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SUMMARY

The input admittance of a rectangular waveguide opening onto a perfectly conducting ground plane and covered with an inhomogeneous dielectric slab is formulated as a boundary-value problem. As a specific application of the solution, a plasma slab is considered. In the calculations, the plasma is assumed to be homogeneous except in the vicinity of the ground plane, where a local boundary layer of electron density occurs. Two ranges of collision frequencies are selected, and calculations are performed for the input admittance as a function of peak electron density and boundary-layer thickness. The numerical results show that a small boundary-layer reduction of electron density near the ground plane can significantly influence the input admittance of the antenna and causes a substantial lowering of the voltage reflection coefficient when the plasma is overdense.

INTRODUCTION

The input admittance of a homogeneous plasma-covered rectangular waveguide-fed aperture antenna was originally investigated by Galejs (ref. 1) and Villeneuve (ref. 2) and then by Compton (ref. 3). Although calculations have been available for a number of years, only recently have attempts been made to confirm experimentally these theoretical results. Using a Langmuir probe to infer electron density, Taylor (ref. 4) placed a rectangular waveguide-fed aperture antenna into a shock tube and measured the voltage reflection coefficient as a function of electron density. Best preliminary data for overdense plasmas resulted in a measured reflection coefficient which was significantly lower than that predicted by homogeneous plasma calculations. Taylor concluded that the inconsistency was due to a local boundary-layer reduction of electron density at the ground plane. This conjecture was supported by the results of Meltz, Freyheit, and Lustig (ref. 5). They had previously observed a discrepancy between theory and experiment and attributed it to the presence of a thin glass wall between the cylindrical gap antenna and the plasma.

These experimental observations suggested that some provision must be made in the theory to account for variations of the dielectric constant near the antenna. This is done herein by considering the plasma as an inhomogeneous dielectric slab. The derivation of the admittance expression closely follows the method given in reference 3. The fields are expressed in terms of their Fourier transforms, and the boundary-value problem is solved by assuming that the aperture field is the dominant waveguide mode. Parseval's theorem is then used to obtain an admittance expression, consisting of a double integration over mode space, which must be numerically evaluated. The present formulation differs from that given in reference 3 in two respects. First, the fields are developed from single-component electric and magnetic vector potentials (instead of a two-component magnetic potential) and, second, the Helmholtz equation is numerically solved by utilizing the plane wave program of Swift and Evans (ref. 6). The numerical solution is required since the dielectric constant of the slab is assumed to be a complicated continuous function of one of the coordinates.

Calculations are performed for the input admittance as a function of peak electron density and boundary-layer thickness for moderate and low values of the collision frequency. It is concluded that a thin boundary layer over the antenna can have a profound effect on the admittance and, indeed, reduces the reflection coefficient of the rectangular aperture covered with an overdense plasma.

The special case of a homogeneous slab over the aperture is considered in the appendix. The excitation of surface waves in a nonlossy slab is discussed, and the admittance integrals are tabulated into forms suitable for numerical evaluation.

SYMBOLS

$\vec{A}$	electric vector potential
$\overrightarrow{A^*}$	magnetic vector potential
a	short dimension of rectangular waveguide
b	long dimension of rectangular waveguide
b_{in}	input susceptance (normalized with respect to Y_{01})
c	speed of light in free space
$\vec{E}$	electric field intensity

f frequency

$f(\beta,z), g(\beta,z)$ normalized Fourier transforms of vector potentials

g_{in} input conductance $\left(\text{normalized with respect to } Y_{01}\right)$

g_s surface-wave conductance $\left(\text{normalized with respect to } Y_{01}\right)$

$\vec{H}$ magnetic field intensity

$j = \sqrt{-1}$

k_0 wave number in free space, $\omega\sqrt{\epsilon_0\mu_0}$

k_x, k_y, k_z Cartesian components of wave number

N complex index of refraction

N_e electron density

$N_{e,o}$ peak value of electron density

P power

V_o aperture voltage

x, y, z Cartesian coordinates

Y_{in} input admittance

Y_0 characteristic admittance of free space, $\sqrt{\epsilon_0/\mu_0}$

Y_{01} characteristic admittance of the TE_{01} mode of the rectangular waveguide,

$$Y_0\sqrt{1 - \left(\frac{\pi}{k_0 b}\right)^2}$$

y^{TE} transverse electric component of admittance $\left(\text{normalized with respect to } Y_{01}\right)$

y^{TM} transverse magnetic component of admittance (normalized with respect to Y_{01})

y_{in} input admittance (normalized with respect to Y_{01})

z_o thickness of dielectric slab

α_p attenuation constant of the plasma,

$$\frac{k_0}{\sqrt{2}}\left\{-1+\frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2}+\sqrt{\left[1-\frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2}\right]^2+\left[\frac{\nu}{\omega}\,\frac{\left(\frac{\omega_p}{\omega}\right)^2}{1+\left(\frac{\nu}{\omega}\right)^2}\right]^2}\right\}^{1/2}$$

α,β polar components of k_x/k_0 and k_y/k_0

β_n surface wave pole

Γ voltage reflection coefficient

ϵ permittivity

λ_0 wavelength in free space, c/f

μ_0 permeability of free space

ν collision frequency

ϕ electric scalar potential

ϕ^* magnetic scalar potential

ω frequency of propagation

ω_p plasma frequency, $2\pi \times 8.97 \times 10^3\sqrt{N_e}$

$\omega_{p,o}$ peak plasma frequency

Subscripts:

I	in dielectric slab
0	in free space
i	imaginary part
r	real part
x,y,z	vector component in direction indicated by subscript

Superscripts:

I	vector component in dielectric slab
II	vector component in free space
TE	transverse electric
TM	transverse magnetic

A double bar over a symbol indicates the double Fourier transform. A prime denotes the derivative with respect to the z-coordinate. An asterisk denotes complex conjugate except when used with magnetic potentials.

THEORY

The geometry of the problem is shown in figure 1. A rectangular waveguide, excited in the dominant TE_{01} mode, opens onto a perfectly conducting ground plane which is covered with an inhomogeneous dielectric slab. The dielectric constant of the slab is assumed to vary continuously in the z-direction for $0 \leqq z \leqq z_o$ (region I), and air is assumed to fill the remaining half space for $z \geqq z_o$ (region II).

It can be shown that the field components in both regions may be constructed from a set of vector potentials $A_z^{I,II}$ and $\left(A_z^*/\epsilon\right)^{I,II}$, such that

$$\left.\begin{aligned}
H_x^{I,II} &= \frac{1}{\mu_0}\frac{\partial A_z^{I,II}}{\partial y} - \frac{1}{j\omega\mu_0}\frac{\partial^2}{\partial x\,\partial z}\left(\frac{A_z^*}{\epsilon}\right)^{I,II} \\
H_y^{I,II} &= -\frac{1}{\mu_0}\frac{\partial A_z^{I,II}}{\partial x} - \frac{1}{j\omega\mu_0}\frac{\partial^2}{\partial y\,\partial z}\left(\frac{A_z^*}{\epsilon}\right)^{I,II} \\
H_z^{I,II} &= \frac{1}{j\omega\mu_0}\left[\frac{\partial^2}{\partial x^2}\left(\frac{A_z^*}{\epsilon}\right)^{I,II} + \frac{\partial^2}{\partial y^2}\left(\frac{A_z^*}{\epsilon}\right)^{I,II}\right] \\
E_x^{I,II} &= -\frac{\partial}{\partial y}\left(\frac{A_z^*}{\epsilon}\right)^{I,II} - \frac{1}{j\omega\mu_0\epsilon_{I,0}}\frac{\partial^2 A_z^{I,II}}{\partial x\,\partial z} \\
E_y^{I,II} &= \frac{\partial}{\partial x}\left(\frac{A_z^*}{\epsilon}\right)^{I,II} - \frac{1}{j\omega\mu_0\epsilon_{I,0}}\frac{\partial^2 A_z^{I,II}}{\partial y\,\partial z} \\
E_z^{I,II} &= \frac{1}{j\omega\mu_0\epsilon_{I,0}}\left[\frac{\partial^2 A_z^{I,II}}{\partial x^2} + \frac{\partial^2 A_z^{I,II}}{\partial y^2}\right]
\end{aligned}\right\} \tag{1}$$

where a time harmonic variation of the form $e^{-j\omega t}$ is implicit and the vector potentials in region I satisfy the partial differential equations

$$\left.\begin{aligned}
&\nabla^2 A_z^{I} - \frac{1}{\epsilon_I}\frac{d\epsilon_I}{dz}\frac{d}{dz}A_z^{I} + k_0^2\frac{\epsilon_I}{\epsilon_0}A_z^{I} = 0 \\
&\nabla^2\left(\frac{A_z^*}{\epsilon}\right)^{I} + k_0^2\frac{\epsilon_I(z)}{\epsilon_0}\left(\frac{A_z^*}{\epsilon}\right)^{I} = 0
\end{aligned}\right\} \tag{2}$$

The vector potentials A_z and A_z^* are those defined by Stratton (ref. 7). It should be noted that the scalar potentials ϕ and ϕ^* are eliminated by the Lorentz conditions $j\omega\mu_0\epsilon\phi = \frac{\partial A_z}{\partial z}$ and $j\omega\mu_0\phi^* = \frac{\partial}{\partial z}\left(\frac{A_z^*}{\epsilon}\right)$. The structure is unbounded in the x-y plane; hence a solution to the potentials is given by (see, for example, ref. 8, p. 145)

$$\left.\begin{aligned} A_z^{I,II} &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} G^{I,II}\left(k_x, k_y, z\right) e^{j\left(k_x x + k_y y\right)} dk_x \, dk_y \\ \left(\frac{A_z^*}{\epsilon}\right)^{I,II} &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} F^{I,II}\left(k_x, k_y, z\right) e^{j\left(k_x x + k_y y\right)} dk_x \, dk_y \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} G^{II}\left(k_x, k_y, z\right) &= A\left(k_x, k_y\right) e^{jk_z{}^{II} z} \\ F^{II}\left(k_x, k_y, z\right) &= B\left(k_x, k_y\right) e^{jk_z{}^{II} z} \end{aligned}\right\} \tag{4}$$

The roots of $k_z{}^{II}$ chosen in accordance with the radiation condition at infinity are

$$\left.\begin{aligned} k_z{}^{II} &= \sqrt{k_0{}^2 - k_x{}^2 - k_y{}^2} \qquad \left(k_0{}^2 > k_x{}^2 + k_y{}^2\right) \\ k_z{}^{II} &= j\sqrt{k_x{}^2 + k_y{}^2 - k_0{}^2} \qquad \left(k_0{}^2 < k_x{}^2 + k_y{}^2\right) \end{aligned}\right\} \tag{5}$$

If equations (3) are substituted into equations (1) and the boundary conditions are applied at $z = z_o$ (i.e., continuity of the transforms of H_x, H_y, E_x, and E_y), the ratios $\frac{F^I(z_o)}{F^{II}(z_o)}$, $\frac{G^I(z_o)}{G^{II}(z_o)}$, and their derivatives can then be determined. By defining $f(z) = \frac{F^I(z)}{F^{II}(z_o)}$ and $g(z) = \frac{G^I(z)}{G^{II}(z_o)}$, the resultant boundary conditions are

$$\left.\begin{aligned} f(z_o) &= \frac{F^I(z_o)}{F^{II}(z_o)} = 1 \qquad f'(z_o) = \frac{d}{dz}\left[\frac{F^I(z)}{F^{II}(z_o)}\right]_{z=z_o} = jk_z{}^{II} \\ g(z_o) &= \frac{G^I(z_o)}{G^{II}(z_o)} = 1 \qquad g'(z_o) = \frac{d}{dz}\left[\frac{G^I(z)}{G^{II}(z_o)}\right]_{z=z_o} = jk_z{}^{II} \frac{\epsilon_I(z_o)}{\epsilon_0} \end{aligned}\right\} \tag{6}$$

where the roots of $k_z{}^{II}$ are given by equations (5). Equations (3) also reduce equations (2) to

$$\frac{d^2f}{dz^2} + \left[k_0^2 \frac{\epsilon_I(z)}{\epsilon_0} - k_x^2 - k_y^2\right] f = 0 \tag{7a}$$

$$\frac{d^2g}{dz^2} - \frac{1}{\epsilon_I(z)} \frac{d\epsilon_I(z)}{dz} \frac{dg}{dz} + \left[k_0^2 \frac{\epsilon_I(z)}{\epsilon_0} - k_x^2 - k_y^2\right] g = 0 \tag{7b}$$

where $f(z) = \dfrac{F^I(z)}{F^{II}(z_o)}$ and $g(z) = \dfrac{G^I(z)}{G^{II}(z_o)}$.

Since equations (6) and (7) are identical in form to the plane-wave expressions given by Swift and Evans in reference 6, the procedure given therein may be used to determine $g(z)$, $f(z)$, and their derivatives at any point in the slab as a function of β, where $k_0{}^2\beta^2 = k_x{}^2 + k_y{}^2$. The fields in the inhomogeneous slab are therefore functions only of the mode constants $G^{II}(z_o)$ and $F^{II}(z_o)$ which can be solved by applying the boundary conditions at $z = 0$.

If it is assumed that the total aperture field is the same as the incident TE_{01} waveguide mode, then the transform pair at the aperture is

$$\left.\begin{aligned} E_x &= \frac{V_o}{a} \cos \frac{\pi y}{b} && (E_y = 0) \\ \overline{\overline{E}}_x &= 2\pi b V_o \frac{\sin\left(\frac{k_x a}{2}\right) \cos \frac{k_y b}{2}}{\left(\frac{k_x a}{2}\right)\left[\pi^2 - (k_y b)\right]^2} && (\overline{\overline{E}}_y = 0) \end{aligned}\right\} \tag{8}$$

From equations (1) and (8), the boundary conditions at $z = 0$ therefore result in the following relationships:

$$\left.\begin{aligned} \overline{\overline{E}}_y &= 0 = j\left\{F^{II}(z_o) k_x f(0) - \frac{k_y G^{II}(z_o)}{j\omega\mu_0\epsilon_I(0)} g'(0)\right\} \\ \overline{\overline{E}}_x &= -j\left\{F^{II}(z_o) k_y f(0) + \frac{k_x G^{II}(z_o) g'(0)}{j\omega\mu_0\epsilon_I(0)}\right\} = 2\pi b V_0 \frac{\sin\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)} \frac{\cos \frac{k_y b}{2}}{\left[\pi^2 - (k_y b)^2\right]} \end{aligned}\right\} \tag{9}$$

which allows $F^{II}(z_o)$ and $G^{II}(z_o)$ to be expressed in terms of V_o, as follows:

$$F^{II}(z_o) = \frac{jk_y}{\left(k_x{}^2 + k_y{}^2\right)} 2\pi b V_o \frac{\sin\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)} \frac{\cos\left(\frac{k_y b}{2}\right)}{\left[\pi^2 - (k_y b)^2\right]} \frac{1}{f(0)} \tag{10a}$$

$$G^{II}(z_o) = -\frac{\omega\mu_0\epsilon_I(0)}{\left(k_x^2 + k_y^2\right)} 2\pi b V_o \frac{\sin\left(\frac{k_x a}{2}\right)}{\left(\frac{k_x a}{2}\right)} \frac{\cos\left(\frac{k_y b}{2}\right)}{\left[\pi^2 - \left(k_y b\right)^2\right]} \frac{k_x}{g'(0)} \tag{10b}$$

Therefore, the transforms of all the field components at the aperture plane are known to within a constant V_o, which is implicit in the transform of E_x. By substituting equations (10) into equations (1), the transform of H_y at $z = 0$ may be evaluated in terms of $\bar{\bar{E}}_x$. The result is

$$\bar{\bar{H}}_y^{\,I} = j\frac{Y_0\bar{\bar{E}}_x k_0^{\,2}}{\left(k_x^{\,2} + k_y^{\,2}\right)}\left\{\left(\frac{k_x}{k_0}\right)^2 \frac{\epsilon_I(0)}{\epsilon_0}\left[\frac{k_0 g(0)}{g'(0)}\right] - \left(\frac{k_y}{k_0}\right)^2\left[\frac{f'(0)}{k_0 f(0)}\right]\right\} \tag{11}$$

The complex conjugate of P at the aperture in terms of the field quantities in region I is given by

$$P^* = \frac{1}{2}\iint_{\text{aperture}} E_x^* H_y^I dx\, dy = \frac{1}{2}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \bar{\bar{E}}_x^{\,*}\bar{\bar{H}}_y^{\,I} dk_x\, dk_y \tag{12}$$

where the dual integral representation is a statement of Parseval's theorem (ref. 8, p. 458).

The complex conjugate of P at the aperture in terms of the fields inside the waveguide is given by

$$P^* = \frac{1}{2} Y_{01} \frac{\left|V_o\right|^2}{a^2} \frac{(1 - \Gamma)}{(1 + \Gamma)} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \cos^2 \frac{\pi y}{b}\, dx\, dy = \frac{1}{2} Y_{01}\left|V_o\right|^2 \frac{(1 - \Gamma)}{(1 + \Gamma)} \frac{b}{2a} \tag{13}$$

From continuity of power across the aperture, equations (11) and (12) give

$$\frac{Y_{in}}{Y_{01}} = y_{in} = g_{in} - jb_{in} = \frac{(1 - \Gamma)}{(1 + \Gamma)} = \frac{2a}{b}\frac{1}{Y_{01}}\left[\frac{1}{\left|V_o\right|^2}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \bar{\bar{E}}_x^{\,*}\bar{\bar{H}}_y^{\,I}\, dk_x\, dk_y\right] \tag{14}$$

The substitution of equation (11) and the explicit form of $\bar{\bar{E}}_x$, as given by equations (8), into equation (14), leads to the following expression for the input admittance:

$$
\left.
\begin{aligned}
y_{in} &= y^{TE} + y^{TM} \\
y^{TM} &= j\,\frac{2\left(k_0 a\right)\left(k_0 b\right)}{Y_{01}/Y_0}\int_0^{\infty}\int_0^{2\pi}\left[\frac{\cos\left(\frac{\beta k_0 b}{2}\sin\alpha\right)}{\pi^2 - \left(k_0 b\beta\sin\alpha\right)^2}\right]^2\left[\frac{\sin\left(\frac{k_0 a\beta}{2}\cos\alpha\right)}{\left(\frac{k_0 a\beta\cos\alpha}{2}\right)}\right]^2\left[\frac{k_0 g(\beta,0)}{g'(\beta,0)}\right]\frac{\epsilon_I(0)}{\epsilon_0}\cos^2\alpha\,\beta\,d\beta\,d\alpha \\
y^{TE} &= -j\,\frac{2\left(k_0 a\right)\left(k_0 b\right)}{Y_{01}/Y_0}\int_0^{\infty}\int_0^{2\pi}\left[\frac{\cos\left(\frac{\beta k_0 b}{2}\sin\alpha\right)}{\pi^2 - \left(k_0 b\beta\sin\alpha\right)^2}\right]^2\left[\frac{\sin\left(\frac{k_0 a\beta}{2}\cos\alpha\right)}{\left(\frac{k_0 a\beta\cos\alpha}{2}\right)}\right]^2\left[\frac{f'(\beta,0)}{k_0 f(\beta,0)}\right]\sin^2\alpha\,\beta\,d\beta\,d\alpha
\end{aligned}
\right\}
\qquad (15)
$$

where a transformation to polar coordinates has been made such that $k_x = k_0\beta\cos\alpha$ and $k_y = k_0\beta\sin\alpha$. The values of $g(\beta,0)$, $g'(\beta,0)$, $f(\beta,0)$, and $f'(\beta,0)$ are provided by the numerical integration of equations (7), subject to the initial conditions (6). The admittance has been broken up into TM and TE components to denote that y^{TM} is derived from the potential A_z and that y^{TE} is derived from the potential A_z^*/ϵ. If surface waves are excited, it is convenient to study y^{TM} and y^{TE} separately. (See the appendix.)

If the losses within the slab are moderate, the integrands of equations (15) are well behaved functions and the numerical evaluation of the input admittance is straightforward. However, if the losses are small, it is possible that the integrands may become large (or infinite if there is no loss) at discrete values of β. In the limit of zero loss, these points become poles of order one, and their residues become a measure of the power which is confined within the slab as surface waves. The discussion of this surface-wave component is deferred to the appendix.

NUMERICAL PARAMETERS

The differential equations (7) were numerically solved as a function of β by using the Runge-Kutta method. These numbers were stored on tape and were then used as input information for the integration of equations (15). Approximately 250 values of β were selected in the interval $0 \leqq \beta \leqq 6$. From the analysis of a similar problem (ref. 9), it was believed that the termination of the integration at $\beta = 6$ would yield a susceptance error of about 1 percent or less. To test this supposition, a numerical check against the methods described in reference 3 was initiated, and the results are discussed later in this section. Including the time needed to solve the differential equations (7), each admittance point required approximately 45 minutes of machine time.

The plasma-covered antenna was of immediate interest; therefore, the dielectric constant of the inhomogeneous slab was assumed to be given by

$$\frac{\epsilon_I(z)}{\epsilon_0} = 1 - \frac{\left[\frac{\omega_p(z)}{\omega}\right]^2}{1 + \left[\frac{\nu(z)}{\omega}\right]^2} + j\,\frac{\left[\frac{\nu(z)}{\omega}\right]\left[\frac{\omega_p(z)}{\omega}\right]^2}{1 + \left[\frac{\nu(z)}{\omega}\right]^2} \tag{16}$$

which is based upon a point application of the Lorentz model (ref. 7, pp. 325-327) of wave interaction with electrons in the plasma. With the dielectric constant defined by equation (16), the admittance of an aperture covered with a homogeneous plasma was computed as a function of $\left(\omega_p/\omega\right)^2 \geqq 0.7$ with $\nu/\omega = 0.4$ and $z_0 = 3.5$ cm. On the basis of an empirical criterion[1] of $e^{-2\alpha_p z_0} > 0.1$, where α_p is the attenuation constant, it was believed that the admittance would be independent of thickness over the computational range. The criterion was, indeed, satisfied because the slab computations were within $1\frac{1}{2}$ percent of the corresponding half-space calculations performed by R. C. Rudduck.[2] This agreement was considered to be a satisfactory check of the slab program.

In order to observe the influence of an inhomogeneous plasma slab on the input admittance of the antenna, the series of electron density profiles shown in figure 2 were considered. Such profiles are typically produced whenever a plasma flows tangentially over a flat plate, as illustrated in the inset of figure 2. The variation of the electron density near the ground plane is due to viscous boundary-layer effects.

With the aperture dimensions fixed at 0.4 × 0.9 inch (inside dimensions of a standard RG 52/U waveguide) and a fixed propagating frequency of 10.0 GHz, admittance calculations were performed as a function of $\left(\frac{\omega_{p,o}}{\omega}\right)^2 = \left(\frac{8.97 \times 10^3}{f}\right)^2 N_{e,o}$ for each of the distributions shown in figure 2. In this way, it was possible to systematically observe the changes which can occur in the input admittance as a result of changes both in the peak electron density and the extent of the boundary layer.

The thickness of the slab was again chosen to be 3.5 cm, and calculations were performed over a range of parameters which satisfied the empirical criterion.

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[1]This criterion was deduced from the inspection of similar calculations (ref. 9) pertaining to long slots on plasma-covered ground planes.

[2]The calculations were made in fulfillment of the grant reported in reference 10 but were not included in the final published report.

RESULTS

The calculated input admittance for $\nu/\omega = 0.4$ is presented as a function of peak plasma frequency and boundary-layer thickness in figure 3(a). The curves are labeled "profile 2a," "profile 2b," and "profile 2c" to indicate that the computations apply to each of the three electron density distributions given in figure 2. Figure 3(a) shows that a substantial reduction of the susceptance can occur as the boundary-layer thickness grows from zero to a small fraction of a free-space wavelength. At $\left(\omega_{p,o}/\omega\right)^2 = 10$, for example, the normalized input susceptance resulting from the homogeneous profile 2a is -3.37 as opposed to -1.90 from the inhomogeneous profile 2b; yet the half-maximum of the electron density of profile 2b occurs only $\frac{1}{30}\lambda_0$ from the ground plane. A further, but less pronounced, decrease of the susceptance is noted as the boundary-layer thickness increases to $\frac{1}{15}\lambda_0$ (profile 2c).

Figure 3(a) also shows that the conductance is relatively insensitive to the detailed distribution of the plasma if the boundary-layer thickness is $\approx\frac{1}{30}\lambda_0$ which indicates that homogeneous slab approximation adequately defines the conductance if the preceding condition is satisfied.

The voltage reflection coefficient for $\nu/\omega = 0.4$ is given as a function of peak plasma frequency in figure 3(b). The results show that a definite lowering of the reflection coefficient occurs as the boundary-layer thickness increases, which is consistent with Taylor's experiments (ref. 4). It is of further interest to note that the slope $\frac{d|\Gamma|}{d\left(\omega_{p,o}/\omega\right)^2}$ decreases in the region $1 \leqq \left(\frac{\omega_{p,o}}{\omega}\right)^2 \leqq 10$ as the boundary layer grows. The determination of electron density within a given percentage of error therefore requires a more accurate measurement of $d|\Gamma|$ as the boundary-layer thickness increases. In the limit as $\frac{d|\Gamma|}{d\left(\omega_{p,o}/\omega\right)^2} \rightarrow 0$, phase information is required to deduce the electron density.

The collision frequency ratio ν/ω was reduced to 0.06, and admittance computations were performed by using profile 2c of figure 2. Calculations were initiated at $\left(\omega_{p,o}/\omega\right)^2 = 2.0$ where the 3.5-cm slab is thick enough to appear as an infinite half space in accordance with the criterion $e^{-2\alpha_p z_o} > 0.1$. The results of these admittance computations are shown in figure 4(a). For reference, homogeneous-half-space calculations (ref. 10) for $\nu/\omega = 0.04$ are also given.

The behavior of the susceptance is similar to the higher collision results of figure 3(a), which indicates that the susceptance is relatively independent of the collision frequency. On the other hand, the conductance decreases with a decrease in collision frequency.

It is interesting to note that a finite standoff distance of the plasma prevents g_{in} from approaching zero when the collision frequency is low. This observation is important, since this small increase in g_{in} is the prime reason for the reduction in the reflection coefficient shown in figure 4(b). It is also interesting to note that $|\Gamma|$ is relatively insensitive to changes in electron density when $\left(\omega_{p,o}/\omega\right)^2 > 1$. It is therefore concluded that phase information is required for the determination of electron density when the collision frequency is low and the plasma is overdense.

Finally, it should be noted that the integrand of y^{TM} was inspected in order to note any surface-wave excitation. No evidence of surface waves was observed in the region $1 < \beta < 6$. It is possible that a surface-wave pole occurs beyond the range of the numerical integration (i.e., $\beta > 6$); however, it is believed that such a surface wave would be weakly excited. Hence, the observations noted previously are not influenced by the question of surface waves propagating in the boundary layer.

CONCLUDING REMARKS

The input admittance expression of a rectangular waveguide-fed aperture antenna covered with an inhomogeneous plasma slab has been formulated, and numerical results have shown that a finite boundary layer over the ground plane can significantly influence the input admittance of the antenna. In particular, a substantial reduction of the inductive susceptance occurs as the plasma becomes more overdense and the boundary-layer thickness increases from zero to a small fraction of a free-space wavelength. A corresponding decrease in the reflection coefficient occurs for a collision frequency ratio ν/ω of 0.4.

When the collision frequency ratio is reduced to approximately 0.06, a decrease in the reflection coefficient is also noted. For this low collision frequency ratio, the decrease in the reflection coefficient is primarily due to the fact that the boundary layer prevents the conductance from approaching zero. It is further noted that, when the plasma is overdense, the reflection coefficient is relatively insensitive to changes in the peak plasma frequency, and phase information is thereby required for plasma diagnostics.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 12, 1967,
129-01-03-03-23.

APPENDIX

ADMITTANCE EXPRESSIONS FOR HOMOGENEOUS DIELECTRICS

If the dielectric constant is independent of position, the Helmholtz equation (7) may be solved to give

$$\left.\begin{aligned} f(z) &= Ae^{jk_z{}^I z} + Be^{-jk_z{}^I z} \\ g(z) &= Ce^{jk_z{}^I z} + De^{-jk_z{}^I z} \end{aligned}\right\} \tag{A1}$$

for $N^2 > 0$, where

$$\left.\begin{aligned} \frac{k_z{}^I}{k_0} &= \pm\sqrt{N^2 - \beta^2} & \left(N^2 > \beta^2\right) \\ &= \pm j\sqrt{\beta^2 - N^2} & \left(N^2 < \beta^2\right) \end{aligned}\right\} \tag{A2a}$$

and for $N^2 < 0$,

$$\frac{k_z{}^I}{k_0} = \pm j\sqrt{\beta^2 + N^2} \tag{A2b}$$

The unknown coefficients A, B, C, and D in equations (A1) are determined from the boundary conditions (6) at $z = z_o$, and explicit forms of $\left[\frac{k_0 g(\beta,0)}{g'(\beta,0)}\right]$ and $\left[\frac{f'(\beta,0)}{k_0 f(\beta,0)}\right]$ may then be substituted into the admittance expressions (15) to give

$$y^{TM} = \frac{4j(k_0a)(k_0b)}{Y_{01}/Y_0} \int_0^{\infty} \int_0^{\pi} \left[\frac{\cos\left(\frac{\beta k_0 b}{2} \sin\alpha\right)}{\pi^2 - \left(k_0 b\beta \sin\alpha\right)^2}\right]^2 \left[\frac{\sin\left(\frac{k_0 a\beta}{2} \cos\alpha\right)}{\left(\frac{k_0 a\beta}{2} \cos\alpha\right)}\right]^2$$

$$\times \left\{\frac{N^2\left[1 - j\,\frac{N^2\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}} \tan k_0 z_o\sqrt{N^2 - \beta^2}\right]}{\sqrt{N^2 - \beta^2}\left[\tan k_0 z_o\sqrt{N^2 - \beta^2} + j\,\frac{N^2\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}}\right]}\right\} \cos^2\alpha\,\beta\,d\beta\,d\alpha \tag{A3}$$

$$y^{TE} = \frac{-4j\left(k_0 a\right)\left(k_0 b\right)}{Y_{01}/Y_0} \int_0^{\infty} \int_0^{\pi} \left[\frac{\cos\left(\dfrac{\beta k_0 b}{2} \sin \alpha\right)}{\pi^2 - \left(k_0 b\beta \sin \alpha\right)^2}\right]^2 \left[\frac{\sin\left(\dfrac{k_0 a\beta}{2} \cos \alpha\right)}{\left(\dfrac{k_0 a\beta}{2} \cos \alpha\right)}\right]^2$$

$$\times \left\{\frac{\sqrt{N^2 - \beta^2}\left[\tan k_0 z_o \sqrt{N^2 - \beta^2} + j \dfrac{\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}}\right]}{1 - j \dfrac{\sqrt{1 - \beta^2}}{\sqrt{N^2 - \beta^2}} \tan k_0 z_o \sqrt{N^2 - \beta^2}}\right\} \sin^2\alpha \, \beta \, d\beta \, d\alpha \tag{A4}$$

The input admittance is the sum of the foregoing TE and TM components – that is,

$$y_{in} = y^{TE} + y^{TM} \tag{A5}$$

If the slab is lossy,

$$N^2 = \frac{\epsilon_{I,r}}{\epsilon_0} + j \frac{\epsilon_{I,i}}{\epsilon_0}$$

and

$$\sqrt{N^2 - \beta^2} = \frac{1}{\sqrt{2}}\left[\frac{\epsilon_{I,r}}{\epsilon_0} - \beta^2 + \sqrt{\left(\frac{\epsilon_{I,r}}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_{I,r}}{\epsilon_0}\right)^2}\right]^{1/2}$$

$$+ \frac{j}{\sqrt{2}}\left[\beta^2 - \frac{\epsilon_{I,r}}{\epsilon_0} + \sqrt{\left(\frac{\epsilon_{I,r}}{\epsilon_0} - \beta^2\right)^2 + \left(\frac{\epsilon_{I,i}}{\epsilon_0}\right)^2}\right]^{1/2}$$

and no difficulty is presented in the evaluation of equations (A3) and (A4), other than a proper root change of $\sqrt{1 - \beta^2}$, in accordance with equations (5).

If the dielectric slab is nonlossy, the integration over β in equations (A3) and (A4) should be broken up in accordance with the root conditions in k_z^{I}/k_0 and k_z^{II}/k_0, as summarized in tables I and II for the following cases:

Case 1: $N^2 > 1$

Case 2: $0 < N^2 < 1$

Case 3: $N^2 < 0$

APPENDIX

It is noted in the tables that poles occur in the integrands of y^{TE} and y^{TM} for $N^2 > 1$ and in the integrand of y^{TM} for $N^2 < -1$. These poles are of order one and occur at those values of $\beta = \beta_n$ which are the roots of the following transcendental equations:

TM component:

$$\left.\begin{aligned}
&\tan k_0 z_o \sqrt{N^2 - \beta^2} - \frac{N^2\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}} = 0 \qquad \left(N^2 > 1\right) \\
&\text{TE component:} \\
&1 + \frac{\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}} \tan k_0 z_o \sqrt{N^2 - \beta^2} = 0 \qquad \left(N^2 > 1\right)
\end{aligned}\right\} \tag{A6a}$$

TM component:

$$\tanh k_0 z_o \sqrt{\left|N^2\right| + \beta^2} = \frac{\left|N^2\right| \sqrt{\beta^2 - 1}}{\sqrt{\left|N^2\right| + \beta^2}} \qquad \left(N^2 < -1\right) \tag{A6b}$$

Since the integrand is infinite at $\beta = \beta_n$, it follows that the numerical integration of y^{TE} and y^{TM} should be terminated at $\beta_n - \delta$ and commenced again at $\beta_n + \delta$, where δ is an arbitrary small number. This procedure also necessitates adding a residue contribution at $\beta = \beta_n$. The poles are of order one; hence, the residue of y^{TM} for $N^2 > 1$, for example, is given by

$$\mathrm{Res}\left(y^{TM}\right)_{N^2>1} = \frac{N^2\beta_n\left[1 + \dfrac{N^2\sqrt{\beta_n{}^2 - 1}}{\sqrt{N^2 - \beta_n{}^2}} \tan k_0 z_o \sqrt{N^2 - \beta_n{}^2}\right]}{\dfrac{d}{d\beta}\left\{\sqrt{N^2 - \beta^2}\left[\tan k_0 z_o \sqrt{N^2 - \beta^2} - \dfrac{N^2\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}}\right]\right\}\Bigg|_{\beta=\beta_n}} \int_0^{\pi} A\left(\alpha,\beta_n\right)\cos^2\alpha \, d\alpha \tag{A7}$$

where $1 < \beta_n < N$, and $A(\alpha,\beta)$ for $\beta = \beta_n$ is defined in table I.

From use of equations (A6a), it immediately follows that

$$1 + \frac{N^2\sqrt{\beta_n{}^2 - 1}}{\sqrt{N^2 - \beta_n{}^2}} \tan k_0 z_o \sqrt{N^2 - \beta_n{}^2} = \sec^2 k_0 z_o \sqrt{N^2 - \beta_n{}^2} \tag{A8}$$

which simplifies the numerator of equation (A7). The derivative of the denominator of equation (A7) is given by

$$\frac{d}{d\beta}\left\{\sqrt{N^2 - \beta^2}\left[\tan k_0 z_o \sqrt{N^2 - \beta^2} - \frac{N^2\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}}\right]\right\}\Bigg|_{\beta=\beta_n}$$

$$= - \frac{\beta_n}{\sqrt{N^2 - \beta_n^2}}\left[\tan k_0 z_o \sqrt{N^2 - \beta_n^2} - \frac{N^2\sqrt{\beta_n^2 - 1}}{\sqrt{N^2 - \beta_n^2}}\right]$$

$$- \beta_n\left[k_0 z_o \sec^2 k_0 z_o \sqrt{N^2 - \beta_n^2} + \frac{N^2}{\sqrt{\beta_n^2 - 1}} + \frac{N^2\sqrt{\beta_n^2 - 1}}{N^2 - \beta_n^2}\right] \tag{A9}$$

From equations (A6a) the first term on the right-hand side of equation (A9) is, by definition, zero. Further use of equations (A6a) gives

$$\frac{N^2}{\sqrt{\beta_n^2 - 1}} + \frac{N^2\sqrt{\beta_n^2 - 1}}{N^2 - \beta_n^2} = \frac{\left(N^2 - 1\right)\tan k_0 z_o \sqrt{N^2 - \beta_n^2}}{\left(\beta_n^2 - 1\right)\sqrt{N^2 - \beta_n^2}} \tag{A10}$$

It is now easy to show that

$$\frac{d}{d\beta}\left\{\sqrt{N^2 - \beta^2}\left[\tan k_0 z_o \sqrt{N^2 - \beta^2} - \frac{N^2\sqrt{\beta^2 - 1}}{\sqrt{N^2 - \beta^2}}\right]\right\}\Bigg|_{\beta=\beta_n}$$

$$= -\beta_n k_0 z_o \sec^2 k_0 z_o \sqrt{N^2 - \beta^2}\left[1 + \frac{\left(N^2 - 1\right)}{\left(\beta_n^2 - 1\right)} \frac{\sin\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}{\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}\right] \tag{A11}$$

and that the residue is

$$\operatorname{Res}\left(y^{TM}\right)_{N^2>1} = - \frac{N^2 \int_0^{\pi} A\left(\alpha,\beta_n\right)\cos^2\alpha \, d\alpha}{k_0 z_o\left[1 + \frac{\left(N^2 - 1\right)}{\left(\beta_n^2 - 1\right)} \frac{\sin\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}{\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}\right]} \tag{A12}$$

Therefore, the pole contribution is

$$\pi j \operatorname{Res}\left(y^{TM}\right)_{N^2>1} = g_{s,n}^{TM} = \frac{4\left(k_0 a\right)\left(k_0 b\right)\pi N^2}{\frac{Y_{01}}{Y_0}\left(k_0 z_o\right)\left[1 + \frac{\left(N^2 - 1\right)}{\left(\beta_n{}^2 - 1\right)} \frac{\sin\left(2k_0 z_o\sqrt{N^2 - \beta_n{}^2}\right)}{\left(2k_0 z_o\sqrt{N^2 - \beta_n{}^2}\right)}\right]}$$

$$\times \int_0^{\pi} \left[\frac{\cos\left(\frac{k_0 b\beta_n}{2} \sin\alpha\right)}{\pi^2 - \left(k_0 b\beta_n \sin\alpha\right)^2}\right]^2 \left[\frac{\sin\left(\frac{k_0 a\beta_n}{2} \cos\alpha\right)}{\left(\frac{k_0 a\beta_n}{2} \cos\alpha\right)}\right]^2 \cos^2\alpha \, d\alpha \qquad \text{(A13)}$$

The right-hand side of equation (A13) is a real number, contributing only to the conductance, and is a measure of the amount of power which remains trapped in the dielectric sheet as a surface wave.

The onset of a TM surface wave occurs when $\beta = 1$ and $\tan k_0 z_o\sqrt{N^2 - \beta^2}$ is a positive number, that is, where

$$m\pi = k_0 z_o\sqrt{N^2 - 1} \qquad (m = 0, 1, 2, \ldots) \qquad \text{(A14)}$$

The total surface-wave contribution is given by

$$g_s^{TM} = \sum_n g_{s,n}^{TM} \qquad \text{(A15)}$$

where the sum includes all the poles as determined from equation (A6). The total TM admittance is therefore given by

$$y^{TM} = \int_0^{\pi} d\alpha \cos^2\alpha \left[\int_0^1 A(\alpha,\beta) F_1(\beta)\beta \, d\beta \right.$$

$$\left. + P\int_1^N A(\alpha,\beta) F_2(\beta)\beta \, d\beta + \int_N^{\infty} A(\alpha,\beta) F_3(\beta)\beta \, d\beta\right] + \sum_n g_{s,n}^{TM} \qquad \text{(A16)}$$

where the explicit functional forms of F_1, F_2, and F_3 are given in table I.

The surface-wave conductance for $y^{TE}\big|_{N^2>1}$ and $y^{TM}\big|_{N^2<0}$ is similarly derived, and explicit forms are given in table III.

If a large number of surface-wave poles exist in the region $1 < \beta < N$, $F_2(\beta)$ becomes a rapidly varying function. As a consequence, it may be difficult to evaluate the principal part of the integral. If such difficulties are encountered, it would be wise to convert the integration over the real axis to one along the branch cut, as indicated in figure 5.[3]

If the medium is inhomogeneous and nonlossy, one should inspect $g'(\beta,0)$ and $f(\beta,0)$. If they go to zero, a surface wave has been excited, and one should numerically compute the values of $\dfrac{k_0 g(\beta_n,0)}{\left.\dfrac{d}{d\beta} g'(\beta,0)\right|_{\beta=\beta_n}}$ and $\dfrac{f'(\beta_n,0)}{\left.k_0 \dfrac{d}{d\beta} f(\beta,0)\right|_{\beta=\beta_n}}$ and numerically enter the residue contribution.

[3]It should be noted that only the forward wave poles are included when the contour is closed in the upper half plane. When the contour is closed in the lower half plane, the backward wave poles must be included. These backward wave poles and the branch line which appear in the lower half plane are not shown in figure 5.

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TABLE I.- TM COMPONENT OF ADMITTANCE EXPRESSION

	Preferred form for the numerical evaluation of y^{TM}	$F_1(\beta)$	$F_2(\beta)$	$F_3(\beta)$	Comments
Case 1: $N^2 > 1$	$y^{TM} = \int_0^{\pi} d\alpha \cos^2\alpha \left[\int_0^1 A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_1^N A(\alpha,\beta)F_2(\beta)\beta\, d\beta + \int_N^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{N^2\left[1 - j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0 z_o\sqrt{N^2-\beta^2}\right]}{\sqrt{N^2-\beta^2}\left[\tan k_0 z_o\sqrt{N^2-\beta^2} + j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right]}$	$\dfrac{N^2\left[1 + \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\tan k_0 z_o\sqrt{N^2-\beta^2}\right]}{\sqrt{N^2-\beta^2}\left[\tan k_0 z_o\sqrt{N^2-\beta^2} - \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right]}$	$\dfrac{-N^2\left[1 + \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0 z_o\sqrt{\beta^2-N^2}\right]}{\sqrt{\beta^2-N^2}\left[\tanh k_0 z_o\sqrt{\beta^2-N^2} + \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right]}$	(1) Poles of $F_2(\beta)$ occur where $\tan k_0 z_o\sqrt{N^2-\beta^2} = \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}$ (2) Principal parts of the integrals which contain $F_2(\beta)$ and $F_3(\beta)$ contribute only to the susceptance.
Case 2: $0 < N^2 < 1$	$y^{TM} = \int_0^{\pi} d\alpha \cos^2\alpha \left[\int_0^N A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_N^1 A(\alpha,\beta)F_2(\beta)\beta\, d\beta + \int_1^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{N^2\left[1 - j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0 z_o\sqrt{N^2-\beta^2}\right]}{\sqrt{N^2-\beta^2}\left[\tan k_0 z_o\sqrt{N^2-\beta^2} + j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right]}$	$\dfrac{-N^2\left[1 - j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\tanh k_0 z_o\sqrt{\beta^2-N^2}\right]}{\sqrt{\beta^2-N^2}\left[\tanh k_0 z_o\sqrt{\beta^2-N^2} - j\dfrac{N^2\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\right]}$	$\dfrac{-N^2\left[1 + \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0 z_o\sqrt{\beta^2-N^2}\right]}{\sqrt{\beta^2-N^2}\left[\tanh k_0 z_o\sqrt{\beta^2-N^2} + \dfrac{N^2\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right]}$	(1) No poles occur anywhere in the range of integration. (2) The integral which contains $F_3(\beta)$ contributes only to the susceptance.
Case 3: $N^2 < 0$	$y^{TM} = \int_0^{\pi} d\alpha \cos^2\alpha \left[\int_0^1 A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_1^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{\lvert N^2\rvert\left[1 + j\dfrac{\lvert N^2\rvert\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0 z_o\sqrt{\lvert N^2\rvert+\beta^2}\right]}{\sqrt{\lvert N^2\rvert+\beta^2}\left[\tanh k_0 z_o\sqrt{\lvert N^2\rvert+\beta^2} + j\dfrac{\lvert N^2\rvert\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right]}$	. . .	$\dfrac{\lvert N^2\rvert\left[1 - \dfrac{\lvert N^2\rvert\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0 z_o\sqrt{\lvert N^2\rvert+\beta^2}\right]}{\sqrt{\lvert N^2\rvert+\beta^2}\left[\tanh k_0 z_o\sqrt{\lvert N^2\rvert+\beta^2} - \dfrac{\lvert N^2\rvert\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right]}$	(1) *Pole of* $F_3(\beta)$ *occurs where* $\tanh k_0 z_o\sqrt{\lvert N^2\rvert+\beta^2} = \dfrac{\lvert N^2\rvert\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}$ $\left(\lvert N^2\rvert > 1\right)$ (2) The integral which contains $F_3(\beta)$ *contributes only to* the susceptance.

$$A(\alpha,\beta) = \frac{4j(k_0a)(k_0b)}{Y_{01}/Y_0}\left[\frac{\cos\left(\frac{k_0b\beta}{2}\sin\alpha\right)}{\pi^2 - \left(k_0b\beta\sin\alpha\right)^2}\right]^2\left[\frac{\sin\left(\frac{k_0a\beta}{2}\cos\alpha\right)}{\left(\frac{k_0a\beta}{2}\cos\alpha\right)}\right]^2$$

Note that $\left[\dfrac{\sin\left(\frac{k_0a\beta}{2}\cos\alpha\right)}{\left(\frac{k_0a\beta}{2}\cos\alpha\right)^2}\right]^2 \approx 1$ for $\beta\cos\alpha \approx 0$ and $\left[\dfrac{\cos\left(\frac{k_0b\beta}{2}\sin\alpha\right)}{\pi^2 - \left(k_0b\beta\sin\alpha\right)^2}\right]^2 \approx \dfrac{1}{16\pi^2}$ for $k_0b\beta\sin\alpha \approx \pi$.

TABLE II.- TE COMPONENT OF ADMITTANCE EXPRESSION

	Preferred form for the numerical evaluation of y^{TE}	$F_1(\beta)$	$F_2(\beta)$	$F_3(\beta)$	Comments
Case 1: $N^2 > 1$	$y^{TE} = -\int_0^{\pi} d\alpha \sin^2\alpha \left[\int_0^1 A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_1^N A(\alpha,\beta)F_2(\beta)\beta\, d\beta + \int_N^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{\sqrt{N^2-\beta^2}\left[\tan k_0z_0\sqrt{N^2-\beta^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right]}{1 - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0z_0\sqrt{N^2-\beta^2}}$	$\dfrac{\sqrt{N^2-\beta^2}\left[\tan k_0z_0\sqrt{N^2-\beta^2} - \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\right]}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{N^2-\beta^2}}\tan k_0z_0\sqrt{N^2-\beta^2}}$	$\dfrac{-\sqrt{\beta^2-N^2}\left[\tanh k_0z_0\sqrt{\beta^2-N^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right]}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0z_0\sqrt{\beta^2-N^2}}$	(1) Poles of $F_2(\beta)$ occur where $\tan k_0z_0\sqrt{N^2-\beta^2} = -\dfrac{\sqrt{N^2-\beta^2}}{\sqrt{\beta^2-1}}$ (2) Principal parts of the integrals which contain $F_2(\beta)$ and $F_3(\beta)$ contribute only to the susceptance.
Case 2: $0 < N^2 < 1$	$y^{TE} = -\int_0^{\pi} d\alpha \sin^2\alpha \left[\int_0^N A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_N^1 A(\alpha,\beta)F_2(\beta)\beta\, d\beta + \int_1^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{\sqrt{N^2-\beta^2}\left[\tan k_0z_0\sqrt{N^2-\beta^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\right]}{1 - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{N^2-\beta^2}}\tan k_0z_0\sqrt{N^2-\beta^2}}$	$\dfrac{\sqrt{\beta^2-N^2}\left[-\tanh k_0z_0\sqrt{\beta^2-N^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\right]}{1 - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\beta^2-N^2}}\tanh k_0z_0\sqrt{\beta^2-N^2}}$	$\dfrac{-\sqrt{\beta^2-N^2}\left[\tanh k_0z_0\sqrt{\beta^2-N^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\right]}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\beta^2-N^2}}\tanh k_0z_0\sqrt{\beta^2-N^2}}$	(1) No poles occur anywhere in the range of integration. (2) The integral which contains $F_3(\beta)$ contributes only to the susceptance.
Case 3: $N^2 < 0$	$y^{TE} = -\int_0^{\pi} d\alpha \sin^2\alpha \left[\int_0^1 A(\alpha,\beta)F_1(\beta)\beta\, d\beta + \int_1^{\infty} A(\alpha,\beta)F_3(\beta)\beta\, d\beta\right]$	$\dfrac{\sqrt{\lvert N^2\rvert+\beta^2}\left[-\tanh k_0z_0\sqrt{\lvert N^2\rvert+\beta^2} + j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right]}{1 - j\dfrac{\sqrt{1-\beta^2}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0z_0\sqrt{\lvert N^2\rvert+\beta^2}}$	. . .	$\dfrac{-\sqrt{\lvert N^2\rvert+\beta^2}\left[\tanh k_0z_0\sqrt{\lvert N^2\rvert+\beta^2} + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\right]}{1 + \dfrac{\sqrt{\beta^2-1}}{\sqrt{\lvert N^2\rvert+\beta^2}}\tanh k_0z_0\sqrt{\lvert N^2\rvert+\beta^2}}$	(1) No poles occur anywhere in the range of integration. (2) The integral which contains $F_3(\beta)$ contributes only to the susceptance.

TABLE III.- TE AND TM SURFACE-WAVE CONDUCTANCE EXPRESSIONS

	Expression for surface-wave conductance	Transcendental equation for the location of the roots β_n	Comments
Case 1: $N^2 > 1$	$g_s^{TM} = \frac{-\pi j}{(k_0 z_o)} \sum_n \frac{N^2}{\left[1 + \frac{(N^2 - 1)}{(\beta_n^2 - 1)} \frac{\sin\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}{\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}\right]} \int_0^\pi A(\alpha,\beta_n)\cos^2\alpha \, d\alpha$	$\tan k_0 z_o \sqrt{N^2 - \beta_n^2} = \frac{N^2 \sqrt{\beta_n^2 - 1}}{\sqrt{N^2 - \beta_n^2}}$	(1) The sum on n of g_s includes all finite roots of the transcendental equation. (2) The onset of a surface wave occurs at thicknesses in wavelengths $\frac{z_o}{\lambda_0} = 0, \frac{1}{2\sqrt{N^2 - 1}}, \frac{1}{\sqrt{N^2 - 1}}, \frac{3}{2\sqrt{N^2 - 1}}, \ldots$
	$g_s^{TE} = \frac{-\pi j}{(k_0 z_o)} \sum_n \frac{\left(N^2 - \beta_n^2\right)}{\left[1 - \frac{(N^2 - 1)}{(\beta_n^2 - 1)} \frac{\sin\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}{\left(2k_0 z_o \sqrt{N^2 - \beta_n^2}\right)}\right]} \int_0^\pi A(\alpha,\beta_n)\sin^2\alpha \, d\alpha$	$\tan k_0 z_o \sqrt{N^2 - \beta_n^2} = -\frac{\sqrt{N^2 - \beta_n^2}}{\sqrt{\beta_n^2 - 1}}$	(1) The sum on n of g_s includes all finite roots of the transcendental equation. (2) The onset of a surface wave occurs at thicknesses in wavelengths $\frac{z_o}{\lambda_0} = \frac{1}{4\sqrt{N^2 - 1}}, \frac{3}{4\sqrt{N^2 - 1}}, \frac{5}{4\sqrt{N^2 - 1}}, \ldots$ (3) No TE surface waves occur for $\frac{z_o}{\lambda_0} < \frac{1}{4\sqrt{N^2 - 1}}$.
Case 3: $N^2 < 0$	$g_s^{TM} = \frac{\pi j}{(k_0 z_o)} \frac{\lvert N^2\rvert}{\left[1 - \frac{(\lvert N^2\rvert + 1)}{(\beta_o^2 - 1)} \frac{\sinh\left(2k_0 z_o \sqrt{\lvert N^2\rvert + \beta_o^2}\right)}{\left(2k_0 z_o \sqrt{\lvert N^2\rvert + \beta_o^2}\right)}\right]} \int_0^\pi A(\alpha,\beta_o)\cos^2\alpha \, d\alpha$.	$\tanh k_0 z_o \sqrt{\lvert N^2\rvert + \beta_o^2} = \frac{\lvert N^2\rvert \sqrt{\beta_o^2 - 1}}{\sqrt{\lvert N^2\rvert + \beta_o^2}}$ $(\lvert N^2\rvert > 1)$	(1) Only one surface wave pole occurs.

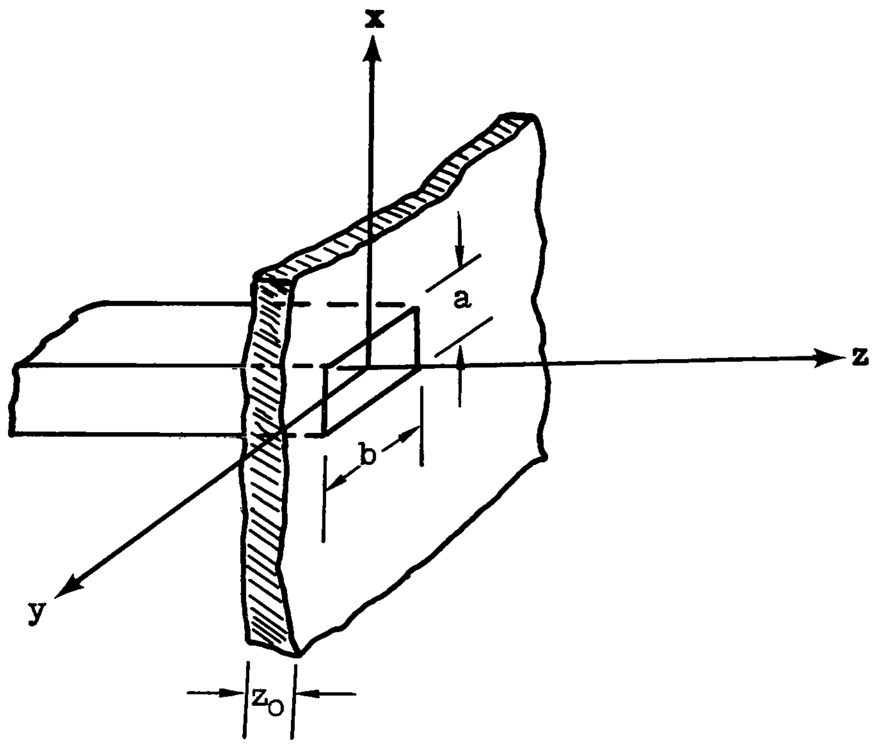


Figure 1.- Rectangular aperture coated with inhomogeneous plasma.

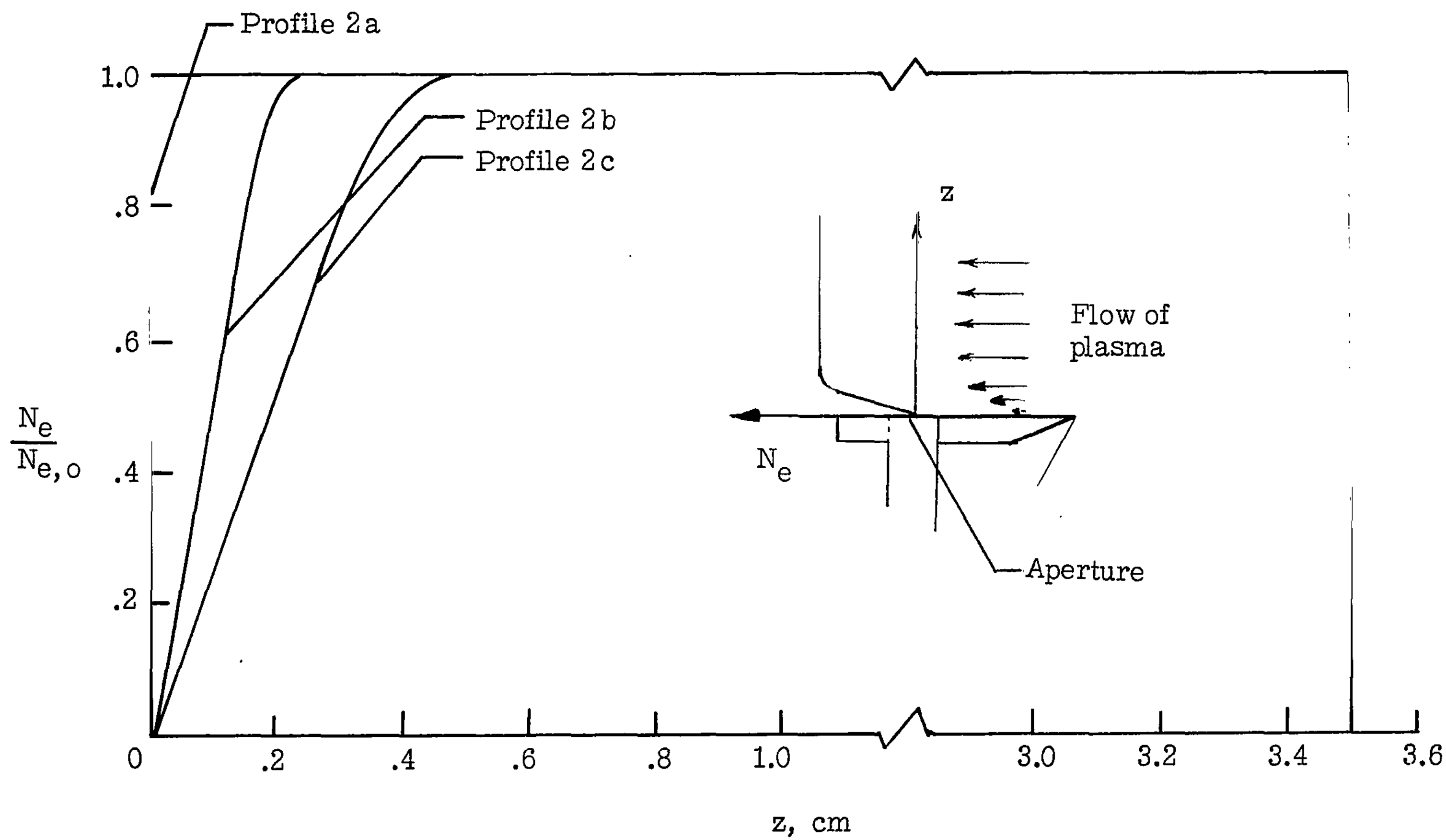
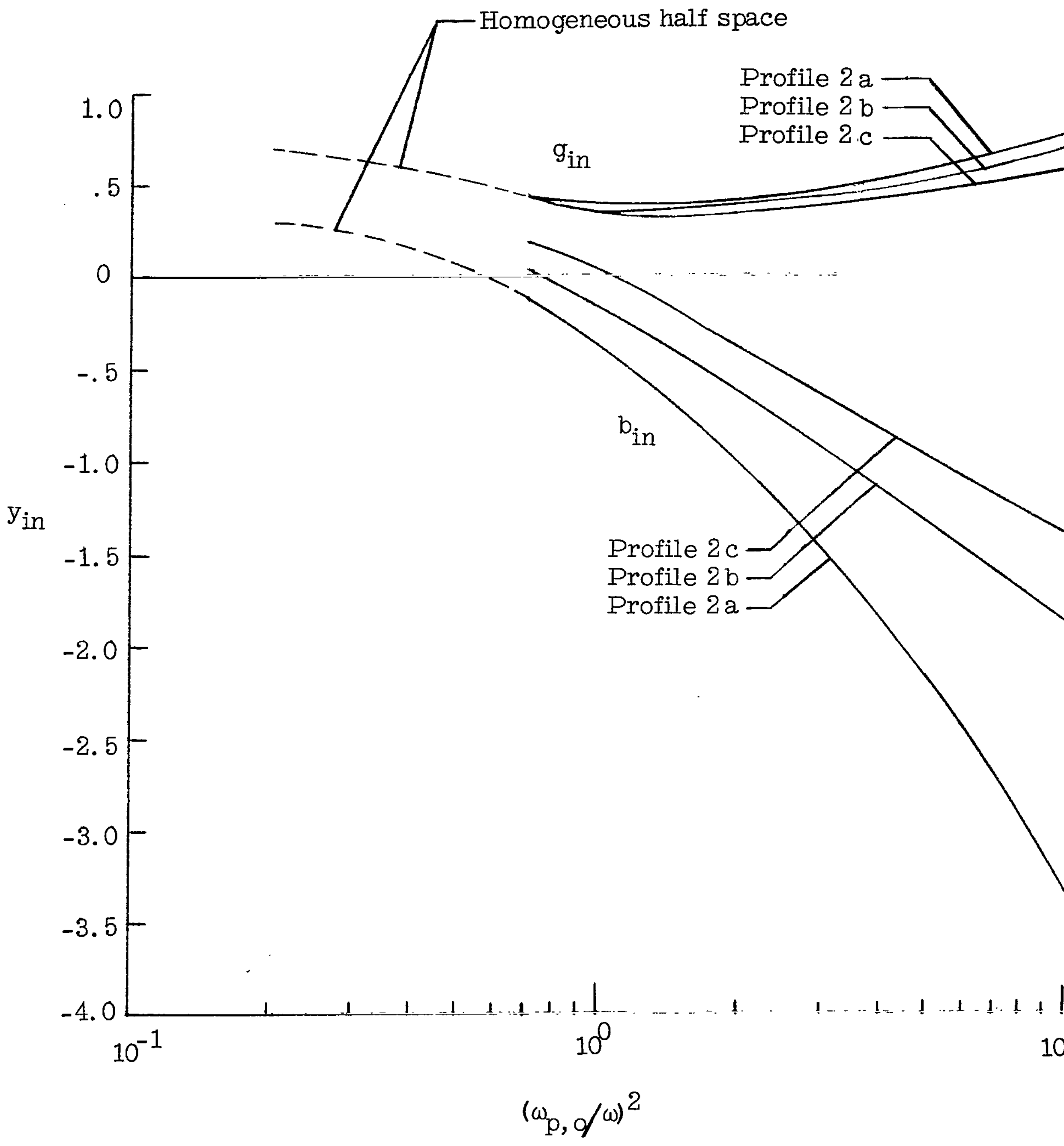
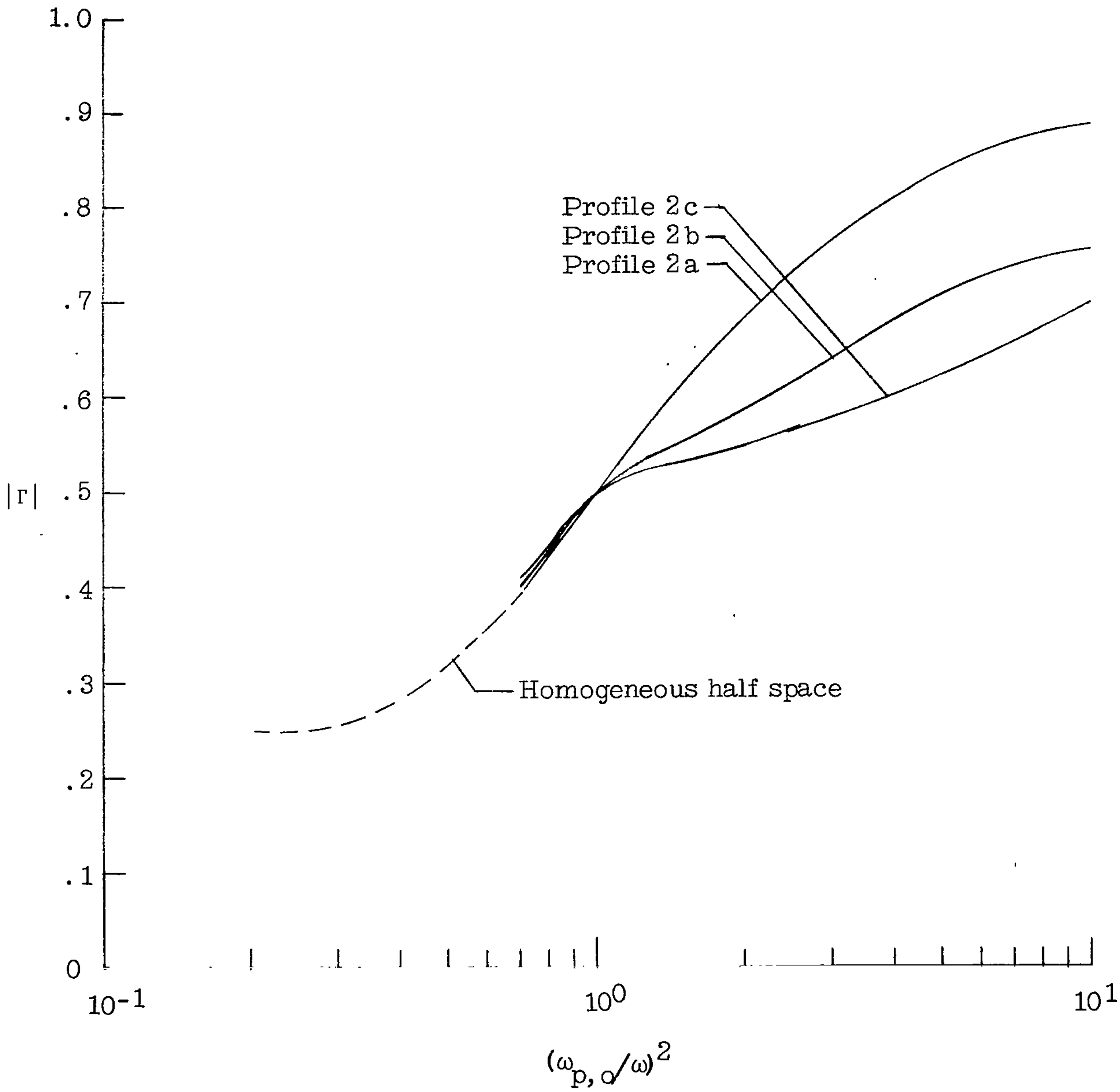


Figure 2.- Normalized electron density profiles used in admittance calculations.



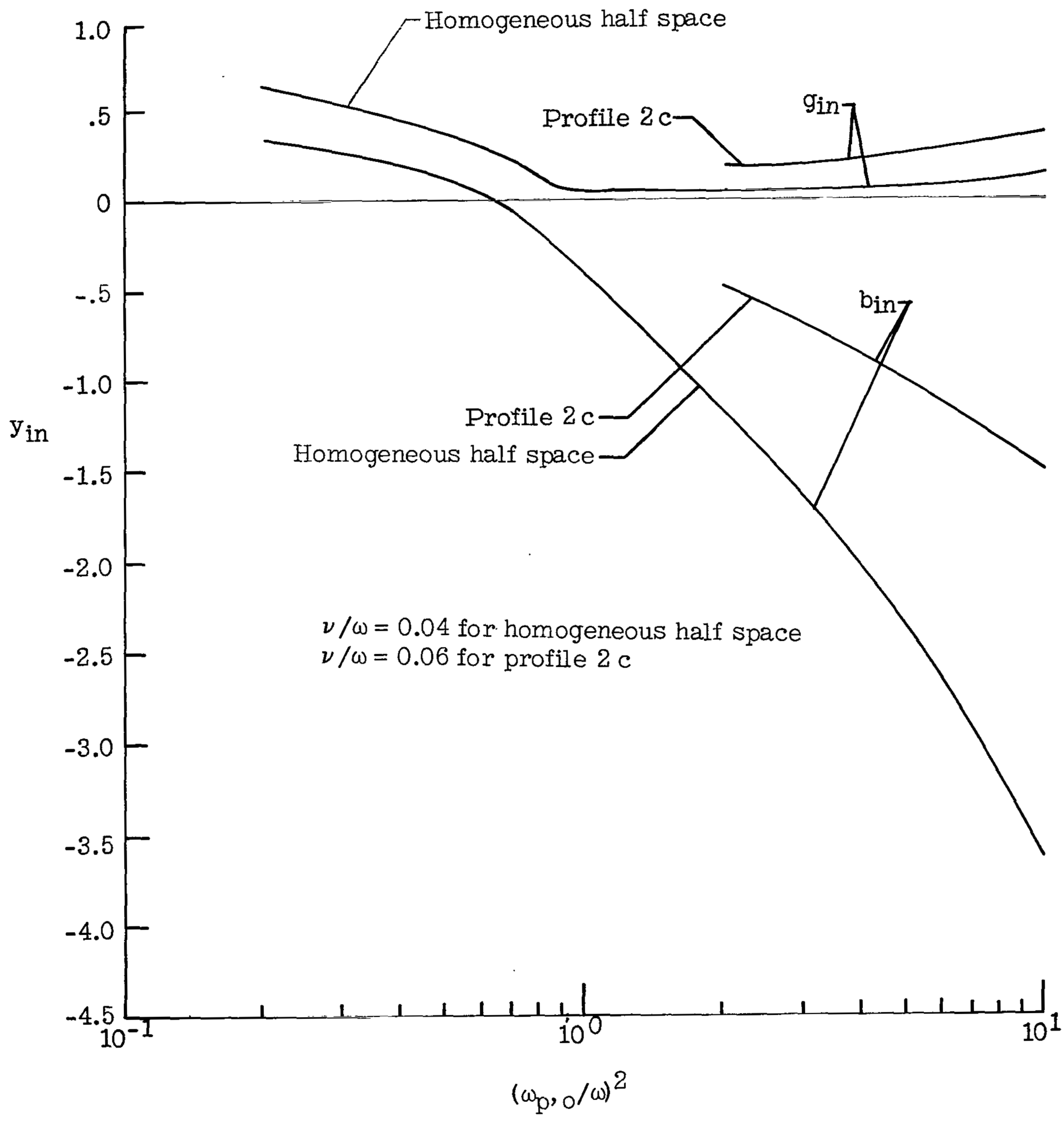
(a) Input admittance.

Figure 3.- Admittance and reflection coefficient of rectangular aperture antenna, coated with an inhomogeneous plasma for $\nu/\omega = 0.4$.



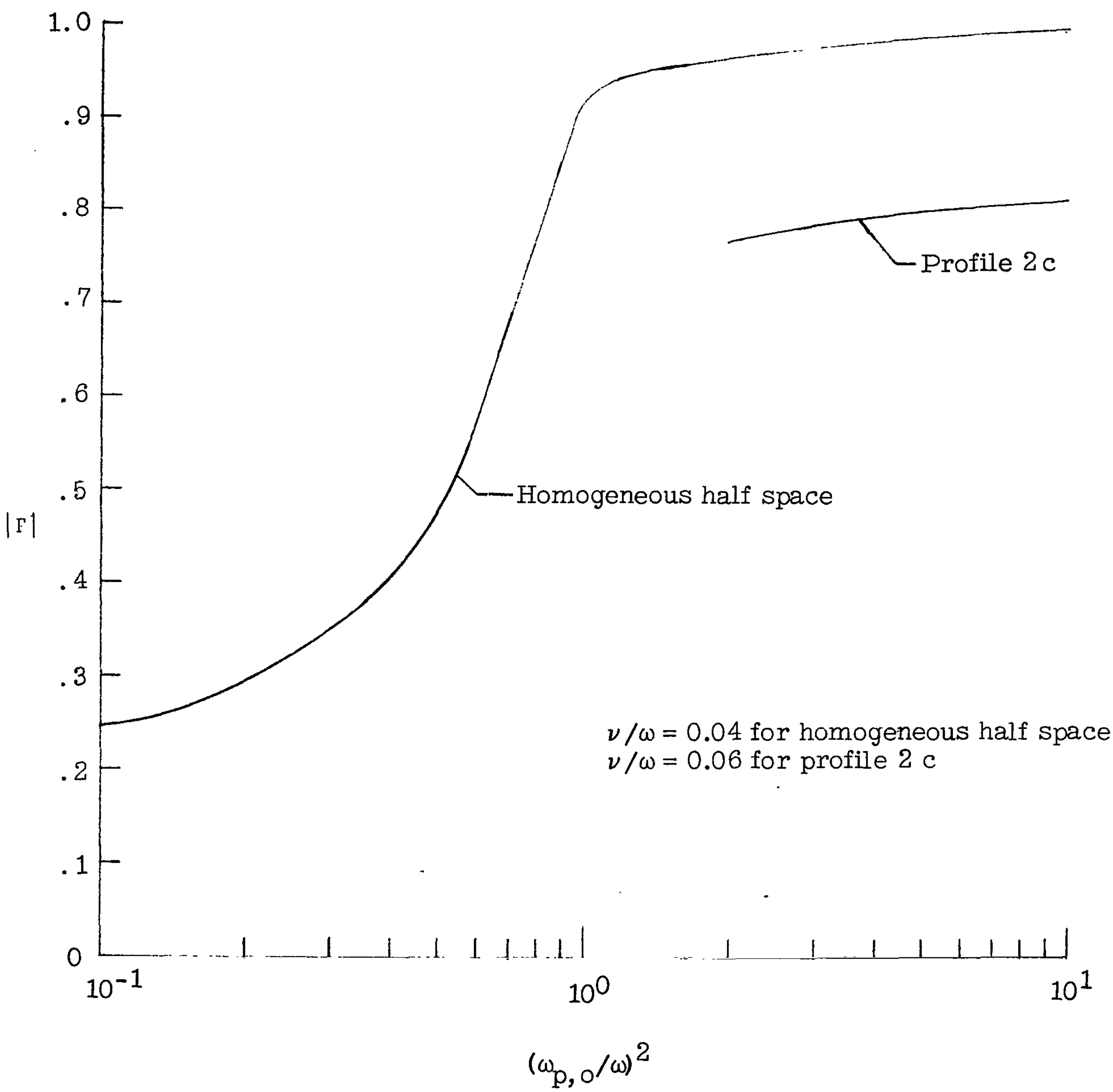
(b) Voltage reflection coefficient.

Figure 3.- Concluded.



(a) Input admittance.

Figure 4.- Admittance and reflection coefficient of rectangular aperture antenna coated with an inhomogeneous plasma for low collision frequency.



(b) Voltage reflection coefficient.

Figure 4.- Concluded.

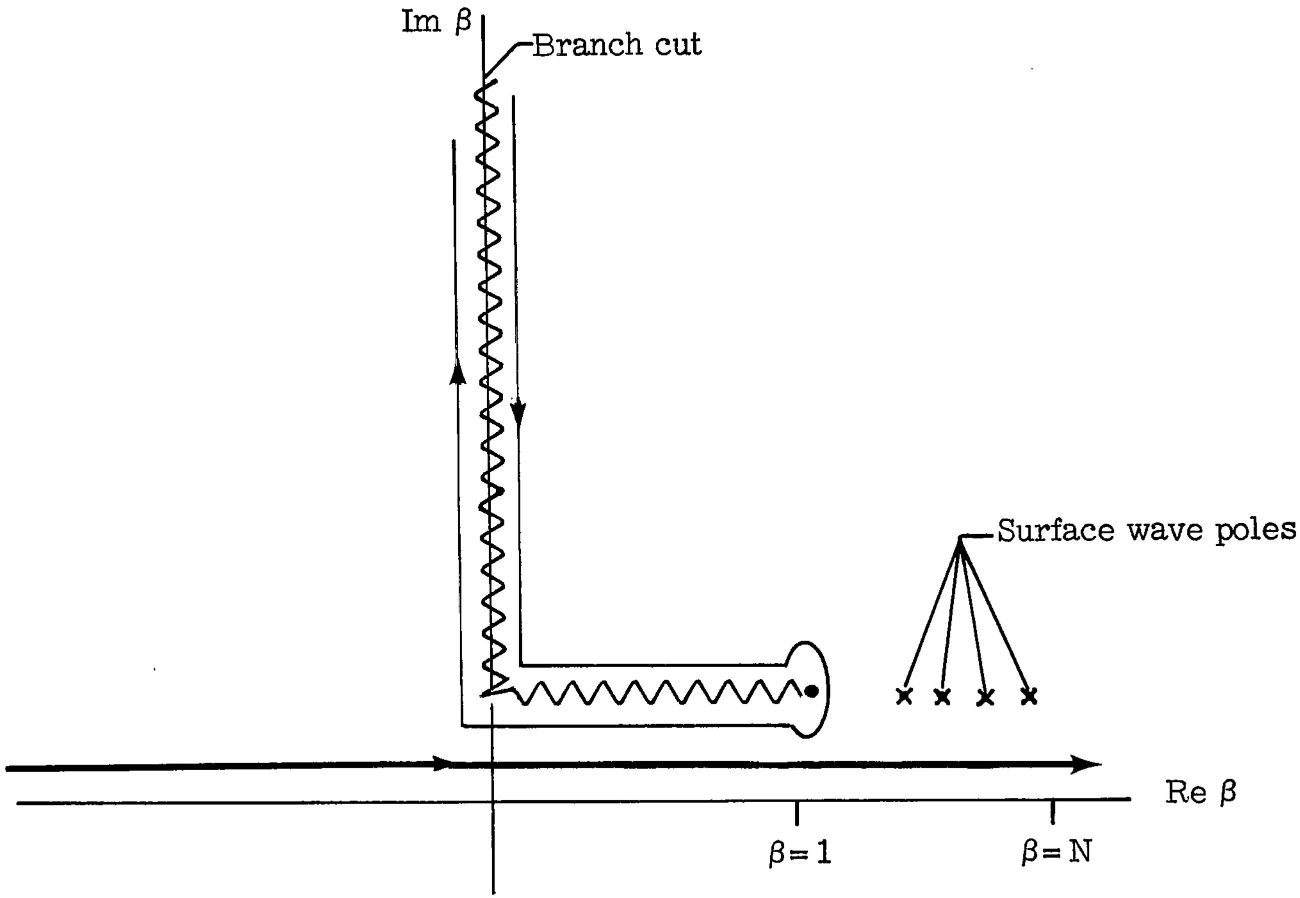


Figure 5.- Contour of integration for the evaluation of the admittance integrals.

NASA-Langley, 1967 — 7 L-5579

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