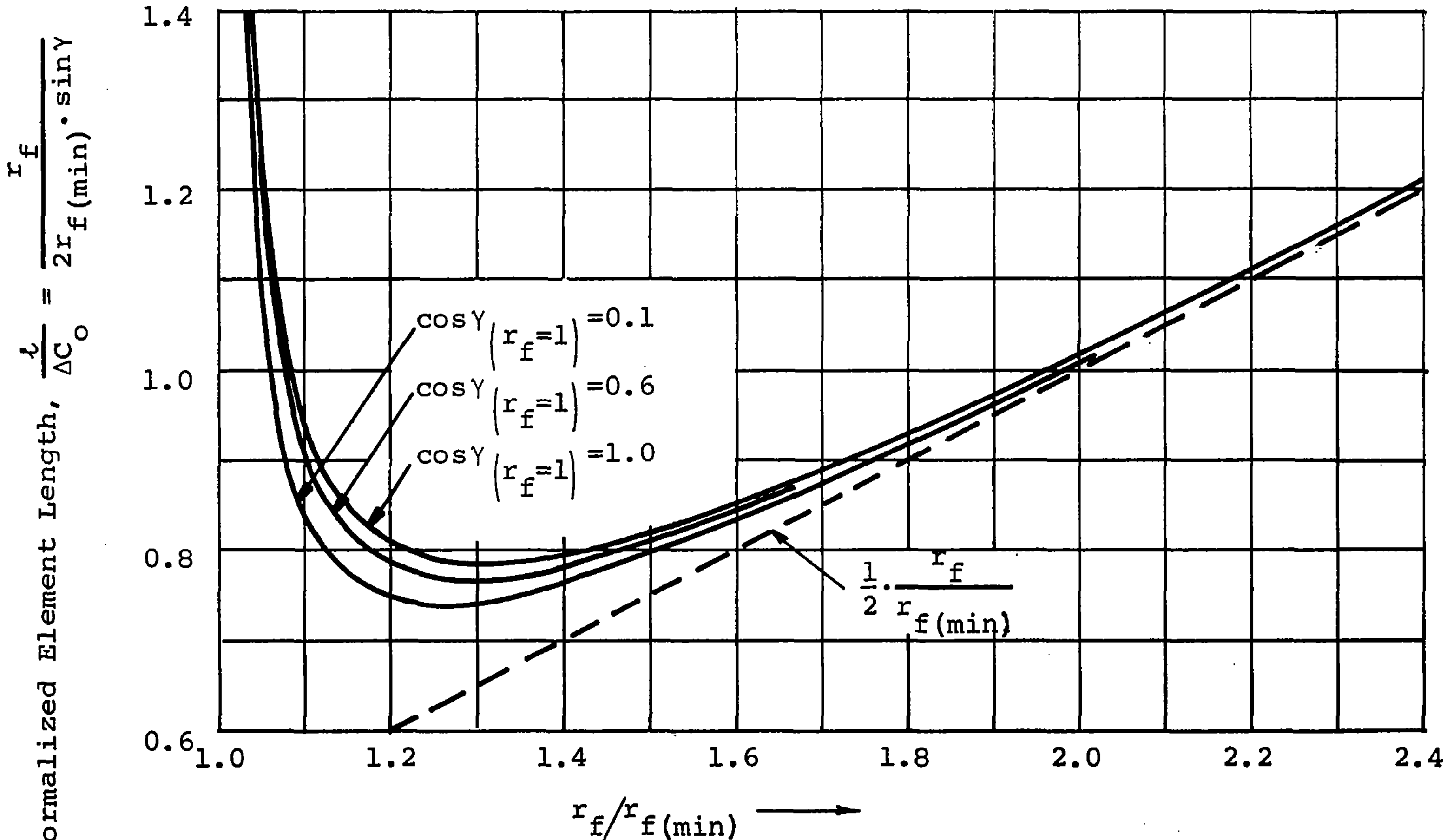

Figure 8. — Normalized Element Length in Paraboloid of Two Families of Fibers

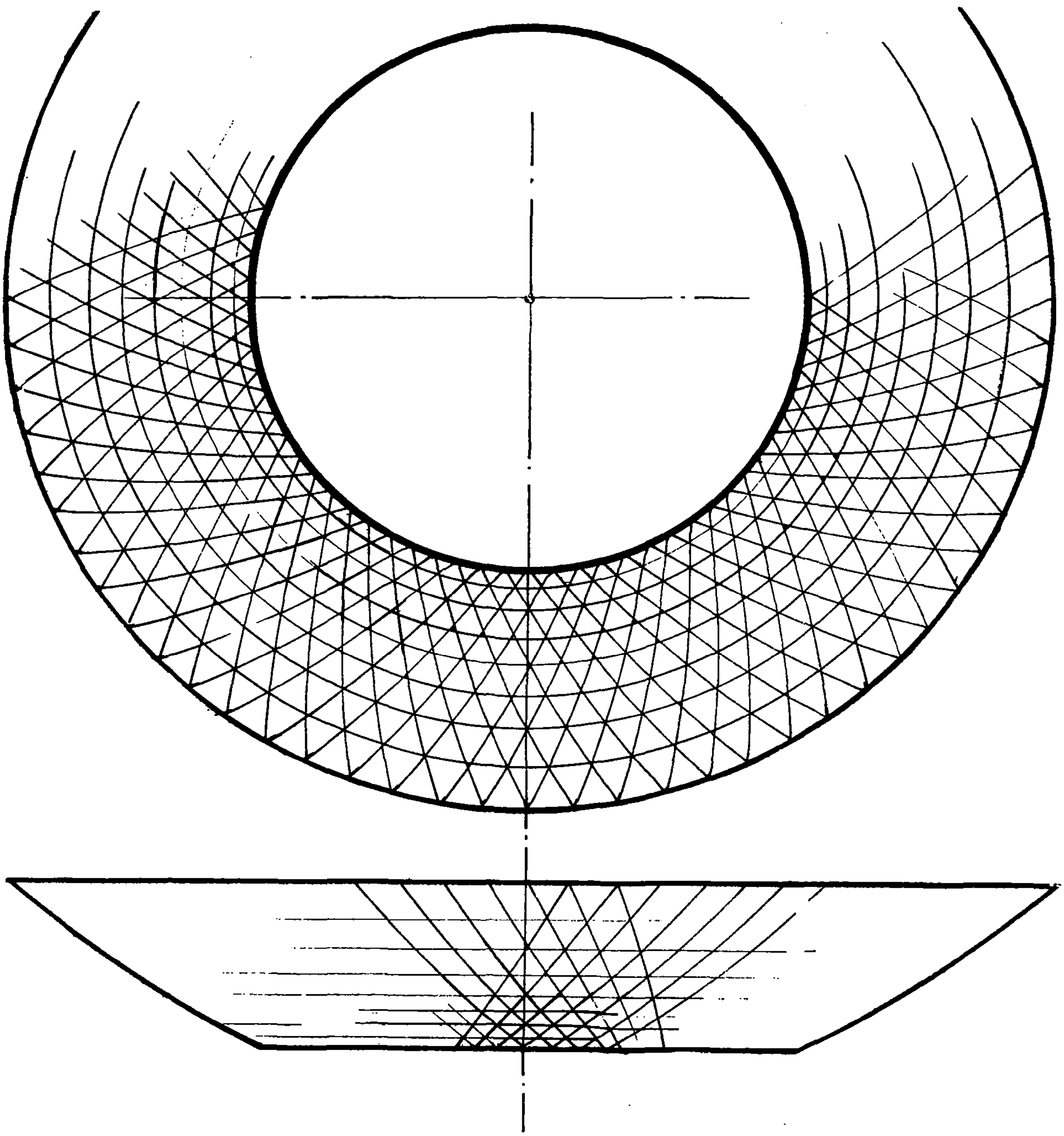

Figure 9. — Overall View of Paraboloid of Three Families of Fibers

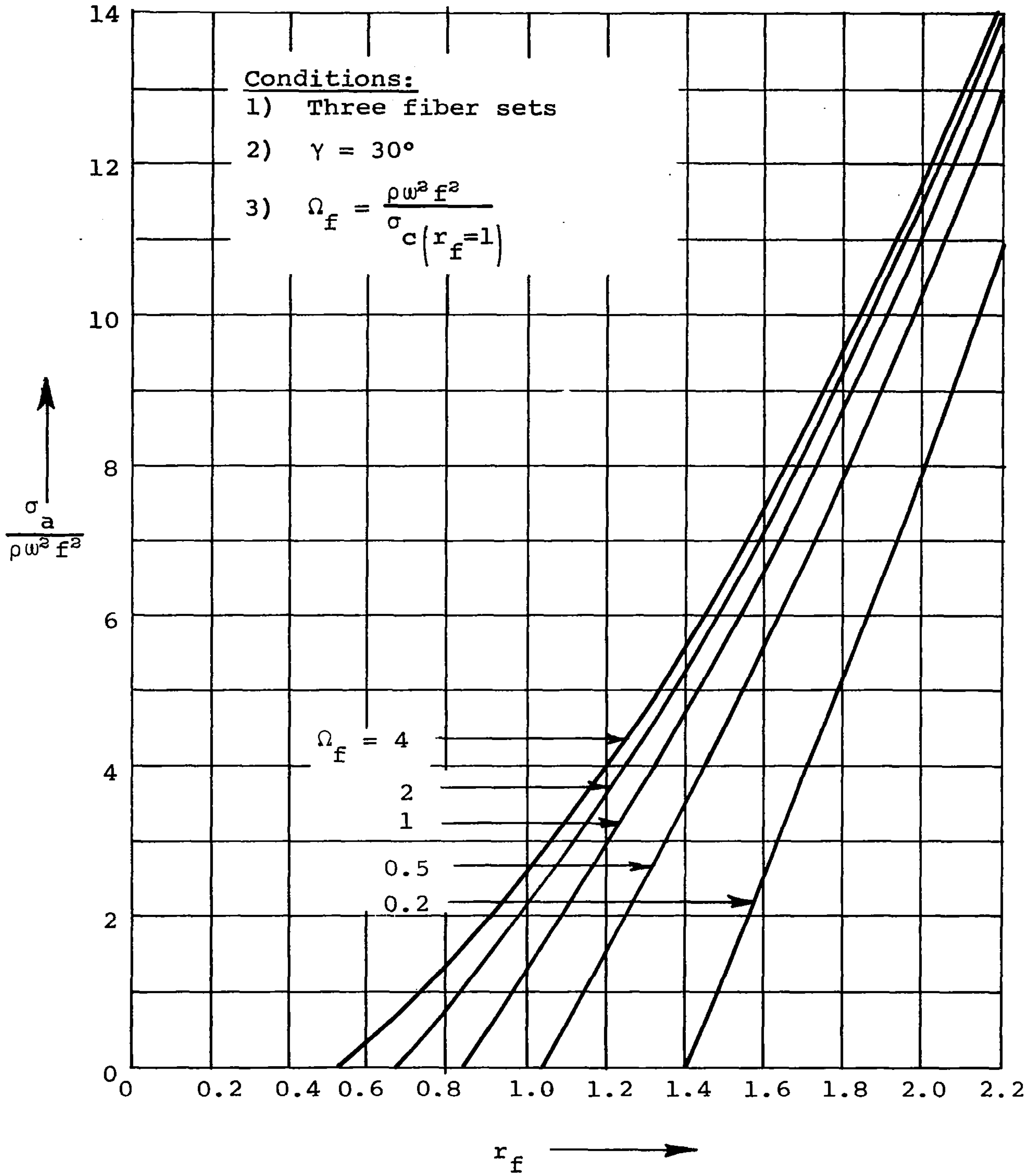

Figure 10. — Normalized Stress in Parallel-Circle Fibers

Conditions:

1) Three fiber sets

2) $\gamma = 30°$

3) $\Omega_f = \dfrac{\rho \omega^2 f^2}{\sigma_c(r_f = 1)}$

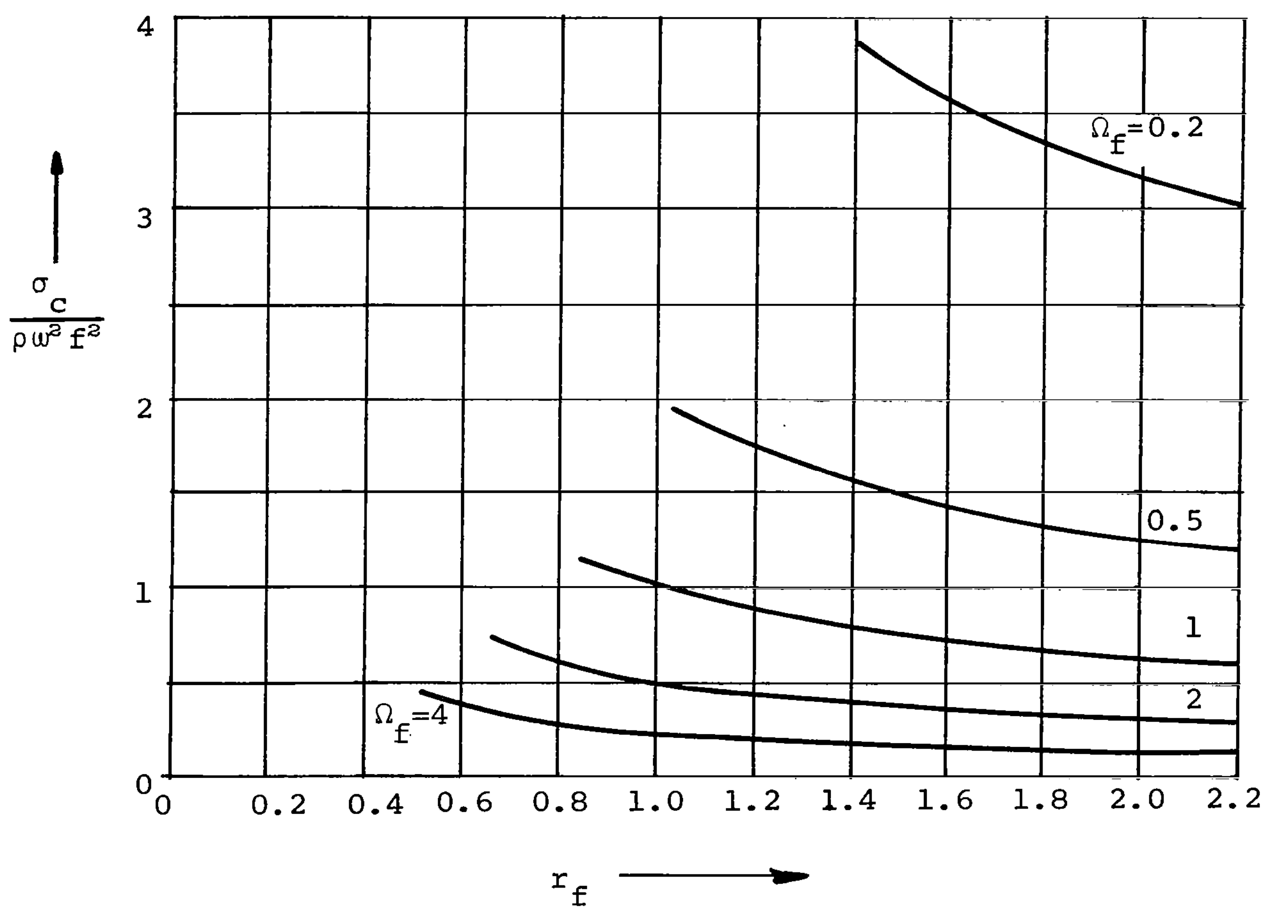

Figure 11. — Normalized Stress in Spiral Fibers

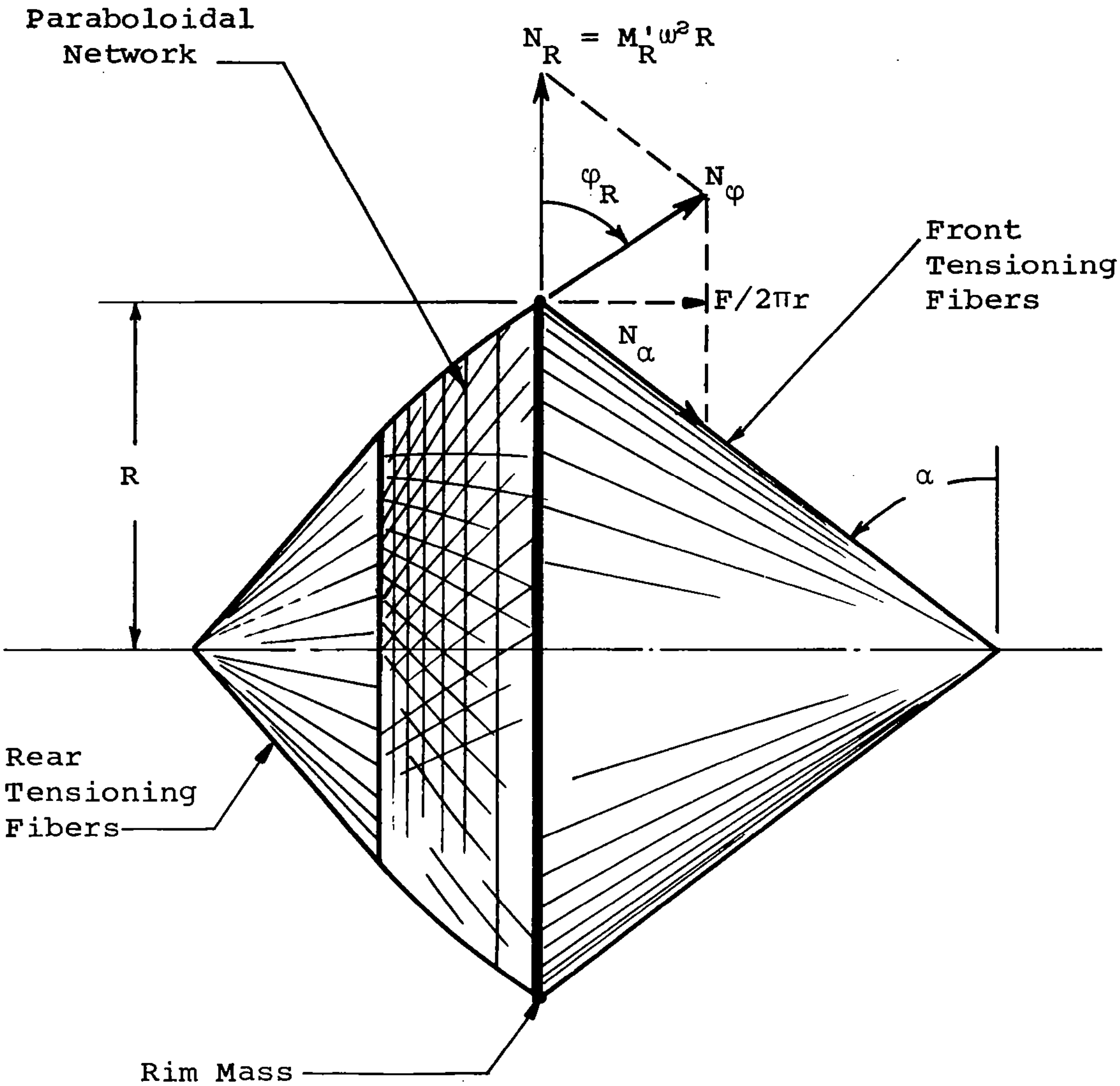

Figure 12. — Forces at Rim

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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