NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1162

Performance of Phase-Coherent Receivers Preceded by Bandpass Limiters

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September 15, 1967

JPL TECHNICAL REPORT 32-1162

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

Acknowledgment

The author wishes to express his thanks to M. Easterling of the Jet Propulsion Laboratory for his helpful comments and for the motivation and inspiration he has provided.

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Abstract

In phase-coherent communication systems, where bandpass limiters precede the RF carrier tracking loop, it is of interest to understand how the noisy RF carrier reference affects system performance. This report characterizes a model probability distribution for the RF phase error and uses this to predict the performance of phase-shift keyed and differentially coherent systems of the *Mariner* and *Pioneer* types. For these systems, two physical situations are considered: (1) system performance when the phase error is constant over the duration of one bit, and (2) system performance when the phase error is allowed to vary over the duration of one bit.

Performance of Phase-Coherent Receivers Preceded By Bandpass Limiters

I. Introduction

Recently, attempts have been made to understand how a bandpass limiter affects the performance of one-way locked, coherent (phase-shift keyed, PSK) and differentially coherent (DPSK) data demodulators. Such demodulators are typical of transmission-data detection systems used in the *Mariner* and *Pioneer* Projects. The objective of this report is to develop mathematical models for overall system performance as a function of well-defined system parameters. These parameters are defined such that measurements taken from various passes of the spacecraft may be used to evaluate and predict system performance at various times after launch. The results are also useful in predicting system performance prior to launch, and in evaluating the performance of a particular laboratory simulation. To avoid 3-dB discrepancies in practice versus theory, it is necessary that the parameters in the test setup (or spacecraft-to-DSIF link) be compatible, in definition, with those that follow. This frequent error is the prime motivation behind this report.

II. System Model

Briefly, the transmitter emits a low-index phase-modulated wave such that, out of the total radiated power of P watts, P_c watts remain in the carrier component for purposes of tracking, and S watts are allocated to the data signal. Therefore, the total transmitter power is:

$$P = P_c + S + P_l \tag{1}$$

where P_l is any losses which may occur because of the modulation process.

If the signaling states are assumed to occur with equal probability and the data signals are negatively correlated and contain equal energies, the conditional bit error probability, conditioned upon a fixed RF carrier loop phase error, ϕ, of a PSK system may be shown (Ref. 1) to be given by

$$P_E(\phi) = \frac{1}{(2\pi)^{1/2}} \int_{(2R)^{1/2}\cos\phi}^{\infty} \exp[-x^2/2]\, dx \qquad \text{(PSK)} \tag{2}$$

where

$$\begin{aligned}
R &= ST_b/N_0 \\
S &= \text{signal power} \\
N_0 &= kT^\circ = \text{single-sided spectral density of the channel noise} \\
k &= \text{Boltzmann's constant} \\
T^\circ &= \text{system temperature in degrees Kelvin} \\
T_b &= \text{duration of the data bits}
\end{aligned} \tag{3}$$

The average bit error probability is obtained easily by averaging over the probability distribution $p(\phi)$ of the phase error. This distribution has been characterized in Ref. 2 and will be defined in a later section of this report.

In the case of detecting DPSK signals, the conditional bit error probability is easily shown (Refs. 3 and 4) to be given by

$$P_E(\phi) = \frac{1}{2} \exp[-R\cos^2\phi] \qquad \text{(DPSK)} \tag{4}$$

where the parameter R is defined in Eq. (3).

The average bit error probability becomes

$$P_E = \int_{-\pi}^{\pi} p(\phi)\, P_E(\phi)\, d\phi \tag{5}$$

where the substitution of Eq. (3) for $P_E(\phi)$ into Eq. (5) yields the average performance of a PSK system, while the substitution of Eq. (4) into Eq. (5) yields the average performance of a DPSK system.

III. Probability Distribution Model for $p(\phi)$

To characterize the distribution $p(\phi)$ requires considerable elaboration (beyond the scope of this report) on the response (signal plus noise) of a phase-locked loop preceded by a bandpass limiter. However, the distribution may be modeled on the basis of experimental and theoretical evidence given in Refs. 2, 5, 6, and 7. From these references, the distribution for $p(\phi)$ is approximated by

$$p(\phi) = \frac{\exp[\rho_L \cos\phi]}{2\pi I_0(\rho_L)} \qquad |\phi| < \pi \tag{6}$$

where

$$\rho_L = \frac{2P_c}{N_0 w_{L0}} \cdot \frac{1}{\Gamma}\left(\frac{1+r_0}{1+r_0/\mu}\right) \tag{7}$$

and the parameters w_{L0}, r_0, and μ are defined from the closed loop transfer function $H(s)$ of the carrier tracking loop,

$$H(s) = \frac{1 + \left(\dfrac{r_0+1}{2w_{L0}}\right)s}{1 + \left(\dfrac{r_0+1}{2w_{L0}}\right)s + \dfrac{\mu}{r_0}\left(\dfrac{r_0+1}{2w_{L0}}\right)^2 s^2} \tag{8}$$

Here, μ is taken to be the ratio of the limiter suppression factor α_0 at the loop's design point (threshold) to the limiter suppression, say α, at any other point, i.e., $\mu = \alpha_0/\alpha$. This assumes that the filter in the carrier tracking loop is of the form

$$F(s) = \frac{1+\tau_1 s}{1+\tau_2 s} \tag{9}$$

in which case

$$r_0 = \alpha_0 K \tau_2^2/\tau_1 \tag{10}$$

and K is the equivalent simple-loop gain (Ref. 5). The subscripts 0 refer to the values of the parameters at the loop design point. The parameter w_{L0} is defined by

$$w_{L0} = \frac{1+r_0}{2\tau_2(1+\tau_2/r_0\tau_1)} \tag{11}$$

The loop bandwidths are conveniently defined by w_L and b_L through the relationship

$$w_L = 2b_L = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} |H(s)|^2\, ds \tag{12}$$

Substitution of Eq. (8) into Eq. (12) yields

$$w_L = w_{L0}\left[\frac{1+r_0/\mu}{1+r_0}\right] = 2b_L \tag{13}$$

The relation $w_{L0} = 2b_{L0}$ may be defined in a similar way in which Eq. (13) becomes

$$2b_L = (2b_{L0})\left[\frac{1 + r_0/\mu}{1 + r_0}\right] \tag{14}$$

This is the usual definition of loop bandwidth. Lower case letters denote these bandwidths so as to conform to the standard set by Tausworthe (Ref. 5). The factor Γ is approximated (Ref. 5) by

$$\Gamma = \frac{1 + 0.345\,\rho_H}{0.862 + 0.690\,\rho_H} \tag{15}$$

where ρ_H is the signal-to-noise ratio at the output of the receivers IF amplifier

$$\rho_H = \frac{2P_c}{N_0 w_H} \tag{16}$$

The parameter w_H is the two-sided bandwidth of the second IF amplifier in the double-heterodyne receiver. In one-sided bandwidth notation, $w_H = 2b_H$ and

$$\rho_H = \frac{P_c}{N_0 b_H} \tag{17}$$

The parameter ρ_H is also the signal-to-noise ratio at the input to the bandpass limiter.

The remaining parameter to define is the factor $\mu = \alpha_0/\alpha$. It may be shown that limiter suppression α is given by

$$\alpha = \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\rho_H}{2}\right)^{1/2}\exp\left[-\frac{\rho_H}{2}\right]\left\{I_0\left(\frac{\rho_H}{2}\right) + I_1\left(\frac{\rho_H}{2}\right)\right\} \tag{18}$$

where $I_k(z)$, $k = 1, 2$, is the modified Bessel function of argument z and order. To specify α_0, the parameter ρ_H is rewritten as follows:

$$\rho_H = \frac{P_c}{N_0 b_H}\cdot\frac{b_{L0}}{b_{L0}} = \frac{P_c}{N_0 b_{L0}}\cdot\frac{b_{L0}}{b_H} = zy \tag{19}$$

where

$$z = \frac{P_c}{N_0 b_{L0}} \quad \text{and} \quad y = \frac{b_{L0}}{b_H} \tag{20}$$

In practice, the parameters of the carrier tracking loop are specified at the loop design-point or threshold. If the design point is defined as $z_0 = \Upsilon_0 =$ constant, then the parameter α_0 is given by

$$\alpha_0 = \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\Upsilon_0 y}{2}\right)^{1/2}\exp\left[-\frac{\Upsilon_0 y}{2}\right]\left\{I_0\left(\frac{\Upsilon_0 y}{2}\right) + I_1\left(\frac{\Upsilon_0 y}{2}\right)\right\} \tag{21}$$

Therefore, it is clear, from Eq. (21), that system performance, P_E, depends upon the choice of Υ_0. In the Deep Space Network (DSN) this choice is usually $\Upsilon_0 = 2$ so that

$$z_0 = \frac{P_{c0}}{(kT^\circ)(b_{L0})} = 2 \tag{22}$$

or, equivalently,

$$\frac{P_{c0}}{(kT^\circ)(2b_{L0})} = 1$$

at the design point. The next section presents the dependence of P_E upon z, and R for the case where $\Upsilon_0 = 1$ and $\Upsilon_0 = 2$, $r_0 = 2$, and $y = 1/400$.

The variance of the distribution $p(\phi)$ is given by

$$\sigma_\phi^2 = \int_{-\pi}^{\pi} \phi^2 p(\phi)\, d\phi \tag{23}$$

Substituting Eq. (6) into Eq. (23) and carrying out the integration yields

$$\sigma_\phi^2 = \frac{\pi^2}{3} + \frac{4}{I_0(\rho_L)}\sum_{k=1}^{\infty}\frac{(-1)^k I_k(\rho_L)}{k^2} \tag{24}$$

where the functions $I_k(u)$ are imaginary Bessel functions. The variance of the phase error is plotted in Fig. 1 for various values of z with $r_0 = 2$, $\Upsilon_0 = 1$ and $y = 1/400$, $y = 1/60$. This variance is in close agreement with that predicted from Tausworthe's linear spectral method (Ref. 5).

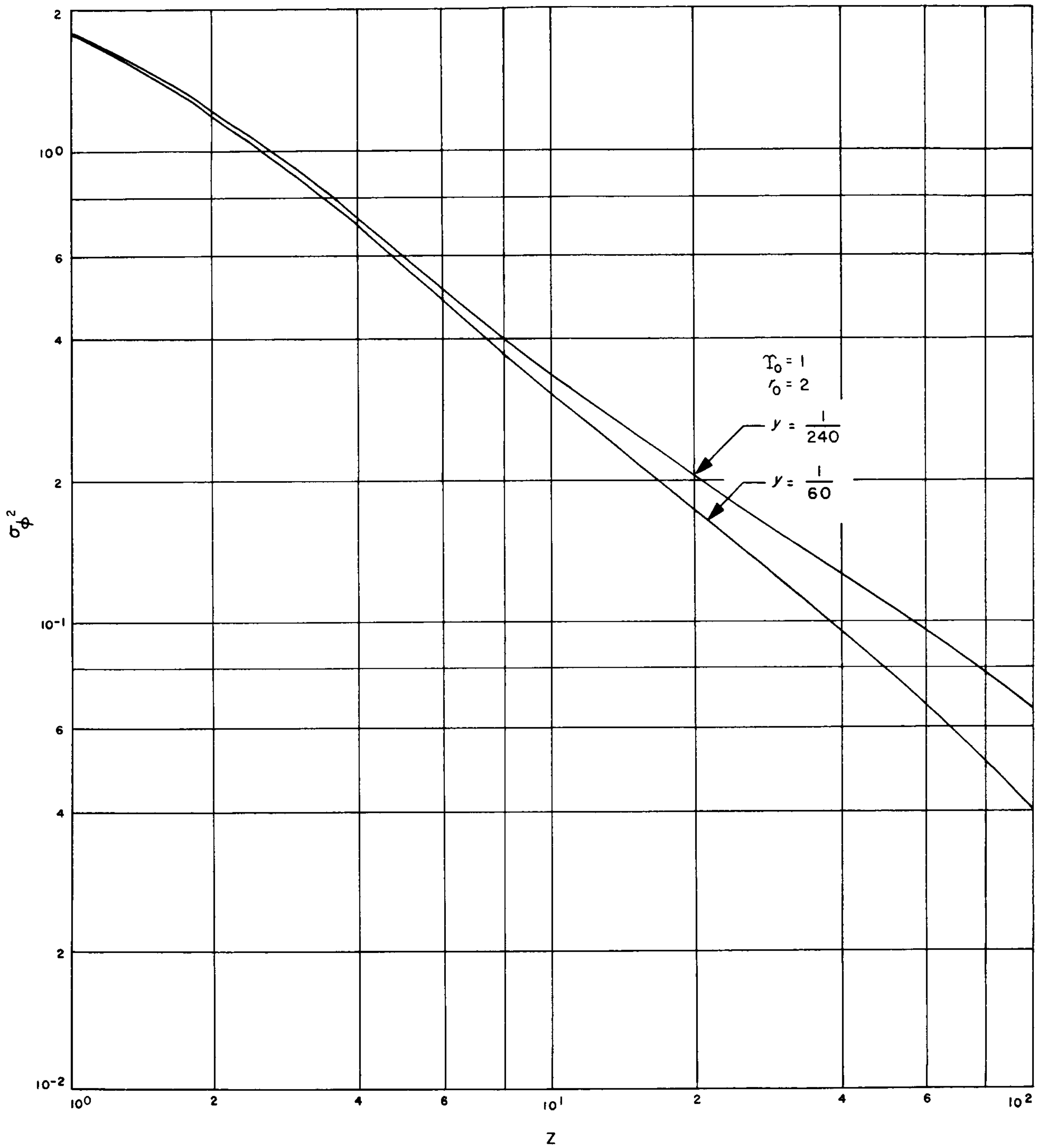

Fig. 1. Variance of the phase error versus the signal-to-noise ratio z in the carrier tracking loop. The value of *r* is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0 b_{L0}$, i.e., $T_0 = 1$.

IV. Error Rates in PSK and DPSK Detectors

If it is assumed that $r_0 = 2$ (which corresponds to a damping factor of 0.707 in the loop as determined from linear phase-locked loop theory), the error probability for PSK systems is easily determined from the material and definitions given in previous sections as follows:

$$P_E = \int_0^\pi \frac{\exp(\rho_L \cos\phi)}{\pi I_0(\rho_L)} Erfc\,[(2R)^{1/2}\cos\phi]\,d\phi \tag{25}$$

where

$$Erfc(\beta) = \int_\beta^\infty \frac{1}{(2\pi)^{1/2}} \exp(-z^2/2)\,dz \tag{26}$$

$$\rho_L = \frac{3z}{\Gamma(1+2/\mu)}, \qquad R = \frac{ST_b}{N_0} \tag{27}$$

$$\Gamma = \frac{1 + 0.345\,zy}{0.862 + 0.690\,zy} \tag{28}$$

$$z = \frac{P_c}{N_0 b_{L0}}, \qquad y = \frac{b_{L0}}{b_H} \tag{29}$$

$$\mu = \frac{(\Upsilon_0)^{1/2} \exp\left(-\frac{\Upsilon_0 y}{2}\right)\left[I_0\left(\frac{\Upsilon_0 y}{2}\right) + I_1\left(\frac{\Upsilon_0 y}{2}\right)\right]}{(z)^{1/2}\exp\left(-\frac{zy}{2}\right)\left[I_0\left(\frac{zy}{2}\right) + I_1\left(\frac{zy}{2}\right)\right]} \tag{30}$$

$$\Upsilon_0 = \frac{P_{c0}}{N_0 b_{L0}} \tag{31}$$

The integration in Eq. (25) may be carried out (Ref. 1); however, the resulting infinite series of Bessel functions does not shed appreciable light upon its behavior as a function of x and R. To illustrate this behavior, Eq. (25) has been evaluated numerically on a digital computer for various values of the parameters R, $x = z/2$ for $\Upsilon_0 = 1$ and $\Upsilon_0 = 2$ with $r_0 = 2$, $y = 1/400$. These results are given in Figs. 2 and 3.

Similarly, the performance of DPSK systems is given by

$$P_E = \int_0^\pi \frac{\exp(\rho_L \cos\phi - R\cos^2\phi)}{2\pi I_0(\rho_L)}\,d\phi \tag{32}$$

where ρ_L is defined in Eq. (27). Again, the integration may be carried out; however, computer results are best obtained by numerical integration of Eq. (32). Results of these computations are illustrated in Figs. 4 and 5 with $\Upsilon_0 = 1$, $\Upsilon_0 = 2$, $r_0 = 2$, and $y = 1/400$ for various values of $R = ST_b/N_0$ and $x = P_c/(kT^\circ)(2b_{L0})$.

V. Error Rates in *Pioneer*-Type Data Detectors

The conditional error probability in the *Pioneer* system mechanization is such that (Ref. 8) the total error probability is given by

$$P_{E_t} = 2\int_{-\pi}^{\pi} p(\phi)\,P_E(\phi)\,[1 - P_E(\phi)]\,d\phi \tag{33}$$

where $P_E(\phi)$ is given by Eq. (2). Performing the integration numerically on the IBM 7094 computer yields the results shown in Fig. 6 for $\Upsilon_0 = 2$, $y = 1/400$ and various values of x, and $R = ST_b/N_0$. The results shown in Fig. 6 can be compared with those given in Fig. 2 for a *Mariner*-type system. Figures 2 and 6 illustrate that the performance of a DPSK system (Fig. 4) is superior to either the *Mariner* or *Pioneer*-type system. This is because a DPSK system makes correct decisions if the frequency of the VCO remains at the frequency of the incoming RF carrier, i.e., phase lock is not as significant in a system employing differentially coherent detection.

VI. System Performance When the Phase Error is Not Constant Over the Duration of the Signal

The previous sections presented the performance of various communications systems where the phase error is constant over the duration of the modulation, i.e., T_b seconds. In certain situations, e.g., command systems and low-rate telemetry systems, the assumption that the phase error of the system remains constant for T_b seconds becomes suspect and it is therefore of interest to understand how system performance changes. The basic system parameter, which is a measure of how stable the phase error is over the signal duration, is the ratio of the data rate to the bandwidth of the carrier tracking loop (Ref. 1). In Ref. 1, it is shown that the decision variables for a correlation receiver are given by

$$q_k = \int_0^T \cos\phi(t)\,x_k(t)\,[x_2(t) - x_1(t)]\,dt + \int_0^{T_b} n'(t)\,dt \qquad k = 1, 2 \tag{34}$$

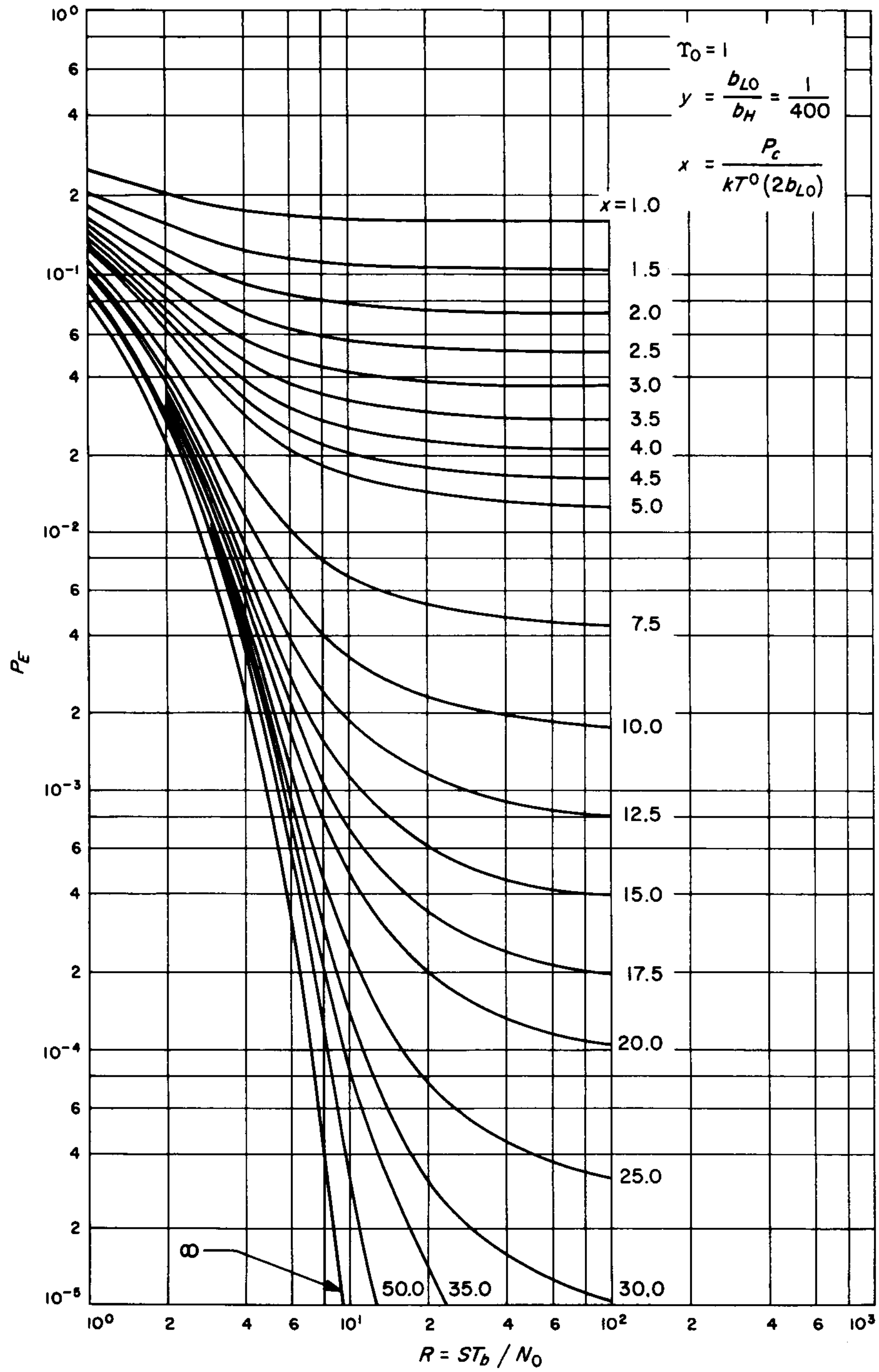

Fig. 2. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0\,(2b_{L0})$, i.e., $\Upsilon_0 = 2$. PSK signaling is assumed.

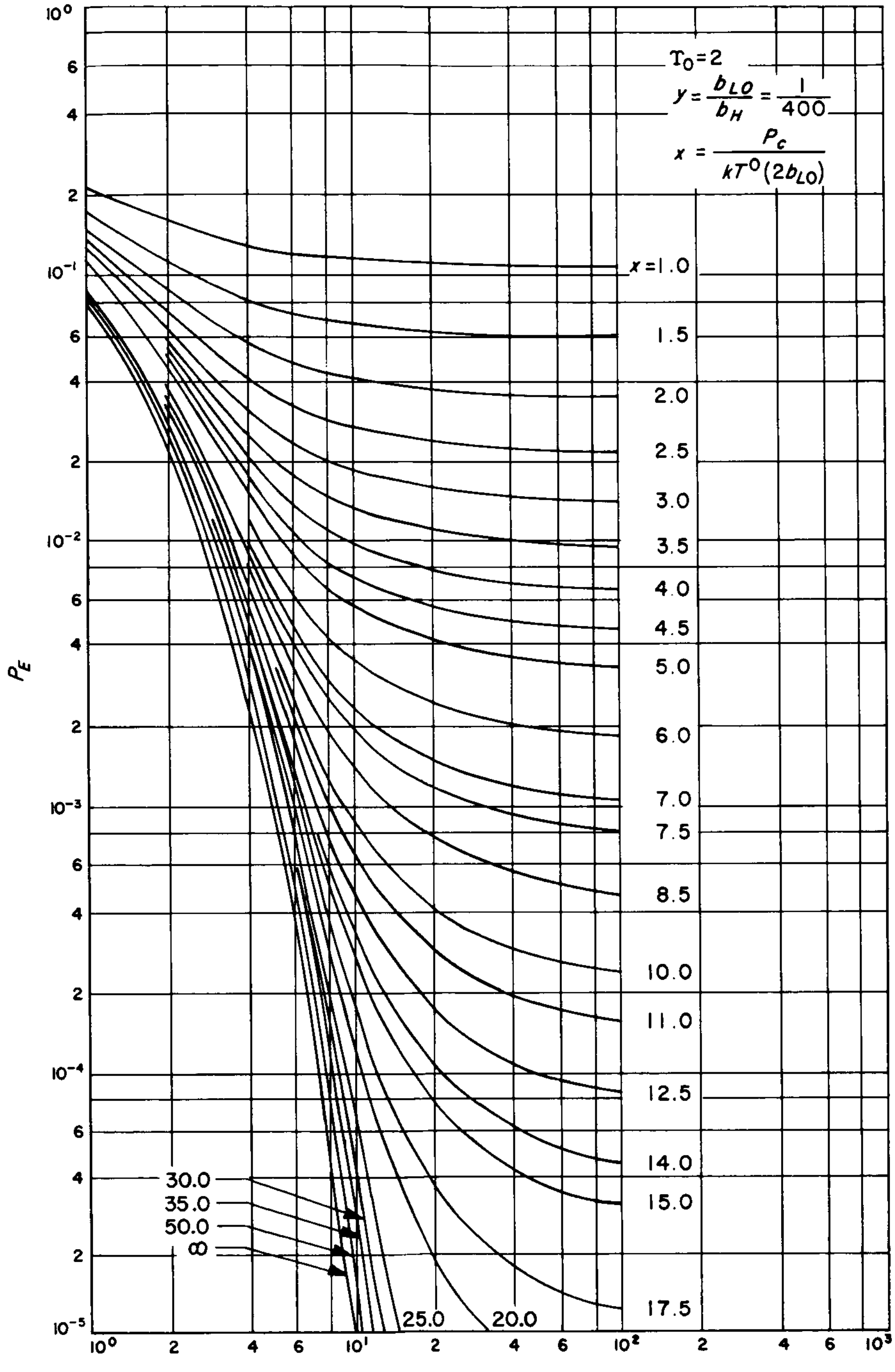

Fig. 3. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level of $P_{c0} = N_0 b_{L0}$, i.e., $\Upsilon_0 = 1$. PSK signaling is assumed.

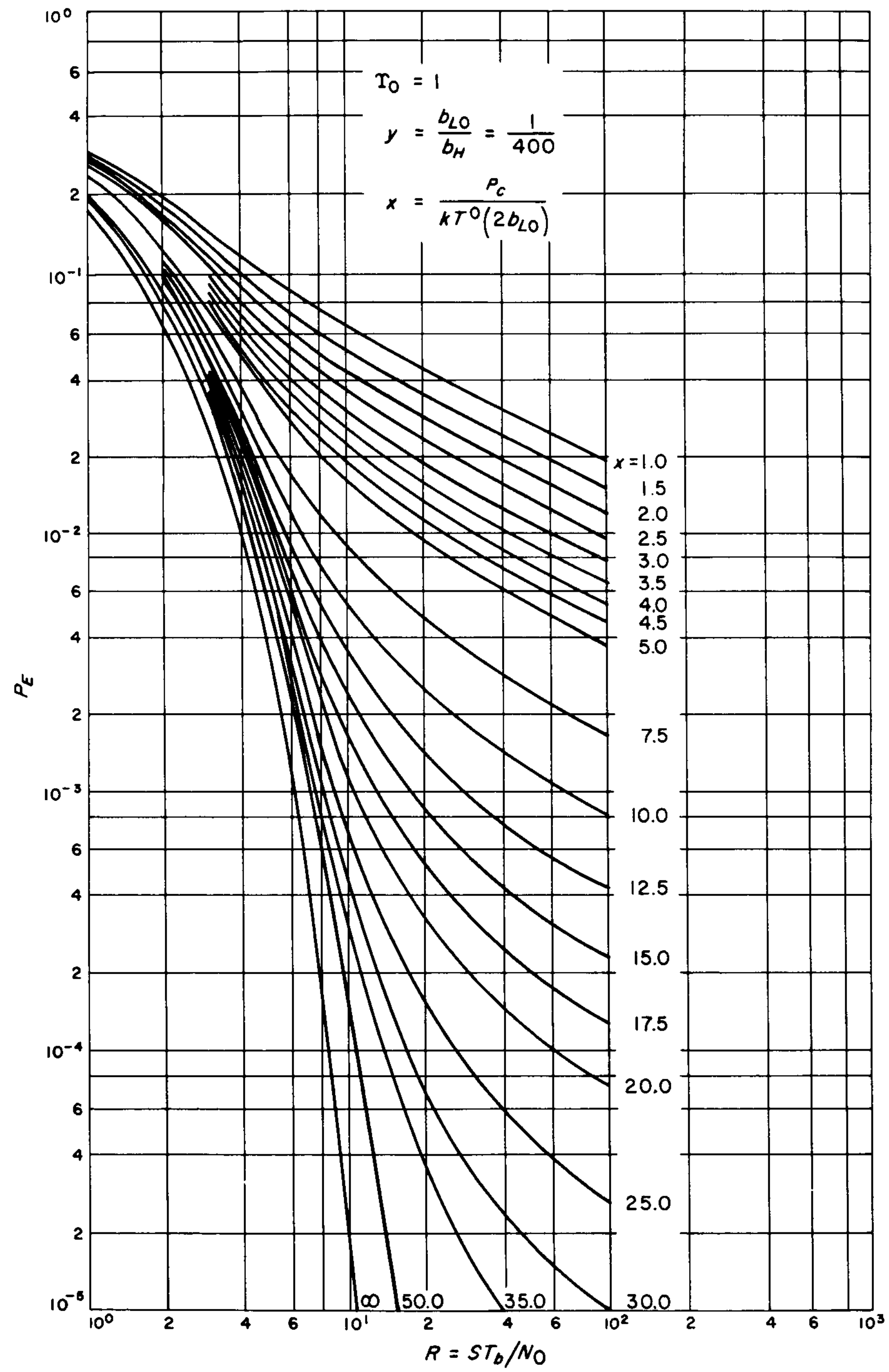

Fig. 4. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level of $P_{c0} = N_0\,(2b_{L0})$, i.e., $\Upsilon_0 = 2$. DPSK signaling is assumed.

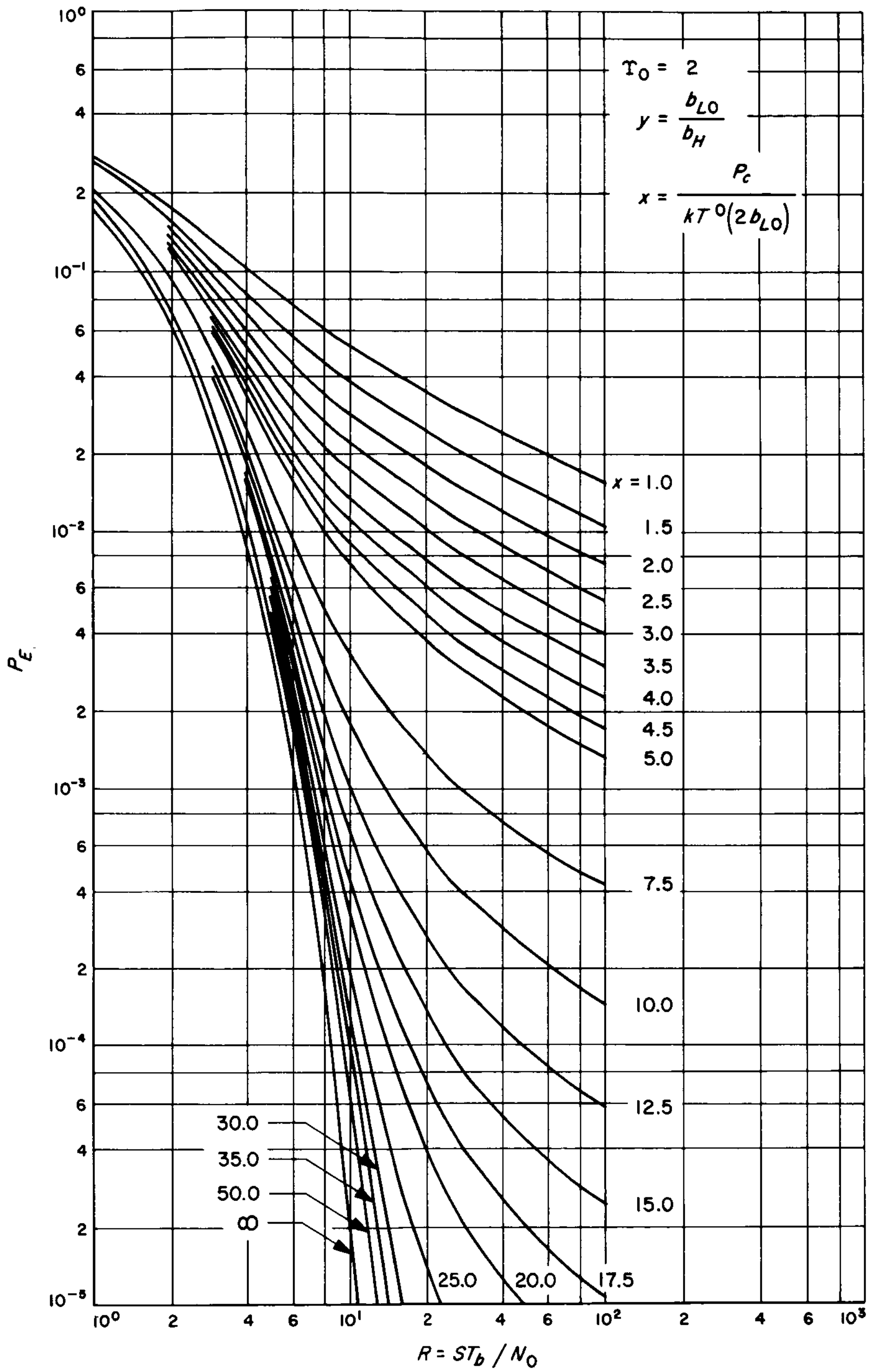

Fig. 5. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level of $P_{c0} = N_0 b_{L0}$, i.e., $\Upsilon_0 = 1$. DPSK signaling is assumed.

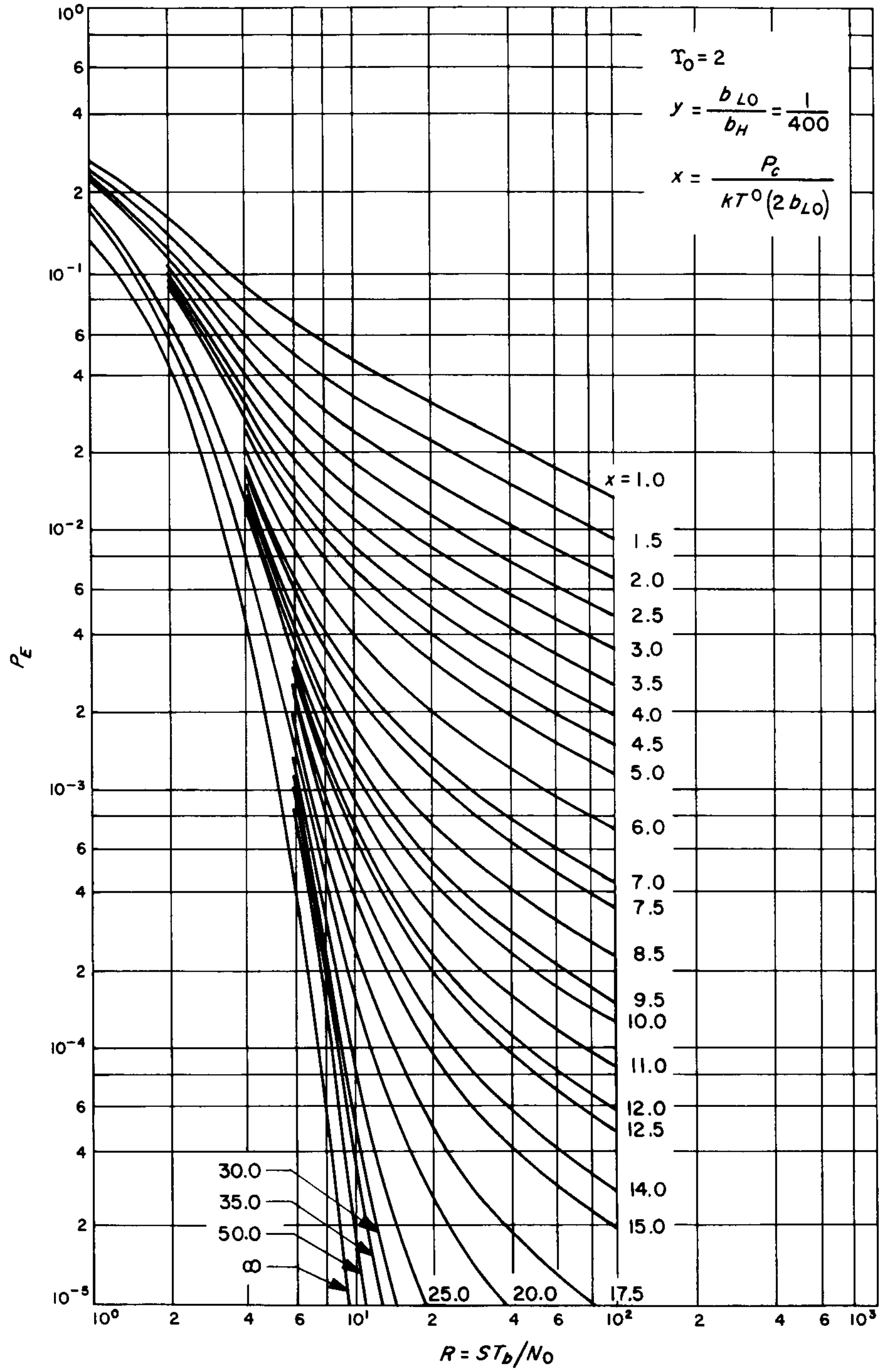

Fig. 6. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0\,(2b_{L0})$, i.e., $\Upsilon_0 = 2$. Pioneer-type system. Phase error is assumed constant over the duration of one bit.

where $x_1(t)$ and $x_2(t)$ are the transmitted signals and $n'(t)$ is additive white Gaussian noise with single-sided spectral density of N_0 watts/cycle. To determine the error probability, the probability density of $p(q_k)$ must be determined and, from these density functions, the probability that q_k is less than zero must be computed, given that $x_2(t)$ was transmitted and vice versa. In general this is a complex problem; however, the best upper bound can be obtained by computing the mean and variance of q_k on the basis that q_k is Gaussian. Such a procedure was used by this author in determining the performance of linear analog demodulators (Ref. 9). By using the procedure outlined in Ref. 9, it can be shown that the mean of q is given by

$$q_2 = 2E\,(\overline{\cos\phi}) \tag{35}$$

and, from Eq. (6),

$$\overline{\cos\phi} = \frac{I_1(\rho_L)}{I_0(\rho_L)} = \eta \tag{36}$$

This function has been studied in detail in Ref. 11. The variance of q_2 is given by

$$\sigma^2_{q_2} = 2ST_bN_0 \tag{37}$$

Similarly, the mean of q_1 becomes

$$q_1 = -2ST_b\,(\overline{\cos\phi})$$

and the variance of q_1 is

$$\sigma^2_{q_1} = 2ST_bN_0 = \sigma^2_{q_2} \tag{38}$$

If the statistics of $p(q_k)$ are characterized as Gaussian and equal probable signals are assumed, then the error probability for PSK systems is given by

$$P_E = \frac{1}{(2\pi)^{1/2}} \int_{(2\eta^2 R)^{1/2}}^{\infty} \exp(-y^2/2)\,dy \qquad \text{(PSK)} \tag{39}$$

This result agrees with that obtained by F. Reed (Ref. 10) and presented in unpublished work performed for Motorola under Contract 951700. For differentially coherent detection,

$$P_E = \frac{1}{2}\exp[-\eta^2 R] \qquad \text{(DPSK)} \tag{40}$$

and the error rate for a *Pioneer*-type system becomes

$$P_{E_t} = 2P_E(1 - P_E) \tag{41}$$

Equation (39) is plotted in Fig. 7 for various values of x, R with $y = 1/400$, and $\Upsilon_0 = 2$. Figure 8 illustrates a similar plot for DPSK systems while Fig. 9 is a plot of Eq. (41). Therefore, any practical system has upper and lower bounds on the error probability. The upper bound is for use in systems where the ratio of the data rate to the carrier tracking loop bandwidth is large, i.e., the phase error is essentially constant over the duration of the signal. The lower bound is to be used in predicting the performance of systems where the ratio of the data rate to the carrier tracking loop bandwidth is small, i.e., the phase error varies over the duration of the signal. For intermediate values of this ratio, system performance lies between these two curves.

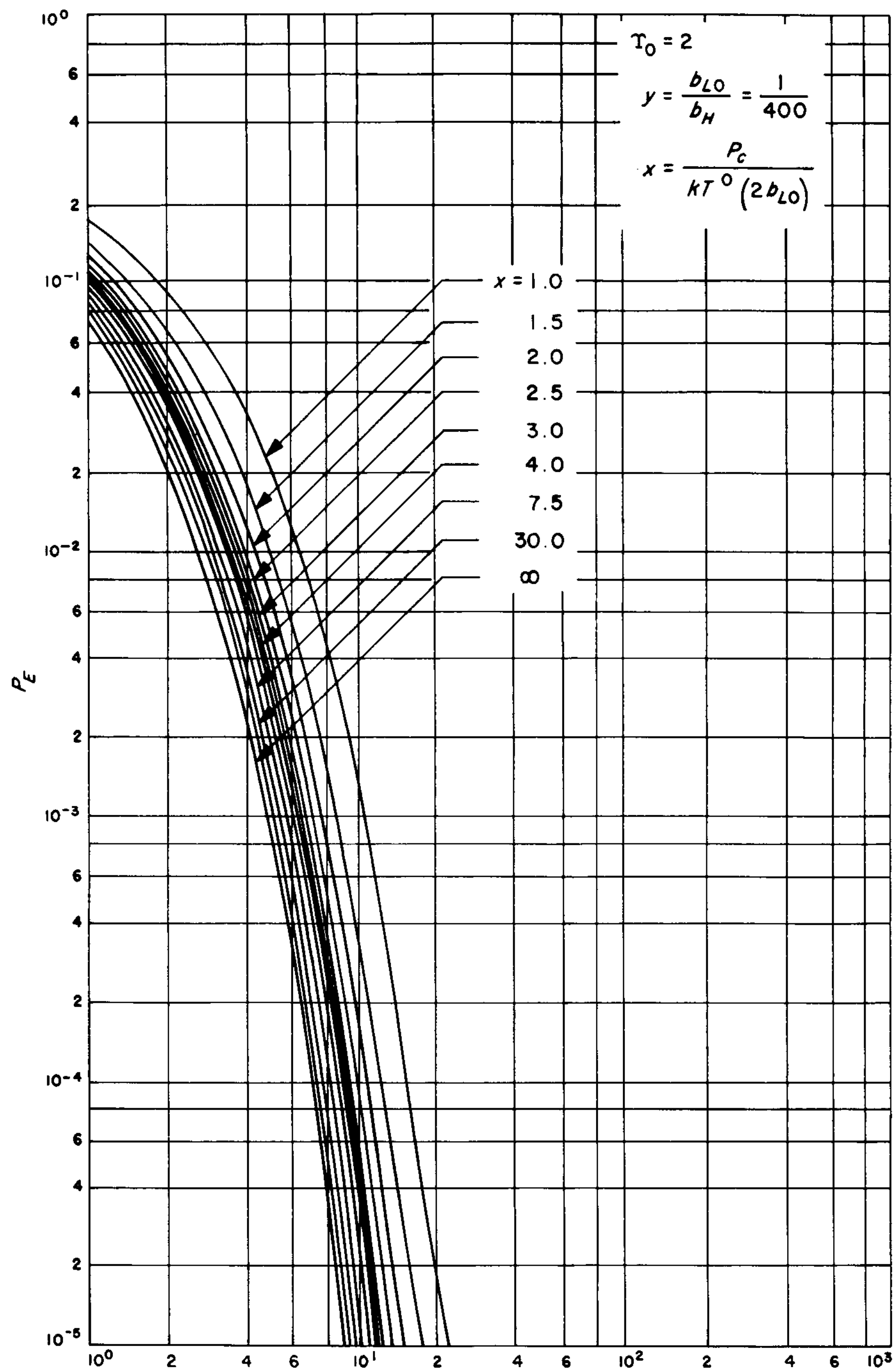

Fig. 7. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0\,(2b_{L0})$, i.e., $\Upsilon_0 = 2$. Phase-error is assumed to vary during the transmission of one bit. PSK signaling is assumed.

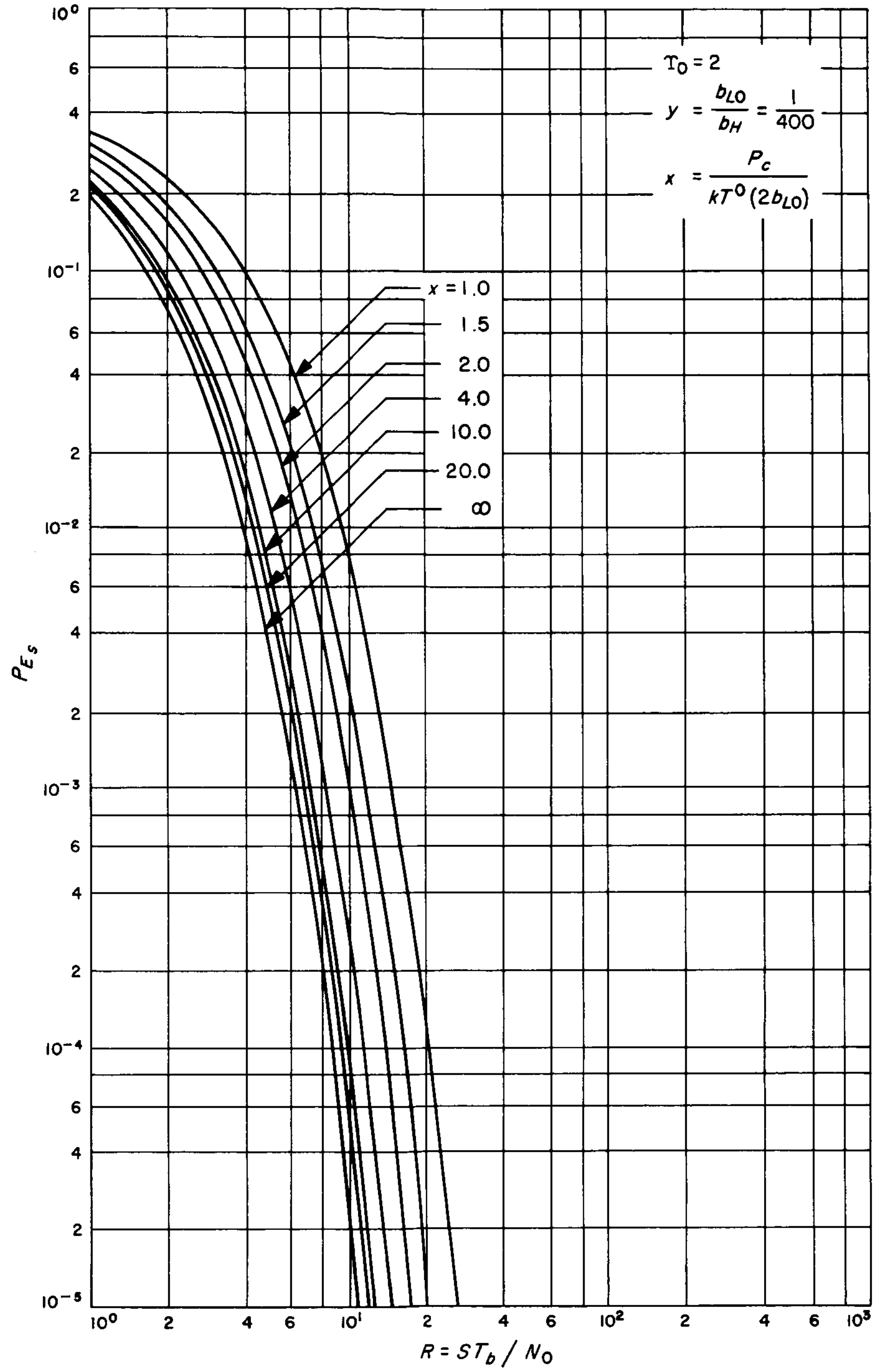

Fig. 8. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0\,(2b_{L0})$, i.e., $\Upsilon_0 = 2$. Phase error is assumed to vary during the transmission of one bit. DPSK signaling is assumed.

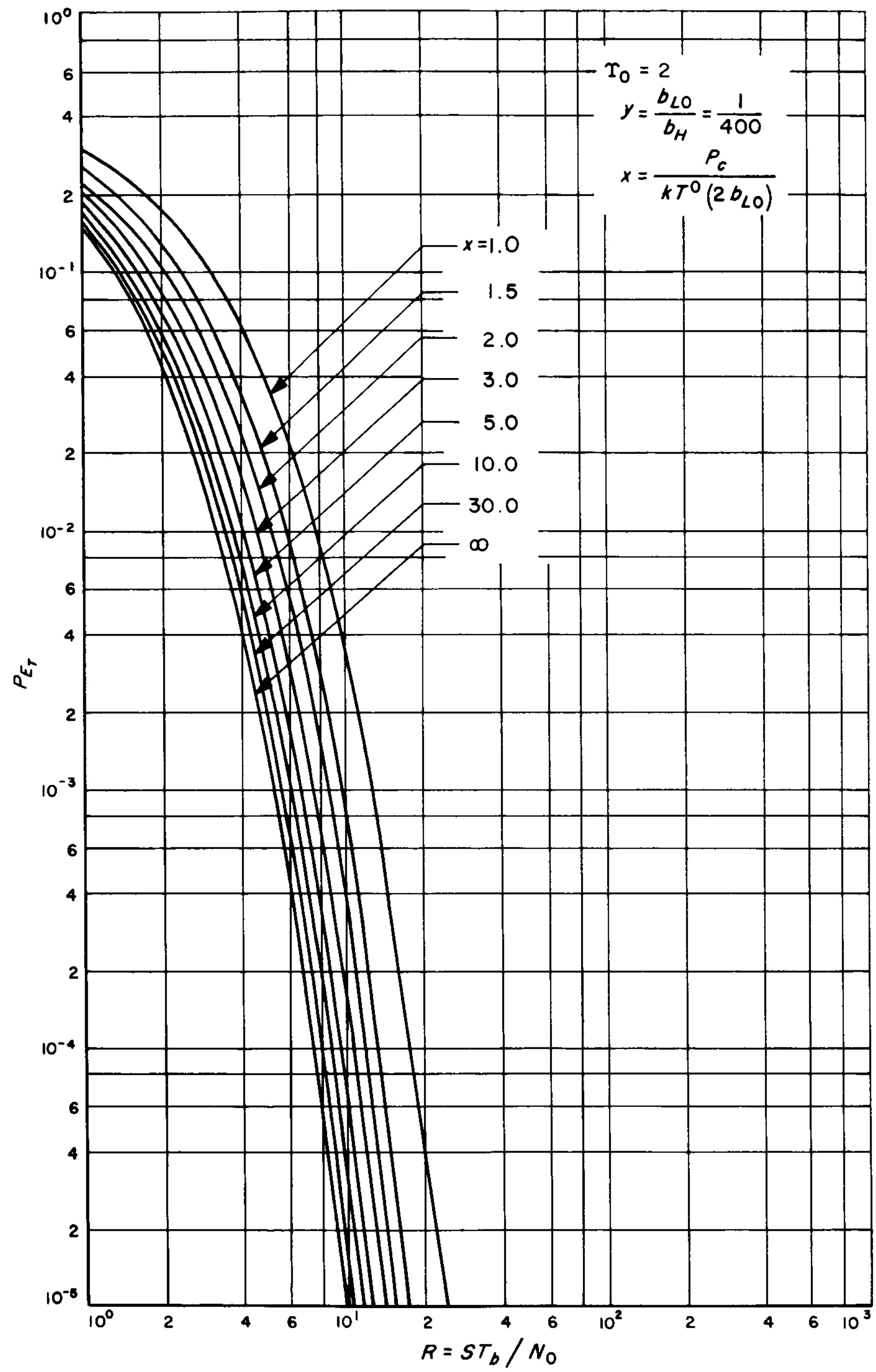

Fig. 9. Error probability versus signal-to-noise ratio in the data ST_b/N_0. The value of r is taken as $r_0 = 2$ at a design point signal level $P_{c0} = N_0(2b_{L0})$, i.e., $\Upsilon_0 = 2$. *Pioneer*-type system is assumed. The phase error is assumed to vary during the transmission of one bit.

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