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COLLISIONAL MODEL OF METEOROIDS

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SUMMARY

The meteoroid environment is of considerable scientific and aerospace interest. With the current interest in manned interplanetary missions with aphelion inside the asteroidal belt, an estimation of the deep space meteoroid environment is necessary. Inasmuch as inadequate information exists regarding the number density of meteoroids in the asteroidal belt, this study was undertaken in an effort to make a constribution in this direction.

More specifically, the physical significance of population index type particle distributions is examined. These are distributions where the number density f(m)dm of particles per unit volume with a mass in the range m to m + dm is given by

$$f(m)dm = Am^{-\alpha}\, dm \qquad (S-1)$$

where A and α are constants, the latter known as the population index.

Because of its simplicity, eq. S-1 is very useful since if f(m) is known at two mass values, then both A and α are defined and extrapolation is possible. In the absence, however, of a priori physical reason for the existence of populations of the form eq. S-1, such extrapolations are questionable at best. The purpose of the present study is to discuss a particular circumstance under which a population of meteoroids is correctly described by eq. S-1, and to compare the results with available experiment.

The following equation constitutes the formulation of the physical model considered, and will be called the collision equation:

Summary (cont'd)

time rate of change in the number density of particles having a mass in the range m to m + dm	=

	number of particles removed (per unit time) from the mass range m to m + dm due to the erosive influence of collisions with comparatively small particles*
−	

(S-2)

	number of particles removed (per unit time) from the mass range m to m + dm due to catastrophic collisions (these result in the complete disruption of the "test" objects with masses in the range m to m + dm)
−	

	number of particles, in the mass range m to m + dm, created (per unit time) due to collisional fragmentation between objects with sufficiently large masses
+	

It is assumed that the population has reached a steady-state value, i.e., the time derivative term on the left hand side of eq. S-2 is zero, or very small. It is then shown, in the text, that a solution of the form eq. S-1 satisfies the collision equation provided that the population index α is given by

$$1.75 < \alpha < 2 \tag{S-3}$$

in the mass range

$$\mu\Gamma' << m << \Lambda M_\infty/\Gamma' \tag{S-4}$$

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*This term also includes the converse effect of slightly larger particles eroding into the mass range m to m + dm.

Summary (cont'd)

where μ is the mass of the smallest object not blown away by radiation pressure, $\mu\Gamma'$ is the largest mass completely shattered by a projectile with mass μ and $\Lambda M_\infty/\Gamma'$ is the mass of the largest fragment when the largest mass in the sample, M_∞, is shattered catastrophically.

The collision equation eq. S-2 cannot be satisfied by trial solutions of the form eq. S-1 for values of α beyond the limits given by eq. S-3. For $\alpha > 2$ erosion by the smallest particles dominates and for $\alpha < 1.75$ collision products from the most massive objects cause evolution of the mass distribution with time.

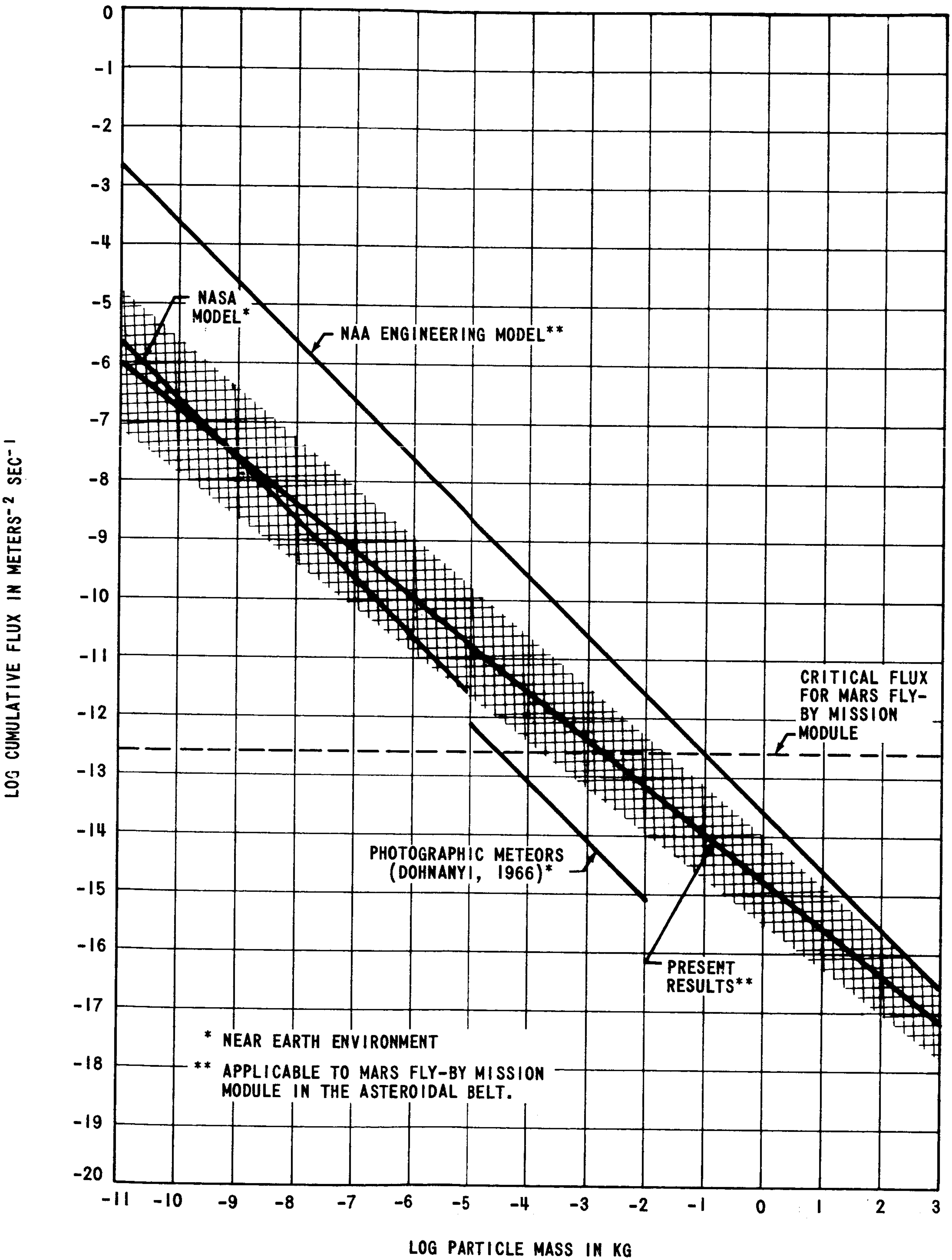
Comparison of this result with observational information regarding the near earth environment is given in Fig. 1. It can be seen, from this figure, that the gross features of the distribution are reproduced by the present model.

The distribution of asteroids catalogued by Kuiper et al is given in Fig. 2. A least squares fit to the number density of the observed asteroids gives

$$\alpha = 1.80 \pm .04 \qquad \text{(S-5)}$$

in good agreement with our result, eq. S-3. Comparison with the distribution of lunar craters and meteorites is also found to be favorable.

It therefore appears that an extrapolation of the number density of asteroids into mass ranges of much smaller objects is a reasonable first approximation to their distribution. This extrapolation is sketched in the summary chart Fig. S-1. It can be seen, from the figure, that the number density of asteroidal debris as obtained by the present study is considerably lower than the NAA engineering model.



LOG CUMULATIVE FLUX IN METERS^{-2} SEC^{-1}
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-13
-14
-15
-16
-17
-18
-19
-20
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
NASA MODEL*
NAA ENGINEERING MODEL**
CRITICAL FLUX FOR MARS FLY-BY MISSION MODULE
PHOTOGRAPHIC METEORS (DOHNANYI, 1966)*
PRESENT RESULTS**
* NEAR EARTH ENVIRONMENT
** APPLICABLE TO MARS FLY-BY MISSION MODULE IN THE ASTEROIDAL BELT.
LOG PARTICLE MASS IN KG

FIGURE S-1

ABSTRACT

A collisional model of meteoroids is formulated. An equation is derived which describes the evolution of a system of particles under the processes of particle destruction due to collisions and particle creation due to fragmentation during collisions. If the system of particles is assumed to have reached steady-state conditions it is found that a particle number density function of the simple form $Am^{-\alpha}\,dm$ (m is particle mass, A and α are constants) satisfies the equation provided that α has a value in the range $\frac{1}{2}(\eta + 5/3) < \alpha < 2$ where η is a material parameter having a value of about 1.8. Comparison with near earth meteoroid fluxes indicates that the gross features of the observed distribution are reproduced by the present model. Agreement is also found with the distribution of the catalogued asteroids.

COLLISIONAL MODEL OF METEOROIDS

1.0 INTRODUCTION

The distribution of interplanetary debris is of considerable scientific and engineering interest. As the smallest members of the solar system, these objects are of interest to the astronomer and since they can collide with a spacecraft, they are of equally strong interest to the aerospace engineer. With the current interest in a manned Mars flyby mission which is planned to enter the asteroidal belt at aphelion, the estimation of the distribution of debris in the asteroidal belt is one of the most important environmental problems to be solved. This paper treats theoretically the collective dynamical properties of a population of debris particles in orbit around the sun undergoing mutual collisions and subsequent fragmentation; a physical basis for estimating particle distributions is therefore established. Application of the present model to the flux of meteoroids near earth serves as an independent check; the gross features of the particle flux into the earth's atmosphere are found to be reproduced with the present model.

Section 2 of this paper is a discussion of the fragmentation of rock during hypervelocity impact. A mathematical formulation of the fragmentation process, based on results by Gault, Shoemaker and Moore (1963) is developed.

The collision equation governing the dynamics of a collection of particles undergoing mutual collisions and fragmentation is developed in Section 3. This is an integro differential equation for the number density function of particles having a mass in the range of m to m + dm.

In Section 4, a trial solution of a population index form

$$f(m)dm = Am^{-\alpha}\,dm \qquad (1)$$

where $f(m)dm$ is the number density function of particles per unit volume having a mass in the range m to m + dm and α is the population index, is then substituted into the collision equation. It is found that a solution of this form (eq. 1) solves the collision equation only if the latter has reached a steady-state condition in time, i.e., when the particle removal rate equals the particle creation rate.

In Section 5, the results of Section 4 are compared with the distribution of near earth meteoroids of presumably cometary origin. It is found that the gross features of the observed distribution are reproduced with the present model.

Section 6 is an application of the model for the estimation of the number density of debris in the asteroidal belt. In Section 7 agreement is sought and found between the solution of the collision equation and the distribution of known asteroids. Comparison with the distribution of lunar craters and of meteorites is also found to be favorable.

2.0 THE CRUSHING LAW

Interplanetary space contains a very large number of objects having different masses and orbits. These objects are believed to frequently collide with each other inelastically. When such a collision occurs at a sufficiently high relative velocity, fragmentation results. In the present study, the relative velocities will be comparable to those of dust particles in space traveling in different but intersecting orbits. This means that the impact velocity will be of the order of kilometers per second and, hence, sufficiently high to cause fragmentation.

Regarding the mass distribution of fragments produced during impact, the following type of crushing will be assumed:

$$g(m;\ M,M_2)dm = C(M,M_2)m^{-\eta}\ dm \tag{2}$$

Here, $g(m;\ M,M_2)dm$ is the number of particles having a mass between m and m + dm produced during the impact of a mass M with another, larger mass M_2. The coefficient $C(M,M_2)$ is a function of the colliding masses and η is a constant.

We first consider the case when the target mass is very great compared with the projectile particle, i.e., $(M_2/M) \to \infty$. This is the frequently considered problem of high velocity impact into a semi-infinite target. The quantity g is now a function of M and m only and eq. 2 becomes

$$g(m;\ M,\infty)dm = C(M,\infty)m^{-\eta}\ dm \tag{2'}$$

This particular crushing law for a semi-infinite target is based on experiment (Gault, Shoemaker and Moore, 1963) and observation over a limited number of cases. Use of a particular crushing law is one of the major assumptions in this paper. However, since evidence supports a crushing law of the general form of equation 2' during hypervelocity impact, it will be adopted here to estimate the distribution of particles resulting from inelastic collisions at orbital velocities.

In order to estimate $C(M,\infty)$ the following integral has to be evaluated

$$M_e = \int_{\mu}^{M_b} m\ g(m;\ M,\infty)dm \tag{3}$$

where M_e is the total ejected mass, M_b is the largest and μ the smallest fragment produced; $g(m;\ M,\infty)$ is given by eq. 2. Substitution of eq. 2' into eq. 3 and subsequent integration gives

$$C(M,\infty) = \frac{(2-\eta)M_e}{M_b^{\ 2-\eta} - \mu^{2-\eta}} \tag{4}$$

Gault, Shoemaker and Moore (1963) found that η is about 1.8; μ is a submicron particle (Gault and Heitowit, 1963) corresponding to a mass of the order of 10^{-14} kg or smaller. In the present treatment, the particle masses in the range of interest will be very much larger than 10^{-14} kg; one therefore obtains (since $\eta < 2$)

$$M_b^{\ 2-\eta} >> \mu^{2-\eta} \tag{5}$$

The denominator in equation 4 can, therefore, be simplified and one obtains

$$C(M,\infty) = (2-\eta)M_e M_b^{\ \eta-2} \tag{6}$$

Both the total ejected mass M_e and the largest fragment M_b are functions of the mass M of the colliding particle and will be assumed to have the form

$$M_e = \Gamma M << M_2$$

$$M_b = \Lambda M << M_2 \qquad (7)$$

where Γ and Λ are both functions of the impact velocity and material properties of the target as well as the projectile but not of their masses. M_2 is the mass impacted by M and the double inequality sign reflects the fact that the coefficients Γ and Λ refer to a semi-infinite target. It can be seen from eq. 7 that Γ is the total ejected mass per unit projectile mass and Λ is the mass of the largest fragment per unit projectile mass.

The use of eq. 7 is based on results from hypervelocity experiments discussed by Gault et al (1963). These authors find that the total ejected mass as well as the mass of the largest fragment during hypervelocity cratering into basalt is proportional to the projectile kinetic energy, and hence, to the projectile mass. These experiments were conducted at impact velocities over a range not exceeding 10 Km/sec and over a range of projectile kinetic energies from 10 joules to 10^4 joules, approximately.

If

$$\Gamma M \approx M_2$$

then eq. 7 breaks down because hypervelocity impact into a relatively small target differs from the former (semi-infinite target) situation since the shock formed during impact will be reflected back toward the impact area rather than propagated away to infinity (i.e., dissipated). This is particularly significant for stones fracturing easily under tension. For these objects, a mass

$$M_2 >> \Gamma M$$

can still be completely shattered by the shock wave (generated during the event) which is reflected at the free surfaces and propagated inward as a tension wave.

In the absence of sufficient factual information describing this catastrophic process, the following will be assumed:

(i) the largest mass M_2 completely shattered by M is given by

$$M_2 = \Gamma' M$$

with $M_e \equiv M + M_2$ and $\Gamma' > \Gamma$

(ii) when

$$M_2 > \Gamma' M$$

the semi-infinite target relations are valid.

Using these constants in eq. 6, one obtains an explicit expression for $C(M,M_2)$ in terms of M

$$C(M,M_2) \cong C(M,\infty) = (2-\eta)\Gamma\Lambda^{\eta-2} M^{\eta-1} , \quad \Gamma' M \leq M_2 \tag{8}$$

For impacts between two particles where $\Gamma' M$ is greater than M_2, we take the total available mass $M + M_2$ to be equal M_e

$$M_e = M + M_2 \tag{9}$$

and obtain

$$C(M,M_2) = (2-\eta)\Lambda^{\eta-2} (M + M_2)M^{\eta-2}, \quad \Gamma' M \geq M_2 \tag{10}$$

This relation, together with eq. 2 and 8 defines the model crushing law employed in this study.

Approximate numerical values for Γ and Λ, based on hypervelocity impact experiments into basalt by Gault et al (1963) are given in Table I at several impact velocities.

TABLE I

V(Km/sec)	Γ	Λ
5	1.3×10^2	1.3×10
10	5×10^2	5×10
15	1.1×10^3	1.1×10^2
20	2×10^3	2×10^2

The value of Γ' is more difficult to estimate. Gault (private communication) observed that a basalt particle is completely shattered by a projectile 10^{-3} times its mass, moving at 2 km/sec. Since Γ is about 20 at this velocity,

$$\Gamma' = 50\ \Gamma$$

for this case.

3.0 COLLISIONAL MODEL

In this section the mathematical formulation of the evolution of a system of colliding particles is developed. To be specific, given that $f(m,t)dm$ is the number of particles having a mass between m and $m + dm$ at a time t, this function will change as a result of collisions between the particles because many new particles are constantly created and others are destroyed. The system itself possesses a "sink" in the sense that sufficiently small particles are removed by the Poynting Robertson effect and still smaller ones are almost instantly blown out of the solar system by radiation pressure.

In what follows, the system will be assumed sufficiently random that an effective average collisional velocity is meaningful; the collision cross-section is taken as the cross-sectional area of the colliding particles. This assumption is equivalent to the process of finding the motion of the center of mass of the system of particles, then switching to the center of mass coordinate system; the particle velocities will then be random, to a first approximation. Here we have invoked the analogy of a system of gas molecules in a box, when the box itself undergoes translation or rotation.

Assuming spherical particles, the probability of collision between two particles with radii r_1 and r_2 is proportional to $\pi(r_1 + r_2)^2$. In what follows, the particle masses rather than particle radii will be taken as the independent variable and whence, the probability of collision per unit time between two particles is proportional to

$$K\left(M_1^{1/3} + M_2^{1/3}\right)^2 \qquad (11)$$

where

$$K = \overline{V}\left[3\pi^{1/2}/4\rho\right]^{2/3}$$

Here $\overline{V}$ is the average relative velocity of the particles and ρ is the material density of the particles. K will be taken to be a constant, to a first approximation. This means that the velocity distribution is taken to be independent of the mass distribution and all particles are assumed to have the same material density. If the expression, eq. 11, is multiplied by the number density per unit volume of particles in the mass range M_2 to $M_2 + dM_2$, the resulting expression

$$K\left(M_1^{1/3} + M_2^{1/3}\right)^2 f(M_2,t)dM_2 \qquad (11')$$

is proportional to the total number of collisions (per unit time) of an individual particle with mass M_1 with other particles in the mass range M_2 to $M_2 + dM_2$. In other words, the expression eq. 11 is the "influx" per unit time of particles in the mass range M_2 to $M_2 + dM_2$ "into" a particular object of mass M_1.

If now the expression, eq. 11', is multiplied by the particle number density per unit volume in the mass range M_1 to $M_1 + dM_1$ the resulting expression

$$K\left(M_1^{1/3} + M_2^{1/3}\right)^2 f(M_2, t)dM_2 \, f(M_1, t)dM_1 \qquad (11')$$

is the total number of collisions, per unit volume and unit time, of particles in the mass range M_1 to $M_1 + dM_1$ with particles in the mass range M_2 to $M_2 + dM_2$.

A collision equation defining the collective evolution of our system of particles can now be defined. The time rate of change of the number of particles in a mass range of m to m + dm is given, in a schematic form, by the following expression (individual terms are explained below):

$$\frac{\partial f(m,t)}{\partial t} dm = \overset{\mathrm{I}}{\boxed{\begin{array}{l}\text{rate of change of the number of}\\ \text{particles per unit volume and}\\ \text{unit time in mass range m to}\\ \text{m + dm due to erosion}\end{array}}} + \overset{\mathrm{II}}{\boxed{\begin{array}{l}\text{rate of change, because of "ca-}\\ \text{tastrophic" collisions, of the}\\ \text{number of particles per unit}\\ \text{volume and unit time in the mass}\\ \text{range m to m + dm}\end{array}}} + \overset{\mathrm{III}}{\boxed{\begin{array}{l}\text{number of particles in the mass}\\ \text{range m to m + dm, created per}\\ \text{unit time and unit volume by}\\ \text{collisional crushing}\end{array}}} + \overset{\mathrm{IV}}{\boxed{\begin{array}{l}\text{number of particles in the mass}\\ \text{range m to m + dm removed per}\\ \text{unit time and unit volume by}\\ \text{the Poynting Robertson effect}\end{array}}} \qquad (12)$$

Term I is the rate of change of the number of particles per unit volume and unit time in the mass range m to m + dm due to the fact that the masses are themselves changing in time. This is caused by collisional processes which erode particles into and out of the mass range m to m + dm with the passage of time. Mathematically, the problem is to find the rate of change of the number density of particles in mass range m to m + dm at <u>fixed mass</u>, given that the masses themselves change at a prescribed rate dm/dt. It can be shown, as is done in Appendix A, that the resulting expression for Term I, is

$$(\mathrm{I}) = \left.\frac{\partial f(m,t)}{\partial t}\right|_{\text{erosion}} = -\frac{\partial f(m,t)}{\partial m}\frac{dm}{dt} - f(m,t)\frac{\partial}{\partial m}\frac{dm}{dt} \qquad (13)$$

For dm/dt, which is the rate at which the mass of a particle changes in time, we use the mass removed per unit time by collisions with mass M not large enough to completely disintegrate our mass m. The amount of mass removed during a single collision with a mass M is, according to equation 7,

$$\Gamma M \qquad (14)$$

The number of collisions that one mass m will experience (per unit time) with particles in a mass range M to M + dM is (cf discussion preceding eq. 11')

$$K f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM \qquad (15)$$

The total mass removed from m (per unit time) due to collisions with particles in the mass range from M to M + dM is

$$K\Gamma M\, f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM \qquad (16)$$

Whence, the mass removal rate due to collisions with a finite mass range of particles is

$$\frac{dm}{dt} = - \Gamma k \int_1^2 M\ f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM \qquad (17)$$

For the lower limit we take particles with mass μ which are the smallest particles present. For the upper limit we take particles having a mass just large enough to completely break up one mass m. As a criterion for "catastrophic" encounter, we take (see discussion preceding eq. 8)

$$m = \Gamma' M \qquad (18)$$

as the limiting mass M still included in the integral (17). Therefore, the expression for the mass loss becomes

$$\frac{dm}{dt} = - \Gamma K \int_\mu^{m/\Gamma'} M\ f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM \qquad (19)$$

and term (I) of the collision equation (12) is:

$$(I) = \frac{\partial f(m,t)}{\partial m} \Gamma K \int_\mu^{m/\Gamma'} M\ f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM$$

$$+ f(m,t)\ \Gamma K \frac{\partial}{\partial m} \int_\mu^{m/\Gamma'} M\ f(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM \qquad (20)$$

This completes our derivation of term (I) in the collision equation.

We now consider term (II), catastrophic collisions, of eq. 12, which can be derived by noting that the probability of collision per unit volume and unit time between two particles in a mass range m to m + dm and M to M + dM is (cf discussion accompanying eq. 11'')

$$f(m,t)\ dm\ f(M,t)dM\ K\left(m^{1/3} + M^{1/3}\right)^2 \qquad (21)$$

The probability for a collision per unit volume and unit time between a particle with mass m and some other particle in a finite mass range is then

$$Kf(m,t) \int_1^2 f(M,t)\left(m^{1/3} + M^{1/3}\right)^2 dM \qquad (22)$$

Since the effect of collisions with particles that do not completely destroy m has been accounted for in term (I), we evaluate the integral in eq. 22 over all "catastrophic" collisions. The result is term (II):

$$(II) = - Kf(m,t) \int_{m/\Gamma'}^{M_\infty} f(M,t)\left(m^{1/3} + M^{1/3}\right)^2 dM \quad (23)$$

where the minus sign is used to denote a particle removal process.

Piotrowsky (1953), in an earlier study, obtained a term similar to eq. 23. We disagree with him, however, inasmuch as Piotrowsky's formulation is equivalent to

$$-Kf(m,t)\ m^{2/3} \int_{m/\Gamma'}^{M_\infty} f(M,t)\ dM$$

i.e., the collision cross-section factor $(m^{1/3} + M^{1/3})^2$ is replaced by $m^{2/3}$. This approximation is invalid for $M > m$, and for arbitrary distributions can introduce a serious error near the upper limit.

Term (III) of the collision equation (eq. 12) can be derived by noting that for each event whose probability is given by eq. 22 there are $g(m; M,M_2)dm$ "secondary" particles produced in the mass range between m and m + dm, where $g(m;M,M_2)$ is the crushing law eq. 3 and where M is the smaller one of the two colliding masses and is assumed to be completely broken up during impact. One, therefore, has

$$dmK \int dM \int dM_2\ g(m;M,M_2)\ f(M,t)\ f(M_2t)\left(M^{1/3} + M_2{}^{1/3}\right)^2$$

$$= Km^{-n}\ dm \int dM \int dM_2\ C(M,M_2)\ f(M,t)\ f(M_2t)\left(M^{1/3} + M_2{}^{1/3}\right)^2 \qquad (24)$$

which is the rate at which particles having a mass between m and m + dm are created (per unit volume and unit time) due to the crushing of mass during an impact between sufficiently large particles to produce fragments of mass m, and where $C(M,M_2)$ is given by equations 8 or 10.

The limits on the double integral in equation 24 can be obtained by noting that the dummy M is always smaller than dummy M_2, otherwise, the argument of $C(M,M_2)$ should be replaced by M_2,M (this follows from the definition of the crushing law, discussed in the previous section). The lower limit of the integral for M_2 is, therefore, M. Since we are interested in all possible collisions, the upper limit for M_2 is M_∞. The lower limit for M is defined by the condition that the largest fragment M_b produced during impact must be equal to or larger than m whence, by equation 7.

$$m \leq M_b = \Lambda M \qquad (25)$$

and

$$M \geq m/\Lambda \qquad (26)$$

Using appropriate upper limits and dropping the differential dm, equation 24 becomes term (III) in the defining equation 12:

$$(III) = K(2-\eta)m^{-\eta}\ \Lambda^{\eta-2}\ A^2\ x$$

$$x\left\{\int_{m/\Lambda}^{M_\infty/\Gamma'} dM \int_{M}^{\Gamma' M} dM_2 (M+M_2)\left(M^{1/3}+M_2^{1/3}\right)^2 M^{\eta-2} f(M,t) f(M_2,t)\right.$$

$$+\ \Gamma \int_{m/\Lambda}^{M_\infty/\Gamma'} dM \int_{\Gamma' M}^{M_\infty} dM_2 \left(M^{1/3}+M_2^{1/3}\right)^2\ M^{\eta-1} f(M,t) f(M_2,t)$$

$$\left. +\int_{M_\infty/\Gamma'}^{M_\infty} dM \int_{M}^{M_\infty} dM_2 (M+M_2)\left(M^{1/3}+M_2^{1/3}\right)^2 M^{\eta-2} f(m,t) f(M_2,t)\right\} \tag{27}$$

where the first and third integrals refer to catastrophic collisions between masses M and M_2 such that both are totally disrupted

$$\Gamma' M \geq M_2 \quad , \tag{28}$$

the second integral refers to erosive collisions between masses M and M_2 such that M_2 behaves as an infinite target

$$\Gamma' M \leq M_2$$

and the mass redistributed is just ΓM.

The third integral refers to catastrophic collisions between objects in the mass range M_∞/Γ' to M_∞. This means that for test masses m

$$m/\Lambda > M_\infty/\Gamma'$$

the first two integrals are zero and only the third integral is retained* with lower limit of m/Λ replacing M_∞/Γ'.

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*See Appendix B, for details.

The last term (i.e., term (IV)) in eq. 12 is the rate at which particles are being lost due to the Poynting Robertson effect (Robertson, 1937) due to radiation damping. According to Robertson's analysis, a particle in an approximately circular orbit will move from a mean distance from the sun of R_o(AU) to a distance of R(AU) from the sun, during a time t given by

$$t = 10^8 \; (R_o{}^2 - R^2) \; m^{1/3} \text{ years} \tag{29}$$

where a particle specific gravity of 3.5 has been assumed, and where m is expressed in Kilograms.

For an effective particle lifetime one may take the time required for moving 1AU closer to the sun ($R = R_o - 1$ although this is arbitrary to some extent), in which case, one has

$$\tau = 10^8 \; (2R_o - 1) \; m^{1/3} \text{ years}$$

If R_o = 1AU, then τ becomes

$$\tau = 10^8 \; m^{1/3} \text{ years} \equiv \tau_o \; m^{1/3} \tag{30}$$

The rate at which the population per unit volume of particles having a mass between m and m + dm decreases in time can then be represented approximately as

$$(IV) = - \frac{f(m,t)}{\tau} = \frac{-m^{-1/3} f(m,t)}{\tau_o} \tag{31}$$

where use has been made of eq. 30, and where the differential dm has been dropped.

4.0 SOLUTION OF THE COLLISION EQUATION FOR SMALL PARTICLES

The collision equation derived in the previous section is complicated. Simplifications can, however, be obtained by restricting one's attention to special cases. We shall, in what follows, consider special cases where the collision equation can be solved. We will limit our attention to the evolution of particles having a mass **m** very much smaller than M_∞, but very much larger than μ which is the smallest particle mass present and is determined by the radiation pressure limit.

More specifically, particles with a mass smaller than $\Gamma'\mu$ will no longer be eroded according to this model, since any collision they experience will be 'catastrophic'. Our present model is, therefore, expected to be valid only for masses m in the range

$$\Gamma'\mu << m << \Lambda M_\infty/\Gamma' \tag{32}$$

where $\Gamma'\mu$ is the largest mass completely disrupted by a collision with the smallest surviving particle μ; this means that a mass $\leq \Gamma'\mu$ is completely disrupted by every collision and the erosion term for particles in this mass range does not apply. $\Lambda M_\infty/\Gamma'$ is the largest mass that can be created by a catastrophic collision and therefore the particle creation term (III) does not apply to the distribution of masses greater than $\Lambda M_\infty/\Gamma'$.

The collision eq. 12 expresses the time rate of change of the particle number density function in terms of the individual collision processes. We shall choose the simplest possible case and as a first approximation assume that the particle distribution has reached "steady-state", i.e., the rate at which particles having a mass m are produced is balanced by the rate of their removal by collisions.

Mathematically, the steady-state assumption implies that a source exists which steadily regenerates the large objects. This appears to be the case with cometary meteoroids, since comets are believed to be constantly giving off particles. While the rate of generating particles by comets as a function of particle size is not precisely known, one has some information on the subject by considering stream meteors which are believed to have been generated comparatively recently.

Radio work indicates that the abundance of faint meteors in streams is small by comparison with the background sporadic flux, while nearly half of the bright photographic meteors move in identifiable streams (Whipple, 1963). It, therefore, appears that the meteoroids generated by comets are comparatively "rich" in large particles and poor in small ones.

We shall, to a first approximation, assume that the only effect of the meteoroid production by comets on our model distribution is one of keeping $f(M_\infty)$ constant in time, i.e., keeping the "top masses" of the distribution constant.

In the case of asteroids, the situation is more speculative since large asteroids are not believed to be regenerated at the present time. If, however, we assume that the processes of particle creation by fragmentation and particle removal by collisions are approximately equal then the present theory is a reasonable first approximation. Among the catalogued asteroids, certain groups exist (Hirayama groups) that are believed to have resulted from collisions by "parent" asteroids. Such collisions between very large asteroids are believed rare, with a lifetime of perhaps many millions of years. As the size of the asteroids diminishes, however, their frequency increases with a corresponding increase in the number of collisions.

We now turn our attention to the individual terms of the collision equation (eq. 12). Their properties will be discussed in detail and appropriate simplifications will be introduced.

Term (I) (eq. 20) of the collision equation expresses the rate of change due to erosion in the particle number density function and has the mathematical form:

$$(\mathrm{I}) = \frac{\partial f(m,t)}{\partial m}\Gamma K \int_{\mu}^{m/\Gamma'} Mf(M,t)\left(M^{1/3} + m^{1/3}\right)^2 dM$$

$$+ f(m,t)\Gamma K \frac{m}{(\Gamma')^2} f\left(\frac{m}{\Gamma'},t\right)\left[(\Gamma')^{-1/3} + 1\right]^2 m^{2/3} \tag{33}$$

$$+ f(m,t)\Gamma K \frac{2}{3} m^{-2/3} \int_{\mu}^{m/\Gamma'} Mf(M,t)\left(M^{1/3} + m^{1/3}\right) dM$$

where the last two terms are the result of carrying out the partial differentiation with respect to m, as indicated in eq. 20.

Since Γ' is of the order of 10^3 to 10^4, the collisional cross-sectional area can be taken to be

$$K\left(M^{1/3} + m^{1/3}\right)^2 \approx Km^{2/3} \quad . \tag{34}$$

The error thereby committed in the upper limit is,

$$\left(\left(\frac{m}{\Gamma'}\right)^{1/3} + m^{1/3}\right)^2 - m^{2/3} = m^{2/3}\ (2\Gamma^{-1/3}) \approx 10\%\ \text{of}\ m^{2/3} \tag{35}$$

and the error committed in the lower limit is completely negligible. We therefore write, approximately

$$\begin{aligned}(I) \approx\ & \frac{\partial f(m,t)}{\partial m}\Gamma K\ m^{2/3} \int_{\mu}^{m/\Gamma'} Mf(M,t)dM \\ & + f(m,t)\ f(m/\Gamma',t)\ \Gamma K/(\Gamma')^2\ \ m^{5/3} \\ & + f(m,t)\Gamma K\ (2/3)\ m^{-1/3} \int_{\mu}^{m/\Gamma'} Mf(M,t)dM\end{aligned} \tag{36}$$

where the above approximation is a slight "under estimation" of the process to the extent that the geometrical cross-section of the smaller particle is neglected. The approximation is, however, far less serious than may appear on the surface, because the cross-section

$$K\left(M^{1/3} + m^{1/3}\right)^2$$

includes the case of "grazing incidence"; and, therefore, overestimates the probability of mass removal during cratering. We are, therefore, satisfied that equation (36) defines process (I) with reasonable accuracy.

We now consider the rate of "catastrophic" collisions given by term II of the collision equation (eq. 23):

$$-Kf(m,t)\int_{m/\Gamma'}^{M_\infty} f(M,t)\left(m^{1/3}+M^{1/3}\right)^2 dM \qquad (23)$$

Using a power law steady-state distribution

$$f(m,t) = f(m) = Am^{-\alpha} \qquad (1)$$

and eq. 23 integrates to give (multiplying out the squared parenthesis):

$$\begin{aligned}- KA^2\, m^{-\alpha}\Big\{ & m^{2/3}\left[M_\infty^{-\alpha+1} - (\Gamma')^{\alpha-1}\, m^{-\alpha+1}\right](-\alpha+1)^{-1} \\ & + 2m^{1/3}\left[M_\infty^{-\alpha+4/3} - (\Gamma')^{\alpha-4/3} m^{-\alpha+4/3}\right](-\alpha+4/3)^{-1} \qquad (23') \\ & + \left[M_\infty^{-\alpha+5/3} - (\Gamma')^{\alpha-5/3} m^{-\alpha+5/3}\right](-\alpha+5/3)^{-1}\Big\}\end{aligned}$$

provided that $\alpha \neq 5/3$, $4/3$ or 1. If now $\alpha > 5/3$ and $\frac{\Lambda M_\infty}{\Gamma' m} \to \infty$, this expression simplifies to

$$- KA^2\, m^{-\alpha}\, m^{-\alpha+5/3} \text{ x constant}$$

and if

$$\alpha < 5/3\ ,\ \frac{\Lambda M_\infty}{\Gamma' m} \to \infty$$

one obtains, instead,

$$- KA^2\, m^{-\alpha}\, M_\infty^{-\alpha+5/3} \text{ x constant.}$$

It therefore appears that for a population of a power law type the collisional lifetime of an object is only a function of the more numerous small objects for a population index $\alpha > 5/3$ and for a population index of $\alpha < 5/3$ it is given by the $-\alpha + 5/3$ power of the largest object in the sample.

The collision equation (eq. 12) can be further simplified when the relative importance of radiation damping (term IV) is considered. We shall show, in what follows, that this term can be neglected altogether by comparison with the influence of collisional processes on the particle number density function. The lifetime of a particle limited by the Poynting Robertson effect is, according to equation 30,

$$\tau_{PR} = 10^8 \, m^{1/3} \text{ year} \tag{38}$$

where the subscript PR stands for Poynting Robertson and where m is the particle mass in Kilograms.

The lifetime of a particle of mass m due to catastrophic collisions by another particle with mass m/Γ' or larger, can be estimated by noting that

$$\tau_{CC} = \frac{1}{Na} \tag{39}$$

where the subscript CC stands for catastrophic collisions, N is the flux of particles having a mass of m/Γ' Kg or larger per meter^2 sec and a is the cross-sectional area of the particle in meter^2. Taking $\Gamma' \approx 10^4$, particle density of 10^3 kg/meter^3 and the NASA model* cometary flux of

$$N = 10^{-16.6} \, m^{-1} \tag{40}$$

one obtains

$$\tau_{CC} \approx 2 \times 10^6 \, m^{1/3} \text{ years}$$

where m is the mass in Kg. This τ_{CC} is 100 times shorter than τ_{PR}.

*Natural Environment and Physical Standards for the Apollo Program, April 1965 - revised.

Another check on the relative importance of radiation damping may be performed by estimating the lifetime of a particle τ_E limited by erosion. The mass erosion rate of a cometary particle has been estimated (Whipple, 1963) as 7×10^{-12} gm/cm^{2}sec. This means that we may take, in MKS units,

$$\frac{dm}{dt} = - b \times 4\pi r^2 = - b \times 5 \times 10^{-2} \ m^{2/3} \qquad (41)$$

where dm/dt is the rate of mass loss due to etching, b is the rate of mass loss per unit surface area and where a particle density of 10^3 Kg/m^3 (1 gm/cm^3) has been assumed.

This equation is easily solved for the time τ_E required to erode the particle:

$$\tau_E \simeq \frac{3 \times m^{1/3}}{5b \times 10^{-2}} \qquad (42)$$

Using $b = 7 \times 10^{-12}$ Kg/meter^2 sec, we get

$$\tau_E \simeq 10^{13} \text{ sec} = 3 \times 10^6 \ m^{1/3} \text{ year} \qquad (43)$$

again yielding a process two orders of magnitude faster than radiation damping. These results are in agreement with an earlier discussion by Whipple (1963) where he treated the lifetime of photographic meteors.

We are, therefore, justified in disregarding term IV in the collision equation for particle distributions near earth. In order to avoid the introduction of special and arbitrary assumptions, the status quo of the meteoroid environment in the near earth space will also be extended into the asteroidal belt under the present model; this means that the lifetime of the asteroidal debris is taken to be their collisional lifetime rather than their lifetime due to radiation damping. Term IV in the collision equation is thereby disregarded altogether.

Using the steady-state assumption and equations 36, 23 and 27, substituted into equation 12, the collision equation can now be expressed as:

$$\frac{\partial f(m,t)}{\partial t} = 0 = Km^{2/3}\,\Gamma\,\frac{\partial f(m)}{\partial m}\int_{\mu}^{m/\Gamma'} Mf(M)dM$$

$$+ (2/3)\, f(m)\Gamma K\, m^{-1/3}\int_{\mu}^{m/\Gamma'} Mf(M)dM$$

$$+ f(m)\,\Gamma(\Gamma')^{-2}\,K\, f(m/\Gamma')\, m^{5/3}$$

$$- K\, f(m)\int_{m/\Gamma'}^{M_\infty} f(M)\left(m^{1/3} + M^{1/3}\right)^2 dM \qquad (44)$$

$$+ K(2-\eta)\, m^{-\eta}\,\Lambda^{\eta-2}\times\left\{\int_{m/\Lambda}^{M_\infty/\Gamma'} dM\int_{M}^{\Gamma' M} dM_2\; Q(M,M_2,m)\right.$$

$$+ \Gamma\int_{m/\Lambda}^{M_\infty/\Gamma'} dM\int_{\Gamma' M}^{M_\infty} dM_2\; R(M,M_2,m)$$

$$\left.+ \int_{M_\infty/\Gamma'}^{M_\infty} dM\int_{M}^{M_\infty} dM_2\; Q(M,M_2,m)\right\}$$

where the time dependence has been dropped from f(m,t) and where

$$Q(M,M_2,m) = (M+M_2)\left(M^{1/3} + M_2^{\,1/3}\right)^2 M^{\eta-2}\, f(M)f(M_2)$$

$$(45)$$

$$R(M,M_2,m) = \left(M^{1/3} + M_2^{\,1/3}\right)^2 M^{\eta-1}\, f(M)f(M_2)$$

Seeking a simple power law solution of the form $m^{-\alpha}$ we take

$$f(m) = A\ m^{-\alpha} \tag{46}$$

$$m << \frac{\Lambda M_\infty}{\Gamma'}$$

and substitute it into equation 12.

Performing the integrations, one obtains,

$$\begin{aligned} 0 = {} & \frac{K\Gamma A^2(2/3-\alpha)}{2-\alpha}\ m^{-\alpha-1/3}\left[(m/\Gamma')^{2-\alpha} - \mu^{2-\alpha}\right] \\ & + K\Gamma(\Gamma')^{\alpha-2}\ A^2\ m^{-2\alpha+5/3} \\ & - \frac{KA^2(\Gamma')^{\alpha-1}}{\alpha-1}\ m^{-2\alpha+5/3} \\ & + KA^2\ C'\ m^{-2\alpha+5/3} \end{aligned} \tag{47}$$

provided that both of the following inequalities are satisfied:

$$\alpha > 5/3 \tag{48}$$

$$\alpha > \frac{1}{2}\ (\eta + 5/3) \tag{49}$$

The quantity C' is a lengthy expression involving α, η, Γ, Γ' and Λ.

If $\alpha < 5/3$, no solution of a population index type exists, as can easily be shown; the dynamic processes of particle creation and catastrophic collision then depend on the largest masses M_∞ present in the distribution and the linearity of eq. 47 in $m^{-2\alpha+5/3}$ is destroyed. A similar thing happens when $\alpha > 2$; the term $\mu^{2-\alpha}$ in the first term of eq. 47 will dominate because $\mu << m/\Gamma'$ and we obtain, for this erosion process

$$\text{const} \times m^{-\alpha-1/3}\ (0 - \mu^{2-\alpha}) \tag{50}$$

Equating each power of m in eq. 47, one obtains

$$-2\alpha+5/3 = -\alpha - 1/3$$

resulting in

$$\alpha = 2$$

yielding the absurd result that if $\alpha > 2$, then $\alpha = 2$, indicating that for $\alpha > 2$ no solution of a population index type exists.

Experimental work by Gault et al (1963) on hypervelocity cratering into basalt indicates that $\eta \gtrsim 1.8$. Some cases with slightly higher η (but still less than 2) have been observed together with lower values for η. For very small particles, the value for η appears to be still lower (Gault et al 1965, Gault and Heitowit, 1963). While these experiments refer to basalt only, the writer is not aware of any other determination of the size distribution for crushed debris during hypervelocity cratering. It is a significant but not necessarily serious mathematical limitation of the present model that if the parameter η is equal to or is larger than two, the form of the crushed particle distribution changes, as can be seen from equation 12 and 13 when $\eta > 2$ is substituted. In that case, the quantity $C(M,\infty)$ becomes, approximately

$$C(M,\infty) = \frac{(\eta-2)M_e}{\mu^{2-\eta}} = \frac{(\eta-2)\Gamma M}{\mu^{2-\eta}} \qquad (51)$$

and the discussion in this paper is no longer applicable.

We now consider, briefly, the time dependent collision equation (eq. 44, with $\partial f/\partial t \neq 0$). When a separable population index type of solution

$$f(m,t) = A(t)\, m^{-\alpha}$$

is substituted into eq. 44, we obtain (with the help of eq. 47)

$$m^{-\alpha} \frac{dA(t)}{dt} = \text{constant} \times A^2(t) \times m^{-2\alpha+5/3} \quad .$$

This equation is satisfied only if $\alpha = 5/3$. It can, however, be shown readily that the particle creation term III (eq. 27), in this case becomes (after iteration with $\alpha = 5/3$ and $\eta > 5/3$).

$$A^2(t) \times (\text{constant} \times m^{-\eta} + \text{terms in } m^{-5/3} \text{ and } m^{-5/3} \times \ln m)$$

and the collision equation is obviously not satisfied. We therefore conclude that no separable population index type of function exists (other than zero) which satisfies the time dependent collision equation employed in the present study.

It is therefore concluded that within the present model, a population index type solution exists if and only if the population has reached a steady-state distribution and that

$$\Gamma'\mu << m$$

$$m << \frac{\Lambda M_\infty}{\Gamma'} \rightarrow \infty \tag{52}$$

$$\frac{1}{2}(\eta+5/3) \approx 1.75 < \alpha < 2$$

in which case eq. 47 reduces to

$$0 = \text{constant} \times m^{-2\alpha+5/3} \tag{53}$$

In this equation, the constant factor is an algebraic as well as transcendental expression in α and the physical parameters. Knowledge of the latter then permits one to calculate the former. Such a study together with a discussion of the stability of the solution is presently under preparation.

An interesting property of the solution in the mass range specified by eq. 52 is that, the particular mass ranges which dominate dynamically the number density of objects in the mass range m to m + dm are:

Term (I): (Erosion) is "dominated" by objects of mass m/Γ'

Term (II): (Catastrophic collision) is "dominated" by objects of mass m/Γ' (53)

Term (III): (Particle creation) is "dominated" by masses of the size m/Λ impacting masses of the size $\Gamma'm$.

In other words, the dynamics of the collisional process described by the present model is to a first approximation independent from the size of the "cutoff" objects μ and M_∞. This is an important conclusion. It indicates that slow depletion of the largest objects (e.g., asteroids) will not invalidate the collisional model in a first order (of approximation) for masses within the range $\Gamma'\eta << m << \frac{\Gamma' M_\infty(t)}{\Lambda}$ where $M_\infty(t)$ designates the mass of the largest objects present as a function of time.

5.0 APPLICATION TO THE NEAR EARTH METEOROID ENVIRONMENT

The results of this paper are compared with experimental information in Figure 1. The figure is a plot of the exponent α as obtained from the present model

$$1.7 < \alpha < 2 \tag{54}$$

versus particle mass.

The small mass limit of the model is

$$m >> \Gamma'\mu \quad . \tag{55}$$

Gault (private communication) observed that a basalt particle is broken up catastrophically by a projectile 10^{-3} times its mass moving at 2km/sec. Since meteoroids move with a relative velocity of about 15 to 20 km/sec and since cometary particles are believed to break up more easily than basalt, we take tentatively

$$\Gamma' \simeq 10^5 \tag{56}$$

The limiting mass with a density of 10^3 kg/m^3 not blown away by radiation pressure is about 10^{-15} kg. We therefore have,

$$m >> 10^{-10} \text{ kg} \tag{57}$$

This means that for masses

$$m \lessgtr 10^{-10} \text{ kg} \tag{58}$$

the present model is not applicable.

It is interesting to note that in precisely this range of masses a definite change in the distribution is observed (Figure 1). The "slope" of the logarithmic distribution changes to smaller absolute values. Near the radiation pressure limit, the slope should move toward a positive value, indicating that the number density of particles increases with increasing mass and at the radiation pressure limit we expect a singularity ($\alpha \to \infty$) reflecting the fact that no particle should exist having a mass equal to the radiation pressure limit or smaller.

For masses

$$m \geq 10^{-9} \text{ kg}$$

the radar results are a lower bound in the range of values for $-\alpha$ obtained in this model. The value of $\alpha = 2.34$ suggested by Hawkins and Upton (1958) is somewhat high, but another analysis (Dohnanyi, 1966 and 1967) of comparable photographic meteor data (McCrosky and Posen, 1961) indicates a slope $\alpha = 2$, in agreement with the present model and irons (Hawkins, 1960) have $\alpha = 1.5$ which is somewhat low.

It is significant that the range of values for α in the present model includes the experimental value of the population index as its upper limit. It is strongly implied that cometary meteoroids undergo the collisional processes discussed in this paper and our model is a first approximation to their distribution.

6.0 APPLICATION TO ASTEROIDAL DEBRIS

This section is a comparison of the present results with the distribution of the known asteroids. While a detailed comparison requires a discussion of the time dependent problem (because the largest asteroids are no longer being created) which is beyond the scope of the present study, it is interesting to note the comparison between the present results and observation.

Figure 2 is a logarithmic plot of the cumulative number of catalogued asteroids as given by Kuiper et al (1958) versus absolute photographic magnitude. An albedo of .1 and a density of 3.5×10^3 Kgm/m^3 have been assumed in associating an asteroidal mass with a given absolute photographic magnitude g. The following relationship can then easily be derived

$$m = 10^{24-.6g} \tag{59}$$

where m is in Kg.

The straight line, in this figure, is the result of a least squares fit to the unnormalized distribution where the 3 largest asteroids have been disregarded. The population index α of the fit is

$$\alpha = 1.80 \pm .04 \tag{60}$$

The dashed line indicates the "true number" of asteroids estimated* by Kuiper et al on the basis of their analysis of selection effects influencing the probability of detection. The reason why the distribution approaches a horizontal line for sufficiently faint asteroids is believed to be due to such an effect.

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*The "true number" of asteroids has been estimated by Kuiper et al over a much larger range of asteroids (to an absolute magnitude of g = 13), but in view of uncertainties, we only consider the estimated distribution for $g \leq 11$.

It can be seen, from the figure, that a straight line is a good fit to the data in the region

$$6 \leq g \leq 11 \tag{61}$$

if we correct the distribution for selection effects in the region of $10 \leq g \leq 11$ as suggested by Kuiper et al. The fit is good, even if data in the region

$$6 \leq g \leq 9.5 \tag{62}$$

are only used. As seen from Fig. 2 the three largest asteroids deviate from the trend exhibited by the 647 others considered. Since for large asteroids no sources of generation are known to exist, the steady-state assumption could hold only for masses considerably smaller than the largest ones present. Thus, the distribution of the known asteroids appears to be consistent with the results of the present model.

The distribution of the lunar craters* yields, on the average, a population index

$$1.6 < \alpha < 1.7 \tag{63}$$

which is close to but a trifle lower than the result

$$1.75 < \alpha < 2 \tag{64}$$

obtained in this study.

Hawkins (1960) has studied the distribution of meteorites and obtained

$$\alpha = 2 \text{ and } \alpha = 1.5 \tag{65}$$

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*See, for example, Fielder (1963), Dodd et al (1963), Brinkman (1966) Hartmann (1964) and Baldwin (1964).

depending on the kind of meteorite (i.e., stones or irons, respectively). Stones are therefore seen to have a population index in agreement with the present model and irons have a somewhat lower population index.

7.0 CONCLUSION

A collisional model of interplanetary debris is formulated and solutions of a population index type are sought, i.e., a solution for the number density of the time independent form

$$f(m)dm = m^{-\alpha}\, dm \tag{66}$$

is substituted into the collision equation (eq. 12).

It is found that eq. 66 indeed solves the collision equation in the mass range of

$$\Gamma'\mu \ll m \ll \Lambda M_\infty/\Gamma' \tag{67}$$

provided that the population has reached a steady-state distribution. The quantity μ is the largest object completely shattered by a particle of mass Γ and $\Lambda M_\infty/\Gamma'$ is the largest fragment produced when an object having a mass M_∞ is "catastrophically" shattered.

The population index α must satisfy

$$\frac{1}{2}\,(\eta+5/3) = 1.75 < \alpha < 2 \quad ; \tag{68}$$

beyond the range of α, the collision equation (eq. 44) has no population index type solutions. For $\alpha > 2$ erosion by the smallest particles dominates and depletes the mass distribution with time. For $\alpha < \frac{1}{2}\,(\eta+5/3)$ collision products from the most massive objects cause evolution of the mass distribution with time. It is furthermore shown that, if the population is evolving rapidly so that $\partial f(m,t)/\partial t \neq 0$ in eq. 44, no separable population index type of solution exists, that would satisfy the time dependent collision equation.

Application of the present results to cometary meteoroids yields satisfactory agreement in both α and the trend of the distribution near the small mass end of cometary meteoroids. This is indicated in Figure 1. We note that the population index for these small particles obtained experimentally is in the neighborhood of

$$\alpha = 2 \tag{69}$$

i.e., at the upper limit of α as obtained in this study.

A least squares fit to the distribution of asteroids catalogued by Kuiper et al (1958) yields (Fig. 2)

$$\alpha = 1.8 \pm .04 \tag{70}$$

which is within the range of values for α permitted by this model.

Lunar craters, having a distribution with a population index in the approximate range

$$1.6 < \alpha < 1.7$$

are a trifle below the lower limit of α obtained in this study.

The result of this model are in reasonable agreement with the distribution of stony meteorites as obtained by Hawkins (α = 2) and the iron meteorites (α = 1.5) have a rather low population index.

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GREEK CHARACTERS

α = population index.

Γ' = threshold mass of finite target completely shattered, per unit projectile mass.

$$M_2 = \Gamma' M$$

where

$$M_2$$

is the target mass and M is the projectile mass.

Γ = total ejected mass from a semi-infinite target per unit mass of the incident particle.

$$M_e = \Gamma M \quad \text{(eq. 7)}$$

where M is the mass of the incident particle.

Λ = mass of largest fragment per unit mass of incident particle.

$$M_b = \Lambda M \quad \text{(eq. 7)}$$

η = population index of ejected particles, (see eq. 2).

μ = mass of smallest object in the solar system not blown away by radiation pressure.

ρ = material density of the particles.

τ = particle lifetime.

LIST OF SYMBOLS

a = particle cross-sectional area (m^2).

b = coefficient, defined by eq. 41.

$C(M,M_2)$ = normalization coefficient of the comminution law.

$f(m,t)dm$ = particle number density function, i.e., number of particles in the interval m to m + dm per unit volume of real space at a time t.

g = absolute photographic magnitude.

$g(m;M,M_2)dm$ = comminution law for ejecta (eq. 2).

k = parameter, equal to $\overline{V}(3\pi^{1/2}/4\rho)^{2/3}$.

m,M = particle mass (in Kg).

M_b = mass of the largest ejected fragment.

M_e = total ejected mass during cratering by impact.

M_∞ = mass of largest object in the distribution.

r = particle radius.

R = distance from the sun in AU.

t = time.

$\overline{V}$ = average relative velocity of colliding particles.

APPENDIX A

The purpose here is to derive an expression for the partial derivative with respect to time of the particle number density function due to the process of erosion (i.e., term I in the collision equation). More specifically, given that:

$$f(m,t)dm \tag{A-1}$$

is the number of particles having a mass in the range m to m + dm at a time t, and that the masses change with time according to the relation

$$\frac{dm}{dt} = \xi(m,t) \tag{A-2}$$

The problem is to find

$$\lim_{dt \to 0} \frac{f(m,t+dt) - f(m,t)}{dt} = \frac{\partial f(m,t)}{\partial t} \tag{A-3}$$

due to the process defined by eq. A-2.

In order to obtain an expression for eq. A-3 we consider the following equalities:

$f(m,t)\Delta m$ = number density of particles per unit volume in the mass range m to m + Δm (A-4)

$f(m,t)\xi(m,t)dt$ = number of particles (per unit volume) which have left the mass range m to m + Δm during a time dt (A-5)

$-f(m+\Delta m,t)\xi(m+\Delta m,t)dt$ = number of particles (per unit volume) which have "entered" the mass range m to m + Δm during dt provided that $\Delta m > \xi(m+\Delta m,t)dt = dm$. Since, however, Δm is arbitrary, we choose Δm such that it satisfies this condition. (A-6)

Appendix A (cont'd)

Using eq. A-3, 5 and 6, we have

$$\frac{\partial f(m,t)}{\partial t} \Delta m = \lim_{dt \to 0} \frac{-f(m+\Delta m,t)\xi(m+\Delta m,t)dt + f(m,t)\xi(m,t)dt}{dt} \qquad (A-7)$$

Since Δm can be taken to be small, we can write

$$f(m+\Delta m,t) = f(m,t) + \frac{\partial f(m,t)}{\partial m} \Delta m \qquad (A-8)$$

and

$$\xi(m+\Delta m) = \xi(m,t) + \frac{\partial \xi(m,t)}{\partial m} \Delta m \qquad (A-9)$$

Substituting eq's A-8 and 9 into A-7 one has, after expanding the product and dropping the term in $\overline{\Delta m}^2$

$$\frac{\partial f(m,t)}{\partial t} = \frac{-\partial}{\partial m}\left[f(m,t)\xi(m,t)\right] = \frac{-\partial}{\partial m}\left[f(m,t)\ \frac{dm}{dt}\right] \quad (A-10)$$

which is the desired result.

APPENDIX B

This appendix is a discussion of some of the properties of the integrals occurring in term III (eq. 27) of the collision equation.

First we derive the limits of integration of the expression eq. 24:

$$Km^{-n} \int dM \int dM_2 \; P(M,M_2,m) \tag{B-1}$$

where

$$\begin{aligned} P(M,M_2,m) &= Q(M,M_2,m) \quad \text{when } M > M_2/\Gamma' \\ &= R(M,M_2,m) \quad \text{when } M < M_2/\Gamma' \end{aligned} \tag{B-2}$$

with Q and R given by eq. 46 in the text.

We first note that we want to integrate over all M, and M_2 for

$$M \leq M_2 \tag{B-3}$$

such that particles of mass m are produced.

This means that

$$M \leq M_2 \leq M_\infty \tag{B-4}$$

and

$$\frac{m}{\Lambda} \leq M \leq M_\infty \tag{B-5}$$

Appendix B (cont'd)

Since, however, the value of the kernel under the integral eq. B-1 has the dependence indicated in eq. B-2, some additional restrictions exist. The situation is indicated in Fig. B-1, which is a plot of the regions of integration in M and M_2. Dashed lines through the origin are the loci of points where

$$M_2 = \Gamma' M \qquad \text{(b-6)}$$

and

$$M_2 = M \qquad \text{(B-7)}$$

These lines separate the M_2 x M space into regions (with the bounds on M and M_2 indicated) with the following significance:

$$\begin{aligned} &\text{Region 1,} \quad M_2 < \Gamma' M \quad P = Q \\ &\text{Region 2,} \quad M_2 > \Gamma' M \quad P = R \qquad \text{(B-8)} \\ &\text{Region 3,} \quad M_2 < \Gamma' M \quad P = Q \end{aligned}$$

whence, integrating over the various regions, the integral eq. B-1 becomes

$$K\,m^{-\eta}\int dM \int dM_2\, P = K\,m^{-\eta}\left\{\int_{m/\Lambda}^{M_\infty/\Gamma'} dM \int_{M}^{\Gamma' M} dM_2\, Q \right. \qquad \text{(B-9)}$$

$$\left. + \,\Gamma \int_{m/\Lambda}^{M_\infty/\Gamma'} dM \int_{\Gamma' M}^{M_\infty} dM_2\, R + \int_{M_\infty/\Gamma'}^{M_\infty} dM \int_{M}^{M_\infty} dM_2\, Q \right\}$$

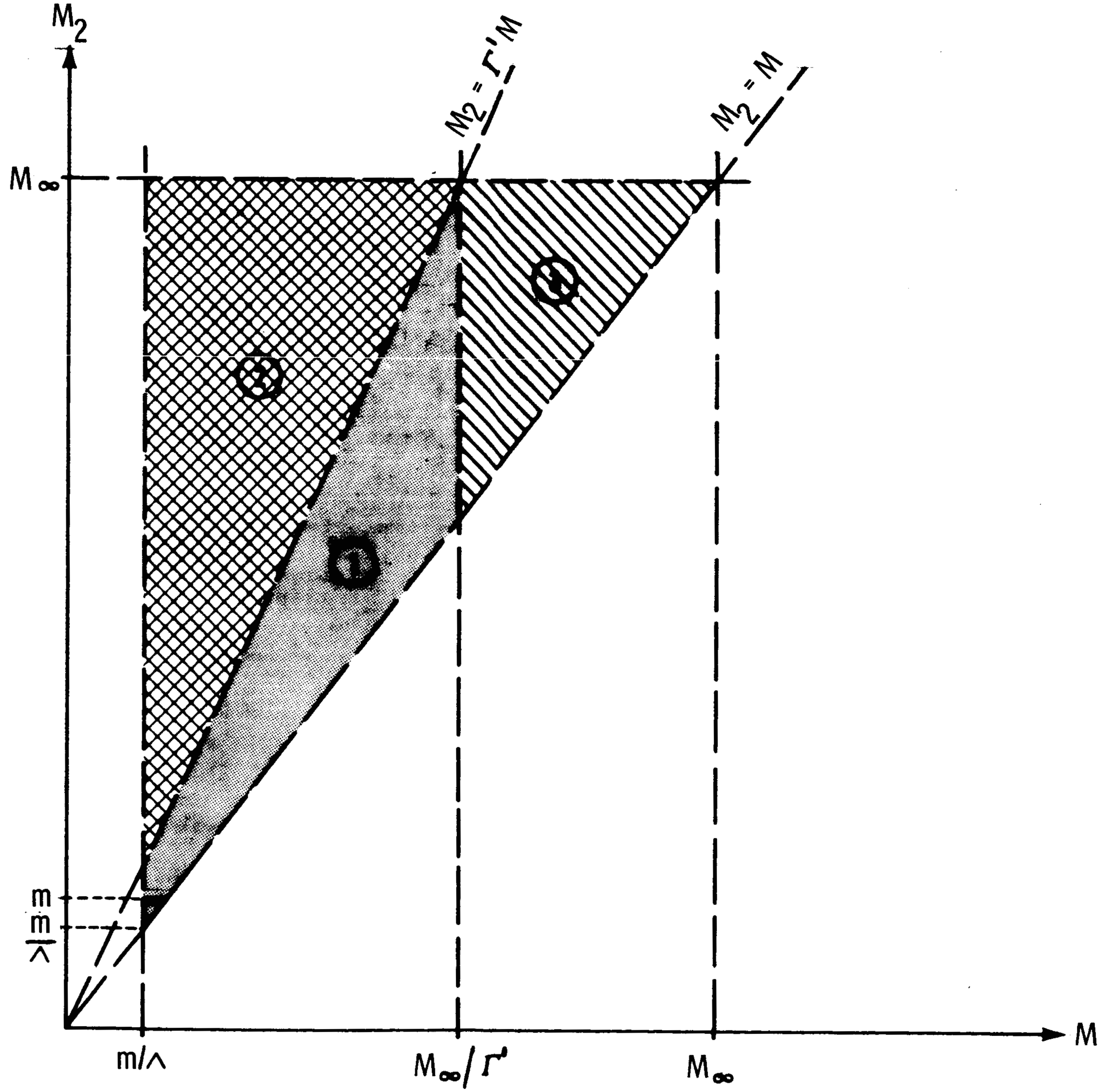
provided that

$$m/\Lambda < M_\infty/\Gamma' \qquad \text{(B-10)}$$

as can be seen from the figure.

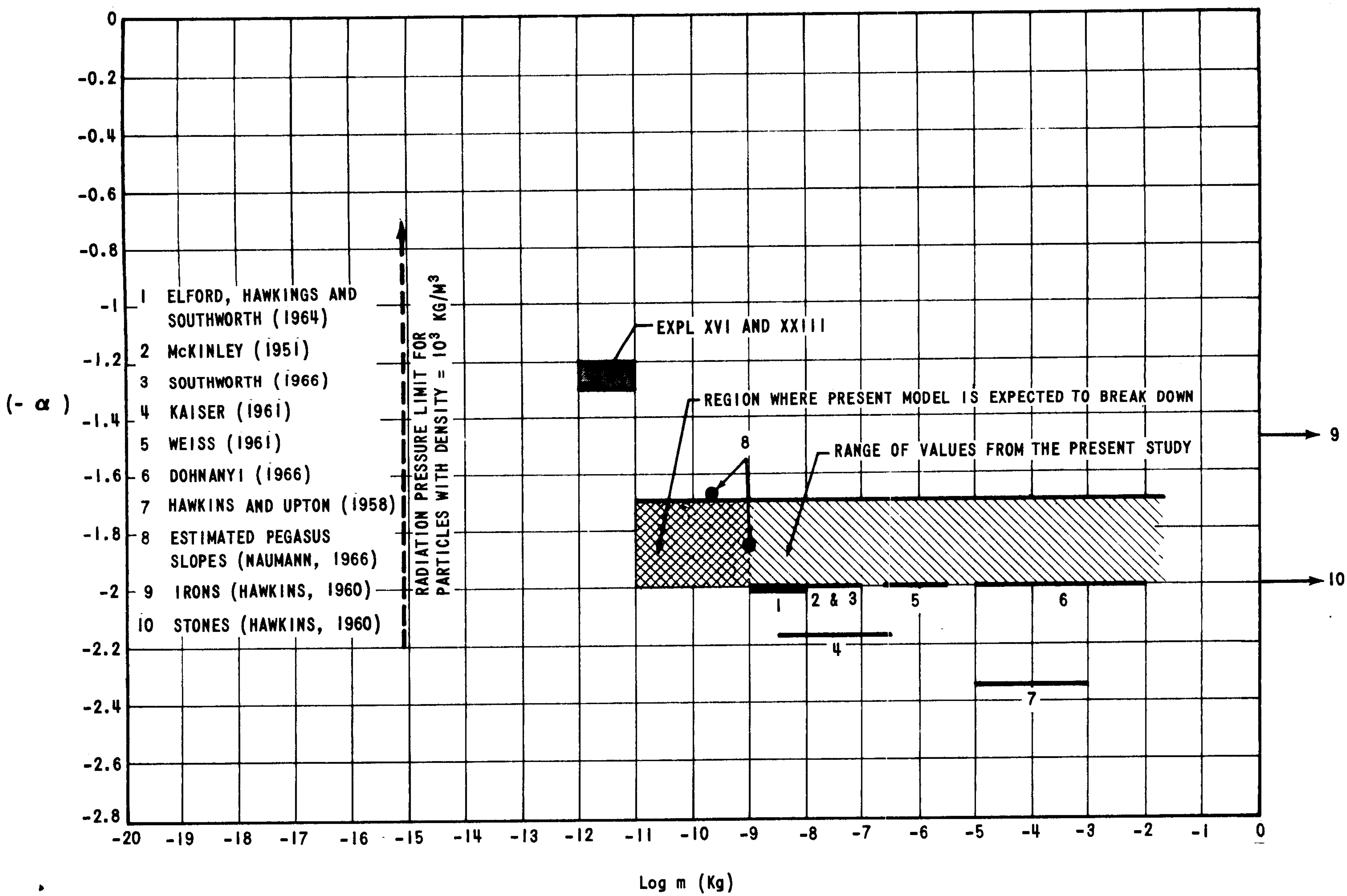
Appendix B (cont'd)

Eq. B-9 overestimates the mass production process at the lower limit in region 1 inasmuch we have included processes where two objects, both having a mass of m/Λ, will create fragments of the size m which is an absurd result. A more detailed treatment would consider the precise expression for the mass of the largest fragment when two objects of similar mass collide.



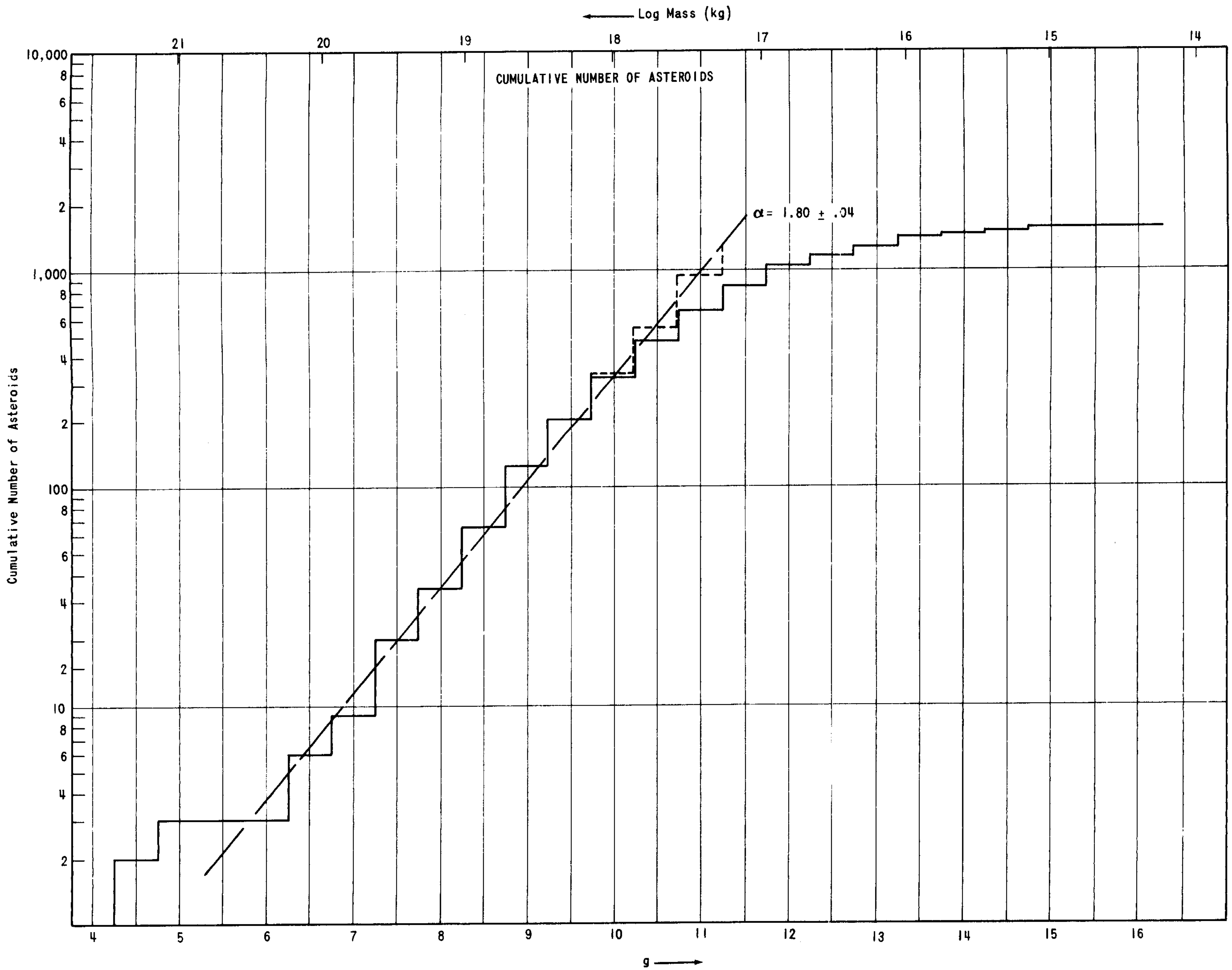
M_2
$M_2 = \Gamma' M$
$M_2 = M$
M_∞
m
$\frac{m}{\Lambda}$
m/Λ
M_∞/Γ'
M_∞
M

FIGURE B-1



1 ELFORD, HAWKINGS AND SOUTHWORTH (1964)
2 McKINLEY (1951)
3 SOUTHWORTH (1966)
4 KAISER (1961)
5 WEISS (1961)
6 DOHNANYI (1966)
7 HAWKINS AND UPTON (1958)
8 ESTIMATED PEGASUS SLOPES (NAUMANN, 1966)
9 IRONS (HAWKINS, 1960)
10 STONES (HAWKINS, 1960)
RADIATION PRESSURE LIMIT FOR PARTICLES WITH DENSITY = 10^3 KG/M^3
EXPL XVI AND XXIII
REGION WHERE PRESENT MODEL IS EXPECTED TO BREAK DOWN
RANGE OF VALUES FROM THE PRESENT STUDY
(- α)
Log m (Kg)

FIGURE 1



Log Mass (kg)
21
20
19
18
17
16
15
14
CUMULATIVE NUMBER OF ASTEROIDS
$\alpha = 1.80 \pm .04$
Cumulative Number of Asteroids
10,000
8
6
4
2
1,000
8
6
4
2
100
8
6
4
2
10
8
6
4
2
4
5
6
7
8
9
10
11
12
13
14
15
16
g

FIGURE 2

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