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6 Allan M. Krall 9
Assistant Professor of Mathematics
The Pennsylvania State University
University Park, Pennsylvania 3

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Stability Techniques

for

Continuous Linear Systems

Allan M. Krall
Assistant Professor of Mathematics
The Pennsylvania State University
University Park, Pennsylvania

Preface

This collection of stability techniques was begun in 1959 to provide a masters thesis for the author at the University of Virginia. It was almost immediately evident that at that time such a collection was impossible. Some significant gaps in the theory were still unfilled. Access to Russian techniques had not yet been provided. In the intervening years these obstacles have been eliminated.

The theory of functions of a complex variable provides most of the necessary background for understanding these techniques, and so a brief discussion is provided in Chapter I. This chapter only discusses those aspects of the subject needed in subsequent chapters, and is not intended to be a thorough treatment. However, a great many theorems, which are not discussed in the text, are presented as problems at the end of the chapter.

Subsequent chapters first discuss the background necessary for the techniques and then the techniques themselves. The Laplace transform is used for convenience, although it is not essential to the subject.

This material has been presented as a course of one semester length at the Pennsylvania State University. It was not necessary to provide many problems of a numerical nature, since the students had no trouble at all providing their own. It is for this reason the problems at the end of each chapter are for the most part theoretical.

We finally remark that sampled data systems are not included, since many excellent books on that subject are now appearing, which provide a much more thorough discussion than would be possible here.

I would like to express many thanks to Mr. Robert Fornaro, who wrote the computer program mentioned in Appendix I; to Mr. Gerald Kraus and Mr. Robert

McNeil, who helped with the examples in Appendix II and with the proof-reading, and to Mrs. Robert D. Moyer, who did a beautiful job of typing the manuscript. Finally, my thanks go to the National Aeronautics and Space Administration who gave me support through NASA Grant NGR-39-009-041.— 29ACY

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Allan M. Krall

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I. Functions of a Complex Variable

Preliminary Remarks: We assume that the reader is familiar with the following basic properties of the complex numbers and complex functions. If $z = x + iy$ with $x = r \cos \theta$ and $y = r \sin \theta$, then $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$. Further, $r = |z|$ and $\theta = \arg z$.

Similarly if

$$f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$$

where $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ are real valued, then

$$|f(z)| = [(\operatorname{Re} f(z))^2 + (\operatorname{Im} f(z))^2]^{\frac{1}{2}} ,$$

$$\arg f(z) = \tan^{-1} [\operatorname{Im} f(z)/\operatorname{Re} f(z)]$$

and

$$f(z) = |f(z)| \, e^{i \arg f(z)} .$$

If

$$w = z_1 \cdot z_2 \cdot z_3 \ldots z_n = \prod_{j=1}^{n} z_j$$

and

$$z_j = r_j e^{i\theta_j} ,$$

then

$$w = \prod_{j=1}^{n} r_j \cdot \exp\left[i \sum_{j=1}^{n} \theta_j\right] .$$

Thus

$$\arg w = \sum_{j=1}^{n} \theta_j$$

and

$$|w| = \prod_{j=1}^{n} r_j .$$

Finally

$$\log w = \sum_{j=1}^{n} \log r_j + i \sum_{j=1}^{n} \theta_j ,$$

so

$$\log w = \log |w| + i \arg w .$$

Definition: Let $z = x + iy$ where x and y are real valued. Let D be a set in the complex plane K. If for all z in D, $f(z)$ is uniquely defined and in K, then $f(z)$ is a function of the complex variable z whose domain is D.

In what follows we will assume that D is a simply connected open set in K.

Definition: $f(z)$ is continuous at z_0 in D if for every $\epsilon > 0$, there is a $\delta(= \delta(\epsilon, z_0))$ such that if z is in D and

$$|z - z_0| < \delta ,$$

then

$$|f(z) - f(z_0)| < \epsilon .$$

Another way of saying this is that

$$\lim_{z \to z_0} f(z) = f(z_0) .$$

Note that z can approach z_0 in any manner whatsoever.

Definition: $f(z)$ is differentiable at z_0 if there is a number $f'(z_0)$ such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) .$$

Again note that the path z takes in approaching z_o is arbitrary. A function differentiable in this sense is analytic.

Definition: The real part of $f(z)$, $u(x,y)$, is

$$u(x,y) = [f(z) + \overline{f(z)}]/2 \quad .$$

The imaginary part of $f(z)$, $v(x,y)$, is

$$v(x,y) = [f(z) - \overline{f(z)}]/2i \quad .$$

Note that

$$f(z) = u(x,y) + iv(x,y) \quad .$$

Theorem 1.1: If $f(z)$ is analytic at z_0, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; , \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

at $x = x_0$, $y = y_0$.

These equations are the Cauchy-Riemann equations.

Proof: If we compute the derivative at z_0, we find that

$$f'(z_0) = \lim_{x \to x_0} \left[\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right] \; ,$$

$$f'(z_0) = \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right]_{x = x_0, \; y = y_0}$$

if z is restricted to the values $z = x + iy_0$.

$$f'(z_0) = \lim_{y \to y_0} \left[\frac{u(x_0,y) - u(x,y_0)}{iy - iy_0} + i\,\frac{v(x_0,y) - v(x_0,y_0)}{iy - iy_0}\right] ,$$

$$f'(z_0) = \left[- i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right]_{x = x_0,\ y = y_0} ,$$

if z is restricted to values $z = x_0 + iy$. Comparing these expressions completes the proof.

Theorem 1.2. If the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist in a region D, are continuous in D and satisfy the Cauchy-Riemann equations, then $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D.

Proof:

$$\begin{aligned} f(z) - f(z_0) &= [u(x,y) - u(x_0,y_0)] + i[v(x,y) - v(x_0,y_0)] \\ &= [u(x,y) - u(x,y_0) + u(x,y_0) - u(x_0,y_0)] \\ &\quad + i[v(x,y) - v(x,y_0) + v(x,y_0) - v(x_0,y_0)] . \end{aligned}$$

Since the partial derivatives are continuous, there exist α, β, γ, δ, functions of x and y, whose limits as $x \to x_0$, $y \to y_0$ are 0, such that

$$\begin{aligned} f(z) - f(z_0) &= (\frac{\partial u}{\partial y} + \alpha)(y - y_0) + (\frac{\partial u}{\partial x} + \beta)(x - x_0) \\ &\quad + i(\frac{\partial v}{\partial y} + \gamma)(y - y_0) + i(\frac{\partial v}{\partial x} + \delta)(x - x_0) \end{aligned}$$

where the partial derivatives are evaluated at (x_0,y_0). Substituting the Cauchy-Riemann equations, we find

$$f(z) - f(z_0) = \frac{\partial u}{\partial x}(z - z_0) + i\frac{\partial v}{\partial x}(z - z_0)$$

$$+ (\alpha + i\gamma)(y - y_0) + (\beta + i\delta)(x - x_0) \quad .$$

Since

$$\left|\frac{x - x_0}{z - z_0}\right| \leq 1$$

and

$$\left|\frac{y - y_0}{z - z_0}\right| \leq 1 \quad ,$$

we see by dividing by $z - z_0$, that

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + o(1)^*$$

as $(x,y) \to (x_0,y_0)$. Thus

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and equals

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad .$$

---

*A function is $o(1)$ as $(x,y) \to (x_0,y_0)$ if the limit of that function is 0.

Definition: Let $f(z)$ be analytic in D. Let C be a contour* in D. We define

$$\int_C f(z)dz$$

as

$$\lim_{\substack{n \to \infty \\ \sup|z_i - z_{i+1}| \to 0}} \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

where $\{z_j\}$ divides C into subarcs, ξ_j is a point of C between z_j and z_{j-1}.

If

$$f(z) = u(x,y) + iv(x,y) ,$$

$$z_j = z_j + iy_j ,$$

$$\xi_j = \sigma_j + i\tau_j ,$$

then

$$\sum_{j=1}^{n} f(\xi_i)(z_i - z_{j-1}) =$$

$$\sum_{j=1}^{n} (u(\sigma_j,\tau_j) + iv(\sigma_j,\tau_j))(x_j + iy_j - x_{j-1} - iy_{j-1}) ,$$

which consists of four sums of real terms.

The usual additive and multiplicative (by constants) properties for integrals hold.

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*A contour is a continuous curve consisting of a finite number of arcs. An arc can be expressed parametrically by two differentiable functions.

EXAMPLES:

1. Let C be any path from $z = a$ to $z = b$. Let $f(z) = K$, a constant. Then

$$\sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1}) = K([z_1 - a] + [z_2 - z_1] + \cdots + [b - z_{n-1}])$$

$$= K(b - a) \quad .$$

Thus

$$\int_a^b K dz = K(b - a) \quad .$$

Note this is independent of the path C.

2. Let C be any path from $z = a$ to $z = b$. Let $f(z) = z$. Then

$$\sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

can be written two ways; first if $\xi_j = z_j$, second if $\xi_j = z_{j-1}$. We find

$$\sum_{j-1}^{n} z_j(z_j - z_{j-1}). = ([z_1^2 - z_1 a] + [z_2^2 - z_2 z_1] + \cdots + [b^2 - bz_{n-1}]) \quad ,$$

$$\sum_{j=1}^{n} z_{j-1}(z_j - z_{j-1}) = ([z_1 a - a^2] + [z_1 z_2 - z_1^2] + \cdots + [bz_{n-1} - z_{n-1}^2]) \quad .$$

One half the sum of these is $\frac{b^2 - a^2}{2}$. Since both approach $\int_a^b zdz$ along C, we find

$$\int_a^b zdz = (b^2 - a^2)/2 \quad .$$

Again the result is independent of C.

3. Let C be a circle of radius ρ.

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = 2\pi i \quad .$$

The integral is independent of ρ. The path of integration is counter-clockwise.

It can be proved in general that the integral of an analytic function always exists. (See Knopp, "Theory of Functions", Vol. 1).

Theorem 1.3: (Cauchy's Theorem). *If* $f(z)$ *is analytic in a simply connected region* D *and* C *is a simple closed contour in* D, *then*

$$\int_C f(z)dz = 0 \quad .$$

Proof: We divide the interior of C by series of squares formed by lines parallel to the real and imaginary axes. D then consists of a number of squares S_n of width ℓ_n and a number of pieces T_n formed by parts of squares and pieces of the curve C. Then

$$\int_C f(z)dz = \Sigma \int_{T_n} f(z)dz + \Sigma \int_{S_n} f(z)dz \quad ,$$

the integration over the interior parts cancelling due to the counter-clockwise path of integration.

We now assume that for any $\varepsilon > 0$, there exists a choice of S_n, T_n so that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

in each S_n, T_n. Thus

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + N(z)(z - z_0)$$

where $|N(z)| < \varepsilon$ in each S_n, T_n.

In S_n,

$$\int_{S_n} f(z)dz =$$

$$\int_{S_n} f(z_0)dz + \int_{S_n} f'(z_0)(z - z_0)dz + \int_{S_n} N(z)(z - z_0)dz .$$

$$\left|\int_{S_n} f(z)dz\right| \leq 0 + 0 + \varepsilon\sqrt{2}\, \ell_n \cdot 4\ell_n .$$

In T_n,

$$\int_{T_n} f(z)dz =$$

$$\int_{T_n} f(z_0)dz + \int_{T_n} f'(z_0)(z - z_0)dz + \int_{T_n} N(z)(z - z_0)dz .$$

$$\left|\int_{T_n} f(z)dz\right| \leq 0 + 0 + \varepsilon\sqrt{2}\, \ell_n[4\ell_n + C_n]$$

where C_n is a part of C.

Thus

$$\left|\int_{C} f(z)dz\right| \leq \varepsilon \Sigma 4\sqrt{2}\, \ell_n^{\,2} + \varepsilon \Sigma 4\sqrt{2}\, \ell_n^{\,2} + \varepsilon \Sigma \sqrt{2}\, \ell_n C_n .$$

$$\left|\int_{C} f(z)dz\right| \leq \varepsilon[A + \ell|C|]\, 4\sqrt{2}$$

where A is the area of a rectangle containing C, ℓ is the length of a side of the rectangle and $|C|$ is the length of C. Since ε is arbitrary,

$$\int_C f(z)dz = 0 .$$

Suppose now our assumption is not valid. Then there would be a region S_n or T_n where it does not hold. Call it R_1. If R_1 is subdivided in fourths, there would have to be a subregion R_2 where it does not hold, etc. This process could not terminate. Thus there exist $R_1 \supset R_2 \supset R_3 \ldots$ In the intersection of $R_1 \ldots R_n \ldots$, there is a point z_0 in common. Since $f(z)$ is analytic at z_0, for $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

when $|z - z_0| < \delta$. Surely the sup of $|z - z_0|$ in R_n is ultimately less than δ. This gives a contradiction.

<u>Theorem 1.4</u>: (<u>Cauchy's formula</u>). <u>Let</u> $f(z)$ <u>be analytic in and on a simple closed contour</u> C. <u>Let</u> z <u>be an interior point of</u> C. <u>Then</u>

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw .$$

<u>Proof</u>: Since

$$\frac{1}{2\pi i} \int_C \frac{1}{w - z} dw = 1 ,$$

$$f(z) - \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw = \frac{1}{2\pi i} \int_C \frac{f(z) - f(w)}{w - z} dw .$$

Since

$$\frac{f(z) - f(w)}{w - z}$$

is analytic except at $w = z$, we may replace C by a circle of arbitrarily small radius ρ centered at z. Since $f(w)$ is continuous, for arbitrary $\epsilon > 0$, there is a $\delta > 0$ so that if

$$|w - z| = \rho < \delta ,$$

then

$$|f(w) - f(z)| < \epsilon .$$

Thus

$$|f(z) - \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw| = |\frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z) - f(w)}{w - z}\,dw|$$

$$< \frac{1}{2\pi}\,\epsilon/\rho \cdot 2\pi\rho = \epsilon$$

and the proof is complete.

Remark: If

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw$$

is differentiable, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z)^2}\,dw .$$

This can be made rigorous.

$$\frac{f(z + h) - f(z)}{h} = \frac{1}{2\pi i}\int_C \left[\frac{1}{w - z - h} - \frac{1}{w - z}\right]\frac{f(w)}{h}\,dw$$

$$= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z - h)(w - z)}\,dw .$$

Now

$$\frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w - z - h)(w - z)} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z)^2}\,dw$$

$$= \frac{1}{2\pi i}\int_C \frac{h\,f(w)\,dw}{(w - z - h)(w - z)^2} .$$

Let $M = \max |f(w)|$ when w is on C. Let $\delta = \min |w - z|$ when w is on C. Let ℓ be the length of C. If $|h| < \delta$, let $\delta_1 = \min |w - z - h|$ when w is on C. Then

$$\frac{1}{2\pi}\,\frac{hM\ell}{\delta_1 \delta^2}$$

is an over estimate for the last integral, which approaches 0 as h approaches 0. Thus

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists and equals

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{(w - z)^2}\,dw .$$

<u>Remark</u>: This process can be continued indefinitely.

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w - z)^{n+1}}\,dw .$$

Thus an analytic function possesses derivatives of all orders.

Theorem 1.5: (Taylor's Series). If $f(z)$ is analytic in a region D and on its boundary C, and z_0 is an interior point of D, then $f(z)$ can be expanded in a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

whose radius of convergence is at least the minimum distance from z_0 to C, and whose coefficients are given by

$$a_n = \frac{1}{n!} f^{(n)}(z_0) .$$

Proof: We note that

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{w - z_0}}$$

If $|z - z_0| < |w - z_0|$ for all w (on C) then

$$\frac{1}{w - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

which converges uniformly. Since, after multiplying this equation by $f(w)$, integration is possible term by term, we find

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} dw \, (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n .$$

Theorem 1.6: Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

have radius of convergence ρ_1 and ρ_2, respectively. If $f(z)$ and $g(z)$ coincide on an infinite number of points including z_0 as a limit in a region around z_0 within both the domains of convergence $|z - z_0| < \rho_1$, $|z - z_0| < \rho_2$, then $f(z) = g(z)$.

Proof: Letting z approach z_0, we see $a_0 = b_0$. Then

$$\sum_{n=1}^{\infty} a_n(z - z_0)^{n-1} = \sum_{n=1}^{\infty} b_n(z - z_0)^{n-1}$$

on an infinite set with limit z_0. Letting $z \to z_0$ on those points shows $a_1 = b_1$. Ultimately considering

$$\sum_{n=m}^{\infty} a_n(z - z_0)^{n-m} = \sum_{n=m}^{\infty} b_n(z - z_0)^{n-m}$$

shows $a_m = b_m$. Thus $f(z) = g(z)$.

Definition: Let $f(z)$ be analytic in a region D except at a point z_0 where $f(z)$ is undefined. If there exists an integer m so that $(z - z_0)^m f(z)$ can be extended analytically but $(z - z_0)^{m-1} f(z)$ cannot, then $f(z)$ has a pole of order m at z_0.

Note

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n .$$

So

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{n=1}^{\infty} \frac{\beta_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

Definition: *If* $f(z)$ *is analytic except for poles, then* $f(z)$ *is a meromorphic function.*

Definition: *If*

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n$$

in a neighborhood of z_0, *the residue of* $f(z)$ *at* z_0 *is* a_{-1}.

Theorem 1.7: *(Residue Theorem). If* $f(z)$ *is analytic in and on a simple closed contour* C *except for a finite number of singularities* $z_1 \ldots z_n$ *with residues* $R_1 \ldots R_n$ *then*

$$\frac{1}{2\pi i} \int_C f(z)dz = \sum_1^n R_j .$$

Proof: We choose circles C_j centered at z_j with radii ε sufficiently small so that they don't overlap. These circles are connected to C by simple non-intersecting arcs A_i. If the contour defined by C, $\{C_j\}$ and the arcs $\{A_j\}$ traversed in both directions is considered, by Cauchy's theorem the integral of $f(z)$ over this path is 0. The integrals along $\{A_j\}$ cancel.

The integrals about each C_j yields $2\pi i\ R_j$, and the result follows.

Theorem 1.8: (Argument Principle). Let $f(z)$ be analytic in and on a simple closed contour C except for a finite number of poles within C. Let N be the number of zeros of $f(z)$ within C (a zero of order m counted m times). Let P be the number of poles within C (a pole of order m counted m times). If $f(z)$ is not zero on C and $\Delta_C \arg f(z)$ denotes the change in the argument of $f(z)$ as z varies over C, then

$$\frac{1}{2\pi} \Delta_C \arg f(z) = N - P \quad .$$

Proof: We write

$$f(z) = g(z) \prod_1^N (z - a_j) \Big/ \prod_1^P (z - b_j)$$

where $g(z)$ is analytic and non-zero in and on C, $\{a_j\}$ are the zeros of $f(z)$ in C and $\{b_j\}$ are the poles of $f(z)$ in C.

We consider

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz \quad .$$

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{N} \frac{1}{z - a_j} - \sum_{j=1}^{P} \frac{1}{z - b_j} + \frac{g'(z)}{g(z)} \quad .$$

Integrating around C we find the integral equal to $N - P$ since the residues in the sums are all 1 and $g'(z)/g(z)$ is analytic. Now

$$f'(z)/f(z) = \frac{d}{dz}(\log f(z)) = \frac{d}{dz} \log |f(z)| + i \frac{d}{dz} \arg f(z) \quad .$$

Since $|f(z)|$ returns to the original value upon traversing C, we find the integral is

$$\frac{1}{2\pi} \Delta_C \arg f(z)$$

and the result follows.

Theorem 1.9: (Rouché's Theorem). If $f(z)$ and $g(z)$ are analytic in and on a simple closed contour C and $|f(z)| > |g(z)|$ on C, then $F(z) = f(z) + g(z)$ and $f(z)$ have the same number of zeros in C.

Proof: Write

$$F(z) = f(z)\left(1 + \frac{g(z)}{f(z)}\right) .$$

Since

$$\left|1 + \frac{g(z)}{f(z)}\right| \geq 1 - \left|\frac{g(z)}{f(z)}\right| > 0$$

on C, $\Delta_C \arg F(z) = \Delta_C \arg f(z)$. ($\left[1 + \frac{g(z)}{f(z)}\right]$ always remains in quadrants 1 and 4. Thus it contributes nothing to $\Delta_C \arg F(z)$). Since neither $f(z)$ or $F(z)$ has poles, the argument principle shows that these functions have the same number of zeros in C.

Theorem 1.10: (Hurwitz's Theorem). Let $f_n(z)$ be a sequence of analytic functions converging uniformly to the analytic function $f(z)$ in a region bounded by a simple closed contour. Let z_0 be an interior point. If z_0 is a limit point of the zeros of the functions $f_n(z)$ then z_0 is a zero of $f(z)$. If z_0 is an m-fold zero of $f(z)$, then for every small region containing z_0, there is an N so that if $n > N$, $f_n(z)$ has precisely m zeros in that small region.

<u>Proof</u>: Choose $\varepsilon > 0$. There exists an N such that if $n > N$, then $|f_n(z) - f(z)| < \varepsilon/3$ for all z in C. Choose $\delta > 0$ so that $|f(z) - f(z_0)| < \varepsilon/3$ when $|z - z_0| < \delta$. There is an N_1 such that if $n > N_1$, then $f_n(z)$ has a zero z_1 such that $|z_1 - z_0| < \delta$. Finally in $|z - z_0| < \delta$ there is a δ_1 such that $|f_n(z) - f_n(z_1)| < \varepsilon/3$ when $|z - z_1| < \delta_1$.

Thus for all $n > N$ and N_1 we have

$$\begin{aligned} |f(z_0)| &\le |f(z_0) - f(z)| \\ &+ |f(z) - f_n(z)| \\ &+ |f_n(z) - f_n(z_1)| \\ &+ |f_n(z_1)|, \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 + 0, \end{aligned}$$

when $|z - z_1| < \delta_1$ and $|z - z_0| < \delta$. Thus $f(z_0) = 0$.

To prove the second half of the theorem we consider

$$f(z) = f_n(z) + [f(z) - f_n(z)]$$

on a circle with center z_o and radius ρ, where ρ is small enough so that $f(z)$ is not zero in and on the circle except at z_o. Since $f_n(z)$ converges uniformly to $f(z)$, we choose n sufficiently large to ensure $|f(z)| > |f(z) - f_n(z)|$ on the circle. The result follows from Rouche's theorem.

EXERCISES

1. Show z^n, e^z, $\sin z$, $\cos z$ are analytic functions.

2. Show $\left|\int_C f(z)\,dz\right| < M\ell$, where $|f(z)| < M$ and ℓ is the length of C.

3. Prove Morera's Theorem: If $f(z)$ is continuous on a simply connected region D and $\int_C f(z)\,dz = 0$ for every simple closed contour C in D, then $f(z)$ is analytic in D.

4. Show that

$$\sum_1^\infty a_n z^n$$

represents an analytic function within its region of convergence.

5. Show that if

$$f(z) = \sum_1^\infty a_n z^n$$

in a circle of radius ρ with $|f(z)| < M$ on the circle, then $|a_n| < M/\rho^n$. (Cauchy's Inequality.)

6. Prove Liouville's Theorem: An analytic function which is bounded for all z is a constant.

7. Prove the Fundamental Theorem of Algebra: If $f(z)$ is a polynomial of degree n, then $f(z)$ has n zeros.

8. Show that if $f(z)$ is analytic in an annular region $\rho_1 < |z - z_0| < \rho_2$ and on the boundary, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0| = \rho_1} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad \text{if} \qquad n < 0 \ ,$$

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0| = \rho_2} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad \text{if} \qquad n \geq 0 \ .$$

(Laurent Expansion).

9. Evaluate the following integrals by contour integration:

$$\int_0^\infty \frac{dx}{x^4 + 1} \ , \quad \int_0^\infty \frac{\sin x}{x}\, dx \ , \quad \int_0^\infty \frac{\cos x}{1 + x^2}\, dx \ , \quad \int_0^\infty \frac{\sin^2 x}{x^2}\, dx \ .$$

References

1. Lars V. Ahlfors, "Complex Analysis", McGraw Hill Book Co., Inc., New York, 1953.

2. Konrad Knopp, "The Theory of Functions", Vol. 1, Dover Publications, New York, 1945.

3. E. C. Titchmarsh, "The Theory of Functions", Oxford University Press, New York, 1952.

II. The Laplace Transform

We will consider those functions $f(t)$ which are piecewise continuous on $[0,\infty)$ and which for some real σ_0, $e^{-\sigma_0 t} f(t)$ is absolutely integrable. For all $s = \sigma + i\tau$ with $\sigma \geq \sigma_0$, $e^{-st}f(t)$ is absolutely integrable.

Definition: The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}f(s)$, is given by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for all $s = \sigma + i\tau$ with $\sigma \geq \sigma_0$.

Theorem 2.1: If f and g have Laplace transforms $\mathcal{L}f$ and $\mathcal{L}g$, respectively, then for all complex numbers a and b, $af + bg$ has a Laplace transform and

$$\mathcal{L}(af + bg) = a\mathcal{L}f + b\mathcal{L}g \quad .$$

Note that if f has a Laplace transform for $\mathrm{Re}(s) \geq \sigma_0$ and g has a Laplace transform for $\mathrm{Re}(s) \geq \sigma_1$, then both f and g will have Laplace transforms for $\mathrm{Re}(s) \geq \mathrm{Max}\,(\sigma_o,\sigma_1)$.

Theorem 2.2: If f and f' have Laplace transforms, then

$$\mathcal{L}(f') = s\mathcal{L}f - f(0) \quad .$$

If $f, f' \ldots f^{(n)}$ have Laplace transforms, then

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}f - \sum_{k=0}^{n-1} f^{(k)}(0)\, s^{n-1-k} \quad .$$

The proof is by induction. When $n = 1$, the result follows upon integration by parts. For arbitrary n,

$$\mathcal{L}(f^{(n)}) = \int_0^\infty e^{-st} f^{(n)}(t)dt$$

$$= e^{-st} f^{(n-1)}(t)\Big|_0^\infty + \int_0^\infty e^{-st} f^{(n-1)}(t)dt$$

$$- f^{(n-1)}(0) - s\left[\sum_{k=0}^{n-2} f^{(k)}(0)s^{n-2-k}\right] + s\left[s^{n-1}\mathcal{L} f\right]$$

$$= s^n \mathcal{L} f - \sum_{k=0}^{n-1} f^{(k)}(0)s^{n-1-k} \quad .$$

Theorem 2.3: Let

$$f^{(-n)} = \underbrace{\int^t \int^{t_n} .. \int^{t_3} \int^{t_2}}_{n} f(t_1)dt_1 dt_2 dt_3 \ldots dt_n$$

then

$$\mathcal{L}(f^{(-n)}) = \mathcal{L}(f)/s^n + \sum_{k=0}^{n-1} f^{(k-n)}(0)/s^{k+1} \quad .$$

Proof: Replace f in Theorem 2.2 by $f^{(-n)}$, multiply by s^{-n} and solve for $\mathcal{L}(f^{(-n)})$.

To show that if f has a Laplace transform so does $f^{(-1)}$ is fairly straight forward upon integration by parts. (See Kaplan [2]).

Theorem 2.4: If $f(t) = 0$ when $t < 0$, then

$$\mathcal{L}(f(t - a)) = e^{-as}\mathcal{L} f \quad .$$

Proof:

$$\int_0^\infty e^{-st} f(t-a)dt = \int_{-a}^\infty e^{-s(a+u)} f(u)du = e^{-as} \int_0^\infty e^{-su} f(u)du \quad .$$

Theorem 2.5: If f has a Laplace transform for $\mathrm{Re}(s) \geq \sigma_0$ and $f(t) = 0$ if $t < 0$, then $e^{at} f$ has a Laplace transform for $\mathrm{Re}(s) \geq \sigma_0 + \mathrm{Re}(a)$ and

$$\mathcal{L}(e^{at} f)(s) = \mathcal{L} f(s - a) \quad .$$

Proof:

$$\mathcal{L}(e^{at} f)(s) = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \mathcal{L} f(s - a) \quad .$$

Theorem 2.6: If f has a Laplace transform for $\mathrm{Re}(s) \geq \sigma_0$, then $t^n f$, $n = 1,2,3, \ldots$ has a Laplace transform for $\mathrm{Re}(s) > \sigma_0$ and

$$\mathcal{L}(t^n f)(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L} f(s) \quad .$$

Proof: Let $\mathrm{Re}(s) = \sigma$. Choose $\sigma = \sigma_0 + \alpha$.

$$\int_0^\infty e^{-\sigma t} t^n |f(t)|dt = \int_0^\infty e^{-\sigma_0 t} t^n e^{-\alpha t} |f(t)|dt < \infty \quad ,$$

which shows $\mathcal{L}(t^n f)(s)$ exists. The result is then formally achieved by differentiation of $\mathcal{L} f(s)$.

<u>Theorem 2.7</u>: <u>(Initial value Theorem). If</u> $|f(t)| < ke^{at}$ <u>for some real values</u> $k > 0$ <u>and</u> $a > 0$, <u>and</u>

$$\lim_{t \to 0} f(t) = f(0) \quad ,$$

<u>then</u>

$$\lim_{s \to \infty} s \mathcal{L} f(s) = f(0) \quad .$$

<u>Proof</u>: We consider

$$I(s) = s \int_0^\infty e^{-st} [f(t) - f(0)]dt \quad .$$

We restrict s so that $s > a$, let $st = u$ and eliminate t in $I(s)$. Thus

$$I(s) = \int_0^\infty e^{-u} [f(u/s) - f(0)]du \quad .$$

Choose $\epsilon > 0$. If u_0 is sufficiently large, then

$$\int_{u_o}^\infty e^{-u(1-a/s)} du < \epsilon/4k \quad .$$

Then

$$\left| \int_{u_o}^\infty e^{-u} [f(u/s) - f(0)] du \right| < \int_{u_o}^\infty e^{-u} 2k\, e^{a\, u/s}\, du < \epsilon/2 \quad .$$

We now choose s such that if u is in $[0,u_0]$

$$|f(u/s) - f(0)| < \epsilon/2u_0 \quad .$$

Then

$$\left|\int_0^{u_o} e^{-u}[f(u/s) - f(0)]du\right| < \int_0^{u_o} e^{-u}(\varepsilon/2u_0)du < \varepsilon/2 .$$

Thus if s is sufficiently large, $|I(s)| < \varepsilon$. Since

$$s\int_0^\infty e^{-st}f(0)dt = f(0) ,$$

we have shown

$$\underset{s \to \infty}{\text{limit}} \; s\mathcal{L}f(s) = f(0) .$$

<u>Theorem 2.8</u>: <u>(Final value theorem). If</u> $|f(t)| < k$ <u>for some real value</u> $k > 0$ <u>and</u>

$$\underset{t \to \infty}{\text{limit}} \; f(t) = f(\infty)$$

<u>exists, then</u>

$$\underset{s \to 0}{\text{limit}} \; s\mathcal{L}f(s) = f(\infty) .$$

<u>Proof</u>: We consider

$$J(s) = s\int_0^\infty e^{-st}[f(t) - f(\infty)]dt .$$

If $st = u$, then

$$J(s) = \int_0^\infty e^{-u}[f(u/s) - f(\infty)]du .$$

Choose $\varepsilon > 0$. If u_0 is small enough so that

$$1 - e^{-u_o} < \varepsilon/2[k + |f(\infty)|] ,$$

then

$$\left|\int_0^{u_o} e^{-u}[f(u/s) - f(\infty)]du\right| < \int_0^{u_o} e^{-u}[k + |f(\infty)|]du \quad ,$$

$$< (1 - e^{-u_o})[k + |f(\infty)|]$$

$$< \varepsilon/2 \quad .$$

Now we choose s small enough so that $|f(u/s) - f(\infty)| < \varepsilon/2\, e^{u_o}$. Then

$$\left|\int_{u_o}^{\infty} e^{-u}[f(u/s) - f(\infty)]du\right| < \int_{u_o}^{\infty} e^{-u}e^{u_o}(\varepsilon/2)du = \varepsilon/2 \quad .$$

Thus if s is sufficiently small, $|J(s)| < \varepsilon$. Since

$$s\int_0^{\infty} e^{-st}f(\infty)dt = f(\infty) \quad ,$$

we have shown

$$\lim_{s \to 0} s\mathcal{L}f(s) = f(\infty) \quad .$$

<u>Theorem 2.9</u>: <u>(Convolution theorem). If</u> $f(t)$ <u>and</u> $g(t)$ <u>have Laplace transforms</u> $F(s)$ <u>and</u> $G(s)$ <u>respectively, then</u>

$$F(s) \cdot G(s) = \mathcal{L}\left(\int_0^t f(\xi)g(t - \xi)d\xi dt\right) \quad .$$

<u>Proof</u>:

$$\int_0^{\infty} e^{-st}\int_0^t f(\xi)g(\xi - t)d\xi dt = \int_0^{\infty}\int_{\xi}^{\infty} f(\xi)g(t - \xi)e^{-s\xi}e^{-s(t-\xi)}dtd\xi \quad ,$$

$$= \int_0^{\infty} e^{-s\xi}f(\xi)\int_0^{\infty} g(u)e^{-su}dud\xi \quad ,$$

$$= F(s) \cdot G(s) \quad .$$

Theorem 2.10: (The Inversion theorem). Let $f(t)$ be piecewise smooth (f and f' be continuous except for a finite number of finite jumps) with Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad , \qquad \mathrm{Re}(s) \geq \sigma_0 \ .$$

Then

$$\tfrac{1}{2}[f(t+0) + f(t-0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\omega) e^{(\sigma+i\omega)t} d\omega$$

for all $\sigma > \sigma_0$.

Proof:

$$\frac{1}{2\pi} \int_{-n}^{n} F(\sigma + i\omega) e^{(\sigma+i\omega)t} d\omega = \frac{1}{2\pi i} \int_{\sigma-in}^{\sigma+in} F(s) e^{st} ds \quad ,$$

$$= \frac{1}{2\pi i} \int_{\sigma+in}^{\sigma+in} \int_0^\infty e^{-su} f(u)\,du\ e^{st} ds \quad ,$$

$$= \frac{1}{2\pi i} \int_0^\infty \int_{\sigma-in}^{\sigma+in} f(u) e^{-s(u-t)} ds\,du \quad ,$$

$$= \frac{1}{\pi} \int_0^\infty f(u) e^{-\sigma(u-t)} \frac{\sin n(u-t)}{u-t} du \quad ,$$

$$= \frac{1}{\pi} \int_{-t}^{\infty} f(t+v) e^{-\sigma v} \frac{\sin nv}{v} dv \quad .$$

The last integral is a well known Dirichlet integral whose limit as $n \to \infty$ is $\tfrac{1}{2}[f(t+0) + f(t-0)]$.

EXAMPLES: Let $f(t) = 0$ if $t < 0$.

1. $f(t) = 1, \quad t > 0$ ———— $F(s) = 1/s, \quad Re(s) > 0.$
2. $f(t) = e^{at}, \quad t > 0$ ———— $F(s) = 1/(s - a), \quad Re(s - a) > 0.$
3. $f(t) = t^n, \quad t > 0$ ———— $F(s) = n!/s^{n+1}, \quad Re(s) > 0.$
4. $f(t) = t^n e^{at}, \quad t > 0$ ———— $F(s) = n!/(s - a)^{n+1}, \quad Re(s-a) > 0.$
5. $f(t) = \cos at, \quad t > 0$ ———— $F(s) = s/(s^2 + a^2), \quad Re(s) > |Im(a)|.$
6. $f(t) = \sin at, \quad t > 0$ ———— $F(s) = 1/(s^2 + a^2), \quad Re(s) > |Im(a)|.$

See tables for Laplace transforms and tables for inverse transforms.

EXAMPLES: Evaluation of the inversion formula by contour integration.

1. Let

$$F(s) = \frac{1}{(s - a)(s - b)} .$$

If F is expanded by partial fractions,

$$F(s) = \left[\frac{1}{s - a} - \frac{1}{s - b}\right] \frac{1}{a - b} .$$

We choose σ_0 greater than a or b and compute

$$\frac{1}{2\pi i} \int_{C_n} F(s)e^{ts}ds$$

where C_n is the contour consisting of the line segment from $\sigma_0 - in$ to $\sigma_0 + in$ and the semicircle to the left of the segment with center at $\sigma_0 + i0$ and radius n. For large n both a and b will be enclosed, so

$$\frac{1}{2\pi i} \int_{C_n} F(s)e^{ts}ds = \frac{1}{a - b}[e^{at} - e^{bt}] .$$

As n increases both the integrals

$$\frac{1}{2\pi i} \int [F(s)/(s - a)]ds$$

and

$$\frac{1}{2\pi i} \int [F(s)/(s - b)]ds$$

taken on the semicircle approach 0. Thus

$$f(t) = \frac{1}{a - b} [e^{at} - e^{bt}] \quad .$$

2. Let

$$F(s) = \frac{1}{(s - a)^{n+1}} \quad .$$

We consider the same contour as in example 1 with $\sigma_0 > a$

$$f(t) = \frac{1}{2\pi i} \int_{C_n} \frac{e^{st}}{(s - a)^{n+1}} ds \quad .$$

This is the formula for

$$\frac{1}{n!} \frac{d^n}{dz^n} e^{zt} \bigg|_{z = a}$$

which is $(t^n/n!)\, e^{at}$. Again as $n \to \infty$, the integral along the semicircle vanishes.

The inversion of any rational function may be computed by this method.

Note that in both examples if $t < 0$, the contour must be closed to the right and the integrals give 0.

EXERCISES

1. Give another proof of the initial and final value theorems when f' and $\mathcal{L}(f')$ exist.

2. Give a direct proof that

$$\int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

when $f(t)$ and $g(t)$ are zero for $t < 0$.

3. Prove 2 using the Laplace transform. (Assume that if $\mathcal{L}f = \mathcal{L}g$, then $f = g$.)

References

1. G. Doetsch, "Theorie und Anwendung der Laplace Transformation", Springer-Verlag, Berlin, 1937.

2. Wilfred Kaplan, "Operational Methods for Linear Systems", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

3. D.V. Widder, "The Laplace Transform", Princeton University Press, Princeton, N. J., 1941.

III. Linear Differential Equations, Stability

Let $A(t)$ be an n by n matrix of continuous functions defined for all $t \geq t_0$. Let $Y(t)$ be an n by 1 matrix, Y_0 be an n by 1 matrix of constants and $B(t)$ be an n by 1 matrix of functions which are continuous for all $t \geq t_0$.

<u>Theorem 3.1</u>: <u>(The Existence Theorem). The problem</u>

$$Y' = A(t) + B(t) \quad ,$$

$$Y(t_0) = Y_0$$

<u>has a unique solution</u> $Y(t)$ <u>whose elements are continuously differentiable for all</u> $t \geq t_0$.

<u>Proof</u>: Let us define a sequence of n by 1 matrices by

$$Y_m(t) = Y_0 + \int_{t_o}^{t} [A(t)Y_{m-1}(t) + B(t)]dt$$

with $Y_0(t) = Y_0$. The elements of this sequence satisfies $Y_m'(t) = A(t)Y_{m-1}(t) + B(t)$, $Y_m(t_0) = Y_0$. Further

$$Y_m(t) = Y_0 + (Y_1(t) - Y_0) + \cdots + (Y_m(t) - Y_{m-1}(t)) \quad .$$

In any finite interval $[t_0,T]$ each element is uniformly bounded since $A(t)$ and $B(t)$ have continuous functions as components. Since $Y_m'(t) - Y_{m-1}'(t) = A(t)(Y_{m-1}(t) - Y_{m-2}(t))$, we have by induction that each element of $Y_m(t) - Y_{m-1}(t)$ is less than $k[n\alpha(T - t_0)]^{m-1}/(m-1)!$ where k is a bound for the components of $Y_1(t) - Y_0$ and α is a bound for the components of $A(t)$. Since the series

$$\sum_{m=0}^{\infty} k[n\alpha(T - t_0)]^m/m!$$

converges to $k \exp [n\alpha(T - t_0)]$, the sequence $Y_m(t)$ converges uniformly for all t to a function $Y(t)$ which is itself continuous. Since the convergence is uniform

$$Y(t) = Y_0 + \int_{t_o}^{t} [A(t)Y(t) + B(t)]dt \quad .$$

The right side is differentiable, thus $Y(t)$ is, and

$$Y'(t) = A(t)Y(t) + B(t) \quad ,$$

$$Y(t_0) = Y_0 \quad .$$

To show that $Y(t)$ is unique, we suppose there exist $Y(t)$ and $Y^*(t)$ satisfying the differential equation and the boundary condition. Then $U(t) = Y(t) - Y^*(t)$ satisfies

$$U'(t) = A(t)U(t) \quad ,$$

$$U(t_0) = 0 \quad .$$

We choose an arbitrary $\varepsilon > 0$ and require $|t - t_0| < \varepsilon$. Then since $U'(t) = A(t)U(t)$, if m denotes the largest component of $U(t)$, every component $u'(t)$ of $U'(t)$ satisfies

$$|u'(t)| \le n\alpha m \quad .$$

Upon integration

$$|u(t)| \le n\alpha m|t - t_0| \le n\alpha m\varepsilon \quad .$$

Since $|u(t)| \le m$,

$$m(1 - n\alpha\epsilon) \le 0 \quad .$$

But since ϵ is arbitrary, $\epsilon < \frac{1}{n\alpha}$ yields a contradiction unless $m = 0$. Thus $U(t) = 0$, and $Y(t)$ is unique.

The equation $Y' = A(t)Y + B(t)$ is called a linear equation. If $B(t) = 0$, the equation is homogeneous. If $B(t) \neq 0$, it is non-homogeneous. Note that the nth order equation

$$a_0(t)y^{(n)} + \cdots + a_n(t)y = f(t) \quad ,$$

$$y(t_0) = \alpha_0, \ \ldots \ y^{(n-1)}(t_0) = \alpha_{n-1} \quad ,$$

can be put in matrix form. Let us write

$$y' = 0\,y + y_1 \ \ldots\ldots\ldots\ldots\ldots + 0,$$

$$y_1' = 0\,y + 0y_1 + y_2 \ \ldots\ldots\ldots\ldots + 0,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y_{n-2}' = 0y + 0y_1 \ \ldots + \cdots + y_{n-1} + 0,$$

$$y_{n-1}' = -\frac{a_n(t)}{a_0(t)}\,y - \frac{a_{n-1}(t)}{a_0(t)}\,y_1 \ \ldots \ \frac{a_1(t)}{a_0(t)}\,y_{n-1} + \frac{b(t)}{a_0(t)} \quad .$$

When $a_i(t)$ are continuous over an interval $[a,b]$ and $a_0(t) \neq 0$, if we let

$$Y = \begin{pmatrix} y \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ -\frac{a_n(t)}{a_0(t)} & & \cdots & & -\frac{a_1(t)}{a_0(t)} \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{b(t)}{a_0(t)} \end{pmatrix}$$

$$Y_0 = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} \end{pmatrix},$$

then the nth order equation with boundary conditions at t_0 is put in the form of a matrix equation.

If $A(t) = A$, a constant matrix, the solution to

$$Y' = AY + B(t)$$

$$Y(t_0) = Y_0$$

is quite simple to find. Let us denote by

$$e^{At} = \sum_{n=0}^{\infty} [A^n/n!]t^n \quad .$$

This series converges uniformly for all bounded t. It is easy to show that $e^{-At}e^{At} = e^{At}e^{-At} = I$, the identity matrix, as well as other properties usually associated with the ordinary exponential function. Now, multiplying $Y' - AY = B(t)$ by e^{-At} yields

$$e^{-At}Y' - Ae^{-At}Y = e^{-At}B(t) \quad ,$$

$$[e^{-At}Y]' = e^{-At}B(t) \quad .$$

Integrating from t_0 to t,

$$e^{-At}Y(t) - Y_0 e^{-At_0} = \int_{t_o}^{t} e^{-A\tau}B(\tau)d\tau$$

and

$$Y(t) = e^{A(t-t_0)}Y_0 + \int_{t_o}^{t} e^{+A(t-\tau)}B(\tau)d\tau \quad .$$

Note that the integral is a convolution.

If the nth order equation

$$a_0 y^{(n)} + \cdots + a_n y = f(t) \quad ,$$

$$y(0) = \alpha_0 \cdots y^{(n-1)}(0) = \alpha_{n-1} \quad ,$$

where a_i are constants, $a_o \neq 0$ is considered, we find

$$y = \sum_{i=1}^{n} C_i y_i(t) + \int_{o}^{t} G(t - \tau)f(\tau)d\tau$$

is a solution where $y_i(t)$ are independent solutions to the homogeneous problem, C_i are appropriate coefficients and $G(t - \tau)$ is the appropriate component of $e^{A(t-\tau)}$. $G(t - \tau)$ is called the Green's function associated with the nth order problem.

It is easy to show that $t^{m-1}e^{rt}$ is a solution of $a_0 y^{(n)} + \cdots + a_n y = 0$ if and only if r is at least an m-fold zero of $F(z) = a_0 z^n + \cdots + a_n = 0$. Further the collection $\{t^{m-1}e^{rt}\}$ where r ranges over the zeros of $F(z)$

and m ranges from 1 to the multiplicity of the zero r form a linearly independent set of solutions. Thus

$$y = \sum_{k=1}^{\text{No. of zeros}} \sum_{j=1}^{\text{Mult. of } r_k} C_{jk} t^{j-1} e^{r_k t} + \int_0^t G(t - \tau) f(\tau) d\tau \ ,$$

where again C_{jk} are the appropriate coefficients.

Linear systems in vector notation can be solved by the Laplace transform. If

$$Y' = AY + B(t), \quad Y(0) = 0 \ ,$$

then

$$\mathcal{L}(Y') = s\mathcal{L}(Y) - Y(0) \ ,$$

and

$$s\mathcal{L}(Y) - Y(0) = A\mathcal{L}Y + \mathcal{L}B \ .$$

Thus

$$[Is - A]\mathcal{L}Y = Y(0) + \mathcal{L}B \ .$$

If s is larger than the eigenvalues of A, $[Is - A]^{-1}$ exists, and

$$\mathcal{L}Y = [Is - A]^{-1}Y(0) + [Is - A]^{-1}\mathcal{L}B \ .$$

Since

$$\mathcal{L}(e^{At}) = \int_0^\infty e^{-st} e^{At} dt = \int_0^\infty e^{-[Is-A]t} dt$$

$$= [Is - A]^{-1} \ ,$$

$$Y = e^{At}Y(0) + \int_0^t e^{A(t-\tau)} B(\tau) d\tau \ ,$$

the same formula as before.

The nth order differential equation with constant coefficients can also be solved directly using the Laplace transform. We consider

$$a_0 y^{(n)} + \cdots a_n y = f(t) ,$$

$$y(0) = \alpha_0, \ \ldots \ y^{(n-1)}(0) = \alpha_{n-1} ,$$

where $f(t)$ possesses a Laplace transform $\mathcal{L}f(s)$. Taking the Laplace transform of both sides,

$$\sum_{i=0}^{n} a_{n-i} \left[s^i \quad y - \sum_{k=0}^{i-1} \alpha_k s^{i-1-k} \right] = \mathcal{L}f(s) .$$

If

$$F(z) = \sum_{i=0}^{n} a_{n-i} z^i ,$$

$$F(s)\mathcal{L}y = \mathcal{L}f(s) + \sum_{i=0}^{n} \sum_{k=0}^{i-1} a_{n-i}\alpha_k s^{i-1-k} ,$$

$$\mathcal{L}y = \left[\mathcal{L}f(s) + \sum_{i=0}^{n} \sum_{k=0}^{i-1} a_{n-i}\alpha_k s^{i-1-k}\right]/F(s) .$$

$$y = \mathcal{L}^{-1}\left[\mathcal{L}f(s)/F(s) + \sum_{i=0}^{n} \sum_{k=0}^{i-1} a_{n-i}\alpha_k s^{i-1-k}/F(s)\right] ,$$

$$y = \mathcal{L}^{-1}\left[\mathcal{L}f(s)/F(s)\right] + \mathcal{L}^{-1}\left[\sum_{i=0}^{n} \sum_{k=0}^{i-1} a_{n-i}\alpha_k s^{i-1-k}/F(s)\right] .$$

If we let $G(t)$ have Laplace transform $1/F(s)$, $y = \int_0^t G(t - \tau)f(t)dt$ + terms having the appearance $C_{ij}t^j e^{r_i t}$ where $j = 0,1,\ldots m_i - 1$ and r_i is an m_i fold root of $F(z)$. The coefficients are chosen so that the initial conditions are satisfied. If $\alpha_0 = 0, \ldots a_{n-1} = 0$ then

$$y = \int_0^t G(t - \tau)f(\tau)d\tau \quad .$$

Thus the Green's function previously mentioned has as its Laplace transform $1/F(s)$.

Definition: A solution

$$Y(t) = \begin{pmatrix} y_1(t) \\ \cdot \\ \cdot \\ \cdot \\ y_n(t) \end{pmatrix} ,$$

of

$$Y' = A(t)Y + B(t) \quad ,$$

is said to be stable if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if any other solution

$$X(t) = \begin{pmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}$$

satisfies

$$\sup_{i=1,\ldots n} |x_i(t_0) - y_i(t_0)| \leq \delta \quad ,$$

then

$$\sup_{i=1,\ldots n} |x_i(t) - y_i(t)| \leq \epsilon$$

for $t > t_0$.

If

$$\lim_{t \to \infty} \sup_{i=1,\ldots n} |x_i(t) - y_i(t)| = 0 ,$$

$Y(t)$ *is asymptotically stable*.

If

$$\sup_{i=1,\ldots n} |x_i(t) - y_i(t)|$$

is merely bounded, as $t \to \infty$, $Y(t)$ *is marginally stable*.

Theorem 3.2: *The solutions of the linear differential equation with constant coefficients*

$$Y' = AY + B(t)$$

are stable if and only if the zeros of

$$F(z) = |Iz - A|$$

are either simple with non-positive real parts or have negative real parts.

The solutions are asymptotically stable if and only if the zeros of $F(z)$ *have negative real parts*.

Proof: The difference between any two solutions will be of the form $e^{At}C$ which will always behave only when e^{At} does.

If the zeros of $F(z)$ are simple, there exists a matrix T so that

$$A = T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} T^{-1}$$

where $\{\lambda_i\}_1^n$ are the zeros of $F(z)$. Then

$$e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}T^{-1},$$

which is bounded if and only if the real parts of the zeros λ_i are non-positive, which approaches zero if and only if the real parts are negative.

If the zeros of $F(z)$ are not simple, there exist matrices T, D and N so that

$$A = T\left[D + N\right]T^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_n & \\ & & & & & \ddots \\ & & & & & & \lambda_n \end{bmatrix},$$

N satisfies $N^m = 0$ for some m, and $ND = DN$. Then

$$e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & & & & & & \\ & \ddots & & & & & \\ & & e^{\lambda_1 t} & & & & \\ & & & \ddots & & & \\ & & & & e^{\lambda_n t} & & \\ & & & & & \ddots & \\ & & & & & & e^{\lambda_n t} \end{bmatrix}e^{Nt}T^{-1}$$

where

$$e^{Nt} = I + Nt + \cdots + \frac{N^{m-1}t^{m-1}}{(m-1)!} .$$

Clearly e^{At} remains bounded and approaches zero if and only if the real parts of the zeros $\{\lambda_i\}_1^n$ have negative real parts.

Theorem 3.3: The solutions of the linear differential equation with constant coefficients

$$a_0 y^{(n)} + \cdots \quad a_n y = f(t)$$

are stable if and only if the zeros of

$$F(z) = a_o z^n + \cdots + a_n$$

are either simple with non-positive real parts or have negative real parts.

The solutions are asymptotically stable if and only if the zeros of $F(z)$ all have negative real parts.

Proof: Clearly the difference between any two solutions will be a sum whose terms are of the form $t^{m-1}e^{rt}$ which is bounded only when $m = 1$, $\mathrm{Re}(r) \leq 0$ or when $\mathrm{Re}(r) < 0$, which approaches 0 as $t \to \infty$ only when $\mathrm{Re}(r) < 0$.

Note that stability is independent of the initial conditions at $t = 0$ in each case.

EXAMPLES:

1. Solution of $xy'' + y' = x$ by using the Laplace Transform.

The transformed equation is

$$(-1)\frac{d}{ds}[s^2F - sy(0) - y'(0)] + sF - y(0) = s^{-2}$$

where $F = \mathcal{L}y$. Simplifying, we find

$$sF' + F = -s^{-3} ,$$

$$sF = (2s^2)^{-1} + C ,$$

$$F = (2s^3)^{-1} + Cs^{-1} ,$$

$$y = \tfrac{1}{4}x^2 + C .$$

The general solution is $y = C_1 + C_2 \ln x + \frac{1}{4}x^2$. Since $\ln x$ does not possess a Laplace transform, it is ignored by this method.

2. Solution of $y'' - 3y' + 2y = f(t)$ by using the Laplace transform.

The transformed equation is

$$[s^2F - sy(0) - y'(0)] - 3[sF - y(0)] + 2[F] = \mathcal{L}f(s) ,$$

$$[s^2 - 3s + 2]F = \mathcal{L}f(s) + s[y(0)] + [y'(0) - 3y(0)] .$$

Dividing and using a partial fraction expansion,

$$F = \frac{(2y(0) - y'(0))}{s - 1} + \frac{(-y(0) + y'(0))}{s - 2}$$

$$- \left(\frac{1}{s - 1}\right)\mathcal{L}f(s) + \left(\frac{1}{s - 2}\right)\mathcal{L}f(s) ,$$

$$y = [2y(0) - y'(0)]e^t + [-y(0) + y'(0)]e^{2t}$$

$$- e^t * f + e^{2t} * f ,$$

$$y = y(0)[2^{et} - e^{2t}] + y'(0)[e^{2t} - e^t]$$

$$- \int_0^t e^{t-u}f(u)du + \int_0^t e^{2(t-u)}f(u)du .$$

3. The approximation procedure for

$$y' = F(x) \cdot y \quad , \qquad y(0) = y_0$$

where $F(x)$ is continuous.

We have

$$y_0(x) = y_0 \quad .$$

$$y_1(x) = y_0 + \int_o^x F(\xi) y_0 d\xi \quad ,$$

$$= y_0\left[1 + \int_o^x F(\xi) d\xi\right] \quad .$$

$$y_2(x) = y_0 + \int_o^x F(\xi) y_0 \left[1 + \int_o^\xi F(z) dz\right] d\xi \quad ,$$

$$= y_0\left[1 + \int_o^x F(\xi) d\xi + \left(\int_o^x F(\xi) d\xi\right)^2 / 2!\right] \quad .$$

. . . .

$$y_n(x) = y_0\left[\sum_{j=0}^{n} \left(\int_o^x F(\xi) d\xi\right)^j / j!\right] \quad .$$

As $n \to \infty$, $y_n(x) \to y(x)$, where

$$y(x) = y_0 \exp\left[\int_o^x F(\xi) d\xi\right] \quad .$$

EXERCISES

1. Solve $y'' - 4y' + 4y = e^{2t}$ by the Laplace transform method.

2. Find the matrix T so that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T^{-1}$$

for some λ_1, λ_2. Use T to solve

$$\begin{aligned} x_1' &= 2x_1 + x_2 + b_1(t) \\ x_2' &= x_1 + 2x_2 + b_2(t) \end{aligned} \quad , \quad \begin{aligned} x_1(0) &= 1 \\ x_2(0) &= 0 \end{aligned} \quad .$$

3. If $y' = 2ty$, $y(0) = a$, find the nth approximation of the solution y as given by the procedure in theorem 1.

4. Show that the characteristic equation for the nth order equation remains unchanged if the nth order equation is put into matrix form.

References

1. Richard Bellman, "Stability theory of Differential Equations", McGraw-Hill Book Co., Inc., 1953.

2. Earl A. Coddington and Normal Levinson, "Theory of Ordinary Differential Equations", McGraw-Hill Book Co., Inc., 1955.

3. Charles W. Curtis, "Linear Algebra, An Introductory Approach", Allyn and Bacon, Inc., Boston, Mass., 1963.

4. E. L. Ince, "Ordinary Differential Equations", Dover Publications, Inc., 1956.

5. Wilfred Kaplan, "Ordinary Differential Equations", Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958.

6. Wilfred Kaplan, "Operational Methods for Linear Systems", Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962.

IV. Bode Diagrams

4.1: The following notation is sometimes convenient. Let L and M be linear differential operators with constant coefficients with the degree of L greater than or equal to the degree of M, and $Ly = Mf$ with f given. Then f is called an input and y an output. $Ly = Mf$ is transformed by the Laplace transform into

$$g(s)O = h(s)I + k(s)$$

where $g(s)$, $h(s)$ and $k(s)$ are polynomials with degree $g(s)$ = order L, degree $h(s)$ = order M, degree $k(s) <$ degree $g(s)$, $I = \mathcal{L}f$, $O = \mathcal{L}y$.

If the initial conditions for y and f are zero,

$$O = [h(s)/g(s)]I \ .$$

This may be symbolically represented by a "black" box with transfer function

$$F(s) = [h(s)/g(s)] \ .$$

That is the input I is transformed by the transfer function F into the output O.

4.2: Bode Diagrams. A Bode diagram is the graph of $20 \log_{10} F(s)$ with $s = i\omega$ versus $\log_{10}(\omega)$. That is, the graphs of $20 \log_{10} |F(i\omega)|$ and $\arg F(i\omega)$ are drawn with respect to $\log_{10}(\omega)$. The graphs are easily constructed on log-log or semi-log paper.

Bode diagrams have various uses. Among them are

1. If F is unknown, it can be found experimentally by the following procedure.

Let $f(t) = Ae^{i\omega t}$ be induced by some mechanical means.* Then

$$I = A(s - i\omega)^{-1} ,$$

and

$$O = AF(s)(s - i\omega)^{-1} .$$

If initial conditions are ignored and a steady state condition is reached, then

$$y = AF(i\omega)e^{i\omega t}$$

and

$$F(i\omega) = y(t)/f(t) .$$

As ω varies from 0 to ∞, a graph of $20 \log_{10} |F(i\omega)|$ and $\arg F(i\omega)$ can be found by considering $|y(t)/f(t)|$ and $\arg y(t) - \arg f(t)$. From this F can be determined.

2. If $f(t) = e^{i\omega t}$ is induced either experimentally or mathematically and a Bode diagram is drawn, data can be found from it to construct the graphs necessary for some of the stability criteria to follow, e.g. the Nyquist and Michailov criteria.

Definition: The amplitude of a quantity A in decibels is

$$A_{db} = 20 \log_{10} A .$$

Definition: An octave, associated with a variable ω, is the interval in which ω doubles itself.

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*The actual input may be the real or imaginary part of $e^{i\omega t}$ or any combination. We are only interested in the fact that it is sinusoidal.

Then

$$|y(t)/f(t)|_{db} = 20 \log_{10} |h(i\omega)| - 20 \log_{10} |g(i\omega)| \quad .$$

The graph of $|y(t)/f(t)|_{db}$ will consist of the sum and difference of the graphs of the individual factors of the form $20 \log_{10} |i\omega - \omega_0|$ and $20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$ coming from $h(i\omega)$ and $g(i\omega)$. We discuss these components separately. Having learned how the components behave, the graph of F is easily constructed, or the form of F can be easily approximated.

Theorem 4.2.1: Let $(z - \omega_0)$ be an arbitrary factor of $h(z)$ or $g(z)$ with real ω_0. Then $20 \log_{10} |i\omega - \omega_0|$ becomes asymptotic to $20 \log_{10} |\omega_0|$ as $\omega \to 0$, and becomes asymptotic to $20 \log_{10} |\omega|$ as $\omega \to \infty$.

Proof: Let

$$f = 20 \log_{10} |i\omega - \omega_0| - 20 \log_{10} |\omega_0| \quad .$$

Then

$$f = 20 \log_{10} ([\omega/\omega_0]^2 + 1)^{\frac{1}{2}}$$

and

$$\lim_{\omega \to 0} f = 0 \quad .$$

Let

$$f = 20 \log_{10} |i\omega - \omega_0| - 20 \log_{10} |\omega| \quad .$$

Then

$$f = 20 \log_{10} ([\omega_0/\omega]^2 + 1)^{\frac{1}{2}},$$

and

$$\lim_{\omega \to \infty} f = 0 \quad .$$

Theorem 4.2.2: The rate of change of $20 \log_{10} |i\omega - \omega_0|$ per octave approaches 0 as $\omega \to 0$, and approaches $20 \log_{10} 2 \approx 6$ as $\omega \to \infty$.

Proof: Let $f = 20 \log_{10} |i\omega - \omega_0|$. Then the rate of change per octave is given by $df/d(\log_2 \omega)$. Differentiating, we have

$$\frac{df}{d\omega} = 20\, \omega(\omega^2 + \omega_0^2)^{-1} \log_{10} e \quad .$$

$$\frac{d \log_2 \omega}{d\omega} = \frac{1}{\omega} \log_2 e \quad .$$

Thus

$$\frac{df}{d(\log_2 \omega)} = 20 \log_{10} 2/[1 + (\omega_0/\omega)^2] \quad .$$

The results follow by taking limits.

Definition: A frequency ω_0 at which the asymptotes of a given factor intersect is called a corner frequency.

Theorem 4.2.3: Let $z^2 + 2\delta z\omega_0 + \omega_0^2$ be an arbitrary factor of $g(z)$ or $h(z)$ with complex zeros, i.e., $\delta^2 < 1$, $\omega_0 \neq 0$, then

$$20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$$

becomes asymptotic to $20 \log_{10} |\omega_0^2|$ as $\omega \to 0$ and becomes asymptotic to

$20 \log_{10} |\omega^2|$ as $\omega \to \infty$.

The proof is similar to theorem 1.

Theorem 4.2.4: The rate of change of

$$20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$$

per octave approaches zero as $\omega \to 0$, and approaches $40 \log_{10} 2 \approx 12$ as $\omega \to \infty$.

The proof is similar to that of theorem 2.

Theorem 4.2.5: When $0 \leq |\delta| \leq \sqrt{2}/2$,

$$20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$$

has a minimum at $\omega = \omega_0(1 - 2\delta^2)^{\frac{1}{2}}$. The value of

$$20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$$

at the minimum is

$$20 \log_{10} (4\delta^2[1 - \delta^2])^{\frac{1}{2}} \quad .$$

When $|\delta| \geq \sqrt{2}/2$,

$$20 \log_{10} |-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2|$$

has a minimum only at zero.

Proof: Differentiation yields

$$\omega(\omega^2 + \omega_0^2[2\delta^2 - 1]) = 0$$

at a critical point. If the function has a minimum not at $\omega = 0$, then

$\omega = \omega_0(1 - 2\delta^2)^{\frac{1}{2}}$ which is real if $0 \leq \sqrt{2}/2$. The second derivative being positive confirms that it is a minimum. If $|\delta| \geq \sqrt{2}/2$, the only critical point is at $\omega = 0$. Since the first derivative is positive for $\omega > 0$, $\omega = 0$ is a minimum.

Theorem 4.2.6: The

$$\lim_{\omega \to 0} \arg (i\omega - \omega_0)$$

is 0, $\frac{\pi}{2}$ or π depending upon whether ω_0 is negative, zero or positive.

The

$$\lim_{\omega \to \infty} \arg (i\omega - \omega_0)$$

is $\frac{\pi}{2}$ for all ω_0.

Proof: These results follow by looking at the vector $[-\omega_0] + i[\omega]$.

Theorem 4.2.7: If $\omega_0 \neq 0$, the rate of change of $\arg (i\omega - \omega_0)$ per octave approaches 0 as ω approaches 0 or ∞.

Proof: Let $f = \arg (i\omega - \omega_0) = \tan^{-1} (-\omega/\omega_0)$. Then

$$\frac{df}{d\omega} = \frac{-\omega_0}{\omega^2 + \omega_0^2} ,$$

$$\frac{d \log_2 \omega}{d\omega} = \frac{1}{\omega} \log_2 e ,$$

and

$$\frac{df}{d \log_2 \omega} = - \frac{1}{\log_2 e} \left[\frac{\omega\omega_0}{\omega^2 + \omega_0^2} \right]$$

which approaches zero as $\omega \to 0$ or ∞.

<u>Theorem 4.2.8</u>: <u>If</u> $|\delta| < 1$, $\omega_0 > 0$,

$$\lim_{\omega \to 0} \arg(-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2) = 0 ,$$

$$\lim_{\omega \to \infty} \arg(-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2) = \begin{cases} \pi & \text{if } \delta > 0 \\ -\pi & \text{if } \delta < 0 \end{cases} .$$

<u>Proof</u>: These results follow from looking at the vector

$$V = [\omega_0^2 - \omega^2] + i[2\delta\omega\omega_0] .$$

<u>Theorem 4.2.9</u>: <u>The rate of change of</u> $\arg(-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2)$ <u>per octave approaches</u> 0 <u>as</u> ω <u>approaches</u> 0 <u>or</u> ∞. <u>For fixed</u> ω, <u>the rate of change per octave approaches</u> 0 <u>as</u> δ <u>approaches</u> 0 <u>when</u> $\omega \neq \omega_0$, <u>approaches</u> ∞ <u>as</u> δ <u>approaches</u> 0 <u>when</u> $\omega = \omega_0$, <u>approaches</u> 0 <u>as</u> δ <u>approaches</u> ∞.

<u>Proof</u>: If $f = \arg(-\omega^2 + 2i\delta\omega\omega_0 + \omega_0^2)$ then

$$\frac{df}{d(\log_2 \omega)} = \frac{2\delta\omega\omega_0(\omega^2 + \omega_0^2)}{(-\omega^2 + \omega_0^2)^2 + (2\delta\omega\omega_0)^2} \cdot \frac{1}{\log_2 e} .$$

The results follow by taking limits.

EXERCISES

1. Solve $y'' = f' - f$ by the Laplace Transform. Show that y depends upon $f(0)$.

2. Draw a Bode diagram for

$$F(s) = (s^2 - 3s + 2)/(s^2 - 4s + 8)(s - 1) \quad .$$

3. Draw a graph of $F(s) = (s^2 - 3s + 2)/(s^2 - 4s + 8)(s + 1)$ as s varies from 0 to $i\infty$ by taking the data from the Bode diagrams in problem 2.

References

1. Floyd E. Nixon, "Principles of Automatic Controls", Prentice-Hall, Inc., Englewood Cliffs, N.J., 1953.

2. C. J. Savant, Jr., "Basic Feedback Control System Design", McGraw-Hill, New York, 1958.

V. Stability Techniques for Linear Systems

5.1: It has already been shown that a linear differential system with constant coefficients is stable only when the roots of the characteristic equation do not lie in the right half plane. In practice, roots upon the imaginary axis do not occur, so that it is desirable to know when the roots all lie in the left half plane. There are various techniques for determining this. We examine them. Let the characteristic equation in question be

$$F(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0 .$$

Theorem 5.1.1: All of the zeros of

$$F(z) = a_0 z^n + \cdots + a_n$$

lie inside a circle of radius $\rho + M + 1$ centered at the origin where

$$M = \sup_{i=1,\dots n} \{a_i/a_0\} .$$

Proof: If $|z| > \rho$, then

$$\frac{|F(z)|}{|a_0|} \geq |z|^n - \sum_{i=1}^{n} |a_i/a_0||z|^{n-i}$$

$$\geq |z|^n - M \sum_{i=1}^{n} |z|^{n-i}$$

$$= |z|^n - M(|z|^n - 1)/(|z| - 1)$$

$$= \frac{|z|^n[|z| - 1 - M] + M}{|z| - 1}$$

$$> 0 .$$

<u>Theorem 5.1.2</u>: <u>Let</u>

$$F(z) = a_0 z^n + \cdots + a_n = a_0 \prod_{j=1}^{p} (z - z_j)^{m_j}$$

<u>where</u>

$$\sum_{j=1}^{p} m_j = n$$

<u>and</u> $a_0 \neq 0$,

$$F_\varepsilon(z) = a_0 z^n + (a_1 + \epsilon_1) z^{n-1} + \cdots + (a_n + \epsilon_n) \quad ,$$

C_k <u>be a circle centered at</u> z_k <u>with radius</u> r_k <u>satisfying</u>

$$0 < r_k < \min_{\substack{j=1,\ldots n \\ j \neq k}} |z_k - z_j| \quad ,$$

<u>then there exists an</u> $\epsilon > 0$ <u>such that if</u> $|\epsilon_i| < \epsilon$ <u>for</u> $i = 1, \ldots n$, <u>then</u> C_k <u>contains precisely</u> m_k <u>zeros of</u> $F_\varepsilon(z)$.

<u>Proof</u>: Note that on C_k, $F(z)$ is not zero. Thus there exists $M_k > 0$ so that $|F(z)| > M_k$ on C_k. Further, on C_k,

$$|F_\varepsilon(z) - F(z)| = |\varepsilon_1 z^{n-1} + \cdots + \epsilon_n|$$

$$< \varepsilon[1 + \rho + \cdots + \rho^{n-1}]$$

where $\rho = r_k + |z_k|$. Thus if

$$\epsilon < M_k/[1 + \cdots + \rho^{n-1}] \quad ,$$

$|F(z)| > |F_\varepsilon(z) - F(z)|$ on C_k, and the result follows by Rouché's Theorem.

This theorem shows that the roots of a polynomial are in a very general way continuous functions of the coefficients.

Theorem 5.1.3: *Let* L *be a line on which* $F(z)$ *has no zeros,* $\Delta_L \arg F(z)$ *be the change in* $\arg F(z)$ *as* z *varies along* L *in a specific direction,* p *be the number of zeros of* $F(z)$ *on the right of* L *and* q *be the number of zeros of* $F(z)$ *on the left of* L. *Then*

$$q - p = \frac{1}{\pi} \Delta_L \arg F(z) \quad ,$$

and

$$p = \tfrac{1}{2}[n - \frac{1}{\pi} \Delta_L \arg F(z)] \quad ,$$

$$q = \tfrac{1}{2}[n + \frac{1}{\pi} \Delta_L \arg F(z)] \quad .$$

Proof: As z varies along L for each zero z_j on the right, $\arg(z - z_j)$ has a net decrease of π, while a zero on the left has a net increase of π. The increase in $\arg F(z)$ is the sum of all these. Thus

$$q - p = \frac{1}{\pi} \Delta_L \arg F(z) \quad .$$

Using this with $q + p = n$ completes the proof.

Theorem 5.1.4: *If the zeros of*

$$F(z) = a_0 z^n + \cdots + a_n$$

have only negative real parts and $a_0, a_1, \ldots a_n$ *are all real, then* $a_0, a_1, \ldots a_n$ *all have the same sign.*

Proof: Suppose $a_0 > 0$. Then

$$F(z) = a_0 \Pi(z + z_j) \cdot \Pi((z + x_j)^2 + y_j^2)$$

where $-z_j$ are real zeros and $-x_j \pm iy_j$ are the complex zeros. The result follows from noting that $F(z)$ consists of factors all having only positive signs.

The converse is not true.

5.2 Hurwitz Criterion: Let

$$F(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

have real coefficients with $a_i > 0$, $i = 1,2, \ldots n$. Let

$$G(z) = a_0 z^n + a_2 z^{n-2} + \cdots ,$$

$$H(z) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots .$$

Then since

$$(-1)^n F(-z) = a_0 z^n - a_1 z^{n-1} \cdots (-1)^n a_n ,$$

$$G(z) = \tfrac{1}{2} [F(z) + (-1)^n F(-z)] ,$$

$$H(z) = \tfrac{1}{2} [F(z) - (-1)^n F(-z)] .$$

Definition: A polynomial is said to be stable if its zeros all lie in the left half plane.

<u>Lemma 5.2.1</u>: <u>Let</u>

$$I(z) = \frac{F(z) + (-1)^n F(-z)}{F(z) - (-1)^n F(-z)} = \frac{G(z)}{H(z)} .$$

<u>Then</u> $F(z)$ <u>is a stable polynomial if and only if</u>

1. $I(z)$ <u>is irreducible</u>,

2. $\mathrm{Re}(I(z)) > 0$ <u>when</u> $\mathrm{Re}(z) > 0$,

3. $\mathrm{Re}(I(z)) < 0$ <u>when</u> $\mathrm{Re}(z) < 0$.

<u>Proof</u>: Let

$$F(z) = \prod_{1}^{n} (z - z_j)$$

have all its zeros in the left half plane. If $G(z)$ and $H(z)$ had a common factor, then

$$F(z) + (-1)^n F(-z) = 0 ,$$

and

$$F(z) - (-1)^n F(-z) = 0 .$$

Thus $F(z) = 0$ and $F(-z) = 0$. This is impossible since $F(z)$ has zeros only in the left half plane.

If z is in the left half plane,

$$|F(z)| < |F(-z)| ,$$

since each term $|z - z_k|$ of $|F(z)|$ is less than each term $|z + z_j|$ of $|F(-z)|$. Now

$$I(z) = \frac{F(z) + (-1)^n F(-z)}{F(z) - (-1)^n F(-z)} \cdot \frac{\overline{F(z)} - (-1)^n \overline{F(-z)}}{\overline{F(z)} - (-1)^n \overline{F(-z)}} ,$$

$$= \frac{|F(z)|^2 - |F(-z)|^2 + (-1)^n F(-z)\,\overline{F(z)} - (-1)^n F(z)\,\overline{F(-z)}}{|F(z) - (-1)^n F(-z)|^2} ,$$

So

$$\mathrm{Re}(I(z)) = \frac{|F(z)|^2 - |F(-z)|^2}{|F(z) - (-1)^n F(-z)|^2} .$$

The remainder being imaginary, since $F(-z)\,\overline{F(z)} - F(z)\,\overline{F(-z)}$ is purely imaginary. Thus $\mathrm{Re}(I(z)) < 0$.

Similarly, if z is in the right half plane, $\mathrm{Re}(I(z)) > 0$.

Conversely, let conditions 1, 2 and 3 be satisfied. Then $|F(z)| > |F(-z)|$ when $\mathrm{Re}(z) > 0$, $|F(z)| < |F(-z)|$ when $\mathrm{Re}(z) < 0$, and $F(z)$ and $F(-z)$ have no common zeros. If $F(z)$ had a zero in the right half plane, then $0 > |F(-z)| > 0$, a contradiction. If $F(z)$ had a zero on the imaginary axis, then in a neighborhood of that point in the right half plane $|F(z)| < |F(-z)|$, which again gives a contradiction.

Lemma 5.2.2: $F(z)$ is a stable polynomial if and only if the poles of $I(z)$ lie on the imaginary axis, are simple, and the residues at these poles are positive.

Proof: If f is stable, then by Lemma 5.2.1 $|F(z)| \neq |F(-z)|$ off the imaginary axis, and hence $H(z) \neq 0$. If $(z - z_j)^k$ is a zero of $H(z)$ on the imaginary axis, then

$$I(z) = \frac{1}{(z - z_j)^k} [A + B(z - z_j) + \cdots + \varphi(z)(z - z_j)^k]$$

in a neighborhood of z_j where $A \neq 0$ and $\varphi(z)$ is analytic at z_j. Let $z - z_j = \rho e^{i\theta}$, $\alpha = \arg A$. Then

$$\arg I(z) = \alpha - k\theta + \arg\left[1 + \frac{B}{A}(z - z_j) + \cdots + \frac{\varphi(z)}{A}(z - z_j)^k\right] ,$$

and as $\rho \to 0$, $\arg I(z) \to \alpha - k\theta$. Now when

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} ,$$

$\mathrm{Re}(I(z)) > 0$, so

$$-\frac{\pi}{2} < \alpha - k\theta < \frac{\pi}{2} .$$

From this we conclude

$$\frac{1}{k}\left(-\frac{\pi}{2} + \alpha\right) < \theta < \frac{1}{k}\left(\frac{\pi}{2} + \alpha\right)$$

and

$$\frac{1}{k}\left(-\frac{\pi}{2} + \alpha\right) = -\frac{\pi}{2} ,$$

$$\frac{1}{k}\left(\frac{\pi}{2} + \alpha\right) = \frac{\pi}{2} .$$

Or

$$-\frac{\pi}{2}(1 - k) + \alpha = 0 ,$$

$$\frac{\pi}{2}(1 - k) + \alpha = 0 .$$

From this we conclude $k = 1$ and $\alpha = 0$.

Conversely, if the poles of $I(z)$ lie on the imaginary axis, are simple, and have positive residues, then

$$I(z) = \frac{a_0}{a_1} z + B + \sum_{j=1}^{n-1} \frac{A_j}{z - z_j} , \qquad A_j > 0 .$$

Letting $z \to \infty$ shows that $B = 0$, thus

$$I(z) = \frac{a_0}{a_1} z + \sum_{j=1}^{n-1} \frac{A_j}{z - z_j} .$$

Clearly if $Re(z) > 0$, $Re(I(z)) > 0$, and if $Re(z) < 0$, $Re(I(z)) < 0$. Since the poles are all simple, if $I(z)$ were reducible this would mean that as z approached the zero common to $G(z)$ and $H(z)$, $I(z)$ would approach a finite limit. This is equivalent to saying that the residue is zero, which is a contradiction. Thus $I(z)$ is irreducible.

Let

$$J(z) = I(z) - \frac{a_0}{a_1} z = \sum_{j=0}^{n-1} \frac{A_j}{z - z_j} ,$$

$$= \frac{a_1 G(z) - a_0 z H(z)}{a_1 H(z)}$$

($a_1 \neq 0$ since $-a_1$ is the sum of the zeros of $F(z)$).

Let

$$I_1(z) = 1/J(z) .$$

Let

$$F_1(z) = a_1 H(z) + a_1 G(z) - a_0 z H(z) .$$

Lemma 5.2.3: $F_1(z)$ is related to $I_1(z)$ in the same way as $F(z)$ is to $I(z)$. $I_1(z)$ satisfies the conditions of Lemma 5.2.1 if and only if $I(z)$ does.

Proof: The first part is trivial. Since $I(z) = J(z) + (a_0/a_1)z$ the proof in Lemma 5.2.2 shows that $I(z)$ satisfies the conditions of Lemma 1 if and only if $J(z)$ does. If

$$J(z) = R(\cos\varphi + i\sin\varphi) ,$$

then

$$I_1(z) = R^{-1}(\cos\varphi - i\sin\varphi) ,$$

so $\mathrm{Re}(I_1(z))$ and $\mathrm{Re}(J(z))$ have the same sign. Thus $\mathrm{Re}(I(z))$ and $\mathrm{Re}(I_1(z))$ have the same sign.

Thus we have shown

Lemma 5.2.4: Let

$$F(z) = G(z) + H(z) ,$$

where

$$G(z) = a_0 z^n + a_2 z^{n-2} + \cdots ,$$

$$H(z) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots$$

and

$$F_1(z) = a_1 H_1(z) + (a_1 G(z) - a_0 z H(z)) .$$

Then $F(z)$ is a stable polynomial if and only if $F_1(z)$ is stable.

We note that

$$F(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n ,$$

$$\begin{aligned} F_1(z) &= a_1^2 z^{n-1} + (a_1 a_2 - a_0 a_3) z^{n-2} \\ &+ a_1 a_3 z^{n-3} + (a_1 a_4 - a_0 a_5) z^{n-4} \\ &+ a_1 a_{2m-1} z^{n-2m+1} + (a_1 a_{2m} - a_0 a_{2m+1}) z^{n-2m} \cdots . \end{aligned}$$

If we let

$$\Delta_k = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & & \\ a_3 & a_2 & a_1 & a_0 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_4 & & \\ \vdots & \vdots & \vdots & \vdots & & \\ & & & & \cdots & a_k \end{vmatrix}$$

and

$$\delta_k = \begin{vmatrix} a_1 a_2 - a_0 a_3 & a_1^{\,2} & 0 & & \cdots & \\ a_1 a_4 - a_0 a_5 & a_1 a_3 & a_1 a_2 - a_0 a_3 & a_1 & 0 & \cdots \\ \vdots & \vdots & a_1 a_4 - a_0 a_5 & a_3 & & \\ a_1 a_{2m} - a_0 a_{2m+1} & \vdots & \vdots & & & \\ \vdots & & & & & b_k \end{vmatrix} ,$$

where $b_k = a_1 a_{k+1}$ when k is even or $b_k = a_1 a_{k+1} - a_0 a_{k+2}$ when k is odd, we have

Lemma 5.2.5:

$$\delta_k = a_1^{k-1} \Delta_{k+1} .$$

Proof:

$$a_0 a_1 \delta_k = \begin{vmatrix} a_0a_1 & 0 & 0 & \cdots & & & \\ a_0a_3 & a_1a_2 - a_0a_3 & a_1^2 & 0 & & \cdots & \\ a_0a_5 & a_1a_4 - a_0a_5 & a_1a_3 & a_1a_2 - a_0a_3 & a_1^2 & 0 & \cdots \\ \vdots & \vdots & a_1a_5 & a_1a_4 - a_0a_5 & a_1a_3 & & \\ & & \vdots & \vdots & \vdots & & \\ & & & & & & b_k \end{vmatrix} ,$$

where a new row and column have been added. To the second column we add the first column. To the fourth column we add a_0/a_1 times the third column, etc. Factoring a_1 from each column completes the proof.

Theorem 5.2.6: (Hurwitz Criterion). If

$$F(z) = a_0 z^n + \cdots + a_n$$

and $a_0 > 0$, then $F(z)$ is a stable polynomial if and only if

$$\Delta_1 > 0, \Delta_2 > 0 \ldots \Delta_n > 0 .$$

Proof: If $n = 1$ the theorem is certainly true. Suppose the theorem is true for $n - 1$.

If $F(z)$ is stable, so is $F_1(z)$, and thus

$$\delta_1 > 0,\ \delta_2 > 0 \ \dots\ \delta_{n-1} > 0 \quad .$$

From Lemma 5 we see

$$\Delta_2 > 0,\ \Delta_3 > 0 \ \dots\ \Delta_n > 0 \quad .$$

Since $\Delta_1 = a_1$ and $-a_1/a_0$ is the sum of the zeros of $F(z)$, $\Delta_1 > 0$ also.

Conversely, if

$$\Delta_1 > 0 \ \dots\ \Delta_n > 0 \quad ,$$

then

$$\delta_1 > 0 \ \dots\ \delta_{n-1} > 0$$

and $F_1(z)$ is stable. By Lemma 4 $F(z)$ is stable. Hence the theorem is true for n, and the induction is complete.

Hurwitz' criterion has been extended to polynomials with complex coefficients. The proof and statement of the criterion are more complicated. (See Marden, "The Geometry of the Zeros of a Polynomial in a Complex Variable".)

5.3 Routh Criterion: Let

$$F(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n \quad ,$$

where a_i is real for $i = 1,2, \dots n$ and $a_0 > 0$. If $z = iy$, then

$$\tan(\arg F(iy)) = -\frac{a_1 y^{n-1} - a_3 y^{n-3} + \cdots}{a_0 y^n - a_2 y^{n-2} + \cdots}$$

when n is even, and

$$\tan(\arg F(iy) - \pi/2) = -\frac{a_1 y^{n-1} - a_3 y^{n-3} + \cdots}{a_0 y^n - a_2 y^{n-2} + \cdots}$$

when n is odd. Let $\{y_i\}_1^n$ be the points where $a_o y^n - a_2 y^{n-2} + \cdots$ is zero, and let $\psi = \arg F(iy)$ when n is even, $\psi = \arg F(iy) - \pi/2$ when n is odd.

Lemma 5.3.1: *Let e be the number of times $\tan\psi$ changes from positive values to negative values at the points $\{y_i\}$ minus the number of times $\tan\psi$ changes from negative to positive values at the points $\{y_i\}$ as y varies from $-\infty$ to ∞. If F(z) has no zeros on the imaginary axis, then*

$$q - p = e ,$$

where p is the number of zeros with positive real parts and q is the number of zeros with negative real parts.

Proof: Let us consider the graph of $\tan\psi$ as y varies from $-\infty$ to ∞ and determine what happens to ψ.

At two consecutive points y_i and y_{i+1}, where the sign of $\tan\psi$ changes, the following occurs. If the changes are + to - and + to -, then $\Delta\psi$ increases by π over (y_i, y_{i+1}). If + to - and - to +, or if - to + and + to -, then $\Delta\psi = 0$. If - to + and - to +, then $\Delta\psi = -\pi$. In these cases Δe is either 1, 0, 0 or -1. Since

$$e = \Sigma \Delta e = \frac{1}{\pi} \Sigma \Delta \psi = \frac{1}{\pi} \Delta_L \psi \quad ,$$

and since

$$\frac{1}{\pi} \Delta_L \psi = \frac{1}{\pi} \Delta_L \arg F(iy) \quad ,$$

we see by Lemma 5.1.3 that $q - p = e$. The points where $\tan \psi$ does not change sign can be ignored since they contribute nothing to $\Delta\psi$.

Corollary:

$$p = \tfrac{1}{2}(n - e) \quad , \qquad q = \tfrac{1}{2}(n + e) \quad .$$

Lemma 5.3.2: At the points $\{y_i\}$ where

$$a_0 y^n - a_2 y^{n-2} + \cdots = 0 \quad ,$$

$\tan \psi$ changes from positive to negative values as y increases when

$$a_0 y^n - a_2 y^{n-2} + \cdots$$

and

$$a_1 y^{n-1} - a_3 y^{n-3} + \cdots$$

change from unlike to like signs, and from negative to positive values when

$$a_0 y^n - a_2 y^{n-2} + \cdots$$

and

$$a_1 y^{n-1} - a_3 y^{n-3} + \cdots$$

change from like to unlike signs.

Proof: This follows from observing that

$$\tan\psi = -\frac{a_1 y^{n-1} - a_3 y^{n-3} + \cdots}{a_0 y^n - a_2 y^{n-2} + \cdots} \quad .$$

We now form the following sequence. Let

$$f_1(y) = a_0 y^n - a_2 y^{n-2} + \cdots \quad ,$$

$$f_2(y) = a_1 y^{n-1} - a_3 y^{n-3} + \cdots \quad ,$$

$$f_3(y) = f_2(y) \cdot \frac{a_0}{a_1} y - f_1(y) \quad ,$$

$$= \frac{a_1 a_2 - a_0 a_3}{a_1} y^{n-2} - \frac{a_1 a_4 - a_0 a_3}{a_1} y^{n-4} + \cdots \quad ,$$

$$f_4(y) = f_3(y) \cdot \frac{a_1^{\,2}}{a_1 a_2 - a_0 a_3} y - f_2(y) \quad ,$$

$$= b_3 y^{n-3} - b_5 y^{n-5} + \cdots \quad ,$$

$$f_5(y) = f_4(y) \cdot \frac{a_1 b_3}{a_1 a_2 - a_0 a_4} y - f_3(y) \quad ,$$

$$= c_4 y^{n-4} - c_6 y^{n-6} + \cdots \quad ,$$

. . . .

$$f_{n+1}(y) = K \quad ,$$

a constant.

Note the degree of $f_m = n - m + 1$. All terms of f_m are even or odd depending upon where $n - m + 1$ is even or odd.

We will temporarily assume that none of the initial coefficients is zero and that no two consecutive functions f_m vanish for the same value of y. Under these assumptions, we have

Lemma 5.3.3: As y increases, except for $m = 1$, if one of the functions f_m changes sign, the total number of sign changes in $f_1, f_2, \dots f_{n+1}$ remains unchanged.

Proof: If f_m changes sign at y_0, then $f_m(y_0) = 0$. At y_0 then

$$f_{m+1}(y_0) + f_{m-1}(y_0) = 0 ,$$

so $f_{m+1}(y_0)$ and $f_{m-1}(y_0)$ differ in sign. The change in sign of f_m does not affect the total.

Lemma 5.3.4: The number of sign changes in $f_1, f_2, \dots f_{n+1}$ increases when f_1 changes sign so that the signs of f_1 and f_2 change from like to unlike signs, and decreases when f_1 changes sign so that the signs of f_1 and f_2 change from unlike to like signs.

Proof: This is obvious.

Lemma 5.3.5: The number of sign changes in $f_1 \dots f_{n+1}$ increases when $\tan \psi$ changes from negative to positive values and decreases when $\tan \psi$ changes from positive to negative values at the points $\{y_i\}$.

Proof: This follows from the definition of $\tan \psi$ and Lemma 5.3.4.

Lemma 5.3.6: The increase in the number of sign changes of $f_1, f_2, \ldots f_{n+1}$ as y varies from $-\infty$ to ∞ is equal to $-e$.

Proof: This follows from Lemmas 5.3.1 and 5.3.5.

Theorem 5.3.7: If none of the leading coefficients $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$ of $f_1(y), f_2(y), \ldots f_{n+1}(y)$ vanishes and no two consecutive functions $f_m(y)$ and $f_{m+1}(y)$ vanish at the same time, then the number of sign changes in $\alpha_1 \ldots \alpha_{n+1}$ is equal to the number of zeros of $F(z)$ with positive real parts.

Proof: For very large positive y the sign of f_m is determined by α_m, and for large negative y the sign of f_m is $(-1)^{n-m+1}\alpha_m$. Hence the number of sign changes in $f_1, f_2, \ldots f_{n+1}$ is given by the number of sign changes in $\alpha_{n+1}, \alpha_n, \ldots \alpha_1$ when y is large and positive and by the number of sign changes in $\alpha_{n+1}, -\alpha_n, \alpha_{n-1}, \ldots (-1)^n\alpha_1$ when y is large and negative. We see that every sign change in the first sequence becomes a sign similarity in the second and vice-versa. Thus if the first sequence has k changes in sign and $n - k$ similarities, the second has $n - k$ similarities and k changes. The net increase in sign changes is $k - (n - k) = 2k - n$. By Lemma 5.3.6 $e = n - 2k$. The Corollary of Lemma 5.3.1 shows that p, the number of zeros of $F(z)$ with positive real parts, is k.

Corollary: (Routh Criterion). Under the conditions of Theorem 5.3.7, $F(z)$ is a stable polynomial if and only if the initial coefficients $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$ of $f_1, f_2, \ldots f_{n+1}$ are all positive.

The initial coefficients can be easily calculated. Let

$$f_m(y) = \alpha_m y^{n-m+1} - \beta_m y^{n-m-1} + \gamma_m y^{n-m-3} - \cdots .$$

Then dividing $f_m(y)$ by $f_{m+1}(y)$ we find

$$f_{m+2}(y) = \frac{\alpha_{m+1}\beta_m - \alpha_m\beta_{m+1}}{\alpha_{m+1}} y^{n-m-1} - \frac{\alpha_{m+1}\gamma_m - \alpha_m\gamma_{m+1}}{\alpha_{m+1}} y^{n-m-3} + \cdots .$$

Thus

$$\alpha_{m+2} = \frac{\alpha_{m+1}\beta_m - \alpha_m\beta_{m+1}}{\alpha_{m+1}} ,$$

$$\beta_{m+2} = \frac{\alpha_{m+1}\gamma_m - \alpha_m\gamma_{m+1}}{\alpha_{m+1}} ,$$

etc. If we form the following array

$$\begin{array}{lllll}
\alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \cdots \\
\alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \cdots \\
\vdots & & & & \\
\alpha_n & \beta_n & & & \\
\alpha_{n+1} & & & &
\end{array}$$

we see that any row can be computed from the two preceding it by cross multiplying first up and to the right and then subtracting the multiple down and to the right, then dividing by the initial coefficient of the lower row.

$\nearrow - \searrow \div$ initial term.

Of course, $\alpha_1 = a_0$, $\alpha_2 = a_1$, $\beta_1 = a_3$, $\beta_4 = a_4$, $\gamma_1 = a_5$, $\gamma_2 = a_6$, etc.

If an initial coefficient in one of the functions $f_m(y)$ is zero while the remaining coefficients in the function are not all zero, it is sometimes possible to apply the test to $(z + a)F(z)$, where a is chosen almost arbitrarily.

To clear up what happens when one of the functions vanishes identically, we present

Theorem 5.3.8: 1. <u>If two consecutive functions $f_m(y)$ ever vanish at a point y_0, then all that are defined at y_0 do also. In particular $f_{n+1} \equiv 0$ if it is defined</u>.

2. <u>$f_1(y)$ and $f_2(y)$ have a common factor if and only if there exists a function $f_m(y) \equiv 0$.</u>

3. <u>$f_1(y)$ and $f_2(y)$ have a common factor if and only if $F(z)$ has at least two zeros which are equal but opposite in sign</u>.

Proof: 1 and 2 follow easily from the definition of the sequence $\{f_m(y)\}$. To prove 3 we note

$$i^n f_1(-iz) = a_0 z^n + a_2 z^{n-2} + \cdots ,$$

$$i^{n-1} f_2(-iz) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots .$$

These have a common factor if and only if they vanish seperately. This can happen if and only if $F(z) = 0$ and $F(-z) = 0$.

If $f_m(y)$ is the last of the sequence of functions which is not identically zero, then $i^m f_m(-z)$ is a factor common to both $a_0 z^n + a_2 z^{n-2} + \cdots$ and $a_1 z^{n-1} + a_3 z^{n-3} + \cdots$. To find the nature of the zeros of $F(z)$ we can write

$$F(z) = i^m f_m(-z) \cdot R(z)$$

and consider these two functions $i^m f(-z)$ and $R(z)$ separately.

Note that zeros occurring on the imaginary axis occur as pairs which are equal but opposite in sign.

Note that there can be no common factor if all the zeros of $F(z)$ lie in the left half plane.

To find the number of zeros of $i^m f(-iz)$ on the imaginary axis it is convenient to use Sturm's Theorem. (See Conkwright, "Theory of Equations".)

5.4 Michailov Criterion: Let

$$F(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

have real coefficients with $a_0 > 0$. Let

$$X(y) = a_n - a_{n-2} y^2 + \cdots ,$$

$$Y(y) = a_{n-1} y - a_{n-3} y^3 + \cdots .$$

Finally let

$$V(y) = X(y) + iY(y) .$$

Theorem 5.4.1: Let y vary from 0 to ∞. If $V(y_0) = 0$ then $\pm iy_0$ are zeros of $F(z)$. Thus $F(z)$ contains at least two zeros (unless $y_0 = 0$) for each time $V(y) = 0$.

If $V(y)$ is never zero, then $V(y)$ varies through $n - 2p$ quadrants where p is the number of zeros of $F(z)$ with positive real parts.

Proof: As y varies from 0 to ∞, each real zero of $F(z)$ contributes $\pi/2$ or $-\pi/2$ to the increase in $\arg F(z)$, depending upon whether the zero is negative or positive. Each pair of complex zeros contributes π or $-\pi$ depending upon whether the real part of these zeros is negative or positive. Since $F(iy) = V(y)$, if $F(z)$ contains p zeros with positive real parts, the net increase in $\arg F(z)$ is $(n - p)\pi/2 - p\pi/2$, or $(n - 2p)\pi/2$, and $V(y)$ varies through $n - 2p$ quadrants.

Corollary: $F(z)$ is a stable polynomial if and only if $V(y)$ varies through n quadrants as y increases from 0 to ∞.

Theorem 5.4.2: $F(z)$ is a stable polynomial if and only if the zeros of $X(y)$ and $Y(y)$ are all real and separate each other.

Proof: Clearly if z is a zero of X or Y, then $-z$ is also. If $F(z)$ is stable, then $V(y)$ varies through n quadrants. Hence Y is initially zero, then X is, then Y is, etc., n times.

If $F(z)$ is unstable, then V cannot vary through n quadrants. Hence either X or Y has complex zeros or the zeros do not separate each other.

This criterion is thus graphical. To test for stability, the graph of V or the graphs of X and Y are drawn, and whether or not the conditions for stability are satisfied is observed by inspection.

5.5 Meerov Criterion for Aperiodic Stability: Sometimes it is desirable to know if a polynomial has only zeros with negative real parts. To find out, the following method has been derived.

Lemma 5.5.1: If $F(z) = a_0 z^n + a_1 z^{n-1} + \cdots$ has only simple zeros, then $F(z)$ has only zeros which are real and negative if and only if $\mathcal{F}(z) = F(z^2) + zF'(z^2)$ has only zeros with negative real parts.

<u>Proof</u>: To $\mathcal{F}(z)$ we apply the Michailov criterion. $\mathcal{F}(z)$ has only zeros with negative real parts if and only if the zeros of $\operatorname{Re} \mathcal{F}(iy) = F(-y^2)$ and $\operatorname{Im} \mathcal{F}(iy) = yF'(-y^2)$ are all real and separate each other.

When $F(z)$ has only simple zeros, it is easy to see that $F(-z)$ has all its zeros on the positive real axis if and only if the zeros of $F(-z)$ and $F'(-z)$ are all real, positive and separate each other.

By taking square roots it is obvious that the zeros of $F(-z)$ and $F'(-z)$ are all real, positive and separate each other if and only if the zeros of $F(-y^2)$ and $yF'(-y^2)$ are all real and separate each other.

<u>Theorem 5.5.2</u>: (<u>Meerov Criterion). If the zeros of</u> $F(z)$ <u>are all simple</u>, <u>then the zeros of</u> $F(z)$ <u>are all real and negative if and only if</u>

1. <u>the coefficients of</u> $F(z)$ <u>are all positive</u>.
2. <u>The Hurwitz determinants for</u> $\mathcal{F}(z) = F(z^2) + zF'(z^2)$ <u>are all positive</u>.

<u>Proof</u>: This follows directly from Lemma 5.5.1.

To see that the theorem is not true if $F(z)$ contains multiple zeros, we only need to try $F(z) = (z + 2)^2$. Here $\Delta_1 = 2$, $\Delta_2 = 4$, $\Delta_3 = 0$, $\Delta_4 = 0$.

Multiple zeros can be reduced to simple zeros by dividing $F(z)$ by the factor common to $F(z)$ and $F'(z)$.

EXERCISES

1. Establish stability criteria for first, second, third, fourth, fifth and sixth degree polynomials.

2. Establish a relationship between the Hurwitz determinants and the initial coefficients in the Routh criterion.

3. Investigate Sturm's theorem and Sturm's functions concerning the number of real roots of a polynomial in an interval.

4. Give an example of a polynomial whose coefficients are all positive but which has zeros in the left half plane.

5. Extend Michailov's criterion to cover polynomials with complex coefficients.

1. F.R. Alex, "A Survey of the Russian Control Literature", Convair Report ZU-044, Part 1.

2. N.B. Conkwright, "Introduction to the Theory of Equations", Ginn and Company, New York, 1941.

3. Adolph Hurwitz, "Über die Bedingunen, unter welchen eine Gleichung nur Wurzeln mit negativen reelen Theilen besitzt", Math. Annalen, Vol. 46, pp. 273-284, 1895.

4. Wilfred Kaplan, "Operational Methods for Linear Systems", Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962.

5. Morris Marden, "The Geometry of the Zeros of a Polynomial in a Complex Variable", American Math. Society, New York, 1949.

6. M.W. Meerov, "Principles of Control of Electrical Machinery", Gosisdat, Moscow, 1952.

7. A.V. Michailov, "Harmonic Analysis in the Theory of Automatic Control", Automatika i Telemkhanika, Moscow, 1938.

8. E.P. Popov, "The Dynamics of Automatic Control Systems", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

9. E.J. Routh, "Dynamics of a System of Rigid Bodies, Part 2", MacMillan and Co., London, 1892.

VI. Stability Techniques for Feedback Systems

6.1: In many instances one or more parameters in a system may be varied. The problem of stability then is resolved into one of choosing those parameters so that the roots lie in the left half plane.

One particular type of system where parameter variation frequently occurs is a feedback system, which is best described in terms of transfer functions. Before, the input I was operated on by a transfer function $F(s) = h(s)/g(s)$ to give the output 0. That is,

$$0 = [h(s)/g(s)]\, I \quad .$$

In a feedback system, however, the output 0 is "fed back" and compared to form an error term. Specifically, 0 is multiplied by $e^{i\theta}$ and added to I to form an error term E.

$$E = I + e^{i\theta} 0 \quad .$$

In this situation $\theta = \pi$ is called negative feedback while $\theta = 0$ is positive feedback. E is then operated on by a transfer function of the form $F(s) = Kh(s)/g(s)$, where K is the parameter to be chosen, to yield 0.

$$0 = [Kh(s)/g(s)]\, E \quad .$$

If E is eliminated from these equations, we find

$$0 = [Kh(s)/(g(s) - Ke^{i\theta}h(s))]\, I \quad ,$$

which is a stable system if and only if the zeros of $g(z) - Ke^{i\theta}h(z)$ lie in the left half plane with simple zeros perhaps lying on the imaginary axis.

$$Kh(s)/[g(s) - Ke^{i\theta}h(s)]$$

is called the closed loop transfer function while

$$Kh(s)/g(s)$$

is the open loop transfer function.

The techniques which follow are concerned with the zeros of a polynomial of the form

$$F(z) = g(z) - Ke^{i\theta}h(z) ,$$

where θ is fixed, and K varies over non-negative values.

6.2. Nyquist Criterion: Let

$$F(z) = g(z) - Ke^{i\theta}h(z) ,$$

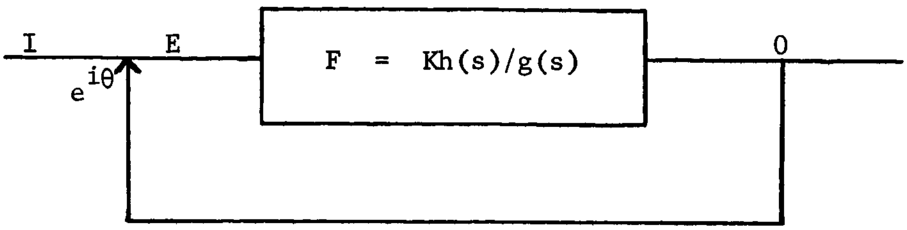
where

$$g(z) = z^n + az^{n-1} + \cdots ,$$

$$h(z) = z^m + bz^{m-1} + \cdots ,$$

with no common factors, $n \geq m$, θ and K are real constants.

Theorem 6.2.1: *Let* R *be any number large enough so that a circle of radius* R *centered at the origin contains the zeros of* $F(z)$ *and* $g(z)$. *Let* C_R *be a semi-circular contour going along the imaginary axis from* $-R$ *to* R *avoiding the zeros of* $g(z)$ *on the imaginary axis by arbitrarily small semi-circles centered at those zeros, then along the circle from* $(0,R)$ *to* $(0,-R)$ *in a clockwise manner. Then the number of times* $Kh(z)/g(z)$ *passes through* $e^{-i\theta}$ *as* z *varies around* C_R *is equal to the number of imaginary zeros of* $F(z)$. *If* $F(z)$ *has no imaginary zeros, let* P *be the number of*



A Feedback System

zeros of $g(z)$ with positive real parts, Z be the number of zeros of $F(z)$ with positive real parts, N be the number of counter-clockwise encirclements of $e^{-i\theta}$ by $Kh(z)/g(z)$ as z varies around C_R. Then $Z = P - N$.

Proof: If $Kh(z)/g(z)$ passes through $e^{-i\theta}$, then for some z_0

$$Kh(z_0)/g(z_0) = e^{-i\theta} .$$

Then

$$g(z_0) - Ke^{i\theta}h(z_0) = 0 .$$

Since this process is reversible, the result follows.

If $F(z)$ has no imaginary zeros, the number of encirclements of the origin by $F(z)/g(z)$ is $P - Z$, since the contour is taken in a clockwise manner. This is the same as the number of encirclements of the origin by

$$e^{-i\theta}F(z)/g(z) = e^{-i\theta} - Kh(z)/g(z) ,$$

which is the same as the number of encirclements of $e^{-i\theta}$ by $Kh(z)/g(z)$.

Corollary: Under the conditions of Theorem 6.2.1 a necessary and sufficient condition that $F(z)$ be a stable polynomial is that $N = P$.

Since $n \geq m$, $Kh(z)/g(z)$ approaches a finite limit, either K or 0, as z becomes large. Thus the contour may be closed by letting R approach infinity.

If $g(z)$ has no zeros on the imaginary axis, $Kh(z)/g(z)$ remains in a bounded portion of the complex plane.

The theorem is still valid if the zeros of $g(z)$ on the imaginary axis are by-passed by small semi-circles in the right half plane.

If $\theta = 0$ or π and $g(z)$ and $h(z)$ have real coefficients, then the path of $Kh(z)/g(z)$ is symmetric with respect to the real axis, the path when $Im(z) > 0$ being the image of that when $Im(z) < 0$.

K enters into the problem as a factor only of the amplitude. The usual procedure is to draw the graph of $Kh(z)/g(z)$ when $K = 1$, then by appropriately adjusting the scale of measurement to achieve stability, an acceptable value of K is found.

M and N Circles: In conjunction with the Nyquist criterion, we note as z varies along the imaginary axis, both the amplitude and the argument of the closed loop transfer function

$$Kh(z)/[g(z) - Ke^{i\theta}h(z)]$$

change.

If $g(z) - Ke^{i\theta}h(z)$ has a zero near the imaginary axis, then the amplitude becomes relatively large along that portion of the imaginary axis nearest that zero. Hence not only is stability desirable, but, in order for the homogeneous solution to have sufficient damping, the closed loop transfer function should not become too large as z varies along the imaginary axis.

It can be shown that for the system

$$y'' + 2\delta y' + y = 1 ,$$

$$y(0) = 0 , \quad y'(0) = 0 ,$$

the error $y_e = y - 1$ satisfies

$$\int_0^\infty y_e^2 dt = \text{minimum}$$

when $\delta = \frac{1}{2}$ and that the function $|-\omega^2 + 2i\delta\omega + 1|^{-1}$ achieves a maximum when

$$\omega = \pm \sqrt{1 - 2\delta^2} \quad .$$

When $\delta = \frac{1}{2}$ this maximum is $(2/3)\sqrt{3} \approx 1.15$.

It has been found experimentally that the maximum amplitude of the closed loop transfer function should be between 1.15 and 1.5.

In a lesser way the arguments of $Kh(z)/[g(z) - Ke^{i\theta}h(z)]$ as z varies along the imaginary axis are also useful. If the input of a system is constant or sinusoidal, then the phase shift between the input and output is determined by the closed loop transfer function.

The paths of constant amplitude and constant argument of $Kh(z)/[g(z) - Ke^{i\theta}h(z)]$ in the $Kh(z)/g(z)$ plane are circles, as we now show.

Let $Kh(z)/g(z) = X + iY$, and let $|Kh(z)/[g(z) - Ke^{i\theta}h(z)]| = M$. Then dividing the numerator and denominator of the fraction by $g(z)$, we find

$$|(X + iY)/[1 - e^{i\theta}(X + iY)]| = M \quad ,$$

which reduces to

$$\left[X - \frac{M^2}{M^2 - 1} \cos\theta\right]^2 + \left[Y + \frac{M^2}{M^2 - 1} \sin\theta\right]^2 = \left[\frac{M^2}{M^2 - 1}\right]^2 \quad .$$

It is easy to show that $e^{-i\theta}$ is inside these circles as long as $M > 1$.

Let $\arg\,(Kh(z)/[g(z) - Ke^{i\theta}h(z)]) = \alpha$, and let $N = \tan\alpha$. Then

$$N = \frac{\operatorname{Im}\left(\dfrac{X + iY}{1 - e^{i\theta}[X + iY]}\right)}{\operatorname{Re}\left(\dfrac{X + iY}{1 - e^{i\theta}[X + iY]}\right)} ,$$

$$N = \frac{Y + \sin\theta\,(X^2 + Y^2)}{X - \cos\theta\,(X^2 + Y^2)} ,$$

which reduces to

$$\left[X - \frac{N}{2(\sin\theta + N\cos\theta)}\right]^2 + \left[Y + \frac{1}{2(\sin\theta + N\cos\theta)}\right]^2 = \frac{N^2 + 1}{4(\sin\theta + N\cos\theta)^2} .$$

These circles can be superimposed on a Nyquist diagram to aid in determining an appropriate choice for the parameter K.

6.3 The Root Locus Method: Let

$$F(z) = g(z) - Ke^{i\theta}h(z) ,$$

where

$$g(z) = z^n + az^{n-1} + \cdots ,$$

$$h(z) = z^m + bz^{m-1} + \cdots ,$$

with no common factors, $n \geq m$, θ and K are real parameters and θ is fixed. If K varies, the zeros of $F(z)$ vary in a continuous manner in the complex plane. Let us consider what happens. If

$$g(z) - Ke^{i\theta}h(z) = 0 ,$$

then

$$\frac{Kh(z)}{g(z)} = e^{-i\theta}$$

and

$$\arg h(z) - \arg g(z) = -\theta - \arg K \pm 2k\pi$$

where k is an integer. If

$$g(z) = \prod_{i=1}^{m} (z - p_i)^{\alpha_i} ,$$

$$h(z) = \prod_{i=1}^{n} (z - z_i)^{\beta_i} ,$$

then

$$\sum_{i=1}^{m} \beta_i \arg (z - z_i) - \sum_{i=1}^{n} \alpha_i \arg (z - p_i) = -\theta - \arg K \pm 2k\pi .$$

Thus the locus of the zeros of $F(z)$ for all values of K consist of those points z where the arguments just defined equal $-\theta - \arg K \pm 2k\pi$. With the aid of a device called a Spirule* this locus can be quickly found and a suitable choice of the parameter K easily chosen. The theorems which follow give guidance in constructing the "root-locus".

Definition: The root-locus of $F(z)$ is the set of all points z such that z is a zero of $h(z)$ or for which there is a real number K, $-\infty < K < \infty$, such that $F(z) = 0$.

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*Write The Spirule Company, 9728 El Venado, Whittier, California.

<u>Definition</u>: <u>The positive root-locus of</u> $F(z)$ <u>is the set of all points</u> z <u>such that</u> z <u>is a zero of</u> $h(z)$ <u>or for which there is a real number</u> K, $0 \leq K < \infty$, <u>such that</u> $F(z) = 0$.

The zeros of $h(z)$ are included in the root-locus, since they are limit points of the zeros of $F(z)$ for all appropriate choices of K, i.e. they are the zeros of $F(z)$ when $K = \infty$.

<u>Theorem 6.3.1</u>: <u>Let</u> z <u>be a point in the complex plane. The following statements are equivalent</u>.

i. z <u>is on the root-locus</u>.

ii. $\cos\theta \, \mathrm{Im}(h(z)\overline{g(z)}) + \sin\theta \, \mathrm{Re}(h(z)\overline{g(z)}) = 0.$

<u>Proof</u>: Suppose z is on the root-locus. If $g(z) \neq 0$, then for some $K \neq 0$,

$$Ke^{i\theta}h(z)/g(z) = 1 \quad .$$

Thus

$$h(z)/g(z) = K^{-1}[\cos\theta - i\sin\theta] \quad ,$$

and

$$h(z)\overline{g(z)} = K^{-1}|g(z)|^2 [\cos\theta - i\sin\theta] \quad .$$

Since K is real, equating real and imaginary parts,

$$\mathrm{Re}(h(z)\overline{g(z)}) = K^{-1}|g(z)|^2 \cos\theta \quad ,$$

$$\mathrm{Im}(h(z)g(z)) = -K^{-1}|g(z)|^2 \sin\theta \quad .$$

Multiplying the first by $\sin\theta$, the second by $\cos\theta$ and adding completes the proof. So i. implies ii.

Conversely, if ii. is satisfied, then $\mathrm{Im}(e^{i\theta}h(z)\overline{g(z)}) = 0$, so $e^{i\theta}h(z)\overline{g(z)} = R(z)$, where $R(z)$ is real. If $R(z) = 0$, then either $h(z) = 0$ or $g(z) = 0$, and z is on the root-locus. If $R(z) \neq 0$, let $K = |g(z)|^2/R(z)$. If $K = 0$, then $g(z) = 0$, and z is on the root-locus. If $K \neq 0$, then $Ke^{i\theta}h(z)/g(z) = 1$, and $g(z) - Ke^{i\theta}h(z) = 0$. So z is on the root-locus. ii. implies i.

Note that K can be found by

$$K = |g(z)|^2 \cos\theta/\mathrm{Re}(h(z)\overline{g(z)})$$

or by

$$K = -|g(z)|^2 \sin\theta/\mathrm{Im}(h(z)\overline{g(z)}) \quad .$$

More can be said if $g(z)$ and $h(z)$ have real coefficients.

Lemma 6.3.2: *If* $z = x + iy$, *then*

$$h(z) = h(x+iy) = \sum_{j=0}^{m} h^{(j)}(x)(iy)^j/j! \quad ,$$

$$\overline{g(z)} = g(x-iy) = \sum_{j=0}^{n} g^{(j)}(x)(-iy)^j/j! \quad .$$

Proof: These are MacLaurin expansions.

Lemma 6.3.3:

$$h(z)\overline{g(z)} = \sum_{j=0}^{n+m} \frac{(iy)^j}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} h^{(k)}(x) g^{(j-k)}(x) \quad ,$$

$$\mathrm{Re}(h(z)\overline{g(z)}) = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \sum_{i=0}^{2k} \binom{2k}{i} (-1)^{2k-i} h^{(i)}(x) g^{(2k-i)}(x) \quad ,$$

$$\mathrm{Im}(h(z)\overline{g(z)}) = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{2k+1-i} h^{(i)}(x) g^{(2k+1-i)}(x) \quad .$$

Proof: This follows from Lemma 6.3.2. and then separating real and imaginary parts.

Theorem 6.3.4: If $g(z)$ and $h(z)$ have real coefficients, then z is on the root-locus if and only if

$$\cos\theta \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{2k+1-i} h^{(i)}(x) g^{(2k+1-i)}(x)$$

$$+ \sin\theta \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \sum_{i=0}^{2k} \binom{2k}{i} (-1)^{2k-i} h^{(i)}(x) g^{(2k-i)}(x) = 0 \quad .$$

Theorem 6.3.5: If $g(z)$ and $h(z)$ have real coefficients, then the root-locus contains the entire real line $(y = 0)$ if and only if $\theta = 0$ or $\theta = \pi$.

Proof: If the real axis is contained in the root-locus, then $y = 0$ satisfies the equation of Theorem 6.3.4. Letting $y = 0$, we find that $\sin\theta = 0$, and $\theta = 0$ or π. The converse is trivial.

Definition: A point on the root-locus is a multiple point if for some value of K the point is a multiple zero of $F(z)$ or if the point is a multiple zero of $h(z)$.

Theorem 6.3.6: The multiple points of the root-locus are isolated, finite in number, and satisfy

$$h(z)g'(z) - g(z)h'(z) = 0 \quad .$$

Proof: If z is a multiple zero of $F(z)$, then $F(z) = 0$ and $F'(z) = 0$. Eliminating $Ke^{i\theta}$ results in the above equation, a polynomial of degree $n + m - 1$.

Theorem 3.6.7: The points on the root-locus of $F(z)$ are continuous functions of K.

Proof: This follows directly from Hurwitz's theorem. Note since the multiple points are isolated, that if a point is on the root-locus with multiplicity α, then α distinct zeros of $F(z)$ approach the point for some values of K.

Theorem 6.3.8: With the exception of the multiple points, the points on the root-locus are differentiable functions of K.

Proof: Let z_0 be a simple zero of $F(z)$ when $K = K_0$. We need to show that

$$\lim_{K \to K_0} \frac{z - z_0}{K - K_0}$$

exists when z is a point on the root-locus and $z \to z_0$ as $K \to K_0$. We have

$$0 = g(z) - Ke^{i\theta}h(z) \quad ,$$

$$0 = g(z) - K_0e^{i\theta}h(z) - (K - K_0)e^{i\theta}h(z) \quad ,$$

$$0 = (z - z_0)W(z) - (K - K_0)e^{i\theta}h(z) \quad ,$$

where $W(z_0) \neq 0$ and

$$W(z_0) = \left.\frac{dF}{dz}\right|_{\substack{z = z_0 \\ K = K_0}} \quad .$$

From this we find that

$$\lim_{K \to K_0} \frac{z - z_0}{K - K_0} = e^{i\theta}h(z_0)/[g'(z_0) - K_0e^{i\theta}h'(z_0)] \quad .$$

<u>The Positive Root-Locus</u>:

<u>Theorem 6.3.9</u>: <u>If</u> $n > m$, <u>the zeros of</u> $F(z)$ <u>become asymptotic to</u> $n - m$ <u>rays starting from</u> $-\frac{a - b}{n - m}$ <u>and passing through</u>

$$z = -\frac{a - b}{n - m} + e^{i\left(\frac{\theta+2k\pi}{n-m}\right)}$$

<u>for</u> $k = 0,1,2, \ldots n\text{-}m\text{-}1$ <u>as</u> K <u>becomes large. Furthermore</u>

$$z = -\frac{a - b}{n - m} + K^{\frac{1}{n-m}} e^{i\left(\frac{\theta+2k\pi}{n-m}\right)}$$

<u>with</u> $k = 0,1,2 \ldots n\text{-}m\text{-}1$ <u>become arbitrarily good estimates for</u> $n - m$ <u>zeros of</u> $F(z)$.

Proof: The method of proof is straight forward but rather long and goes as follows. We first perform a translation so that $-\frac{a-b}{n-m}$ is at the origin. We then choose one of the points

$$K^{\frac{1}{n-m}} e^{i\left(\frac{\theta+2k\pi}{n-m}\right)}$$

and perform a rotation so that it lies on the positive real axis. We then perform a contraction so that it lies at the point 1. We show that a zero of the resulting function lies arbitrarily near 1 and reverse the steps.

Let $z = t - \frac{a-b}{n-m}$. Then

$$F(z) = t^n + \frac{nb-ma}{n-m} t^{n-1} + \cdots - Ke^{i\theta}\left(t^m + \frac{nb-ma}{n-m} t^{m-1} + \cdots\right) .$$

Choose k. Letting $A = \frac{nb-ma}{n-m}$ and

$$t = \omega e^{i\left(\frac{\theta+2k\pi}{n-m}\right)}$$

we find after multiplication by

$$e^{-in\left(\frac{\theta+2k\pi}{n-m}\right)}$$

that $F(z) = 0$ is equivalent to

$$\omega^n + A\omega^{n-1} e^{-i\left(\frac{\theta+2k\pi}{n-m}\right)} + \sum_{j=2}^{n} \alpha_j \omega^{n-j}$$

$$- \left(K \omega^m + A\omega^{m-1} e^{-i\left(\frac{\theta+2k\pi}{n-m}\right)} + \sum_{j=2}^{m} \beta_j \omega^{m-j}\right) = 0 ,$$

where α_j and β_j are the appropriate coefficients.

Define

$$M = \sum_{j=2}^{n} |\alpha_j| (3|A| + 1)^{-j+2} (4/3)^{n-j} + \sum_{j=2}^{m} |\beta_j| (3|A| + 1)^{-j+2} (4/3)^{m-j} ,$$

$$\rho = K^{\frac{1}{n-m}} ,$$

and

$$\epsilon = \frac{3^m M}{\rho^2} = \frac{3^m M}{K^{2/(n-m)}} .$$

If K is greater than the maximum of

$$(3|A| + 1)^{n-m} , \qquad (3^{m+1} M)^{\frac{n-m}{2}}$$

and

$$[3^m M / \{\tfrac{1}{2} |1 - e^{\pm \pi i/2(n-m)}|\}]^{\frac{n-m}{2}} ,$$

then $\rho > 3|A| + 1$, $\epsilon < 1/3$ and $\epsilon < \frac{1}{2}|1 - e^{\pm \pi i/2(n-m)}|$. Letting $\xi = \omega/\rho$ and dividing by ρ^n, we find $F(z) = 0$ is equivalent to

$$\xi^n + \frac{A}{\rho} e^{-i\left(\frac{\theta+2k\pi}{n-m}\right)} \xi^{n-1} + \sum_{j=2}^{n} \alpha_j \rho^{-j} \xi^{n-j}$$

$$- \left(\xi^m + \frac{A}{\rho} e^{-i\left(\frac{\theta+2k\pi}{n-m}\right)} \xi^{m-1} + \sum_{j=2}^{m} \beta_j \rho^{-j} \xi^{m-j} \right) = 0 .$$

Thus

$$\xi^{m-1} \left(\xi + \frac{A}{\rho} e^{-i\left(\frac{\theta+2k\pi}{n-m}\right)} \right) (\xi^{n-m} - 1)$$

$$+ \frac{1}{\rho^2} \left(\sum_{j=2}^{n} \alpha_j \rho^{-j+2} \xi^{n-j} - \sum_{j=2}^{m} \beta_j \rho^{-j+2} \xi^{m-j} \right) = 0 .$$

Let

$$\varphi(\xi) = \xi^{m-1}\left(\xi + \frac{A}{\rho} e^{-i(\frac{\theta+2k\pi}{n-m})}\right)(\xi^{n-m} - 1)$$

and

$$\psi(\xi) = \frac{1}{\rho^2}\left(\sum_{j=2}^{n} \alpha_j \rho^{-j+2}\xi^{n-j} - \sum_{j=2}^{m} \beta_j \rho^{-j+2}\xi^{m-j}\right) .$$

The zeros of $\xi^{n-m} - 1$ are spread out at equal intervals on the unit circle; the first after 1 in either direction are $e^{\pm \pi i/2(n-m)}$. Consider a circle, Γ, of radius ϵ (previously defined). On Γ the real part of ξ^j for $j = 0,1, \ldots ,n-m-1$ is positive. So $|\xi^{n-m-1} + \cdots + 1| > 1$. On Γ we then have

$$|\varphi(\xi)| > (1 - \epsilon)^{m-1} \cdot \frac{1}{3} \cdot 1 \cdot \epsilon > (\frac{1}{3})^m \epsilon .$$

Also on Γ,

$$|\psi(\xi)| \leq M/\rho^2 = (1/3)^m \epsilon .$$

So on Γ, $|\varphi(\xi)| > |\psi(\xi)|$. By Rouché's Theorem $\varphi(\xi)$ and $\varphi(\xi) + \psi(\xi)$ have the same number of zeros within Γ, namely 1 zero. Thus there is a zero, ξ_j, of $\varphi(\xi) + \psi(\xi)$ such that $|\xi_j - 1| < \epsilon$. There is a zero, ω_j, of

$$\omega^n + A\omega^{n-1}e^{-i(\frac{\theta+2k\pi}{n-m})} + \cdots - K\left(\omega^m + A\omega^{m-1}e^{-i(\frac{\theta+2k\pi}{n-m})} + \cdots\right)$$

such that $|\omega_j - \rho| < \rho\epsilon = 3^m M/\rho$. Thus there is a zero, t_j, of $F(t - \frac{a - b}{n - m})$ such that

$$\left|t_j - K^{\frac{1}{n-m}} e^{i(\frac{\theta+2k\pi}{n-m})}\right| < 3^m M/K^{\frac{1}{n-m}} .$$

Finally there is a zero, z_j, of $F(z)$ such that

$$\left| z_j - \left\{ -\frac{a-b}{n-m} + K^{\frac{1}{n-m}} e^{i\left(\frac{\theta+2k\pi}{n-m}\right)} \right\} \right| < 3^m M / K^{\frac{1}{n-m}} .$$

It is evident that, for K sufficiently large, once the zero comes within a certain distance of

$$-\frac{a-b}{n-m} + K^{\frac{1}{n-m}} e^{i\left(\frac{\theta+2k\pi}{n-m}\right)} ,$$

it remains within that distance for all large K.

<u>Definition</u>: <u>The center of gravity of</u>

$$F(z) = (z^n + az^{n-1} + \cdots) - Ke^{i\theta}(z^m + bz^{m-1} + \cdots)$$

<u>when</u> $n > m$ <u>is</u>

$$-\frac{(a-b)}{(n-m)} .$$

<u>Corollary</u>: <u>When</u> $n > m$, <u>the asymptotes of the root-locus pass through the center of gravity</u>.

<u>Corollary</u>: <u>When</u> $n > m$, <u>the angles subtended by the asymptotes of the root-locus from a line parallel to the positive real axis are</u>

$$\frac{\theta + 2k\pi}{n-m}$$

<u>for</u> $k = 0, 1, \ldots, n-m-1$.

The following is attributed to Robert M. Stewart of the California Institute of Technology by W.R. Evans.

Corollary: When $n > m$ the center of gravity is given by

$$\frac{1}{n-m}[\Sigma \text{ zeros of } g(z) - \Sigma \text{ zeros of } h(z)] \quad .$$

Proof: The sum of the zeros of $g(z)$ is $-a$. The sum of the zeros of $h(z)$ is $-b$.

Theorem 6.3.10: Let

$$g(z) = \prod_j (z - z_j)^{\beta_j} \quad \text{and} \quad h(z) = \prod_j (z - p_j)^{\alpha_j}$$

where

$$\Sigma_j \alpha_j = n$$

and

$$\Sigma_j \beta_j = m \quad .$$

As K approaches 0, α_j distinct branches of the positive root-locus of $F(z)$ approach each zero, p_j, of $g(z)$. As K approaches ∞, β_j distinct branches of the positive root-locus of $F(z)$ approach each zero, z_j, of $h(z)$.

Proof: This follows directly from Hurwitz's Theorem.

Definition: The angle of departure (arrival) of the root-locus of $F(z)$ at z_0 is the angle made at z_0 by the tangent to the root-locus for increasing (decreasing) K.

Theorem 6.3.11: If p_j is a zero of $g(z)$ of order α_j, then the positive root-locus of $F(z)$ departs from p_j making angles

$$\varphi_j = (1/\alpha_j)\left(\sum_i \beta_i \arg(p_j - z_i) - \sum_{i\neq j} \alpha_i \arg(p_j - p_i) + \theta - 2k\pi\right),$$

<u>where</u> $k = 0, 1, \ldots, \alpha_j - 1$.

<u>If</u> z_j <u>is a zero of</u> $h(z)$ <u>of order</u> β_j, <u>then the positive root-locus of</u> $F(z)$ <u>arrives at</u> z_j <u>making angles</u>

$$\theta_j = (1/\beta_j)\left(\sum_i \alpha_i \arg(z_j - p_i) - \sum_{i\neq j} \beta_i \arg(z_j - z_i) - \theta + 2k\pi\right),$$

<u>where</u> $k = 0, 1, \ldots, \beta_j - 1$.

<u>Proof</u>: Consider one of the branches of the positive root-locus which departs from p_j. Choose K close to zero, and let z be on that branch for that value of K. Then

$$Ke^{i\theta} \prod_i (z - z_i)^{\beta_i} / \prod_i (z - p_i)^{\alpha_i} = 1 .$$

Taking arguments,

$$\sum \beta_i \arg(z - z_i) - \sum \alpha_i \arg(z - p_i) + \theta = 2k\pi .$$

Solving for those terms involving p_j, (or z_j), and letting K approach 0, (or ∞), completes the proof.

<u>Theorem 6.3.12</u>: <u>Let</u> z_0 <u>be any point on the real axis</u>, $h(z)$ <u>and</u> $g(z)$ <u>have real coefficients</u>, $\{z_i\}_1^r$ <u>and</u> $\{p_i\}_1^s$ <u>be the real zeros of</u> $h(z)$ <u>and</u> $g(z)$ <u>greater than</u> z_0, <u>and let</u> $\theta = 0$ $(\theta = \pi)$. <u>Then</u> z_0 <u>is contained in the positive root-locus if and only if</u>

$$\Sigma_1^r \beta_i + \Sigma_1^s \alpha_i$$

<u>is even (odd)</u>.

<u>Proof</u>: Consider $\theta = 0$. Since $g(z)$ and $h(z)$ have real coefficients, zeros of $g(z)$ and $h(z)$ occur in conjugate pairs when complex. On the real axis, if z_1 and $\bar{z}_1$ are complex conjugates, then

$$\arg (z - z_1) + \arg (z - \bar{z}_1) = 0 .$$

Now consider $Kh(z_0)/g(z_0)$. On the real axis

$$\arg [Kh(z_0)/g(z_0)] = \sum_i \beta_i \arg(z_0 - z_i) - \sum_i \alpha_i \arg(z_0 - p_i) ,$$

where the sums are taken over real zeros greater than z_0, complex zeros adding to give 0, zeros less than z_0 giving 0. Each of the arguments left equaling π.

If $\Sigma_i^r \beta_i + \Sigma_i^s \alpha_i$ is even, then $\Sigma_i^r \beta_i - \Sigma_i^s \alpha_i$ is even and z_0 is on the positive root-locus. If $\Sigma_i^r \beta_i + \Sigma_i^s \alpha_i$ is odd, then $\Sigma_i^r \beta_i - \Sigma_i^s \alpha_i$ is odd and z_0 is not on the positive root-locus. The case $\theta = \pi$ is similar.

<u>Theorem 6.3.13</u>: <u>If the coefficients of</u> $g(z)$ <u>and</u> $h(z)$ <u>are real</u>, $\theta = 0$ <u>(or</u> π), <u>and</u> $F(z)$ <u>has a zero of order</u> m <u>at</u> $z = a$ <u>on the real axis</u> <u>for</u> $K = K_0$, $0 < K_0 < \infty$, <u>then the positive root-locus arrives at</u> $z = a$ <u>making angles</u>

$$\theta_k = 2k\pi/m , \qquad k = 0,1, \ldots, m - 1 ,$$

<u>and departs making angles</u>

$$\varphi_k = (2k + 1)\pi/m , \qquad k = 0,1, \ldots, m - 1 ,$$

<u>when the number of zeros of</u> $g(z)$ <u>plus zeros of</u> $h(z)$ <u>to the right of</u> a <u>is odd (even) or arrives making angles</u>

$$\theta_k = (2k+1)\pi/m \quad , \qquad k = 0,1, \ldots, m-1 \quad ,$$

and departs making angles

$$\varphi_k = 2k\pi/m \quad , \qquad k = 0,1, \ldots m-1 \quad ,$$

when the number of zeros of $g(z)$ plus zeros of $h(z)$ to the right of a is even (odd).

Proof: Since the coefficients of $g(z)$ and $h(z)$ are real, for $\theta = 0$ (or π), if zeros of $F(z)$ leave or arrive at the real axis, they do so in conjugate pairs as K varies from 0 to K_0. The evenness or oddness of the number of zeros of $g(z)$ plus zeros of $h(z)$ to the right of $z = a$ is the same as that of $g(z) - K_0 e^{i\theta} h(z)$ and $h(z)$.

Write $F(z) = g(z) - K_0 e^{i\theta} h(z) - (K - K_0) e^{i\theta} h(z) = 0$, where z is on one of the branches of the positive root-locus near a. Let $g(z) - K_0 e^{i\theta} h(z) = (z - a)^m G(z)$ where $G(a) \neq 0$. Then $G(z)$ is real on the real axis and has the same number of zeros to the right of a as $g(z) - K_0 e^{i\theta} h(z)$. We have $(z - a)^m G(z) - (K - K_0) e^{i\theta} h(z) = 0$. Dividing by the second term and taking arguments,

$$m \arg (z - a) + \arg G(z) - \arg (K - K_0) - \theta - \arg h(z) = 2k\pi \quad .$$

$$\arg (z - a) = (1/m)[- \arg G(z) + \arg h(z) + \theta + 2k\pi + \arg (K - K_0)] \quad .$$

Now $- \arg G(z) + \arg h(z) + \theta + 2k\pi$ approaches either an even (odd) or odd (even) multiple of π as z approaches a, depending upon whether the number of zeros of $g(z)$ plus zeros of $h(z)$ is even or odd. Since $G(z)$ and $h(z)$ are non-zero and real on the real axis. Arg $(K - K_0)$ is either 0 or π depending upon whether $K > K_0$ or $K < K_0$. Letting K approach K_0

completes the proof.

6.4. J. I. Neimark's D - Partitions: The method of D-partitions is essentially an inside-out Nyquist diagram. It gives direct information concerning the acceptable values of the parameter K. We will prove several theorems outlining various properties of the method and then formally state the procedure at the end of the section. Let $g(z) = z^n + az^{n-1} + \cdots$, $h(z) = z^m + bz^{m-1} + \cdots$, $n \geq m$ and θ be a constant.

Definition: Let $K = e^{-i\theta} g(z)/h(z)$. As z varies from $-i\infty$ to $i\infty$ along the imaginary axis, K describes a series of paths, approaching ∞ only at the imaginary zeros of $h(z)$ or as $|z| \to \infty$. These paths are called the D-contour.

Theorem 6.4.1: The D-contour divides the plane up into a finite number of regions.

Theorem 6.4.2: $g(z) - Ke^{i\theta}h(z)$ has the same number of zeros in the right half plane and in the left half plane for all values of K lying within the same region.

Proof: Suppose K_0 and K_1 are in the same region but $g(z) - K_0e^{i\theta}h(z)$ and $g(z) - K_1e^{i\theta}h(z)$ do not have the same number of zeros in the two half planes. If K_0 and K_1 are connected by a simple arc lying within the region, then for some point K between K_0 and K_1 on the arc $g(z) - Ke^{i\theta}h(z)$ has an imaginary zero. Then $K = e^{-i\theta}g(z)/h(z)$ is on the D-contour and not an interior point.

<u>Theorem 6.4.3</u>: <u>The image of the left-plane under</u> $K = e^{-i\theta}g(z)/h(z)$ <u>lies locally to the left of the path taken by</u> K <u>on the</u> D-<u>contour</u>.

<u>Proof</u>: If z_0 is on the imaginary axis and $h(z_0) \neq 0$, we express K locally as a power series. If $K_0 = e^{-i\theta}g(z_0)/h(z_0)$, then

$$K - K_0 = \left[\frac{K^{(m)}(z_0)}{m!} + o(1)\right](z - z_0)^m \quad ,$$

where $K^{(m)}$ is the first non-zero derivative, and $o(1)$ can be made arbitrarily small by choosing z sufficiently close to z_0. Taking arguments, we have

$$\arg(K - K_0) = \arg(z - z_0) + \arg\left[\frac{K^{(m)}(z_0)}{m!} + o(1)\right] .$$

If $o(1)$ is small enough,

$$\arg\left[\frac{K^{(m)}(z_0)}{m!} + o(1)\right]$$

varies little, remaining close to

$$\arg\left[\frac{K^{(m)}(z_0)}{m!}\right] .$$

We thus see that a small change in the direction from z_0 to z results in m times that change taken by K. Thus that part of the complex z - plane just to the left of the imaginary axis is mapped just to the left of the D-contour in the K-plane.

We therefore shade the region just to the left of the D-contour.

<u>Theorem 6.4.4</u>: <u>Let</u> K <u>vary from</u> K_0 <u>in a region</u> D_0 <u>to</u> K_1 <u>in a region</u> D_1, <u>cutting the</u> D-<u>contour only at</u> K^*. <u>At</u> K^* <u>let there be</u> p <u>shadings in the region</u> D_0 <u>and</u> q <u>shadings in</u> D_1. <u>Then as</u> K <u>varies from</u> K_0 <u>to</u> K_1 <u>through</u> K^*, $g(z) - Ke^{i\theta}h(z)$ <u>has a net increase of</u> $p - q$ <u>zeros in the right half plane</u>.

<u>Proof</u>: Consider one of the zeros of $g(z) - Ke^{i\theta}h(z)$ which is on the imaginary axis when $K = K^*$. If K_0 is in the region which is shaded because of this zero and K_1 is not, then this zero moves from the left to right half plane as K varies from K_0 to K_1.

If K_0 is unshaded and K_1 is, the zero moves from right to left. If both are shaded, the zero stays in the left half plane. If both are unshaded, the zero stays in the right half plane. The conclusion follows from considering the net effect.

The method of D-partitions is thus as follows:

1. The graph of $K = e^{-i\theta}g(z)/h(z)$ is drawn as z varies from $-i\infty$ to $i\infty$.

2. The left side of the graph is shaded locally.

3. The number of zeros of $F(z) = g(z) - Ke^{i\theta}h(z)$ in the right half plane is determined in one of the regions, e.g. for $K = 0$.

4. The number of zeros of $F(z)$ in the right half plane in the other regions is determined from the one already determined by considering the shadings on the boundary between adjacent regions.

5. The stable situations, if any, are then found by inspection.

The power of the method of D-partitions lies in the fact that it tells directly which values of K are acceptable and which are not. While the root-locus method gives excellent information concerning the zeros of $F(z)$, it is fairly difficult to compute which values of K are acceptable for stability purposes.

The Nyquist criterion sheds more light on K than does the root-locus method, but again only indirectly.

The method of D-partitions has been extended to polynomials involving two parameters, but is more complex. See Chapter X.

EXERCISES

1. Consider the feedback system with transfer function $1/[s(s + 2\delta)]$ in the open loop with $0 \le \delta \le 1$.

a. Show that

$$\frac{0}{I} = \frac{1}{s^2 + 2\delta s + 1} .$$

b. If the input of 1 show that the output y satisfying $y(0) = 0$, $y'(0) = 0$ is

$$y = 1 + \frac{1}{(1 - \delta^2)^{\frac{1}{2}}} e^{-\delta t} \sin\left(\sqrt{1 - \delta^2}t - \psi\right)$$

where $\tan \psi = \sqrt{1 - \delta^2}/(-\delta)$.

c. If $y_e = 1 - y$, show that

$$\int_0^\infty y_e^2 dt = \frac{1 + 4\delta^2}{\delta}$$

which has a minimum when $\delta = \frac{1}{2}$.

d. Show that

$$\left|\frac{1}{-\omega^2 + 2i\delta\omega + 1}\right| = \frac{1}{[(1 - \omega^2)^2 + 4\delta^2\omega^2]^{\frac{1}{2}}}$$

is maximal when $\omega = \pm \sqrt{1 - 2\delta^2}$ and that the maximum is

$$\frac{1}{2\delta\sqrt{1 - \delta^2}} .$$

e. Show that if $\delta = \frac{1}{2}$ this maximum is $2/3\sqrt{3} \approx 1.15$. Hence the maximum of

$$\left|\frac{1}{-\omega^2 + 2i\delta\omega + 1}\right|$$

when

$$\int_0^\infty y_e^{\,2}\,dt$$

is minimal under zero initial conditions with a unit step input is 1.15.

2. If $F(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has no imaginary zeros, show that $F(D)y = \sin \omega t$ has $y = F(i\omega)^{-1} \sin \omega t$ as a solution. Thus the phase shift between the input $\sin \omega t$ and the output y is determined by the transfer function evaluated at $i\omega$.

3. The root-locus of $F(z)$ has asymptotes if $n > m$ or if $n < m$. If $n = m$, does the root-locus ever follow asymptotic lines toward ∞? If so, describe the asymptotes.

4. Show that the equations of the tangent lines of the root-locus approach the equations of the asymptotes of the root-locus as the root-locus approaches the asymptotes.

5. Prove Theorem 6.4.1. To show a countable number of components is easy, since the complex plane is separable. If a valid proof is found for showing only a finite number of components, the author would like a copy.

References

1. Lars V. Ahlfors, "Complex Analysis", McGraw-Hill Book Co., Inc., New York, 1953.

2. F.R. Alex, "A Survey of Russian Control Literature", Convair Report ZU-044, Part 1.

3. Gerald Berman and R.G. Stanton, "The Asymptotes of the Root-Locus", Soc. Ind. Appl. Math. Review, Vol. 5, 1963, pp. 209-218.

4. W.R. Evans, "Graphical Analysis of Control Systems", Trans. Amer. Inst. Elec. Engrs., 67, (1948), pp. 547-551.

5. __________, "Control System Synthesis by the Root-Locus Method", Trans. Amer. Inst. Elec. Engrs., 69, (1950), pp. 66-69.

6. __________, "Control System Dynamics", McGraw-Hill Book Co., Inc., New York, 1954.

7. Wilfred Kaplan, "Operational Methods for Linear Systems", Addison-Wesley Publishing Co., Reading, Mass., 1962.

8. Allan M. Krall, "An Extension and Proof of the Root-Locus Method", Jour. Soc. Ind. Appl. Math., Vol. 9, (1961), pp. 644-653.

9. ______________, "A closed Expression for the Root-Locus Method", Jour. Soc. Appl. Math., Vol. 11, (1963), pp. 700-704.

10. J.I. Neimark, "On the Distribution of Roots of Polynomials", Dokl. Akad. Nauk, (58), 1947.

11. ____________, "On the Structure of the D-Partitions of Polynomials", Dokl. Akad. Nauk, (59), 1948.

12. ____________, "On Parameters for Stable Systems", Automatika i Telemekhanika, (9), 1948.

13. ____________, "On the Structure of Stable Systems", Automatika i Telemekhanika, (11), 1950.

14. ____________, "Stability of Linear Systems", Leningrad, 1949.

15. Floyd E. Nixon, "Principles of Automatic Controls", Prentice-Hall, Inc., Englewood Cliffs, N.J., 1953.

16. H. Nyquist, "Regeneration Theory", Bell System Tech. Jour., Vol. 11, 1932.

17. E.P. Popov, "The Dynamics of Automatic Control Systems", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

18. C.J. Savant, "Basic Feedback Control System Design", McGraw-Hill Book Co., Inc., New York, 1958.

19. John G. Truxal, "Control System Synthesis", McGraw-Hill Book Co., Inc., New York, 1955.

VII. Differential-Difference Equations, Time Lags

7.1 Existence: If there is a time lag between the input and output of one of the components of a system, the system can no longer be adequately described by a differential equation, but rather must be described by a differential-difference equation. Differential-difference equations share many of the properties associated with differential equations. There are existence and uniqueness theorems, matrix forms for systems of equations and characteristic equations for equations with constant coefficients. There are also, however, substantial differences between differential and differential-difference equations as we shall see.

In the interests of simplicity we will assume that there is only one time lag. For those who are interested in more than one time lag, we recommend the books listed at the end of the chapter.

Let us consider the equation

$$\sum_{i=0}^{n} a_{n-i}(t)y^{(i)}(t) + \sum_{i=0}^{m} b_{m-i}(t)y^{(i)}(t-\tau) = f(t) \quad ,$$

where $a_i(t)$, $b_i(t)$ and $f(t)$ are continuous functions of t, and put the equation in matrix form.

Let $y' = y_1$, $y_1' = y_2, \ldots, y_{n-2}' = y_{n-1}$. If $n = m$, we write

$$
\begin{aligned}
&y' + 0y - y_1 + y'(t-\tau) + 0y(t-\tau) - y_1(t-\tau) = 0\,,\\
&y_1' + 0y + 0y_1 - y_2 + y_1'(t-\tau) + 0y(t-\tau) + 0y_1(t-\tau) - y_2(t-\tau) = 0\,,\\
&\vdots\\
&y_{n-2}' + 0y + 0y_1 + \cdots - y_{n-1} + y_{n-2}'(t-\tau) + 0y(t-\tau) + 0y_1(t-\tau) \cdots - y_{n-1}(t-\tau) = 0\,,\\
&a_0y_{n-1}' + a_ny + a_{n-1}y_1 + \cdots + a_1y_{n-1} + b_0y_{n-1}'(t-\tau) + b_ny(t-\tau) + b_{n-1}y_1(t-\tau)\\
&\qquad \cdots + b_1y_{n-1}(t-\tau) = f(t)\,.
\end{aligned}
$$

If

$$
A_0 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; a_0 \end{pmatrix}, \qquad
B_0 = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 \\ a_n & a_{n-1} & \cdots & & a_1 \end{pmatrix},
$$

$$
A_\tau = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; b_0 \end{pmatrix}, \qquad
B_\tau = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & & -1 \\ b_n & b_{n-1} & \cdots & & b_1 \end{pmatrix},
$$

$$
Y = \begin{pmatrix} y \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}, \qquad
F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix},
$$

then

$$A_0 Y' + B_0 Y + A_\tau Y'(t - \tau) + B_\tau Y(t - \tau) = F$$

is equivalent to the nth order equation.

If $n > m$, let $A_\tau = 0$,

$$B_\tau = \begin{pmatrix} 0 & 0 & \cdots & & 0 & \cdots \\ 0 & 0 & \cdots & & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots \end{pmatrix} .$$

The matrix equation is again equivalent to the nth order equation.

If $n < m$, let $A_0 = 0$,

$$B_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & \cdots & a_0 & \cdots \end{pmatrix}$$

and the equations are again equivalent. The system when $n < m$ has some fundamental difficulties, however, if solutions are desired for increasing t. If 0 is the initial point under consideration, then in the interval $(\tau, 2\tau)$ Y depends upon the behavior of Y' in $(0,\tau)$. In general, in $(n\tau, (n+1)\tau)$, Y depends upon the behavior of $Y^{(n)}$ in $(0,t)$. Y may not be infinitely differentiable in $(0,\tau)$, especially if F is not. So we should not expect a general existence theorem when $n < m$.

When $n \geq m$ we note that A_0 is non-singular if $a_0 \neq 0$. We make that assumption. The following theorem, stated for a system of equations clearly contains the nth order equation with $n \geq m$ as a special case.

Theorem 7.1.1: *Let* $\tau > 0$. *Let* A_0, B_0, A_τ *and* B_τ *be continuous* n *by* n *matrices. Let* F *be a continuous* n *by* 1 *matrix and* G *be a continuously differentiable* n *by* 1 *matrix. Let* $|A_0| \neq 0$.

1. *There is a unique function* $Y(t)$, *continuous for all* $t \geq 0$, *satisfying*

$$A_0Y' + B_0Y + A_\tau Y'(t - \tau) + B_\tau Y(t - \tau) = F(t) ,$$

$$Y(t) = G(t) , \qquad 0 \leq t \leq \tau ,$$

for all $t > \tau$, $t \neq nt$, $n = 1,2,3, \ldots$.

2. $Y(t)$ *is continuously differentiable for all* $t > 0$, $t \neq n\tau$, $n = 1,2,3, \ldots$.

3. $Y(t)$ *is continuously differentiable for all* $t > 0$ *and satisfies*

$$A_0Y' + B_0Y + A_\tau Y'(t - \tau) + B_\tau Y(t - \tau) = F(t) ,$$

$$Y(t) = G(t) , \qquad 0 \leq t \leq \tau ,$$

for all $t > \tau$ *if and only if*

$$A_0(\tau)G'(\tau) + B_0(\tau)G(\tau) + A_\tau(\tau)G'(0) + B_\tau(\tau)G(0) = F(\tau) .$$

Proof: Since $|A_0| \neq 0$, A_0^{-1} exists. By multiplication by A_0^{-1} we have a similar equation with A_0 now the identity. Thus we assume $A_0 = I$.

In the interval $[\tau,2\tau]$, let

$$V = F - A_\tau G'(t - \tau) - B_\tau G(t - \tau) .$$

Our differential-difference equation is of the form

$$Y' + B_0 Y = V(t) \quad , \qquad Y(\tau) = G(\tau) \quad .$$

By Theorem 3.1, this possesses a unique solution which agrees with G on $[0,\tau]$.

In the interval $[2\tau,3\tau]$, let

$$V = F - A_\tau Y'(t - \tau) - B_\tau G(t - \tau)$$

and solve

$$Y' + B_0 Y = V(t) \quad , \qquad Y(2\tau) = Y(2\tau) \quad ,$$

etc.

Now Y' is continuous at $t = \tau$ if and only if $Y'(\tau+) = G'(\tau-)$. Since $Y(\tau+) = G(\tau)$, this happens if and only if

$$A_0(\tau)G'(\tau) + B_0(\tau)Y(\tau) + A_\tau(\tau)G'(0) + B_\tau(\tau)G(0) = F(\tau) \quad .$$

Y' is continuous at $t = 2\tau$ if and only if $Y'(2\tau+) = Y'(2\tau-)$. Looking at the differential difference equation, we see that this happens if and only if Y' is continuous at $t = \tau$, etc.

When A_0, B_0, A_τ and B_τ are constant matrices, it is easily seen that for an appropriate choice of C, $Y = Ce^{\lambda t}$ is a solution of

$$A_0 Y' + B_0 Y + A_\tau Y'(t - \tau) + B_\tau Y(t - \tau) = 0$$

whenever

$$|A_0\lambda + B_0 + A_\tau \lambda e^{-\lambda\tau} + B_\tau e^{-\lambda\tau}| = 0 \quad .$$

The general solution to the non-homogeneous equation would be of the form

$$Y = \sum_i C_i(t)e^{\lambda_i t} + Y_0(t) \quad ,$$

where λ_i are the roots of the characteristic determinant above, $C_i(t)$ are appropriate matrices with elements polynomials in t and Y_0 is any solution of the non-homogeneous equation. The equation will be stable if and only if all the λ_i's are simple zeros and have non-positive real parts or have negative real parts.

7.2 Solution by the Laplace Transform: If A_0, B_0, A_τ and B_τ are constant and $|A_0| \neq 0$, it is possible to solve

$$A_0 Y' + B_0 Y + A_\tau Y'(t - \tau) + B_\tau Y(t - \tau) = F(t)$$

when $F(t)$ does not grow faster than an exponential function. If $Y(t) = G(t)$ on $[0,\tau]$, we multiply by e^{-st} and integrate from τ to ∞.

$$\int_\tau^\infty Y(s)e^{-\tau s}ds =$$

$$[A_0 s + B_0 + A_\tau se^{-\tau s} + B_\tau e^{-\tau s}]^{-1} [\int_\tau^\infty F(t)e^{-st}dt + H(s)]$$

where

$$H(s) = [A_0 G(\tau) + A_\tau G(0) - (A_\tau s + B_\tau)e^{-s\tau} \int_0^\tau G(t)e^{-st}dt] \quad .$$

$Y(t)$ can then be found by evaluating the residues at the zeros of

$$|A_0 s + B_0 + A_\tau se^{-\tau s} + B_\tau e^{-\tau s}| \quad .$$

This however is not a simple task. A knowledge of the location of these zeros is necessary.

7.3 Zeros of the characteristic equation, constant coefficients: In the interests of further simplicity we examine the characteristic equation of the nth order system which is

$$\left(\sum_{i=0}^{n} a_{n-i} z^i\right) + \left(\sum_{i=0}^{m} b_{m-i} z^i\right) e^{-\tau z} = 0 \ .$$

It is convenient to write this equation in the form

$$(z^n + az^{n-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots) = 0 \ ,$$

where K is non-negative and real.

Theorem 7.3.1: The function

$$F(z) = (z^n + az^{n-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots)$$

has an infinite number of zeros of the form

$$z = (1/\tau)[\{\log K + (n - m)\log \tau - (n - m)\log 2k\pi\}$$

$$+ i\{2k\pi + \theta \mp (n - m)\tfrac{\pi}{2}\}] + o(1) \ ,$$

$k = 0, \pm 1, \pm 2, \ldots$.

Proof: If we substitute $z = (1/\tau)[\ ] + \omega$ into $F(z)$ and divide by $(2k\pi i/\tau)^n$, we find

$$F(z)/(2k\pi i/\tau)^n = [1 - e^{-\tau\omega}] + o(1) \ .$$

Having suitably restricted ourselves to any sufficiently small neighborhood of $\omega = 0$, we see that if k is large enough, Rouché's theorem guarantees exactly one zero within the neighborhood.

This theorem immediately rules out stability when $n < m$ or when $n = m$, $K > 1$.

Lemma 7.3.2: Let φ, $0 < \varphi < \pi/2$, be arbitrary but fixed. There exists at most a finite number of zeros of

$$F(z) = (z^n + az^{n-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots)$$

in the regions

$$-\frac{\pi}{2} + \varphi < \arg z < \frac{\pi}{2} - \varphi ,$$

$$\frac{\pi}{2} + \varphi < \arg z < \frac{3\pi}{2} - \varphi .$$

Proof: In the first region $e^{-\tau z}$ approaches 0 as $|z| \to \infty$, so z^n dominates $F(z)$ and $|F(z)| \to \infty$ uniformly. In the second region $e^{-\tau z}$ approaches ∞ as $|z| \to \infty$, and $e^{-\tau z}z^m$ dominates making $|F(z)| \to \infty$ uniformly.

Lemma 7.3.3: Let $G(z) = e^{\tau z}(1 + o(1)) - Ke^{i\theta}(1 + o(1))$. All but a finite number of zeros of $G(z)$ take the form

$$z = (1/\tau)[\log K + i(\theta + 2k\pi)] + o(1) ,$$

$k = 0, \pm 1, \ldots$.

Proof: Choose any real a. To the left of a, $e^{\tau z}$ is bounded by $e^{\tau a}$, and the zeros of $G(z)$ become arbitrarily close to those of $e^{\tau z} - Ke^{i\theta}$. To the right of a, $e^{-\tau z}$ is bounded by $e^{-\tau a}$, and the zeros of $e^{-\tau z}G(z)$ (and thus of $G(z)$) become arbitrarily close to those of $1 - Ke^{i\theta}e^{-\tau z}$.

Theorem 7.3.4: Let $\{z_k\}$ be an infinite sequence of zeros of

$$F(z) = (z^n + az^{n-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^m + bz^{n-1} + \cdots)$$

Then the sequence can be represented by

$$z_n = (1/\tau)[\{\log K - (n - m) \log 2k\pi + (n - m) \log \tau\}$$

$$+ i\{\theta + 2k\pi \mp (n - m) \pi/2\}] + o(1) ,$$

as $k = 0, \pm 1, \pm 2, \ldots$.

Proof: We can write

$$z^{-m}e^{\tau z}F(z) = e^{\tau z}z^{n-m}(1 + o(1)) - Ke^{i\theta}(1 + o(1))$$

for large $|z|$. If we let $s = z + \frac{n - m}{\tau} \log z$ we see that $|z|$ increases if and only if $|s|$ does. Thus $z^{-m}e^{\tau z}F(z)$ becomes

$$G(s) = e^{\tau s}(1 + o(1)) - Ke^{i\theta}(1 + o(1)) ,$$

where $o(1) \to 0$ as $|s| \to \infty$. Lemma 7.3.3 tells us that all but a finite number of zeros of $G(s)$ have the form

$$s_k = (1/\tau)[\log K + i(\theta + 2k\pi)] + o(1) .$$

If

$$z_k = x_k + iy_k , \quad s_k = u_k + iv_k ,$$

we have

$$u_k = x_k + \frac{n - m}{\tau} \log |x_k + iy_k| ,$$

$$v_k = y_k + \frac{n - m}{\tau} \arg (x_k + iy_k) .$$

Or

$$(1/\tau)\ \log K + o(1) = x_k + \frac{n - m}{\tau} \log |x_k + iy_k| \ ,$$

$$(1/\tau)(\theta + 2k\pi) + o(1) = y_k + \frac{n - m}{\tau} \arg (x_k + iy_k) \ .$$

Now we see by Lemma 7.3.2 that $\arg (x_k + iy_k) \to \pm \frac{\pi}{2}$ as $k \to \infty$. Hence

$$y_k = (1/\tau)[\theta + 2k\pi \mp (n - m) \tfrac{\pi}{2}] + o(1) \ .$$

Let x_k^1 satisfy

$$\frac{1}{\tau} \log K = x_k^1 + \frac{n - m}{\tau} \log |y_k| \ .$$

Since

$$\frac{1}{\tau} \log K = x_k + \frac{n - m}{\tau} \log |x_k + iy_k| \ ,$$

we see

$$|x_k - x_k^1| = \left|\frac{n - m}{\tau} \log \left|\frac{x_k + iy_k}{y_k}\right|\right|$$

$$= \left|\frac{n - m}{\tau}\right| \left|\log [1 + o(1)]\right| \ ,$$

since y_k must increase relatively faster than x_k as was shown in Lemma 7.3.2. Thus

$$x_k = x_k^1 + o(1) \ .$$

Thus

$$x_k = (1/\tau)[\log K - (n-m)\log |(1/\tau)(\theta + 2k\pi \mp (n-m)\tfrac{\pi}{2}) + o(1)|] + o(1) \quad ,$$

$$x_k = (1/\tau)[\{\log K - (n-m)\log 2k\pi + (n-m)\log\tau\}] + o(1) \quad ,$$

and

$$z_k = (1/\tau)[\{\log K - (n-m)\log 2k\pi + (n-m)\log\tau\}$$

$$+ i\{\theta + 2k\pi \mp (n-m)\tfrac{\pi}{2}\}] + o(1) \quad .$$

Note that this is the same result found in Theorem 7.3.1.

Similar statements can be made about the zeros of exponential polynomials with more than one exponential function. Zeros occur in chains as just exhibited, but there may be more than one chain. Chains may go to the right, left , vertically or in any combination.

Note finally that the zeros become ultimately separated by a distance which approaches 2π.

<u>7.4 Solution of the nth order equation by the Laplace Transform</u>: We solve the equation

$$\sum_{i=0}^{n} a_{n-i}y^{(i)}(t) + \sum_{i=0}^{m} b_{m-i}y^{(i)}(t-\tau) = f(t) \quad ,$$

$$y(t) = g(t) \quad , \qquad 0 \le t \le \tau \quad ,$$

where $n \ge m$, $g(t)$ is n times differentiable and $g(t)$, $g'(t)$, $\ldots, g^{(n)}(t)$ and $f(t)$ possess Laplace Transforms.

It is an easy computation to see that

$$\int_{\tau}^{\infty} y^{(i)}(t)e^{-st}dt = s^{i}Y(s) - e^{-s\tau}\left[\sum_{j=0}^{i-1} g^{(j)}(\tau)s^{i-1-j}\right] ,$$

and

$$\int_{\tau}^{\infty} y^{(i)}(t-\tau)e^{-st}dt = e^{-s\tau}\left[s^{i}Y(s) + s^{i}\int_{0}^{\tau} e^{-st}g(t)dt - \sum_{j=0}^{i-1} g^{(i)}(0)s^{i-1-j}\right] ,$$

where

$$Y(s) = \int_{\tau}^{\infty} y(t)e^{-st}dt .$$

Applying the Laplace Transform to the original equation, we find

$$Y(s) = G(s)/F(s) ,$$

where

$$F(s) = \sum_{i=0}^{n} a_{n-i}s^{i} + e^{-\tau s}\sum_{i=0}^{m} b_{m-i}s^{i} ,$$

$$G(s) = \mathcal{L}f(s) + e^{-\tau s}\left[\sum_{i=0}^{n}\sum_{j=0}^{i-1} a_{n-i}g^{(j)}(\tau)s^{i-1-j}\right.$$

$$+ \sum_{i=0}^{m}\sum_{j=0}^{i-1} b_{m-i}g^{(j)}(0)s^{i-1-j}$$

$$\left. - \left(\sum_{i=0}^{m} b_{m-i}s^{i}\right)\int_{0}^{\tau} g(t)e^{-st}dt\right] .$$

Thus for all $t > \tau$,

$$y(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [e^{st}G(s)/F(s)]ds$$

for some value of c sufficiently large. To evaluate the integral we choose c so that all the zeros of $F(s)$ lie to the left of $\mathrm{Re}(s) = c$ and then a sequence of semi-circular contours as follows. If R is a large real number, we go from $(c,-R)$ to (c,R) along a vertical line. Then from (c,R) to $(c,-R)$ along a semi-circle centered at $(c,0)$ lying to the left of the vertical line. We choose a sequence of R's so that the contours are uniformly bounded away from the zeros of $F(s)$ and so that $R \to \infty$.

It is an easy matter to show that the integral of $e^{st}G(s)/F(s)$ along the semi-circle approaches 0 as $R \to \infty$. Since $e^{st}G(s)/F(s)$ is bounded, for any $\epsilon > 0$ the integrals from (c,R) to the point $(c + R\cos\theta, R\sin\theta)$, $\frac{\pi}{2} \le \theta \le \pi$, and from $(c + R\cos\varphi, R\sin\varphi)$ to $(c,-R)$, $\pi \le \varphi \le \frac{3\pi}{2}$, can be made less than $\epsilon/4$ if θ is close to $\frac{\pi}{2}$ and φ is close to $\frac{3\pi}{2}$. The integral over the remainder can be made small (because of e^{ts}) by choosing R sufficiently large. Thus

$$y(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} [e^{ts}G(s)/F(s)]ds = \Sigma[\text{residues of } e^{ts}G(s)/F(s)] .$$

If $\{s_i\}$ are the zeros of $F(s)$ and $\{\alpha_i(t)e^{s_i t}\}$ are the residues of $e^{ts}G(s)/F(s)$, then

$$y(t) = \sum_{i=1}^{\infty} \alpha_i(t)e^{s_i t} .$$

Again it is easy to see that stability occurs only when all of the zeros of $F(s)$ lie in the left half plane.

EXERCISES

1. Show that if each term of $F(t)$ is less than ce^{at} and

$$A_0 Y' + B_0 Y + A_1 Y'(t - \tau) + B_\tau Y(t - \tau) = F(t) ,$$

$|A_0| \neq 0$, then there exist constants d and b such that each term of $Y(t)$ is less than de^{bt}. Thus the Laplace Transform is applicable.

(See Bellman and Cooke, "Differential-Difference Equations".)

2. State and prove an existence theorem for

$$a_1 y(t) + b_0 y'(t - \tau) + b_1 y(t - \tau) = f(t) ,$$

$$y(t) = g(t) , \qquad 0 \leq t \leq \tau ,$$

when $f(t)$ and $g(t)$ are infinitely differentiable.

References

1. Richard Bellman and Kenneth Cooke, "Differential-Difference Equations", Academic Press, New York, 1963.

2. Edmund Pinney, "Ordinary Difference-Differential Equations", Univ. of California Press, Los Angeles, 1958.

3. L.S. Pontrjagin, "On the Zeros of Some Elementary Trancendental Functions", Amer. Math. Soc. Transl., Ser. 2, Vol. 1, 1955, pp. 95-110.

VIII. Stability Techniques for Time-Lag Systems

8.1: There are two techniques for determining stability in general time-lag systems: one due to L. S. Pontrjagin, and an extension of Michailov's criterion due to A. A. Sokolov and N. N. Miasnikov which is referred to in Popov's book. The two criteria are very closely related. However, Pontrjagin's is somewhat more complicated and harder to apply. It is for this reason we present Michailov's criterion and merely state Pontrjagin's criterion leaving the proof as an exercise.

It is obvious that the Hurwitz and Routh criteria are not applicable.

8.2 Preliminary Remarks: For the sake of convenience we again consider the function

$$F(z) = (z^n + az^{n-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^m + bz^{m-1} + \cdots) ,$$

where $K \geq 0, \theta$ and $\tau \geq 0$ are fixed constants.

Theorem 8.2.1: If $n > m$ or $n = m$, $K < 1$, then $F(z)$ has only a finite number of zeros lying in the right half plane; all lying within a semi-circle of radius $\rho = M + 1$ where

$$M = \sup[\{|a_i| + K|b_i|\}_1^m \quad \text{and} \quad \{|a_i|\}_{m+1}^n] \quad \text{if} \quad n > m ,$$

and

$$M = \sup[\{(|a_i| + K|b_i|)/(1 - K)\}_1^n] \quad \text{if} \quad n = m .$$

__Proof__: If $|z| > \rho$ and $\mathrm{Re}(z) \geq 0$, then when $n > m$,

$$|F(z)| \geq |z|^n - \sum_{i=1}^{n} |a_i||z|^{n-i} - K \sum_{i=1}^{m} |b_i||z|^{n-i},$$

$$\geq |z|^n - M \sum_{i=1}^{n} |z|^{n-i},$$

$$= |z|^n - M(|z|^n - 1)/(|z| - 1),$$

$$= \{|z|^n[|z| - (1 + M)] + M\}/(|z| - 1),$$

$$> 0 .$$

When $n = m$,

$$|F(z)| \geq (1 - K)|z|^n - \sum_{i=1}^{n} (|a_i| + K|b_i|)|z|^{n-i},$$

$$\geq (1 - K)[|z|^n - M \sum_{i=1}^{n} |z|^{n-i}],$$

$$> 0 .$$

Note that the radius ρ is independent of τ and θ as long as $\tau \geq 0$.

8.3 The Michailov Criterion:

__Theorem 8.3.1__: _If_ $n > m$, _as_ z _varies from_ $-i\infty$ _to_ $i\infty$ _along the imaginary axis_, $F(z)$ _passes through the origin each time_ z _passes through a zero of_

F(z). <u>If</u> F(z) <u>has no imaginary zeros, the number of zeros of</u> F(z) <u>in the right half plane</u>, N, <u>satisfies</u>

$$N = \frac{n}{2} - \frac{1}{2\pi}\Delta(-i\infty, i\infty) \arg F(z) \quad ,$$

<u>where</u> $\Delta(-i\infty, i\infty) \arg F(z)$ <u>is the increase in</u> $\arg F(z)$ <u>as</u> z <u>varies along the imaginary axis from</u> $-i\infty$ <u>to</u> $i\infty$.

<u>Proof</u>: Choose any y greater than ρ of Theorem 8.2.1. The number of zeros of F(z) in the right half plane is given by

$$N = \frac{1}{2\pi} \Delta \arg F(z) \quad ,$$

where z varies along the contour from iy to -iy along the imaginary axis and from -iy to iy along a semi-circle in the right half plane centered at the origin. We write

$$F(z) = z^n(1 + \varphi(z)) \quad ,$$

where

$$\varphi(z) = (az^{-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(z^{m-n} + bz^{m-n-1} + \cdots) \quad .$$

Then

$$\arg F(z) = n \arg z + \arg(1 + \varphi(z)) \quad .$$

As z varies along the semi-circle $\Delta \arg z = \pi$. As $z \to \infty$ in the right half plane $\varphi(z) \to 0$ uniformly, and $\arg(1 + \varphi(z)) \to 0$ uniformly. The result follows by substitution.

<u>Corollary</u>: <u>If</u> $n > m$ <u>and</u> F(z) <u>has no imaginary zeros, then</u> $N = 0$ <u>if and only if</u>

$$\Delta(-i\infty, i\infty) \arg F(z) = n\pi \quad .$$

<u>Corollary</u>: <u>If</u> $n > m$, $F(z)$ <u>has no imaginary zeros and all of the coefficients of</u> $F(z)$ <u>are real</u>,

$$N = \frac{n}{2} - \frac{1}{\pi}\Delta(0,i\infty)\arg F(z) \quad .$$

<u>Corollary</u>: <u>If</u> $n > m$, $F(z)$ <u>has no imaginary zeros and all of the coefficients of</u> $F(z)$ <u>are real, then</u> $N = 0$ <u>if and only if</u>

$$\Delta(0,i\infty)\arg F(z) = \frac{n\pi}{2} \quad .$$

<u>Theorem 8.3.2</u>: <u>If</u> $n = m$, $K < 1$, <u>as</u> z <u>varies from</u> $-i\infty$ <u>to</u> $i\infty$ <u>along the imaginary axis</u>, $F(z)$ <u>passes through the origin each time</u> z <u>passes through a zero of</u> $F(z)$. <u>If</u> $F(z)$ <u>has no imaginary zeros, the number of zeros of</u> $F(z)$ <u>in the right half plane</u>, N, <u>satisfies</u>

$$N = \frac{n}{2} - \frac{1}{2\pi}\Delta(-iy,iy)\arg F(z)$$

$$+ \frac{1}{2\pi}\arg(1 - Ke^{i\theta}e^{-i\tau y}) - \frac{1}{2\pi}\arg(1 - Ke^{i\theta}e^{\tau iy}) + o(1)$$

<u>for all sufficiently large</u> y.

<u>Proof</u>: If $F(z)$ has no imaginary zeros, we write

$$F(z) = z^n(1 - Ke^{i\theta}e^{-\tau z})(1 + \varphi(z)) \quad ,$$

where

$$\varphi(z) = [(az^{-1} + \cdots) - Ke^{i\theta}e^{-\tau z}(bz^{-1} + \cdots)]/(1 - Ke^{i\theta}e^{-\tau z}) \quad .$$

Then

$$\arg F(z) = n\arg z + \arg(1 - Ke^{i\theta}e^{-\tau z}) + \arg(1 + \varphi(z)) \quad .$$

In the right half plane $\varphi(z) \to 0$ uniformly as $|z| \to \infty$, so $\arg(1+\varphi(z)) \to 0$ uniformly. Since $K < 1$, $1 - Ke^{i\theta}e^{-\tau z}$ remains in the right half plane and cannot encircle the origin. Thus if we proceed around a semi-circular contour as in Theorem 7.3.1., along the semi-circle

$$\Delta \arg(1 - Ke^{i\theta}e^{-\tau z}) = \arg(1 - Ke^{i\theta}e^{-\tau iy}) - \arg(1 - Ke^{i\theta}e^{\tau iy}) ,$$

and

$$N = \frac{n}{2} - \frac{1}{2\pi}\Delta(-iy,iy)\arg F(z) + \frac{1}{2\pi}\arg(1 - Ke^{i\theta}e^{-\tau iy}) - \frac{1}{2\pi}\arg(1 - Ke^{i\theta}e^{\tau iy}) + o(1) .$$

Note that when $y = k\pi/\tau$ the terms $\arg(1 - Ke^{i\theta}e^{-\tau iy})$ and $\arg(1 - Ke^{i\theta}e^{\tau iy})$ are equal and

$$N = \frac{n}{2} - \frac{1}{2\pi}\Delta(-2k\pi i/\tau,\ 2k\pi i/\tau)\arg F(z) + o(1) ,$$

when k is sufficiently large.

<u>Corollary</u>: <u>If</u> $n = m$, $K < 1$ <u>and</u> $F(z)$ <u>has no imaginary zeros, then</u> $N = 0$ <u>if and only if</u>

$$\Delta(-iy,iy)\arg F(z) = n\pi + \arg(1 - Ke^{i\theta}e^{-\tau iy}) - \arg(1 - Ke^{i\theta}e^{\tau iy}) + o(1)$$

<u>for all large</u> y. $N = 0$ <u>if and only if for all large</u> k

$$\Delta(-k\pi i/\tau,\ k\pi i/\tau)\arg F(z) = n\pi + o(1) .$$

Corollary: *If* $n = m$, $K < 1$, $F(z)$ *has no imaginary zeros and all of the coefficients of* $F(z)$ *are real, then*

$$N = \frac{n}{2} - \frac{1}{\pi}\Delta(0,iy) \arg F(z)$$

$$+ \frac{1}{\pi} \arg (1 - Ke^{i\theta}e^{-i\tau y}) + o(1)$$

for all large y.

$$N = \frac{n}{2} - \frac{1}{\pi}\Delta(0,k\pi i/\tau) \arg F(z) + o(1)$$

for all large k.

Corollary: *If* $n = m$, $K < 1$, $F(z)$ *has no imaginary zeros and all of the coefficients of* $F(z)$ *are real, then* $N = 0$ *if and only if*

$$\Delta(0,iy) \arg F(z) = \frac{n\pi}{2} + \arg (1 - Ke^{i\theta}e^{-\tau iy}) + o(1)$$

for all large y. $N = 0$ *if and only if for all large* k

$$\Delta(0,k\pi i/\tau) \arg F(z) = \frac{n\pi}{2} + o(1) .$$

8.4 Pontrjagin's Criterion: We multiply $F(z)$ of the previous section by $e^{\tau z}$ and consider the resulting function

$$G(z) = e^{\tau z}(z^n + az^{n-1} + \cdots) - Ke^{i\theta}(z^m + bz^{m-1} + \cdots)$$

where $n > m$ or $n = m$, $K < 1$. Clearly the zeros are left unchanged by this multiplication.

Theorem 8.4.1: *If* $n > m$ *or* $n = m$, $K < 1$, *as* z *varies from* $-i\infty$ *to* $i\infty$ *along the imaginary axis*, $G(z)$ *passes through the origin each time* z *passes through a zero of* $G(z)$. *When* $G(z)$ *has no imaginary zeros*, $G(z)$ *has no zeros in the right half plane if and only if*

$$\Delta(-k\pi i/\tau, \quad k\pi i/\tau) \arg G(z) \quad = \quad 2k\pi \quad + \quad n\pi \quad + \quad o(1)$$

for all large k.

The proof is left as an exercise.

As in the previous section various corollaries can be stated under special circumstances.

EXERCISES

1. The characteristic equation for a general linear differential-difference equation is of the form

$$F(z) = \sum_{j=0}^{n} \sum_{i=1}^{m} a_{ij} z^j e^{-\omega_i z} ,$$

$$0 \le \omega_1 < \omega_2 \cdots < \omega_m .$$

Extend the Michailov and Pontrjagin criteria to exponential polynomials of this form.

References

1. Richard Bellman and Kenneth L. Cooke, "Differential-Difference Equations", Academic Press, New York, 1963.

2. Allan M. Krall, "On the Michailov Criterion for Exponential Polynomials", unpublished manuscript.

3. L.S. Pontrjagin, "On the Zeros of Some Elementary Transcendental Functions", Amer. Math. Soc. Transl., Ser, 2, Vol. 1, 1955, pp. 95-110.

4. E. P. Popov, "The Dynamics of Automatic Control Systems:, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

5. A.A. Sokolov, "The Stability Criterion in Linear Regulation Systems with Distributed Parameters and its Application", Inzhenernyi Sbornik 2, Vol. 2, 1946.

IX. Stability Techniques for Time Lag Feedback Systems

9.1 Preliminary Remarks: We consider a linear feedback system with constant coefficients where the open loop transfer function has a time lag, that is, the open loop transfer function is of the form

$$F(s) = Ke^{-\tau s}h(s)/g(s),$$

where τ is a time lag, and $g(z) = z^n + az^{n-1} + \cdots$ and $h(s) = z^m + bz^{m-1} + \cdots$ are relatively prime polynomials. If the feedback is multiplied by $e^{i\theta}$ and added to the input, the closed loop transfer function is found to be

$$Ke^{-\tau s}h(s)/[g(s) - Ke^{i\theta}e^{-\tau s}h(s)] .$$

The characteristic equation is thus

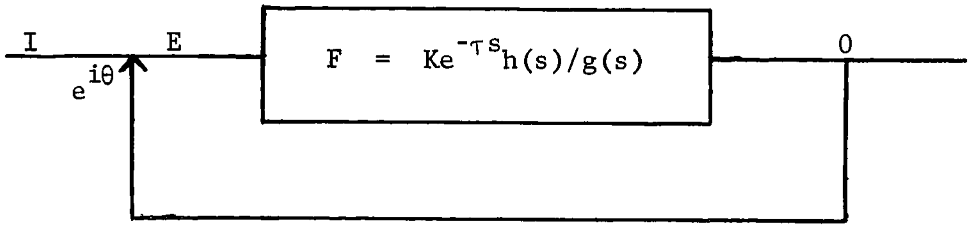
$$F(z) = g(z) - Ke^{i\theta}e^{-\tau z}h(z) = 0 ,$$

the same function considered in sections 7.3 and 7.4. It was shown there that stability is impossible unless $n > m$ or $n = m$, $K < 1$.

The techniques discussed in the previous chapter for testing for stability are concerned with $F(z)$ when K is fixed. In this chapter we not only wish to test for stability, we also wish to determine which values of K, if any, lead to stable systems.

The methods of Chapter VI can be extended to time lag systems. We discuss them each in turn.

9.2 Nyquist Criterion: Let $F(z) = g(z) - Ke^{i\theta}e^{-\tau z}h(z)$, where $g(z) = z^n + az^{n-1} + \cdots$, $h(z) = z^m + bz^{m-1} + \cdots$ have no common factor, θ and τ are real constants and K is a real parameter.



A Feedback System with Time-Lag

Theorem 9.2.1: Let $n > m$ or $n = m$, $K < 1$; let R be any number greater than ρ of Theorem 8.2.1. Let C_R be a semi-circular contour varying along the imaginary axis from $-R$ to R avoiding the zeros of $g(z)$ on the imaginary axis by arbitrarily small semi-circles centered at those zeros, and then in the right half plane from $(0,R)$ to $(0,-R)$ along a circle of radius R centered at the origin. The number of times $Ke^{-\tau z}h(z)/g(z)$ passes through $e^{-i\theta}$ as z varies around C_R is equal to the number of imaginary zeros of $F(z)$. If $F(z)$ has no imaginary zeros, let P be the number of zeros of $g(z)$ with positive real parts, Z be the number of zeros of $F(z)$ with positive real parts, N be the number of counter-clockwise encirclements of $e^{-i\theta}$ by $Ke^{-\tau z}h(z)/g(z)$ as z varies around C_R. Then $Z = P - N$.

Proof: The number of encirclements of the origin by a function as z varies in a counter-clockwise manner around a contour is equal to the number of zeros minus the number of poles of the function within the contour. As z varies around C_R, the number of encirclements of $e^{-i\theta}$ by $Ke^{-\tau z}h(z)/g(z)$ is the same as the number of encirclements of the origin by $e^{-i\theta} - Ke^{-\tau z}h(z)/g(z)$. This is the same as the number of encirclements of the origin by $1 - Ke^{i\theta}h(z)/g(z)$, which is the same as the number of encirclements of the origin by $F(z)/g(z)$. Since C_R is a clockwise contour, $Z = P - N$.

Corollary: Under the conditions of Theorem 9.2.1. a necessary and sufficient condition that $F(z)$ have no zeros with positive real parts is that $N = P$.

Notice that in constructing the path of $Ke^{-\tau z}h(z)/g(z)$ as z varies along the imaginary axis, the magnitude of $Ke^{-\tau z}h(z)/g(z)$ is the same as when $\tau = 0$. Only the argument is changed by an amount $-\tau\omega$ when $z = i\omega$.

Further note that M and N circles may be used the same as when $\tau = 0$.

If $n > m$, $Ke^{-\tau z}h(z)/g(z)$ approaches 0 as $|z|$ approaches ∞. The semi-circular part of the contour has its image in an arbitrarily small neighborhood of the origin when R is sufficiently large. The semi-circular part of the contour thus does not affect the number of encirclements of $e^{-i\theta}$ when R is sufficiently large.

If $n = m$, $K < 1$, the modulus of $Ke^{-\tau z}h(z)/g(z)$ approaches K if $|z|$ is sufficiently large. Since $Ke^{-\tau z}h(z)/g(z)$ is then bounded away from $e^{-i\theta}$ on the semi-circle when R is sufficiently large, again the semi-circular part of the contour cannot affect the number of encirclements of $e^{-i\theta}$.

If $g(z)$ has no zeros on the imaginary axis, then $Ke^{-\tau z}h(z)/g(z)$ remains in a bounded portion of the complex plane.

If $\theta = 0$ or π and $g(z)$ and $h(z)$ have real coefficients, then the path of $Ke^{-\tau z}h(z)/g(z)$ is symmetric with respect to the real axis.

K enters into the problem as a factor only of the amplitude. The procedure to follow then is to draw the contour for $K = 1$, and then by appropriately adjusting the scale of measurement to achieve stability, an acceptable value of K is found.

There is an alternate method which may be used which involves only the Nyquist contour with $\tau = 0$. This procedure was first used by A. A. Sokolov and N. N. Miasnikov while considering the Michailov criterion.

<u>Theorem 9.2.2</u>: <u>Let</u> $N(\tau)$ <u>be the number of counter-clockwise encirclements of</u> $e^{-i\theta}$ <u>by</u> $Ke^{-\tau z}h(z)/g(z)$ <u>as</u> z <u>varies over</u> C_R (<u>see Theorem 9.2.1.</u>). <u>Then if the path of</u> $Kh(z)/g(z)$ <u>does not intersect the unit circle as</u> z <u>varies over</u> C_R, $N(\tau) = N(0)$ <u>for all</u> $\tau \geq 0$.

Proof: If $N(\tau) \neq N(0)$ for some $\tau \neq 0$, then, since the Nyquist contour is continuous in τ, there must be a τ_0, $0 < \tau_0 < \tau$, for which the Nyquist contour passes through $e^{-i\theta}$. Thus there is an ω such that

$$Ke^{-i\tau_0\omega}h(i\omega)/g(i\omega) = e^{-i\theta}$$

and $|Kh(i\omega)/g(i\omega)| = 1$, which is impossible.

If the path of $Kh(z)/g(z)$ does intersect the unit circle, let

$$e^{i\alpha_1}, e^{i\alpha_2}, \ldots e^{i\alpha_j}, \ldots$$

be the points of intersection. For each α_j let $i\omega_j$ be a point on the imaginary axis such that

$$Kh(i\omega_j)/g(i\omega_j) = e^{i\alpha_j} .$$

Then if $Ke^{-\tau z}h(z)/g(z)$ is to pass through $e^{-i\theta}$, we must have

$$Ke^{-i\tau\omega_j}h(i\omega_j)/g(i\omega_j) = e^{i(\alpha-\tau\omega_j)} = e^{i\theta},$$

or

$$\tau = (1/\omega_j)(\theta + \alpha_j + 2k\pi) ,$$

where k is an integer. Let these non-negative values of τ for all j, k be arranged in an increasing sequence $\tau_1, \tau_2, \tau_3, \ldots$. We then have

Theorem 9.2.3: If t_1 and t_2 are in the same open interval (τ_i, τ_{i+1}), then $N(t_1) = N(t_2)$.

Tests for stability may be conducted by considering the ordinary Nyquist diagram with $\tau = 0$.

<u>9.3 The Root-Locus Method</u>: Although the Nyquist criterion is left relatively unchanged for systems with a delay, the root-locus diagrams are radically altered. This is to be expected, since the characteristic equation has an infinite number of roots. We will see, however, that only a small part of the root-locus diagram is important, and with the aid of some construction rules, that part may be easily found.

Let $F(z) = g(z) - Ke^{i\theta}e^{-\tau z}h(z)$, where $g(z) = z^n + az^{n-1} + \cdots$, $h(z) = z^m + bz^{m-1} + \cdots$ have no common factor, θ and τ are real constants and K is a real parameter.

<u>Definition</u>: <u>The root-locus of</u> $F(z)$ <u>is the set of all points</u> z <u>such that</u> z <u>is a zero of</u> $h(z)$, <u>or for which there is a real number</u> K, $-\infty < K < \infty$, <u>such that</u> $F(z) = 0$.

<u>The positive root-locus of</u> $F(z)$ <u>is the set of all points</u> z <u>such that</u> z <u>is a zero of</u> $h(z)$, <u>or for which there is a real number</u> K, $0 \leq K < \infty$, <u>such that</u> $F(z) = 0$.

The zeros of $h(z)$ are included in the root-locus, since they are the limit points of the zeros of $F(z)$ for all the appropriate choices of K, i.e., they are the zeros of $F(z)$ when $K = \infty$.

The negative root-locus can be similarly defined. It is easy to see that the negative root-locus for θ is the positive root-locus for $\pi + \theta$.

<u>Theorem 9.3.1</u>: <u>Let</u> z <u>be a point in the complex plane. The following statements are equivalent</u>.

(i). z <u>is on the root-locus of</u> $F(z)$.

(ii). $\cos(\theta - \tau y)\mathrm{Im}(h(z)\overline{g(z)}) + \sin(\theta - \tau y)\mathrm{Re}(h(z)\overline{g(z)}) = 0$.

<u>Proof</u>: Suppose z is on the root-locus. If $g(z) \neq 0$, then for some $K \neq 0$, $Ke^{i\theta}e^{-\tau z}h(z)/g(z) = 1$. Thus

$$h(z)/g(z) = K^{-1}e^{\tau x}[\cos(\theta - \tau y) - i \sin(\theta - \tau y)] .$$

$$h(z)\overline{g(z)} = K^{-1}e^{\tau x}|g(z)|^2[\cos(\theta - \tau y) - i \sin(\theta - \tau y)] .$$

Since K, τ, x are real,

$$\mathrm{Re}(h(z)\overline{g(z)}) = K^{-1}e^{\tau x}|g(z)|^2 \cos(\theta - \tau y) ,$$

$$\mathrm{Im}(h(z)\overline{g(z)}) = - K^{-1}e^{\tau x}|g(z)|^2 \sin(\theta - \tau y) .$$

Multiplying the first by $\sin(\theta - \tau y)$, the second by $\cos(\theta - \tau y)$ and adding achieves (ii). So (i) implies (ii).

Conversely, if (ii) is satisfied, then $\mathrm{Im}(e^{i\theta}e^{-\tau z}h(z)\overline{g(z)}) = 0$. So $e^{i\theta}e^{-\tau z}h(z)\overline{g(z)} = R(z)$, where $R(z)$ is real. If $R(z) = 0$, then either $h(z) = 0$ or $g(z) = 0$, and z is on the root-locus. If $R(z) \neq 0$, let $K = |g(z)|^2/R(z)$. If $K = 0$, then $g(z) = 0$ and z is on the root-locus. If $K \neq 0$, then $Ke^{i\theta}e^{-\tau z}h(z)/g(z) = 1$ and $F(z) = 0$. So (ii) implies (i).

Note that K can be found by

$$K = e^{\tau x}|g(z)|^2 \cos(\theta - \tau y)/\mathrm{Re}(h(z)\overline{g(z)})$$

or by

$$K = - e^{\tau x}|g(z)|^2 \sin(\theta - \tau y)/\mathrm{Im}(h(z)\overline{g(z)}) .$$

More can be said if $g(z)$ and $h(z)$ have real coefficients.

<u>Theorem 9.3.2</u>: <u>If</u> $g(z)$ <u>and</u> $h(z)$ <u>have real coefficients, then</u> z <u>is on the root-locus if and only if</u>

$$\cos(\theta - \tau y) \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{2k+1-i} h^{(i)}(x) g^{(2k+1-i)}(x)$$

$$+ \sin(\theta - \tau y) \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} \sum_{i=0}^{2k} \binom{2k}{i} (-1)^{2k-i} h^{(i)}(x) g^{(2k-i)}(x) = 0 .$$

<u>Proof</u>: This follows from Lemma 6.3.3. and Theorem 9.2.1.

<u>Theorem 9.3.3</u>: <u>If</u> $g(z)$ <u>and</u> $h(z)$ <u>have real coefficients, the root-locus contains the entire real line</u> $(y = 0)$ <u>if and only if</u> $\theta = 0$ <u>or</u> $\theta = \pi$.

<u>Proof</u>: If the x-axis is contained in the root-locus, then $y = 0$ is a solution of the equation in Theorem 9.3.2. Thus $\sin\theta = 0$ and $\theta = 0$ or $\theta = \pi$.

The converse is trivial.

<u>Definition</u>: <u>A point on the root-locus is a multiple point if for some value of</u> K <u>the point is a multiple zero of</u> $F(z)$ <u>or if the point is a multiple zero of</u> $h(z)$.

<u>Theorem 9.3.4</u>: <u>The multiple points of the root-locus are isolated and satisfy</u>

$$h(z)[g'(z) + \tau g(z)] - g(z)h'(z) = 0 .$$

<u>Proof</u>: If z is a multiple zero of $F(z)$, then $F(z) = 0$ and $F'(z) = 0$. Eliminating $Ke^{i\theta}e^{-\tau z}$ results in the above equation, a polynomial of degree $n + m$.

<u>Theorem 9.3.5</u>: <u>The points on the root-locus of</u> $F(z)$ <u>are continuous functions of</u> K.

<u>Proof</u>: This follows from Hurwitz's Theorem.

<u>Theorem 9.3.6</u>: <u>With the exception of the multiple points, the points on the root-locus of</u> $F(z)$ <u>are differentiable functions of</u> K.

<u>Proof</u>: Let z_0 be a simple zero of $F(z)$ when $K = K_0$. We need to show that

$$\lim_{K \to K_0} \frac{z - z_0}{K - K_0}$$

exists when z is a point on the root-locus and $z \to z_0$ as $K \to K_0$. We have

$$0 = g(z) - Ke^{i\theta}e^{-\tau z}h(z) \quad ,$$

$$0 = g(z) - K_0e^{i\theta}e^{-\tau z} - (K - K_0)e^{i\theta}e^{-\tau z}h(z) \quad ,$$

$$0 = (z - z_0)W(z) - (K - K_0)e^{i\theta}e^{-\tau z}h(z) \quad ,$$

where $W(z_0) \neq 0$ and

$$W(z_0) = \left.\frac{dF}{dz}\right|_{\substack{z=z_0 \\ K=K_0}} \quad .$$

From this we find that

$$\lim_{K \to K_0} \frac{z - z_0}{K - K_0} = \frac{e^{i\theta}e^{-\tau z_0}h(z_0)}{g'(z_0) - Ke^{i\theta}e^{-\tau z_0}(h'(z_0) - \tau h(z_0))} \quad .$$

<u>The Positive Root-Locus</u>:

<u>Theorem 9.3.7</u>: <u>As</u> x <u>becomes arbitrarily large to the right, the positive root-locus of</u> $F(z)$ <u>approaches</u>

$$y = (1/\tau)(\theta + 2k\pi)$$

<u>in the right half plane asymptotically, where</u> $k = 0, \pm 1, \pm 2, \ldots$. <u>Further</u>, $K \to \infty$ <u>as</u> $x \to \infty$.

<u>As</u> x <u>becomes arbitrarily large to the left, the positive root-locus of</u> $F(z)$ <u>approaches</u>

$$y = (1/\tau)(\theta - (n - m)\pi + 2k\pi)$$

<u>in the right half plane asymptotically, where</u> $k = 0, \pm 1, \pm 2, \ldots$. <u>Further</u>, $K \to 0$ <u>as</u> $x \to -\infty$.

<u>Proof</u>: Let

$$u(z) = e^{i\theta}e^{-\tau z}h(z)/g(z)$$

and consider only those values of z greater in absolute value than the zeros of $g(z)$ and $h(z)$. For those values of z the positive root-locus of $F(z)$ consists of all points where $u(z)$ is real and $u(z) > 0$, i.e., $\arg u(z) = 2k\pi$ for some integer k.

Now

$$\arg u(z) = \theta - \tau y + \arg h(z) - \arg g(z) \quad .$$

For bounded y, as $x \to \infty$, $\arg h(z) \to 0$ and $\arg g(z) \to 0$. Thus for bounded y, as $x \to \infty$, $\arg u(z) = \theta - \tau z + o(1)$.

Choose any $\epsilon > 0$ and then any $y = (1/\tau)(\theta + 2k\pi) - (1/\tau)\epsilon$, where k is an integer. If $z = x + iy$, $\arg u(z) = 2k\pi + \epsilon + o(1)$. By choosing $x > x_0$ so that $|o(1)| < \epsilon/2$, we see that $\arg u(z) = 2k\pi + \gamma$, where γ is between $\epsilon/2$ and $3\epsilon/2$. Similarly if $x > x_1$ and $y = (1/\tau)(\theta + 2k\pi) + (1/\tau)\epsilon$, $\arg u(z) = 2k\pi - \delta$, where δ is between $\epsilon/2$ and $3\epsilon/2$. Choose x so that $x > x_0$ and $x > x_1$. Consider a straight line between

$$z_0 = x + i(1/\tau)(\theta + 2k\pi) + i(1/\tau)\epsilon$$

and

$$z_1 = x + i(1/\tau)(\theta + 2k\pi) - i(1/\tau)\epsilon \quad .$$

Since $\arg u(z)$ is continuous in z, at some point between z_0 and z_1, $\arg u(z) = 2k\pi$ and $u(z) > 0$.

Note that as $x \to \infty$, $|\arg h(z) - \arg g(z)| \to 0$ so that ϵ may be chosen arbitrarily small. Further note that for z approaching the asymptotes K is given by

$$K = e^{-i\theta} e^{\tau z} g(z)/h(z) = e^{\tau x} x^{n-m}(1 + o(1)) \quad ,$$

as $x \to \infty$, so that $K \to \infty$ as $x \to \infty$.

The second part of the theorem follows by replacing z by $-z$.

Note that those values of K for which the root-locus crosses the imaginary axis increase as the root-locus becomes farther away from the origin. This means that for fixed K, most of the zeros of $F(z)$ lie in the left half plane, and also that it takes a larger value of K to force more to cross the imaginary axis. Thus only a finite part of the complex plane near the origin needs to be considered.

Theorem 9.3.8: Let

$$g(z) = \prod_j (z - p_j)^{\alpha_j}$$

and

$$h(z) = \prod_j (z - z_j)^{\beta_j} \quad ,$$

where $\Sigma_j \alpha_j = n$ and $\Sigma_j \beta_j = m$. As K approaches 0 α_j distinct branches of the positive root-locus of $F(z)$ approach each zero, p_j, of $g(z)$. As K approaches ∞, β_j distinct branches of the positive root-locus of $F(z)$ approach each zero, z_j, of $h(z)$.

Proof: This follows directly from Hurwitz's Theorem.

Definition: The angle of departure (arrival) of the root-locus of $F(z)$ at z_0 is the angle made at z_0 by the tangent to the root-locus for increasing (decreasing) K.

Theorem 9.3.9: If p_j is a zero of $g(z)$ of order α_j, then the positive root-locus of $F(z)$ departs from p_j making angles

$$\varphi_j = (1/\alpha_j)\left(\sum_j \beta_i \arg (p_j - z_i) - \sum_{i \neq j} \alpha_j \arg (p_j - p_i) + \theta - \tau y_j - 2k\pi\right) ,$$

where $k = 0, 1, \ldots, \alpha_j - 1$ and $y_j = \mathrm{Im}(p_j)$.

If z_j is a zero of $h(z)$ of order β_j, then the positive root-locus of $F(z)$ arrives at z_j making angles

$$\theta_j = (1/\beta_j)\left(\sum_i \alpha_i \arg (z_j - p_i) - \sum_{i \neq j} \beta_j \arg (z_j - z_i) - \theta + \tau y_j + 2k\pi\right) ,$$

where $k = 0, 1, \ldots, \beta_j - 1$ and $y_j = \mathrm{Im}(z_j)$.

<u>Proof</u>: Consider one of the branches of the positive root-locus which departs from p_j. Choose K close to 0, and let z be on that branch for that value of K. Then we have

$$Ke^{i\theta}e^{-\tau z}\prod_i (z - z_i)^{\beta_i} / \prod_i (z - p_i)^{\alpha_i} = 1 \quad .$$

Taking arguments,

$$\Sigma\, \beta_i \arg (z - z_i) - \Sigma\, \alpha_i \arg (z - p_i) + \theta - \tau y = 2k\pi \quad .$$

Solving for those terms involving p_j, and letting K approach 0 completes the first part of the proof. The second part is similar.

<u>Theorem 9.3.10</u>: <u>Let</u> z_0 <u>be any point on the real axis</u>, $h(z)$ <u>and</u> $g(z)$ <u>have real coefficients</u>, $\{z_i\}_1^r$ <u>and</u> $\{p_i\}_1^s$ <u>be the real zeros of</u> $h(z)$ <u>and</u> $g(z)$ <u>greater than</u> z_0 <u>and</u> $\theta = 0$ $(\theta = \pi)$. <u>Then</u> z_0 <u>is contained in the positive root-locus of</u> $F(z)$ <u>if and only if</u>

$$\sum_{i=1}^{r} \beta_i + \sum_{i=1}^{s} \alpha_i$$

<u>is even (odd)</u>.

<u>Proof</u>: Consider the case where $\theta = 0$. Since $g(z)$ and $h(z)$ have real coefficients, zeros of $g(z)$ and $h(z)$, if complex, occur in conjugate pairs. On the real axis, if z_1 and $\bar{z}_1$ are complex conjugates, then

$$\arg (z - z_1) + \arg (z - \bar{z}_1) = 0 \quad .$$

Now as in the proof of Theorem 9.4.1 , consider

$$u(z_0) = e^{-\tau z_0} h(z_0)/g(z_0) \quad .$$

On the real axis

$$\arg u(z_0) = \sum_i \beta_i \arg (z_0 - z_i) - \sum_i \alpha_i \arg (z_0 - p_i) ,$$

where the sums are taken over real zeros greater than z_0, complex zeros adding to give 0, zeros less than z_0 being 0. Each of the arguments left equal π. If

$$\sum_{i=1}^{r} \beta_i + \sum_{i=1}^{s} \alpha_i$$

is even, then

$$\sum_{i=1}^{r} \beta_i - \sum_{i=1}^{s} \alpha_i$$

is even, $u(z_0) > 0$ and z_0 is on the positive root-locus. If

$$\sum_{i=1}^{r} \beta_i + \sum_{i=1}^{s} \alpha_i$$

is odd, then

$$\sum_{i=1}^{r} \beta_i - \sum_{i=1}^{s} \alpha_i$$

is odd, $u(z_0) < 0$ and z_0 is not on the positive root-locus. The case $\theta = \pi$ is similar.

Theorem 9.3.11: *If the coefficients of* $g(z)$ *and* $h(z)$ *are real*, $\theta = 0$ *(or* π*), and* $F(z)$ *has a zero of order* m *at* $z = a$ *on the real axis for* $K = K_0$, $0 < K_0 < \infty$, *then the positive root-locus arrives at* $z = a$ *making angles*

$$\theta_k = 2k\pi/m \ , \qquad k = 0,1, \ldots, m-1 \ ,$$

<u>and departs from</u> $z = a$ <u>making angles</u>

$$\varphi_k = (2k+1)\pi/m \ , \qquad k = 0,1, \ldots, m-1 \ ,$$

<u>or arrives at</u> $z = a$ <u>making angles</u>

$$\theta_k = (2k+1)\pi/m \ , \qquad k = 0,1, \ldots, m-1 \ ,$$

<u>and departs from</u> $z = a$ <u>making angles</u>

$$\varphi_k = 2k\pi/m \ , \qquad k = 0,1, \ldots, m-1 \ .$$

<u>Proof</u>: Since the coefficients of $g(z)$ and $h(z)$ are real, for $\theta = 0$ or π, if the zeros of $g(z) - Ke^{i\theta}e^{-\tau z}h(z)$ leave or arrive at the real axis, they do so in conjugate pairs as K varies from 0 to K_0. Thus the evenness or oddness of the number of zeros of $g(z)$ plus zeros of $h(z)$ to the right of $z = a$ is the same as that of $g(z) - K_0e^{i\theta}e^{-\tau z}h(z)$ and $h(z)$.

Write

$$F(z) = G(z) - K_0e^{i\theta}e^{-\tau z}h(z) - (K - K_0)e^{i\theta}e^{-\tau z}h(z) = 0 \ ,$$

where z is on one of the branches of the positive root-locus near a. Let $g(z) - K_0e^{i\theta}e^{-\tau z}h(z) = (z-a)^mG(z)$, where $G(a) \neq 0$. $G(z)$ is real on the real axis. Then

$$(z-a)^mG(z) - (K - K_0)e^{i\theta}e^{-\tau z}h(z) = 0 \ .$$

Dividing by the second term and taking arguments,

$$m \arg (z - a) + \arg G(z) - \arg (K - K_0) - \theta + \tau y - \arg h(z) = 2k\pi ,$$

where k is an integer. Thus

$$\arg (z - a) = (1/m)(- \arg G(z) + \arg h(z) + \arg (K - K_0) + \theta - \tau y + 2k\pi) .$$

Now $- \arg G(z) + \arg h(z) + \theta - \tau y + 2k\pi$ approaches either an even or odd multiple of π as z approaches a, since $G(z)$ and $h(z)$ are non-zero and real on the real axis and $y = 0$. Arg $(K - K_0)$ is either 0 or π depending upon whether $K > K_0$ or $K < K_0$. Letting K approach K_0 completes the proof.

The most frequent occurrence is when there is a double zero of $F(z)$ on the real axis for some value of K. In this case $\theta_0 = \pi/2$, $\theta_1 = 3\theta/2$ and $\varphi_0 = 0$, $\varphi_1 = \pi$ or $\theta_0 = 0$, $\theta_1 = \pi$ and $\varphi_0 = \pi/2$, $\varphi_1 = 3\pi/2$.

From the preceding theorems it would appear that the root-locus for time lag systems is similar to those with no time lag. This similarity, however, is superficial. The root-locus diagrams become radically altered, as simple examples such as $z - Ke^{-\tau z} = 0$ and $z^2 - Ke^{-\tau z} = 0$ as well as the asymptotic theorem will testify.

A rather easy procedure has been found by Yaohan Chu for constructing time lag root-locus diagrams. It consists of first constructing a diagram with no time lag and using this diagram to construct the time lag diagram. We refer the reader to his paper rather than reproduce it here.

<u>9.4 J. I. Neimark's D-Partitions</u>: While the Nyquist and root-locust are changed in some way by the inclusion of a time lag, the method of D-Partitions is unaltered with the exception of changing the number of components from finite to countable. Statement of the various properties of the D-contour appear

in Section 6.4. The proofs offered there are still valid. It is for this reason that we only state the procedure.

The method of D-partitions is as follows.

1. The graph of $K = e^{-i\theta}e^{\tau z}g(z)/h(z)$ where $g(z) = z^n + az^{n-1} + \cdots$, $h(z) = z^m + bz^{m-1} + \cdots$ have no common factor and θ and τ are real constants is drawn as z varies from $-i\infty$ to $i\infty$ along the imaginary axis.

2. The left side of the graph is shaded locally.

3. The number of zeros of $F(z) = g(z) - Ke^{i\theta}e^{-\tau z}h(z)$ in the right half plane is determined in one of the regions, e.g., for $K = 0$.

4. The number of zeros of $F(z)$ in the right half plane in the other regions is determined from the one already determined by considering the shadings on the boundary between adjacent regions.

5. The stable situations, if any, are then found by inspection.

As with the ordinary method of D-partitions the real power of the method lies in the information it gives concerning acceptable values of K.

EXERCISES

1. Show that the equations of the tangent lines of the time lag root-locus approach the equations of the asymptotes of the time lag root-locus as the root-locus approaches the asymptotes.

References

1. Richard Bellman and Kenneth L. Cooke, "Differential-Difference Equations", Academic Press, New York, 1963.

2. Yaohan Chu, "Feedback Control Systems with a Dead-Time Lag or Distributed Lag by the Root-Locus Method", Trans. A.I.E.E., Vol. 71, 1952, pp. 291-296.

3. Allan M. Krall, "On the Real Parts of Zeros of Exponential Polynomials", Bull. Amer. Math. Soc., Vol. 70, 1964, pp. 291-292.

4. ______________, "Stability Criteria for Feedback Systems with a Time Lag", J. Soc. Ind. Appl. Math., Ser. A, Cont., Vol. 2.2, 1965, pp. 160-170.

5. ______________, "The Asymptotes of the Time Lag Root-Locus", J. Soc. Ind. Appl. Math., Ser. A, Cont., Vol 2.3, 1965.

6. E.P. Popov, "The Dynamics of Automatic Control Systems", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

X. Stability of Two Parameter Systems, Neimark's D-Partitions

In this chapter we consider the behavior of the zeros of a polynomial with two linear parameters. The method used is due to J. I. Neimark and is similar to his method when there is only one parameter involved, but with some substantial changes. While the method of D-partitions with one parameter has sufficient freedom so that the parameter can be considered as a function of a complex variable, if two parameters are involved, there is only enough freedom to permit them to be real valued. Further, there are times when the method is not applicable.

As in the one parameter case, the image of the imaginary axis in the z plane divides the parameter plane up into various regions. In each region the characteristic polynomial has the same number of zeros in the right and left half-planes.

A shading rule is found enabling the user to determine which points are to the left or right of the imaginary axis in the z plane.

Let $F(z) = \mu P(z) + \nu Q(z) + R(z)$, where

$$P(z) = P_1(z) + iP_2(z) \quad ,$$

$$Q(z) = Q_1(z) + iQ_2(z) \quad ,$$

and

$$R(z) = R_1(z) + iR_2(z)$$

are polynomials in z; P_1, P_2, Q_1, Q_2, R_1 and R_2 are real valued. Then $F(z) = 0$ is equivalent to

$$\mu P_1(z) + \nu Q_1(z) + R_1(z) = 0 ,$$

$$\mu P_2(z) + \nu Q_2(z) + R_2(z) = 0 .$$

If we let

$$\Delta(z) = \begin{vmatrix} P_1(z) & Q_1(z) \\ P_2(z) & Q_2(z) \end{vmatrix} ,$$

$$\Delta_\mu(z) = \begin{vmatrix} R_1(z) & Q_1(z) \\ R_2(z) & Q_2(z) \end{vmatrix} ,$$

$$\Delta_\nu(z) = \begin{vmatrix} P_1(z) & R_1(z) \\ P_2(z) & R_2(z) \end{vmatrix} ,$$

and $\Delta(z) \neq 0$, then

$$\mu = -\Delta_\mu(z)/\Delta(z) , \qquad \nu = -\Delta_\nu(z)/\Delta(z) .$$

If $\Delta(z) = 0$, then the two equations are dependent, determining a line in the (μ,ν) plane, when $\Delta_\mu(z) = \Delta_\nu(z) = 0$, or the two equations are inconsistent, and no solution exists when $\Delta_\mu(z) \neq 0$ and $\Delta_\nu(z) \neq 0$.

It may happen that $\Delta(z) \equiv 0$ for all z. In this case our method fails. We exclude this possibility from consideration.

Definition: If $\Delta(z) = \Delta_\mu(z) = \Delta_\nu(z) = 0$, the line generated in the (μ,ν) plane by $F(z) = 0$ is called a line of singularity.

Definition: Under the mapping

$$\mu = -\Delta_{\mu}(z)/\Delta(z) \quad , \qquad \nu = -\Delta_{\nu}(z)/\Delta(z) \quad ,$$

the image in the (μ,ν) plane of the imaginary axis in the z plane together with the limit points of the image is called the graph of ordinary points.

Definition: The lines of singularity and the graph of ordinary points constitute the D-contour.

Theorem 10.1: The D-contour divides the complex plane up into a finite number of regions.

Theorem 10.2: $F(z)$ has a zero on the imaginary axis if and only if (μ,ν) is on the D-contour.

Proof: The D-contour is determined by letting z take on all values $i\omega$, and then finding out which points in the (μ,ν) plane $F(i\omega) = 0$ determines.

Theorem 10.3: In each region in the (μ,ν) plane determined by the D-contour $F(z)$ has the same number of zeros in the right and left half-planes.

Proof: Suppose there were two points within the same region with different numbers of zeros in the left and right half-planes. Then along any arc within that region connecting the points there must be a point where some of the zeros of $F(z)$ lie on the imaginary axis. This point must lie on the D-contour, giving a contradiction.

Theorem 10.4: The image of the left half-plane lies locally to the left of the path taken by (μ,ν) along the graph of ordinary points as z varies from $-i\infty$ to $+i\infty$, when $\Delta(z) > 0$, and lies locally to the right of the path *taken by* (μ,ν) *along the graph of ordinary points as* z *varies from* $-i\infty$ *to* $+i\infty$, *when* $\Delta(z) < 0$.

Proof: The mapping from the z plane into the w plane, defined by

$$w = -R(z) \quad ,$$

is orientation preserving. (See the proof of Theorem 6.4.3.) Now, the mapping from the w plane into the (μ,ν) plane, defined by

$$w = \mu P(z) + \nu Q(z) \quad ,$$

or

$$\operatorname{Re} w = \mu P_1(z) + \nu Q_1(z) \quad ,$$

$$\operatorname{Im} w = \mu P_2(z) + \nu Q_2(z) \quad ,$$

is known to be orientation preserving if

$$\frac{\partial(\operatorname{Re} w, \operatorname{Im} w)}{\partial(\mu,\nu)} > 0 \quad ,$$

and orientation reversing if

$$\frac{\partial(\operatorname{Re} w, \operatorname{Im} w)}{\partial(\mu,\nu)} < 0 \quad .$$

(See Taylor, "Advanced Calculus", page 430.) Since

$$\frac{\partial(\operatorname{Re} w, \operatorname{Im} w)}{\partial(\mu,\nu)} = \begin{vmatrix} \dfrac{\partial \operatorname{Re} w}{\partial\mu} & \dfrac{\partial \operatorname{Im} w}{\partial\mu} \\[2ex] \dfrac{\partial \operatorname{Re} w}{\partial\nu} & \dfrac{\partial \operatorname{Im} w}{\partial\nu} \end{vmatrix} = \Delta(z) \quad ,$$

the result follows.

We therefore introduce the following shading rule: As z varies from

$-i\infty$ to $+i\infty$, and (μ,ν) varies along the graph of ordinary points, we shade locally to the left in the (μ,ν) plane, when $\Delta(z) > 0$, and locally to the right, when $\Delta(z) < 0$. The shaded area then corresponds to points lying near the imaginary axis but in the left half plane.

The lines of singularity must sometimes also be shaded. The rule here is not quite so simple. These lines must be shaded so that they are compatable with the shading given the graph of ordinary points.

For example, if the graph of ordinary points passes through (μ_0,ν_0), and $\Delta(z)$ changes sign so that the shading changes, then the line of singularity through (μ_0,ν_0) must be shaded in the opposite direction from the graph of ordinary points. In this case, as in all other cases, if a circle is drawn around (μ_0,ν_0), the net increase in zeros in the respective half-planes as (μ,ν) varies around the circle is zero.

The method of D-partitions with two parameters is as follows.

1. Construct the graph of ordinary points.

2. Fill in the lines of singularity.

3. Shade the appropriate side of the graph of ordinary points.

4. Shade the lines of singularity so that the net increase in zeros of $F(z)$ in the right and left half-planes is zero, when (μ,ν) follows any closed contour in the (μ,ν) plane.

5. Find the number of zeros of $F(z)$ in the respective half-planes for one of the regions in the (μ,ν) plane determined by the D-contour.

6. Determine the zeros of $F(z)$ in the respective half-planes for the other regions in the (μ,ν) planes by considering the shadings.

7. Choose values μ and ν so that $F(z)$ has no zeros in the right half-plane.

The following remarks are in order. The precedure can be extended to problems with non-linear parameters. In this case the requirement that $\Delta(z) \not\equiv 0$ is replaced by the requirement that the Jacobian

$$\frac{\partial(\mathrm{Re}\ F,\ \mathrm{Im}\ F)}{\partial(\mu,\nu)} \not\equiv 0 \ .$$

The fact that $P(z)$, $Q(z)$ and $R(z)$ were polynomials had very little to do with the method. This procedure works just as well for exponential polynomials, for example.

The points at $z = \pm i\infty$ are to be considered as well as the finite points in the z plane. $\pm i\infty$ are not to be ignored.

EXERCISES

1. Give a proof of Theorem 10.1. Send the author a copy.

2. Extend the method of D-partitions to differential-difference systems.

Reference

1. F.R. Alex, "A Survey of Russian Control Literature", Convair Report ZU-044, Part 1.

2. J. I. Neimark, "On the Distribution of Roots of Polynomials", Dokl. Akad. Nauk, (58), 1947.

3. ____________, "On the Structure of the D-Partitions of Polynomials", Dokl. Akad. Nauk, (59), 1948.

4. ____________, "On Parameters for Stable Systems", Automatika i Telemekhanika, (9), 1948.

5. ____________, "On the Structure of Stable Systems", Automatika i Telemekhanika, (11), 1950.

6. ____________, "Stability of Linear Systems", Leningrad, 1949.

7. Angus E. Taylor, "Advanced Calculus", Ginn and Company, 1955.

XI. A General Stability Criterion for Feedback Systems

11.1 Introduction: It seems that no author writing about this subject can resist the temptation to include a section on non-linear systems. Usually discussed are Laipounoff's second method, the Poincaré-Bendixson phase plane analysis, describing functions or certain other linearization techniques. We prefer, instead, to present a rather general stability criterion, which applies to both linear and non-linear systems. This criterion is very powerful and easy to use. It gives results which, even in the linear case, are not available by the other techniques. It does, however, require a more general setting than we have previously used -- that of a Banach Space.

11.2 Banach Space: Let X be a set of elements f, g, h, etc. We say X is a linear space if $\alpha f + \beta g$ is in X whenever f and g are in X, and α and β are arbitrary complex numbers.

A norm is a function denoted by $\|\cdot\|$ with domain X and range the non-negative real numbers, satisfying the following:

1. $\|f\| \geq 0$,
2. $\|f\| \neq 0$ if and only if $f \neq 0$,
3. $\|\alpha f\| = |\alpha| \, \|f\|$,
4. $\|f + g\| \leq \|f\| + \|g\|$,

for any f and g in X and α any complex number. If X has a norm associated with it, it is a normed linear space.

A Cauchy sequence in X is a set of elements $f_1, f_2, \ldots, f_n, \ldots$ in X such that for any $\varepsilon > 0$, there is an N such that, if m and n are greater than N, then $\|f_n - f_m\| < \varepsilon$.

The space X is complete if every Cauchy sequence $f_1, f_2, \ldots, f_n, \ldots$ in X has a limit, f, in X. That is,

$$\lim_{n \to \infty} \left\| f_n - f \right\| = 0 \ .$$

A Banach space is a complete, normed, linear space.

11.3 Examples of Banach Spaces: The simplest example of a Banach space is, of course, the real number system, where the norm of a real number is its absolute value. Other simple examples include the plane E^2 with its Euclidean norm, three dimensional space E^3, or even n dimensional space E^n. The norm of $x = (x_1, x_2, \ldots, x_n)$ in E^n is given by

$$\|x\| = \left(\sum_1^n x_i^{\,2} \right)^{\frac{1}{2}} \ .$$

Other possible norms include

$$\|x\| = \left(\sum_1^n |x_2|^p \right)^{1/p} , \qquad 1 \le p < \infty \ ,$$

or

$$\|x\| = \sup_{i=1,\ldots,n} |x_i| \ , \qquad p = \infty \ .$$

Actually all these norms are equivalent.

Another set of not quite so simple Banach spaces is the ℓ^p spaces, $1 \le p \le \infty$. $x = (x_1, x_2, \ldots, x_n, \ldots)$ is in ℓ^p when its norm,

$$\|x\| = \left(\sum_1^\infty |x_i|^p \right)^{1/p} , \qquad 1 \le p < \infty \ ,$$

or

$$\|x\| = \sup_{i=1,\ldots,n} |x_i| \quad , \qquad p = \infty \quad ,$$

is finite. In this case the norms are not equivalent.

A still more interesting class of Banach spaces is the function spaces. Let Ω be an arbitrary interval. If we use the Lebesque integral and agree to identify functions differing only on sets of measure zero, then the set of all functions, f, satisfying

$$\|f\| = \left(\int_{\Omega} |f|^p dx \right)^{1/p} < \infty \quad ,$$

form a Banach space $\mathcal{L}^p$, $1 \le p < \infty$. $\mathcal{L}^\infty$ is defined by using the norm

$$\|f\| = \operatorname*{ess\ sup}_{x \in \Omega} |f(x)| \quad .$$

These spaces are all infinite dimensional. The norms are not equivalent.

Another pair of interesting Banach spaces use the ordinary sup norm. Let Ω denote an arbitrary interval. Then $B(\Omega)$, the set of all bounded functions on Ω, and $C(\Omega)$, the set of all bounded, continuous functions on Ω, form Banach spaces.

11.4 Operators on a Banach Space:

Definition: Let X be a Banach space. A function, A, with domain and range in X, is called an operator.

Definition: If

$$A(f + g) = Af + Ag \quad ,$$

and

$$A(\alpha f) = \alpha Af ,$$

for any f and g in X and any complex number α, then A is a linear operator.

If A fails to satisfy one of the preceding conditions, it is non-linear.

Definition: The operator A is bounded if there exists a constant K such that

$$\|Af\| \leq K\|f\| ,$$

for all f in X. The smallest such number K for which this is true is called the norm of A and is denoted by $\|A\|$.

It is easy to see that

$$\|A\| = \sup_{\|f\| \neq 0} \|Af\| / \|f\| .$$

Definition: The operator A satisfies a Lipschitz condition if there exists a number K such that

$$\|Af - Ag\| \leq K\|f - g\| ,$$

for all f and g in X.

Definition: The operator A is continuous at f if

$$\lim_{n \to \infty} Af_n = Af ,$$

whenever

$$\lim_{n \to \infty} f_n = f ,$$

that is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|Af_n - Af\| < \epsilon$, whenever $\|f_n - f\| < \delta$. An operator, continuous for all f in X, is continuous.

The following theorem is of special interest in uniting these concepts.

Theorem 11.4.1: If A is a linear operator the following are equivalent.

1. A is a bounded operator.

2. A satisfies a Lipschitz condition.

3. A is continuous at 0 in X.

4. A is continuous.

Proof: We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$1 \Rightarrow 2$: If A is bounded, then

$$\|Af\| \le \|A\| \|f\| ,$$

for all f in X. Replacing f by $f - g$,

$$\|Af - Ag\| = \|A(f - g)\| \le \|A\| \|f - g\| ,$$

and

$$\|Af - Ag\| \le \|A\| \|f - g\| .$$

$2 \Rightarrow 3$: Suppose $\{f_n\} \to 0$ as $n \to \infty$. Since A is linear, $A(0) = 0$. The Lipschitz condition guarantees

$$\|Af_n - 0\| \le K\|f_n - 0\| .$$

From this it is obvious that $Af_n \to 0$, as $f_n \to 0$, and A is continuous at 0.

<u>3 ⇒ 4</u>: Suppose $\{f_n\} \to f$ as $n \to \infty$. Since A is continuous at 0, we see

$$\lim_{n \to \infty} A(f_n - f) = 0 \quad .$$

But this is equivalent to

$$\lim_{n \to \infty} Af_n = Af \quad ,$$

since A is linear.

<u>4 ⇒ 1</u>: Suppose A were not bounded. Then there would exist a sequence $\{f_n\}$ in X such that

$$\|Af_n\| > n\|f_n\| \quad ,$$

for each n. Since A is linear, we can divide by $\|f_n\|$ to arrive at

$$\|Ag_n\| > n \quad ,$$

for all n, where $g_n = f_n/\|f_n\|$. Thus $\|g_n\| = 1$. But if $h_n = g_n/n$, we have

$$\lim_{n \to \infty} h_n = 0 \quad ,$$

since $\|h_n\| = 1/n$, and $\|Ah_n\| > 1$. This is a contradiction of the assumption of continuity of A.

One final statement of passing interest is the "closed graph theorem", which states that any linear operator, defined for all elements in X, is continuous, and hence, Lipschitzian and bounded. We shall not prove it here. The theorem is false for non-linear operators.

11.5 Examples of Operators on Banach Spaces: The simplest example of a linear operator L is to multiply by a constant. That is,

$$Lf = \alpha f ,$$

for some complex number α. Obviously the norm of L is $|\alpha|$.

Another more interesting example is that of multiplying any element x in E^n by a matrix. If

$$x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} , \qquad M = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} ,$$

then $Lx = Mx$ is a bounded linear operator no matter which norm is used.

If X is a function space, we have some examples which are quite useful in physical situations. If f is in X, such operators as multiplication by an arbitrary function $m(x)$, that is,

$$L_1 f(x) = m(x)f(x) ,$$

convolution,

$$L_2 f(x) = \int_\Omega m(x - t)f(t)dt ,$$

or more generally an integration of the form

$$L_3 f(x) = \int_\Omega m(x,t)f(t)dt ,$$

and finally a time-lag or shift operator,

$$L_4 f(x) = f(x - a) \quad ,$$

are frequently encountered.

It is easy to see that L_1 is bounded by

$$\operatorname*{ess\,sup}_{x \in \Omega} |m(x)|$$

in $\mathcal{L}^p(\Omega)$, $1 \le p \le \infty$, $B(\Omega)$ and $C(\Omega)$.

L_2 is bounded by

$$\int_\Omega |m(x)|dx$$

in $\mathcal{L}^1(\Omega)$, and in certain instances by

$$\int_\Omega |m(x)|dx$$

in $\mathcal{L}^2(\Omega)$. In $\mathcal{L}^\infty(\Omega)$, $B(\Omega)$ or $C(\Omega)$, L_2 is again bounded by

$$\int_\Omega |m(x)|dx \quad .$$

L_3 is bounded by

$$\int_\Omega \operatorname*{ess\,sup}_x |m(x,t)|dt$$

in $\mathcal{L}^1(\Omega)$, by

$$\left[\int_\Omega \int_\Omega |m(x,t)|^2 dxdt\right]^{\frac{1}{2}}$$

in $\mathcal{L}^2(\Omega)$, and in $\mathcal{L}^\infty(\Omega)$, $B(\Omega)$ or $C(\Omega)$ by

$$\int_\Omega \operatorname*{ess\,sup}_t |m(x,t)|dt \quad .$$

L_4, when appropriately defined, has 1 for a bound.

Some non-linear operators frequently encountered are the squaring operator,

$$Qf = f^2 \quad ,$$

which is unbounded, the dead zone operator, Zf, and the saturation operator, Sf. The last two are affected only by the modulus of f. The slope of any line lying above these graphs in the first quadrant will serve as a bound for these operators.

11.6 A Simple Feedback System in a Banach Space: Let us consider the following simple feedback system, $\mathcal{F}$, in X. The "input" or forcing function, f, is operated on by the operator, F, giving the "output" or response, r. r itself is operated on by G and the result, Gr, is subtracted from f giving an "error", e. e is then "fed back" to be operated on by F. Mathematically $\mathcal{F}$ can be described by

$$f - G(r) = e \quad ,$$

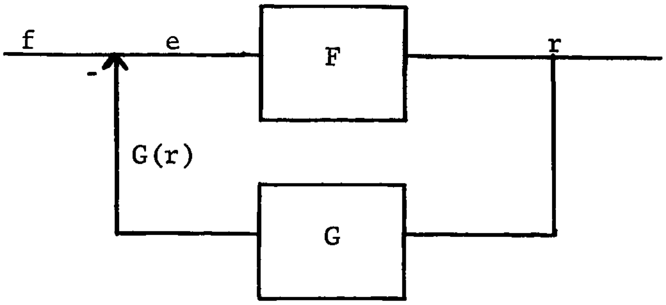
$$F(e) = r \quad ,$$

or by the single equation

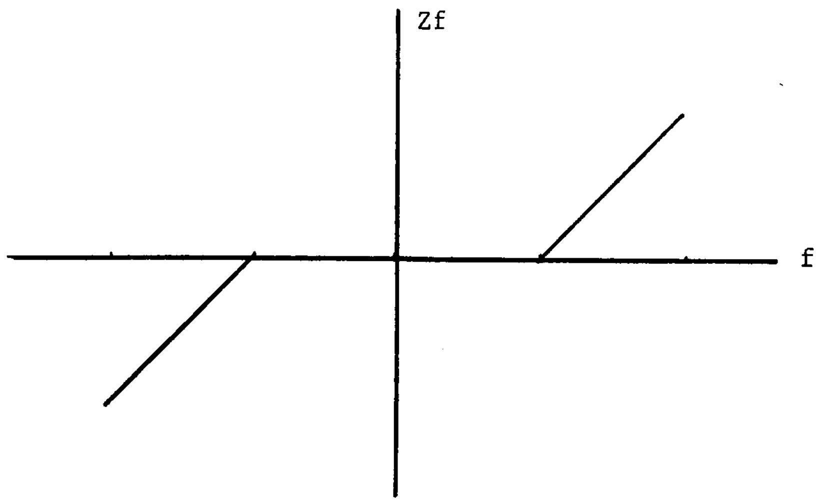
$$F(f - G(r)) = r \quad .$$

If f is in X, the question of whether or not there is an r in X is a natural one. It is also sometimes valuable to have an idea of the norm of r, given that of f. We present two theorems giving answers to these questions. Note we do NOT assume F and G are linear operators.

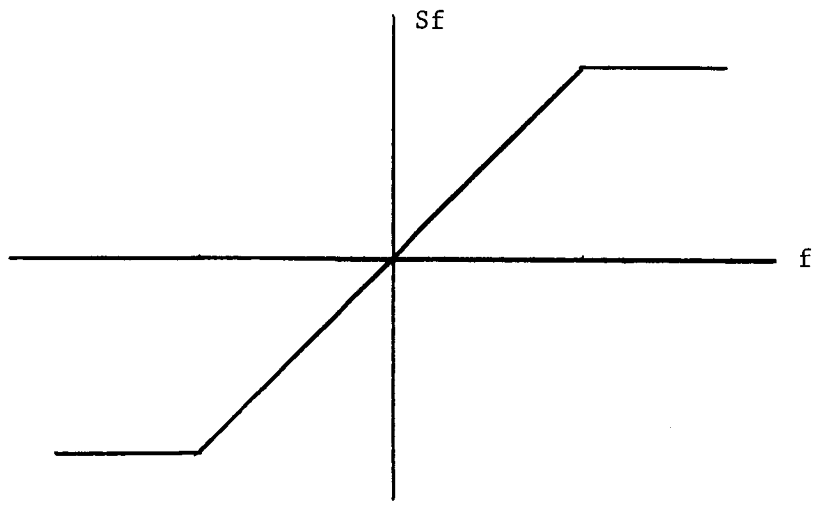
Theorem 11.6.1: If F and G satisfy a Lipschitz condition with constants K and L, respectively, and $KL < 1$, then $F(f - G(r)) = r$ has a unique



The Feedback System $\mathcal{F}$



Dead Zone Operator Z



Saturation Operator S

<u>solution</u> r <u>in</u> X.

<u>Proof</u>: We develop an iterative procedure to give a sequence of elements in X which converge to r. Let

$$r_1 = F(f) \quad ,$$

$$r_2 = F(f - G(r_1)) \quad ,$$

and in general

$$r_{n+1} = F(f - G(r_n)) \quad .$$

Each r_n is in X, and

$$\begin{aligned}
\|r_n - r_{n-1}\| &= \|F(f - G(r_{n-1})) - F(f - G(r_{n-2}))\| \quad , \\
&\le K\|[f - G(r_{n-1})] - [f - G(r_{n-2})]\| \quad , \\
&= K\|G(r_{n-1}) - G(r_{n-2})\| \quad , \\
&\le KL\|r_{n-1} - r_{n-2}\| \quad .
\end{aligned}$$

We see by induction that

$$\|r_n - r_{n-1}\| \le (KL)^{n-1}\|F(f)\| \quad ,$$

and

$$\|r_{n+m} - r_n\| \le (KL)^n\|F(f)\|/[1 - KL] \quad .$$

Thus the sequence $\{r_n\}$ is a Cauchy sequence in X and has a limit, r, in X.

Further,

$$\left\| r - F(f - G(r)) \right\| = \left\| r - r_n + F(f - G(r_{n-1})) - F(f - G(r)) \right\| ,$$

$$\leq \left\| r - r_n \right\| + KL \left\| r - r_{n-1} \right\| ,$$

which approaches 0 as $n \to \infty$. Thus

$$r = F(f - G(r)) .$$

If r and r' are both solutions in X, then

$$\left\| r - r' \right\| \leq KL \left\| r - r' \right\| < \left\| r - r' \right\| ,$$

which is a contradiction unless $r = r'$.

As with ordinary differential equations, this method of successive approximations gives a procedure for calculating r as well as an estimate of the error involved.

It is also important to note that this theorem guarantees uniqueness only in X. It is quite possible for $\mathcal{F}$ to have solutions not in X, and in so doing destroy "stability", which we have not yet defined in this setting.

Theorem 11.6.2: *If* F *and* G *are bounded, respectively, by* K *and* L, $KL < 1$, *and* $F(f - G(r)) = r$ *has a solution in* X, *then*

$$\left\| r \right\| \leq [K/(1 - KL)] \left\| f \right\| .$$

Proof:

$$\|r\| = \|F(f - G(r))\| ,$$

$$\le K\|f - G(r)\| ,$$

$$\le K\|f\| + K\|G(r)\| ,$$

$$\le K\|f\| + KL\|r\| .$$

Thus

$$\|r\|(1 - KL) \le K\|f\| ,$$

and the result follows.

Corollary: If F and G satisfy a Lipschitz condition with constants K and L, respectively, $KL < 1$, and $F(0) = 0$, $G(0) = 0$, then $F(f - G(r)) = r$ has a unique solution r in X, and

$$\|r\| \le [K/(1 - KL)]\|f\| .$$

Proof: Since $F(0) = 0$, the Lipschitz condition

$$\|Ff - Fg\| \le K\|f - g\| ,$$

with $g = 0$, shows that K is a bound for F. Similarly, L is a bound for G. The results follow from Theorems 11.6.1 and 11.6.2.

11.7 Stability in a Banach Space: The following definition is a convenient choice. It is motivated by the rather loose definition "bounded inputs have bounded outputs", which is sometimes seen.

Definition: The system, $\mathcal{F}$ described by

$$F(f - G(r)) = r \quad ,$$

is stable if, for all forcing functions, f, in X, the response, r, is always an element of X.

It is entirely possible for f to be in X with r not. Such systems are unstable.

Some of the Banach spaces most frequently encountered while working with feedback systems are $\mathcal{L}^p(0,\infty)$, $1 \le p \le \infty$, $C(0,\infty)$ and $B(0,\infty)$. In practical applications the elements in these spaces are functions of time, t, which varies from 0 to ∞. If negative times are considered, the elements are assumed to be zero when $t < 0$.

Let $X(0,N)$ denote any of the spaces $\mathcal{L}^p(0,N)$, $1 \le p \le \infty$, $B(0,N)$ or $C(0,N)$, where $0 < N \le \infty$. We denote the norm in $X(0,N)$ by $\|\cdot\|_N$ unless $N = \infty$. In that case we denote it by $\|\cdot\|$.

Stability Theorem 11.7.1: Let f be in $X(0,\infty)$, and let r be in $X(0,N)$ for all finite $N > 0$. Suppose for all g and h in $X(0,N)$, for all $N > 0$, that Fg, Fh, Gg and Gh are in $X(0,N)$, and satisfy

$$\|Fg - Fh\|_N \le K\|g - h\|_N \quad ,$$

$$\|Gg - Gh\|_N \le L\|g - h\|_N \quad ,$$

with $KL < 1$. Then r is in $X(0,\infty)$ and is identical with that of Theorem 11.6.1.

The proof is identical to the uniqueness part of Theorem 11.6.1.

Stability Theorem 11.7.2: Let f be in $X(0,\infty)$, and let r be in $X(0,N)$ for all finite $N > 0$. Suppose for all g in $X(0,N)$, for all $N > 0$, Fg and Gg are in $X(0,N)$, and satisfy

$$\|Fg\|_N \leq K\|g\|_N ,$$

$$\|Gg\|_N \leq L\|g\|_N ,$$

with $KL < 1$. Then r is in $X(0,\infty)$, and

$$\|r\| \leq [K/(1 - KL)]\|f\| .$$

Proof: We have $F(f - G(r)) = r$. Thus

$$\|r\|_N \leq K\|f - G(r)\|_N ,$$

$$\leq K\|f\|_N + KL\|r\|_N ,$$

and

$$\|r\|_N \leq [K/(1 - KL)]\|f\|_N .$$

Letting $N \to \infty$ completes the proof.

11.8 Applications: Again let $X(0,\infty)$ denote any of the spaces $\mathcal{L}^p(0,\infty)$, $1 \leq p \leq \infty$, $B(0,\infty)$ or $C(0,\infty)$.

In the feedback system we considered in Chapter VI, F was expressed as $h(D)/g(D)$, where h and g were polynomials with constant coefficients and G was a multiplication by $-e^{i\theta}$. In this case, in X, the norm of G is 1, while F is not uniquely defined. However, if we require Ff and a sufficient number of derivatives to vanish at 0, Ff is unique, and can be expressed in terms of a Green's function as

$$\int_0^t \Gamma(t - u)f(u)du \quad .$$

The norm of F on $X(0,\infty)$ is therefore

$$\int_0^\infty |\Gamma(t)|dt \quad .$$

As an example, let $F = K(D + a)^{-1}$, $a > 0$, and $G = 1$.

$$Ff(t) = K \int_0^t e^{-a(t-u)} f(u)du \quad .$$

The norm of F is $|K|/a$. Theorems 11.7.1 and 11.7.2 tell us that if $|K| < a$, the solution, r, of

$$K \int_0^t e^{-a(t-u)}[f(u) - r(u)]du = r(t) \quad ,$$

or of

$$(D + a + K)r(t) = f(t) \quad , \qquad r(0) = 0 \quad ,$$

is in $X(0,\infty)$ when f is, and further that

$$\left\| r \right\| \le \frac{K}{a - K} \left\| f \right\| \quad .$$

If F also involves a time lag, τ, i.e., $F = Ke^{-\tau D}(D + a)^{-1}$, the results are still valid.

As a final example, let us consider the system, $\mathfrak{F}$, in which $F = F_1 \cdot S$, $G = G_1 \cdot Z$, where

$$F_1 = K[(D + a_1)^{n_1}(D + a_2)^{n_2}(D + a_3)^{n_3}]^{-1} \quad ,$$

$a_i > 0$, S is a saturation operator,

$$Sf = \begin{cases} f, & |f| < 1, \\ \text{sign } f, & |f| \geq 1, \end{cases}$$

$G_1 = L[(D + b_1)^{m_1}(D + b_2)^{m_2}]^{-1}$, $b_i > 0$, Z is a dead zone operator,

$$Zf = \begin{cases} f + 1, & f < -1, \\ 0, & |f| < 1, \\ f - 1, & f > 1. \end{cases}$$

Then F_1 is bounded by

$$K/a_1^{n_1}a_2^{n_2}a_3^{n_3},$$

S by 1, G_1 by

$$L/b_1^{m_1}b_2^{m_2}$$

and Z by 1. Theorems 11.7.1 and 11.7.2 tell us that $\mathcal{F}$ is stable when

$$KL/a_1^{n_1}a_2^{n_2}a_3^{n_3}b_1^{m_1}b_2^{m_2} < 1,$$

and in that case

$$\|r\| \leq \frac{[K/a_1^{n_1}a_2^{n_2}a_3^{n_3}]}{[1 - KL/a_1^{n_1}a_2^{n_2}a_3^{n_3}b_1^{m_1}b_2^{m_2}]}\|f\|.$$

EXERCISES

1. Let A be a bounded linear operator on a Banach space, X. Show that

$$\|A\| = \sup_{\|b\|=1} \|Af\| = \sup_{\|f\|\leq 1} \|Af\|.$$

2. Find a bound for $(D^2 + 2D + 4)^{-1}$.

References

1. W. Kaplan, "Operational Methods for Linear Systems", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

2. Ja. Kudrevic, "Stability of Non-linear Systems with Feedback", Automat. i Telemeh., Vol. 25, 1964, pp. 1145-1155.

3. B. C. Kuo, "Automatic Control Systems", Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.

4. E. J. McShane and T. A. Botts, "Real Analysis", D. Van Nostrand Company, Inc., Princeton, N. J., 1959.

5. A. R. Stubberud, "Analysis and Synthesis of Linear Time-Variable Systems", Univ. of California Press, Berkeley, Calif., 1964.

6. J. C. Truxal, "Automatic Feedback Control System Synthesis", McGraw-Hill Book Co., Inc., New York, 1955.

7. G. Zames, "On the Stability of Non-linear Time Varying Feedback Systems", Proc. Nat. Elect. Conf., Inc., Chicago, Ill., 1964, pp. 725-730.

Appendix I. Computer Program for the Ordinary and Time-Lag Root-Locus Methods

To have a computer give data for or actually draw a Michailov, Nyquist or Neimark diagram is a relatively easy task. The calculations are straight foreward computation. The computer is ideally suited for such a purpose.

The root-locus diagrams are somewhat more complicated. Upon first inspection it seems appropriate to let K take on various values and then factor $g(z) - Ke^{i\theta}h(z)$, but if this is attempted, certain problems immediately arise. First, what range should K have? Second, in factoring the polynomials, how much error is incurred? Each successive factor incurs more error. Third, in what part of the complex plane will the zeros of the polynomials to be factored lie? Most factoring techniques need a good first estimate to work at all. For reasons such as these, factoring $g(z) - Ke^{i\theta}h(z)$ is rejected as a reasonable procedure.

Let us consider the case where $\theta = \pi$ and the coefficients of $g(z)$ and $h(z)$ are all real, since this is what occurs in physical applications. In this instance the formula of Theorem 6.3.4,

$$\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{2k+1-i} h^{(i)}(x) g^{(2k+1-i)}(x) = 0 ,$$

provides an excellent method of attack. For each fixed value of x, the left side of this equation is a polynomial in y. The real zeros of the polynomial in y, together with the fixed value of x, determine points on the root-locus. A computer is well designed to permit x to range over a certain set of points, for instance from -10 to 0 in tenths and then find the zeros of the resulting polynomial in y, thus finding the root-locus in that interval on the x-axis.

Rather than factor the resultant polynomials in y, it is more convenient to let y range over a set of values, such as the integers, and look for sign changes. When an interval is found with the resulting polynomial in y taking on different signs at the ends, the interval is itself split into pieces and examined for sign changes. In this manner each point on the root-locus may be accurately found without accumulating errors.

Various attempts have been made to find accurate extimates for the largest zero of the resulting polynomial in y. None have proved reasonable. It seems better to arbitrarily restrict y just as x is. Usually only the root-locus near the origin is of interest, so this does not seem unnecessarily restrictive.

The procedure used in programming the time-lag root-locus is the same. The formula used is found in Theorem 9.3.2 and is

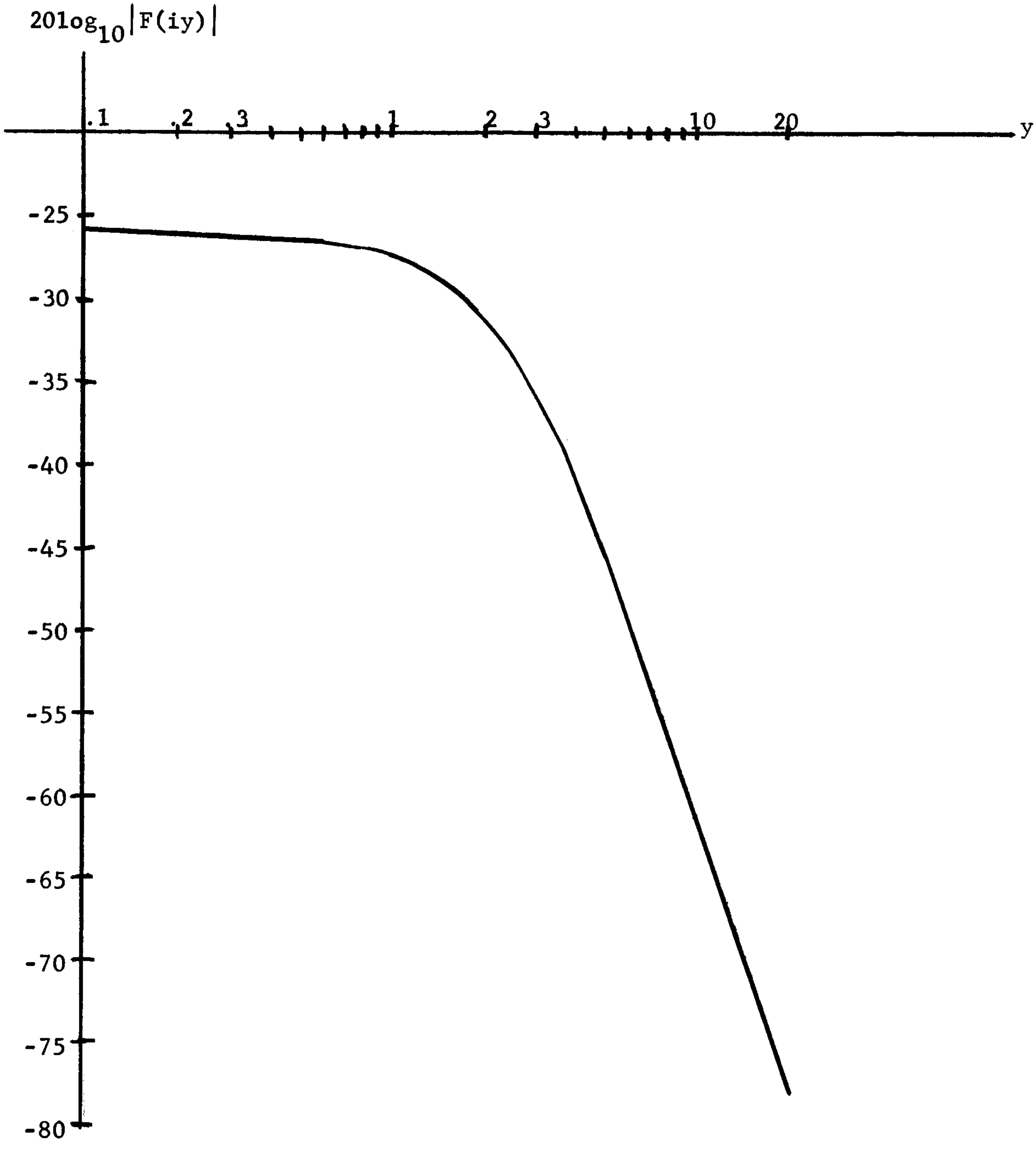
$$\cos \tau y \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^{2k+1-i} h^{(i)}(x) g^{(2k+1-i}(x)$$

$$- \sin \tau y \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k+1)!} \sum_{i=0}^{2k} \binom{2k}{i} (-1)^{2k-i} h^{(i)}(x) g^{(2k-i)}(x) = 0 .$$

A program has been written at the Pennsylvania State University with very satisfactory results.

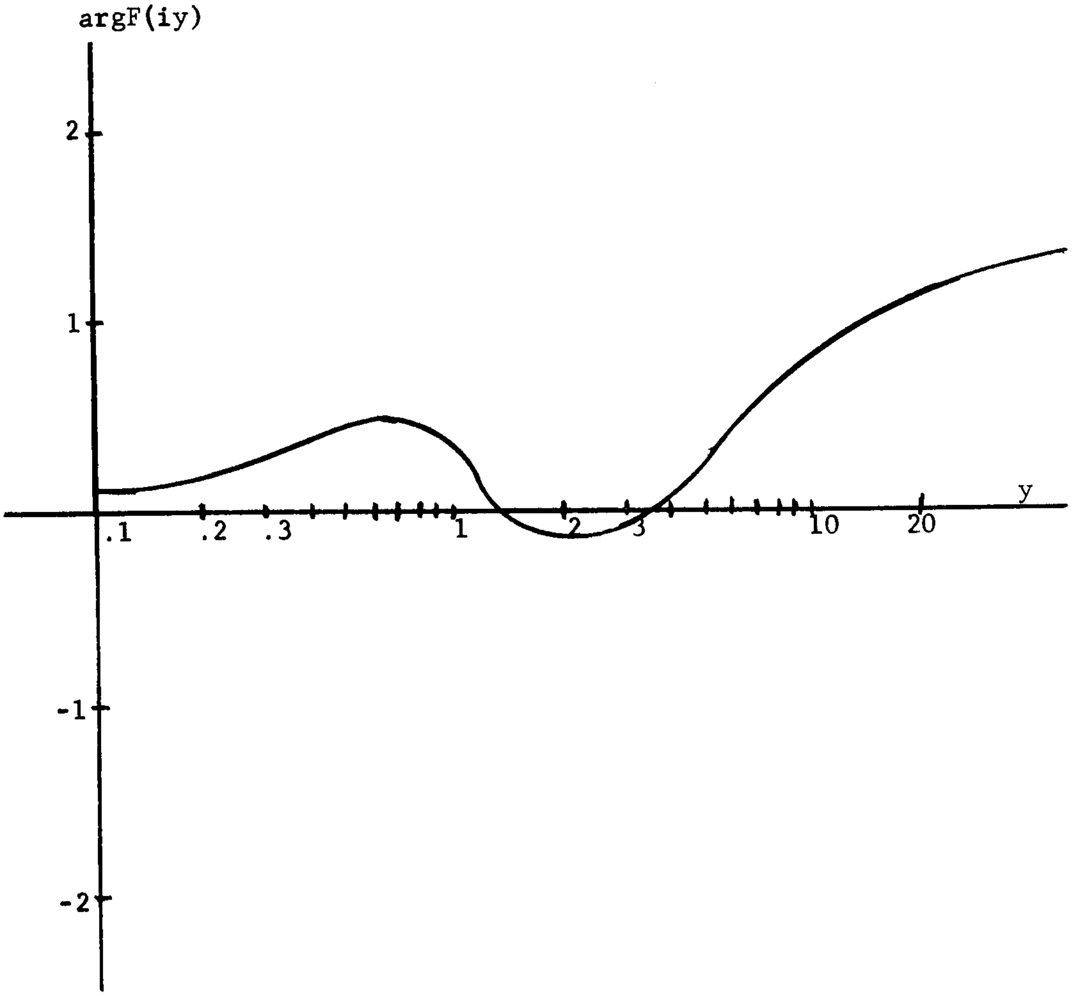
Appendix II. Examples for the Graphical criteria

Bode Diagram



$$F(z) = \frac{(z+1)(z+2)(z+3)(z^2-2z+2)}{(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)}$$

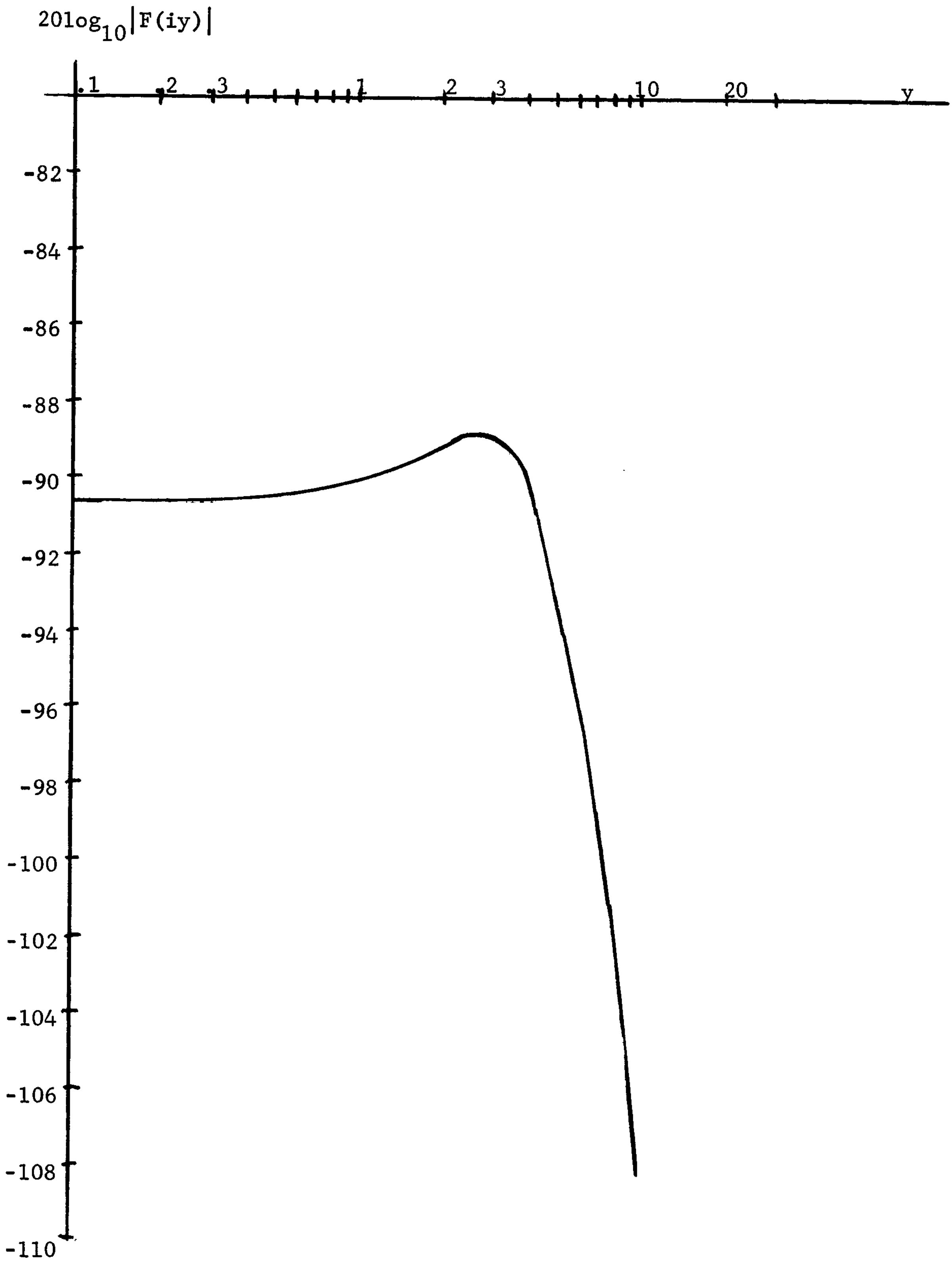
Bode Diagram



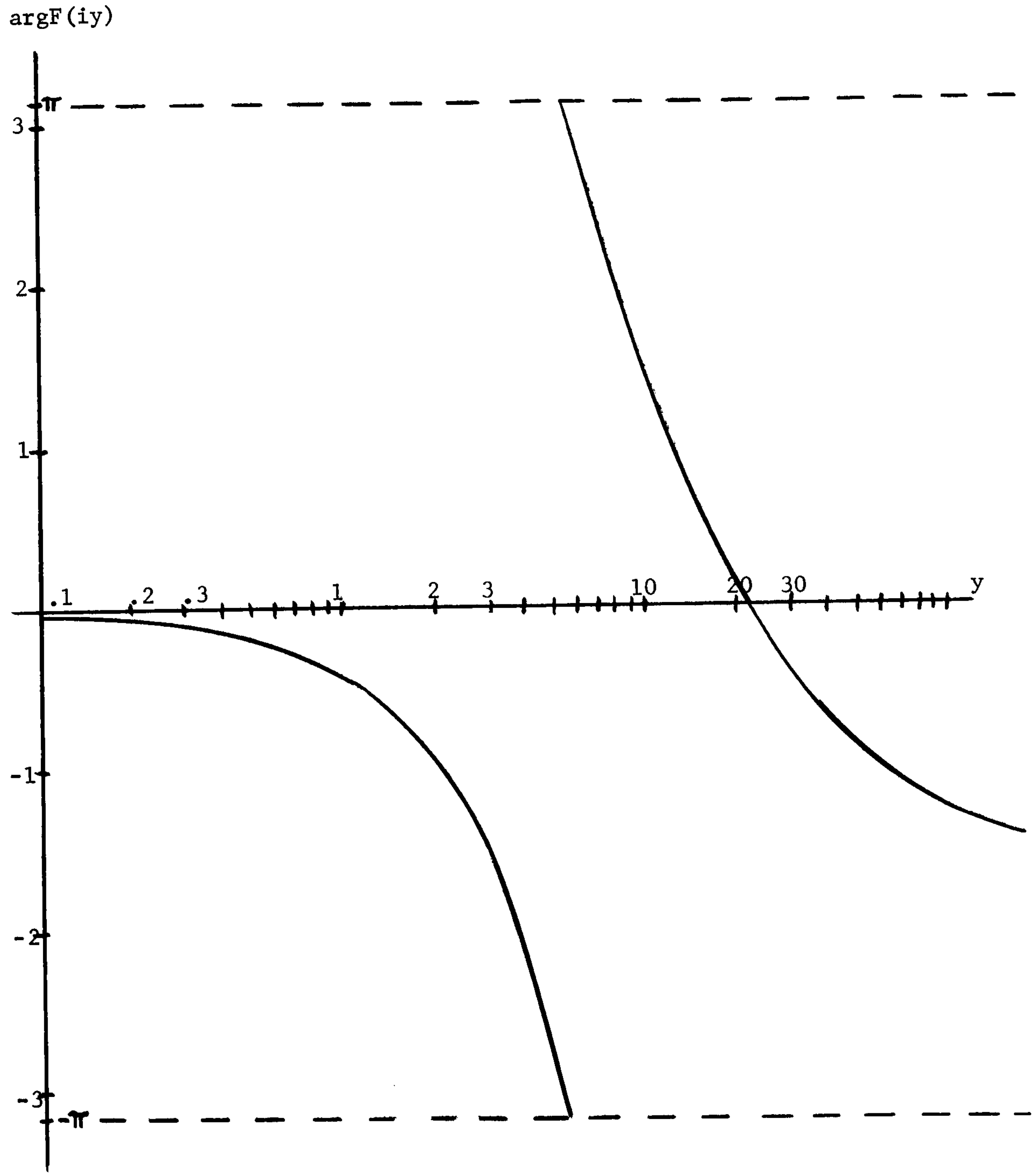
$$F(z) = \frac{(z+1)(z+2)(z+3)(z^2-2z+2)}{(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)}$$

Bode Diagram

$$F(z) = \frac{(z+5)(z+7)(z+9)}{(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)}$$ 183

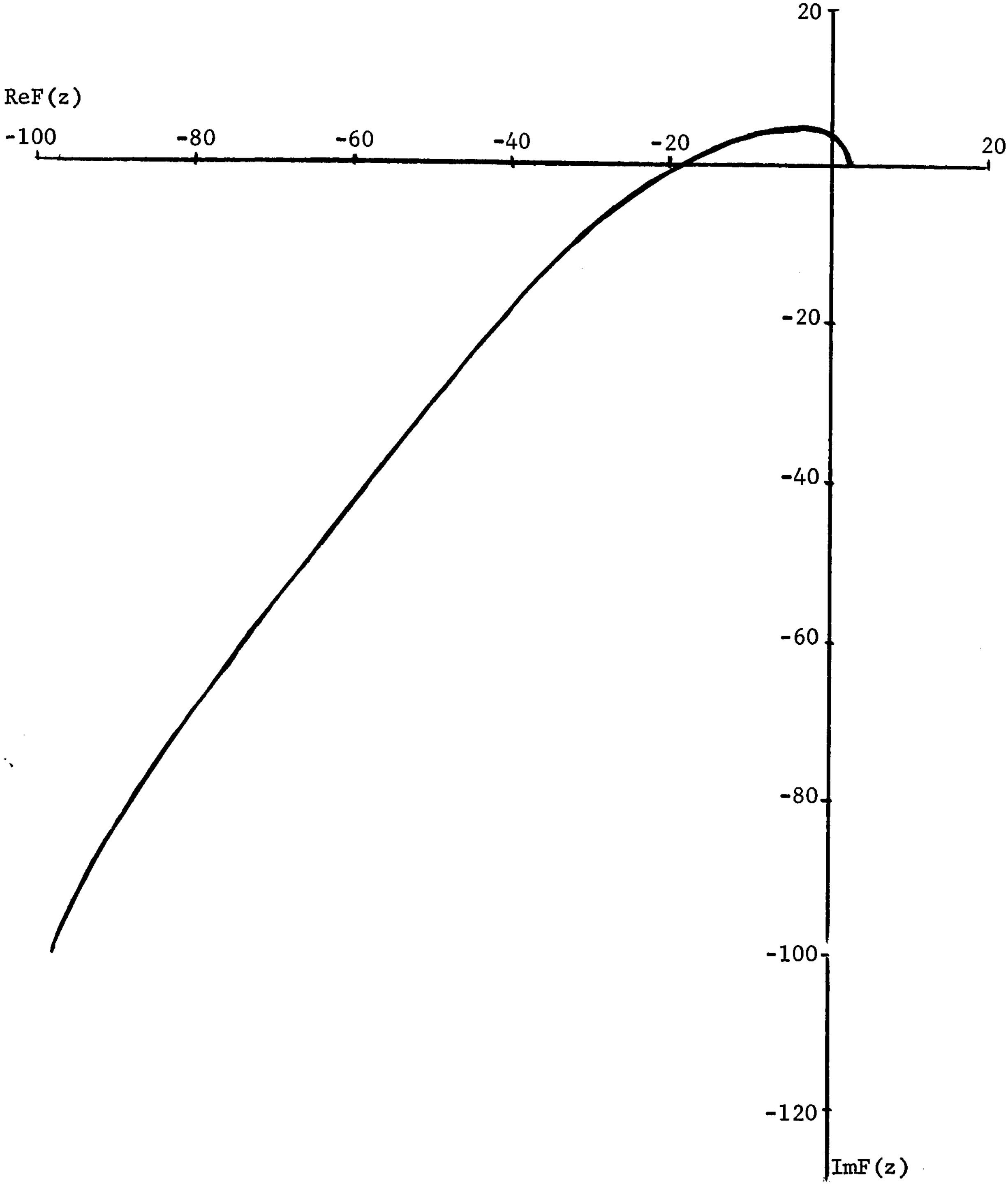


Bode Diagram



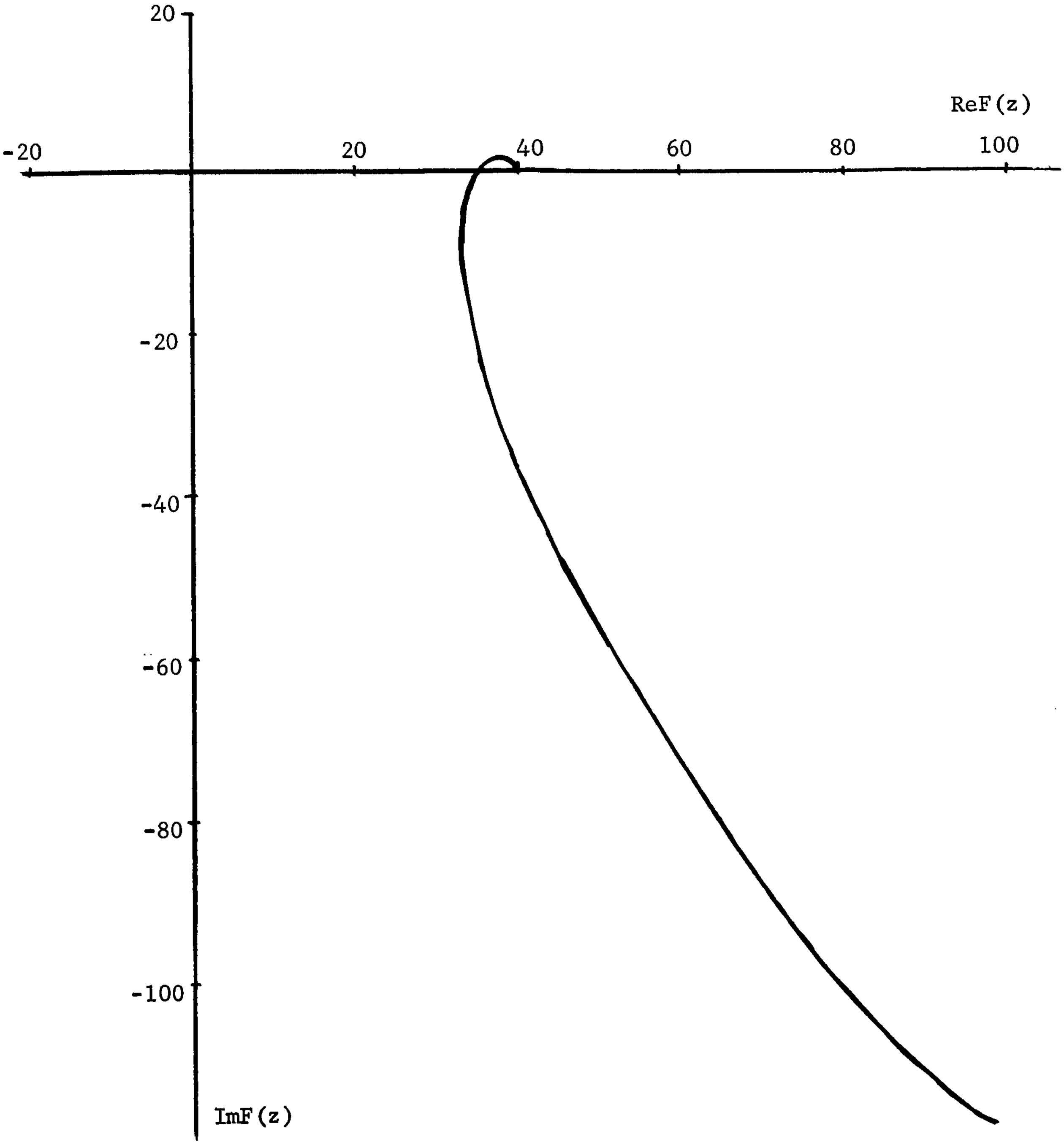
$$F(z) = \frac{(z+5)(z+7)(z+9)}{(z+10)9z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)}$$

Michailov Diagram



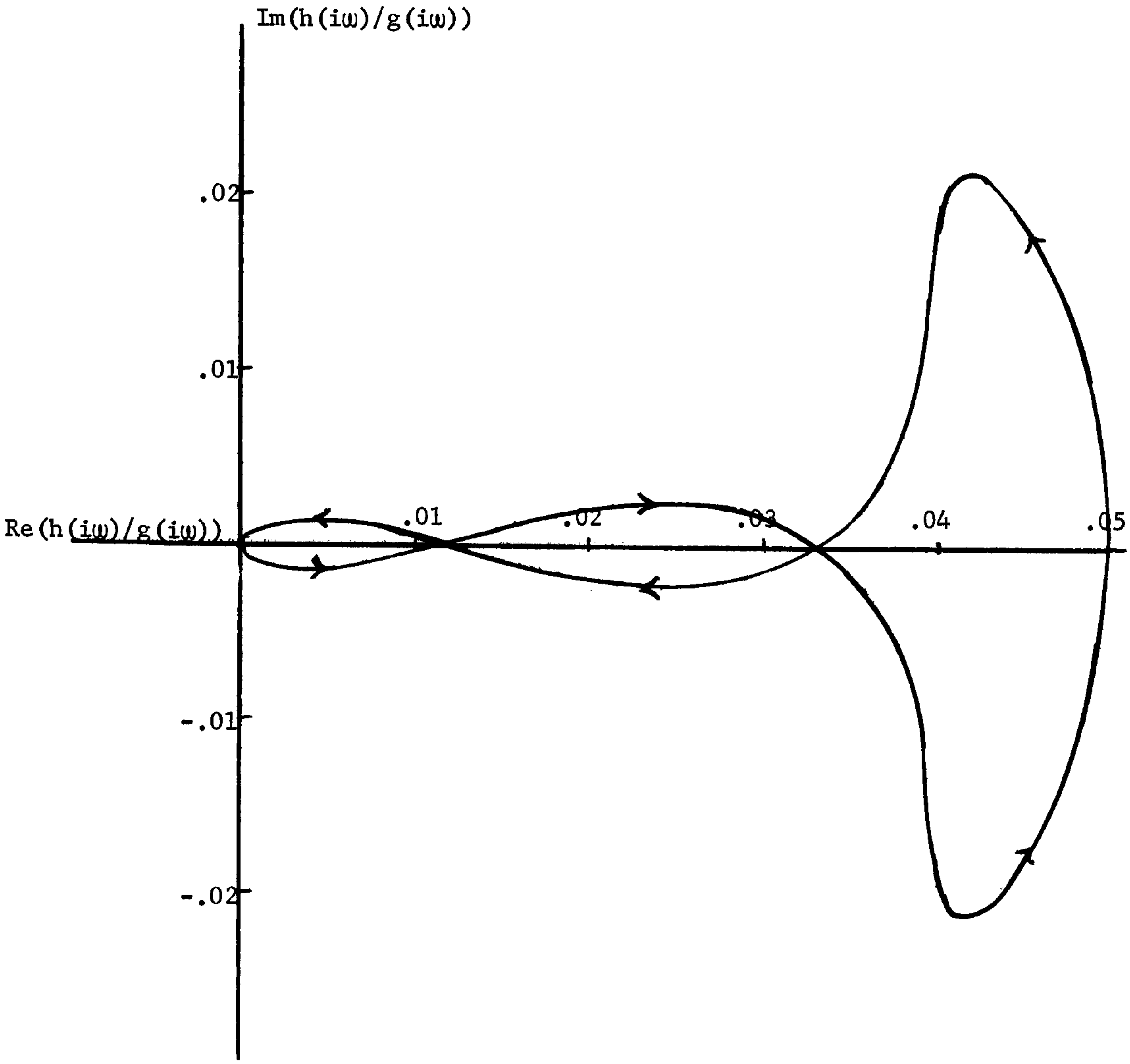
$F(z) = (z+1)^2(z+2)$

Michailov Diagram



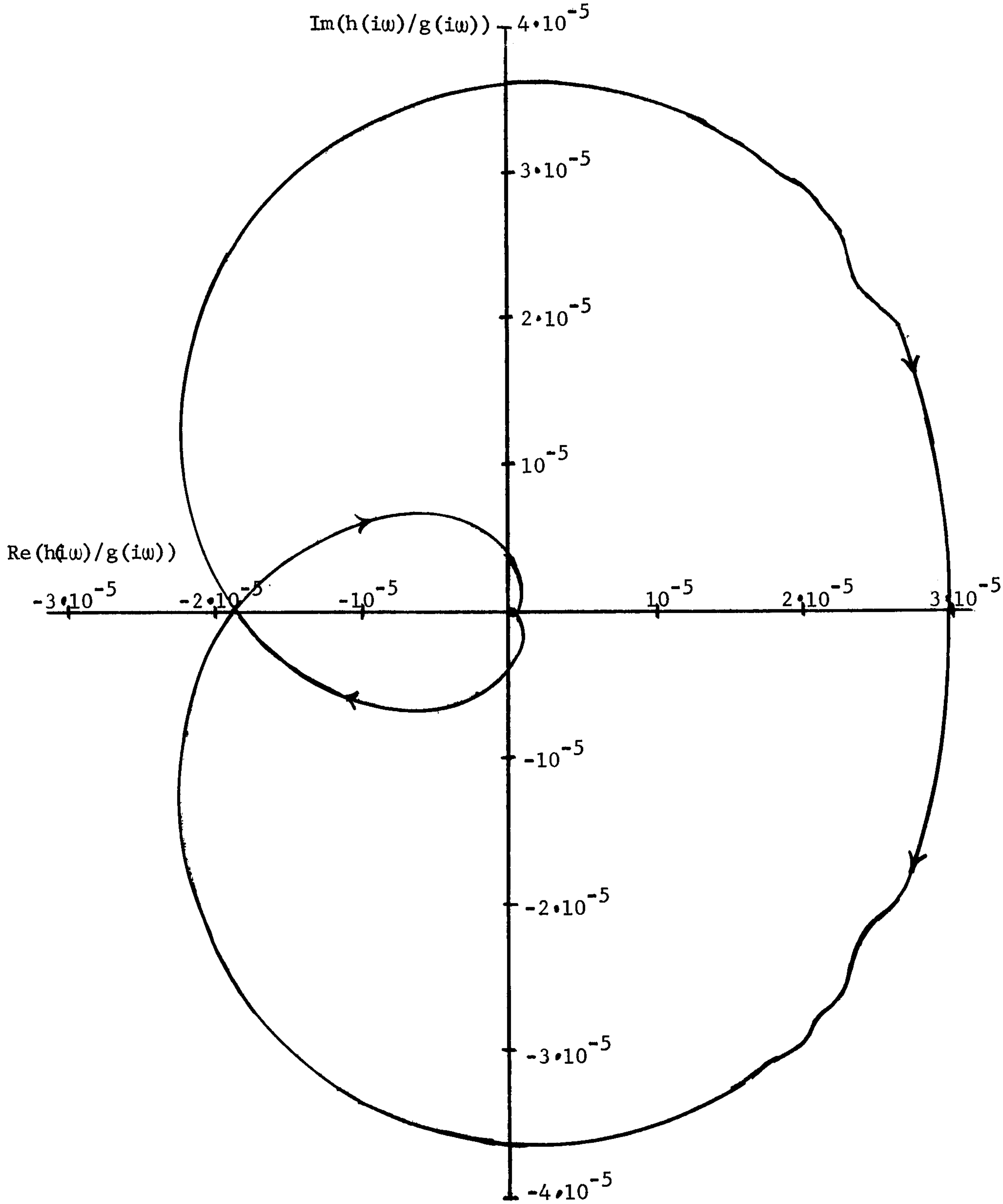
$$F(z) = (z^2+4z+8)(z^2-2z+5)$$

Nyquist Diagram



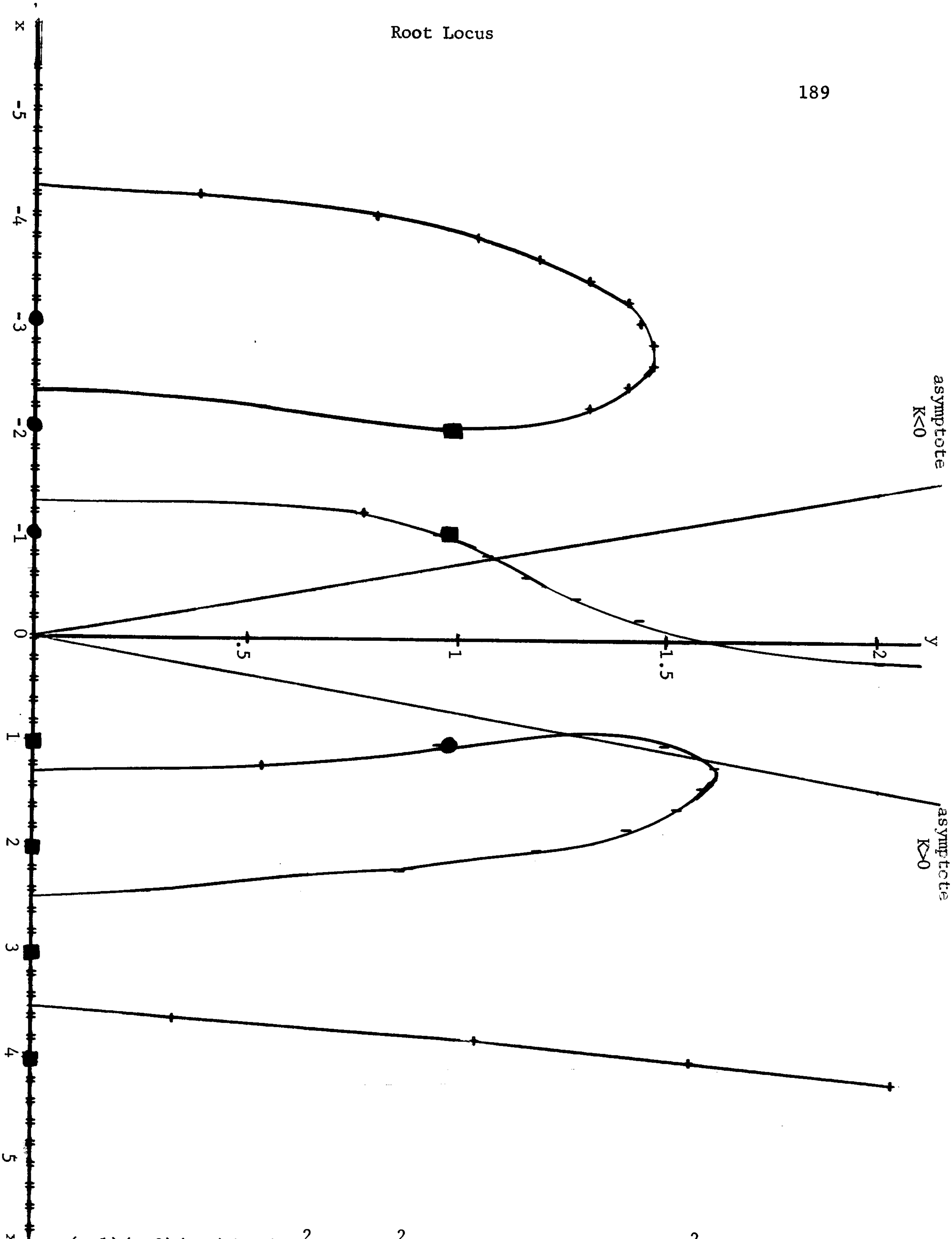
$$\frac{h(z)}{g(z)} = \frac{(z+1)(z+2)(z+3)(z^2-2z+2)}{(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)}$$

Nyquist Diagram



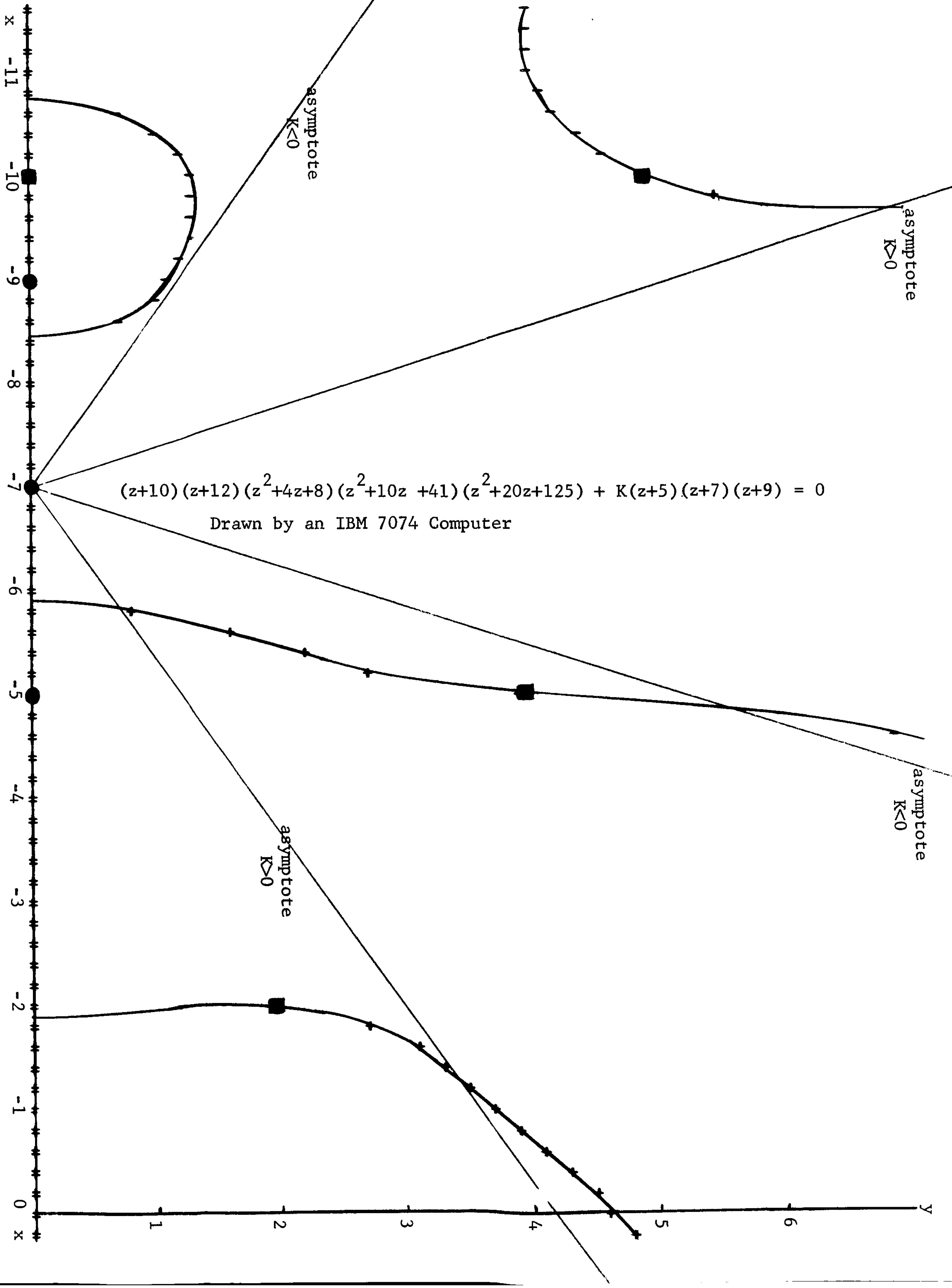
$h(z)/g(z) = (z+5)(z+7)(z+9)/(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)$

Root Locus

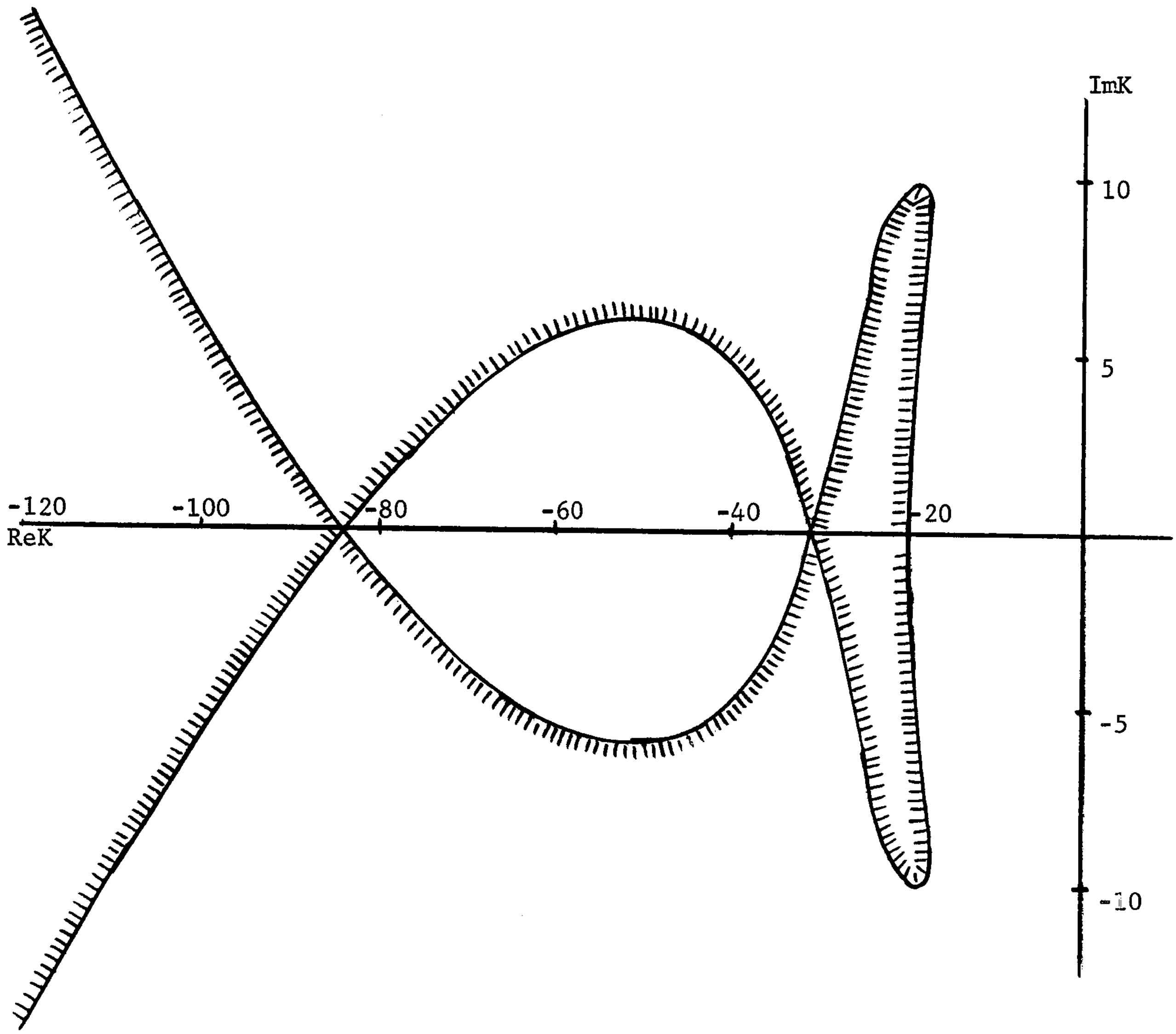


$(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5) + K(z+1)(z+2)(z+3)(z^2-2z+2) = 0$

Drawn by an IBM 7074 Computer

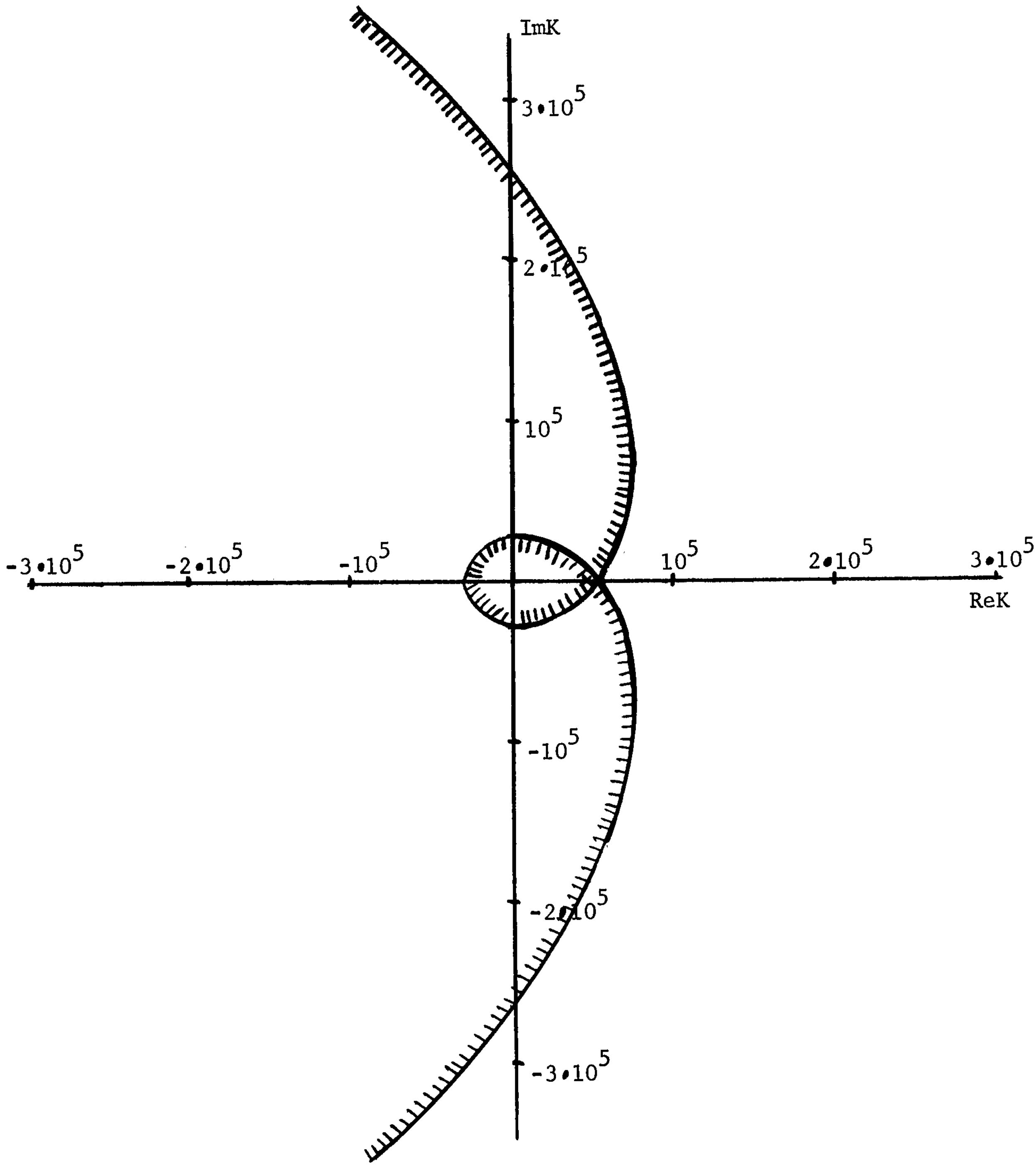

asymptote K<0
asymptote K>0
asymptote K<0
asymptote K>0
(z+10)(z+12)(z^2+4z+8)(z^2+10z +41)(z^2+20z+125) + K(z+5)(z+7)(z+9) = 0
Drawn by an IBM 7074 Computer
x
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
y

Neimark D-Partition

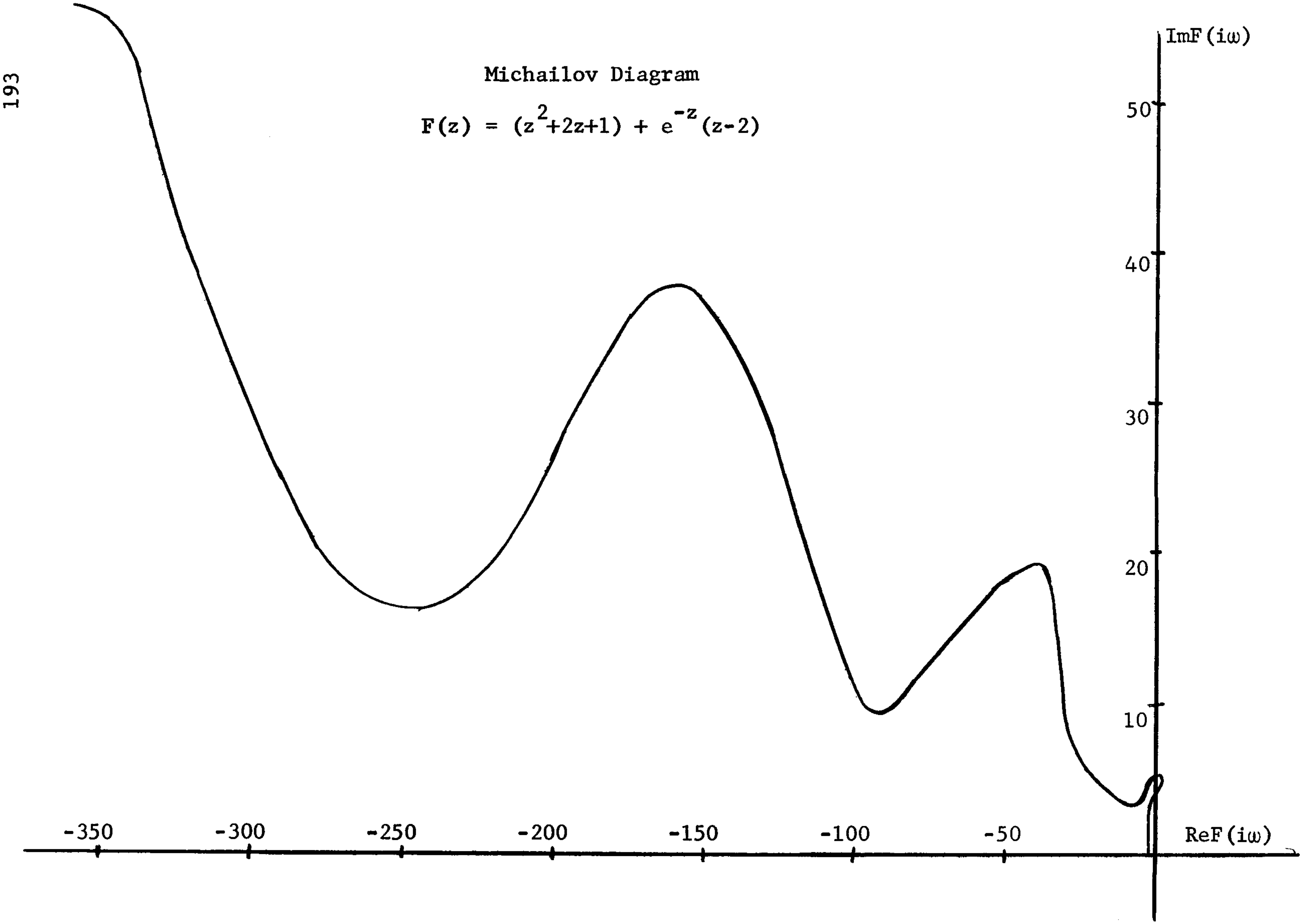


$$K = -\frac{(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)}{(z+1)(z+2)(z+3)(z^2-2z+2)}$$

Neimark D-Partition



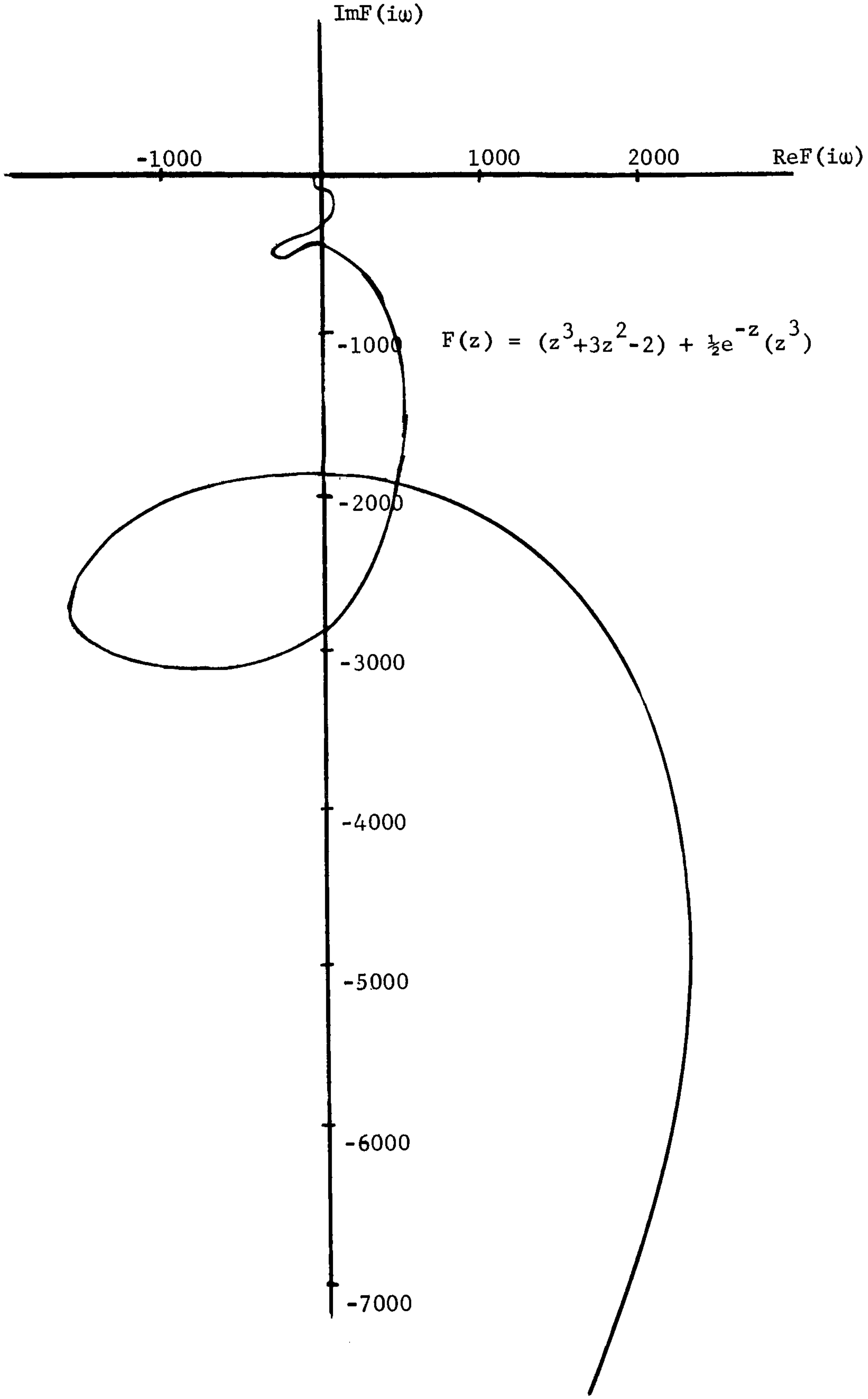
$$K = -\frac{(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)}{(z+5)(z+7)(z+9)}$$



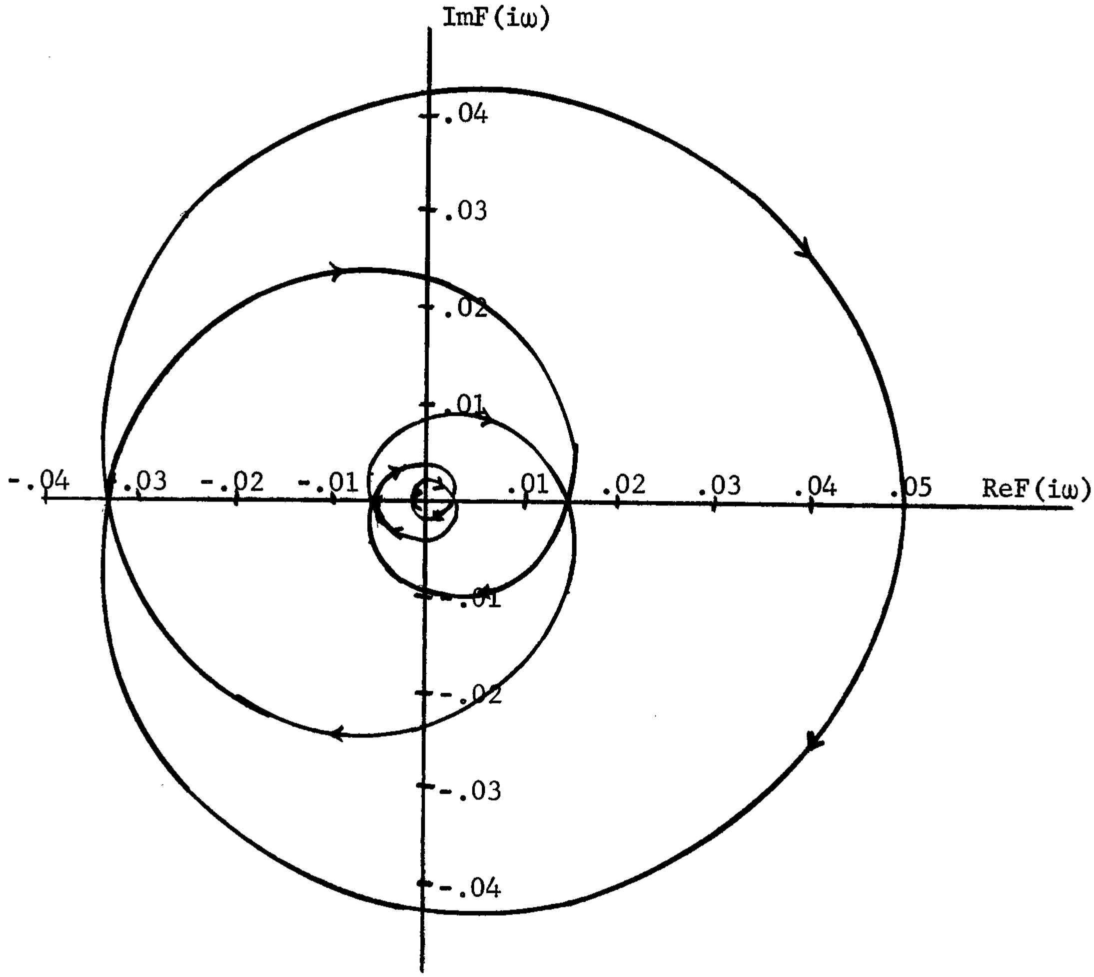
Michailov Diagram

$F(z) = (z^2+2z+1) + e^{-z}(z-2)$

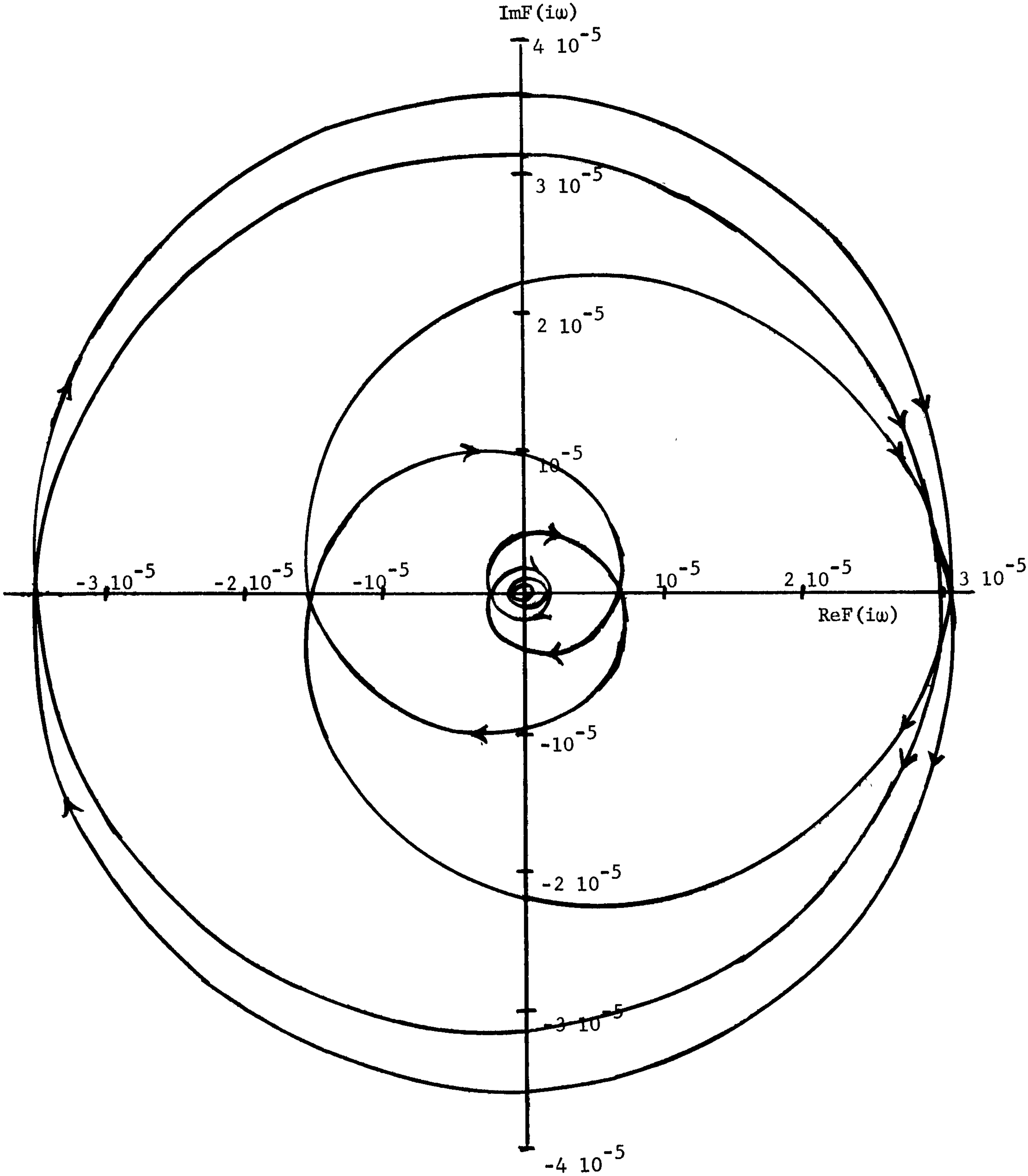
Michailov Diagram



Nyquist Diagram



$$F(z) = \frac{e^{-2z}(z+1)(z+2)(z+3)(z^2-2z+2)}{(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)}$$

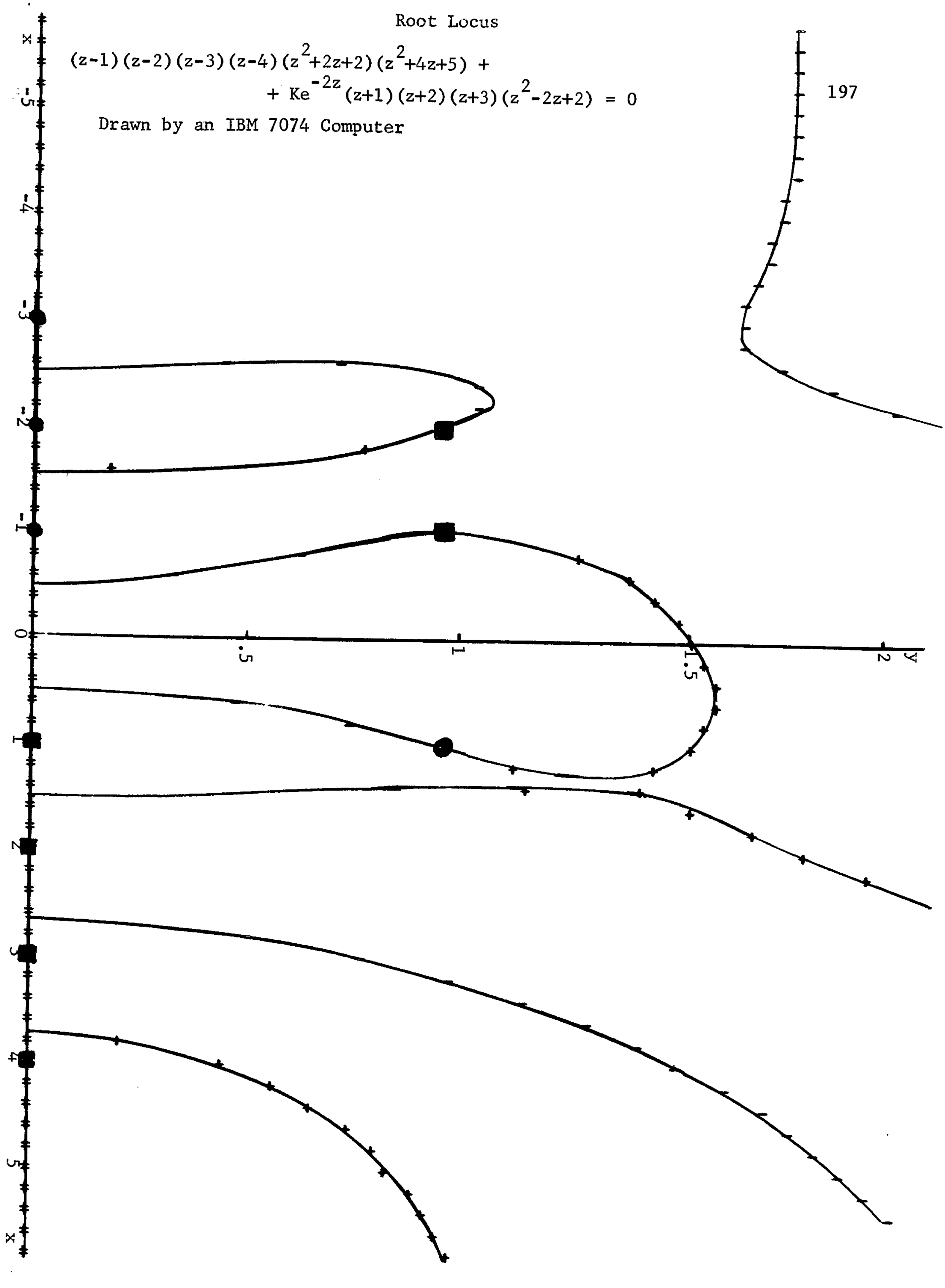


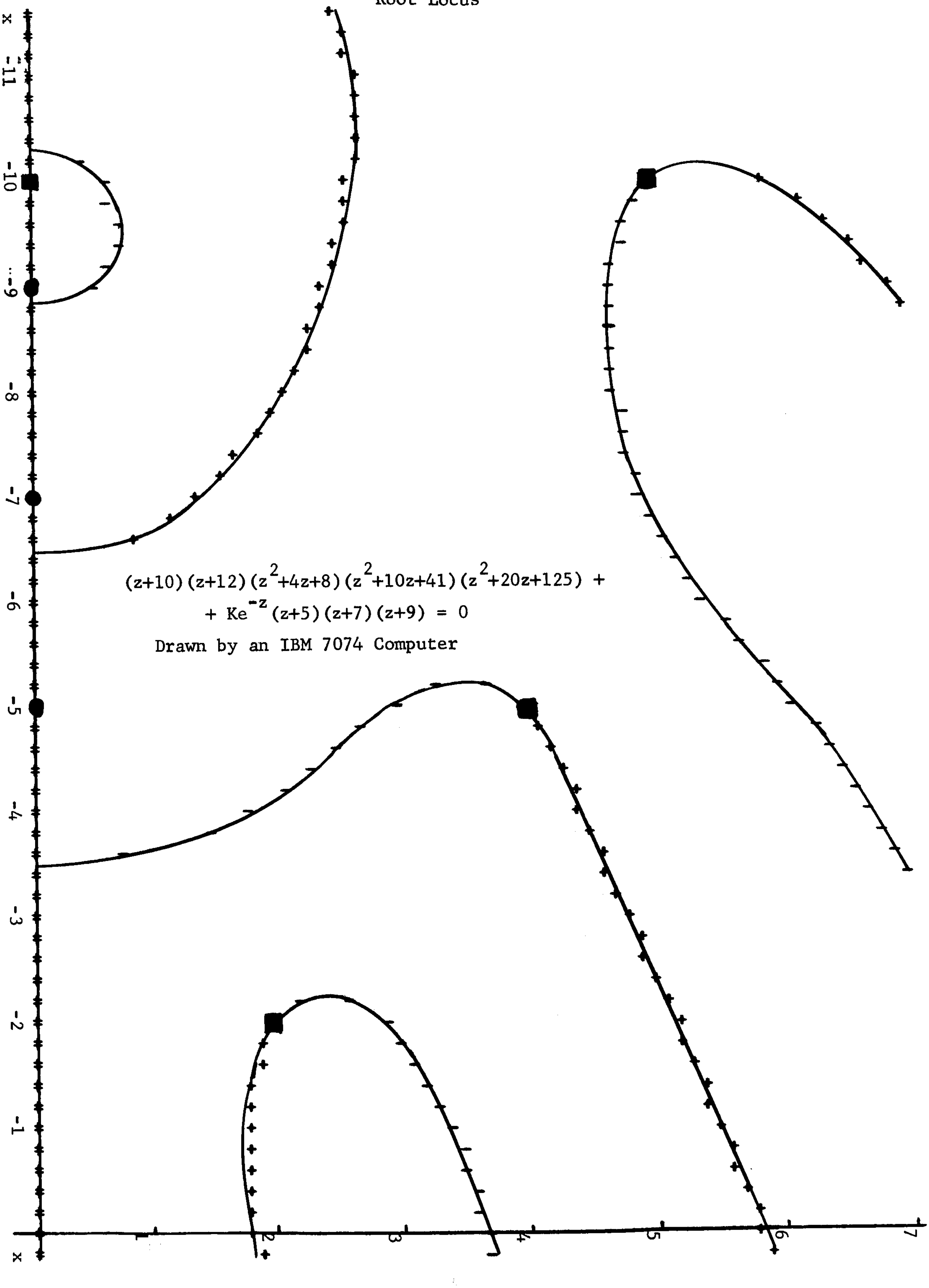
$$F(z) = e^{-z}(z+5)(z+7)(z+9)/(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)$$

Root Locus

$$(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5) +$$
$$+ Ke^{-2z}(z+1)(z+2)(z+3)(z^2-2z+2) = 0$$

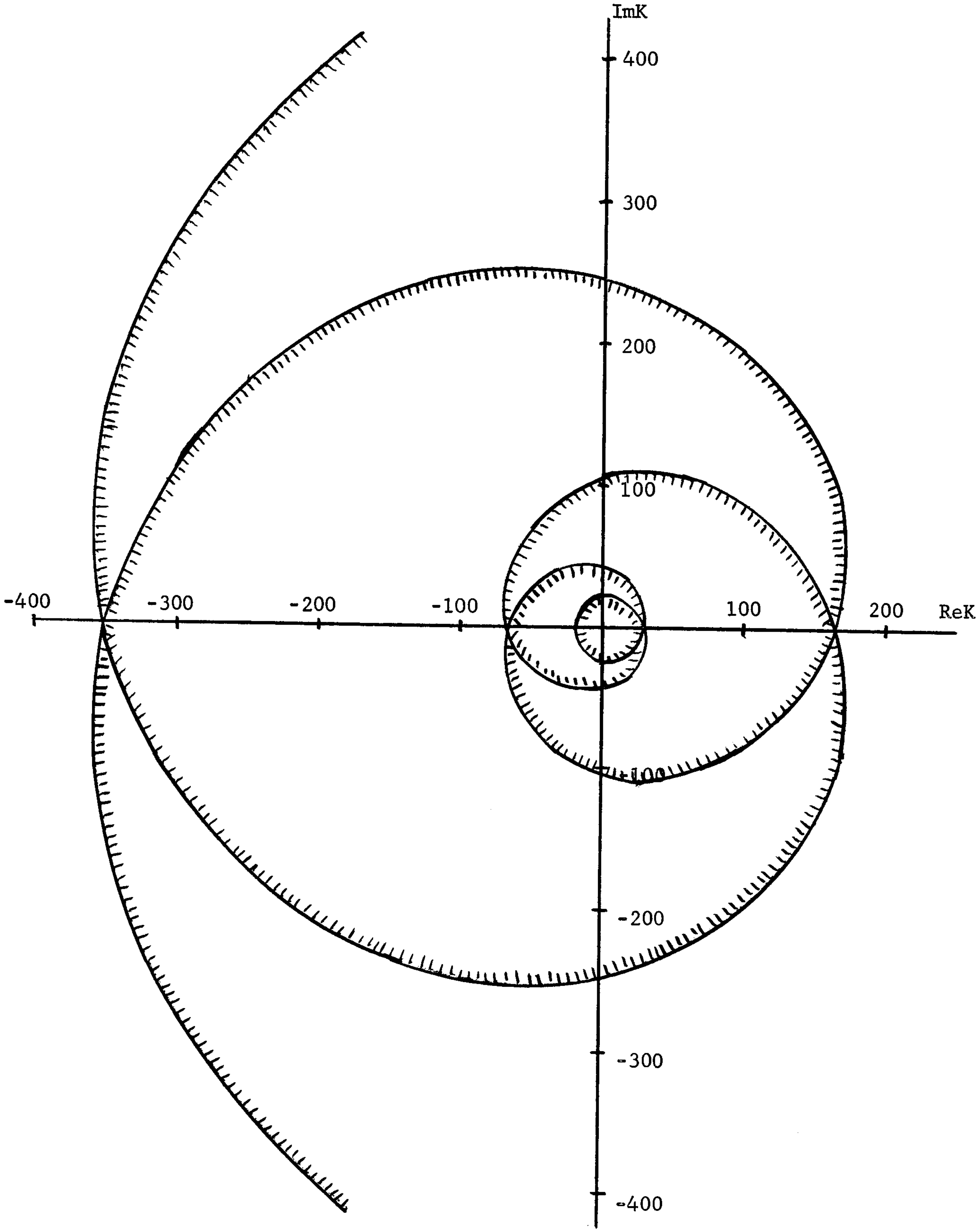
Drawn by an IBM 7074 Computer



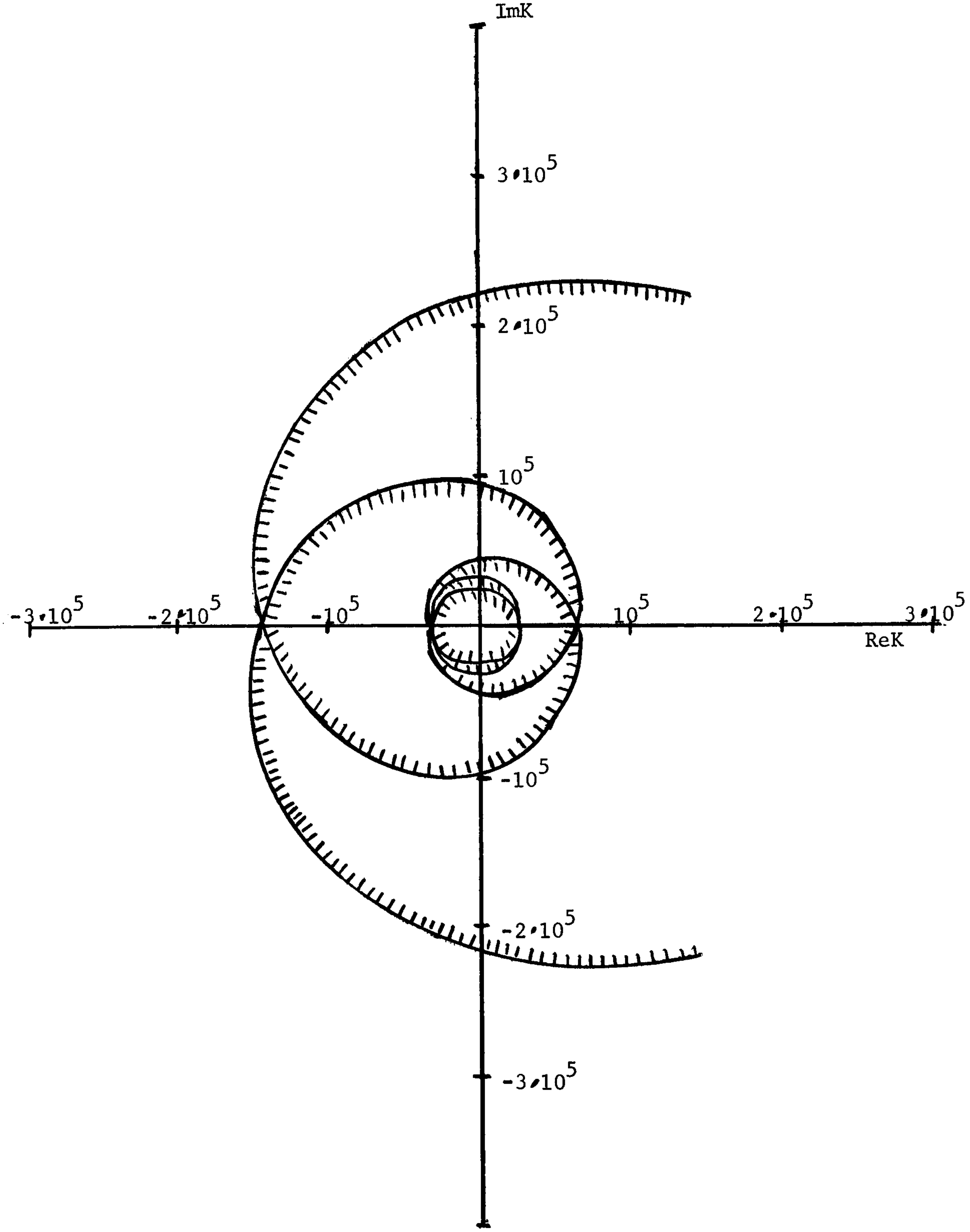


$$(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125) +$$
$$+ Ke^{-z}(z+5)(z+7)(z+9) = 0$$

Drawn by an IBM 7074 Computer



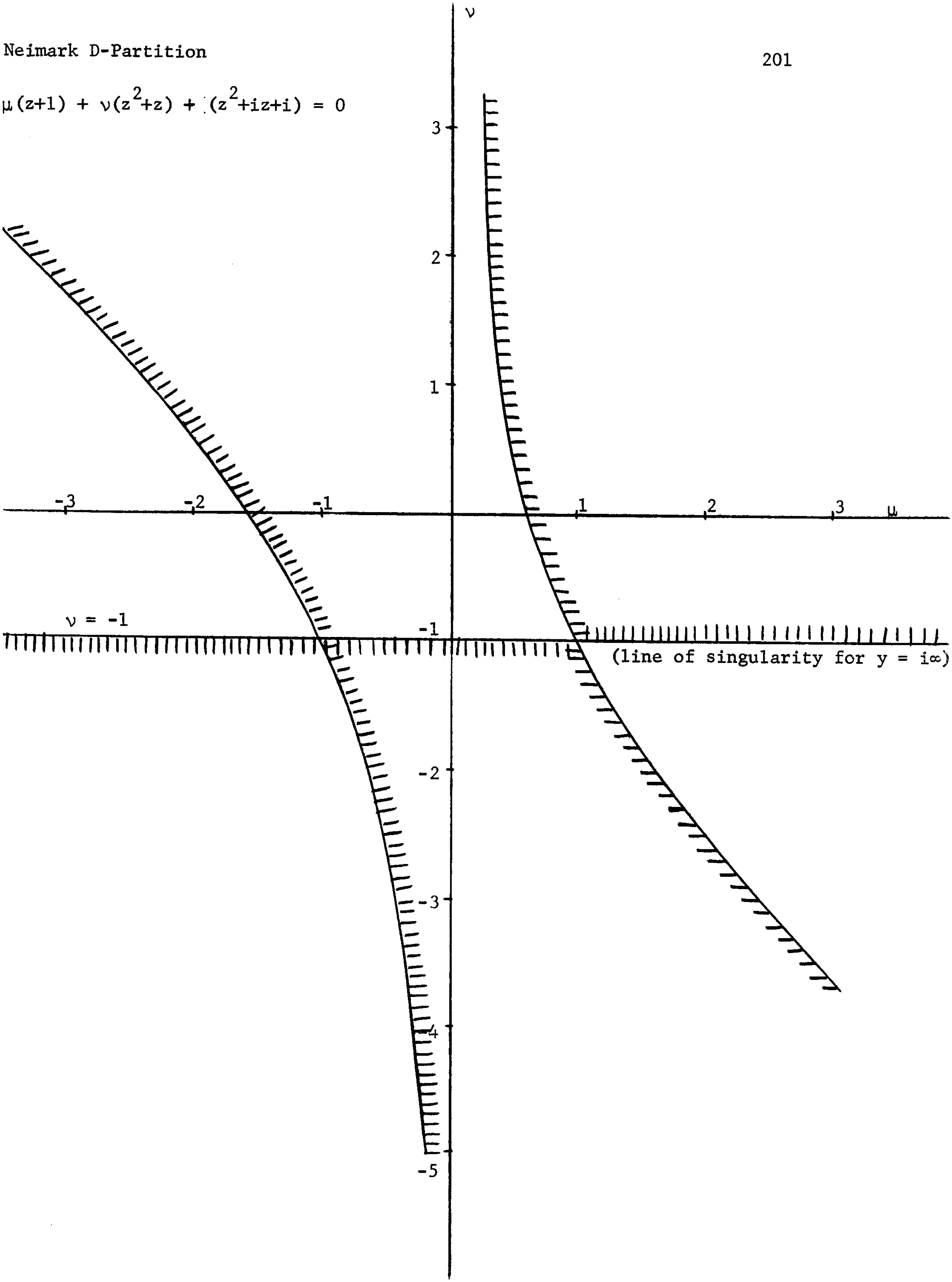
$$K = -e^{2z}(z-1)(z-2)(z-3)(z-4)(z^2+2z+2)(z^2+4z+5)/(z+1)(z+2)(z+3)(z^2-2z+2)$$

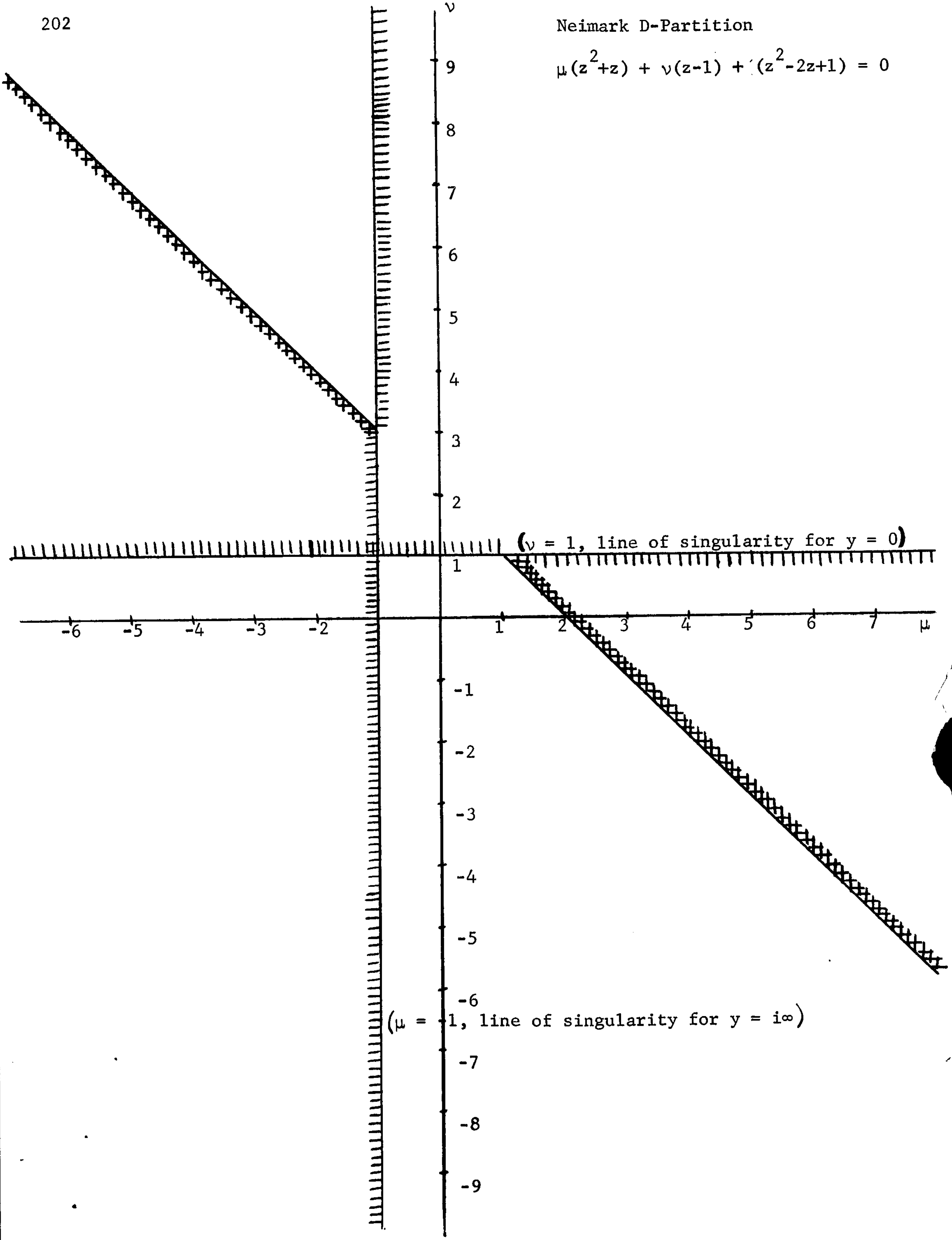


$$K = -e^z(z+10)(z+12)(z^2+4z+8)(z^2+10z+41)(z^2+20z+125)/(z+5)(z+7)(z+9)$$

Neimark D-Partition

$\mu(z+1) + \nu(z^2+z) + (z^2+iz+i) = 0$




Neimark D-Partition
$\mu(z^2+z) + \nu(z-1) + (z^2-2z+1) = 0$
(ν = 1, line of singularity for y = 0)
(μ = -1, line of singularity for y = i∞)
ν
μ
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-9
-6
-5
-4
-3
-2
1
2
3
4
5
6
7

Errata to "Stability Techniques for Continuous Linear Systems"

Page 1: Insert between paragraphs 2 and 3:

Note that arg z and arg f(z) are multiple valued functions. A principal determination can be made by choosing that value between $-\pi$ and π. Throughout the remainder of the book statements concerning arguments will be made for each fixed, but arbitrary, value of the function involved.

Page 3: Replace Line 2 by:

A function differentiable in this sense over a region D is said to be analytic in D.

Page 4: Start Theorem 1.2 with:

Let $f(z) = u(z) + iv(z)$ be defined in a region D. Omit "a region" found later in the statement of the theorem.

Page 6, the footnote. Let it read:

We will mean by a contour a continuous curve consisting of a finite number of simple rectifiable arcs joined end to end. A simple arc is the image of a closed finite interval under a continuous mapping such that no point on the arc is the image of more than one point in the interval.

Page 10, Theorem 1.4. Let it read:

Theorem 1.4: (Cauchy's formula) Let f(z) be analytic in a region D and let C be a simple closed contour in D. Let z be on the interior of the region with boundary C. Then...

Page 13, Theorem 1.5. Let it read:

---- D with boundary C, ---

Page 14, The Definition, Let it read:

--- where $\lim_{z \to z_o} f(z) = \infty$ --- --- can be extended analytically to z_o but ---

Page 15: Definition, Let it read:

... except for a finite number of poles in a region D, then f(z) is meromorphic in D.

Page 15, Theorem 1.7. Replace " singularities" with "poles".

After "z_1 -- z_n", insert "interior to C".

Page 19: Exercise 5. Let it read:

-- of radius $\rho > 0$ with ---

Exercise 8. Let it read:

--- $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-z_o)^n$, valid for $\rho_1 < |z - z_o| < \rho_2$, ---