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The Precession and Nutation of Deformable Bodies II: The Form of Distorted Configurations

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ABSTRACT

In the preceding report of this series (Boeing document D1-82-0590) the Eulerian equations for the precession and nutation of self-gravitating deformable bodies have been set up, with coefficients expressible in terms of the displacement components arising from deformation which may oscillate with the time. The aim of the present report will be to derive an explicit form of the equations which govern the deformations of self-gravitating viscous bodies in an external field of force; and to solve them in the particular case of a homogeneous compressible configuration of arbitrarily high viscosity. A possession of such a solution then opens the way for the evaluation of the coefficients of the generalized equations (4-38) - (4-40) for the precession and nutation of deformable bodies, and to their eventual solution.

An application of the results presented herewith to the physical librations of the Moon, or the luni-solar precession and nutation of the Earth will be given in subsequent reports of this series. The sections (as well as equations) of the present report are numbered consecutively to those of Report I.

V. DEFORMATION OF SELF-GRAVITATING VISCOUS FLUIDS IN EXTERNAL FIELD OF FORCE: FUNDAMENTAL EQUATIONS

The fundamental equations in viscous fluid motion in Cartesian coordinates have already been stated in Section II of our previous report (hereafter referred to as Report I). In order to use them for studies of the motions invoked by *tides* in self-gravitating systems--i.e., deformations of configurations which, in the absence of external force would be spherical in form--we find it expedient to change over from rectangular coordinates x,y,z to spherical polar coordinates r,θ,ϕ, related with the former by

$$\left.\begin{aligned} x &= r \cos\phi \sin\theta, \\ y &= r \sin\phi \sin\theta, \\ z &= r \cos\theta. \end{aligned}\right\} \tag{5-1}$$

If so, then--at a price of some loss of symmetry--equations (2-1) - (2-11) of Section II can be rewritten as

$$\begin{aligned} \rho \frac{DU}{Dt} - \rho \frac{V^2+W^2}{r} = \rho \frac{\partial\Omega}{\partial r} - \frac{\partial P}{\partial r} + \mu\Big[\nabla^2 U + \frac{1}{3}\frac{\partial\Delta}{\partial r} - \frac{2}{r^2}\frac{\partial V}{\partial\theta} - \frac{2}{r^2 \sin\theta}\frac{\partial W}{\partial\phi} \\ - \frac{2U}{r^2} - \frac{2V\cot\theta}{r^2}\Big] + 2\frac{\partial\mu}{\partial r}\Big[\frac{\partial U}{\partial r} - \frac{\Delta}{3}\Big] \\ + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\Big[\frac{1}{r}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r} - \frac{V}{r}\Big] + \frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} + \frac{\partial W}{\partial r} - \frac{W}{r}\Big], \end{aligned} \tag{5-2}$$

$$\begin{aligned} \rho \frac{DV}{Dt} + \rho \frac{UV}{r} - \rho \frac{W^2\cot\theta}{r} = \frac{1}{r}\Big[\rho\frac{\partial\Omega}{\partial\theta} - \frac{\partial P}{\partial\theta}\Big] + \mu\Big[\nabla^2 V + \frac{1}{3r}\frac{\partial\Delta}{\partial\theta} \\ + \frac{2}{r^2}\frac{\partial U}{\partial\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial W}{\partial\phi} - \frac{V}{r^2\sin^2\theta}\Big] \end{aligned} \tag{5-3}$$

$$+ \frac{\partial\mu}{\partial r}\left[\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{V}{r}\right] + \frac{2}{r}\frac{\partial\mu}{\partial\theta}\left[\frac{1}{r}\frac{\partial V}{\partial\theta} + \frac{U}{r} - \frac{\Delta}{3}\right]$$

$$+ \frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} + \frac{1}{r}\frac{\partial W}{\partial\theta} - \frac{W\cot\theta}{r}\right],$$

$$\rho\frac{DW}{Dt} + \rho\frac{W(U+V\cot\theta)}{r} = \frac{1}{r\sin\theta}\left[\rho\frac{\partial\Omega}{\partial\phi} - \frac{\partial P}{\partial\phi}\right] + \mu\left[\nabla^2 W + \frac{1}{3r\sin\theta}\frac{\partial\Delta}{\partial\phi}\right.$$

$$\left. + \frac{2}{r^2\sin\theta}\frac{\partial U}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V}{\partial\phi} - \frac{W}{r^2\sin^2\theta}\right] \tag{5-4}$$

$$+ \frac{\partial\mu}{\partial r}\left[\frac{\partial W}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} - \frac{W}{r}\right] + \frac{1}{r}\frac{\partial\mu}{\partial\theta}\left[\frac{1}{r}\frac{\partial W}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} - \frac{W\cot\theta}{r}\right]$$

$$+ \frac{2}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial W}{\partial\phi} + \frac{U+V\cot\theta}{r} - \frac{\Delta}{3}\right]$$

where the polar velocity components U,V,W are related with the rectangular velocities u,v,w of Report I by

$$U = \dot{r} = u\cos\phi\sin\theta + v\sin\phi\sin\theta + w\cos\theta, \tag{5-5}$$

$$V = r\dot{\theta} = u\cos\phi\cos\theta + v\sin\phi\cos\theta - w\sin\theta, \tag{5-6}$$

$$W = (r\sin\theta)\dot{\phi} = -u\sin\phi + v\cos\phi; \tag{5-7}$$

the operators

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{5-8}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(V\sin\theta) + \frac{\partial W}{\partial\phi}\right\} \tag{5-9}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}; \tag{5-10}$$

and all other symbols possess the same meaning as in Section II of Report I.

In order to study the displacements governed by the foregoing system of equations, we shall hereafter assume that all three velocity-components U,V,W of viscous motion are small enough for their squares and cross-products to be negligible. Let, moreover, the pressure P, density ρ, and gravitational potential Ω characterizing the internal structure of our configuration be expressible as

$$P = P_0(r) + P'(r,\theta,\phi;t), \tag{5-11}$$

$$\rho = \rho_0(r) + \rho'(r,\theta,\phi;t), \tag{5-12}$$

$$\Omega = \Omega_0(r) + \Omega'(r,\theta,\phi;t), \tag{5-13}$$

where P_0, ρ_0, and Ω_0 refer to the respective properties of our configuration in its stationary (equilibrium) state; and P',ρ',Ω' stand for their changes brought about by motion with the velocity components U,V,W.

In the state of equilibrium (when $U = V = W = 0$), equation (5-2) reveals at once that

$$\frac{\partial P_0}{\partial r} - \rho_0 \frac{\partial \Omega_0}{\partial r} = -g\rho_0, \tag{5-14}$$

where the gravitational acceleration

$$g = G\,\frac{m(r)}{r^2} = \frac{4\pi G}{r^2}\int_0^r \rho_0 r^2 dr. \tag{5-15}$$

If, moreover, we regard the coefficient μ of viscosity to be a function of r only, and assume that the primed functions P', ρ', Ω' are--like the velocity components U, V, W --small enough for their squares and cross-products to be negligible, the fundamental equations (5-2) - (5-4) of motion reduce to their linearized forms

$$\rho \frac{\partial U}{\partial t} = \rho \frac{\partial \Omega'}{\partial r} - \frac{\partial P'}{\partial r} - g\rho' + \frac{\mu}{r}\left[\nabla^2(rU) - 2\Delta + \frac{r}{3}\frac{\partial \Delta}{\partial r}\right] + 2\frac{\partial \mu}{\partial r}\left[\frac{\partial U}{\partial r} - \frac{\Delta}{3}\right], \tag{5-16}$$

$$\rho \frac{\partial V}{\partial t} = \frac{1}{r}\left[\rho \frac{\partial \Omega'}{\partial \theta} - \frac{\partial P'}{\partial \theta}\right] + \frac{\mu}{r^2}\left[r^2\nabla^2(V) + 2\frac{\partial U}{\partial \theta} - \frac{V}{\sin^2\theta}\right.$$
$$\left. - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial W}{\partial \phi} + \frac{r}{3}\frac{\partial \Delta}{\partial \theta}\right] + \frac{\partial \mu}{\partial r}\left[\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial V}{\partial \theta} - \frac{V}{r}\right], \tag{5-17}$$

$$\rho \frac{\partial W}{\partial t} = \frac{1}{r\sin\theta}\left[\rho \frac{\partial \Omega'}{\partial \phi} - \frac{\partial P'}{\partial \phi}\right] + \frac{\mu}{r^2\sin^2\theta}\left[(r^2\sin^2\theta)\nabla^2 W + 2\frac{\partial U}{\partial \phi}\right.$$
$$\left. + 2\cos\theta\frac{\partial V}{\partial \phi} - W + \frac{1}{3}(r\sin\theta)\frac{\partial \Delta}{\partial \phi}\right] \tag{5-18}$$
$$+ \frac{\partial \mu}{\partial r}\left[\frac{\partial W}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} - \frac{W}{r}\right],$$

where we have dropped for simplicity--there should be no danger of confusion--the zero subscript of ρ_0.

Moreover, the equation (2-12) of continuity and the Poisson equation (2-13) can be similarly linearized to yield

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho}{\partial r} + \rho\Delta = 0 \tag{5-19}$$

and

$$\nabla^2\Omega' = -4\pi G\rho', \tag{5-20}$$

respectively.

The foregoing equations (5-16) - (5-20) constitute a simultaneous linear system of five relations between six dependent variables: U,V,W; P',ρ' and Ω'. In order to render this system determinate, an additional relation between them must be sought; and this can be deduced from the principle of the conservation of energy. If, for this purpose, we assume the changes in the state variables to be adiabatic, the respective equation can be shown to assume the form

$$\frac{DP}{Dt} = a^2 \frac{D\rho}{Dt} , \tag{5-21}$$

where

$$a^2 = \gamma \frac{P}{\rho} \tag{5-22}$$

denotes the square of the velocity of sound in the material characterized by a ratio γ of specific heats; and which for small changes in state can be replaced by its linearized version

$$\frac{\partial P'}{\partial t} + U \frac{\partial P_0}{\partial r} = a_0^2 \left\{ \frac{\partial \rho'}{\partial t} + U \frac{\partial \rho_0}{\partial r} \right\} , \tag{5-23}$$

which by use of (5-14) and (5-19) can be rewritten to assert that

$$\frac{\partial P'}{\partial t} = \rho(gU - a^2\Delta), \tag{5-24}$$

where zero subscripts of ρ and a^2 have likewise been omitted.

VI. SPHEROIDAL DEFORMATIONS

In order to proceed further, let us hereafter assume that the anticipated deformation of our configuration is *spheroidal*--which implies that the velocity components U,V,W are constrained to be of the form

$$U = u(r,t)Y_j^i(\theta,\phi), \tag{6-1}$$

$$V = v(r,t)\frac{\partial Y_j^i}{\partial\theta}, \tag{6-2}$$

$$W = \frac{v(r,t)}{\sin\theta}\frac{\partial Y_j^i}{\partial\phi}, \tag{6-3}$$

where $u(r,t)$ as well as $v(r,t)$ are functions of r and t only, while the $Y_j^i(\theta,\phi)$'s are surface harmonics of index i and order j, obeying the differential equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + j(j+1)Y = 0. \tag{6-4}$$

If, furthermore, we abbreviate

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) - j(j+1)\frac{v}{r} = y \tag{6-5}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{u}{r} = z, \tag{6-6}$$

it follows by insertion of (6-1) - (6-3) in (5-9) that

$$\Delta = yY_j^i; \tag{6-7}$$

while the linearized equation (5-19) of continuity will assume the form

$$\frac{\partial \rho'}{\partial t} = -\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 u) - j(j + 1) \frac{\rho v}{r} \right\} Y_j^i = -fY_j^i; \tag{6-8}$$

and the energy equation (5-24) transforms into

$$\frac{\partial P'}{\partial t} = \rho(gu - a^2 y)Y_j^i = -a^2 hY_j^i. \tag{6-9}$$

As a result

$$\frac{\partial}{\partial t} \left\{ \frac{\partial P'}{\partial r} + g\rho' \right\} = - \left\{ gf + \frac{\partial}{\partial r} (a^2 h) \right\} Y_j^i \tag{6-10}$$

and

$$\frac{\partial}{\partial t} \left(\frac{\partial P'}{\partial \theta} \right) = -a^2 h \frac{\partial Y}{\partial \theta}, \qquad \frac{\partial}{\partial t} \left(\frac{\partial P'}{\partial \phi} \right) = -a^2 h \frac{\partial Y}{\partial \phi}. \tag{6-11}$$

On the other hand, an insertion of (6-1) - (6-3) on the right-hand sides of equations (5-16) - (5-18) reveals that the viscous terms transform into

$$\frac{\mu}{r}\left\{\nabla^2(rU) - 2\Delta + \frac{r}{3}\frac{\partial\Delta}{\partial r}\right\} + 2\frac{\partial\mu}{\partial r}\left\{\frac{\partial U}{\partial r} - \frac{\Delta}{3}\right\}$$

$$= \left\{2\left[\mu\frac{\partial y}{\partial r} + \frac{\partial\mu}{\partial r}\frac{\partial u}{\partial r}\right] + \frac{j(j+1)}{r}\mu z - \frac{2}{3}\frac{\partial}{\partial r}(\mu y)\right\} Y_j^i$$

$$\equiv F(r,t)Y_j^i(\theta,\phi) \qquad (6\text{-}12)$$

while

$$\mu\left\{\nabla^2 V + \frac{2}{r^2}\frac{\partial U}{\partial\theta} - \frac{V}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial W}{\partial\phi} + \frac{1}{3r}\frac{\partial\Delta}{\partial\theta}\right\}$$

$$+ \frac{\partial\mu}{\partial r}\left\{\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial U}{\partial\theta} - \frac{V}{r}\right\} = \left\{\frac{1}{r}\frac{\partial}{\partial r}(r\mu z)\right. \qquad (6\text{-}13)$$

$$\left. + \frac{4\mu}{3r}y + \frac{2}{r}\frac{\partial\mu}{\partial r}(u-v)\right\}\frac{\partial Y}{\partial\theta} \equiv G(r,t)\frac{\partial Y}{\partial\theta}$$

and

$$\mu\left\{\nabla^2 W + \frac{2}{r^2\sin^2\theta}\frac{\partial U}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V}{\partial\phi} - \frac{W}{r^2\sin^2\theta} + \frac{1}{3r\sin\theta}\frac{\partial\Delta}{\partial\phi}\right\}$$

$$+ \frac{\partial\mu}{\partial r}\left\{\frac{\partial W}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} - \frac{W}{r}\right\} = G(r,t)\frac{\partial Y}{\partial\phi}\,. \qquad (6\text{-}14)$$

If, lastly, we set

$$\frac{\partial\Omega'}{\partial t} = R(r,t)Y_j^i(\theta,\phi), \qquad (6\text{-}15)$$

a differentiation of the first fundamental equation (5-16) of motion with respect to the time renders the latter to assume, for spheroidal deformations, the explicit form

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho \frac{\partial R}{\partial r} + \frac{\partial}{\partial r}(a^2 h) + gf + \frac{\partial F}{\partial t} ; \tag{6-16}$$

while equations (5-17) and (5-18) similarly treated can both be reduced to

$$\rho r \frac{\partial^2 v}{\partial t^2} = \rho R + a^2 h + \frac{\partial G}{\partial t} , \tag{6-17}$$

where the quantities f and h are defined by equations (6-8) and (6-9); the viscous terms F and G by (6-12) and (6-13) and R, by (6-15).

The foregoing equations (6-16) and (6-17) constitute a simultaneous set of partial differential equations for the unknown functions $u(r,t)$ and $v(r,t)$ introduced by (6-1) - (6-3). Before, however, we solve for them it is desirable to eliminate the potential function $R(r,t)$ between them; and this can be done in the following manner.

First, divide both sides of equation (6-17) by ρ, differentiate with respect to r, and then eliminate $\partial R/\partial r$ between it and equation (6-16): the outcome will assume the form

$$\frac{\partial^2}{\partial t^2}(rz) + \frac{\partial}{\partial t}\left\{ \frac{F}{\rho} - \frac{\partial}{\partial r}\left(\frac{G}{\rho}\right)\right\} + a^2 A y = 0, \tag{6-18}$$

where y and z continue to be defined by (6-5) and (6-6), and where we have abbreviated

$$A = \frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{1}{\gamma P}\frac{\partial P}{\partial r} , \tag{6-19}$$

so that, by (5-14) and (5-22),

$$a^2 A = g + \frac{a^2}{\rho} \frac{\partial \rho}{\partial r} . \tag{6-20}$$

In order to obtain a second independent relation between u and v which does not involve R, recourse must be had to the Poisson equation (5-20). Differentiating the latter with respect to the time and inserting from (6-15) we find that the radial part of $\partial\Omega'/\partial t$ should satisfy the differential equation

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r} \frac{\partial R}{\partial r} - j(j + 1) \frac{R}{r^2} = -4\pi G \frac{\partial \rho'}{\partial t} = 4\pi G f \tag{6-21}$$

by (6-8). Now multiply (6-16) by r^2/ρ and differentiate with respect to r: the result will be

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = \frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial r} (r^2 u)\right\} - \frac{\partial}{\partial r}\left\{\frac{r^2}{\rho}\left[\frac{\partial}{\partial r} (a^2 h + gf)\right]\right\} - \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial r}\left(\frac{r^2 F}{\rho}\right)\right\}, \tag{6-22}$$

which inserted in (6-21) together with (6-17) reveals that

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} = 4\pi G f &+ \frac{1}{r^2} \frac{\partial}{\partial r}\left\{\frac{r^2}{\rho} \frac{\partial}{\partial r} (a^2 h)\right\} + \frac{1}{r^2} \frac{\partial}{\partial r}\left\{\frac{r^2 g f}{\rho}\right\} \\ &- \frac{j(j+1)}{r^2} \frac{a^2 h}{\rho} + \frac{1}{r^2} \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial r}\left(\frac{r^2 F}{\rho}\right) - j(j + 1)\frac{G}{\rho}\right\} \end{aligned} \tag{6-23}$$

where, it may be remembered,

$$4\pi G\rho = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 g) \tag{6-24}$$

by (5-15).

If we eventually insert for F and G from (6-12) and (6-13), our fundamental simultaneous equations (6-18) and (6-23) for u and v will assume the more explicit forms

$$\frac{\partial^2}{\partial t^2}(rz) + a^2 Ay = \frac{1}{\rho}\frac{\partial}{\partial t}\left\{\left[\rho\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{r^2}\right](\mu rz) + 2\left(y - \frac{\partial u}{\partial r}\right)\frac{\partial \mu}{\partial r} + 2\rho\frac{\partial}{\partial r}\left(\frac{u-v}{\rho}\frac{\partial \mu}{\partial r}\right) - \frac{4}{3}\left(\frac{\mu}{\rho}\frac{\partial \rho}{\partial r}\right)y\right\} \tag{6-25}$$

and

$$\frac{\partial^2}{\partial t^2}(r^2 y) - 4\pi G r^2 f - \frac{\partial}{\partial r}\left\{\frac{r^2}{\rho}\left[\frac{\partial}{\partial r}(a^2 h) + gf\right] + j(j+1)\frac{a^2 h}{\rho}\right\} = \frac{\partial}{\partial t}\left\{\frac{4}{3}\left[\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{\rho}\right](\mu y) - 2\frac{\partial}{\partial r}\left[\frac{r^2}{\rho}\left(y - \frac{\partial u}{\partial r}\right)\frac{\partial \mu}{\partial r}\right] - \frac{j(j+1)}{\rho}\left[2(u-v)\frac{\partial \mu}{\partial r} + \frac{1}{\rho}\frac{\partial \rho}{\partial r}(\mu rz)\right]\right\}. \tag{6-26}$$

The foregoing system (6-25) - (6-26) of two simultaneous linear differential equations for u and v is evidently one of *fourth* order in t, and of *sixth* order in r. If viscosity μ were absent, the right-hand sides of both (6-25) and (6-26) would vanish identically;

and their left-hand sides equated to zero would constitute equations each of which would be of second order with respect to the time. However, (6-25) so truncated would reduce to a first-order equation with respect to r; while (6-25) would remain one of third order in the spatial variable. The appearance of even a constant viscosity μ would raise the order of (6-25) from one to three; but the order of (6-26) would remain unaltered.

VII. BOUNDARY CONDITIONS

Before we can proceed with the solution of the equations established in the preceding section, let us specify the boundary conditions which such solutions will be called upon to satisfy. Since our general system (6-25) - (6-26) has proved to be one of sixth order, six boundary conditions are obviously necessary for a complete specification of a desired particular solution; and in what follows we shall enumerate these in turn.

First, the obvious requirement that there be no displacement at the center necessitates that

$$u(0,t) = v(0,t) = 0 \tag{7-1}$$

at all times.

Next, let us require that there be no variation in pressure at the center $(r = 0)$ and over the distorted surface $(r = r_*)$. The vanishing of $\partial P'/\partial t$ as given by (6-9) at the center will obviously be fulfilled if, in addition to $u(0,t) = 0$, also

$$\left(\frac{\partial u}{\partial r}\right)_0 = 0; \tag{7-2}$$

while on the surface, where

$$\rho(r_*) = P(r_*) = 0, \tag{7-3}$$

the vanishing on the right-hand side of equation (6-9) is automatically satisfied provided only that $u(r_*)$ remains bounded.

It is of interest to note that if, in addition to (7-2),

$$\lim_{r\to 0} \frac{v}{r} = \left(\frac{dv}{dr}\right)_0 = 0 \tag{7-4}$$

the linearized equation (6-8) of continuity ensures that there will also be no variation of density at the center as well.

At the boundary $r = r_*$ of a self-gravitating configuration of viscous fluid we may require the vanishing of the radial viscous stress components

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0. \tag{7-5}$$

The explicit forms of the six components σ_{ij} of viscous stresses in Cartesian coordinates have already been given by equations (2-5) - (2-10) of Report I. Rewriting these in spherical polar coordinates we find that

$$\sigma_{rr} = \mu \frac{\partial U}{\partial r} = \mu \left(\frac{\partial u}{\partial r}\right) Y^i_j, \tag{7-6}$$

$$\sigma_{r\theta} = \mu \left\{ \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} \right\} = \mu \left\{ \frac{\partial v}{\partial r} + \frac{u-v}{r} \right\} \frac{\partial Y}{\partial \theta}, \tag{7-7}$$

$$\begin{aligned} \sigma_{r\phi} &= \mu \left\{ \frac{1}{r \sin\theta}\frac{\partial U}{\partial \phi} + \frac{\partial W}{\partial r} - \frac{W}{r} \right\} \\ &= \mu \left\{ \frac{\partial v}{\partial r} + \frac{u-v}{r} \right\} \frac{1}{\sin\theta}\frac{\partial Y}{\partial \phi} \end{aligned} \tag{7-8}$$

by (6-1) - (6-3); so that all three radial components of the viscous stress tensor will vanish on the surface provided that

$$\left(\frac{\partial u}{\partial r}\right) = 0 \tag{7-9}$$

and

$$\frac{\partial v}{\partial r} = \frac{v-u}{r} \tag{7-10}$$

for $r = r_*$.

The last type of the boundary conditions which we must investigate consists of the requirement that the total gravitational potential and its normal derivative (i.e., gravitational acceleration) must be continuous across the free boundary of the distorted configuration; and this can be enforced in the following manner.

Let the total potential W of all forces acting upon any point of our configuration be expressed as the sum

$$W = \Omega + V_T, \tag{7-11}$$

where Ω denotes (as before) the potential arising from the mass of the respective body, and

$$V_T = \sum_{i,j} C_{i,j}(t) r^j Y_j^i(\theta,\phi) \tag{7-12}$$

stands for a potential of the forces causing the distortion (such as the attraction of external masses, for instance), specified by a set of the functions $C_{i,j}(t)$.

The total potential W must obviously satisfy the Poisson equation

$$\nabla^2 W = -4\pi G\rho. \tag{7-13}$$

If, moreover, the potential Ω arising from the mass and the internal distribution of density ρ are likewise expansible in the form

$$\Omega = \sum_{i,j} \mathcal{P}_{i,j}(r,t) Y_j^i(\theta,\phi) \tag{7-14}$$

and

$$\rho = \sum_{i,j} \mathcal{G}_{i,j}(r,t) Y_j^i(\theta,\phi), \tag{7-15}$$

where the Y_j^i's are surface harmonics satisfying equation (6-4) and $\mathcal{P}_{i,j}(r,t)$, $\mathcal{G}_{i,j}(r,t)$ are functions as yet unspecified, it follows from (7-14) and (7-15) that equation (7-13) can be expressed more explicitly in the form

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} \mathcal{P}_{i,j} = -4\pi G \mathcal{G}_{i,j}. \tag{7-16}$$

This linear nonhomogeneous equation for $\mathcal{P}$ can, in turn, be shown (by standard methods) to admit of the solution

$$\begin{aligned} \mathcal{P}_{i,j}(r,t) = C_{i,j}(t) r^j + \frac{4\pi G}{2j+1} \Bigg\{ & \frac{1}{r^{j+1}} \int_0^r \mathcal{G}_{i,j}(r,t) r^{j+2} dr \\ & + r^j \int_r^\infty \mathcal{G}_{i,j}(r,t) r^{1-j} dr \Bigg\}, \end{aligned} \tag{7-17}$$

the particular integral of which consists of a sum of the interior and exterior potential arising from the mass itself, and the complementary function represents the disturbing potential (7-12). The reader may note that, for $j = 0$ (i.e., in the case of spherical symmetry), the foregoing equation (7-17) reduces to

$$\mathfrak{X}_0 = \frac{4\pi G}{r}\int_0^r \rho r^2 dr + 4\pi G\int_r^\infty \rho r dr + \text{a constant}, \tag{7-18}$$

which makes the nature of this solution obvious. Moreover, the infinite upper limit in the second interval on the right-hand side of (7-17) or (7-18) can be replaced by the mean radius r_* of the respective configuration without altering the value of the respective integral; since

$$\rho(r \geq r_*) = 0.$$

Let us differentiate now the equation (7-17) with respect to r: in doing so we find that

$$\begin{aligned}\frac{\partial \mathfrak{X}}{\partial r} = jCr^{j-1} + \frac{4\pi G}{2j+1}\Bigg\{ &-\frac{j+1}{r^{j+2}}\int_0^r \mathcal{G} r^{j+2} dr \\ &+ jr^{j-1}\int_r^{r_*} \mathcal{G} r^{1-j} dr\Bigg\},\end{aligned} \tag{7-19}$$

which combined with (7-17) yields

$$\frac{\partial \mathfrak{X}}{\partial r} + \frac{j+1}{r} = (2j+1)Cr^{j-1} + 4\pi G r^j \int_r^{r_*} \mathcal{G} r^{1-j} dr \tag{7-20}$$

for any value of i and j; and at the surface of our configuration (i.e., for $r = r_*$) equation (7-20) reduces further to

$$\left\{ \frac{\partial \mathcal{Q}}{\partial r} + \frac{j+1}{r} \mathcal{Q} \right\}_{r=r_*} = (2j+1)Cr_*^{j-1}. \qquad (7\text{-}21)$$

Let us next differentiate the foregoing relation with respect to the time. Since, by a comparison of (7-14) with (6-15) it follows that

$$\frac{\partial}{\partial t} \mathcal{Q}(r,t) = R(r,t), \qquad (7\text{-}22)$$

the boundary condition (7-21) can at once be rewritten to assert that

$$\left\{ \frac{\partial}{\partial r} R(r,t) \right\}_{r=r_*} + \frac{j+1}{r_*} R(r_*,t) = (2j+1) \frac{\partial C_j}{\partial t} r_*^{j-1}, \qquad (7\text{-}23)$$

where the surface values of R and $\partial R/\partial r$ can be inserted from (6-16) and (6-17) in terms of the local values of u and v which, in turn, are constrained by the condition (7-5) to satisfy the equations (7-9) and (7-10).

The question can be raised, in this connection, as to the boundary condition which the function $R(r,t)$ must satisfy at the center $r = 0$ of our configuration. In order to ascertain its form, let us return to equation (7-17), divide its both sides by r^j, and differentiate with respect to r: the outcome of these operations discloses that

$$\frac{\partial}{\partial r}\left(\frac{\mathcal{Q}}{r^j} \right) = -\frac{4\pi G}{r^{2j+2}} \int_0^r \mathcal{G} r^{j+2} dr. \qquad (7\text{-}24)$$

Differentiate now again both sides of this equation with respect to t: inasmuch as

$$\frac{\partial \mathcal{G}}{\partial t} = -f(r,t), \tag{7-25}$$

where the function f has already been defined in terms of u and v by equation (6-8), it follows from (7-22) and (7-24) - (7-25) that

$$r\frac{\partial R}{\partial r} - jR = \frac{4\pi G}{r^{j+1}} \int_0^r f(r,t) r^{j+2} dr. \tag{7-26}$$

Since

$$\lim_{r \to 0} \frac{1}{r^{j+1}} \int_0^r f(r,t) r^{j+2} dr = \lim_{r \to 0} \frac{fr^2}{j+1} = 0, \tag{7-27}$$

it follows from (7-26) that

$$R(0,t) = 0, \tag{7-28}$$

which together with (7-23) represents the boundary conditions imposed on the acceptable solutions of equation (6-21).

The reader may note that, in the absence of any external disturbing forces which depend on the time, the boundary conditions (7-23) as well as (7-28) are homogeneous in R, just as equation (6-21) itself is homogeneous in its dependent variables. Should this be the case, the solutions of our equations of Section VI would correspond, e.g., to *free* spheroidal oscillations of compressible self-gravitating configurations of viscous fluid with air *arbitrary* amplitude. However, the physical librations of the Moon (or the tidal phenomena in general) belong to the category of *forced* oscillations, with amplitudes governed by disturbing forces of external origin, the magnitude of which depends on the time. Since the solution of our problem of lunar librations depends on the nature of the attraction exerted on the globe of the Moon by the Earth and the Sun, as determined by the disturbing potential V_T, the time dependence of the quantities $C_{i,j}(t)$ in (7-12) will have to be investigated. Before doing so we wish, however, to consider a possibility of resonance between free and forced oscillations in the Earth-Moon system; and to this task we shall address ourselves in the next three sections of this report.

VIII. SOLUTION FOR HOMOGENEOUS BODIES: FREE OSCILLATIONS

In Sections V - VII the equations have been set up which should govern spheroidal oscillations--free or forced--of self-gravitating configurations of viscous fluid, of which physical librations of the Moon will constitute a particular case. The aim of the present section will be to particularize our problem in one more respect to facilitate applications to the Moon: namely, in taking advantage of the fact that a self-gravitating globe of so small a mass can be expected to be so nearly homogeneous that $\rho_0(r)$ in its interior can, to a good approximation, be regarded as constant. The physical reasons why this should be so have been explained elsewhere (cf., e.g., Kopal, 1966; Chapter VII) and need not be repeated in this place. They entitle us, however, to conclude that, throughout most part of the lunar interior, the assumptions ρ_0 = constant and (though to a lesser degree) μ = constant should represent good approximations. Moreover, their adoption will prove to simplify the solution of our equations of Section VI to such an extent that the underlying problem can be solved in a closed form in terms of certain hypergeometric series which we shall now proceed to construct.

To begin with, let us change over from the physical independent variables r,t of our problem to the corresponding nondimensional variables x,τ defined by the equations

$$\left.\begin{aligned} r &= r_* x, \\ t &= \frac{\tau}{\sqrt{2\pi G\rho_0}}, \end{aligned}\right\} \tag{8-1}$$

normalized so that

$$0 \leq x \leq 1 \tag{8-2}$$

between the center and the surface of our configuration. If so, then for constant ρ_0 equation (5-14) can be readily integrated to

$$P_0 = \frac{2}{3}\pi G\rho_0^2 r_*^2 (1-x^2), \tag{8-3}$$

while (5-15) yields

$$g = \frac{4}{3}\pi G\rho_0 r_* x \tag{8-4}$$

and from (5-22) combined with (8-3),

$$a^2 = \frac{2}{3}\pi G\rho_0 r_* \gamma(1-x^2). \tag{8-5}$$

Moreover, in accordance with (6-19),

$$A = \frac{2x}{\gamma r_*(1-x^2)}\,, \tag{8-6}$$

so that (6-20) assumes the form

$$a^2 A = \frac{4}{3}\pi G\rho_0 r_* x; \tag{8-7}$$

and, lastly, by (6-8) and (6-9)

$$gf = \frac{4}{3}\pi G\rho_0^2 x\tilde{y} \tag{8-8}$$

and

$$a^2 h = \frac{2}{3}\pi G\rho_0^2 r_*\{\gamma(1-x^2)\tilde{y} - 2xu\}, \tag{8-9}$$

where we have abbreviated

$$\tilde{y} = r_* y \quad \text{and} \quad \tilde{z} = r_* z. \tag{8-10}$$

If so, the system (6-25) - (6-26) of our fundamental equations of motion can, for constant μ, be rewritten as

$$\frac{3}{2}\frac{\partial^2\tilde{y}}{\partial\tau^2} = 3\tilde{y} + \frac{1}{x^2}\frac{\partial}{\partial x}(x^3\tilde{y}) + \left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\frac{\gamma}{2}(1-x^2)\tilde{y} - xu\right]$$
$$+ \tilde{\mu}\frac{\partial}{\partial\tau}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\tilde{y}, \tag{8-11}$$

and

$$\frac{3}{2}\frac{\partial^2\tilde{z}}{\partial\tau^2} + \tilde{y} = \frac{3}{4}\tilde{\mu}\frac{\partial}{\partial\tau}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\tilde{z}, \tag{8-12}$$

where

$$\tilde{\mu} = \frac{2\mu}{\rho_0 r_*^2\sqrt{2\pi G\rho_0}} \tag{8-13}$$

stands for a nondimensional constant parameter proportional to the viscosity.

Let us disregard at first the terms factored by $\tilde{\mu}$ in the foregoing equations (8-11) - (8-12) and--anticipating the motion governed by them to be harmonic--set

$$\frac{\partial^2}{\partial\tau^2} = -\tilde{\nu}^2, \tag{8-14}$$

where $\tilde{\nu}$ stands for the (normalized) frequency of the respective motion. If so, the system (8-11) - (8-12) obviously reduces to

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right\}\left[\frac{\gamma}{2}(1-x^2)\tilde{y} - xu\right] + x\frac{\partial\tilde{y}}{\partial x} + (\tilde{\nu}^2 + 6)\tilde{y} = 0 \tag{8-15}$$

and

$$\tilde{y} = \tilde{\nu}^2\tilde{z}, \tag{8-16}$$

respectively.

The foregoing equations (8-15) and (8-16) constitute a fourth-order simultaneous system for the velocity components u and v. However, in this particular case of a homogeneous configuration, their integration can be split up in two stages. First, let us note that, by virtue of equations (6-5) - (6-6) and (8-16),

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{\partial x^2}\right\}(xu) = \frac{1}{x}\frac{\partial}{\partial x}(x^2\tilde{y}) + \frac{j(j+1)}{\tilde{\nu}^2}\tilde{y}, \qquad (8\text{-}17)$$

which on insertion in (8-15) yields

$$x^2(1 - x^2)\frac{\partial^2\tilde{y}}{\partial x^2} + 2x(1 - 3x^2)\frac{\partial\tilde{y}}{\partial x} + [Kx^2 - j(j+1)]\tilde{y} = 0 \qquad (8\text{-}18)$$

where we have abbreviated

$$K = \frac{2}{\gamma}\left\{\tilde{\nu}^2 + 4 - \frac{j(j+1)}{\tilde{\nu}^2}\right\} + (j - 2)(j + 3). \qquad (8\text{-}19)$$

The foregoing relation (8-17) constitutes a second-order differential equation for $\tilde{y}$ which can be integrated as it stands; and once its solution has been obtained, the velocity components u and v can be solved for from the equations

$$\tilde{y} = \frac{\partial u}{\partial x} + \frac{2u}{x} - j(j + 1)\frac{v}{x} = \tilde{\nu}^2\left|\frac{\partial v}{\partial x} + \frac{v-u}{x}\right| \qquad (8\text{-}20)$$

resulting from (8-10) and (8-16).

Let us, however, return now to equation (8-18). If we substitute $x^2 = \xi$, the latter can be rewritten as

$$\xi^2(\xi - 1)\frac{\partial^2\tilde{y}}{\partial\xi^2} + \frac{7\xi-3}{2}\xi\frac{\partial\tilde{y}}{\partial\xi} + \frac{1}{4}\{j(j+1) - K\xi\}\tilde{y} = 0, \qquad (8\text{-}21)$$

and its complete primitive can be expressed as a linear combination of two hypergeometric series of the form

$$\tilde{y} = Ax^{j}F(a,b,c,x^{2}) + Bx^{-j-1}F(a-c+1,b-c+1,2-c,x^{2}), \tag{8-22}$$

where A,B are arbitrary integration constants, and

$$\left.\begin{aligned} a &= \tfrac{1}{4}[2j + 5 \pm \sqrt{25+4K}], \\ b &= \tfrac{1}{4}[2j + 5 \mp \sqrt{25+4K}], \\ c &= j + \frac{3}{2}. \end{aligned}\right\} \tag{8-23}$$

The finiteness of $\tilde{y}$ as given by equation (8-22) at the origin obviously requires that $B = 0$. Moreover, inasmuch as, by (8-23),

$$a + b - c = 1, \tag{8-24}$$

an application of standard tests for the convergence of hypergeometric series discloses that both series on the right-hand side of (8-22) diverge for $x = 1$. The solution (8-22) for $\tilde{y}$ expressed in their terms can, therefore, remain finite at the boundary only if the respective hypergeometric series are made to *terminate* by setting (say)

$$\tfrac{1}{4}\{2j + 5 \mp \sqrt{25+4K}\} = -k, \tag{8-25}$$

where k stands for an arbitary positive integer. Since, however, then

$$a = j + k + \frac{5}{2} \quad \text{and} \quad b = -k, \tag{8-26}$$

the particular solution of (8-21) which remain finite for $0 \leq x \leq 1$ assumes the form

$$\begin{aligned} \tilde{y} &= A_{j,k} x^j F(j + k + \tfrac{5}{2}, -k, j + \tfrac{3}{2}, x^2) \\ &= A_{j,k} x^j G_k(j + \tfrac{5}{2}, j + \tfrac{3}{2}, x^2) \end{aligned} \tag{8-27}$$

where G_k denotes the corresponding Jacobi polynomial of degree k.

Moreover, equation (8-25) implies that

$$(2j + 4k + 5)^2 = 25 + 4K, \tag{8-28}$$

which on insertion for K from (8-19) and after some rearrangement of terms assumes the form

$$\tilde{\nu}^2 + 4 - \frac{j(j+1)}{\tilde{\nu}^2} = \gamma(k + 1)(2j + 2k + 3); \tag{8-29}$$

and the frequency of the corresponding oscillation will be given by

$$\tilde{\nu}^2 = \omega \pm \sqrt{\omega^2 + j(j+1)} \tag{8-30}$$

where we have abbreviated

$$\omega = \gamma(k + 1)(j + k + \tfrac{3}{2}) - 2. \tag{8-31}$$

For $j > 0$, one of the conjugate roots of (8-29) is bound to be negative--implying dynamical instability. If ω (i.e., k) is large, the conjugate roots (8-30) will be led by the terms

$$\tilde{\nu}^2 = 2\omega + \frac{j(j+1)}{2\omega} + \cdots \quad \text{or} \quad - \frac{j(j+1)}{2\omega} + \cdots. \tag{8-32}$$

Thus, for any given j, the requirement that $\tilde{y}$ be finite throughout the interval $0 \leq x \leq 1$ leads to two types of the spectra: one consisting of positive eigenvalues tending towards infinity as k increases; the other of negative eigenvalues tending towards zero.

With the explicit form of $\tilde{y}$ as given by (8-27), the equations (8-20) for u and v assume the explicit form

$$\begin{aligned} \frac{1}{x^2}\left\{ \frac{\partial}{\partial x}(x^2 u) - j(j+1)xv \right\} &= \frac{\tilde{\nu}^2}{x}\left\{ \frac{\partial}{\partial x}(xv) - u \right\} \\ &= A_{j,k}x^j F(a,b,c,x^2), \end{aligned} \tag{8-33}$$

where, as before,

$$\left.\begin{aligned} a &= j + \tfrac{5}{2} + k, \\ b &= -k, \\ c &= j + \tfrac{3}{2}; \end{aligned}\right\} \tag{8-34}$$

and can be integrated to furnish polynomial solutions for u and v in the following manner.

First, we note that an elimination of xv between the first two parts of the equation (8-33) yields a relation of the form

$$\frac{\partial^2}{\partial x^2}(x^2 u) - j(j+1)u = A_{j,k}\left\{\frac{\partial}{\partial x}(x^{j+2}F) + \frac{j(j+1)}{\tilde{\nu}^2}x^{j+1}F\right\}, \qquad (8\text{-}35)$$

with $F \equiv F(a,b,c,x^2)$; and its particular integral which remains finite at the origin becomes

$$u_{j,k}(x) = \frac{A_{j,k}}{2j+1}\left\{j\left[1+\frac{j+1}{\tilde{\nu}^2}\right]x^{j-1}\int_0^x xFdx \right.$$
$$\left. + (j+1)\left[1-\frac{j}{\tilde{\nu}^2}\right]x^{-j-2}\int_0^x x^{2(j+1)}Fdx\right\}. \qquad (8\text{-}36)$$

Since, moreover,

$$xF(a,b,c,x^2) = \frac{c-1}{2(a-1)(b-1)}\frac{\partial}{\partial x}F(a-1,b-1,c-1,x^2) \qquad (8\text{-}37)$$

and

$$(2j+3)x^{2(j+1)}F(a,b,c,x^2) = \frac{\partial}{\partial x}\left\{x^{2j+3}F(a,b,c+1,x^2)\right\} \qquad (8\text{-}38)$$

the integrals on the right-hand side of (8-36) can be evaluated in a closed form to yield

$$u_{j,k}(x) = \frac{A_{j,k}x^{j-1}}{4(a-1)(b-1)\tilde{\nu}^2}\left\{(j+1)(j-\tilde{\nu}^2)F(a-1,b-1,c,x^2)\right.$$
$$\left. + 2(c-1)\tilde{\nu}^2F(a-1,b-1,c-1,x^2) - j(j+\tilde{\nu}^2+1)\right\} \qquad (8\text{-}39)$$

where advantage has been taken of the identity

$$x^2F(a,b,c+1,x^2) = \frac{c(c-1)}{(a-1)(b-1)}\left\{F(a-1,b-1,c-1,x^2) - F(a-1,b-1,c,x^2)\right\}. \tag{8-40}$$

The arguments $b - 1 = -(k + 1)$ in both hypergeometric series on the right-hand side of (8-39) are negative integers; hence, both series are terminating and can be expressed in terms of the Jacobi polynomials

$$F(a-1,b-1,c,x^2) \equiv G_{k+1}(j + \tfrac{1}{2}, j + \tfrac{3}{2}, x^2) \tag{8-41}$$

and

$$F(a-1,b-1,c-1,x^2) \equiv G_{k+1}(j + \tfrac{1}{2}, j + \tfrac{1}{2}, x^2) \tag{8-42}$$

of degrees $2(k + 1)$ in x. Moreover, inasmuch as

$$(j + 1)(j - \tilde{\nu}^2) + 2(c - 1) = j(j + \tilde{\nu}^2 + 1), \tag{8-43}$$

the leading term of the polynomial expression (8-39) will be

$$u_{j,0}(x) = A_{j,0}\left\{\frac{j(j+1) + (j+2)\tilde{\nu}^2}{2(2j+3)\tilde{\nu}^2}\right\}x^{j+1}. \tag{8-44}$$

With a polynomial solution for u thus established, that for v follows algebraically from an inequality of the first and third term of the equation (8-33), disclosing that

$$j(j + 1)v = \frac{1}{x}\frac{\partial}{\partial x}(x^2u) - x^{j+1}F(a,b,c,x^2). \tag{8-45}$$

Inserting in (8-45) for u from (8-39) we find that

$$v_{j,k}(x) = \frac{A_{j,k}x^{j-1}}{4(a-1)(b-1)\tilde{\nu}^2}\Big\{2(c-1)F(a-1,b-1,c-1,x^2) + (\tilde{\nu}^2 - j)F(a-1,b-1,c,x^2) - (j + \tilde{\nu}^2 + 1)\Big\} \tag{8-46}$$

which for $k = 0$ reduces again to

$$v_{j,0}(x) = A_{j,0}\left\{\frac{j+\tilde{\nu}^2+3}{2(2j+3)\tilde{\nu}^2}\right\} x^{j+1}. \tag{8-47}$$

The foregoing equations (8-27) with (8-39) and (8-46) represent closed polynomial solutions for free nonradial oscillations of order j and mode k of a homogeneous configuration of compressible inviscid fluid, with characteristic frequencies $\tilde{\nu}$ as given by equations (8-29) or (3-30). By virtue of (8-14), these solutions are to be multiplied by a harmonic time-factor $\exp i(\tilde{\nu}_{j,k})$ of the respective oscillation; and their arbitrary linear combination

$$u(x,\tau) = \sum_{j,k} u_{j,k}(x)e^{i(\tilde{\nu}_{j,k})\tau} \tag{8-48}$$

or

$$v(x,\tau) = \sum_{j,k} v_{j,k}(x)e^{i(\tilde{\nu}_{j,k})\tau} \tag{8-49}$$

then represent complete solutions of the problem set forth in this section.

IX. THE EFFECTS OF VISCOSITY

Throughout most part of the preceding section we have regarded our configuration to consist of inviscid fluid--a simplifying assumption which enabled us temporarily to ignore the effects of the operator $\tilde{\mu}(\partial/\partial t)$ on the right-hand sides of equations (8-11) and (8-12). The aim of the present section will be now to restore these terms and proceed to construct such solutions of equations (8-11) and (8-12) which take the effects of constant viscosity duly into account.

The principal new feature of this problem which distinguishes it from the one treated in the preceding section is the fact that, for harmonic motion,

$$\frac{\partial}{\partial t} = i\tilde{\nu}; \tag{9-1}$$

in consequence of which the right-hand sides of equations (8-11) and (8-12) become complex; and their solutions must be sought in terms of complex velocity components $u(x,\tau)$ and $v(x,\tau)$ of the form

$$u(x,\tau) = \{u_1(x) + i\tilde{\mu}u_2(x)\}e^{(\kappa+i\lambda)\tau} \tag{9-2}$$

and

$$v(x,\tau) = \{v_1(x) + i\tilde{\mu}v_2(x)\}e^{(\kappa+i\lambda)\tau} \tag{9-3}$$

where $u_{1,2}(x)$, $v_{1,2}(x)$ are real functions of x and κ,λ are real constants. As $\tilde{\mu} \to 0$, the functions $u_1(x)$ and $v_1(x)$ should, moreover, tend to the limits represented by equations (8-34) and (8-35); but the new functions $u_2(x)$ and $v_2(x)$ arising when $\tilde{\mu} \neq 0$ remain yet to be defined and solved for. This can be done by a method initiated by the present writer (cf. Kopal, 1964) for the case of purely radial oscillations $(j = 0)$, and extended by Moutsoulas (1967) to nonradial oscillations, which are of primary interest to us in the present report, in a manner to be developed in this section.

In embarking on this task, consider first the complex frequency $\kappa + i\lambda$ of harmonic oscillations in (9-2) and (9-3), the real part of which represents the rate of damping of the oscillatory motion, due to a viscous dissipation of kinetic energy into heat. As, however, the viscous dissipation function is known to be homogeneous and *quadratic* in the velocity components u and v, its magnitude becomes ignorable within the framework of our linearized theory initiated in Section V; and, hence, consistent with our scheme of approximation κ can hereafter be neglected.

Next, as a consequence of (9-2) and (9-3), the normalized quantities $\tilde{y}$ and $\tilde{z}$ should likewise be expressible as complex functions

$$\tilde{y}(x,\tau) = \{y_1(x) + i\tilde{\mu}y_2(x)\}e^{i\lambda\tau} \tag{9-4}$$

$$\tilde{z}(x,\tau) = \{z_1(x) + i\tilde{\mu}z_2(x)\}e^{i\lambda\tau}, \tag{9-5}$$

where, in accordance with (6-5) - (6-6) and (8-10),

$$xy_{1,2} = x\,\frac{\partial u_{1,2}}{\partial x} + 2u_{1,2} - j(j+1)v_{1,2} \tag{9-6}$$

and

$$xz_{1,2} = x\frac{\partial v_{1,2}}{\partial x} + v_{1,2} - u_{1,2}. \tag{9-7}$$

If we insert the foregoing complex expressions for u, v, and $\tilde{y}$ in (8-11), and separate the real and imaginary parts we find that (8-11) splits up into the following symmetrical pair of simultaneous differential equations

$$3(4+\lambda^2)y_{1,2} + 2x\frac{\partial y_{1,2}}{\partial x} + \left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\gamma(1-x^2)y_{1,2} - 2xu_{1,2}\right]$$

$$\pm 2\lambda\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]y_{2,1}. \tag{9-8}$$

Equations (9-6) and (9-7) reveal that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](xu_{1,2}) = \frac{1}{x}\frac{\partial}{\partial x}(x^2y_{1,2}) + j(j+1)z_{1,2} \tag{9-9}$$

and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](1-x^2)y_{1,2} =$$

$$= \left[(1-x^2)\frac{\partial^2}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial}{\partial x} - \frac{j(j+1)(1-x^2)}{x^2} - 6\right]y_{1,2}, \tag{9-10}$$

which inserted in (9-8) permit us to rewrite the latter as

$$(1-x^2)\frac{\partial^2 y_{1,2}}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_{1,2}}{\partial x} + \left\{\frac{2}{\gamma}(4+\frac{3}{2}\lambda^2) + (j-2)(j+3) - \frac{j(j+1)}{x^2}\right\}y_{1,2} =$$

$$= \frac{2}{\gamma}j(j+1)z_{1,2} \pm 2\frac{\lambda}{\gamma}\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]y_{2,1} \tag{9-11}$$

constituting two relations between the unknown functions $y_{1,2}$ and $z_{1,2}$. The remaining two can, in turn, be obtained by an insertion of (9-5) in (8-12): in doing so and separating the real and imaginary parts we find them to assume the forms

$$\frac{2}{3}\, y_{1,2} = \lambda^2 z_{1,2} \mp \frac{1}{2}\, \lambda \tilde{\mu}^{2,0} \left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2} \right] z_{2,1}, \qquad (9\text{-}12)$$

which together with (9-11) represent the fundamental system of differential equations governing the oscillations of self-gravitating configurations of viscous fluid; for once these have been solved for $y_{1,2}$ and $z_{1,2}$, the corresponding velocity components $u_{1,2}$ and $v_{1,2}$ can be determined from (9-6) - (9-7).

It is of interest to note that these equations are quadratic in $\tilde{\mu}$ -- a fact which underlines the importance of the terms introduced by viscosity. If $\tilde{\mu} \to 0$, only the functions with subscript 1 remain relevant to the problem; and a combination of equations (9-11) and (9-12) reduces then to $y_1 \equiv \tilde{y}$ and $z_1 \equiv \tilde{\nu}^{-2} y$, with y as previously given by (8-18), and $\lambda \equiv \tilde{\nu}$. If, on the other hand, $\tilde{\mu} \to \infty$, the relevant solutions for y_2 and z_2 which remain finite at the origin will both vary as x^j and can differ only in their multiplicative constants; the same being true of u_2 and v_2 which vary as x^{j+1}.

For finite values of $\tilde{\mu}$, equations (9-11) - (9-12) must be treated as a simultaneous system governing four unknown functions $y_{1,2}(x)$ and $z_{1,2}(x)$; a solution of which may also be approached by successive approximations. For insert for $z_{1,2}$ from (9-12) in (9-11): the outcome discloses that

$$(1 - x^2)\,\frac{\partial^2 y_{1,2}}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_{1,2}}{\partial x} + \left\{K - \frac{j(j+1)}{x^2}\right\} y_{1,2} =$$
$$= \pm\, 2\,\frac{\lambda}{\gamma}\,\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\frac{2}{4}\, j(j+1) z_{1,2} + y_{1,2}\right] \tag{9-13}$$

where the constant K continues to be given by (8-19). In consequence, the relation

$$(1 - x^2)\,\frac{\partial^2 y_2}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_2}{\partial x} + \left\{K - \frac{j(j+1)}{x^2}\right\} y_2 =$$
$$= -2\,\frac{\lambda}{\gamma}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\frac{2}{4}\, j(j+1) z_1 + y_1\right] \tag{9-14}$$

becomes independent of viscosity; and the operator on its left-hand side is identical with that of equation (8-18). If the right-hand side of the foregoing equation (9-14) we set equal to zero, the complete primitive of the homogeneous equation for y_2 would indeed be of the form (8-18). If, moreover, we approximate the functions on the right-hand side of (9-14) by polynomial expressions of the form (8-27), the particular integral of the complete nonhomogeneous equation (9-14) for $(y_2)_0$ can be obtained by standard method; and the corresponding expression for $(z_2)_0$ then follows from (9-12) algebraically as

$$(z_2)_0 = \frac{2}{3\lambda^2}\,(y_2)_0 - \frac{1}{2\lambda}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](z_1)_0, \tag{9-15}$$

where, by (8-16), $(z_1)_0 = \lambda^{-2}(y_1)_0$.

As the next step of our approximation procedure, we revert to the second one of the equations (9-11) in the form

$$(1 - x^2)\frac{\partial^2 y_1}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_1}{\partial x} + \left\{K - \frac{j(j+1)}{x^2}\right\}y_1 =$$

$$= 2\frac{\lambda}{\gamma}\tilde{\mu}^2\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\frac{2}{4}\, j(j+1)z_2 + y_2\right], \tag{9-16}$$

where y_2 and z_2 on the right-hand side can be inserted from the solutions of (9-14) and (9-15); and (9-16) regarded as a nonhomogeneous equation for a second approximation to $(y_1)_1$; and

$$(z_1)_1 = \frac{2}{3\lambda^2}(y_1)_1 + \frac{\tilde{\mu}^2}{2\lambda}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](z_2)_0. \tag{9-17}$$

The same procedure can obviously be continued until the expressions for $(y_{1,2})_{i+1}$ or $(z_{1,2})_{i+1}$ cease to differ from $(y_{1,2})_i$ or $(z_{1,2})_i$ by significant amounts.

Suppose that such stabilized solutions for $y_{1,2}$ and $z_{1,2}$ have been established; and from these the velocity components $u_{1,2}$ and $v_{1,2}$ with the aid of (9-2) and (9-3). The complex velocity components $u(x,\tau)$ and $v(x,\tau)$ then follow by insertion of $u_{1,2}$ and $v_{1,2}$ in (9-2) - (9-3); and their *real* parts--which are of interest in connection with our physical problem--should be of the form

$$\mathrm{Re}[u(x,\tau)] = e^{\kappa\tau}\sqrt{u_1^2+u_2^2}\cos\left\{\lambda\tau + \tan^{-1}\tilde{\mu}(u_2/u_1)\right\} \tag{9-18}$$

and, similarly,

$$\mathrm{Re}[v(x,\tau)] = e^{\kappa\tau}\sqrt{v_1^2+v_2^2}\cos\left\{\lambda\tau + \tan^{-1}\tilde{\mu}(v_2/v_1)\right\}, \tag{9-19}$$

whose amplitudes

$$\sqrt{u_1^2(x) + u_2^2(x)}, \qquad \sqrt{v_1^2(x) + v_2^2(x)}$$

are identical with the moduli of the respective complex velocity components, while the angles

$$\varepsilon_u = \tan^{-1}\tilde{\mu}(u_2/u_1) \tag{9-20}$$

and

$$\varepsilon_v = \tan^{-1}\tilde{\mu}(v_2/v_1) \tag{9-21}$$

represent their phase lag.

It is evident from the expressions (9-18) and (9-19) that the phase with which a self-gravitating configuration of a viscous fluid can oscillate--freely or in response to an external field of force--is bound to vary between the center and the surface of our configuration. The respective motion represents, therefore, a travelling rather than a standing wave. Only inviscid configurations can oscillate with constant phase; and the latter is bound to become a function of r as soon as $\tilde{\mu} > 0$. Moreover, our results make it evident that the amount by which the oscillations--free or forced--will lag in phase may be different for different velocity components. In particular, in the case of spheroidal symmetry represented by equations (6-1) - (6-3), the phase lag in the radial velocity component U will be different from that in the angular velocity components V and W.

In conclusion of the present section the following question can be asked how large is the nondimensional viscosity parameter $\tilde{\mu}$ likely to be for the Moon? Since, in this case, our unit of length r_* (identical with the Moon's mean radius) is equal to 1738 km = 1.738×10^8cm; and our unit of the time $(2\pi G\rho_0)^{-\frac{1}{2}} = 845$ sec for $\rho_0 = 3.34$ g/cm^3 and $G = 6.68 \times 10^{-8}$cm^3/g sec^2, it follows from (8-13) that

$$\tilde{\mu} = 1.68 \times 10^{-14}\mu, \tag{9-22}$$

where μ is expressed in g/cm sec. This magnitude of $\tilde{\mu}$ makes it obvious that for $\mu << 10^{14}$g/cm sec, the quantity $\tilde{\mu}$ can be, throughout Section IX, regarded as a small parameter, and its effects treated as perturbations of the inviscid case discussed in Section VIII. If, however, $\mu >> 10^{15}$g/cm sec, the viscosity effects in our equations of motion will tend to become dominant; and their solutions should asymptotically approach their limiting forms $u_{1,2}(x) \sim v_{1,2}(x) \sim x^{j+1}$. In such a case, the angles ε of phase lag will tend to become constant and the dynamical behavior of our configuration will approach--as it should--that of a rigid body. In such a case, it is the reciprocal of $\tilde{\mu}$ which can be treated as a small parameter in our equations of motion; and the perturbations caused by it will measure the departures of the behavior of our configuration from that appropriate for a rigid body.

X. FORCED OSCILLATIONS: DISTURBING POTENTIAL

The solution of the vibrational problem of self-gravitating configurations of a homogeneous but compressible viscous fluid of arbitrary viscosity, as given in the preceding section, applies to small oscillations of any kind--whatever their origin--be they free or forced. In the former case, the underlying differential equations are homogeneous, and the amplitudes of the oscillatory motions governed by them are arbitrary (though small enough for their squares and higher powers to be negligible); but the characteristic frequencies of oscillation are limited to a discrete spectrum defined by equation (8-30) in the inviscid case, and its generalization in Section IX when viscosity is taken in account.

However, inasmuch as the fundamental equations (8-11) and back to (6-25) - (6-26) for heterogeneous configurations--were obtained by an elimination of the gravitational potential R between (6-16) and (6-17) with the aid of (6-21), it follows that the same equations must also equally control all forced oscillations of our configurations--such as, for instance, the librations of the lunar globe, or all bodily lives raised by an external field of force. Moreover, their solutions must likewise be expressible by means of the characteristic functions which specify free oscillations in terms of general series of the form (8-48) and (8-49), provided only that none of the characteristic frequencies $\tilde{\nu}$ or λ of free oscillations coincide with the frequency of the disturbing external force. The characteristic frequencies of free oscillations can be determined by imposing the boundary conditions discussed in Section VII on differential equations set up in Sections VIII and IX; but the periods of forced oscillations impressed by external forces have not so far made their appearance in this report. The aim of the present section will be to deduce them from the structure of the disturbing potential.

In order to specify, quite generally, the nature of the attraction exerted by an external mass m' at a distance Δ from the center of gravity of our configuration, let

$$\left.\begin{aligned} \lambda' &= \cos\phi \sin\theta, \\ \mu' &= \sin\phi \sin\theta, \\ \nu' &= \cos\theta, \end{aligned}\right\} \qquad (10\text{-}1)$$

denote the direction cosines of an arbitrary radius-vector r in the rotating body axes (cf. Section III), and

$$\left.\begin{aligned} \lambda'' &= \cos u \cos\Omega - \sin u \sin\Omega \cos I, \\ \mu'' &= \cos u \sin\Omega + \sin u \cos\Omega \cos I, \\ \nu'' &= \sin u \sin I, \end{aligned}\right\} \qquad (10\text{-}2)$$

be the direction cosines, in the space axes, of the radius-vector Δ joining the centers of mass of the two bodies, where I denotes the inclination of the orbital plane of the disturbing body to the equatorial plane $x'y'$ of the distorted configuration; and u, the true longitude of the mass m' from the longitude Ω of the ascending node in which the equatorial and orbital planes intersect.

If so then, as is well known, the attractive force of mass m' on our configuration will derive from the disturbing potential

$$V_T = \frac{Gm'r^2}{\Delta^3} P_2(\cos \Theta) + \cdots \tag{10-3}$$

where

$$\cos \Theta = \lambda'\lambda'' + \mu'\mu'' + \nu'\nu''; \tag{10-4}$$

so that, by the addition theorem for spherical harmonics,

$$P_j(\cos \Theta) = P_j(\nu')P_j(\nu'') + 2 \sum_{k=1}^{j} \frac{(j-k)!}{(j+k)!} P_j^k(\nu')P_j^k(\nu'')\cos k(\phi - p), \tag{10-5}$$

where

$$p = \Omega + \tan^{-1}(\cos I \tan u). \tag{10-6}$$

In consequence, by rewriting the circular functions involved on the right-hand side of (10-5) in terms of imaginary exponentials, we find that, for $j = 2$,

$$\begin{aligned} P_2(\cos \Theta) = &- \frac{1}{2} Y_2^0(\theta)\{P_2^0(q) - \frac{1}{2} P_2^2(q)e^{\pm 2iu}\} \\ &- \frac{i}{16} Y_2^1(\theta,\phi)\{2qe^{\pm i\Omega} - (1+q)e^{\pm i(\Omega+2u)} + (1-q)e^{\pm i(\Omega-2u)}\} \sqrt{1-q^2} \\ &+ \frac{1}{96} Y_2^2(\theta,\phi)\{2(1-q^2)e^{\pm 2i\Omega} + (1+q)^2 e^{\pm 2i(\Omega+u)} + (1-q)^2 e^{\pm 2i(\Omega-u)}\}, \end{aligned} \tag{10-7}$$

where

$$\left.\begin{aligned} Y_2^0(\theta) &= P_2^0(\nu), \\ Y_2^1(\theta,\phi) &= e^{\mp i\phi} P_2^1(\nu), \\ Y_2^2(\theta,\phi) &= e^{\mp 2i\phi} P_2^2(\nu); \end{aligned}\right\} \tag{10-8}$$

$$\left.\begin{aligned} P_2^0(\nu) &= \tfrac{1}{2}(3\nu^2 - 1), \\ P_2^1(\nu) &= -3\nu\sqrt{1-\nu^2}), \\ P_2^2(\nu) &= 3(1-\nu^2); \end{aligned}\right\} \tag{10-9}$$

$i \equiv \sqrt{-1}$ denotes the imaginary unit; and

$$q = \cos I. \tag{10-10}$$

It is, furthermore, obvious that expressions analogous to (10-7) can be constructed for any solid harmonic $P_j(\cos\Theta)$ that may occur on the right-hand side of equation (10-3) of any order; but the requisite algebra becomes progressively more complicated and may be left as an exercise for the interested reader.

On insertion of (10-7) in (10-3) the latter can evidently be identified with an expansion of the form (7-12), where

$$C_{0,2}(t) = -\frac{Gm'}{2\Delta^3}\left\{1 - \frac{3}{2}(1 + e^{\pm 2iu})\sin^2 I\right\}, \tag{10-11}$$

$$\begin{aligned} C_{1,2}(t) = -i\,\frac{Gm'}{8\Delta^3}\Big\{&\left[e^{\pm i\Omega} - e^{\pm i(\Omega+2u)}\right]\cos^2\frac{1}{2}I \\ &+ \left[e^{\pm i\Omega} - e^{\pm i(\Omega-2u)}\right]\sin^2\frac{1}{2}I\Big\}\sin I, \end{aligned} \tag{10-12}$$

$$C_{2,2}(t) = \frac{Gm'}{24\Delta^3}\left\{ e^{\pm 2i(\Omega+u)}\cos^4\frac{1}{2}I \right.$$

$$+ 2e^{\pm 2i\Omega}\sin^2\frac{1}{2}I\cos^2\frac{1}{2}I \qquad (10\text{-}13)$$

$$\left. + e^{\pm 2i(\Omega-u)}\sin^4\frac{1}{2}I \right\}.$$

Next, let us assume that the (Eulerian) angle θ between the instantaneous position of the space and body axes z and z' be zero, and (ignoring, for the moment, the phenomena of the precession and nutation) that the body axes x',y' rotate about the fixed $z \equiv z'$ axis with a uniform angular velocity ω_z, If so, however, it follows that

$$\Omega = \mp \omega_z t; \qquad (10\text{-}14)$$

the upper or lower sign referring to direct or retrograde rotation, respectively. Moreover, in accordance with the laws of elliptic motion, the true longitude u can be expanded in a Fourier series of the time t in the form

$$\begin{aligned} u &= \tilde{\omega} + v \\ &= \tilde{\omega} + nt + 2e\sin nt + \frac{5}{4}e^2\sin 2nt + \cdots, \end{aligned} \qquad (10\text{-}15)$$

where $\tilde{\omega}$ denotes the longitude of the pericenter (i.e., the angular distance between the nodal and apsidal lines); e, the orbital eccentricity; and n, the mean daily motion of the disturbing body.

In Section VII we have seen that the time functions $C_{i,j}(t)$ enter our problem through the boundary condition (7-23), not as such, but through their time derivatives. If we differentiate now equations (10-11) - (11-13) with respect to the time we find that (for constant Δ) the second-harmonic term $P_2(\cos\Theta)$ in V_T alone will give rise to the following *five* pairs of fundamental frequencies in the disturbing function:

$$\pm\,\dot{\Omega},$$

$$\pm\,2\dot{\Omega},$$

$$\pm\,2\dot{u},$$

$$\pm\,(\dot{\Omega}\pm 2\dot{u}),$$

$$\pm\,2(\dot{\Omega}\pm\dot{u}),$$

where, by a differentiation of (10-14) and (10-15) with respect to the time,

$$\dot{\Omega} = \mp\,\omega_z, \tag{10-16}$$

and

$$\dot{u} = \dot{\omega} + n(1 + 2e\cos nt + \frac{5}{2}e^2\cos 2nt + \cdots). \tag{10-17}$$

If m' were to stand for the mass of the Moon raising tides on the Earth, those of the frequency $2\dot{u}$ would regarded as "fortnightly tides"; and those associated with $(\omega_z \pm 2\dot{u})$ as "diurnal tides"; while those of the frequency $2(\omega_z \pm \dot{u})$ would be termed "semi-diurnal". On the Moon tidally disturbed by the Earth the axial rotation is exactly synchronized with the revolution--i.e., $\omega_z = n$ --and, as a result, the "fortnightly" tide becomes semi-diurnal while the "diurnal" tide will of course last a month; but the "semi-diurnal" becomes weekly (i.e., with a period of one-quarter of the month). The solar tides on the Moon will occur in accordance with a similar scheme--except that, in this case, ω_z as defined by (10-14) continues to be identical with the mean daily motion of the Moon, the quantity n in (10-15) refers to the mean daily motion of the Sun.

A glance at the right-hand side of equation (10-7) reveals that most tides invoked by the individual terms of the disturbing function arise in connection with a finite inclination I of the orbital plane of the disturbing body to the equator of the distorted configuration--an angle which is equal to $6°41'$ for the distortion of the Moon by the Earth, and $1°32'$ for its distortion by the Sun. If I were zero and the orbit of the disturbing body circular, only "semi-diurnal" tides of frequency $2(\mp\omega_z + \dot{u})$ would survive. If, however, $e > 0$, the variation of the radius vector Δ^{-3} in V_T would--to the first power in orbital eccentricity--give rise to three new "diurnal" terms of frequencies

$$\omega_z + n - \dot{\tilde{\omega}},$$

$$\omega_z - n - \dot{\tilde{\omega}},$$

$$\omega_z - 3n + \dot{\tilde{\omega}},$$

and two "semi-diurnal" terms with frequencies

$$2\omega_z - n - \dot{\tilde{\omega}},$$

$$2\omega_z - 3n + \dot{\tilde{\omega}};$$

all of which will possess coefficients factored by the eccentricity e, the mean value of which for the lunar orbit around the Earth is equal to 0.0549 (and 0.01675 for the terrestrial orbit around the Sun), fluctuating between 0.0432 and 0.0666 each month because of evection; and this latter perturbation will, in turn, give rise to partial tides with frequencies

$$2\omega_z - n_{\oplus} - 2n_{\odot} + \dot{\tilde{\omega}},$$

$$2\omega_z - 3n_{\oplus} + 2n_{\odot} - \dot{\tilde{\omega}},$$

where $n_{\oplus}, n_{\odot}$ denote the mean daily motions (as seen from the Moon) of the Earth and of the Sun, respectively.

The number of partial tides associated with harmonics of orders higher than the second in the disturbing potential V_T, or with coefficients factored by higher powers of the orbital eccentricity, becomes so large that no adequate account of them can be given in this place; and the reader must be referred to appropriate literature (e.g., Poincaré, 1910; or Bartels, 1957).

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