NOTES ON THE DISCRIMINATION OF FREE-SURFACE MODES IN A CYLINDRICAL TANK

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Technical Report No. 3
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Prepared for

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
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SOUTHWEST RESEARCH INSTITUTE
SAN ANTONIO HOUSTON

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8500 Culebra Road, San Antonio, Texas 78206

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APPROVED:

H. Norman Abramson, Director
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ABSTRACT

The problem of discrimination of certain cylindrical tank modes of free surface motion arose in the course of an exploratory investigation of the behavior of the fluid in tanks under random excitation. This problem is treated herein for discrimination of symmetric and antisymmetric modes, and several designs for specialized free surface measuring systems are presented.

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NOMENCLATURE

a	-	tank radius
c	-	probe sensitivity constant
$D(m) = \dfrac{\text{dif}\,(m\Delta/2)}{\text{dif}\,(\Delta/2)}$		
$\text{dif}(x) = \dfrac{\sin(x)}{x}$		
E_i E_{2-1}, E_P^k, E_Q^k	}	probe or array signal outputs
$f(m, n, t)$	-	$i\,\text{Exp}\,(i\,\omega_{mn} t)$
G_{mn}	-	coefficients in potential function
g	-	acceleration of gravity
H_{mn}	-	$\dfrac{\omega_{mn}}{g} G_{mn} \cosh \xi_{mn} \left(\dfrac{h}{a}\right)$
h	-	fluid depth
$J_m(\;)$	-	Bessel function of first kind of order m
j, i	-	indices
k	-	index
m, n	-	indices
$P(m, n, t)$ $Q(m, n, t)$	}	convenient functions of $f(m, n, t)$, H_{mn}, $J_m(\xi_m)$ and ϵ_{mn}
r, ϕ, z	-	cylindrical coordinate system
t	-	time

NOMENCLATURE (Cont'd)

$x_i = \cos \phi_i$

$$a_m^k = \frac{\sum_{i=1}^{k} \cos m\phi_i}{\sum_{i=1}^{k} \cos \phi_i}$$

Δ	-	angular extent of averaging probes
ϵ_{mn}	-	spatial phase orienting the mn^{th} mode
η	-	free surface elevation
$\eta^{\Delta}_{\phi=\phi_i}$ or $\eta^{\Delta}_{\phi_i}$	-	average surface elevation on tank circumference between $\phi = \phi_i + \frac{\Delta}{2}$ and $\phi = \phi_i - \frac{\Delta}{2}$
ξ_{mn}	-	roots of $\left[\frac{\partial}{\partial r} J_m\left(\xi \frac{r}{a}\right)\right]_{r=a} = 0$
Φ	-	velocity potential
ω_{mn}	-	modal frequencies (angular)

INTRODUCTION

The problem of discrimination of free surface modes in a cylindrical tank became of interest in the course of preliminary studies of the fluid response to random excitation (Refs. 1, 2). In the particular case of longitudinal excitation (along the axis of symmetry) of a cylindrical tank, what is expected from sinusoidal excitation experiments (Ref. 3) is that the fluid responds in 1/2-subharmonic modes of the excitation. In effect, excitation frequency must be twice the linear modal frequency for that free surface mode to occur. In sinusoidal excitation experiments, only one excitation frequency is present at a time, and identification of the resulting modal fluid response may be done by measuring the fluid-free surface elevation at some point and comparing the response frequency with calculated linear modal frequencies, and by visual inspection of the fluid-free surface.

However, the identification problem is considerably more complicated in the random excitation case. Initially, data on the behavior of the free surface under random longitudinal excitation were nonexistent, and, consequently, some qualitative experiments were carried out. In these experiments, a rigid cylindrical tank was excited longitudinally with a random acceleration having energy content over a relatively broad band of frequency. Visual observation confirmed the presence of symmetric free surface modes and indicated, in addition, that many antisymmetric modes of higher order than the first were present. It was concluded that under random excitation the possible presence of all modes of fluid motion at once must be assumed. Even a casual inspection of the lower order modal frequencies, of the infinite number which are mathematically possible, indicates that there is so much potential ambiguity that identification of the free surface modes present on the basis of frequency alone might be practical only for the first antisymmetric mode, and this marginally.

Since general methods of analyzing nonlinear random systems are not immediately available and were certainly not within the scope of the present project, the question arose as to just what could be explored in the random longitudinal excitation program (Ref. 1). The conclusion was that the most useful objective would be to attempt to find out if the fluid-free surface behavior is essentially what might be expected from sinusoidal experiments; that is, is the fluid motion under random longitudinal excitation a summation of subharmonic response?

The general plan for a first attempt was to vary the frequency distribution of energy in the random excitation and measure the frequency distribution of the fluid-free surface response. If significant fluid surface energy were present at frequencies significantly lower than the band of significant excitation energy, the presence of subharmonic response would be indicated. Unfortunately, however, the modal frequency spacing is so small if all modes are considered that resolution of individual modes by the necessary frequency spectrum analysis would not be possible. The spectrum of fluid response at any point in the free surface would be anticipated to be sufficiently "blurred," and the "sharpness" to which a random signal can be bandpassed is sufficiently gradual that clear distinctions between harmonic and 1/2-subharmonic response were thought to be improbable. However, if by some special measuring technique the contributions of all modes except the symmetric (or the nth antisymmetric) were eliminated, the frequency spectrum of this single remaining mode type might be satisfactorily resolved and might have sufficiently pronounced and identifiable peaks so that distinctions between harmonic and subharmonic response could be made.

Thus, the experimental design commenced with the assumption that the free surface is a summation of normal modes. The problem discussed herein is that of discriminating the contributions of one mode in the presence of all the rest.

Since the axisymmetric modes were the obviously identifiable ones present in the qualitative experiments, discrimination of these modes was considered first. From the point of view of the influence of sloshing on rigid body vehicle motions, the first antisymmetric modes are of most importance, and discrimination of these modes is considered next.

FLUID ELEVATION MODEL

The mathematical model for the free surface as a superposition of normal modes can be taken directly from linearized potential theory. In particular, if the velocity potential is Φ and the dynamic free surface elevation is η,

$$\eta \approx \frac{1}{g}\left[\frac{\partial \Phi}{\partial t}\right]_{z=0} \tag{1}$$

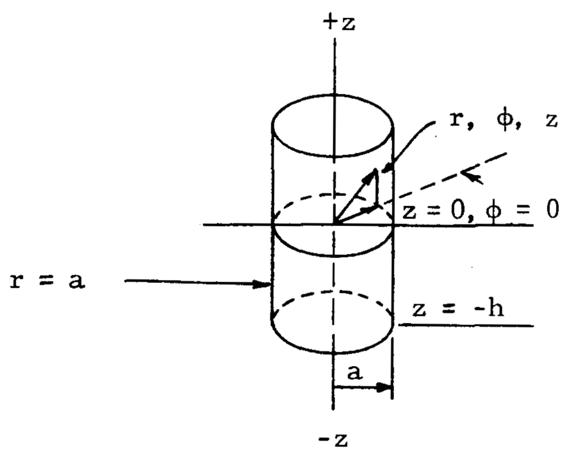
(where the z axis has its origin in the undisturbed free surface).

The usual treatment for a cylindrical tank results in a velocity potential as follows:

$$\Phi = \sum_n \sum_m e^{i\omega_{mn}t} G_{mn} \cosh \xi_{mn} \left(\frac{z}{a} + \frac{h}{a}\right) \cos\left(m\phi + \epsilon_{mn}\right) J_m \left(\xi_{mn}\frac{r}{a}\right)$$

$$m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots \tag{2}$$

where the tank geometry is defined as in the sketch.



The factor $J_m\left(\xi_{mn}\frac{r}{a}\right)$ is the Bessel function of the first kind of order m. The ξ_{mn} are the roots of:

$$\left[\frac{\partial}{\partial r} J_m\left(\xi \frac{r}{a}\right)\right]_{r = a} = 0 \qquad (3)$$

and the frequencies are defined by:

$$\omega_{mn}^2\left(\frac{a}{g}\right) = \xi_{mn} \tanh\left(\xi_{mn}\frac{h}{a}\right) \qquad (4)$$

The phase ϵ_{mn} in the cosine factor is inserted to account for the fact that there is no mathematical basis for orienting the circumferential mode patterns with respect to the tank.

The linearized model for the free surface elevations with respect to z = 0 becomes:

$$\eta = \frac{1}{g}\sum_n \sum_m i\omega_{mn} e^{i\omega_{mn}t} G_{mn} \cosh \xi_{mn}\left(\frac{h}{a}\right) \cdot \cos\left(m\phi + \epsilon_{mn}\right) \cdot J_m\left(\xi_{mn}\frac{r}{a}\right) \qquad (5)$$

Letting:

$$H_{mn} = \frac{\omega_{mn}}{g} G_{mn} \cosh \xi_{mn} \left(\frac{h}{a} \right)$$

$$f(m, n, t) = ie^{i\omega_{mn} t}$$

$$\eta = \sum_n \sum_m f(m, n, t) \, H_{mn} \cos(m\phi + \epsilon_{mn}) \, J_m \left(\xi_{mn} \frac{r}{a} \right)$$

$$m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots \tag{6}$$

In Equation (6), the first two factors for each choice of m, n and tank geometry define amplitude and frequency and the last two factors describe the spatial form of each mode.

DISCRIMINATION OF THE SYMMETRIC MODES

For the symmetric modes, $m = 0$. The Bessel function of the first kind has the property that:

$$J_o(0) = 1$$

$$J_i(0) = 0 \qquad \text{For } i = 1, 2, 3, \ldots \tag{7}$$

Thus, the fluid elevation on the axis of symmetry ($r = 0$) becomes:

$$\eta_{r=0} = \sum_n f(0, n, t) \, H_{0n} \cos(\epsilon_{0n}) \tag{8}$$

and the discrimination of the $m = 0$ modes can be done with a single surface elevation probe on the axis of symmetry of the tank. If the fluid-free surface is a superposition of normal modes the signal from a probe on the axis of symmetry should contain only the symmetric mode components present, and a frequency analysis of this signal should indicate relative amplitudes of these modes.

For present purposes, the ratio $h/a \geq 2$ and the minimum tabulated value of ξ_{mn} is about 1. 8. Thus with better than 0. 1% accuracy:

$$\omega_{mn}^2 \left(\frac{a}{g} \right) = \xi_{mn} \quad , \quad \frac{h}{a} \geq 2 \tag{9}$$

TABLE I

SYMMETRIC MODE FREQUENCIES

n	$\omega_{0n}^2 \left(\frac{a}{g}\right)$
0	3.832
1	7.016
2	10.173
3	13.324
4	16.471
5	19.616
6	22.760
7	25.904
8	29.047
9	32.189

DISCRIMINATION OF NONSYMMETRIC MODES

Preliminary

For convenience, Equation (6) may be expanded:

$$\eta = \sum_n \sum_m f(m, n, t) H_{mn} \left[\cos m\phi \cos \epsilon_{mn} - \sin m\phi \sin \epsilon_{mn}\right] J_m \left(\xi_{mn} \frac{r}{a}\right) \tag{10}$$

The object is to design an array of surface elevation probes which will convert the double sum of Equation (10) to a summation over n for some particular m (in the case of most interest m = 1, the first antisymmetric modes). At first glance, a function of the outputs of an array of probes located at the zeros of the Bessel Function for unwanted modes is enticing. However, there are so many zeros of $J_m \left(\xi_{mn} \frac{r}{a}\right)$ that a probe array along some arbitrary $\phi = \phi_o$ would probably turn into an almost solid bulkhead, if of the surface piercing wire type, and, in any event, would involve a prohibitive number of signals. If it could be assumed that ϵ_{mn} = a constant for all m and n, some possibility might exist for arrays which take advantage of the zeros of

$$\cos m\phi \, J_m \left(\xi_{mn} \frac{r}{a}\right)$$

and

$$\sin m\phi \, J_m \left(\xi_{mn} \frac{r}{a}\right)$$

Unfortunately, there is no justification for such an assumption.

The boundary condition on the cylindrical surface (Eq. 3) insures that the Bessel function in Equation (10) is finite for all modes at r = a. Consequently, probes at or very near the circular tank boundary will contain contributions of all modes. Thus

$$\eta_{r=a} = \sum_n \sum_m f(m,n,t)\, H_{mn} J_m\ (\xi_{mn})\ \cos \epsilon_{mn} \cos m\phi$$
$$- \sum_n \sum_m f(m,n,t)\, H_{mn} J_m\ (\xi_{mn})\ \sin \epsilon_{mn} \sin m\phi \qquad (11)$$

Letting

$$P(m,n,t) = f(m,n,t)\, H_{mn} J_m\ (\xi_{mn})\ \cos \epsilon_{mn}$$
$$Q(m,n,t) = f(m,n,t)\, H_{mn} J_m\ (\xi_{mn})\ \sin \epsilon_{mn} \qquad (12)$$

$$\eta_{r=a} = \sum_n \sum_m P(m,n,t) \cos m\phi - \sum_n \sum_m Q(m,n,t) \sin m\phi$$

$$m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots \qquad (13)$$

Thus, the surface elevation at the circumference of the tank is in the form of Fourier series with time varying coefficients. The coefficients do not change with probe location along the circumference, and Equation (13) suggests that the summations over m may possibly be removed by summing the outputs of a number of probes, in effect a continuous spatial harmonic analysis. If in addition, a procedure can be devised to separate the two double sum terms in Equation (13), an estimate of the unknown phases, ϵ_{mn}, may be possible.

Effect of Probe Averaging

All the expressions for free surface elevation involve the elevation at a point. All practical probes average the free surface elevation over a finite area. However, the type in common use is a wire piercing the surface. Typical diameters of this wire are a/100 or less. It can be seen that for such a small probe to smooth the contribution of any mode, the spatial semiperiod of that mode must be perhaps 3 wire diameters. In the circumferential direction of the tank boundary, a semioscillation of 3a/100 corresponds to about 1/100 of the entire circumference for the case m = 50. Similarly in the radial direction near the boundary, the "semiperiod" for the Bessel functions of high order is approximately

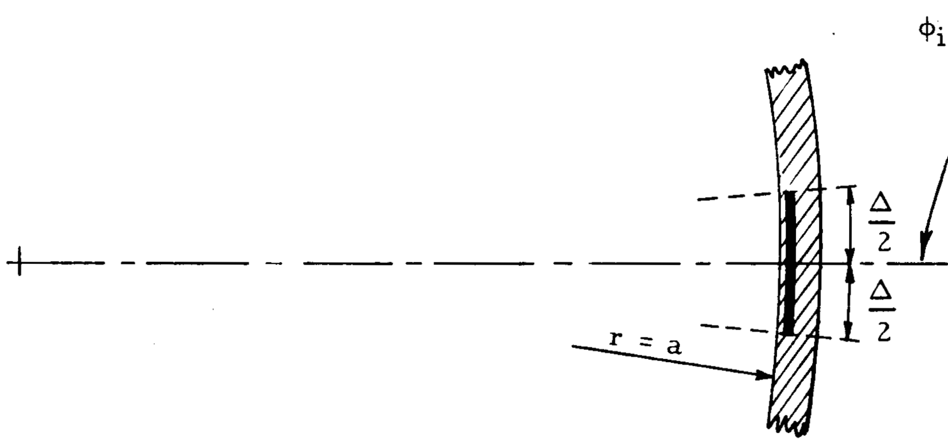
$$3 = \xi_{mn} \frac{\Delta r}{a}$$

Thus

$$\frac{\Delta r}{a} \approx \frac{3}{100} \approx \frac{3}{\xi_{mn}}$$

which implies $\xi_{mn} \approx 100$, and distortion will probably not set in until $n = 30$ for $m = 0, 1, 2, 3$; or $n = 25$ for $m = 16\text{-}18$, for example. Thus for the lower modes of interest, small wire probes are sufficiently close to "point" measurements. Large diameter probes within the confines of the tank cannot be considered for fear of interference with flow.

A type of probe well suited to recording surface elevation at the tank circumference without disturbing the flow would be a capacitance plate imbedded in the tank wall as in the sketch



Such a plate insulated from the tank fluid by a thin plastic tank wall would act as a capacitor and, integrated into a bridge sensing capacitance changes, would have a signal output closely proportional to:

$$E_i \propto \frac{1}{\Delta} \int_{\phi_i - \frac{\Delta}{2}}^{\phi_i + \frac{\Delta}{2}} \eta \, d\phi \qquad (14)$$

The sensitivity of such a probe depends on the thickness of the insulation, and a practical installation for a small lucite tank appears possible on the basis of some crude trial experiments. Capacitance type probes have the additional useful property that a number of probes as in the sketch, if matched in sensitivity and shorted together, will yield a net capacitance change equal to the sum of the integrals (14) for all probes.

A useful quantitative result may be obtained by carrying out the integration of Equation (14) with the free surface elevation Equation (13).

$$E_i \propto \sum_n \sum_m P(m, n, t) \int_{\phi_i - \frac{\Delta}{2}}^{\phi_i + \frac{\Delta}{2}} \frac{\cos m\phi}{\Delta}\, d\phi$$

$$- \sum_n \sum_m Q(m, n, t) \int_{\phi_i - \frac{\Delta}{2}}^{\phi_i + \frac{\Delta}{2}} \frac{\sin m\phi}{\Delta}\, d\phi \qquad (15)$$

Carrying out the integrations

$$E_i \propto \sum_n \sum_m P(m, n, t) \cos (m\phi_i) \operatorname{dif}\left(\frac{m\Delta}{2}\right)$$

$$- \sum_n \sum_m Q(m, n, t) \sin (m\phi_i) \operatorname{dif}\left(\frac{m\Delta}{2}\right) \qquad (16)$$

where

$$\operatorname{dif}\left[\frac{m\Delta}{2}\right] = \frac{\sin\left(\frac{m\Delta}{2}\right)}{\frac{m\Delta}{2}} \qquad (17)$$

The function of Equation (17) is less than 1% different from unity over the range

$$0 \leq \frac{m\Delta}{2} \leq 0.24 \quad \text{radians}$$

or

$$0 \leq \frac{m\Delta}{2} \leq 13.7 \quad \text{degrees}$$

and relatively large plates can be used without serious distortion of the modes corresponding to small m.

A 4-Probe Integrating Array for Discrimination of m = 1 Modes

The properties of the "dif" function of Equation (17) suggested a 4-probe array. In order to eliminate the symmetric modes from the output signal, it is necessary to add and subtract pairs of probes. This can be accomplished by wiring probes into opposite sides of a bridge circuit.

A probe array consisting of two capacitance plates wrapped around the tank to include an angle of Δ radians each and located π radians apart can be assumed. If the signal from a plate probe of width Δ and centered on $\phi_1 + \pi$ is subtracted from the signal from a like probe centered on ϕ_1

$$E_{2-1} \propto \sum_n \sum_m P(m, n, t) \left[\cos m(\phi_1) - \cos m(\phi_1 + \pi)\right] \operatorname{dif}\left(\frac{m\Delta}{2}\right)$$

$$- \sum_n \sum_m Q(m, n, t) \left[\sin m(\phi_1) - \sin m(\phi_1 + \pi)\right] \operatorname{dif}\left(\frac{m\Delta}{2}\right) \qquad (18)$$

Simplifying

$$E_{2-1} \propto \sum_n \sum_m P(m, n, t) \operatorname{dif}\left(\frac{m\Delta}{2}\right) \cos m\phi_1 \; [1 - \cos m\pi]$$

$$- \sum_n \sum_m Q(m, n, t) \operatorname{dif}\left(\frac{m\Delta}{2}\right) \sin m\phi_1 \; [1 - \cos m\pi] \qquad (19)$$

Choosing some probe sensitivity constant c: and $\phi_1 = 0$

$$\frac{E_{2-1}}{c} = \sum_n \sum_m P(m, n, t)(2) \operatorname{dif}\left(\frac{m\Delta}{2}\right) \qquad m = 1, 3, 5, \ldots$$

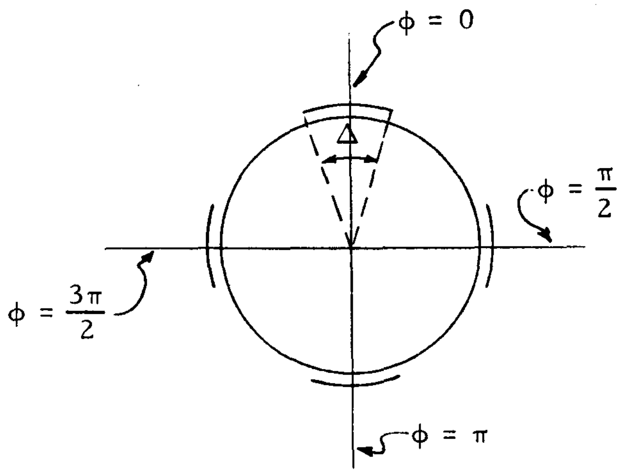
$$= 0 \qquad m = 0, 2, 4, 6, \ldots \qquad (20)$$

if $\phi_1 = \frac{\pi}{2}$

$$\frac{E_{2-1}}{c} = -\sum_n \sum_m Q(m, n, t)(2) \operatorname{dif}\left(\frac{m\Delta}{2}\right) \qquad m = 1, 3, 5, \ldots$$

$$= 0 \qquad m = 0, 2, 4, 6, \ldots \qquad (21)$$

Thus, the array of four probes as in the following sketch wired into two bridges will produce two signals, containing contributions of odd circumferential modes only.



It remains to see how much may be accomplished toward rejection of certain modes by a suitable choice of Δ. The function dif (X) has zeros at

$$X = k\pi \quad , \qquad k = 1, 2, 3, \ldots$$

Consequently, it is not possible to choose $\Delta/2$ so that

$$\frac{m\Delta}{2} = k\pi \qquad \text{for} \qquad m = 1, 3, 5, \ldots$$

for more than one odd mode. For example, if only the $m = 1$ and $m = 5$ modes are present, Δ can be chosen as $2\pi/5$ (72°), and the system will reject the $m = 5$.

For the present case, interest centers on rejecting all odd modes except the first as much as possible. Equations (20) and (21), the two bridge outputs, may be rewritten:

For $\phi_1 = 0$

$$\frac{E_{2-1}}{c} = 2 \text{ dif}\left(\frac{\Delta}{2}\right) \sum_{n} \sum_{m_{odd}} P(m, n, t) \frac{\text{dif}\left(\frac{m\Delta}{2}\right)}{\text{dif}\left(\frac{\Delta}{2}\right)} \tag{22}$$

For $\phi_1 = \frac{\pi}{2}$

$$\frac{E_{2-1}}{c} = -2 \operatorname{dif}\left(\frac{\Delta}{2}\right) \sum_{n} \sum_{m_{odd}} Q(m, n, t) \frac{\operatorname{dif}\left(\frac{m\Delta}{2}\right)}{\operatorname{dif}\left(\frac{\Delta}{2}\right)} \tag{23}$$

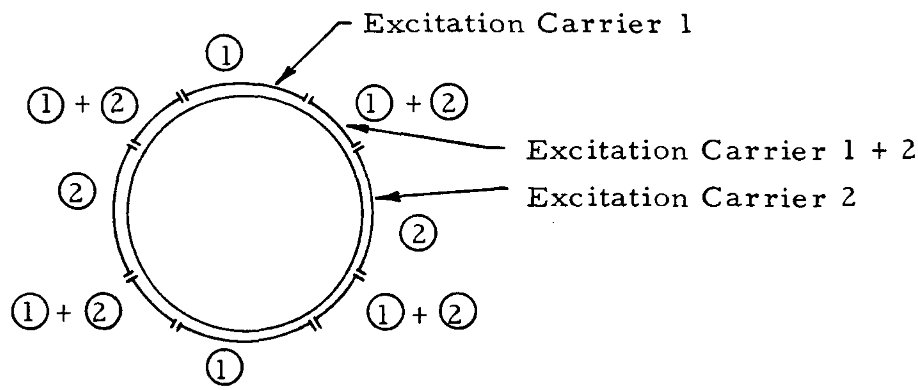
Table II summarizes the relative contributions of the odd modes $\left[\operatorname{dif}\left(\frac{m\Delta}{2}\right) / \operatorname{dif}\left(\frac{\Delta}{2}\right)\right]$ for various values of Δ. It may be seen from the first column that a small Δ does not begin to attenuate any odd mode appreciably to $m = 19$. The second and third columns indicate what happens when Δ is selected so as to reject $m = 7$ and $m = 5$ modes. Columns four and five of Table II are perhaps more interesting. A choice of $\Delta = 2\pi/3$ results in rejection of the $m = 3, 9, 15, \ldots, 3j$ modes but attenuates the $m = 5$ and 7 only 80 or 85%. A choice of $\Delta = 3\pi/4$ is the "compromise" choice to make the relative attenuation of all modes except the first 85% or better. It may be seen that a choice of Δ of either $2\pi/3$ or $3\pi/4$ would not result in an impossibly bad "filter" for experiments where interest was centered in measuring the first odd mode in the presence of small higher mode "noise." Thus, the present array might be useful in studies of rotational instability under lateral sinusoidal excitation, for example.

TABLE II

ODD MODE TRANSMISSION, 4-PROBE INTEGRATING ARRAY

		Δ				
		$2\pi/314$ ($\approx 1.1°$)	$2\pi/7$ ($\approx 51°$)	$2\pi/5$ ($\approx 72°$)	$2\pi/3$ ($\approx 120°$)	$2\pi/4$ ($\approx 135°$)
$2 \operatorname{dif}(\Delta/2)$		2.000	1.932	1.872	1.652	1.568
$\frac{\operatorname{dif}(m\Delta/2)}{\operatorname{dif}(\Delta/2)}$ for	$m = 1$	1.000	1.000	1.000	1.000	1.000
	$m = 3$	1.000	0.747	0.538	0	-0.140
	$m = 5$	1.000	0.362	0	-0.197	-0.083
	$m = 7$	1.000	0	-0.231	0.142	0.141
	$m = 9$	0.999	-0.200	-0.110	0	-0.111
	$m = 11$	0.998	-0.204	0.089	-0.092	0.039
	$m = 13$	0.997	-0.075	0.124	0.077	0.032
	$m = 15$	0.996	0.067	0	0	-0.064
	$m = 17$	0.995	0.131	-0.095	-0.054	0.053
	$m = 19$	0.994	0.094	-0.053	0.052	-0.020

Actual measurements from both bridges simultaneously could yield information on the variation of the spatial orientation of the node (ϵ_{11}) for this case. Since for $\Delta = 2\pi/3$ the plates for the two bridges would overlap, some special electronic measures would have to be taken, probably like multiplexed carrier excitation of the overlapping portions as in the following sketch.*



For the random longitudinal excitation case, however, the first two or three odd modes are likely as not to be of equal magnitude; in fact, it is not impossible that m = 3 mode may be much larger than the m = 1 mode. In this case, the 4-probe integrating array probably does not have adequate discriminating power.

Equispaced Point Arrays

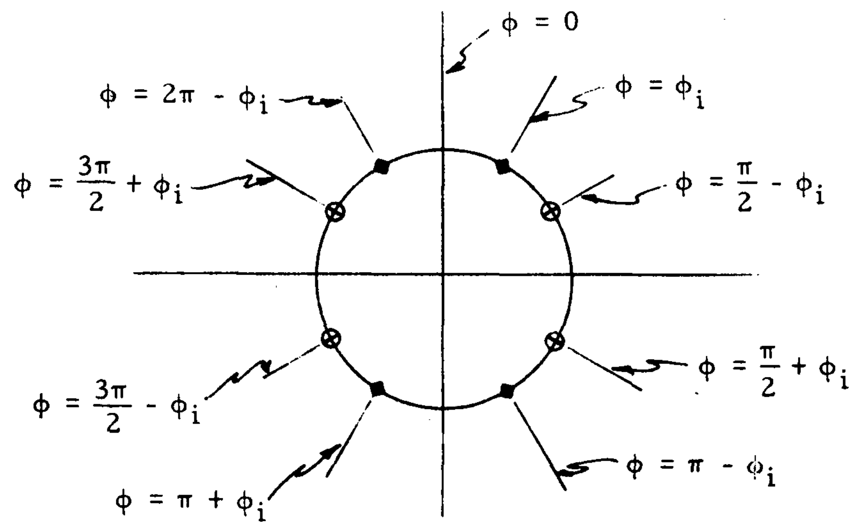
The results of Equation (13) suggest that the signals from multiple point arrays of probes around the tank circumference may be manipulated as in ordinary harmonic analysis to reject at least the lower modes from an estimate for the first. In an ordinary L-point harmonic analysis, the period (circumference in this case) is divided into L equispaced intervals, and the analysis for the fundamental amplitude is free of the 2nd through approximately the (L/2 - 1)th harmonic. In the present case, a 12-point equispaced array would yield an estimate of the m = 1 modal amplitude free of the m = 0, and m = 2, 3, 4 and 5 modes. Higher modes would be present. In addition, in an ordinary L-point harmonic analysis, the ordinates must be multiplied by various positive or negative constants and added. Thus, in order to adapt ordinary harmonic analysis methods to the present problem, the L probes must have at least [L/2 - 1] gain adjustments. This means something like an L/2 amplifier analog computer setup, and a probable like

*Suggested by J. E. Modisette of the Department of Mechanical Sciences, SwRI.

number of carrier amplifiers to convert the probe bridge outputs to reasonable levels. In addition, if simultaneous measurements of each of the double sum terms of Equation (13) are desired, the equipment requirements are doubled (and the adjustment and alignment problem quadrupled). Though possible in principal, the equispaced point arrays similar to ordinary harmonic analysis techniques are not practically appealing.

Doubly Symmetric Arrays (8k Probes)

If each half of the 4-probe integrating array is conceptually split into two parts and account is taken of the effect of the plate width, it can be seen that a doubly symmetric array of special form can both eliminate all even modes ($m = 0, 2, 4, \ldots$) and separate the two double sum terms of Equation (16) for example. No special gain adjustment other than initial trimming would be necessary.



As shown in the sketch, the array, though not necessarily equispaced, is symmetric about the angles $\phi = 0$ and $\phi = \pi/2$. If the eight probes are connected in two bridges so that the two net signals are as follows:

$$E_P \propto [\eta_{\phi=\phi_i}^{\Delta} + \eta_{\phi=2\pi-\phi_i}^{\Delta} - \eta_{\phi=\pi-\phi_i}^{\Delta} - \eta_{\phi=\pi+\phi_i}^{\Delta}]$$
$$E_Q \propto [\eta_{\phi=\pi/2+\phi_i}^{\Delta} + \eta_{\phi=\pi/2-\phi_i}^{\Delta} - \eta_{\phi=3\pi/2-\phi_i}^{\Delta} - \eta_{\phi=3\pi/2+\phi_i}^{\Delta}] \tag{24}$$

where

$$\eta_{\phi=\phi_i}^{\Delta}$$

refers to the surface elevations on the circumference averaged from $\phi = \phi_i + \frac{\Delta}{2}$ to $\phi = \phi_i - \frac{\Delta}{2}$ (Eq. 16), substitution of Equation (13) and simplification results in

$$E_P \propto \sum_n \sum_{m_{odd}} P(m, n, t)(4) \cos m\phi_i \operatorname{dif}\left(\frac{m\Delta}{2}\right)$$

$$E_Q \propto -\sum_n \sum_{m_{odd}} Q(m, n, t)(4) \cos m\phi_i \operatorname{dif}\left(\frac{m\Delta}{2}\right) (-1)^{\frac{m-1}{2}} \tag{25}$$

Thus, the double sum terms of Equation (13) are effectively separated and all modes of even m are rejected. Essentially, the same result is obtained for an array of 8k probes which is defined similar to Equations (24):

$$E_P^k \propto \sum_{i=1}^{k} [\eta^{\Delta}_{\phi_i} + \eta^{\Delta}_{2\pi - \phi_i} - \eta^{\Delta}_{\pi - \phi_i} - \eta^{\Delta}_{\pi + \phi_i}]$$

$$E_Q^k \propto \sum_{i=1}^{k} [\eta^{\Delta}_{\pi/2 + \phi_i} + \eta^{\Delta}_{\pi/2 - \phi_i} - \eta^{\Delta}_{3\pi/2 - \phi_i} - \eta^{\Delta}_{3\pi/2 + \phi_i}] \tag{26}$$

Reduction as before yields:

$$E_P^k \propto \sum_n \sum_{m_{odd}} P(m, n, t) \left[4 \operatorname{dif}\left(\frac{m\Delta}{2}\right)\right] \sum_{i=1}^{k} \cos m\phi_i$$

$$E_Q^k \propto \sum_n \sum_{m_{odd}} Q(m, n, t) \left[(-1)^{\frac{m+1}{2}} (4) \operatorname{dif}\left(\frac{m\Delta}{2}\right)\right] \sum_{i=1}^{k} \cos m\phi_i \tag{27}$$

Odd mode transmission is governed by

$$\sum_{i=1}^{k} \cos m\phi_i \qquad \text{and} \qquad \operatorname{dif}\left(\frac{m\Delta}{2}\right)$$

If, as is desired, the m = 1 mode is to be passed, it is necessary that

$$\operatorname{dif}\left(\frac{\Delta}{2}\right) \sum_{i=1}^{k} \cos \phi_i \neq 0$$

Thus, Equation (27) may be modified:

$$E_P^k \propto \sum_n P(1, n, t) + \sum_n \sum_{m = 3, 5, 7, \ldots} P(m, n, t) \cdot D(m) \cdot a_m^k$$

$$E_Q^k \propto \sum_n Q(1, n, t) + \sum_n \sum_{m = 3, 5, 7, \ldots} Q(m, n, t)(-1)^{\frac{m+1}{2}} D(m)\, a_m^k \tag{28}$$

where

$$D(m) = \frac{\mathrm{dif}\left(\frac{m\Delta}{2}\right)}{\mathrm{dif}\left(\frac{\Delta}{2}\right)}$$

$$a_m^k = \frac{\sum_{i=1}^{k} \cos m\phi_i}{\sum_{i=1}^{k} \cos \phi_i}$$

The first terms of Equation (28) are a summation of all the amplitudes of the m = 1 modes, the result sought. The double sum terms are to be eliminated as far as possible. From the results of a previous section, a limited amount can be done with the factor D(m). Consequently, how many of the odd modes which can be eliminated depend on how many zeros can be achieved with a_m^k. In effect, we require

$$a_m^k = 0 \quad \text{for} \quad m = 3, 5, \ldots \tag{29}$$

If it is assumed that the lowest k of the modes m = 3, 5, 7, . . . can be eliminated with 8k probes, there results k equations in k unknown ϕ's

$$\sum_{i=1}^{k} \cos (2j + 1)\, \phi_i = 0 \quad \text{for} \quad j = 1, 2, \ldots, k \tag{30}$$

(where $\sum_{i=1}^{k} \cos \phi_i \neq 0$)

Even if a solution of Equation (30) for the elimination of the $m = 3, 5, \ldots (2k+1)$ modes can be achieved, there would remain in Equation (28) contributions from the $m = 2(k+1)$ and higher modes. However, if the $2(k+1)^{th}$ mode is sufficiently high, the lowest modal frequency $[m = 2(k+1),\ n = 0]$ might perhaps be sufficiently removed from the lowest frequencies of the first mode ($m = 1,\ n = 0, 1, 2, \ldots$) so that a frequency analysis of the bridge signals of Equation (28) would allow examination of these lowest $m = 1$ modes.

It is heartening to note that the Tchebyscheff-Radau quadrature method for the integral

$$\int_{-\pi}^{\pi} f(\cos\phi)\cos\phi\, d\phi \tag{31}$$

involves the solutions of Equation (30). The above integral may be shown to be equivalent to the Fourier cosine coefficient for a harmonic series, and the development of this section could equally well have been started with Equation (31).

Usable solutions of Equations (30) are not so obvious in general. It may be shown with the aid of the properties of one of the Tchebyscheff polynomials that the system of Equations (30) is equivalent to:

$$\sum_{i=1}^{k} (x_i)^{2j+1} = \left[\frac{[2j+1]^{(j)}}{2^{2j}\, j!}\right] \sum_{i=1}^{k} x_i \qquad j = 1, 2, 3, \ldots, k$$

where

$$x_i = \cos\phi_i$$

and

$$[2j+1]^{(j)} = [2j+1]\,[2j], \ldots, [j+2] \tag{32}$$

The system of Equation (32) is composed of symmetric functions. Unfortunately, since no specification on the sums $\sum_i^k x_i^{2j}$ is made, no systematic way of finding roots for k ordinates could be found.

Particular Exact Solutions: Doubly Symmetric Arrays

Though general nontrivial solutions to the system of Equations (30) were not available, some particular exact solutions could be obtained, and tentative array design specified.

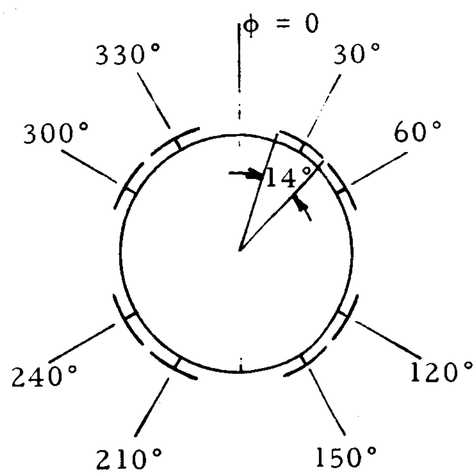
8-Probe array (k = 1). - When k = 1, Equations (30) reduce to:

$$\cos 3\phi_1 = 0 \tag{33}$$

and a useful solution is

$$\phi_1 = \frac{\pi}{6} \tag{34}$$

Since the closest position of probes would be 30°, Δ can be practically chosen as 14°, resulting in the array as in the sketch.



The signals would be:

$$E'_P \propto \left[\eta^{14°}_{30°} + \eta^{14°}_{330°} - \eta^{14°}_{150°} - \eta^{14°}_{210°}\right]$$

$$E'_Q \propto \left[\eta^{14°}_{120°} + \eta^{14°}_{60°} - \eta^{14°}_{240°} - \eta^{14°}_{300°}\right] \tag{35}$$

The odd mode transmission is shown by the function; $D(m)a'_m$; as in Equation (28). This function was evaluated and is tabulated in Table III.

TABLE III

CDD MODE TRANSMISSION, 8-PROBE
ARRAY, Δ = 14°

Mode, m	$D(m)a'_m$
1	1.000
3	0
5	-0.941
7	-0.884
9	0
11	0.727
13	0.631
15	0
17	-0.422
19	-0.315
21	0
23	0.116
25	0.029
27	0
29	0.110

This probe array is similar to the 4-probe integrating array with Δ = 120°, Table II, Column 4. As may be seen from the two tables, the 8-probe array does nowhere near as good a job of rejecting odd modes. However, it has the advantage that it rejects the m = 3 mode without overlapping transducers. This would simplify the electronics considerably.

The initial portion of the anticipated modal frequency spectrum of the outputs of this array may be summarized as in Table IV (where the approximation of Equation (9) is employed and the numerical values are from Table 2.4 of Ref. 4).

Table IV shows that considerable mode ambiguity in the bridge signal frequency content starts at nearly the m = 1, n = 1 mode. Separation of the (m = 1, n = 1) mode from next higher modes would be very difficult if not impossible with normal frequency analyses. The lowest antisymmetric mode (m = 1, n = 0) is separated by only 0.9 octave from the lowest unwanted ($m \neq 1$) mode.

TABLE IV

MODAL FREQUENCY SPECTRUM, 8-PROBE ARRAY
(For $h/a \geq 2$)

Frequency Order	m	n	$\omega_{mn}\sqrt{\frac{a}{g}}$	$\frac{\omega_{nm}}{\omega_{10}}$
1	1	0	1.35	1.000
2	1	1	2.31	1.71
3	5	0	2.53	1.88
4	1	2	2.92	2.16
5	7	0	2.93	2.17
6	5	1	3.24	2.40
7	1	3	3.42	2.52
8	11	0	3.56	2.64
9	7	1	3.57	2.65

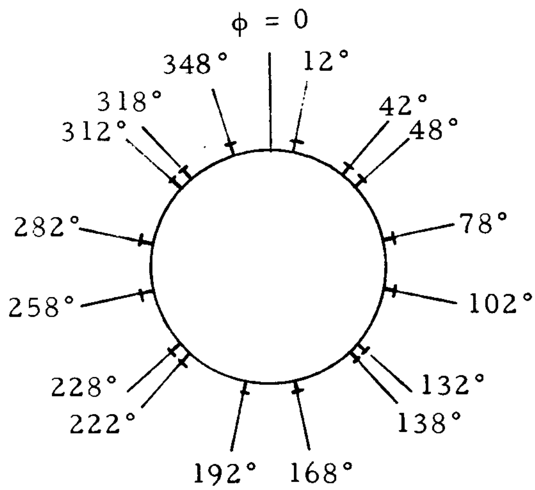
16-Probe array (k = 2). - When k = 2, Equations (30) become:

$$\begin{aligned} \cos 3\phi_1 + \cos 3\phi_2 &= 0 \\ \cos 5\phi_1 + \cos 5\phi_2 &= 0 \end{aligned} \tag{36}$$

Manipulating with half angle identities, a useful nontrivial solution results:

$$\begin{aligned} \phi_1 &= \frac{\pi}{15}: \ (12°) \\ \phi_2 &= \frac{4\pi}{15}: \ (48°) \end{aligned} \tag{37}$$

The closest probe position in this array is 6°, thus a choice of $\Delta = 5°$ may be practical. The arrangement would be as in the sketch:



If corrections are made such that E_p^2 and E_ϕ^2 as defined in Equation (26) ($k = 2$, $\Delta = 5°$, $\phi_1 = 12°$, $\phi_2 = 48°$), the odd mode transmission is again defined by $D(m)a_m^2$ as in Equation (28). This function was evaluated and is tabulated in Table V.

TABLE V

ODD MODE TRANSMISSION, 16-PROBE ARRAY, $\Delta = 5°$

Mode, m	$D(m)a_m^2$
1	1.000
3	0
5	0
7	0.609
9	0
11	-0.962
13	-0.585
15	0
17	-0.563
19	-0.890
21	0
23	0.520
25	0
27	0
29	0.754

As in the 8-probe array, those odd modes which are not rejected are very little attenuated; however, an additional low mode ($m = 5$) is rejected. It is interesting to note from Tables III and V that odd modes which are a multiple of modes rejected are also rejected. (All modes defined by m having a factor of 3 are rejected, as is $m = 25 = 5 \times 5$, Table V).

The initial portion of the anticipated modal frequency spectrum of the outputs is summarized in Table VI.

TABLE VI

MODAL FREQUENCY SPECTRUM, 16-PROBE ARRAY
($h/a \geq 2$)

Frequency Order	m	n	$\omega_{mn}\sqrt{\frac{a}{g}}$	$\frac{\omega_{mn}}{\omega_{10}}$
1	1	0	1.35	1.000
2	1	1	2.31	1.71
3	1	2	2.92	2.16
4	7	0	2.93	2.17
5	1	3	3.42	2.52
6	11	0	3.56	2.64
7	7	1	3.57	2.65
8	13	0	3.84	2.84
9	1	4	3.85	2.85
10	7	2	3.86	2.86
11	11	1	4.18	3.10
12	1	5	4.24	3.14
13	13	1	4.45	3.30

Table IV shows that the frequency content of the bridge signals would be entirely $m = 1$ modes up to one octave above the lowest mode and that the first two antisymmetric modes might be discriminated with a frequency analysis.

32-Probe Array ($k = 4$). -When $k = 4$, the Equations (30) reduce to

$$\begin{aligned}
&\cos 3\phi_1 + \cos 3\phi_2 + \cos 3\phi_3 + \cos 3\phi_4 = 0 \\
&\cos 5\phi_1 + \cos 5\phi_2 + \cos 5\phi_3 + \cos 5\phi_4 = 0 \\
&\cos 7\phi_1 + \cos 7\phi_2 + \cos 7\phi_3 + \cos 7\phi_4 = 0 \\
&\cos 9\phi_1 + \cos 9\phi_2 + \cos 9\phi_3 + \cos 9\phi_4 = 0
\end{aligned} \tag{38}$$

If:

$$\phi_1 = \frac{\pi}{15} - \delta_1$$

$$\phi_2 = \frac{\pi}{15} + \delta_2$$

$$\phi_3 = \frac{4\pi}{15} - \delta_2 \qquad (39)$$

$$\phi_4 = \frac{4\pi}{15} + \delta_1$$

Substitution in Equation (38) and satisfaction of the resulting equations in the δ's yields:

$$\delta_1 = \delta_2 \qquad (40)$$

and:

$$\phi_1 = \frac{\pi}{210} : (0.857°)$$

$$\phi_2 = \frac{29\pi}{210} : (24.857°)$$

$$\phi_3 = \frac{41\pi}{210} : (35.143°) \qquad (41)$$

$$\phi_4 = \frac{71\pi}{210} : (60.857°)$$

The first quadrant of this array is shown in the sketch. The closest approach of two probe centers is 1.714°, but, since the probes in question are effectively added together, they need not be physically separated and Δ may be taken as 1.714°.

Connections would be made so that E_p^4 and E_ϕ^4 as defined by Equation (26) would be made. Odd mode transmission is defined by $D(m)a_m^4$ as previously. This function was evaluated and is tabulated in Table VII. The initial portion of the modal frequency spectrum is tabulated in Table VIII.

TABLE VII

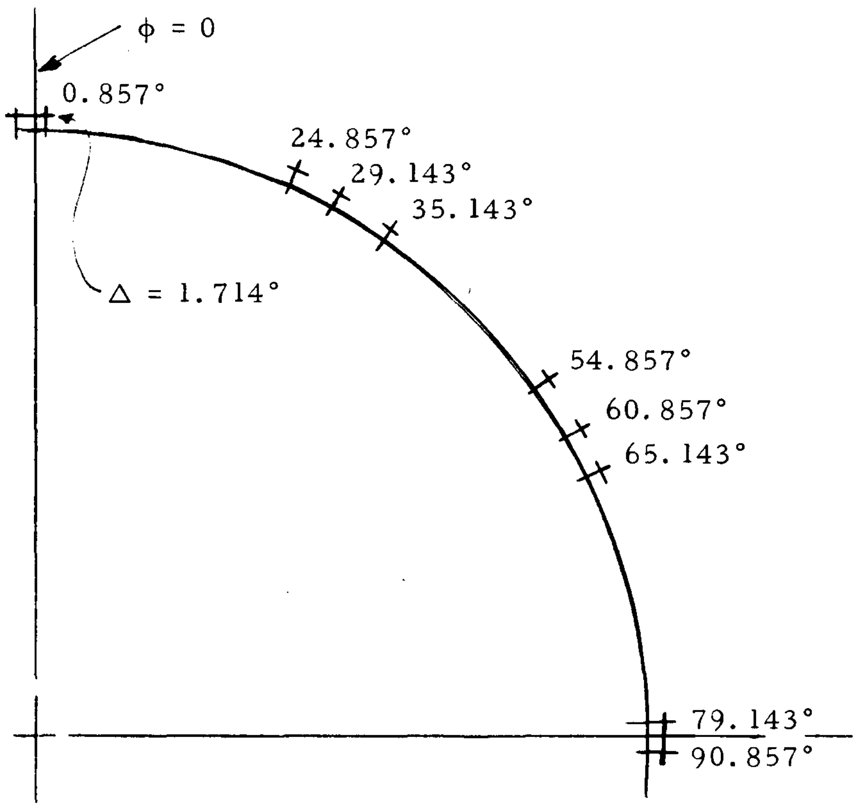
ODD MODE TRANSMISSION, 32-PROBE ARRAY, $\Delta = 1.714°$

Mode, m	$D(m)a_m^4$
1	1.000
3	0
5	0
7	0
9	0
11	0.798
13	0.614
15	0
17	0.490
19	0.439
21	0
23	0.269
25	0
27	0
29	0.967

TABLE VIII

MODAL FREQUENCY SPECTRUM
32-PROBE ARRAY ($h/a \geq 2$)

Frequency Order	m	n	$\omega_{mn}\sqrt{\frac{a}{g}}$	$\frac{\omega_{mn}}{\omega_{10}}$
1	1	0	1.35	1.00
2	1	1	2.31	1.71
3	1	2	2.92	2.16
4	1	3	3.42	2.52
5	11	0	3.56	2.64
6	13	0	3.84	2.84
7	1	4	3.85	2.85
8	11	1	4.18	3.10
9	1	5	4.24	3.14
10	13	1	4.45	3.30



It may be seen from a comparison of Table V for the 16-probe array with Table VII that doubling the number of probes and satisfying Equations (30) for k = 4 result in the rejection of only one additional mode (m = 7) since the m = 9, 15, . . . , 3i modes appear to be rejected if the m = 3 mode is rejected. Thus, if a 24-probe array (k = 3) could be achieved, essentially the same results as in Tables VII and VIII might be expected for 2/3 the number of probes.

Table VIII shows that the frequency content of the bridge signals would be entirely m = 1 modes up to about 1-1/4 octaves above the lowest mode. The m = 1, n = 0, 1 and 2 modes might thus be discriminated with a frequency analysis.

Numerical Approximations: Doubly Symmetric Arrays

The preceding section indicated the potential virtues of a 24-probe (k = 3) array, and it was mentioned that a rational procedure for solution of Equations (30) for any k was not developed. In the process of attempting to provide particular exact solutions, it was noted that crude approximations to solutions could sometimes be obtained graphically. If an initial approximate solution set is defined as:

$$[\phi^\circ]' = [\phi_1^\circ \phi_2^\circ \ldots \phi_k^\circ]$$

and the errors are defined as:

$$[D']' = [\delta_1' \; \delta_2' \; \delta_3' \ldots \delta_k']$$

such that the true solutions are:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{bmatrix} = \begin{bmatrix} \phi_1^\circ \\ \phi_2^\circ \\ \vdots \\ \phi_k^\circ \end{bmatrix} + \begin{bmatrix} \delta_1' \\ \delta_2' \\ \vdots \\ \delta_k' \end{bmatrix} = [\phi^\circ] + [D'] \qquad (42)$$

Now if the initial approximations are sufficiently good, the δ_i' are small angles and:

$$\cos k\delta_i' \approx 1$$
$$\sin k\delta_i' \approx k\delta_i' \qquad (43)$$

Then:

$$\cos(2j+1)\phi_i \approx \cos(2j+1)\phi_i^\circ - (2j+1)\delta_i' \sin(2j+1)\phi_i^\circ \qquad (44)$$

If the approximations ϕ_i° are substituted for ϕ_i in Equations (30), the equations become:

$$\sum_{i=1}^{k} \cos 3\phi_i^\circ = R_3^\circ (\neq 0)$$

$$\sum_{i=1}^{k} \cos 5\phi_i^\circ = R_5^\circ (\neq 0) \qquad (45)$$

$$\ldots$$

$$\sum_{i=1}^{k} \cos(2k+1)\phi_i^\circ = R_{(2k+1)}^\circ (\neq 0)$$

If the approximations of Equation (44) are inserted in Equations (30), the result is:

$$[S^\circ]\,[D'] = [R^\circ] \tag{46}$$

where

$$[S^\circ] = \begin{bmatrix} \sin 3\phi_1^\circ & \sin 3\phi_2^\circ & \cdots & \sin 3\phi_k^\circ \\ \sin 5\phi_1^\circ & \sin 5\phi_2^\circ & \cdots & \sin 5\phi_k^\circ \\ \vdots & \vdots & & \vdots \\ \sin (2k+1)\phi_1^\circ & \cdots & \cdots & + \sin (2k+1)\phi_k^\circ \end{bmatrix} \tag{47}$$

$$[R^\circ] = \begin{bmatrix} R_3^\circ/3 \\ R_5^\circ/5 \\ \vdots \\ R_{(2k+1)}^\circ/(2k+1) \end{bmatrix} \tag{48}$$

$$[D'] = \begin{bmatrix} \delta_1' \\ \delta_2' \\ \vdots \\ \delta_k' \end{bmatrix} \tag{49}$$

The matrices defined by Equations (47) and (48) are defined by the initially assumed values of ϕ_i° and may be evaluated. With the help of a computer, Equation (46) may be solved:

$$[D'] = [S^\circ]^{-1}\,[R^\circ] \tag{50}$$

and

$$[\phi] = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k \end{bmatrix} = [\phi^\circ] + [S^\circ]^{-1} [R^\circ] \tag{51}$$

The result of Equation (51) can only be regarded as a possibly improved estimate of the ϕ_i, depending on how close the initial approximation was. Although convergence is not proven, it was thought possible that replacement of $[\phi^\circ]$ by the "improved" estimate $[\phi]$ of Equation (51) and a repeat of the processes defined by Equations (46) through (51), etc., might with luck result in reasonable solutions.

There is no reason to use the Equations (30) in order in this procedure, for k probes, k equations are needed, and any k of the equations:

$$\sum_{i=1}^{k} \cos (2j + 1)\phi_i = 0 \qquad j = 1, 2, 3, . . . , \infty \tag{52}$$

may be selected, depending on the particular set of k modes it is desired to eliminate. For example, for a 32-probe array, $k = 4$, and the odd modes $m = 3, 5, 7, 11$ might be eliminated by choosing $j = 1, 2, 3, 5$ for the four equations [of Eq. (52)] to be satisfied.

Application of Numerical Procedures to a 24-Probe Array (k = 3)

As mentioned in the previous section, a 24-probe array ($k = 3$) might be useful, and the procedures just outlined were applied to this case.

When $k = 3$, Equations (30) reduce to:

$$\begin{aligned} \cos 3\phi_1 + \cos 3\phi_2 + \cos 3\phi_3 &= 0 \\ \cos 5\phi_1 + \cos 5\phi_2 + \cos 5\phi_3 &= 0 \\ \cos 7\phi_1 + \cos 7\phi_2 + \cos 7\phi_3 &= 0 \end{aligned} \tag{53}$$

After some manipulation Equation 38 becomes:

$$\cos \frac{3}{2}(\phi_1 + \phi_2) \cos \frac{3}{2}(\phi_2 + \phi_3) \cos \frac{3}{2}(\phi_1 + \phi_3) = \frac{1}{4} \cos 3(\phi_1 + \phi_2 + \phi_3)$$

$$\cos \frac{5}{2}(\phi_1 + \phi_2) \cos \frac{5}{2}(\phi_2 + \phi_3) \cos \frac{5}{2}(\phi_1 + \phi_3) = \frac{1}{4} \cos 5(\phi_1 + \phi_2 + \phi_3)$$

$$\cos \frac{7}{2}(\phi_1 + \phi_2) \cos \frac{7}{2}(\phi_2 + \phi_3) \cos \frac{7}{2}(\phi_1 + \phi_3) = \frac{1}{4} \cos 7(\phi_1 + \phi_2 + \phi_3) \tag{54}$$

On intuitive grounds, the sum of the three angles was assumed to be $\pi/2$. Under this assumption, the set of angles:

$$\phi_i = \frac{11\pi}{210}, \frac{31\pi}{210}, \frac{59\pi}{210} \tag{55}$$

is found to satisfy Equation (54). Their sum is only approximately equal to $\pi/2$, however.

The set of angles, $\frac{15\pi}{210}, \frac{35\pi}{210}, \frac{63\pi}{210}$, makes one of the terms in each of the equations zero. And the errors in satisfaction of Equations (53) with these angles are of the same magnitude as result when the angles of Equation (55) were substituted into Equation (53).

It was assumed, therefore, that the angles of Equation (55) might be a sufficiently close approximation to allow the numerical procedure in the preceding section to begin. The computations of Equations (46) through (51) were programmed for the CDC 160A for $k = 3$. The iteration starting with the angles of Equation (55) converged in 4 iterations to the 7 significant digit capability of the machine. The results were:

$$\phi_1 = 0.0648354\pi = 11.67037°$$

$$\phi_2 = 0.1496475\pi = 26.93654°$$

$$\phi_3 = 0.3114235\pi = 56.05624°$$

The following sketch shows the first quadrant of this array. A Δ of 5° appears practical. If connections are made so that E_p^3 and E_Q^3 are produced [Eq. (26)], odd mode transmission is defined by the function $D(m)a_m^3$. This function was evaluated using values of ϕ rounded to the nearest 0.001 degree, and the results are shown in Table IX.

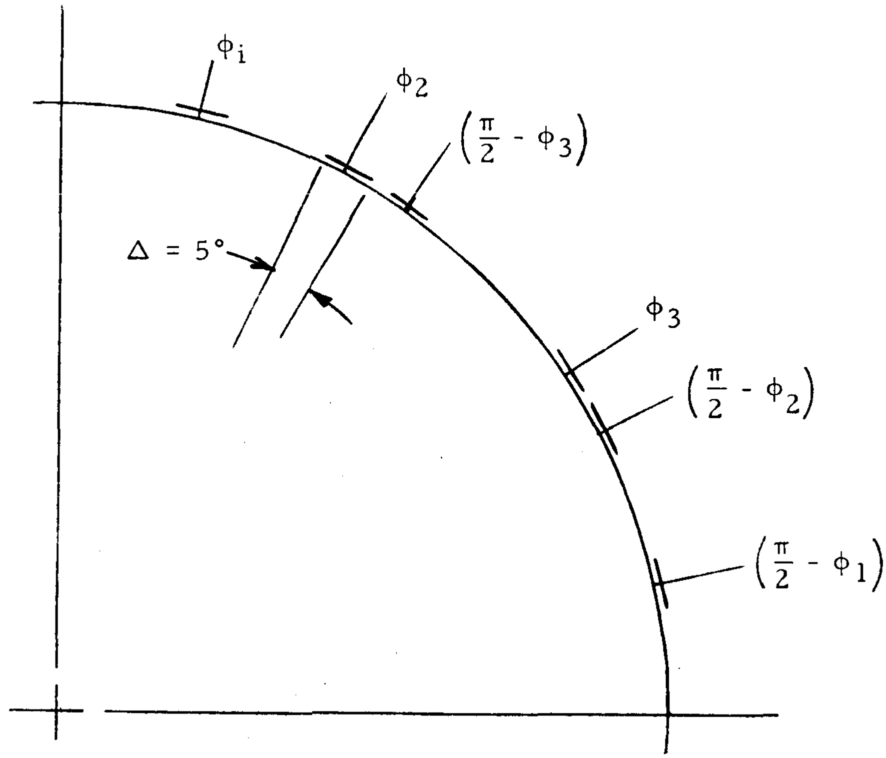


TABLE IX

ODD MODE TRANSMISSION, 24-PROBE ARRAY, $\Delta = 5°$

Mode, m	$D(m)a_m^3$
1	1.0000
3	-0.000017
5	0.000017
7	0.000020
9	-0.6165
11	-0.1621
13	0.4264
15	-0.3025
17	-0.6334
19	-0.2421
21	-0.5162
23	-0.3743
25	0.6158
27	0.6407
29	0.1295

In this case, the results are significantly different from those for previous arrays. The rejection of modes which have m a multiple of three ceases in this solution. In effect, what was hoped for with a 24-probe array was rejection of the m = 9 mode essentially for free, but the 24-probe array merelv does what is expected of it. It is worth noting that an analysis of the errors introduced by errors in probe location will be necessary. Table IX was computed for angles within ± 0.001°; the modes "rejected" are actually attenuated to one part in 50,000 (roughly 100 dB, voltage ratio). This precision can only degrade with the precision of probe location.

The initial portion of the modal frequency spectrum is shown in Table X.

TABLE X

MODAL FREQUENCY SPECTRUM, 24-PROBE ARRAY
$(h/a \geq 2)$

Frequency Order	m	n	$\omega_{mn} \sqrt{\frac{a}{g}}$	$\frac{\omega_{mn}}{\omega_{10}}$
1	1	0	1.35	1.00
2	1	1	2.31	1.71
3	1	2	2.92	2.16
4	9	0	3.27	2.42
5	1	3	3.42	2.52
6	11	0	3.56	2.64
7	13	0	3.84	2.84
8	1	4	3.85	2.85
9	9	1	3.96	2.94
10	11	1	4.18	3.10

In the 24-probe case, the frequency content of the bridge signals would be wholely m = 1 modes to 1-2/10 octaves above the lowest m = 1 mode; the m = 1, n = 0 and 1 modes might be discriminated with a frequency analysis. For practical purposes, the 24-probe array is slightly better than the 16-probe case and slightly worse than the 32-probe case.

PRACTICAL REALIZATIONS

Axisymmetric Modes

Plate I is a photograph of a practical realization for symmetric (m = 0) mode discrimination. The probe is on the tank axis and is a surface piercing capacitance type of No. 26 magnet wire (Niclad insulation). Quite adequate sensitivity (about 2 volts/inch surface elevation) was obtained using a Tektronix "Q" unit and No. 133 power supply in a capacitance measuring mode. This system is that used for the experiments of Reference 1.

Antisymmetric Modes (m = 1)

Table XI summarizes the array designs which were considered suitable for the longitudinal random excitation experiments contemplated in the present program. The five designs are arranged in the estimated order of increasing sensitivity to minor errors in probe position. (Some preliminary computations on sensitivity were made but are not detailed herein.)

The first two columns show two probes having equal discrimination properties (4 probe and 8 probe). The effect in each is to make the separation of the m = 1, n = 0 mode from all others more pronounced. Of the two, the array with overlapping probes has potentially difficult electronic problems, and the 8-probe array is to be preferred. As the number of probes is increased, the number of m = 1 modes which it is estimated could be discriminated by a frequency analysis increases. Unfortunately, though the lowest undesired odd mode for the most complicated array corresponds to m = 9, the lowest unwanted frequency does not increase so much, and, for instance, a fourfold increase in the number of probes provides only a threefold increase in the number of m = 1 modes which may be discriminated with a frequency analysis.

Although the lowest m = 2, 3, 4 modes have been generated in the laboratory, they are difficult to start, maintain and recognize visually. Whether or not the m = 5, 7, 9, . . . modes will appear under longitudinal excitation of any sort is not known. There is some reason to believe that the higher order circumferential modes may not be a problem, due to high damping.

For present purposes, it was felt that the 16-probe array of Table XI was the best all around choice for further development, and a summary of reasons follows:

(1) It is probably desirable to be able to discriminate the lowest two m = 1 modes, and to have at least an octave frequency separation between the m = 1, n = 0 mode and the frequency where the modal ambiguities begin.

(2) There exists the possibility that the m = 7, and higher modes are highly damped and would not appear in any case.

TABLE XI

SUMMARY OF ARRAY DESIGNS

Type of Array	4-Probe Integrating	8(k) Probe Doubly Symmetric			
Number of probes	4	8	16	24	32
Overlapping probes	Yes	No	No	No	No
Width ea probe (deg)	120°	14°	5°	5°	1.714°
Even m modes rejected	Yes	Yes	Yes	Yes	Yes
Lowest undesired odd mode	m = 5	m = 5	m = 7	m = 9	m = 11
Attenuation, this mode	0.197	0.941	0.609	-0.617	0.798
Lowest undesired modal frequency $\left(\omega \sqrt{\frac{a}{g}}\right)$	2.53	2.53	2.93	3.27	3.56
m = 1 modes between above frequency and 0	n = 0, 1	n = 0, 1	n = 0, 1, 2	n = 0, 1, 2	n = 0, 1, 2, 3
m = 1 modes which can probably be discriminated with frequency analysis	n = 0	n = 0	n = 0, 1	n = 0, 1, 2?	n = 0, 1, 2
$\left(\frac{\text{Lowest undesired frequency}}{(m = 1, n = 0) \text{ mode frequency}}\right)$	1.88	1.88	2.17	2.42	2.64
Order: Fabrication difficulty	2	1	3	4	5
Order: Electronic difficulty	5	1	2	3	4

(3) Such an array would probably be the first of its kind. Thus, minimization of mechanical construction and electronic trimming difficulties is highly desirable.

(4) The 16 and higher probe arrays are almost totally unaffected by probe width, Δ. Thus, wire probes could be substituted for the plate type, though a capacitance principle of operation is mandatory.

Plate II is a photograph of a realization of a 16-probe array constructed in the same size tank as was used in Reference 1. Each probe is a brass plate 0.125 ± 0.001 in. wide imbedded in the plastic tank wall flush with the inner tank surface. Insulation is provided by 0.002-in. Mylar pressure sensitive tape. Circumferential location of the probes was easily achieved within $\pm 1/50$ degree, and a sensitivity analysis indicates that odd mode transmission should be as close to that shown in Table V as could be measured. Wiring and checkout of this design were not completed within the limits of the present exploratory program.

SUMMARY

The mode discrimination problem stems from the assumption that the instantaneous free surface in the cylindrical tank may be described by a superposition of normal modes. The array designs developed are built around the usual linear normal mode theory.

The discrimination problem for axially symmetric modes may be solved very simply by a fluid elevation probe on the tank axis. In theory, <u>none</u> of the antisymmetric modes contribute at this point.

The discrimination problem for antisymmetric modes is an order of magnitude more difficult. The emphasis in the present work has been on the "$m = 1$" modes which are the important ones from the rocket booster stability and control viewpoint. The tentative designs achieved in the present work involve determination of the positions which will in effect do a continuous spatial harmonic analysis of the fluid elevation around the tank wall for the (time varying) amplitude of the first few $m = 1$ modes. It was found that the <u>orientation of the nodes</u> for $m = 1$ modes may be obtained with two arrays, the signal of one array being proportional to the cosine and the other to the sine of the angle the node makes with an array reference angle. No way of discriminating all $m = 1$ modes from <u>all</u> others was found, although the separation of all m_{odd} modes from all m_{even} modes is straightforward.

A slightly different probe is suggested for measuring free surface elevations around the tank wall--essentially a curved capacitor imbedded in the wall. This type of probe is vital for one class of arrays visualized, (the "4 or 2-probe integrating" type). Though not considered useful for the longitudinal random vibration case, one of these arrays may be interesting for purposes of "filtering" relatively small amplitude $m = 3, 5, 7, \ldots$ modes and all m_{even} modes from a measurement of $m = 1$ mode amplitudes. Measurement of nodal position is possible only with some special electronic measuring techniques, however.

Equispaced (around the circumference) arrays of probes, the natural start for a harmonic analysis, appear unsuitable in most cases because a gain adjustment must be made for each probe.

A type of array termed "doubly symmetric" was found which has the desired general property of rejecting all modes corresponding to even m and is arranged into two bridges for measurement of nodal position. Four arrays of this type were found which reject certain of the modes having odd m. These arrays do not reject all the modes that could be desired, but, by removing some of the lower odd circumferential modes, the lowest of the $m = 1$ modes may be examined by a frequency analysis.

It is conceivable that the present development could have other applications, shell vibration, for example. The solutions given for doubly symmetric arrays are also directly applicable to harmonic analyses (for the fundamental) as the arrays are essentially Tchebycheff-Radau quadrature formulas.

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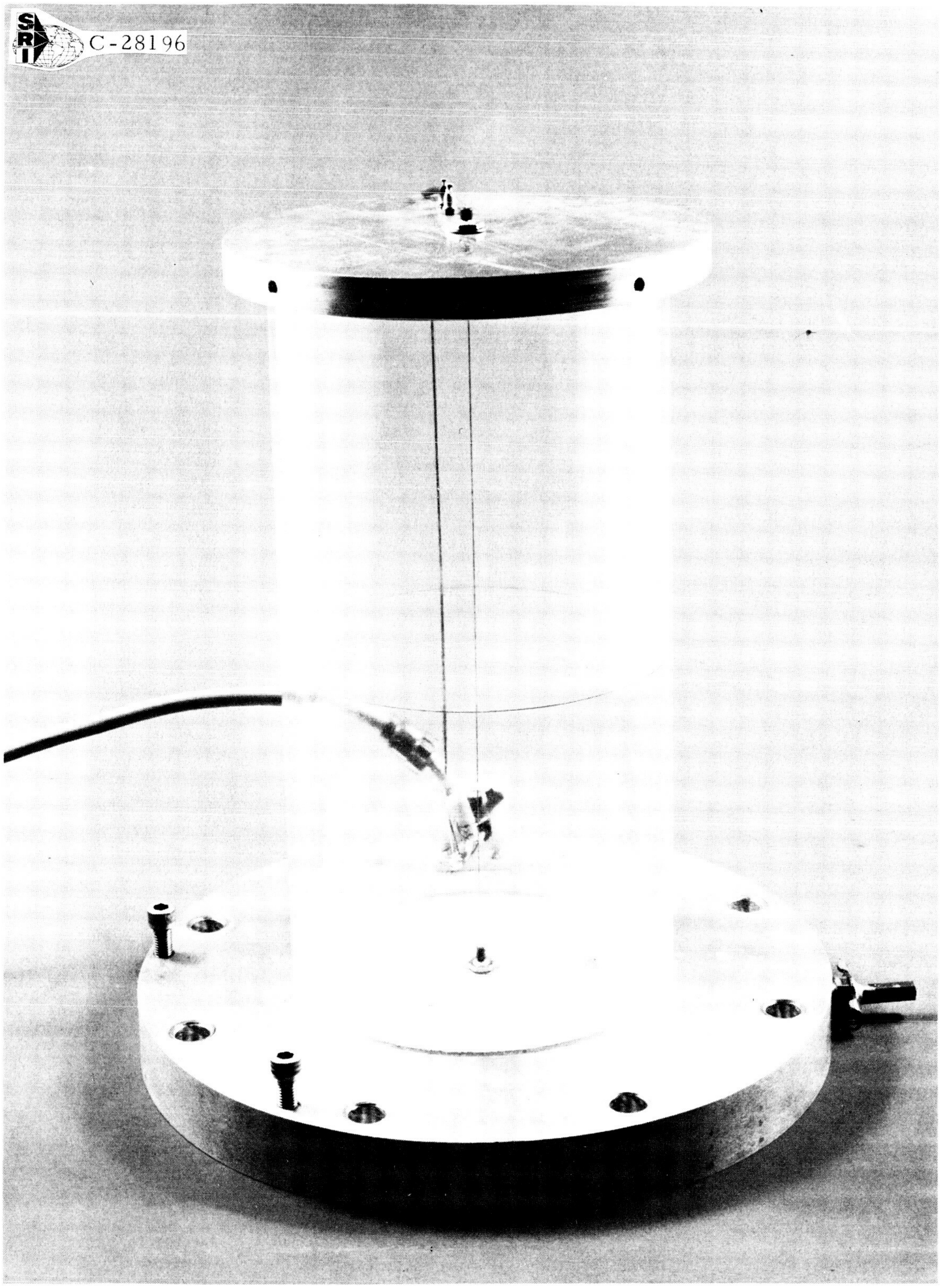

PLATE I. TANK WITH PROBE ON TANK AXIS --
SYMMETRIC MODE DISCRIMINATION

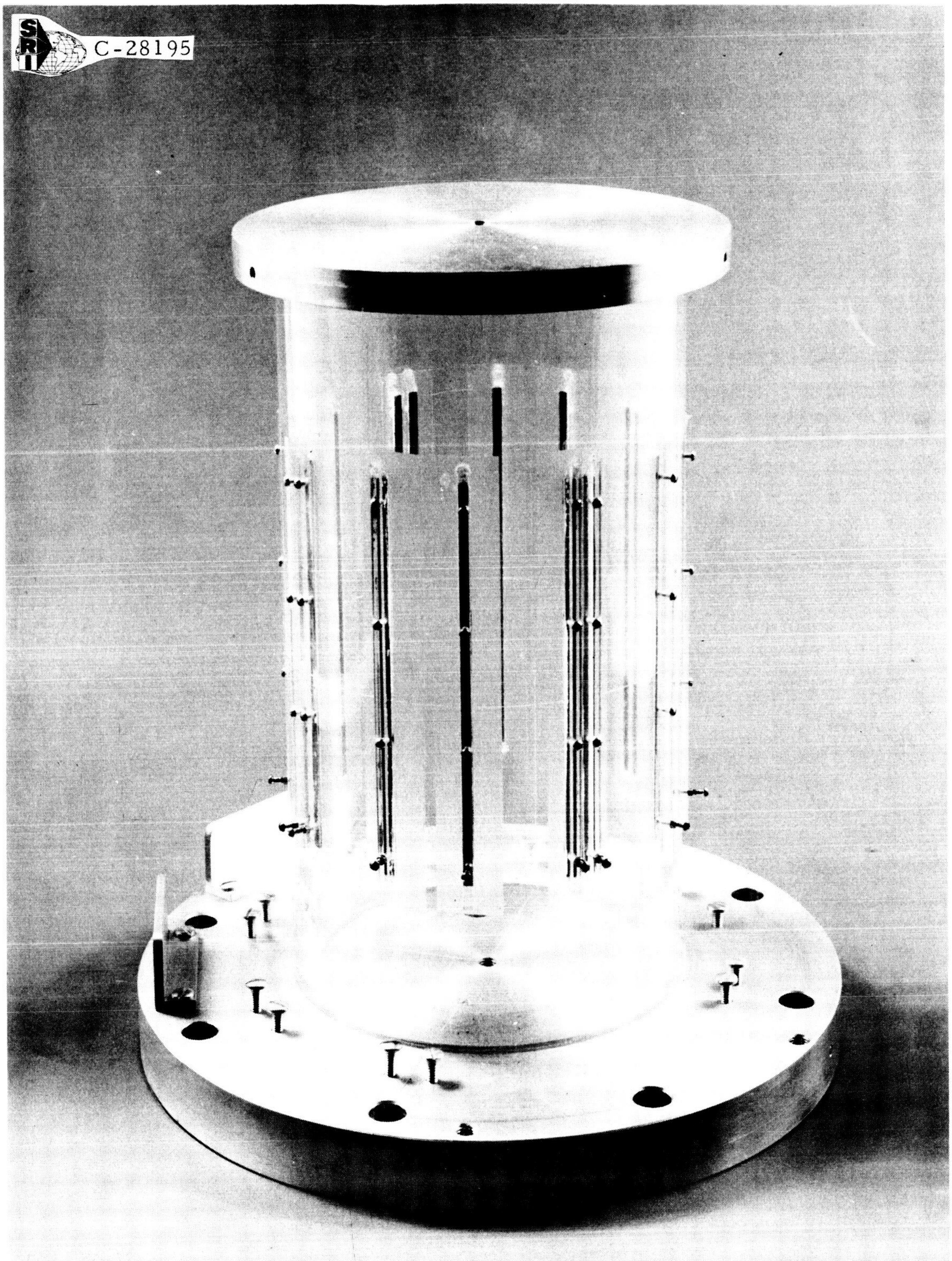

PLATE II. ANTISYMMETRIC MODE DISCRIMINATION--
TANK FITTED WITH 16-PROBE ARRAY