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ABSTRACT

For Kerr's rotating metric, it is shown that $-ma$ is the angular momentum of the body where m is the mass and a is the rotation parameter. This is true even for large m and a. The calculations are carried out using the exterior calculus of E. Cartan.

I. INTRODUCTION

In a previous paper[1] it was shown that when a is sufficiently small so that terms of higher power than the first are negligible, -ma is the angular momentum of a slowly rotating mass shell. It was also pointed out that a slowly rotating mass shell is not the only source for the Kerr metric to first order in a; other sources are e.g. a slowly rotating solid sphere of perfect fluid, many concentric shell, etc. The purpose of this paper is to find the expression for the angular momentum generated by any body which has the Kerr metric exterior to it. This is accomplished by integrating the conservation law over all space-time and applying a generalized form of Stokes theorem to this integral.

II. GENERALIZED STOKES THEOREM

In this section we consider the special case where the generalized Stokes Theorem takes the form of a divergence theorem. This divergence theorem (5) is expressed in tensor notation. Since it is Eq. (5) which is used in the subsequent calculations, the reader not familiar with exterior calculus can go directly to this equation without loss of continuity.

The generalized form of Stokes[2] is

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega \qquad (1)$$

where ω is an n- form, $d\omega$(the exterior derivative of ω) is an n+1 - form, and $\partial\sigma$ is the n dimensional boundary of the n+1 - dimensional surface σ. This form of Stokes Theorem[1] includes as special cases: the fundamental theorem of calculus, Stokes Theorem of vector analysis, Gauss' divergence theorem, etc. Also, this theorem (1) is valid in curved space as well as in flat space.

In finding the conserved angular momentum, it is convenient to express Stokes Theorem(1) as a generalized four dimensional divergence theorem. This is accomplished by considering the differential form ω to be a three-form dual to a one form b:[2,3]

$$\omega = {}^*b_{\mu}\omega^{\mu} = b^{\mu}\varepsilon_{\mu\alpha\beta\gamma}\ \omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}. \qquad \alpha < \beta < \gamma \qquad (2)$$

Exterior differentiation of Eq. (2) yields

$$d\omega = b^{\mu}{}_{;\mu}\ \varepsilon_{\alpha\beta\gamma\delta}\ \omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta} \qquad \alpha < \beta < \gamma < \delta \qquad (3)$$

in a four dimensional space. Substitutions of Eqs. (2) and (3) into Eq. (1) yields

$$\int_{\sigma} b^{\mu}{}_{;\mu} \underset{\alpha<\beta<\gamma<\delta}{\epsilon_{\alpha\beta\gamma\delta}} \omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta} = \int_{\partial\sigma} b^{\mu} \underset{\mu<\alpha<\beta<\gamma}{\epsilon_{\mu\alpha\beta\gamma}} \omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}. \quad (4)$$

The expression multiplying the divergence on the left side of Eq. (4) is the four dimensional volume element dV_4 while the integrand on the right side can be expressed as $b^{\mu}\, d\sigma_{\mu}$ where $d\sigma_{\mu}$ denotes the 3 dimensional surface element normal to b^{μ} as can be seen by inspection of Eq. (4). Hence Eq. (4) takes the form

$$\int_{\sigma_4} b^{\mu}{}_{;\mu}\, dV_4 = \int_{\partial\sigma} b^{\mu} d\sigma_{\mu}, \quad (5)$$

where σ_4 denotes the four dimensional surface over which the integral is carried out. In the same manner, it can be shown that the above result (5) is valid for spaces of arbitrary dimensions; it relates the integral over an n+1 dimensional space to that over its n dimensional boundary.

III. CONSERVATION LAW

The conservation law for general relativity

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{6}$$

can be transformed to a scalar equation by contraction with an arbitrary vectur ξ_μ:

$$\xi_\mu T^{\mu\nu}{}_{;\nu} = 0. \tag{7}$$

Integration of this quantity over all space-time yields

$$0 = \int_\sigma \xi_\mu T^{\mu\nu}{}_{;\nu}\, dV_4 \tag{8}$$

which can be transformed into a divergence and additional term

$$0 = \int_\sigma \left[\xi_\mu T^{\mu\nu} \right]_{;\nu} dV_4 - \int_\sigma \xi_{\mu;\nu}\, T^{\mu\nu} dV_4. \tag{9}$$

The second term on the right vanishes if ξ_μ is a Killing vector since the stress-energy tensor $T^{\mu\nu}$ is symmetric; the first term can be transformed to an integral over the boundary of σ via Eq. (5). Thus, there results

$$0 = \int_{\partial\sigma} \xi_\mu\, T^{\mu\nu}\, d\sigma_\nu. \tag{10}$$

If the source is bounded in space, the integral (10) reduces to the difference of the value of an integral over two different spacelike surfaces. This integral is therefore independent of the spacelike surface, and consequently is a conserved quantity:

$$J = \int_{\Sigma} \xi_{\mu} \, T^{\mu o} d\sigma_{o}. \qquad (11)$$

Here Σ denotes the three dimensional spacelike surface t = constant and $d\sigma_o$ is the three dimensional surface element.

In the special case of the metric and stress-energy tensor of reference (4) and the Killing vector

$\xi_{\mu} = [0,0,0,r\Psi^2\sin\theta]$, the quantity (11) becomes the conserved angular momentum[1]

$$J = \frac{2}{3} m(1+\beta_o) r_o^{\,2} \Psi_o^{\,5} (\omega_s - \Omega)/V_o, \qquad (12)$$

of a slowly rotating mass shell.

Since the stress-energy tensor is related to the Einstein tensor via Einstein's equations, the angular momentum J in Eq. (11) can be expressed completely in terms of geometrical quantities.

If the Cartan frames ω^{μ} are oriented so that the Killing vector ξ_{μ} has only the space-like components

ξ_1, Eq. (11) takes the form

$$8\pi J = \int_{\Sigma} \xi_1 G^{io} d\sigma_o . \tag{13}$$

When G^{io} is expressed in terms of the second fundamental form,[5] it takes the form of a divergence:

$$8\pi J = \int_{\Sigma} \xi_i [P^{ij} - \eta^{ij} P]_{;j} d\sigma_o \tag{14}$$

which becomes

$$8\pi J = \int_{\Sigma} (\xi_i [P^{ij} - \dot{\eta}^{ij} P])_{;j} d\sigma_o \tag{15}$$

because ξ_i is a Killing vector. The right side of Eq. (15) is of the same form as the left side of Eq. (5). Application of the three dimensional form of Eq. (5) to Eq. (15) yields

$$8\pi J = \int_{\partial\Sigma} \xi_i (P^{ij} - \eta^{ij} P) d\sigma_j^{2} \tag{16}$$

where $\partial\Sigma$ denotes the 2-dimensional boundary of the space-like surface t = const. and $d\sigma_j^{2}$ is a two dimensional area element of $\partial\Sigma$. If the space-like surfaces are closed (as in the Friedman universe), the integral (16) vanishes since a closed space has no boundary.

The boundary of an open space does not vanish and the angular momentum J can be found by considering only the asymptotic metric at large distances from the source.[6] At large distances the Kerr metric and the metric for a slowly rotating body coincide. Hence any source which generates the Kerr metric has the angular momentum

$$J = - ma. \tag{17}$$

This is true for large a as well as large m.[8] Also, since many different sources generate the exterior Kerr metric for small a, it is expected that there are many sources which generate the Kerr metric for large a.

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Footnotes

1. J. M. Cohen (to be published in J. Math. Phys.)
2. W. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press (1952); G. de Rham, Variétés Differentiables, Hermann, Paris (1960); D. R. Brill and J. M. Cohen, Selected Topics in General Relativity (to be published); C. Chevally, Theory of Lie Groups, Princeton University Press (1946).
3. C. W. Misner and J. A. Wheeler, Ann. of Phys. 2, 525 (1957)
4. D. R. Brill and J. M. Cohen, Phys. Rev. 143, 1011 (1966)
5. A. Lichnerowicz, Théories Relativistes de la Gravitation et de l'Électromagnétisme, Masson et Cie, Paris (1955); Y. Bruhat, J. Rat. Mech. Anal. 5, 915 (1956); J. M. Cohen (to be published); D. R. Brill, Suppl. Nuovo. Cimento II, 1, 3 (1964).
6. This result is closely related to those of R. Penrose, in Relativity, Groups and Topology, Gordon and Breach (1964) P. 565, and of J. Winicour (to be published).
7. The minus sign was first noticed by Boyer and Price |Proc. Cambridge Phil. Soc. 61, 531 (1965)| who considered the case of both m and a small.

8. By a coordinate transformation similar to that of Eq.(9) of reference 1, the metric of Newman, Couch Chinnapared, Exton, Prakash and Torrence [J. Math. Phys. 6, 918 (1965)] can be put in the form

$$ds^2 = -dt^2 + f\Sigma^{-1}(dt + a\sin^2\theta d\phi)^2$$
$$+ \Sigma(\Delta^{-1}dR^2 + d\theta^2) + (R^2 + a^2)\sin^2\theta d\phi^2$$

where $f = 2mr - e^2$,

$\Sigma = R^2 + a^2\cos^2\theta$,

$\Delta = R^2 + a^2 - f$.

Substitution of this metric into Eq. (16) yields the angular momentum $J = -ma$, the same as that for the Kerr metric. Also, as for the Kerr metric, this result is valid for large a as well as large m. The above form of the rotating charged metric was found independently by Brandon Carter (private communication).