NASA TECHNICAL NOTE

NASA TN D-4216

NASA TN D-4216

A PROCEDURE FOR COMPUTING THE MOTION OF A LUNAR-LANDING VEHICLE DURING THE LANDING IMPACT

by William C. Walton, Jr., and Barbara J. Durling

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • OCTOBER 1967

NASA TN D-4216

A PROCEDURE FOR COMPUTING THE MOTION OF A LUNAR-LANDING VEHICLE DURING THE LANDING IMPACT

By William C. Walton, Jr., and Barbara J. Durling

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

CONTENTS

A PROCEDURE FOR COMPUTING THE MOTION OF A LUNAR-LANDING VEHICLE DURING THE LANDING IMPACT

By William C. Walton, Jr., and Barbara J. Durling
Langley Research Center

SUMMARY

This paper is a description of a procedure for computing the general motions during impact of a vehicle performing a controlled soft landing on the moon. The assumptions involved in idealization of the vehicle and landing surface are stated. The equations of motion of the idealized system are derived. A procedure is set forth for numerical integration of the equations. Detailed and explicit instructions are given for coding a digital computer to carry out the computation of an impact history.

The procedure takes into account experience obtained from extensive correlations of computed landing motions with results of tests of a model incorporating many of the mechanical actions of a realistic vehicle. These correlations are reported in reference 1 (NASA TN D-4215).

The vehicle is idealized as an arbitrary rigid body with up to four legs arbitrarily attached. A leg is composed of three struts in an inverted tripod arrangement. The struts telescope during impact. Telescoping of a strut is resisted by forces which represent the effects of an aluminum honeycomb shock absorber mounted in the strut and the effects of overall system elasticity and damping. Representation of a very general landing surface is allowed.

INTRODUCTION

American designers' conceptions of spacecraft capable of controlled, soft landings on the moon have been characterized by a preference for vehicles with legs. Prominent examples are the Surveyor vehicle (with three legs) for unmanned landing and the Apollo lunar module (with four legs) for manned landing. The legs must be designed to prevent damage to the craft during the landing impact and to bring it to rest in an upright attitude so that no part of its mission such as deployment of instruments or relaunch will be inhibited. Leg designs have very generally been trusses terminating on the lower end in foot pad structures. Load-limiting shock absorbers are built into the trusses.

Evaluation of the design of such a leg system is a considerable task. The main reasons are as follows:

(1) Limitations on present methods of maneuvering vehicles during the approach to landing make it necessary to consider a wide range of attitudes and velocities of the vehicle at the moment of impact

(2) Because of uncertainty regarding the nature of landing sites on the moon, it is necessary to consider very general conditions of topography and composition of the landing surface

(3) The relation between the initial conditions and the outcome of an impact generally is extremely complicated

(4) Impact testing of structurally realistic full-scale vehicles in the earth's gravitational field is seriously hampered by problems which arise in simulating lunar gravity

(5) Because of fabrication difficulties, it does not appear to be feasible to construct models structurally scaled so that the elastic behavior of the model when impacting under earth's gravity simulates that of a realistic full-scale vehicle impacting under lunar gravity.

Faced with these difficulties and yet required to make judgements regarding the merit of different leg designs, a number of organizations (refs. 2 to 7) have settled on the following procedure:

Equations of motion of an impacting vehicle based on simplifying idealizations of the structure and the landing surface are derived. A digital computer program is devised for generating numerical solutions of the equations. A dynamic model suitable for landing tests on earth is constructed, as many features of a realistic vehicle as is feasible being scaled. Landing tests are conducted to determine the behavior of the model during impacts under earth's gravity. Impact motion histories are calculated for the model and are compared with the results of the tests. On the basis of the correlation, the analysis is refined; and when consistent success is achieved in predicting the behavior of the model, the analysis is used to assess the performance of realistic full-scale vehicles impacting under lunar conditions.

Heavy reliance is currently placed upon such procedures in making design decisions affecting manned craft.

The purpose of this paper is to describe in some detail an analytical procedure developed at the Langley Research Center along the lines which have been described. The model which was used, the testing method, and correlations between the analysis and experiment are discussed in reference 1. It is felt that the present paper and reference 1 together will be useful as an aid to ascertaining the current state of the art in analytical prediction of the performance of leg systems, in attaining computing capability on a level with the state of the art, and in planning research to improve the state of the art.

SYMBOLS

t	time
X,Y,Z	space-fixed Cartesian axes
ξ,η,ζ	body-fixed Cartesian axes
g	gravitational constant
ϕ,θ,ψ	Eulerian angles
Q_X,Q_Y,Q_Z	space system components of general vector
Q_ξ,Q_η,Q_ζ	body system components of general vector Q_X,Q_Y,Q_Z
$\delta_{p,q}$	element of matrix transforming space coordinates to body coordinates (see eqs. (2) to (3i))
C_ξ,C_η,C_ζ	body system components of a general vector fixed in body
C_X,C_Y,C_Z	space system components of vector C_ξ,C_η,C_ζ
$\omega_\xi,\omega_\eta,\omega_\zeta$	body system components of angular velocity vector of body
M	total mass of vehicle
I_ξ,I_η,I_ζ	principal moments of inertia about body axes for vehicle in its initial undeformed configuration
X_O,Y_O,Z_O	space system components of center of gravity of body
ξ_P,η_P,ζ_P	body system coordinates of an arbitrary point fixed in body
X_P,Y_P,Z_P	space system coordinates of point ξ_P,η_P,ζ_P

F_X, F_Y, F_Z space system components of total force exerted on body through leg struts

N_ξ, N_η, N_ζ body system components of total torque about center of gravity of body produced by forces through leg struts

V_{OX}, V_{OY}, V_{OZ} space system components of velocity of center of gravity of body

E_K kinetic energy of body

$X_{F,j}, Y_{F,j}, Z_{F,j}$ space coordinates of foot

$\xi_{H,j,k}, \eta_{H,j,k}, \zeta_{H,j,k}$ body coordinates of hard point

$X_{H,j,k}, Y_{H,j,k}, Z_{H,j,k}$ space coordinates of hard point

$S_{j,k}$ instantaneous length of a strut

$U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k}$ space components of unit vector directed along a strut with sense from foot to hard point

$F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k}$ space components of force exerted on body through a strut

$F_{H,j,k}$ magnitude of force exerted on body through a strut, positive when sense is from foot to hard point

$S_{E,j,k}$ amount of contraction of a strut beyond which permanent shortening occurs

$S_{R,j,k}$ length of strut when shock absorber is neither contracted nor extended, permanent shortening due to crushing of shock-absorbing material being taken into account

$S_{O,j,k}$ initial length of a strut

$S_{S,j,k}$ stroke (see eq. (18))

$F_{R,j,k}$ part of magnitude of shock-absorber force termed rate-dependent force

$F_{S,j,k}$ part of magnitude of shock-absorber force termed quasi-static force

$\dot{S}_{EC,j,k}, \dot{S}_{EE,j,k}, \dot{S}_{CC,j,k} \dot{S}_{CE,j,k}$
$F_{EC,j,k}, F_{EE,j,k}, F_{CC,j,k}, F_{CE,j,k}$ — constants determining rate-dependent force (see fig. 2)

$P_{l,j,k}$ coefficient in equation for quasi-static force (see eq. (22))

K_S coefficient in equation for stop force (see eq. (23))

$F_{H\xi,j,k}, F_{H\eta,j,k}, F_{H\zeta,j,k}$ body components of force $F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k}$

$A_{X,j}, A_{Y,j}, A_{Z,j}, A_j$ coefficients in equation describing landing-surface boundary plane associated with jth foot (see eq. (27))

$W_{X,j}, W_{Y,j}, W_{Z,j}, W_j$ normalized coefficients in equation describing landing-surface boundary plane (see eqs. (28) to (29c))

H_j signed distance of a point from jth landing surface plane (see eq. (30))

$N_{p,q,j}, T_{p,q,j}$ elements of matrices for computing components of a vector normal and tangential to a boundary plane, respectively (see eqs. (31a) to (32b))

$F_{FNSX,j}, F_{FNSY,j}, F_{FNSZ,j}$ part of force upon a foot acting normal to landing surface termed quasi-static normal force

$F_{FNS,j}$ magnitude of quasi-static normal force

$K_{i,j}$ coefficient in equation for $F_{FNS,j}$ (see eq. (38))

D_j absolute distance from foot to landing surface plane (see eq. (39))

$F_{FNDX,j}, F_{FNDY,j}, F_{FNDZ,j}$ part of force upon a foot acting normal to landing surface termed dynamic normal force

$R_{N,j}$ coefficient in equation for dynamic normal force (see eq. (40))

$\dot{X}_{FN,j}, \dot{Y}_{FN,j}, \dot{Z}_{FN,j}$ space components of velocity of foot normal to landing surface plane

$F_{FTDX,j} F_{FTDY,j}, F_{FTDZ,j}$ force on a foot acting tangential to landing surface plane termed dynamic tangential force

$R_{T,j}$ coefficient in equation for dynamic tangential force (see eq. (42))

$\dot{X}_{FT,j}, \dot{Y}_{FT,j}, \dot{Z}_{FT,j}$ space components of velocity of foot tangential to landing surface plane

$F_{FLX,j}, F_{FLY,j}, F_{FLZ,j}$ space components of total force on a foot acting through three struts bearing on foot

$F_{FLNX,j}, F_{FLNY,j}, F_{FLNZ,j}$ space components of component of force $F_{FLX,j}, F_{FLY,j} F_{FLZ,j}$ normal to landing surface plane

$F_{FLTX,j}, F_{FLTY,j}, F_{FLTZ,j}$ space components of component of force $F_{FLX,j}, F_{FLY,j}, F_{FLZ,j}$ tangential to landing surface plane

$t^{(f)}$ some time at which a foot has just become free of landing surface

$\xi_{F,j}, \eta_{F,j}, \zeta_{F,j}$ body coordinates of foot

$\xi_{F,j}^{(f)}, \eta_{F,j}^{(f)}, \zeta_{F,j}^{(f)}$ body coordinates of foot becoming free at time $t^{(f)}$

$X_{F,j}^{(f)}, Y_{F,j}^{(f)}, Z_{F,j}^{(f)}$ space coordinates of foot becoming free at time $t^{(f)}$

L some characteristic length of the vehicle

The subscript B on any symbol denotes a nondimensional value. A dot over a symbol denotes differentiation with respect to time t. An asterisk over a symbol denotes differentiation with respect to t_B. The superscript (n) denotes nth iteration. $| \ |$ denotes absolute values.

GENERAL ASPECTS OF IDEALIZATION

Overall Vehicle

The vehicle is treated as an arbitrary rigid body to which there are attached up to four legs, each leg consisting of three struts in an inverted tripod arrangement. (See fig. 1.) The struts are connected to the body by universal joints, and the junction point of the three struts in a leg is also a universal joint. This junction point is called the foot of the leg, and the points where struts attach to the body are called hard points. There is a shock absorber in each strut. The individual struts may shorten or lengthen because of stroking of the shock absorbers, but otherwise they do not deform. Locations of the hard points on the body and the initial positions of the feet relative to the body may be arbitrarily chosen.

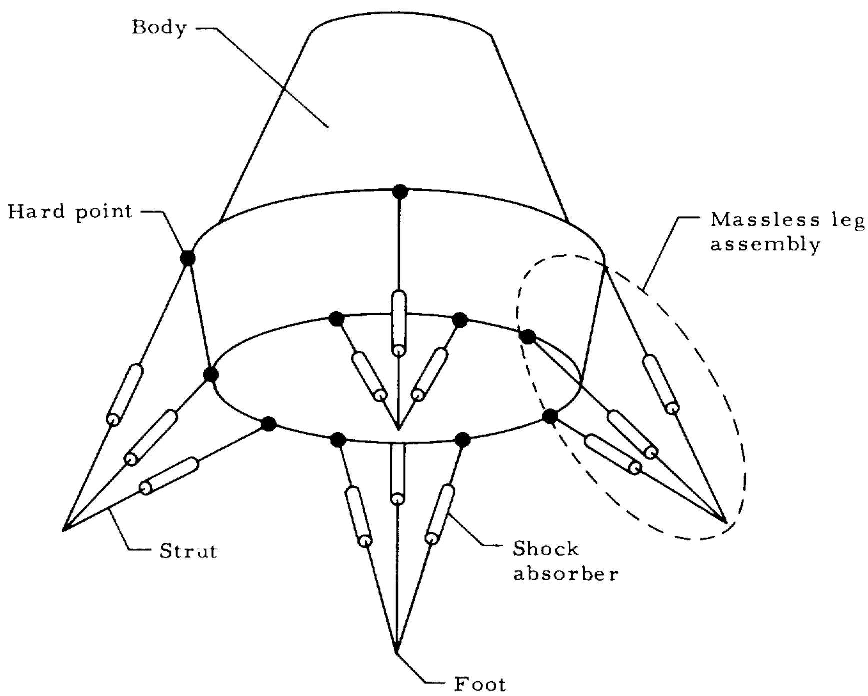


Figure 1.- Idealization of vehicle.

Inertias

In the idealized system the legs are considered to have no mass. The inertial properties of the body are characterized by specification of the total mass, the center of gravity, a set of principal axes, and the moments of inertia taken about the principal axes. In representing an actual vehicle or model, the inertial properties are computed for the system as a whole, including the legs. These inertial properties are then assigned to the body alone in the idealized system. This approach has been adopted because it permits considerable simplification of the equations of motion and reduces the number of equations. The approach is reasonable if the legs are of light construction and/or deformations of the legs due to stroking of the shock absorbers are small; that is, the moments of inertia do not change appreciably because of stroking of the struts.

Shock Absorbers

The shock absorber in a strut is considered to produce simultaneously a force at the hard point and at the foot to which the strut is connected. The forces are considered to be equal in magnitude but opposite in direction and to be directed along the axis of the strut. The magnitude of the shock absorber force is assumed to depend on the change of length of the strut from the initial length and on the rate of change of the length. The specific relation between the force and these variables is described in the section "Forces and Torques on the Body."

Landing Surface

The boundary of the landing surface is represented by a set of arbitrarily oriented planes, one plane associated with each foot. Use of a different plane for each foot allows for the representation of many irregular surfaces. When the foot of a leg is not interacting with the landing surface material, the leg is assumed to move as a rigid extension of the body. When the foot is interacting with the landing surface material, a force acts on the foot, and the shock absorbers may stroke. The force exerted on the foot by the landing surface is assumed to be a function of the position and velocity of the foot. A basic assumption of the analysis, leading to equations of motion of the feet, is that a foot always moves in such a way that this force is kept in balance with the shock-absorber forces bearing on the foot.

In representing the forces generated by the interaction of the foot pad with the landing surface material, the analyst must face two facts. First, knowledge of the properties of the lunar surface which would affect landing performance is as yet limited. Second, soil mechanics has not progressed to the point where one can predict with any confidence the history of forces on an arbitrary body impinging upon or passing through soil even under laboratory conditions. The reaction to these difficulties has been a rather general concentration of initial analytical effort on the case where a foot upon contacting the surface is stopped abruptly, is effectively pinned, and remains in place until there is a tendency for it to lift off. Cases where the feet move substantially through or along the surface are studied by assuming laws for the force on a foot. The procedure given by this paper is organized so that one may construct a variety of laws for the force on a foot by specialization of constants. The details of the computation are given in the section "Equations of Motion of the Feet."

EQUATIONS OF MOTION OF THE BODY

The object in this part of the paper is to set down the differential equations which govern the motion of the central rigid body. Equations of motion of the feet are developed in a subsequent section.

Coordinate Systems

Space and body coordinate systems.- Reference is made to space and body coordinate systems (fig. 2). These systems are both right-handed Cartesian systems. The space system is assumed to be an inertial system, and its axes are denoted by X, Y, and Z. It is oriented so that gravitational forces point along the negative Z-axis. The body system is assumed to be fixed in the body, and its axes are denoted by ξ, η, and ζ. It is placed within the body so that the origin coincides with the center of gravity and the three coordinate axes coincide with principal axes of inertia. Further considerations explained under the heading "Eulerian Angles" also enter into the establishment of the body axes.

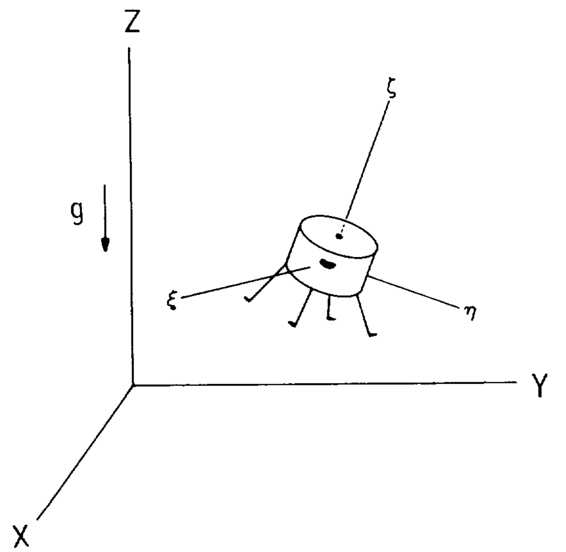


Figure 2.- Space and body coordinate system.

Eulerian angles.- Rotations of the body are specified by use of Eulerian angles ϕ, θ, and ψ as shown in figure 3. (The horizontal orientation of the Z-axis in the sketch is merely for convenience in drawing.) Note that the Eulerian angles are not defined in the usual way; that is, setting the angles equal to zero does not bring the body axes into coincidence with the space axes. Instead, the positive ξ-axis merges with the positive X-axis, the positive η-axis with the negative Z-axis, and the positive ζ-axis with the positive Y-axis.

The following limits are set on the ranges of the Eulerian angles:

$$0 < \theta < \pi \tag{1a}$$

$$-\pi \leqq \phi \leqq \pi \tag{1b}$$

$$-\pi \leqq \psi \leqq \pi \tag{1c}$$

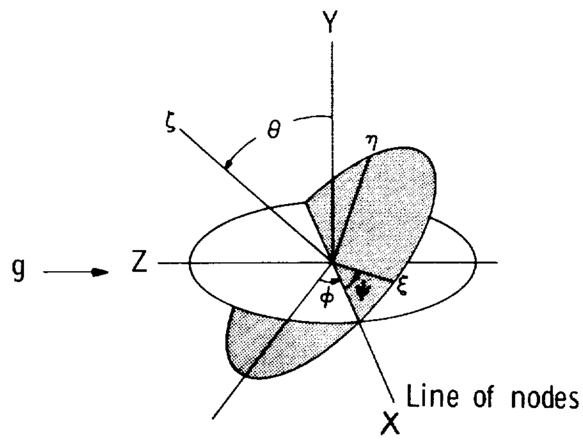


Figure 3.- Eulerian angles.

The limits on ϕ and ψ do not restrict the generality of orientations. The limits on θ are restrictive, however, and are imposed to avoid encountering a mathematical singularity for θ equal to 0 or π as explained in a subsequent section. As θ approaches either of the end values 0 or π, the ζ-axis approaches a horizontal orientation. Therefore, in positioning the body axes within the body, the ζ-axis should be associated with the principal axis of the vehicle which is most nearly longitudinal. If this is done, the restriction on θ causes no practical limitation on motions, because calculation normally stops before the vehicle has tipped so far that the longitudinal axis becomes horizontal.

Transforming vector components between space and body systems.- Let Q_X, Q_Y, and Q_Z be the space system components of a vector, and let Q_ξ, Q_η, and Q_ζ be the body system components of the same vector. As described in reference 8 and other texts on classical mechanics, the components are related by the equation

$$\begin{Bmatrix} Q_\xi \\ Q_\eta \\ Q_\zeta \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} \qquad (p = 1, 2, 3; \; q = 1, 2, 3) \tag{2}$$

where

$$\delta_{11} = \cos\psi \cos\phi - \cos\theta \sin\phi \sin\psi \tag{3a}$$

$$\delta_{12} = \sin\psi \sin\theta \tag{3b}$$

$$\delta_{13} = -\cos\psi \sin\phi - \cos\theta \cos\phi \sin\psi \tag{3c}$$

$$\delta_{21} = -\sin\psi \cos\phi - \cos\theta \sin\phi \cos\psi \tag{3d}$$

$$\delta_{22} = \cos\psi \sin\theta \tag{3e}$$

$$\delta_{23} = \sin\psi \sin\phi - \cos\theta \cos\phi \cos\psi \tag{3f}$$

$$\delta_{31} = \sin\theta \sin\phi \tag{3g}$$

$$\delta_{32} = \cos\theta \tag{3h}$$

$$\delta_{33} = \sin\theta \cos\phi \tag{3i}$$

The matrix $\left[\delta_{p,q}\right]$ appearing in equation (2) is orthogonal which means that its inverse and its transpose are identical. Therefore, the inverse transformation giving space components in terms of body components is

$$\begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} Q_\xi \\ Q_\eta \\ Q_\zeta \end{Bmatrix} \tag{4}$$

the T denoting the transpose. It follows from equation (4) that if C_ξ, C_η, and C_ζ are the constant body system components of a vector fixed in the moving body, and C_X, C_Y, and C_Z are the space components of the vector then

$$\begin{Bmatrix} \dot{C}_X \\ \dot{C}_Y \\ \dot{C}_Z \end{Bmatrix} = \begin{bmatrix} & & \\ & \dot{\delta}_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} C_\xi \\ C_\eta \\ C_\zeta \end{Bmatrix} \tag{5}$$

where a dot denotes differentiation with respect to time.

Relation between Eulerian angles and direction cosines.- The elements of the matrix $\left[\delta_{p,q}\right]$ in equation (2) may be interpreted as the direction cosines of the body axes measured with respect to the space axes. That is, if one imagines a translation of the space axes without rotation such that the origins of the body system and the translated space system are brought into coincidence, δ_{11} is the cosine of the angle between the ξ-axis and the translated X-axis, δ_{12} is the angle between the ξ-axis and the translated Y-axis, and so forth.

It is generally considerably easier to specify the direction cosines associated with an initial orientation of the body than it is to specify the Eulerian angles. Therefore, it is useful to have a procedure for computing Eulerian angles once direction cosines are given. The following scheme which is easily derived from equations (3a) to (3i) has proved to be satisfactory.

$$|\phi| = \text{arc cosine} \frac{\delta_{33}}{\sqrt{(\delta_{31})^2 + (\delta_{33})^2}} \qquad (0 \leqq |\phi| \leqq \pi) \tag{6a}$$

$$\phi = |\phi| \qquad \left(\delta_{31} \geqq 0\right) \tag{6b}$$

$$\phi = -|\phi| \qquad \left(\delta_{31} < 0\right) \tag{6c}$$

$$\theta = \text{arc cosine } \delta_{32} \qquad (0 < \theta < \pi) \tag{6d}$$

$$|\psi| = \text{arc cosine} \frac{\delta_{22}}{\sqrt{\left(\delta_{31}\right)^2 + \left(\delta_{33}\right)^2}} \qquad (0 \leqq |\psi| \leqq \pi) \tag{6e}$$

$$\psi = |\psi| \qquad \left(\delta_{12} \geqq 0\right) \tag{6f}$$

$$\psi = -|\psi| \qquad \left(\delta_{12} < 0\right) \tag{6g}$$

Relation between Eulerian angles and components of angular velocity.- Let ω_ξ, ω_η, and ω_ζ denote the body components of the angular velocity vector of the body. They are related to the rates of change of the Eulerian angles as follows: (See p. 134 of ref. 8.)

$$\begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \tag{7}$$

The inverse relation is

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} \frac{\sin\psi}{\sin\theta} & \frac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\frac{\cos\theta\sin\psi}{\sin\theta} & -\frac{\cos\theta\cos\psi}{\sin\theta} & 1 \end{bmatrix} \begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix} \tag{8}$$

The singularity referred to previously in imposing a restricted range on θ is involved in the factor $1/\sin\theta$ appearing in equation (8). With the range as established, the denominator $\sin\theta$ never vanishes.

Equations of Motion

The following definitions are necessary for expressing the equations of motion of the body:

M	total mass of vehicle
I_ξ, I_η, I_ζ	principal moments of inertia about body axes ξ, η, and ζ
X_O, Y_O, Z_O	space system components of center of gravity of body
ξ_P, η_P, ζ_P ; X_P, Y_P, Z_P	body and space system coordinates, respectively, of an arbitrary point fixed in body

F_X, F_Y, F_Z space system components of total force exerted on body through leg struts

N_ξ, N_η, N_ζ body system components of total torque about center of gravity of body produced by forces through leg struts

V_{OX}, V_{OY}, V_{OZ} space system components of velocity of center of gravity of body

g gravitational constant

By using these definitions, the elementary equations of translation are

$$\dot{X}_O = V_{OX} \tag{9a}$$

$$\dot{Y}_O = V_{OY} \tag{9b}$$

$$\dot{Z}_O = V_{OZ} \tag{9c}$$

$$M\dot{V}_{OX} = F_X \tag{9d}$$

$$M\dot{V}_{OY} = F_Y \tag{9e}$$

$$M\dot{V}_{OZ} = F_Z - Mg \tag{9f}$$

Euler's equations for rotation of the body are

$$I_\xi\left(\dot{\omega}_\xi\right) = \left(\omega_\eta\right)\left(\omega_\zeta\right)\left(I_\eta - I_\zeta\right) + N_\xi \tag{10a}$$

$$I_\eta\left(\dot{\omega}_\eta\right) = \left(\omega_\zeta\right)\left(\omega_\xi\right)\left(I_\zeta - I_\xi\right) + N_\eta \tag{10b}$$

$$I_\zeta\left(\dot{\omega}_\zeta\right) = \left(\omega_\xi\right)\left(\omega_\eta\right)\left(I_\xi - I_\eta\right) + N_\zeta \tag{10c}$$

For completeness, equation (8) is repeated

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} \dfrac{\sin\psi}{\sin\theta} & \dfrac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\dfrac{\cos\theta\sin\psi}{\sin\theta} & -\dfrac{\cos\theta\cos\psi}{\sin\theta} & 1 \end{bmatrix} \begin{Bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{Bmatrix} \tag{11}$$

From equation (4), it follows that

$$\begin{Bmatrix} X_P - X_O \\ Y_P - Y_O \\ Z_P - Z_O \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi_P \\ \eta_P \\ \zeta_P \end{Bmatrix} \tag{12}$$

The preceding equations constitute the equations of motion of the body. A dimensionless form of the equations more suitable for use in computation is given in a subsequent section.

Kinetic Energy

It is sometimes useful to compute a history of the kinetic energy of the system throughout an impact. The kinetic energy E_K is given by the equation:

$$E_K = \frac{M}{2}\left(V_{OX}^2 + V_{OY}^2 + V_{OZ}^2\right) + \frac{1}{2}\left(I_\xi \omega_\xi^2 + I_\eta \omega_\eta^2 + I_\zeta \omega_\zeta^2\right) \tag{13}$$

FORCES AND TORQUES ON THE BODY

The forces which act upon the central rigid body are the gravitational force and the forces through the leg struts produced by the action of the shock absorbers. The object here is to write down rules for computing these forces based on practical considerations which arose in analyzing the behavior of the model of reference 1.

Definitions

It is assumed that the legs have been numbered 1 to 4 in any order and the struts in each leg have been numbered 1 to 3 in any order. Cases where there are less than four legs are handled by setting the shock absorber forces equal to zero in some legs. The following definitions are necessary to subsequent developments:

$X_{F,j}, Y_{F,j}, Z_{F,j}$	instantaneous space coordinates of foot of jth leg
$\left.\begin{matrix} \xi_{H,j,k}, \eta_{H,j,k}, \zeta_{H,j,k} \\ X_{H,j,k}, Y_{H,j,k}, Z_{H,j,k} \end{matrix}\right\}$	body and space coordinates, respectively, of hard point at which the kth strut of jth leg is attached to body
$S_{j,k}$	instantaneous length of kth strut of jth leg
$U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k}$	space components of a unit vector directed along kth strut of jth leg, the sense being from foot to hard point

$F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k}$ space system components of force exerted on body through kth strut of jth leg

$F_{H,j,k}$ magnitude of force $F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k}$ taken positive if the force is directed from foot to hard point and negative if the force is directed from hard point to foot

The following equations may be written on the basis of these definitions:

$$S_{j,k} = \left[\left(X_{H,j,k} - X_{F,j}\right)^2 + \left(Y_{H,j,k} - Y_{F,j}\right)^2 + \left(Z_{H,j,k} - Z_{F,j}\right)^2\right]^{1/2} \tag{14}$$

$$\dot{S}_{j,k} = \frac{1}{S_{j,k}}\left[\left(X_{H,j,k} - X_{F,j}\right)\left(\dot{X}_{H,j,k} - \dot{X}_{F,j}\right) + \left(Y_{H,j,k} - Y_{F,j}\right)\left(\dot{Y}_{H,j,k} - \dot{Y}_{F,j}\right) + \left(Z_{H,j,k} - Z_{F,j}\right)\left(\dot{Z}_{H,j,k} - \dot{Z}_{F,j}\right)\right] \tag{15}$$

$$U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k} = \frac{1}{S_{j,k}}\left[\left(X_{H,j,k} - X_{F,j}\right), \left(Y_{H,j,k} - Y_{F,j}\right), \left(Z_{H,j,k} - Z_{F,j}\right)\right] \tag{16}$$

$$F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k} = F_{H,j,k}\left(U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k}\right) \tag{17}$$

Reference Length

The shock absorbers for the model of reference 1 were cylinders of aluminum honeycomb mounted in the struts. For compressive loading this type of shock absorber has a load-deflection relationship as shown in figure 4.

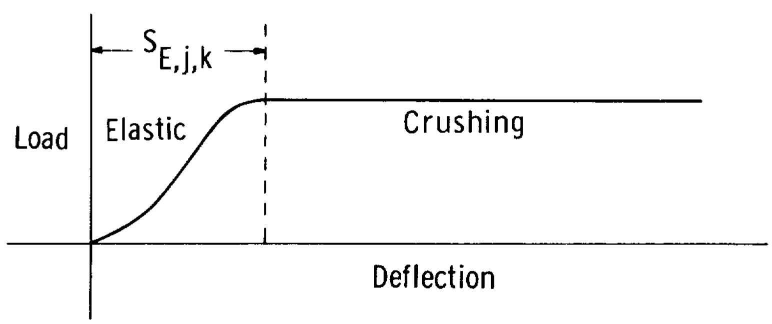


Figure 4.- Load deflection relationship before crushing.

The shock absorber compresses because of elastic deformation until the load reaches the level at which crushing of the honeycomb begins. Compression continues at constant load as the cylinder crushes. The elastic part of the load-deflection diagram may be nonlinear. If crushing occurs, the cylinder becomes permanently shortened, and leaves a gap as shown by figure 5. Actually the shape of the elastic part of the load-deflection diagram may be altered somewhat as a result of crushing; but in this analysis, it is assumed that this shape remains unchanged.

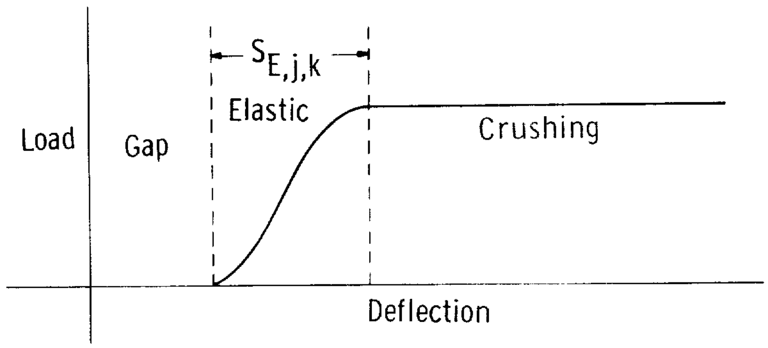


Figure 5.- Load deflection relationship after crushing.

In order to account for permanent shortening, it is convenient to define a variable called the reference length of a strut denoted by the symbol $S_{R,j,k}$. Initially, the reference length is equal to $S_{O,j,k}$ the initial length of the strut. The stroke $S_{S,j,k}$ at any time t is defined to be the reference length minus the length of the strut at time t.

$$S_{S,j,k} = S_{R,j,k} - S_{j,k} \tag{18}$$

The length of the portion of the deflection denoted "elastic" in sketches 3 and 4 is represented by the symbol $S_{E,j,k}$. At the completion of any stroke exceeding $S_{E,j,k}$, the reference length is diminished by the amount by which the stroke exceeded $S_{E,j,k}$.

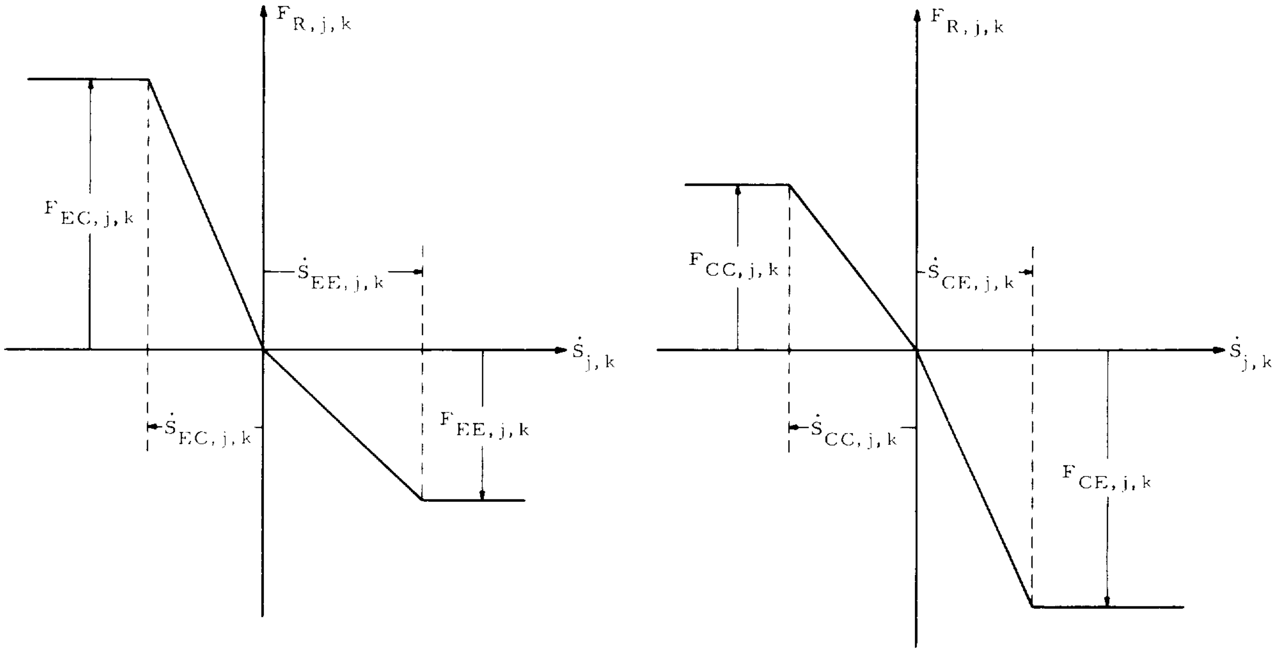
Magnitude of Shock Absorber Force

It is convenient to express $F_{H,j,k}$, the magnitude of the shock absorber force, as the sum of two terms

$$F_{H,j,k} = F_{R,j,k} + F_{S,j,k} \tag{19}$$

The term $F_{R,j,k}$ is spoken of as the rate-dependent force and $F_{S,j,k}$ as the quasi-static force.

Rate-dependent force.- Figure 6 shows the relation between the rate-dependent force and the dynamic state of the strut. With respect to the current reference length, a strut is either extended $\left(S_{R,j,k} - S_{j,k} \leq 0\right)$ or contracted $\left(S_{R,j,k} - S_{j,k} > 0\right)$. In either



(a) Extended strut, $S_{R,j,k} - S_{j,k} \leqq 0$. (b) Contracted strut, $S_{R,j,k} - S_{j,k} \geqq 0$.

Figure 6.- Relationship between rate-dependent force and the dynamic state of the strut.

case the strut may be extending $\left(\dot{S}_{j,k} > 0\right)$ or contracting $\left(\dot{S}_{j,k} \leqq 0\right)$. For each of the four possible combinations of these conditions, the rate-dependent force $F_{R,j,k}$ is described by a separate ramp function of $\dot{S}_{j,k}$, the rate at which the strut length is changing. Eight arbitrary constants associated with each strut establish the shapes of the ramps. These constants are $\dot{S}_{EC,j,k}$, $\dot{S}_{EE,j,k}$, $\dot{S}_{CC,j,k}$, and $\dot{S}_{CE,j,k}$ which bear the units of velocity and $F_{EC,j,k}$, $F_{EE,j,k}$, $F_{CC,j,k}$, and $F_{CE,j,k}$ which bear the units of force. As figure 6 indicates, the velocity constants establish ranges of $\dot{S}_{j,k}$ within which $F_{R,j,k}$ varies linearly with $\dot{S}_{j,k}$ and beyond which $F_{R,j,k}$ is a constant. The force constants establish the heights of the ramps. The subscripts EC, EE, CC, and CE, respectively, denote the conditions extended and contracting, extended and extending, contracted and contracting, and contracted and extending.

Quasi-static force.- The quasi-static force is considered to be zero unless one of the following conditions holds: (1) The strut is contracted with respect to the current reference length and is extending

$$S_{R,j,k} - S_{j,k} > 0 \tag{20a}$$

$$\dot{S}_{j,k} > 0 \tag{20b}$$

(2) The strut is extended beyond its initial length $S_{O,j,k}$ and is extending

$$S_{j,k} - S_{O,j,k} \geqq 0 \tag{21a}$$

$$\dot{S}_{j,k} > 0 \tag{21b}$$

When the first condition holds, the force is assumed to be given by the following third-order polynomial function of the stroke $S_{R,j,k} - S_{j,k}$:

$$F_{S,j,k} = P_{1,j,k} + P_{2,j,k}\left(S_{R,j,k} - S_{j,k}\right) + P_{3,j,k}\left(S_{R,j,k} - S_{j,k}\right)^2 + P_{4,j,k}\left(S_{R,j,k} - S_{j,k}\right)^3 \tag{22}$$

wherein the constants $P_{l,j,k}$ $(l = 1, 2, 3, 4)$ may be arbitrarily selected. For this condition the stroke is always less than or equal to $S_{E,j,k}$ because of the way $S_{R,j,k}$ is reset at the completion of a period of contracting which necessarily precedes a period of extending. Figure 7 shows an example of how the quasi-static force might vary with the stroke $S_{R,j,k} - S_{j,k}$.

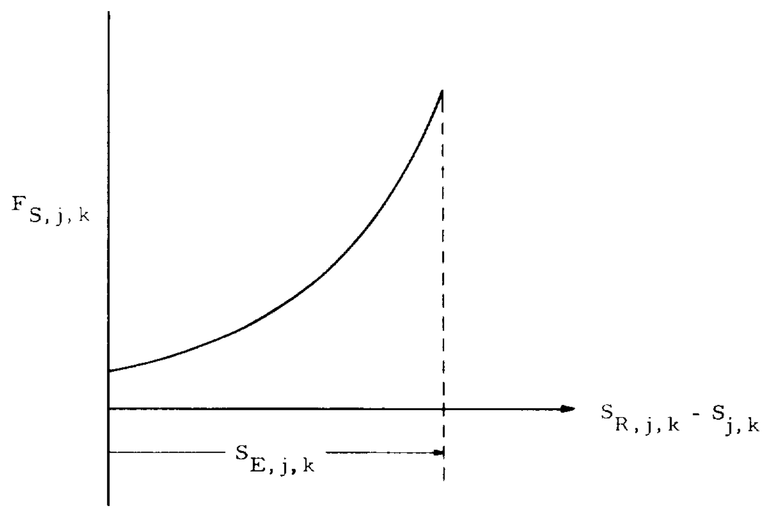


Figure 7.- Quasi-static force diagram for strut contracted but extending.

When the second condition holds, the equation for the quasi-static force is:

$$F_{S,j,k} = -K_S\left(S_{j,k} - S_{O,j,k}\right) \tag{23}$$

where K_S is an arbitrary positive constant.

Representing Forces for the Model of Reference 1

The preceding relations between the magnitude of the shock absorber force and the dynamic state of a strut were established with the following three objectives in mind:

(1) to represent constant force crushing and free extension characteristic of the aluminum honeycomb shock absorbers used on the model of reference 1, (2) to represent measured elastic characteristics of the model, and (3) to represent, in a gross way, energy dissipation due to causes other than crushing of the honeycomb to prevent unrealistic bouncing of the idealized system.

Constant force crushing.- To accomplish the first objective, the constant $F_{CC,j,k}$ of the rate-dependent force is set equal to the force at which the honeycomb crushes, and the constants $F_{EC,j,k}$, $F_{EE,j,k}$, and $F_{CE,j,k}$ are assigned relatively small values for a nominal representation of frictional resistance to stroking produced by the bearings in the strut. The constants $\dot{S}_{EC,j,k}$, $\dot{S}_{EE,j,k}$, $\dot{S}_{CC,j,k}$, and $\dot{S}_{CE,j,k}$ are assigned values greater than zero but small compared with the average stroking rate of the shock absorbers. With these settings, the shock absorbers will operate on the constant-force parts of the rate-dependent force diagrams except when stroking rates are very low. When stroking rates are very low, it is desirable to operate on the sloping part of the rate-dependent curves in order to avoid an annoying problem in numerical integration of the equations of motion. The problem is an oscillatory instability of the calculated stroking rates $\dot{S}_{j,k}$ which is encountered when very large forces occur simultaneously with very small stroking rates so that the sign of a stroking rate may change on one time step and change back on the next time step.

Elasticity.- For the second objective, the constants $S_{E,j,k}$ and $P_{l,j,k}$ are chosen for all struts to represent insofar as possible the elastic behavior of the model. Since there can be significant elastic deformations of the body as well as of the struts and shock absorbers, the validity of such a representation is open to question. This is an important limitation arising out of the simplifications made in the analysis. Reference 1 describes in detail a procedure used to assign these constants for the test model. The procedure involved a series of static load tests of the model with various leg struts removed. In spite of the questionable aspects of the representation, the procedure resulted in good predictions of the outcome of test impacts in which elastic rebound was a significant influence.

Damping.- It is noted that for the idealized system, there may be an axial elastic force in a strut which is contracted with respect to the reference length only when the strut is extending. If a strut is contracted with respect to the reference length and is contracting the full crush force, $F_{CC,j,k}$ resists the stroke unless the stroking rate is less than the constant $\dot{S}_{CC,j,k}$. As discussed previously, $\dot{S}_{CC,j,k}$ is set very low compared with the stroking rates expected throughout most of the impact. The elastic force must be lower than the crush force. Therefore, for any appreciable loading rate, it takes more energy to compress a strut spring than is given up when the spring unloads. As a result, elastic action is damped. This representation is not realistic. A strut containing a crushable aluminum honeycomb shock absorber both loads and unloads elastically with a period of crushing between if the loads get high enough. Energy losses other than those caused by crushing of the honeycomb come about in diverse ways. However, the representation described here has been found to be an easy way to provide some damping for the idealized system during that part of an impact history during which loads have subsided to the point where the honeycomb in the shock absorbers no longer crushes. If such damping is not provided, the idealized system will bounce erratically throughout this period. This result is contrary to the behavior of the model of reference 1 during test impacts. There the motion was quickly damped to that of a smooth, essentially rigid body, pivoting about one or two feet pressed against the landing surface. Unrealistic bouncing can result in erroneous predictions as to whether the model will overturn as the result of an impact. Representation of the elasticity of the model is not marred by removing the elastic forces from the compressive part of the stroke. Elastic action of the model appears to be significant only during a few initial relatively hard impingements of the feet on the landing surface. On the compressive part of the stroke during such an impingement, the strut compresses through the elastic part of the stroke so quickly that it does not matter what representation of the force is used. The primary effect of elastic forces comes as they impart a push to the vehicle during extension of a strut.

Quasi-static force for an extended strut.- The force given by equation (23) is not considered to be of very general interest. It was included in the programing in order to keep struts from extending beyond their original lengths since the struts of the model of reference 1 could not do so. A force which was linear in the extension rather than a step force was chosen to avoid instability in the numerical integration of the type discussed previously. The force acts only if the strut is extending, and it opposes extension. Consequently, it can only absorb energy and can never add energy to the system. This aspect of the programing is important because the constant K_S must be set very high to prevent substantial extension of the strut. If the force is programed as a spring which can return energy to the system, lightly damped high-frequency oscillations may be introduced into the calculated motion and cause difficulties in the numerical integration.

Total Forces and Torques

The space system components of the total force acting on the body are computed by the equation

$$F_X,F_Y,F_Z = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \left(F_{HX,j,k},F_{HY,j,k}F_{HZ,j,k}\right) \tag{24}$$

The body system components of the force acting on the body at a hard point are denoted by $F_{H\xi,j,k}, F_{H\eta,j,k}, F_{H\zeta,j,k}$ and are related to the space system components of the force by the equation

$$\begin{Bmatrix} F_{H\xi,j,k} \\ F_{H\eta,j,k} \\ F_{H\zeta,j,k} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{HX,j,k} \\ F_{HY,j,k} \\ F_{HZ,j,k} \end{Bmatrix} \tag{25}$$

The body system components of the total torque acting on the body are given by the equations:

$$N_\xi = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \left(\eta_{H,j,k} F_{H\zeta,j,k} - \zeta_{H,j,k} F_{H\eta,j,k}\right) \tag{26a}$$

$$N_\eta = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \left(\zeta_{H,j,k} F_{H\xi,j,k} - \xi_{H,j,k} F_{H\zeta,j,k}\right) \tag{26b}$$

$$N_\zeta = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \left(\xi_{H,j,k} F_{H\eta,j,k} - \eta_{H,j,k} F_{H\xi,j,k}\right) \tag{26c}$$

EQUATIONS OF MOTION OF THE FEET

Considerations Related to Landing Surface Planes

Equations of planes.- The landing surface plane associated with the jth foot is described by an equation of the form:

$$A_{X,j}X + A_{Y,j}Y + A_{Z,j}Z + A_j = 0 \tag{27}$$

where $A_{X,j}$, $A_{Y,j}$, $A_{Z,j}$, and A_j are constants which may be artibrarily selected. The equation is rewritten with normalized coefficients as follows in order to avoid ambiguity in specifying the position of a point in space relative to the plane:

$$W_{X,j}X + W_{Y,j}Y + W_{Z,j}Z + W_j = 0 \tag{28}$$

where

$$W_{X,j}, W_{Y,j}, W_{Z,j}, W_j = \frac{\left|A_{X,j}\right|\left(A_{X,j}, A_{Y,j}, A_{Z,j}, \frac{A_j}{L}\right)}{A_{Z,j}\sqrt{\left(A_{X,j}\right)^2 + \left(A_{Y,j}\right)^2 + \left(A_{Z,j}\right)^2}} \qquad \left(A_{Z,j} \neq 0\right) \tag{29a}$$

$$W_{X,j},W_{Y,j},W_{Z,j},W_j = \frac{-\left|A_{Y,j}\right|\left(A_{X,j},A_{Y,j},A_{Z,j},\frac{A_j}{L}\right)}{A_{Y,j}\sqrt{\left(A_{X,j}\right)^2 + \left(A_{Y,j}\right)^2 + \left(A_{Z,j}\right)^2}} \qquad \left(A_{Z,j} = 0, \quad A_{Y,j} \neq 0\right) \tag{29b}$$

$$W_{X,j},W_{Y,j},W_{Z,j},W_j = \frac{-\left|A_{X,j}\right|\left(A_{X,j},A_{Y,j},A_{Z,j},\frac{A_j}{L}\right)}{A_{X,j}\sqrt{\left(A_{X,j}\right)^2 + \left(A_{Y,j}\right)^2 + \left(A_{Z,j}\right)^2}} \qquad \left(A_{Z,j} = 0, \quad A_{Y,j} = 0, \quad A_{X,j} \neq 0\right) \tag{29c}$$

Normals.- The vector $W_{X,j},W_{Y,j},W_{Z,j}$ is normal to the landing surface plane associated with the jth foot. As a result of the normalization procedure just described, the normal is a unit vector directed as follows:

(1) If the jth plane is not vertical, the projection of the normal on the Z-axis points in the positive Z-direction.

(2) If the jth plane is vertical but not parallel to the Y-axis, the projection of the normal on the Y-axis points in the negative Y-direction.

(3) If the jth plane is vertical and parallel to the Y-axis, the normal points in the negative X-direction.

Surface and subsurface sides of a plane.- A landing surface plane divides inertial space into two spaces, one into which the normal is directed and one out of which the normal is directed. Points lying within the space into which the normal is directed are said to be to the surface side of the plane. Points lying within the space out of which the normal is directed are said to be to the subsurface side of the plane.

Distance of a point from a plane.- Let the symbol H_j denote the length of the perpendicular from a general point X,Y,Z to the jth landing surface plane. The distance is considered positive if the point is to the surface side of the plane and negative if the point is to the subsurface side of the plane. The following formula then gives the distance:

$$H_j = W_{X,j}X + W_{Y,j}Y + W_{Z,j}Z + W_j \tag{30}$$

Normal and tangential projections of a vector.- If the space system components Q_X,Q_Y,Q_Z of an arbitrary vector are given, it will be necessary to resolve the vector into the sum of a vector normal to and a vector tangential to the jth landing surface plane. The components in the space system of the normal and tangential vectors are, respectively, $N_{X,j},N_{Y,j},N_{Z,j}$ and $T_{X,j},T_{Y,j},T_{Z,j}$ and are readily computed with use of the following relations:

$$\begin{Bmatrix} N_{X,j} \\ N_{Y,j} \\ N_{Z,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & N_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} \tag{31a}$$

$$\begin{Bmatrix} T_{X,j} \\ T_{Y,j} \\ T_{Z,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & T_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} \tag{31b}$$

where

$$\left[N_{p,q,j}\right] = \begin{bmatrix} \left(W_{X,j}\right)^2 & \left(W_{X,j}\right)\left(W_{Y,j}\right) & \left(W_{X,j}\right)\left(W_{Z,j}\right) \\ \left(W_{X,j}\right)\left(W_{Y,j}\right) & \left(W_{Y,j}\right)^2 & \left(W_{Y,j}\right)\left(W_{Z,j}\right) \\ \left(W_{X,j}\right)\left(W_{Z,j}\right) & \left(W_{Y,j}\right)\left(W_{Z,j}\right) & \left(W_{Z,j}\right)^2 \end{bmatrix} \tag{32a}$$

$$\left[T_{p,q,j}\right] = \begin{bmatrix} \left[\left(W_{Y,j}\right)^2 + \left(W_{Z,j}\right)^2\right] & -\left(W_{X,j}\right)\left(W_{Y,j}\right) & -\left(W_{X,j}\right)\left(W_{Z,j}\right) \\ -\left(W_{X,j}\right)\left(W_{Y,j}\right) & \left[\left(W_{X,j}\right)^2 + \left(W_{Z,j}\right)^2\right] & -\left(W_{Y,j}\right)\left(W_{Z,j}\right) \\ -\left(W_{X,j}\right)\left(W_{Z,j}\right) & -\left(W_{Y,j}\right)\left(W_{Z,j}\right) & \left[\left(W_{X,j}\right)^2 + \left(W_{Y,j}\right)^2\right] \end{bmatrix} \tag{32b}$$

Free and penetrating foot.- The following definitions are made to facilitate concise statements regarding whether a foot is interacting with the landing surface material. The jth foot is said to be free if any of the following conditions hold:

(a) The foot is on the associated landing surface plane

$$W_{X,j}X_{F,j} + W_{Y,j}Y_{F,j} + W_{Z,j}Z_{F,j} + W_j = 0 \tag{33}$$

(b) The foot is to the surface side of the plane

$$W_{X,j}X_{F,j} + W_{Y,j}Y_{F,j} + W_{Z,j}Z_{F,j} + W_j > 0 \tag{34}$$

(c) The foot is to the subsurface side of the plane; and at the same time, the velocity component of the foot normal to the plane tends to carry the foot toward the plane

$$W_{X,j}X_{F,j} + W_{Y,j}Y_{F,j} + W_{Z,j}Z_{F,j} + W_j < 0 \quad (35a)$$

$$W_{X,j}\dot{X}_{F,j} + W_{Y,j}\dot{Y}_{F,j} + W_{Z,j}\dot{Z}_{F,j} > 0 \quad (35b)$$

The jth foot is said to be penetrating if it is to the subsurface side of the plane, and at the same time the velocity component of the foot normal to the plane is either zero or tends to carry the foot away from the plane

$$W_{X,j}X_{F,j} + W_{Y,j}Y_{F,j} + W_{Z,j}Z_{F,j} + W_j < 0 \quad (36a)$$

$$W_{X,j}\dot{X}_{F,j} + W_{Y,j}\dot{Y}_{F,j} + W_{Z,j}\dot{Z}_{F,j} \leq 0 \quad (36b)$$

Forces on a Penetrating Foot

When a foot is penetrating, it is considered to interact with the landing surface material and thus produces a force on the foot. Additional forces act on the foot through the three struts bearing on it. The force caused by the interaction with the landing surface material is considered to be the resultant of three separate forces termed the quasi-static normal force, the dynamic normal force, and the dynamic tangential force.

Quasi-static normal force.- This force acts normal to the landing surface plane associated with a foot and may therefore be expressed for the jth foot as

$$F_{FNSX,j}, F_{FNSY,j}, F_{FNSZ,j} = F_{FNS,j}\left(W_{X,j}, W_{Y,j}, W_{Z,j}\right) \quad (37)$$

Here $F_{FNSX,j}$, $F_{FNSY,j}$, and $F_{FNSZ,j}$ are the space system components of the force, $F_{FNS,j}$ is a scalar, and $W_{X,j}$, $W_{Y,j}$, and $W_{Z,j}$ are the components of the unit normal vector previously defined. The scalar $F_{FNS,j}$ is taken to be a cubic function of the variable D_j which is defined to be the absolute length of the perpendicular from the jth foot to the associated landing surface plane.

$$F_{FNS,j} = K_{1,j}D_j + K_{2,j}D_j^2 + K_{3,j}D_j^3 \quad (38)$$

where the coefficients $K_{i,j}$ (i = 1, 2, 3) may be arbitrarily assigned. From equation (30), D_j may be computed by the formula

$$D_j = \left|W_{X,j}X_{F,j} + W_{Y,j}Y_{F,j} + W_{Z,j}Z_{F,j} + W_j\right| \quad (39)$$

Dynamic normal force.- The space system components of this force are denoted $F_{FNDX,j}$, $F_{FNDY,j}$, and $F_{FNDZ,j}$. The force is proportional to the component of the velocity of the foot normal to the associated landing surface plane and acts in the direction opposite to that of the normal velocity; that is,

$$F_{FNDX,j}, F_{FNDY,j}, F_{FNDZ,j} = \frac{-1}{R_{N,j}}\left(\dot{X}_{FN,j}, \dot{Y}_{FN,j}, \dot{Z}_{FN,j}\right) \tag{40}$$

where $R_{N,j}$ is a positive constant to be arbitrarily assigned and $\dot{X}_{FN,j}$, $\dot{Y}_{FN,j}$, and $\dot{Z}_{FN,j}$ are the space system components of the normal velocity of the foot. By equation (31a),

$$\begin{Bmatrix} \dot{X}_{FN,j} \\ \dot{Y}_{FN,j} \\ \dot{Z}_{FN,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & N_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} \dot{X}_{F,j} \\ \dot{Y}_{F,j} \\ \dot{Z}_{F,j} \end{Bmatrix} \tag{41}$$

The dynamic normal force has the character of a frictional resistance to penetration into the landing surface by the foot. The constant $R_{N,j}$ has the character of the reciprocal of a viscosity constant.

Dynamic tangential force.- This force is the tangential counterpart of the dynamic normal force. The space system components of the force, denoted $F_{FTDX,j}, F_{FTDY,j}, F_{FTDZ,j}$, are given by the equation

$$F_{FTDX,j}, F_{FTDY,j}, F_{FTDZ,j} = \frac{-1}{R_{T,j}}\left(\dot{X}_{FT,j}, \dot{Y}_{FT,j}, \dot{Z}_{FT,j}\right) \tag{42}$$

where $R_{T,j}$ is a positive constant, and $\dot{X}_{FT,j}$, $\dot{Y}_{FT,j}$, and $\dot{Z}_{FT,j}$ are the space system coordinates of the tangential velocity given according to equation (31b) by

$$\begin{Bmatrix} \dot{X}_{FT,j} \\ \dot{Y}_{FT,j} \\ \dot{Z}_{FT,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & T_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} \dot{X}_{F,j} \\ \dot{Y}_{F,j} \\ \dot{Z}_{F,j} \end{Bmatrix} \tag{43}$$

Force through the struts.- The force acting on a foot through a strut is equal in magnitude but opposite in direction to the force $F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k}$ which is exerted through the strut onto the hard point at which the strut is attached to the body. Therefore, the space components $F_{FLX,j}$, $F_{FLY,j}$, and $F_{FLZ,j}$ of the total force on a foot through the three struts bearing on the foot may be computed by the formula

$$F_{FLX,j}, F_{FLY,j}, F_{FLZ,j} = -\sum_{k=1}^{k=3} F_{HX,j,k}, F_{HY,j,k}, F_{HZ,j,k} \tag{44}$$

By using equations (31a) and (31b), this force can be resolved into a force normal to and a force tangential to the associated landing surface plane with space components given by the respective equations:

$$\begin{Bmatrix} F_{FLNX,j} \\ F_{FLNY,j} \\ F_{FLNZ,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & N_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLX,j} \\ F_{FLY,j} \\ F_{FLZ,j} \end{Bmatrix} \tag{45a}$$

$$\begin{Bmatrix} F_{FLTX,j} \\ F_{FLTY,j} \\ F_{FLTZ,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & T_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLX,j} \\ F_{FLY,j} \\ F_{FLZ,j} \end{Bmatrix} \tag{45b}$$

Equation for a Penetrating Foot

Summing the forces on a foot in the normal and tangential directions gives the following equations:

$$\dot{X}_{FN,j}, \dot{Y}_{FN,j}, \dot{Z}_{FN,j} = F_{FNS,j} R_{N,j} \left(W_{X,j}, W_{Y,j}, W_{Z,j} \right) + R_{N,j} \left(F_{FLNX,j}, F_{FLNY,j}, F_{FLNZ,j} \right) \tag{46a}$$

$$\dot{X}_{FT,j}, \dot{Y}_{FT,j}, \dot{Z}_{FT,j} = R_{T,j} \left(F_{FLTX,j}, F_{FLTY,j}, F_{FLTZ,j} \right) \tag{46b}$$

Adding equations (46a) and (46b) gives

$$\begin{aligned} \dot{X}_{F,j}, \dot{Y}_{F,j}, \dot{Z}_{F,j} = {} & F_{FNS,j} R_{N,j} \left(W_{X,j}, W_{Y,j}, W_{Z,j} \right) + R_{N,j} \left(F_{FLNX,j}, F_{FLNY,j}, F_{FLNZ,j} \right) \\ & + R_{T,j} \left(F_{FLTX,j}, F_{FLTY,j}, F_{FLTZ,j} \right) \end{aligned} \tag{47}$$

Equation (47) is the equation of motion for the jth foot when the foot is penetrating.

Equation for a Free Foot

For computing the trajectory of a free foot, it is assumed that the lengths of the three struts connecting the foot to the body remain fixed so that the foot moves as a rigid extension of the body. Let $\xi_{F,j}, \eta_{F,j}, \zeta_{F,j}$ denote the instantaneous body coordinates of the jth foot. From equation (2)

$$\begin{Bmatrix} \xi_{F,j} \\ \eta_{F,j} \\ \zeta_{F,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} X_{F,j} \\ Y_{F,j} \\ Z_{F,j} \end{Bmatrix} \tag{48}$$

Suppose that at a specified time $t^{(f)}$ a foot has just become free. This time may be the initial time or a later time following a period of penetration. The space coordinates of the foot at this time may be presumed to be known and are denoted by $X^{(f)}_{F,j}, Y^{(f)}_{F,j}, Z^{(f)}_{F,j}$. The corresponding body coordinates are denoted by $\xi^{(f)}_{F,j}$, $\eta^{(f)}_{F,j}$, and $\zeta^{(f)}_{F,j}$ and may be computed from equation (48). The body coordinates of the foot remain fixed as long as the foot remains free. Therefore, during this time the space coordinates of the foot may be computed from equation (4) as

$$\begin{Bmatrix} X_{F,j} \\ Y_{F,j} \\ Z_{F,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi^{(f)}_{F,j} \\ \eta^{(f)}_{F,j} \\ \zeta^{(f)}_{F,j} \end{Bmatrix} + \begin{Bmatrix} X_O \\ Y_O \\ Z_O \end{Bmatrix} \tag{49}$$

Equation (49) is the equation of motion of the jth foot which is used when the foot is free.

DIMENSIONLESS EQUATIONS

The purpose of this part of the paper is to convert the equations of motion and certain auxiliary relations into equivalent dimensionless forms. Working with dimensionless equations of motion facilitates the application of results to both model and full-scale versions of a vehicle as is frequently necessary in studies of lunar landing dynamics. Also, replacing time by a dimensionless variable allows one to rely to some extent on previous experience in sizing the time step for numerical integration of the equations of motion.

Definitions of Dimensionless Quantities

The symbol for a dimensionless quantity is generally formed by adding the letter B to the subscript in a corresponding dimensional quantity. (The only exceptions to this rule are the dimensionless quantities defined by equations (56a) to (57) and quantities such as $U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k}$ and $W_{X,j}, W_{Y,j}, W_{Z,j}$ which are dimensionless as defined.)

Time.- Time t is replaced by the dimensionless variable t_B where

$$t_B = \sqrt{\frac{g}{L}}\, t \tag{50}$$

and L represents some characteristic length. An asterisk denotes differentiation with respect to t_B.

Lengths, forces, and velocities.- Lengths, forces, and linear velocities are rendered dimensionless by dividing them, respectively, by L, Mg, and $\sqrt{gL}$. For example,

Lengths:

$$X_{OB} = \frac{1}{L} X_O \tag{51a}$$

Forces:

$$F_{XB} = \frac{1}{Mg} F_X \tag{51b}$$

Velocities:

$$\overset{*}{X}_{FB,j} = \frac{1}{\sqrt{gL}} \dot{X}_{F,j} \tag{51c}$$

The dimensionless quantities arising in this manner (and denoted by subscript B) are as follows:

From lengths	From forces	From velocities
X_B, Y_B, Z_B	F_{XB}, F_{YB}, F_{ZB}	$V_{OXB}, V_{OYB}, V_{OZB}$
ξ_B, η_B, ζ_B	$F_{HXB,j,k}, F_{HYB,j,k}, F_{HZB,j,k}$	$\overset{*}{X}_{OB}, \overset{*}{Y}_{OB}, \overset{*}{Z}_{OB}$
X_{OB}, Y_{OB}, Z_{OB}	$F_{HB,j,k}$	$\overset{*}{S}_{B,j,k}$
$\xi_{PB}, \eta_{PB}, \zeta_{PB}$	$F_{RB,j,k}$	$\overset{*}{X}_{HB,j,k}, \overset{*}{Y}_{HB,j,k}, \overset{*}{Z}_{HB,j,k}$
X_{PB}, Y_{PB}, Z_{PB}	$F_{SB,j,k}$	$\overset{*}{S}_{ECB,j,k}$
$X_{FB,j}, Y_{FB,j}, Z_{FB,j}$	$F_{ECB,j,k}$	$\overset{*}{S}_{EEB,j,k}$
$X_{HB,j,k}, Y_{HB,j,k}, Z_{HB,j,k}$	$F_{EEB,j,k}$	$\overset{*}{S}_{CCB,j,k}$
$\xi_{HB,j,k}, \eta_{HB,j,k}, \zeta_{HB,j,k}$	$F_{CCB,j,k}$	$\overset{*}{S}_{CEB,j,k}$
$S_{B,j,k}$	$F_{CEB,j,k}$	$\overset{*}{X}_{FB,j}, \overset{*}{Y}_{FB,j}, \overset{*}{Z}_{FB,j}$
$S_{EB,j,k}$	$F_{H\xi B,j,k}, F_{H\eta B,j,k}, F_{H\zeta B,j,k}$	
$S_{RB,j,k}$	$F_{FNSB,j}$	
$S_{OB,j,k}$	$F_{FLXB,j}, F_{FLYB,j}, F_{FLZB,j}$	
$S_{SB,j,k}$	$F_{FLNXB,j}, F_{FLNYB,j}, F_{FLNZB,j}$	
$W_{B,j}$	$F_{FLTXB,j}, F_{FLTYB,j}, F_{FLTZB,j}$	
$H_{B,j}$		
$D_{B,j}$		
$\xi_{FB,j}, \eta_{FB,j}, \zeta_{FB,j}$		

Additional quantities.- Miscellaneous additional dimensionless quantities necessary for remaining developments are defined by the following equations:

$$\overset{*}{V}_{OXB},\overset{*}{V}_{OYB},\overset{*}{V}_{OZB} = \frac{1}{g}\left(\dot{V}_{OX},\dot{V}_{OY},\dot{V}_{OZ}\right) \tag{52}$$

$$\omega_{\xi B},\omega_{\eta B},\omega_{\zeta B} = \sqrt{\frac{L}{g}}\left(\omega_{\xi},\omega_{\eta},\omega_{\zeta}\right) \tag{53}$$

$$\overset{*}{\omega}_{\xi B},\overset{*}{\omega}_{\eta B},\overset{*}{\omega}_{\zeta B} = \frac{L}{g}\left(\dot{\omega}_{\xi},\dot{\omega}_{\eta},\dot{\omega}_{\zeta}\right) \tag{54}$$

$$N_{\xi B},N_{\eta B},N_{\zeta B} = \frac{L}{g}\left(\frac{N_{\xi}}{I_{\xi}},\frac{N_{\eta}}{I_{\eta}},\frac{N_{\zeta}}{I_{\zeta}}\right) \tag{55}$$

$$I_{\eta\xi B},I_{\zeta\xi B} = \frac{1}{I_{\xi}}\left(I_{\eta},I_{\zeta}\right) \tag{56a}$$

$$I_{\zeta\eta B},I_{\xi\eta B} = \frac{1}{I_{\eta}}\left(I_{\zeta},I_{\xi}\right) \tag{56b}$$

$$I_{\xi\zeta B},I_{\eta\zeta B} = \frac{1}{I_{\zeta}}\left(I_{\xi},I_{\eta}\right) \tag{56c}$$

$$B_{\xi B},B_{\eta B},B_{\zeta B} = \frac{1}{ML^2}\left(I_{\xi},I_{\eta},I_{\zeta}\right) \tag{57}$$

$$P_{B,l,j,k} = \frac{L^{l-1}}{Mg} P_{l,j,k} \qquad (l = 1, 2, 3, 4) \tag{58}$$

$$K_{SB} = \frac{L}{Mg} K_S \tag{59}$$

$$K_{B,i,j} = \frac{L^i K_{i,j}}{Mg} \qquad (i = 1, 2, 3) \tag{60}$$

$$R_{NB,j},R_{TB,j} = M\sqrt{\frac{g}{L}}\left(R_{N,j},R_{T,j}\right) \tag{61}$$

$$E_{KB} = \frac{1}{MgL} E_K \tag{62}$$

Dimensionless Form of Equations of Motion of Body

The following dimensionless form has been adopted for the equations of motion of the body. These dimensionless equations are readily derived from equations (9a) to (12) with use of the definitions of dimensionless quantities which have been established in this part of the paper.

$$\overset{*}{X}_{OB} = V_{OXB} \tag{63a}$$

$$\overset{*}{Y}_{OB} = V_{OYB} \tag{63b}$$

$$\overset{*}{Z}_{OB} = V_{OZB} \tag{63c}$$

$$\overset{*}{V}_{OXB} = F_{XB} \tag{63d}$$

$$\overset{*}{V}_{OYB} = F_{YB} \tag{63e}$$

$$\overset{*}{V}_{OZB} = F_{ZB} - 1 \tag{63f}$$

$$\overset{*}{\omega}_{\xi B} = \omega_{\eta B}\omega_{\zeta B}\left(I_{\eta\xi B} - I_{\zeta\xi B}\right) + N_{\xi B} \tag{64a}$$

$$\overset{*}{\omega}_{\eta B} = \omega_{\zeta B}\omega_{\xi B}\left(I_{\zeta\eta B} - I_{\xi\eta B}\right) + N_{\eta B} \tag{64b}$$

$$\overset{*}{\omega}_{\zeta B} = \omega_{\xi B}\omega_{\eta B}\left(I_{\xi\zeta B} - I_{\eta\zeta B}\right) + N_{\zeta B} \tag{64c}$$

$$\begin{Bmatrix} \overset{*}{\phi} \\ \overset{*}{\theta} \\ \overset{*}{\psi} \end{Bmatrix} = \begin{bmatrix} \dfrac{\sin\psi}{\sin\theta} & \dfrac{\cos\psi}{\sin\theta} & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\dfrac{\cos\theta\sin\psi}{\sin\theta} & -\dfrac{\cos\theta\cos\psi}{\sin\theta} & 1 \end{bmatrix} \begin{Bmatrix} \omega_{\xi B} \\ \omega_{\eta B} \\ \omega_{\zeta B} \end{Bmatrix} \tag{65}$$

$$\begin{Bmatrix} X_{PB} - X_{OB} \\ Y_{PB} - Y_{OB} \\ Z_{PB} - Z_{OB} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} \xi_{PB} \\ \eta_{PB} \\ \zeta_{PB} \end{Bmatrix} \tag{66}$$

Dimensionless Form of Equations of Motion of Feet

The dimensionless forms adopted for the equations of motion of a penetrating foot and a free foot are, respectively,

$$\overset{*}{X}_{FB,j}, \overset{*}{Y}_{FB,j}, \overset{*}{Z}_{FB,j} = R_{NB,j} F_{FNSB,j}\left(W_{X,j}, W_{Y,j}, W_{Z,j}\right)$$

$$+ R_{NB,j}\left(F_{FLNXB,j}, F_{FLNYB,j}, F_{FLNZB,j}\right)$$

$$+ R_{TB,j}\left(F_{FLTXB,j}, F_{FLTYB,j}, F_{FLTZB,j}\right) \qquad (67)$$

$$\begin{Bmatrix} X_{FB,j} \\ Y_{FB,j} \\ Z_{FB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi^{(f)}_{FB,j} \\ \eta^{(f)}_{FB,j} \\ \zeta^{(f)}_{FB,j} \end{Bmatrix} + \begin{Bmatrix} X_{OB} \\ Y_{OB} \\ Z_{OB} \end{Bmatrix} \qquad (68)$$

These equations are derived from equations (47) and (49).

Dimensionless Form of Auxiliary Relations

To facilitate working with the equations of motion in dimensionless form, a number of the relations established in the preceding sections are rewritten here in equivalent dimensionless forms:

From equation (13):

$$E_{KB} = \frac{1}{2}\left(V^2_{OXB} + V^2_{OYB} + V^2_{OZB}\right) + \frac{1}{2}\left(B_{\xi B}\omega^2_{\xi B} + B_{\eta B}\omega^2_{\eta B} + B_{\zeta B}\omega^2_{\zeta B}\right) \qquad (69)$$

From equation (14):

$$S_{B,j,k} = \left[\left(X_{HB,j,k} - X_{FB,j}\right)^2 + \left(Y_{HB,j,k} - Y_{FB,j}\right)^2 + \left(Z_{HB,j,k} - Z_{FB,j}\right)^2\right]^{1/2} \qquad (70)$$

From equation (15):

$$\overset{*}{S}_{B,j,k} = \frac{1}{S_{B,j,k}}\left[\left(X_{HB,j,k} - X_{FB,j}\right)\left(\overset{*}{X}_{HB,j,k} - \overset{*}{X}_{FB,j}\right)\right.$$

$$\left. + \left(Y_{HB,j,k} - Y_{FB,j}\right)\left(\overset{*}{Y}_{HB,j,k} - \overset{*}{Y}_{FB,j}\right) + \left(Z_{HB,j,k} - Z_{FB,j}\right)\left(\overset{*}{Z}_{HB,j,k} - \overset{*}{Z}_{FB,j}\right)\right]$$

(71)

From equation (16):

$$U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k} = \frac{1}{S_{B,j,k}}\left[\left(X_{HB,j,k} - X_{FB,j}\right), \left(Y_{HB,j,k} - Y_{FB,j}\right), \left(Z_{HB,j,k} - Z_{FB,j}\right)\right] \tag{72}$$

From equation (17):

$$F_{HXB,j,k}, F_{HYB,j,k}, F_{HZB,j,k} = F_{HB,j,k}\left(U_{X,j,k}, U_{Y,j,k}, U_{Z,j,k}\right) \tag{73}$$

From equation (18):

$$S_{SB,j,k} = S_{RB,j,k} - S_{B,j,k} \tag{74}$$

From equation (19):

$$F_{HB,j,k} = F_{SB,j,k} + F_{RB,j,k} \tag{75}$$

From equation (22):

$$\begin{aligned} F_{SB,j,k} = P_{B,1,j,k} &+ P_{B,2,j,k}\left(S_{RB,j,k} - S_{B,j,k}\right) + P_{B,3,j,k}\left(S_{RB,j,k} - S_{B,j,k}\right)^2 \\ &+ P_{B,4,j,k}\left(S_{RB,j,k} - S_{B,j,k}\right)^3 \\ &\qquad \left(0 \leqq S_{RB,j,k} - S_{B,j,k} \leqq S_{EB,j,k};\quad 0 < S^*_{B,j,k}\right) \end{aligned} \tag{76}$$

From equation (23):

$$F_{SB,j,k} = -K_{SB}\left(S_{B,j,k} - S_{OB,j,k}\right) \qquad \left(0 \leqq S_{B,j,k} - S_{OB,j,k};\quad 0 < S^*_{B,j,k}\right) \tag{77}$$

From equation (24):

$$F_{XB}, F_{YB}, F_{ZB} = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \left(F_{HXB,j,k}, F_{HYB,j,k}, F_{HZB,j,k}\right) \tag{78}$$

From equation (25):

$$\begin{Bmatrix} F_{H\xi B,j,k} \\ F_{H\eta B,j,k} \\ F_{H\zeta B,j,k} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{HXB,j,k} \\ F_{HYB,j,k} \\ F_{HZB,j,k} \end{Bmatrix} \tag{79}$$

From equation (26a):

$$N_{\xi B} = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \frac{1}{B_{\xi B}} \left(\eta_{HB,j,k} F_{H\zeta B,j,k} - \zeta_{HB,j,k} F_{H\eta B,j,k} \right) \tag{80a}$$

From equation (26b):

$$N_{\eta B} = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \frac{1}{B_{\eta B}} \left(\zeta_{HB,j,k} F_{H\xi B,j,k} - \xi_{HB,j,k} F_{H\zeta B,j,k} \right) \tag{80b}$$

From equation (26c):

$$N_{\zeta B} = \sum_{j=1}^{j=4} \sum_{k=1}^{k=3} \frac{1}{B_{\zeta B}} \left(\xi_{HB,j,k} F_{H\eta B,j,k} - \eta_{HB,j,k} F_{H\xi B,j,k} \right) \tag{80c}$$

From equation (28):

$$W_{X,j} X_B + W_{Y,j} Y_B + W_{Z,j} Z_B + W_{B,j} = 0 \tag{81}$$

From equation (30):

$$H_B = W_{X,j} X_B + W_{Y,j} Y_B + W_{Z,j} Z_B + W_{B,j} \tag{82}$$

From equation (38):

$$F_{FNSB,j} = K_{B,1,j} D_{B,j} + K_{B,2,j} D_{B,j}^2 + K_{B,3,j} D_{B,j}^3 \tag{83}$$

From equation (39):

$$D_{B,j} = \left| W_{X,j} X_{FB,j} + W_{Y,j} Y_{FB,j} + W_{Z,j} Z_{FB,j} + W_{B,j} \right| \tag{84}$$

From equation (44):

$$F_{FLXB,j}, F_{FLYB,j}, F_{FLZB,j} = - \sum_{k=1}^{k=3} F_{HXB,j,k}, F_{HYB,j,k}, F_{HZB,j,k} \tag{85}$$

From equation (45a):

$$\begin{Bmatrix} F_{FLNXB,j} \\ F_{FLNYB,j} \\ F_{FLNZB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & N_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLXB,j} \\ F_{FLYB,j} \\ F_{FLZB,j} \end{Bmatrix} \tag{86a}$$

From equation (45b):

$$\begin{Bmatrix} F_{FLTXB,j} \\ F_{FLTYB,j} \\ F_{FLTZB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & T_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLXB,j} \\ F_{FLYB,j} \\ F_{FLZB,j} \end{Bmatrix} \tag{86b}$$

NUMERICAL INTEGRATION OF EQUATIONS OF MOTION

A brief discussion of the basic ideas involved in the numerical integration of the equations of motion is given in this section.

Recurrence Equations

The fundamental equations used in the integration are the following equations which form a set of recurrence equations. A superscript (n) indicates that a quantity is computed by or is otherwise associated with the nth of successive applications of the recurrence equations. The quantity $\Delta t_B^{(n)}$ is the increment of dimensionless time used on the nth application.

$$X_{OB}^{(n)},Y_{OB}^{(n)},Z_{OB}^{(n)} = \left(X_{OB}^{(n-1)},Y_{OB}^{(n-1)},Z_{OB}^{(n-1)}\right) + \Delta t_B^{(n)}\left(\overset{*}{X}{}_{OB}^{(n-1)},\overset{*}{Y}{}_{OB}^{(n-1)},\overset{*}{Z}{}_{OB}^{(n-1)}\right) \tag{87a}$$

$$V_{OXB}^{(n)},V_{OYB}^{(n)},V_{OZB}^{(n)} = \left(V_{OXB}^{(n-1)},V_{OYB}^{(n-1)},V_{OZB}^{(n-1)}\right) + \Delta t_B^{(n)}\left(\overset{*}{V}{}_{OXB}^{(n-1)},\overset{*}{V}{}_{OYB}^{(n-1)},\overset{*}{V}{}_{OZB}^{(n-1)}\right) \tag{87b}$$

$$\omega_{\xi B}^{(n)},\omega_{\eta B}^{(n)},\omega_{\zeta B}^{(n)} = \left(\omega_{\xi B}^{(n-1)},\omega_{\eta B}^{(n-1)},\omega_{\zeta B}^{(n-1)}\right) + \Delta t_B^{(n)}\left(\overset{*}{\omega}{}_{\xi B}^{(n-1)},\overset{*}{\omega}{}_{\eta B}^{(n-1)},\overset{*}{\omega}{}_{\zeta B}^{(n-1)}\right) \tag{87c}$$

$$\overset{*}{X}{}_{OB}^{(n)},\overset{*}{Y}{}_{OB}^{(n)},\overset{*}{Z}{}_{OB}^{(n)} = \left(V_{OXB}^{(n)},V_{OYB}^{(n)},V_{OZB}^{(n)}\right) \tag{88a}$$

$$\overset{*}{V}{}_{OXB}^{(n)},\overset{*}{V}{}_{OYB}^{(n)},\overset{*}{V}{}_{OZB}^{(n)} = \left(F_{XB}^{(n)},F_{YB}^{(n)},F_{ZB}^{(n)} - 1\right) \tag{88b}$$

$$\overset{*}{\omega}{}^{(n)}_{\xi B} = \omega^{(n)}_{\eta B}\omega^{(n)}_{\zeta B}\left(I_{\eta\xi B} - I_{\zeta\xi B}\right) + N^{(n)}_{\xi B} \tag{89a}$$

$$\overset{*}{\omega}{}^{(n)}_{\eta B} = \omega^{(n)}_{\zeta B}\omega^{(n)}_{\xi B}\left(I_{\zeta\eta B} - I_{\xi\eta B}\right) + N^{(n)}_{\eta B} \tag{89b}$$

$$\overset{*}{\omega}{}^{(n)}_{\zeta B} = \omega^{(n)}_{\xi B}\omega^{(n)}_{\eta B}\left(I_{\xi\zeta B} - I_{\eta\zeta B}\right) + N^{(n)}_{\zeta B} \tag{89c}$$

If foot j is penetrating at $t_B^{(n-1)}$

$$X^{(n)}_{FB,j}, Y^{(n)}_{FB,j}, Z^{(n)}_{FB,j} = \left(X^{(n-1)}_{FB,j}, Y^{(n-1)}_{FB,j}, Z^{(n-1)}_{FB,j}\right) + \Delta t_B^{(n)}\left(\overset{*}{X}{}^{(n-1)}_{FB,j}, \overset{*}{Y}{}^{(n-1)}_{FB,j}, \overset{*}{Z}{}^{(n-1)}_{FB,j}\right) \tag{90}$$

$$\begin{aligned}\overset{*}{X}{}^{(n)}_{FB,j}, \overset{*}{Y}{}^{(n)}_{FB,j}, \overset{*}{Z}{}^{(n)}_{FB,j} = {} & R_{NB,j}F^{(n-1)}_{FNSB,j}\left(W_{X,j}, W_{Y,j}, W_{Z,j}\right) \\ & + R_{NB,j}\left(F^{(n-1)}_{FLNXB,j}, F^{(n-1)}_{FLNYB,j}, F^{(n-1)}_{FLNZB,j}\right) \\ & + R_{TB,j}\left(F^{(n-1)}_{FLTXB,j}, F^{(n-1)}_{FLTYB,j}, F^{(n-1)}_{FLTZB,j}\right)\end{aligned} \tag{91}$$

If foot j is free at $t_B^{(n-1)}$

$$\begin{Bmatrix} X^{(n)}_{FB,j} \\ Y^{(n)}_{FB,j} \\ Z^{(n)}_{FB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta^{(n)}_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi^{(n-1)}_{FB,j} \\ \eta^{(n-1)}_{FB,j} \\ \zeta^{(n-1)}_{FB,j} \end{Bmatrix} + \begin{Bmatrix} X^{(n)}_{OB} \\ Y^{(n)}_{OB} \\ Z^{(n)}_{OB} \end{Bmatrix} \tag{92a}$$

$$\begin{Bmatrix} \overset{*}{X}{}^{(n)}_{FB,j} \\ \overset{*}{Y}{}^{(n)}_{FB,j} \\ \overset{*}{Z}{}^{(n)}_{FB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \overset{*}{\delta}{}^{(n)}_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi^{(n-1)}_{FB,j} \\ \eta^{(n-1)}_{FB,j} \\ \zeta^{(n-1)}_{FB,j} \end{Bmatrix} + \begin{Bmatrix} V^{(n)}_{OXB} \\ V^{(n)}_{OYB} \\ V^{(n)}_{OZB} \end{Bmatrix} \tag{92b}$$

If foot j is free at $t_B^{(n)}$ and free at $t_B^{(n-1)}$

$$\xi^{(n)}_{FB,j}, \eta^{(n)}_{FB,j}, \zeta^{(n)}_{FB,j} = \xi^{(n-1)}_{FB,j}, \eta^{(n-1)}_{FB,j}, \zeta^{(n-1)}_{FB,j} \tag{93}$$

If foot j is free at $t_B^{(n)}$ and penetrating at $t_B^{(n-1)}$ or if foot j is penetrating at $t_B^{(n)}$

$$\begin{Bmatrix} \xi_{FB,j}^{(n)} \\ \eta_{FB,j}^{(n)} \\ \zeta_{FB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q}^{(n)} & \\ & & \end{bmatrix} \begin{Bmatrix} X_{FB,j}^{(n)} - X_{OB}^{(n)} \\ Y_{FB,j}^{(n)} - Y_{OB}^{(n)} \\ Z_{FB,j}^{(n)} - Z_{OB}^{(n)} \end{Bmatrix} \tag{94}$$

Auxiliary equations, for example, those relating forces and torques to the system variables, are used in the integration. Also, on each step a series of tests is performed to establish the dynamic state of each strut so that operations such as selection of appropriate force equations and proper setting of reference lengths may be performed. The auxiliary computations and tests are not shown here but are described in complete detail in the section "Programing Instructions."

Relation to Euler's Method

The integration scheme is practically identical to the classical straightforward method of Euler. (See ref. 9.) It differs from Euler's method only in the following respect:

Euler's recurrence scheme, stated for simplicity for a single-degree-of-freedom system with dependent variable α

$$\overset{*}{\alpha} = f(\alpha, t_B) \tag{95}$$

amounts to the following:

$$\alpha^{(n)} = \alpha^{(n-1)} + \Delta t_B^{(n)} \overset{*}{\alpha}^{(n-1)} \tag{96a}$$

$$\overset{*}{\alpha}^{(n)} = f\left(\alpha^{(n)}, t_B^{(n)}\right) \tag{96b}$$

However, if rate-dependent forces are considered, the single-degree-of-freedom analog of the dimensionless equations of motion does not take the form of equation (95) but is rather

$$\overset{*}{\alpha} = f\left(\alpha, \overset{*}{\alpha}, t_B\right) \tag{97}$$

The variable $\overset{*}{\alpha}$ on the right appears because the forces and torques may depend upon the rates of change of the strut lengths. Thus, Euler's method is not directly applicable because $\overset{*}{\alpha}$ is not an explicit function of α and t_B.

To get around this difficulty and yet preserve the basic simplicity of Euler's method, the dimensionless foot velocities $\overset{*}{X}^{(n)}_{FB,j}, \overset{*}{Y}^{(n)}_{FB,j}, \overset{*}{Z}^{(n)}_{FB,j}$ for a penetrating foot are not computed as functions of the forces on the foot at $t_B^{(n)}$ as would be required by a strict application of equation (96b) but as functions of the forces at $t_B^{(n-1)}$ as equation (91) shows. Thus, the dimensionless accelerations $\overset{*}{V}^{(n)}_{OXB}, \overset{*}{V}^{(n)}_{OYB}, \overset{*}{V}^{(n)}_{OZB}$ and $\overset{*}{\omega}^{(n)}_{\xi B}, \overset{*}{\omega}^{(n)}_{\eta B}, \overset{*}{\omega}^{(n)}_{\zeta B}$ are partially dependent upon conditions at $t_B^{(n-1)}$ because the forces $F^{(n)}_{XB}, F^{(n)}_{YB}, F^{(n)}_{ZB}$ and the torques $N^{(n)}_{\xi B}, N^{(n)}_{\eta B}, N^{(n)}_{\zeta B}$ depend on $\overset{*}{X}^{(n)}_{FB,j}$, $\overset{*}{Y}^{(n)}_{FB,j}$, and $\overset{*}{Z}^{(n)}_{FB,j}$. In contrast to Euler's method, therefore, the recurrence scheme depends on the information at both $t_B^{(n-1)}$ and $t_B^{(n-2)}$ in computing the system variables $\left(X^{(n)}_{OB}, Y^{(n)}_{OB}, Z^{(n)}_{OB}\right)$, $\left(V^{(n)}_{OXB}, V^{(n)}_{OYB}, V^{(n)}_{OZB}\right)$, and $\left(\omega^{(n)}_{\xi B}, \omega^{(n)}_{\eta B}, \omega^{(n)}_{\zeta B}\right)$.

If rate-dependent forces were suppressed, that is, if

$$F_{EC,j,k}, F_{EE,j,k}, F_{CC,j,k}, F_{CE,j,k} = 0,0,0,0 \tag{98}$$

and if on the right in equation (91) the superscripts n-1 were replaced by n, the recurrence scheme would reduce to Euler's method.

Efficiency and Validity

The method for integration was selected because it is very easy to program and because all the mathematical relations in the scheme may be interpreted physically. Efficiency was not a primary consideration. However, discussions with other workers in analysis of lunar landings have led the authors to believe that the method is relatively efficient in regard to consumption of computer time compared with methods now in use. About 2 minutes are required to compute an impact history with an IBM 7094 digital computer. Although the recurrence equations as given indicate a variable time interval $\Delta t_B^{(n)}$, the authors to date have programed a constant time interval throughout an impact. The computer time could probably be reduced by using a short time interval during the initial part of an impact when accelerations are generally high and a longer one later when accelerations have been reduced.

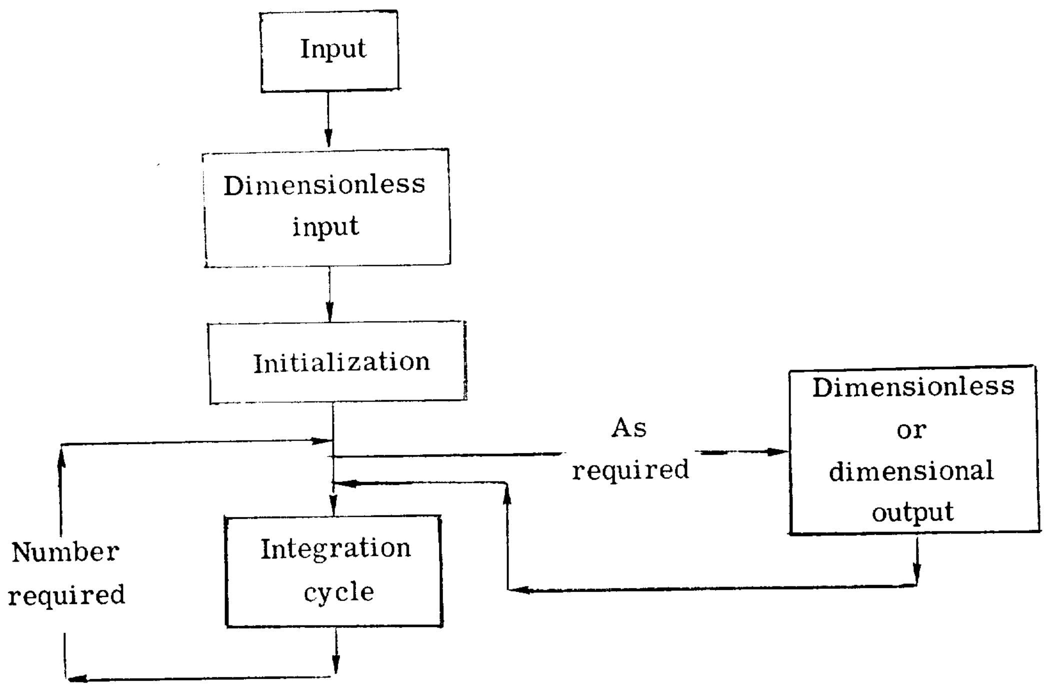
The authors are unable to offer any rigorous demonstration of the validity or accuracy of the integration method. The method has been tried on a number of systems selected so that exact solutions for the motion could be computed, falling bodies, for example. The approximate solutions have consistently been in good agreement with the exact when sufficiently small increments of the dimensionless time are used. Further confidence in the method has developed as a result of the good correlation between analytical and test results reported in reference 1.

PROGRAMING INSTRUCTIONS

The object in this part of the paper is to provide instructions which expedite programing a digital computer to carry out the computation of an impact history. The instructions are based entirely on relations which have been established in preceding parts of the paper. A constant time interval is assumed.

Basic Organization

The suggested basic organization of computing is shown in block diagram form:



A table of input information is read which describes the vehicle, the landing surface, and the initial orientation and velocities of the vehicle. The input is converted to dimensionless form by use of the definitions and relations in section "Dimensionless Equations." Then initial values of all quantities of interest are computed. After initialization, the computation proceeds into the integration cycle. On each pass through the integration cycle, the recurrence equations discussed in "Numerical Integration of the Equations of Motion" are applied to advance computation of the variables of the system one step in time. As often as desired, quantities are extracted from the integration cycle as output. Sometimes, it is convenient to convert the output to dimensional form.

Input

The quantities required as input are listed. The subscripts range as follows: $j = 1,2,3,4$; $k = 1,2,3$; $l = 1,2,3,4$; $i = 1,2,3$; $p = 1,2,3$; $q = 1,2,3$.

Gravitational constant: g

Characteristic length: L

Mass: M

Principal inertias: I_ξ, I_η, I_ζ

Body coordinates of hard points: $\xi_{H,j,k}, \eta_{H,j,k}, \zeta_{H,j,k}$

Initial body coordinates of feet: $\xi^{(0)}_{F,j}, \eta^{(0)}_{F,j}, \zeta^{(0)}_{F,j}$

Shock absorber constants:

$P_{l,j,k}$

$S_{E,j,k}$

$F_{EC,j,k}, F_{EE,j,k}, F_{CC,j,k}, F_{CE,j,k}$

$\dot{S}_{EC,j,k}, \dot{S}_{EE,j,k}, \dot{S}_{CC,j,k}, \dot{S}_{CE,j,k}$

K_S

Boundary plane coefficients: $A_{X,j}, A_{Y,j}, A_{Z,j}, A_j$

Surface impedance coefficients: $R_{N,j}$, $R_{T,j}$, and $K_{i,j}$ (Do not set $R_{N,j}$ or $R_{T,j}$ precisely equal to zero. In the present computing scheme, this procedure will cause the feet to be locked onto the landing surface plane. If infinite viscosity is desired, use very small finite values for $R_{N,j}$ and $R_{T,j}$.)

Initial space coordinates of center of gravity: $X^{(0)}_O, Y^{(0)}_O, Z^{(0)}_O$

Initial space components of velocity center of gravity: $V^{(0)}_{OX}, V^{(0)}_{OY}, V^{(0)}_{OZ}$

Initial direction cosines of body axes with respect to space axes: $\left[\delta^{(0)}_{p,q}\right]$

Initial components of angular velocity referred to body axes: $\omega_\xi^{(0)}, \omega_\eta^{(0)}, \omega_\zeta^{(0)}$

Dimensionless time increment: Δt_B

Conversion of Input to Dimensionless Form

The computations necessary to render the input dimensionless are as follows:

$$I_{\eta\xi B}, I_{\zeta\xi B} = \frac{1}{I_\xi}\left(I_\eta, I_\zeta\right)$$

$$I_{\zeta\eta B}, I_{\xi\eta B} = \frac{1}{I_\eta}\left(I_\zeta, I_\xi\right)$$

$$I_{\xi\zeta B}, I_{\eta\zeta B} = \frac{1}{I_\zeta}\left(I_\xi, I_\eta\right)$$

$$\xi_{HB,j,k}, \eta_{HB,j,k}, \zeta_{HB,j,k} = \frac{1}{L}\left(\xi_{H,j,k}, \eta_{H,j,k}, \zeta_{H,j,k}\right)$$

$$\xi_{FB,j}^{(0)}, \eta_{FB,j}^{(0)}, \zeta_{FB,j}^{(0)} = \frac{1}{L}\left(\xi_{F,j}^{(0)}, \eta_{F,j}^{(0)}, \zeta_{F,j}^{(0)}\right)$$

$$P_{B,l,j,k} = \frac{L^{l-1}}{Mg} P_{l,j,k}$$

$$S_{EB,j,k} = \frac{1}{L} S_{E,j,k}$$

$$F_{ECB,j,k}, F_{EEB,j,k}, F_{CCB,j,k}, F_{CEB,j,k} = \frac{1}{Mg}\left(F_{EC,j,k}, F_{EE,j,k}, F_{CC,j,k}, F_{CE,j,k}\right)$$

$$\overset{*}{S}_{ECB,j,k}, \overset{*}{S}_{EEB,j,k}, \overset{*}{S}_{CCB,j,k}, \overset{*}{S}_{CEB,j,k} = \frac{1}{\sqrt{gL}}\left(\dot{S}_{EC,j,k}, \dot{S}_{EE,j,k}, \dot{S}_{CC,j,k}, \dot{S}_{CE,j,k}\right)$$

$$K_{SB} = \frac{L}{Mg} K_S$$

$$R_{NB,j}, R_{TB,j} = M\sqrt{\frac{g}{L}}\left(R_{N,j}, R_{T,j}\right)$$

$$K_{B,i,j} = \frac{L^i K_{i,j}}{Mg}$$

$$X_{OB}^{(0)}, Y_{OB}^{(0)}, Z_{OB}^{(0)} = \frac{1}{L}\left(X_O^{(0)}, Y_O^{(0)}, Z_O^{(0)}\right)$$

$$V_{OXB}^{(0)}, V_{OYB}^{(0)}, V_{OZB}^{(0)} = \frac{1}{\sqrt{gL}}\left(V_{OX}^{(0)}, V_{OY}^{(0)}, V_{OZ}^{(0)}\right)$$

$$\omega^{(0)}_{\xi B}, \omega^{(0)}_{\eta B}, \omega^{(0)}_{\zeta B} = \sqrt{\frac{L}{g}}\left(\omega^{(0)}_{\xi}, \omega^{(0)}_{\eta}, \omega^{(0)}_{\zeta}\right)$$

Integration Cycle

The integration cycle is discussed before initialization because the necessity for the steps in initialization is much easier to understand once the integration cycle is understood. Upon entering the integration cycle for the first time, all quantities with superscript zero will have been provided as input or computed under initialization. Also the following quantities will have been computed under initialization:

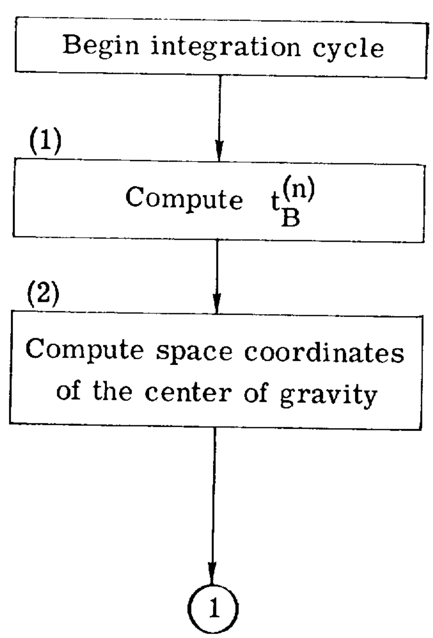
$$W_{X,j}, W_{Y,j}, W_{Z,j}, W_{B,j}$$

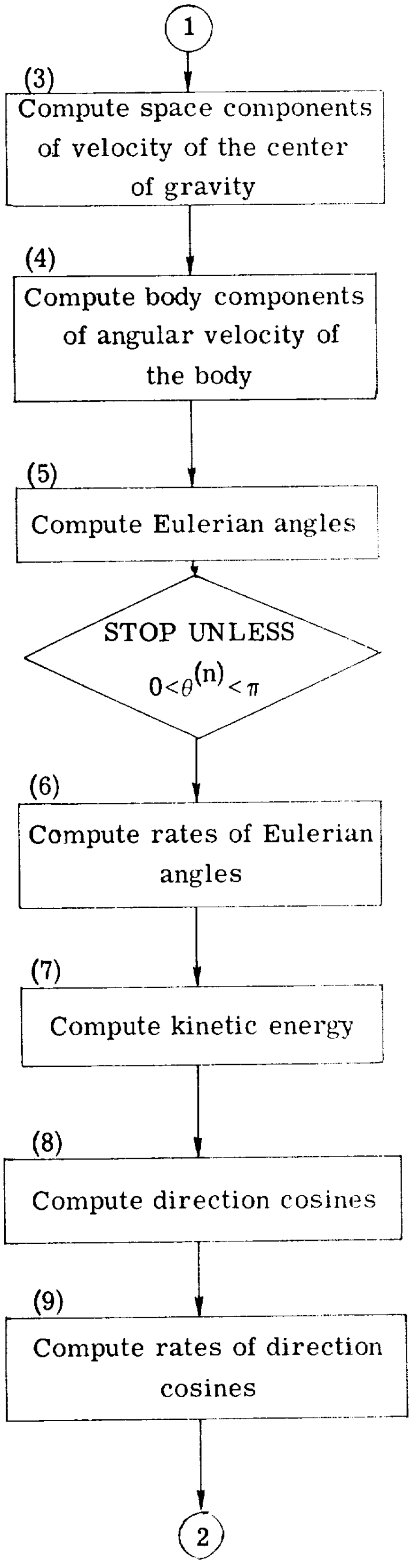
$$\left[N_{p,q,j}\right] \text{ and } \left[T_{p,q,j}\right]$$

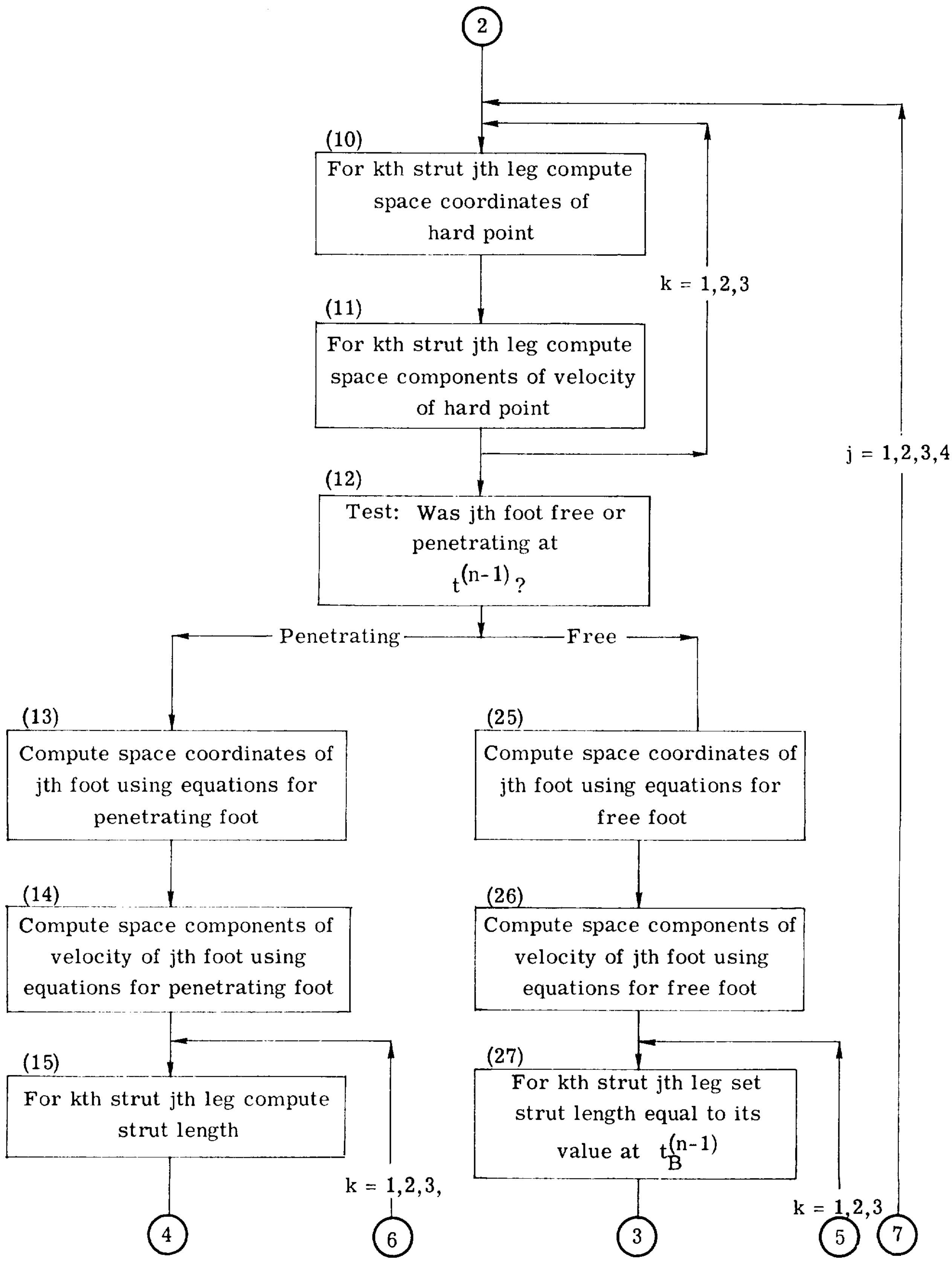
$$B_{\xi B}, B_{\eta B}, B_{\zeta B}$$

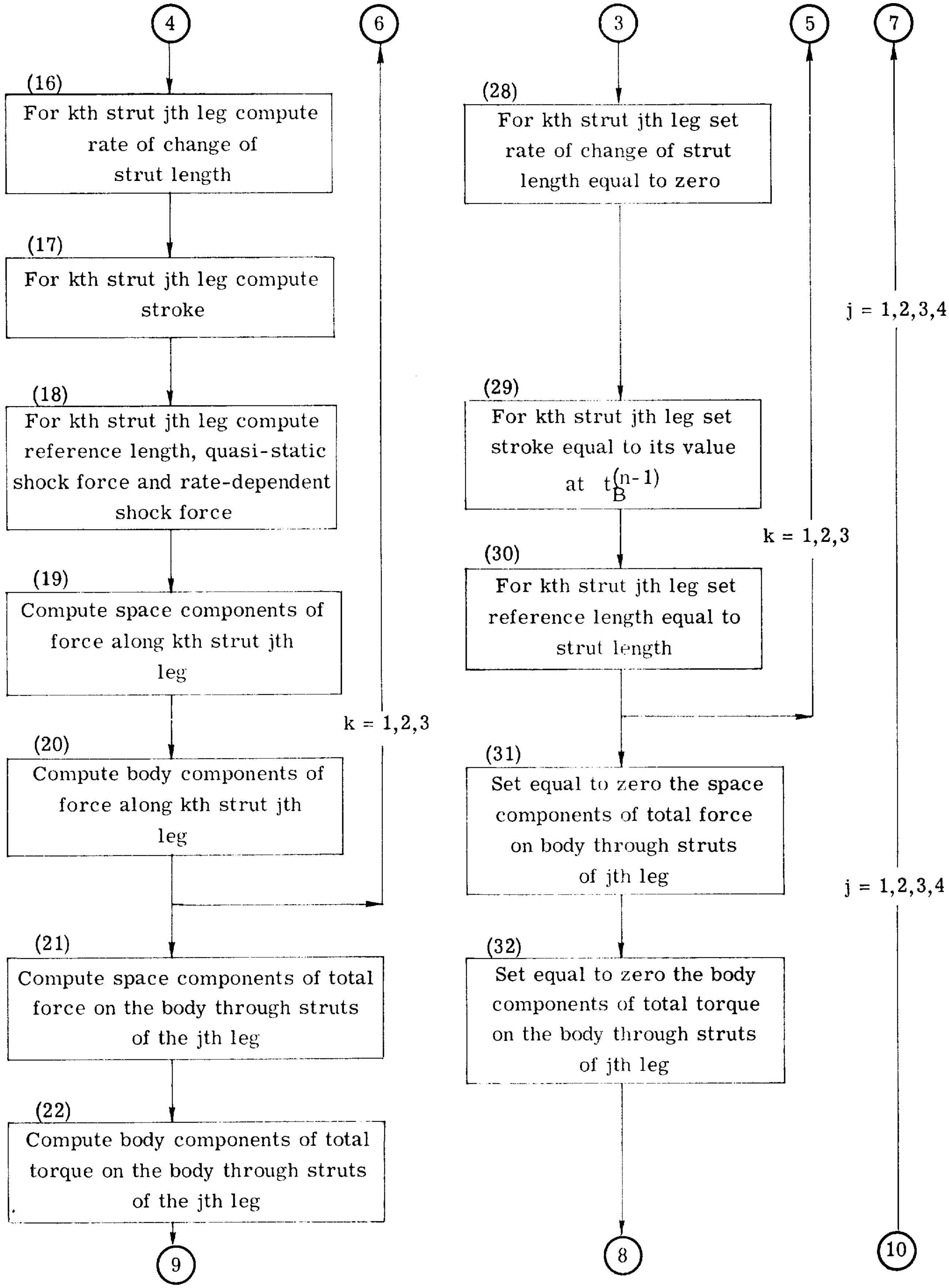
Upon entering the integration cycle for the nth time $(n > 1)$, all quantities with superscript $(n-1)$ will have been computed on the previous pass through the integration cycle.

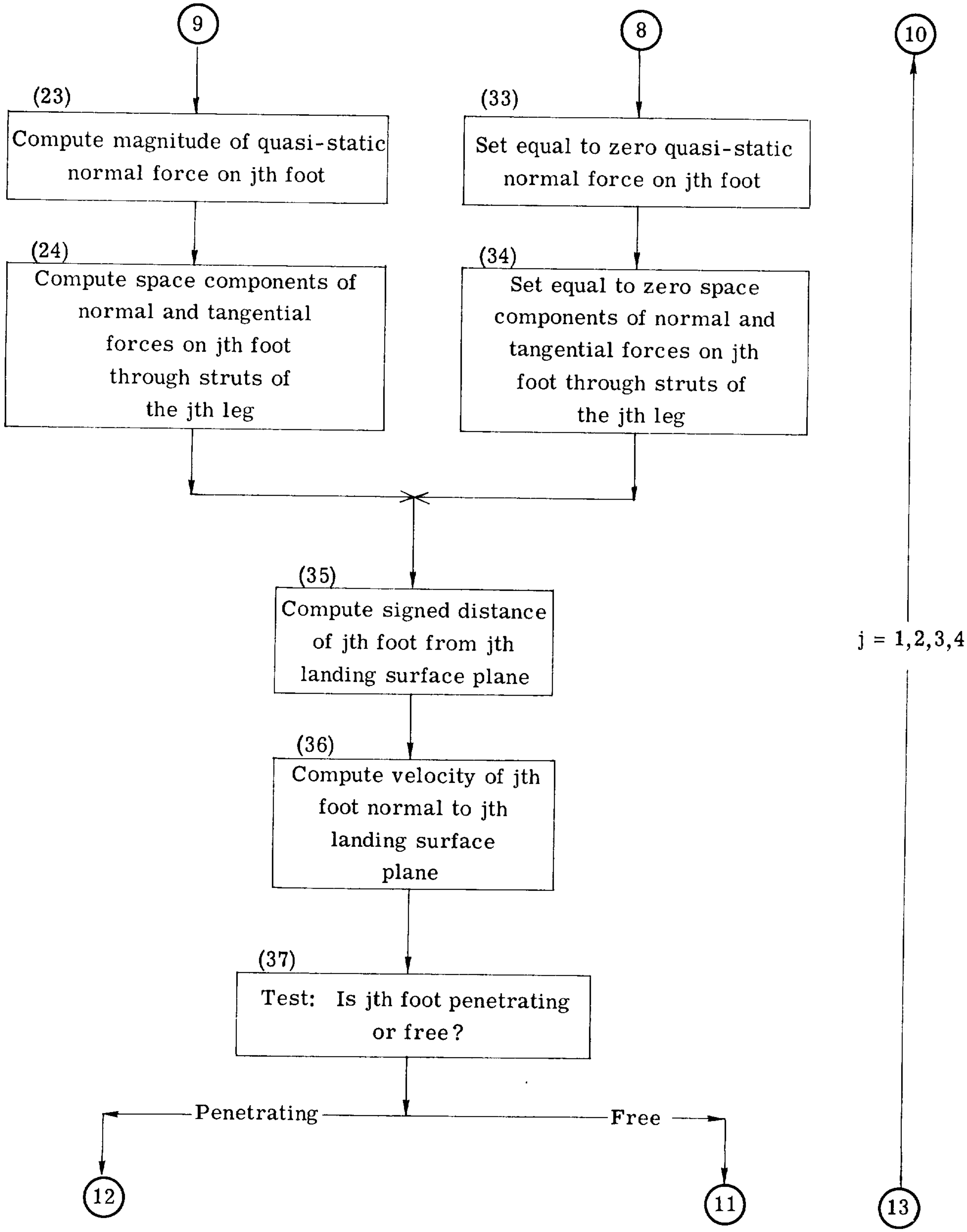
The organization of the integration cycle is shown in the block diagram which follows. The circled numbers indicate the connections to be made from page to page. After the block diagram are listed the relations and tests necessary to perform the operations called for in each block. A block is identified by the number at the upper left-hand corner.

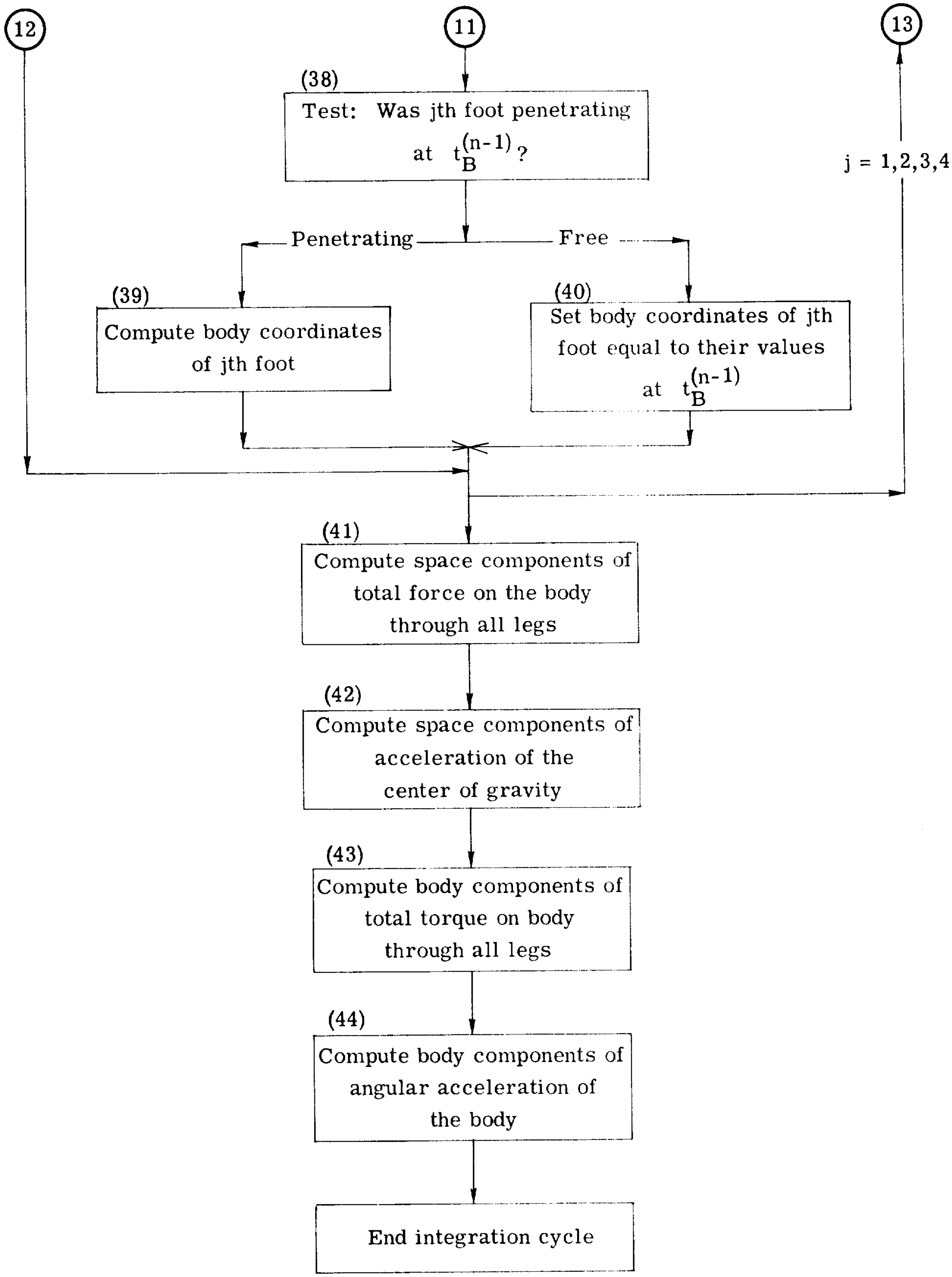



1
(3)
Compute space components of velocity of the center of gravity
(4)
Compute body components of angular velocity of the body
(5)
Compute Eulerian angles
STOP UNLESS $0<\theta^{(n)}<\pi$
(6)
Compute rates of Eulerian angles
(7)
Compute kinetic energy
(8)
Compute direction cosines
(9)
Compute rates of direction cosines
2


2
(10)
For kth strut jth leg compute space coordinates of hard point
k = 1,2,3
(11)
For kth strut jth leg compute space components of velocity of hard point
j = 1,2,3,4
(12)
Test: Was jth foot free or penetrating at $t^{(n-1)}$?
Penetrating
Free
(13)
Compute space coordinates of jth foot using equations for penetrating foot
(25)
Compute space coordinates of jth foot using equations for free foot
(14)
Compute space components of velocity of jth foot using equations for penetrating foot
(26)
Compute space components of velocity of jth foot using equations for free foot
(15)
For kth strut jth leg compute strut length
(27)
For kth strut jth leg set strut length equal to its value at $t_B^{(n-1)}$
k = 1,2,3,
k = 1,2,3
4
6
3
5
7


4
6
3
5
7
(16)
For kth strut jth leg compute rate of change of strut length
(17)
For kth strut jth leg compute stroke
(18)
For kth strut jth leg compute reference length, quasi-static shock force and rate-dependent shock force
(19)
Compute space components of force along kth strut jth leg
(20)
Compute body components of force along kth strut jth leg
k = 1,2,3
(21)
Compute space components of total force on the body through struts of the jth leg
(22)
Compute body components of total torque on the body through struts of the jth leg
9
(28)
For kth strut jth leg set rate of change of strut length equal to zero
(29)
For kth strut jth leg set stroke equal to its value at $t_B^{(n-1)}$
k = 1,2,3
(30)
For kth strut jth leg set reference length equal to strut length
(31)
Set equal to zero the space components of total force on body through struts of jth leg
(32)
Set equal to zero the body components of total torque on the body through struts of jth leg
8
j = 1,2,3,4
j = 1,2,3,4
10


9
8
10
(23)
Compute magnitude of quasi-static normal force on jth foot
(33)
Set equal to zero quasi-static normal force on jth foot
(24)
Compute space components of normal and tangential forces on jth foot through struts of the jth leg
(34)
Set equal to zero space components of normal and tangential forces on jth foot through struts of the jth leg
(35)
Compute signed distance of jth foot from jth landing surface plane
(36)
Compute velocity of jth foot normal to jth landing surface plane
(37)
Test: Is jth foot penetrating or free?
Penetrating
Free
j = 1,2,3,4
12
11
13


12
11
13
(38)
Test: Was jth foot penetrating at $t_B^{(n-1)}$?
j = 1,2,3,4
Penetrating
Free
(39)
Compute body coordinates of jth foot
(40)
Set body coordinates of jth foot equal to their values at $t_B^{(n-1)}$
(41)
Compute space components of total force on the body through all legs
(42)
Compute space components of acceleration of the center of gravity
(43)
Compute body components of total torque on body through all legs
(44)
Compute body components of angular acceleration of the body
End integration cycle

BLOCK 1:

$$t_B^{(n)} = t_B^{(n-1)} + \Delta t_B$$

BLOCK 2:

$$X_{OB}^{(n)}, Y_{OB}^{(n)}, Z_{OB}^{(n)} = \left(X_{OB}^{(n-1)}, Y_{OB}^{(n-1)}, Z_{OB}^{(n-1)}\right) + \Delta t_B \left(V_{OXB}^{(n-1)}, V_{OYB}^{(n-1)}, V_{OZB}^{(n-1)}\right)$$

BLOCK 3:

$$V_{OXB}^{(n)}, V_{OYB}^{(n)}, V_{OZB}^{(n)} = \left(V_{OXB}^{(n-1)}, V_{OYB}^{(n-1)}, V_{OZB}^{(n-1)}\right) + \Delta t_B \left(\overset{*}{V}{}_{OXB}^{(n-1)}, \overset{*}{V}{}_{OYB}^{(n-1)}, \overset{*}{V}{}_{OZB}^{(n-1)}\right)$$

BLOCK 4:

$$\omega_{\xi B}^{(n)}, \omega_{\eta B}^{(n)}, \omega_{\zeta B}^{(n)} = \left(\omega_{\xi B}^{(n-1)}, \omega_{\eta B}^{(n-1)}, \omega_{\zeta B}^{(n-1)}\right) + \Delta t_B \left(\overset{*}{\omega}{}_{\xi B}^{(n-1)}, \overset{*}{\omega}{}_{\eta B}^{(n-1)}, \overset{*}{\omega}{}_{\zeta B}^{(n-1)}\right)$$

BLOCK 5:

$$\phi^{(n)}, \theta^{(n)}, \psi^{(n)} = \left(\phi^{(n-1)}, \theta^{(n-1)}, \psi^{(n-1)}\right) + \Delta t_B \left(\overset{*}{\phi}{}^{(n-1)}, \overset{*}{\theta}{}^{(n-1)}, \overset{*}{\psi}{}^{(n-1)}\right)$$

BLOCK 6:

$$\begin{Bmatrix} \overset{*}{\phi}{}^{(n)} \\ \overset{*}{\theta}{}^{(n)} \\ \overset{*}{\psi}{}^{(n)} \end{Bmatrix} = \begin{bmatrix} \dfrac{\sin \psi^{(n)}}{\sin \theta^{(n)}} & \dfrac{\cos \psi^{(n)}}{\sin \theta^{(n)}} & 0 \\ \cos \psi^{(n)} & -\sin \psi^{(n)} & 0 \\ \dfrac{-\cos \theta^{(n)} \sin \psi^{(n)}}{\sin \theta^{(n)}} & \dfrac{-\cos \theta^{(n)} \cos \psi^{(n)}}{\sin \theta^{(n)}} & 1 \end{bmatrix} \begin{Bmatrix} \omega_{\xi B}^{(n)} \\ \omega_{\eta B}^{(n)} \\ \omega_{\zeta B}^{(n)} \end{Bmatrix}$$

BLOCK 7:

$$E_{KB}^{(n)} = \frac{1}{2}\left[\left(V_{OXB}^{(n)}\right)^2 + \left(V_{OYB}^{(n)}\right)^2 + \left(V_{OZB}^{(n)}\right)^2\right]$$

$$+ \frac{1}{2}\left[B_{\xi B}\left(\omega_{\xi B}^{(n)}\right)^2 + B_{\eta B}\left(\omega_{\eta B}^{(n)}\right)^2 + B_{\zeta B}\left(\omega_{\zeta B}^{(n)}\right)^2\right]$$

BLOCK 8:

$$\delta_{1,1}^{(n)} = \cos \psi^{(n)} \cos \phi^{(n)} - \cos \theta^{(n)} \sin \phi^{(n)} \sin \psi^{(n)}$$

$$\delta_{1,2}^{(n)} = \sin \psi^{(n)} \sin \theta^{(n)}$$

BLOCK 8 – Concluded:

$$\delta_{1,3}^{(n)} = -\cos \psi^{(n)}\sin \phi^{(n)} - \cos \theta^{(n)}\cos \phi^{(n)}\sin \psi^{(n)}$$

$$\delta_{2,1}^{(n)} = -\sin \psi^{(n)}\cos \phi^{(n)} - \cos \theta^{(n)}\sin \phi^{(n)}\cos \psi^{(n)}$$

$$\delta_{2,2}^{(n)} = \cos \psi^{(n)}\sin \theta^{(n)}$$

$$\delta_{2,3}^{(n)} = \sin \psi^{(n)}\sin \phi^{(n)} - \cos \theta^{(n)}\cos \phi^{(n)}\cos \psi^{(n)}$$

$$\delta_{3,1}^{(n)} = \sin \theta^{(n)}\sin \phi^{(n)}$$

$$\delta_{3,2}^{(n)} = \cos \theta^{(n)}$$

$$\delta_{3,3}^{(n)} = \sin \theta^{(n)}\cos \phi^{(n)}$$

BLOCK 9:

$$\begin{aligned}\overset{*}{\delta}{}_{1,1}^{(n)} = &-\left[\cos \psi^{(n)}\sin \phi^{(n)} + \cos \theta^{(n)}\cos \phi^{(n)}\sin \psi^{(n)}\right]\overset{*}{\phi}{}^{(n)} \\ &+ \left[\sin \theta^{(n)}\sin \phi^{(n)}\sin \psi^{(n)}\right]\overset{*}{\theta}{}^{(n)} \\ &- \left[\sin \psi^{(n)}\cos \phi^{(n)} + \cos \theta^{(n)}\sin \phi^{(n)}\cos \psi^{(n)}\right]\overset{*}{\psi}{}^{(n)}\end{aligned}$$

$$\overset{*}{\delta}{}_{1,2}^{(n)} = +\left[\sin \psi^{(n)}\cos \theta^{(n)}\right]\overset{*}{\theta}{}^{(n)} + \left[\cos \psi^{(n)}\sin \theta^{(n)}\right]\overset{*}{\psi}{}^{(n)}$$

$$\begin{aligned}\overset{*}{\delta}{}_{1,3}^{(n)} = &-\left[\cos \psi^{(n)}\cos \phi^{(n)} - \cos \theta^{(n)}\sin \phi^{(n)}\sin \psi^{(n)}\right]\overset{*}{\phi}{}^{(n)} \\ &+ \left[\sin \theta^{(n)}\cos \phi^{(n)}\sin \psi^{(n)}\right]\overset{*}{\theta}{}^{(n)} \\ &+ \left[\sin \psi^{(n)}\sin \phi^{(n)} - \cos \theta^{(n)}\cos \phi^{(n)}\cos \psi^{(n)}\right]\overset{*}{\psi}{}^{(n)}\end{aligned}$$

$$\begin{aligned}\overset{*}{\delta}{}_{2,1}^{(n)} = &\left[\sin \psi^{(n)}\sin \phi^{(n)} - \cos \theta^{(n)}\cos \phi^{(n)}\cos \psi^{(n)}\right]\overset{*}{\phi}{}^{(n)} \\ &+ \left[\sin \theta^{(n)}\sin \phi^{(n)}\cos \psi^{(n)}\right]\overset{*}{\theta}{}^{(n)} \\ &- \left[\cos \psi^{(n)}\cos \phi^{(n)} - \cos \theta^{(n)}\sin \phi^{(n)}\sin \psi^{(n)}\right]\overset{*}{\psi}{}^{(n)}\end{aligned}$$

BLOCK 9 – Concluded:

$$\overset{*}{\delta}{}^{(n)}_{2,2} = \left[\cos\psi^{(n)}\cos\theta^{(n)}\right]\overset{*}{\theta}{}^{(n)} - \left[\sin\psi^{(n)}\sin\theta^{(n)}\right]\overset{*}{\psi}{}^{(n)}$$

$$\begin{aligned}\overset{*}{\delta}{}^{(n)}_{2,3} = &\left[\sin\psi^{(n)}\cos\phi^{(n)} + \cos\theta^{(n)}\sin\phi^{(n)}\cos\psi^{(n)}\right]\overset{*}{\phi}{}^{(n)} \\ &+ \left[\sin\theta^{(n)}\cos\phi^{(n)}\cos\psi^{(n)}\right]\overset{*}{\theta}{}^{(n)} \\ &+ \left[\cos\psi^{(n)}\sin\phi^{(n)} + \cos\theta^{(n)}\cos\phi^{(n)}\sin\psi^{(n)}\right]\overset{*}{\psi}{}^{(n)}\end{aligned}$$

$$\overset{*}{\delta}{}^{(n)}_{3,1} = \left[\sin\theta^{(n)}\cos\phi^{(n)}\right]\overset{*}{\phi}{}^{(n)} + \left[\cos\theta^{(n)}\sin\phi^{(n)}\right]\overset{*}{\theta}{}^{(n)}$$

$$\overset{*}{\delta}{}^{(n)}_{3,2} = -\left[\sin\theta^{(n)}\right]\overset{*}{\theta}{}^{(n)}$$

$$\overset{*}{\delta}{}^{(n)}_{3,3} = -\left[\sin\theta^{(n)}\sin\phi^{(n)}\right]\overset{*}{\phi}{}^{(n)} + \left[\cos\theta^{(n)}\cos\phi^{(n)}\right]\overset{*}{\theta}{}^{(n)}$$

BLOCK 10:

$$\begin{Bmatrix} X^{(n)}_{HB,j,k} \\ Y^{(n)}_{HB,j,k} \\ Z^{(n)}_{HB,j,k} \end{Bmatrix} = \begin{bmatrix} \\ \delta^{(n)}_{p,q} \\ \\ \end{bmatrix}^T \begin{Bmatrix} \xi_{HB,j,k} \\ \eta_{HB,j,k} \\ \zeta_{HB,j,k} \end{Bmatrix} + \begin{Bmatrix} X^{(n)}_{OB} \\ Y^{(n)}_{OB} \\ Z^{(n)}_{OB} \end{Bmatrix}$$

BLOCK 11:

$$\begin{Bmatrix} \overset{*}{X}{}^{(n)}_{HB,j,k} \\ \overset{*}{Y}{}^{(n)}_{HB,j,k} \\ \overset{*}{Z}{}^{(n)}_{HB,j,k} \end{Bmatrix} = \begin{bmatrix} \\ \overset{*}{\delta}{}^{(n)}_{p,q} \\ \\ \end{bmatrix}^T \begin{Bmatrix} \xi_{HB,j,k} \\ \eta_{HB,j,k} \\ \zeta_{HB,j,k} \end{Bmatrix} + \begin{Bmatrix} V^{(n)}_{OXB} \\ V^{(n)}_{OYB} \\ V^{(n)}_{OZB} \end{Bmatrix}$$

BLOCK 12: The foot was free if either $H^{(n-1)}_{B,j} \geqq 0$ or $V^{(n-1)}_{NB,j} > 0$; otherwise, the foot was penetrating.

BLOCK 13:

$$X^{(n)}_{FB,j}, Y^{(n)}_{FB,j}, Z^{(n)}_{FB,j} = \left(X^{(n-1)}_{FB,j}, Y^{(n-1)}_{FB,j}, Z^{(n-1)}_{FB,j}\right) + \Delta t_B \left(\overset{*}{X}{}^{(n-1)}_{FB,j}, \overset{*}{Y}{}^{(n-1)}_{FB,j}, \overset{*}{Z}{}^{(n-1)}_{FB,j}\right)$$

BLOCK 14:

$$\overset{*}{X}{}^{(n)}_{FB,j}, \overset{*}{Y}{}^{(n)}_{FB,j}, \overset{*}{Z}{}^{(n)}_{FB,j} = R_{NB,j} F^{(n-1)}_{FNSB,j} \left(W_{X,j}, W_{Y,j}, W_{Z,j}\right)$$

$$+ R_{NB,j} \left(F^{(n-1)}_{FLNXB,j}, F^{(n-1)}_{FLNYB,j}, F^{(n-1)}_{FLNZB,j}\right)$$

$$+ R_{TB,j} \left(F^{(n-1)}_{FLTXB,j}, F^{(n-1)}_{FLTYB,j}, F^{(n-1)}_{FLTZB,j}\right)$$

BLOCK 15:

$$S^{(n)}_{B,j,k} = \left[\left(X^{(n)}_{HB,j,k} - X^{(n)}_{FB,j}\right)^2 + \left(Y^{(n)}_{HB,j,k} - Y^{(n)}_{FB,j}\right)^2 + \left(Z^{(n)}_{HB,j,k} - Z^{(n)}_{FB,j}\right)^2\right]^{1/2}$$

BLOCK 16:

$$\overset{*}{S}{}^{(n)}_{B,j,k} = \frac{1}{S^{(n)}_{B,j,k}} \left[\left(X^{(n)}_{HB,j,k} - X^{(n)}_{FB,j}\right)\left(\overset{*}{X}{}^{(n)}_{HB,j,k} - \overset{*}{X}{}^{(n)}_{FB,j}\right)\right.$$

$$+ \left(Y^{(n)}_{HB,j,k} - Y^{(n)}_{FB,j}\right)\left(\overset{*}{Y}{}^{(n)}_{HB,j,k} - \overset{*}{Y}{}^{(n)}_{FB,j}\right)$$

$$\left. + \left(Z^{(n)}_{HB,j,k} - Z^{(n)}_{FB,j}\right)\left(\overset{*}{Z}{}^{(n)}_{HB,j,k} - \overset{*}{Z}{}^{(n)}_{FB,j}\right)\right]$$

BLOCK 17:

$$S^{(n)}_{SB,j,k} = S^{(n-1)}_{RB,j,k} - S^{(n)}_{B,j,k}$$

BLOCK 18: The following table gives conditions and the operations to be performed when all conditions in a row of the table hold. Test first to determine whether the first condition holds; all other conditions are mutually exclusive.

Conditions				Do operation
Contracted $S^{(n)}_{SB,j,k} > 0$	Contracting $\overset{*}{S}{}^{(n)}_{B,j,k} < 0$	Elastic $0 < S^{(n)}_{SB,j,k} < S_{EB,j,k}$	Other	
---	No	---	$0 < S^{(n)}_{B,j,k} - S^{(0)}_{B,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = K_{SB}\left(S^{(0)}_{B,j,k} - S^{(n)}_{B,j,k}\right)$ $F^{(n)}_{RB,j,k} = 0$
Yes	Yes	Yes	$\left\|\overset{*}{S}_{B,j,k}\right\| < \overset{*}{S}_{CCB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{CCB,j,k}} F_{CCB,j,k}$
Yes	Yes	Yes	$\left\|\overset{*}{S}_{B,j,k}\right\| \geq \overset{*}{S}_{CCB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = F_{CCB,j,k}$
Yes	Yes	No	$\left\|\overset{*}{S}_{B,j,k}\right\| < \overset{*}{S}_{CCB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{CCB,j,k}} F_{CCB,j,k}$
Yes	Yes	No	$\left\|\overset{*}{S}_{B,j,k}\right\| \geq \overset{*}{S}_{CCB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = F_{CCB,j,k}$
Yes	No	Yes	$\overset{*}{S}_{B,j,k} < \overset{*}{S}_{CEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = P_{B,1,j,k} + P_{B,2,j,k}\left(S^{(n)}_{SB,j,k}\right) + P_{B,3,j,k}\left(S^{(n)}_{SB,j,k}\right)^2 + P_{B,4,j,k}\left(S^{(n)}_{SB,j,k}\right)^3$ $F^{(n)}_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{CEB,j,k}} F_{CEB,j,k}$

Conditions				Do operation
Contracted $S^{(n)}_{SB,j,k} > 0$	Contracting $\overset{*}{S}{}^{(n)}_{B,j,k} < 0$	Elastic $0 < S^{(n)}_{SB,j,k} \leq S_{EB,j,k}$	Other	
Yes	No	Yes	$\overset{*}{S}_{B,j,k} > \overset{*}{S}_{CEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = P_{B,1,j,k} + P_{B,2,j,k}\left(S^{(n)}_{SB,j,k}\right) + P_{B,3,j,k}\left(S^{(n)}_{SB,j,k}\right)^2 + P_{B,4,j,k}\left(S^{(n)}_{SB,j,k}\right)^3$ $F^{(n)}_{RB,j,k} = -F_{CEB,j,k}$
Yes	No	No	$\overset{*}{S}_{B,j,k} < \overset{*}{S}_{CEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n)}_{B,j,k} + S_{EB,j,k}$ $F^{(n)}_{SB,j,k} = P_{B,1,j,k} + P_{B,2,j,k}\left(S_{EB,j,k}\right) + P_{B,3,j,k}\left(S_{EB,j,k}\right)^2 + P_{B,4,j,k}\left(S_{EB,j,k}\right)^3$ $F^{(n)}_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{CEB,j,k}} F_{CEB,j,k}$
Yes	No	No	$\overset{*}{S}_{B,j,k} \geq \overset{*}{S}_{CEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n)}_{B,j,k} + S_{EB,j,k}$ $F^{(n)}_{SB,j,k} = P_{B,1,j,k} + P_{B,2,j,k}\left(S_{EB,j,k}\right) + P_{B,3,j,k}\left(S_{EB,j,k}\right)^2 + P_{B,4,j,k}\left(S_{EB,j,k}\right)^3$ $F^{(n)}_{RB,j,k} = -F_{CEB,j,k}$
No	Yes	---	$\left\lvert\overset{*}{S}_{B,j,k}\right\rvert < \overset{*}{S}_{ECB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{ECB,j,k}} F_{ECB,j,k}$
No	Yes	---	$\left\lvert\overset{*}{S}_{B,j,k}\right\rvert \geq \overset{*}{S}_{ECB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = F_{ECB,j,k}$
No	No	---	$\left\lvert\overset{*}{S}_{B,j,k}\right\rvert < \overset{*}{S}_{EEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F_{RB,j,k} = -\dfrac{\overset{*}{S}{}^{(n)}_{B,j,k}}{\overset{*}{S}_{EEB,j,k}} F_{EEB,j,k}$
No	No	---	$\left\lvert\overset{*}{S}_{B,j,k}\right\rvert \geq \overset{*}{S}_{EEB,j,k}$	$S^{(n)}_{RB,j,k} = S^{(n-1)}_{RB,j,k}$ $F^{(n)}_{SB,j,k} = 0$ $F^{(n)}_{RB,j,k} = -F_{EEB,j,k}$

BLOCK 19:

$$U^{(n)}_{X,j,k}, U^{(n)}_{Y,j,k}, U^{(n)}_{Z,j,k} = \frac{1}{S^{(n)}_{B,j,k}}\left[\left(X^{(n)}_{HB,j,k} - X^{(n)}_{FB,j}\right), \left(Y^{(n)}_{HB,j,k} - Y^{(n)}_{FB,j}\right), \left(Z^{(n)}_{HB,j,k} - Z^{(n)}_{FB,j}\right)\right]$$

$$F^{(n)}_{HB,j,k} = F^{(n)}_{SB,j,k} + F^{(n)}_{RB,j,k}$$

$$F^{(n)}_{HXB,j,k}, F^{(n)}_{HYB,j,k}, F^{(n)}_{HZB,j,k} = F^{(n)}_{HB,j,k}\left(U^{(n)}_{X,j,k}, U^{(n)}_{Y,j,k}, U^{(n)}_{Z,j,k}\right)$$

BLOCK 20:

$$\begin{Bmatrix} F^{(n)}_{H\xi B,j,k} \\ F^{(n)}_{H\eta B,j,k} \\ F^{(n)}_{H\zeta B,j,k} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta^{(n)}_{p,q} & \\ & & \end{bmatrix} \begin{Bmatrix} F^{(n)}_{HXB,j,k} \\ F^{(n)}_{HYB,j,k} \\ F^{(n)}_{HZB,j,k} \end{Bmatrix}$$

BLOCK 21:

$$F^{(n)}_{HXBT,j}, F^{(n)}_{HYBT,j}, F^{(n)}_{HZBT,j} = \sum_{k=1}^{k=3}\left(F^{(n)}_{HXB,j,k}, F^{(n)}_{HYB,j,k}, F^{(n)}_{HZB,j,k}\right)$$

BLOCK 22:

$$N^{(n)}_{\xi BT,j} = \sum_{k=1}^{k=3} \frac{1}{B_{\xi B}}\left(\eta_{HB,j,k} F^{(n)}_{H\zeta B,j,k} - \zeta_{HB,j,k} F^{(n)}_{H\eta B,j,k}\right)$$

$$N^{(n)}_{\eta BT,j} = \sum_{k=1}^{k=3} \frac{1}{B_{\eta B}}\left(\zeta_{HB,j,k} F^{(n)}_{h\xi B,j,k} - \xi_{HB,j,k} F^{(n)}_{H\zeta B,j,k}\right)$$

$$N^{(n)}_{\zeta BT,j} = \sum_{k=1}^{k=3} \frac{1}{B_{\zeta B}}\left(\xi_{HB,j,k} F^{(n)}_{H\eta B,j,k} - \eta_{HB,j,k} F^{(n)}_{H\xi B,j,k}\right)$$

BLOCK 23:

$$D^{(n)}_{B,j} = \left|\left(W_{X,j}\right)\left(X^{(n)}_{FB,j}\right) + \left(W_{Y,j}\right)\left(Y^{(n)}_{FB,j}\right) + \left(W_{Z,j}\right)\left(Z^{(n)}_{FB,j}\right) + W_{B,j}\right|$$

BLOCK 23 – Concluded:

$$F_{FNSB,j}^{(n)} = K_{B,1,j} D_{B,j}^{(n)} + K_{B,2,j}\left(D_{B,j}^{(n)}\right)^2 + K_{B,3,j}\left(D_{B,j}^{(n)}\right)^3$$

BLOCK 24:

$$F_{FLXB,j}^{(n)}, F_{FLYB,j}^{(n)}, F_{FLZB,j}^{(n)} = -\left(F_{HXBT,j}^{(n)}, F_{HYBT,j}^{(n)}, F_{HZBT,j}^{(n)}\right)$$

$$\begin{Bmatrix} F_{FLNXB,j}^{(n)} \\ F_{FLNYB,j}^{(n)} \\ F_{FLNZB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & N_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLXB,j}^{(n)} \\ F_{FLYB,j}^{(n)} \\ F_{FLZB,j}^{(n)} \end{Bmatrix}$$

$$\begin{Bmatrix} F_{FLTXB,j}^{(n)} \\ F_{FLTYB,j}^{(n)} \\ F_{FLTZB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & T_{p,q,j} & \\ & & \end{bmatrix} \begin{Bmatrix} F_{FLXB,j}^{(n)} \\ F_{FLYB,j}^{(n)} \\ F_{FLZB,j}^{(n)} \end{Bmatrix}$$

BLOCK 25:

$$\begin{Bmatrix} X_{FB,j}^{(n)} \\ Y_{FB,j}^{(n)} \\ Z_{FB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q}^{(n)} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi_{FB,j}^{(n-1)} \\ \eta_{FB,j}^{(n-1)} \\ \zeta_{FB,j}^{(n-1)} \end{Bmatrix} + \begin{Bmatrix} X_{OB}^{(n)} \\ Y_{OB}^{(n)} \\ Z_{OB}^{(n)} \end{Bmatrix}$$

BLOCK 26:

$$\begin{Bmatrix} \overset{*}{X}{}_{FB,j}^{(n)} \\ \overset{*}{Y}{}_{FB,j}^{(n)} \\ \overset{*}{Z}{}_{FB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & \overset{*}{\delta}{}_{p,q}^{(n)} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi_{FB,j}^{(n-1)} \\ \eta_{FB,j}^{(n-1)} \\ \zeta_{FB,j}^{(n-1)} \end{Bmatrix} + \begin{Bmatrix} V_{OXB}^{(n)} \\ V_{OYB}^{(n)} \\ V_{OZB}^{(n)} \end{Bmatrix}$$

BLOCK 27:

$$S_{B,j,k}^{(n)} = S_{B,j,k}^{(n-1)}$$

BLOCK 28:

$$\overset{*}{S}_{B,j,k}^{(n)} = 0$$

BLOCK 29:

$$S_{SB,j,k}^{(n)} = S_{SB,j,k}^{(n-1)}$$

BLOCK 30:

$$S_{RB,j,k}^{(n)} = S_{B,j,k}^{(n)}$$

BLOCK 31:

$$F_{HXBT,j}^{(n)}, F_{HYBT,j}^{(n)}, F_{HZBT,j}^{(n)} = 0,0,0$$

BLOCK 32:

$$N_{\xi BT,j}^{(n)}, N_{\eta BT,j}^{(n)}, N_{\zeta BT,j}^{(n)} = 0,0,0$$

BLOCK 33:

$$F_{FNSB,j}^{(n)} = 0$$

BLOCK 34:

$$F_{FLNXB,j}^{(n)}, F_{FLNYB,j}^{(n)}, F_{FLNZB,j}^{(n)} = 0,0,0$$

$$F_{FLTXB,j}^{(n)}, F_{FLTYB,j}^{(n)}, F_{FLTZB,j}^{(n)} = 0,0,0$$

BLOCK 35:

$$H_{B,j}^{(n)} = W_{X,j} X_{FB,j}^{(n)} + W_{Y,j} Y_{FB,j}^{(n)} + W_{Z,j} Z_{FB,j}^{(n)} + W_{B,j}$$

BLOCK 36:

$$V_{NB,j}^{(n)} = W_{X,j} \overset{*}{X}_{FB,j}^{(n)} + W_{Y,j} \overset{*}{Y}_{FB,j}^{(n)} + W_{Z,j} \overset{*}{Z}_{FB,j}^{(n)}$$

BLOCK 37: The foot is free if either $H_{B,j}^{(n)} \geqq 0$ or $V_{NB,j}^{(n)} > 0$; otherwise, the foot is penetrating.

BLOCK 38: The foot was free if either $H_{B,j}^{(n-1)} \geqq 0$ or $V_{NB,j}^{(n-1)} > 0$; otherwise, the foot was penetrating.

BLOCK 39:

$$\begin{Bmatrix} \xi_{FB,j}^{(n)} \\ \eta_{FB,j}^{(n)} \\ \zeta_{FB,j}^{(n)} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta_{p,q}^{(n)} & \\ & & \end{bmatrix} \begin{Bmatrix} X_{FB,j}^{(n)} - X_{OB}^{(n)} \\ Y_{FB,j}^{(n)} - Y_{OB}^{(n)} \\ Z_{FB,j}^{(n)} - Z_{OB}^{(n)} \end{Bmatrix}$$

BLOCK 40:

$$\xi_{FB,j}^{(n)}, \eta_{FB,j}^{(n)}, \zeta_{FB,j}^{(n)} = \xi_{FB,j}^{(n-1)}, \eta_{FB,j}^{(n-1)}, \zeta_{FB,j}^{(n-1)}$$

BLOCK 41:

$$F_{XB}^{(n)}, F_{YB}^{(n)}, F_{ZB}^{(n)} = \sum_{j=1}^{j=4} \left(F_{HXBT,j}^{(n)}, F_{HYBT,j}^{(n)}, F_{HZBT,j}^{(n)} \right)$$

BLOCK 42:

$$\overset{*}{V}{}_{OXB}^{(n)}, \overset{*}{V}{}_{OYB}^{(n)}, \overset{*}{V}{}_{OZB}^{(n)} = F_{XB}^{(n)}, F_{YB}^{(n)}, F_{ZB}^{(n)} - 1$$

BLOCK 43:

$$N_{\xi B}^{(n)}, N_{\eta B}^{(n)}, N_{\zeta B}^{(n)} = \sum_{j=1}^{j=4} \left(N_{\xi BT,j}^{(n)}, N_{\eta BT,j}^{(n)}, N_{\zeta BT,j}^{(n)} \right)$$

BLOCK 44:

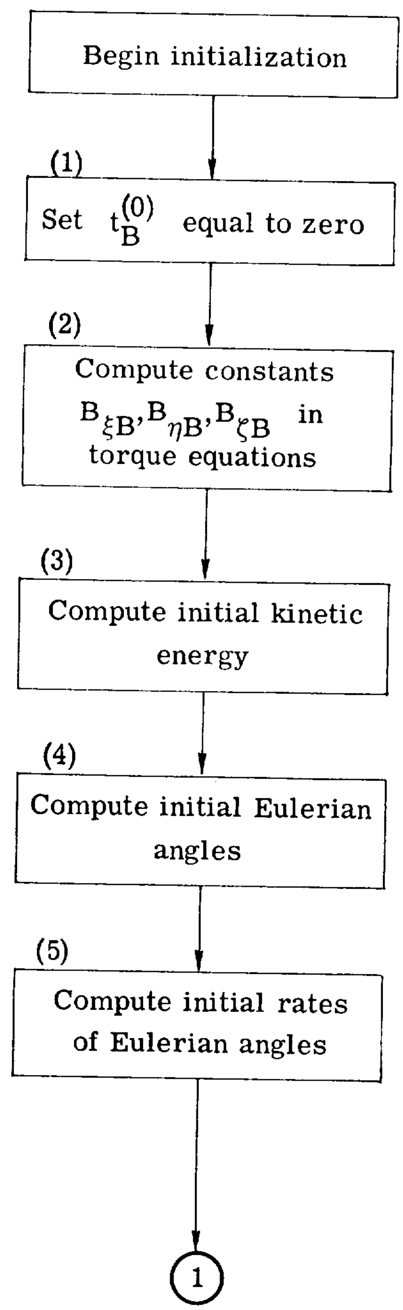
$$\overset{*}{\omega}{}_{\xi B}^{(n)} = \omega_{\eta B}^{(n)} \omega_{\zeta B}^{(n)} \left(I_{\eta \xi B} - I_{\zeta \xi B} \right) + N_{\xi B}^{(n)}$$

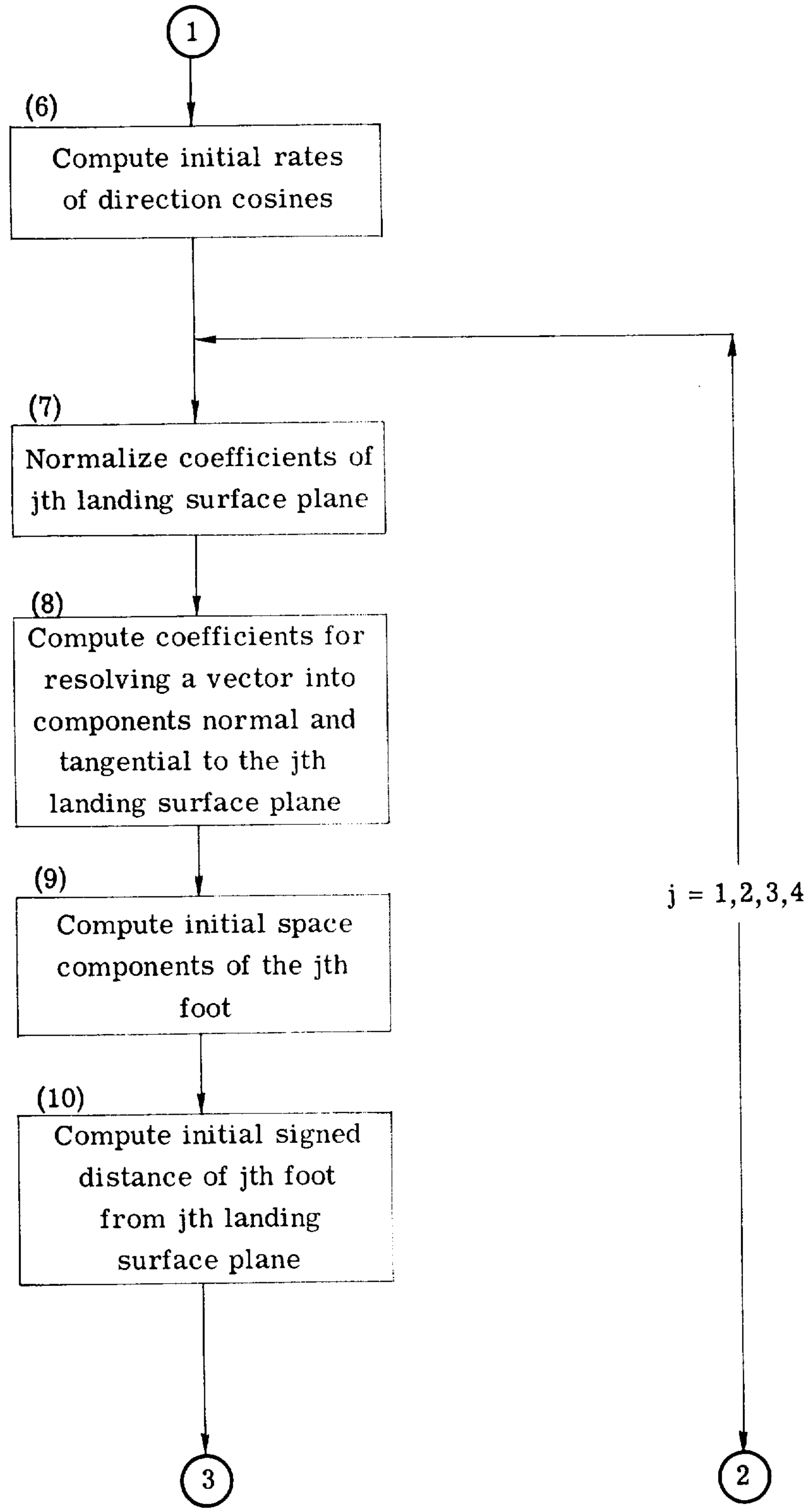
$$\overset{*}{\omega}{}_{\eta B}^{(n)} = \omega_{\zeta B}^{(n)} \omega_{\xi B}^{(n)} \left(I_{\zeta \eta B} - I_{\xi \eta B} \right) + N_{\eta B}^{(n)}$$

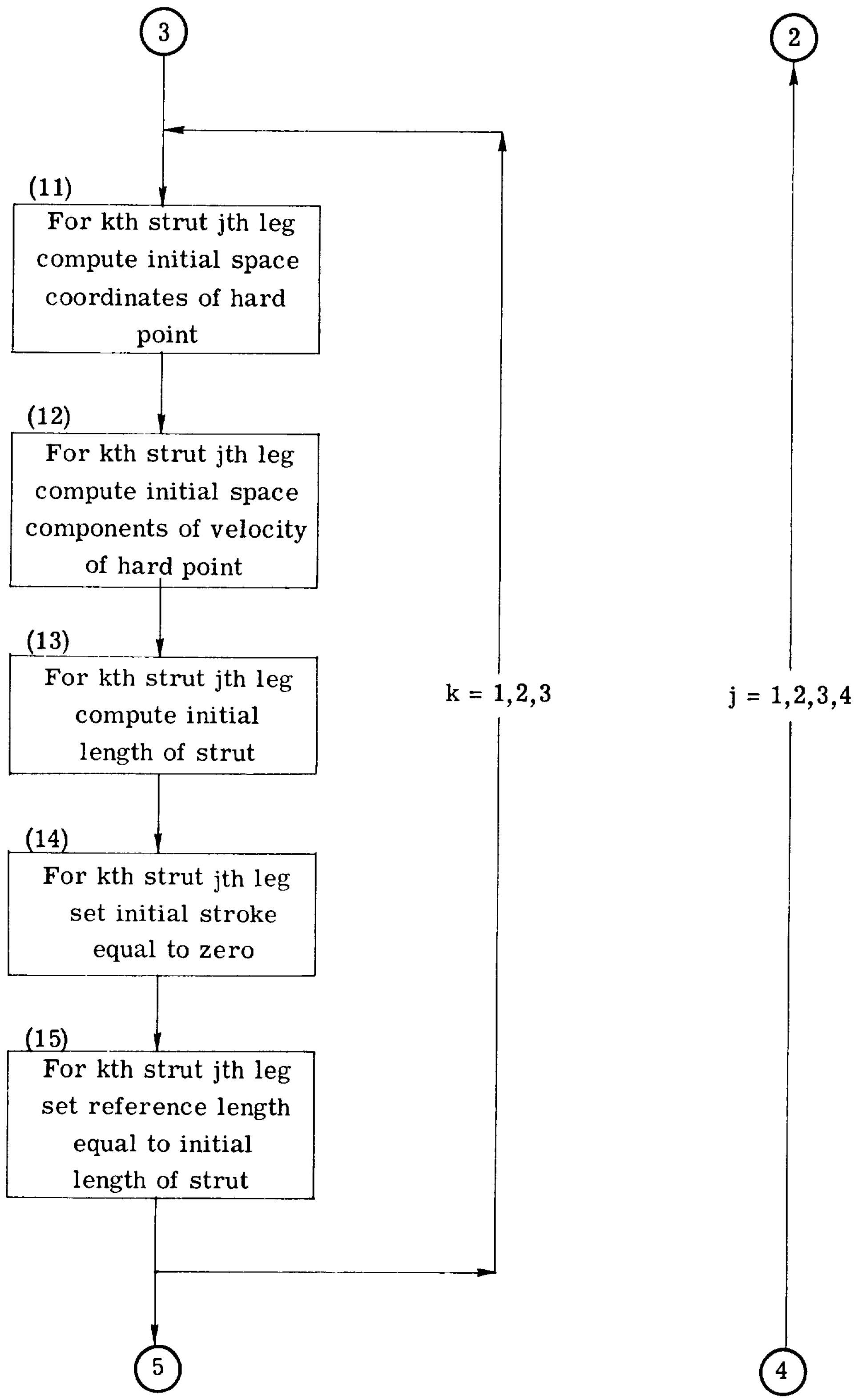
$$\overset{*}{\omega}{}_{\zeta B}^{(n)} = \omega_{\xi B}^{(n)} \omega_{\eta B}^{(n)} \left(I_{\xi \zeta B} - I_{\eta \zeta B} \right) + N_{\zeta B}^{(n)}$$

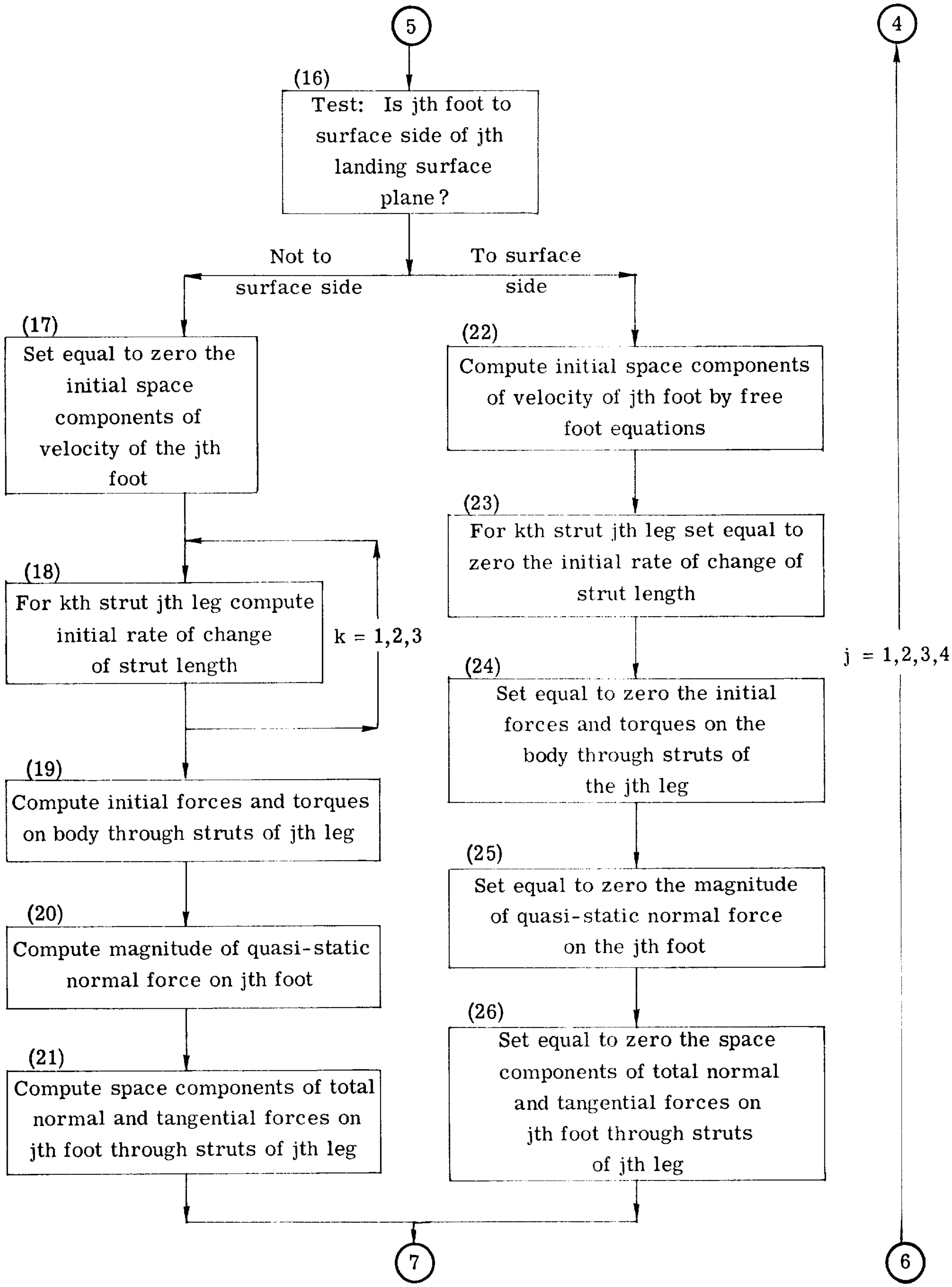
Initialization

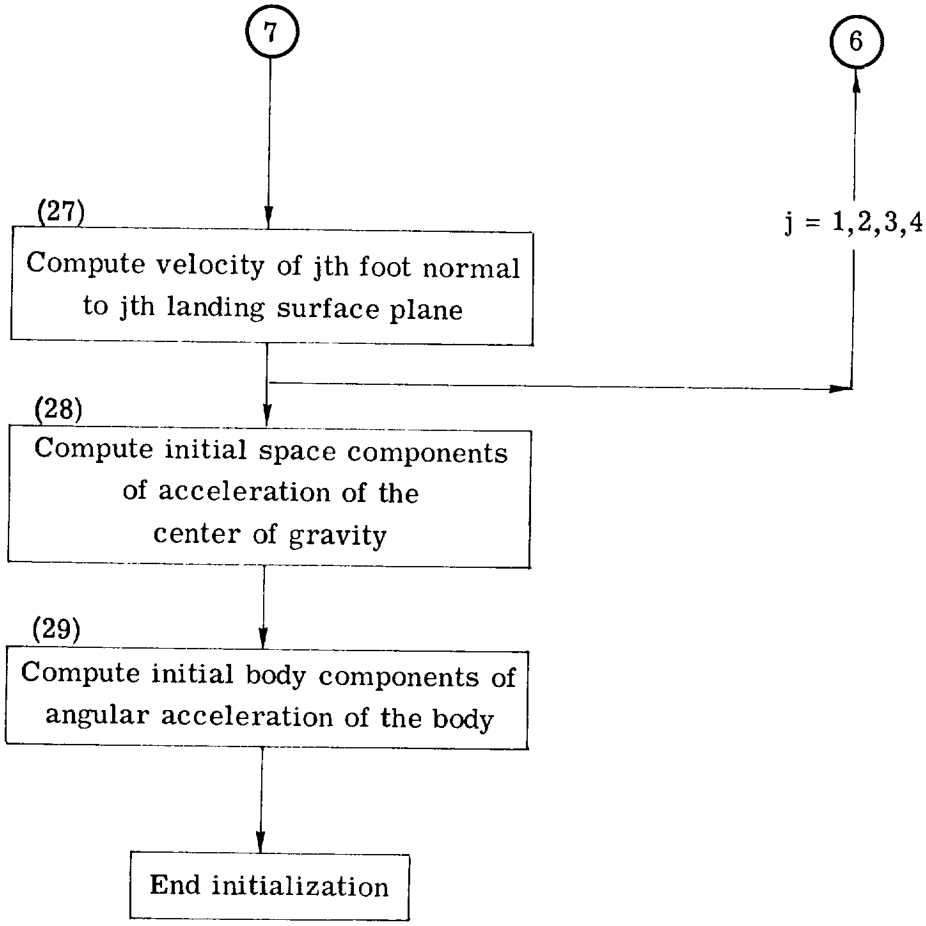
Before entering the integration cycle, it is first necessary to compute certain constants and to compute initial values of those time-dependent variables whose initial values are not given as input. As in the previous section, the computation will be described by a block diagram followed by the details of the steps in each block. When referring to equations written in the description of the integration cycle it is always assumed that the superscript (n) has the value zero.




1
(6)
Compute initial rates of direction cosines
(7)
Normalize coefficients of jth landing surface plane
(8)
Compute coefficients for resolving a vector into components normal and tangential to the jth landing surface plane
(9)
Compute initial space components of the jth foot
(10)
Compute initial signed distance of jth foot from jth landing surface plane
j = 1,2,3,4
3
2


3
2
(11)
For kth strut jth leg compute initial space coordinates of hard point
(12)
For kth strut jth leg compute initial space components of velocity of hard point
(13)
For kth strut jth leg compute initial length of strut
(14)
For kth strut jth leg set initial stroke equal to zero
(15)
For kth strut jth leg set reference length equal to initial length of strut
k = 1,2,3
j = 1,2,3,4
5
4


5
4
(16)
Test: Is jth foot to surface side of jth landing surface plane?
Not to surface side
To surface side
(17)
Set equal to zero the initial space components of velocity of the jth foot
(18)
For kth strut jth leg compute initial rate of change of strut length
k = 1,2,3
(19)
Compute initial forces and torques on body through struts of jth leg
(20)
Compute magnitude of quasi-static normal force on jth foot
(21)
Compute space components of total normal and tangential forces on jth foot through struts of jth leg
(22)
Compute initial space components of velocity of jth foot by free foot equations
(23)
For kth strut jth leg set equal to zero the initial rate of change of strut length
(24)
Set equal to zero the initial forces and torques on the body through struts of the jth leg
(25)
Set equal to zero the magnitude of quasi-static normal force on the jth foot
(26)
Set equal to zero the space components of total normal and tangential forces on jth foot through struts of jth leg
j = 1,2,3,4
7
6



BLOCK 1:

$$t_B^{(0)} = 0$$

BLOCK 2:

$$B_{\xi B}, B_{\eta B}, B_{\zeta B} = \frac{1}{ML^2}\left(I_\xi, I_\eta, I_\zeta\right)$$

BLOCK 3: Use block 7 of integration cycle.

BLOCK 4: Compute

$$\left|\phi^{(0)}\right| = \text{arc cosine } \frac{\delta_{3,3}^{(0)}}{\sqrt{\left(\delta_{3,1}^{(0)}\right)^2 + \left(\delta_{3,3}^{(0)}\right)^2}} \qquad \left(0 \leqq \left|\phi^{(0)}\right| \leqq \pi\right)$$

Set $\phi^{(0)} = -\left|\phi^{(0)}\right|$ if $\delta_{3,1}^{(0)} < 0$; otherwise, set $\phi^{(0)} = \left|\phi^{(0)}\right|$.

Compute

$$\theta^{(0)} = \text{arc cosine } \delta_{3,2}^{(0)} \qquad \left(0 < \theta^{(0)} < \pi\right)$$

Compute

$$\left|\psi^{(0)}\right| = \text{arc cosine } \frac{\delta_{2,2}^{(0)}}{\sqrt{\left(\delta_{3,1}^{(0)}\right)^2 + \left(\delta_{3,3}^{(0)}\right)^2}} \qquad \left(0 \leqq \left|\psi^{(0)}\right| \leqq \pi\right)$$

Set $\psi^{(0)} = -\left|\psi^{(0)}\right|$ if $\delta_{1,2}^{(0)} < 0$; otherwise, set $\psi^{(0)} = \left|\psi^{(0)}\right|$.

If $\delta_{3,1}^{(0)} = 0$, then $\left|\phi^{(0)}\right| = \text{arc cosine } (1) = 0$. However, because of round-off error, it is possible for a computer to generate a number slightly larger than unity for $\delta_{3,3}^{(0)} \Big/ \sqrt{\delta_{3,3}^{(0)\,2}}$, and thus cause an error stop in most subroutines for computing the arc cosine. This possibility should be accounted for in programing the computation of the initial Eulerian angles.

BLOCK 5: Use block 6 of the integration cycle.

BLOCK 6: Use block 9 of the integration cycle.

BLOCK 7: Compute $W_{X,j}, W_{Y,j}, W_{Z,j}$ by (a), (b), or (c).

Conditions			Do operation
$A_{Z,j} \neq 0$	-------	-------	$W_{X,j}, W_{Y,j}, W_{Z,j}, W_{B,j} = \dfrac{\left\|A_{Z,j}\right\|}{A_{Z,j}\sqrt{A_{X,j}^2 + A_{Y,j}^2 + A_{Z,j}^2}}\left(A_{X,j}, A_{Y,j}, A_{Z,j}, \dfrac{A_j}{L}\right)$
$A_{Z,j} = 0$	$A_{Y,j} \neq 0$	-------	$W_{X,j}, W_{Y,j}, W_{Z,j}, W_{B,j} = \dfrac{-\left\|A_{Y,j}\right\|}{A_{Y,j}\sqrt{A_{X,j}^2 + A_{Y,j}^2 + A_{Z,j}^2}}\left(A_{X,j}, A_{Y,j}, A_{Z,j}, \dfrac{A_j}{L}\right)$
$A_{Z,j} = 0$	$A_{Y,j} = 0$	$A_{X,j} \neq 0$	$W_{X,j}, W_{Y,j}, W_{Z,j}, W_{B,j} = \dfrac{-\left\|A_{X,j}\right\|}{A_{X,j}\sqrt{A_{X,j}^2 + A_{Y,j}^2 + A_{Z,j}^2}}\left(A_{X,j}, A_{Y,j}, A_{Z,j}, \dfrac{A_j}{L}\right)$

BLOCK 8: Compute $\left[N_{p,q,j}\right]$ where

$$N_{1,1,j} = W^2_{X,j}$$

$$N_{1,2,j} = N_{2,1,j} = W_{X,j}W_{Y,j}$$

$$N_{1,3,j} = N_{3,1,j} = W_{X,j}W_{Z,j}$$

$$N_{2,2,j} = W^2_{Y,j}$$

$$N_{2,3,j} = N_{3,2,j} = W_{Y,j}W_{Z,j}$$

$$N_{3,3,j} = W^2_{Z,j}$$

Compute $\left[T_{p,q,j}\right]$ where

$$T_{1,1,j} = W^2_{Y,j} + W^2_{Z,j}$$

$$T_{1,2,j} = T_{2,1,j} = -W_{X,j}W_{Y,j}$$

$$T_{1,3,j} = T_{3,1,j} = -W_{X,j}W_{Z,j}$$

$$T_{2,2,j} = W^2_{X,j} + W^2_{Z,j}$$

$$T_{2,3,j} = T_{3,2,j} = -W_{Y,j}W_{Z,j}$$

$$T_{3,3,j} = W^2_{X,j} + W^2_{Y,j}$$

BLOCK 9:

$$\begin{Bmatrix} X^{(0)}_{FB,j} \\ Y^{(0)}_{FB,j} \\ Z^{(0)}_{FB,j} \end{Bmatrix} = \begin{bmatrix} & & \\ & \delta^{(0)}_{p,q} & \\ & & \end{bmatrix}^T \begin{Bmatrix} \xi^{(0)}_{FB,j} \\ \eta^{(0)}_{FB,j} \\ \zeta^{(0)}_{FB,j} \end{Bmatrix} + \begin{Bmatrix} X^{(0)}_{OB} \\ Y^{(0)}_{OB} \\ Z^{(0)}_{OB} \end{Bmatrix}$$

BLOCK 10: Use block 35 of integration cycle.

BLOCK 11: Use block 10 of integration cycle.

BLOCK 12: Use block 11 of integration cycle.

BLOCK 13: Use block 15 of integration cycle.

BLOCK 14:

$$S^{(0)}_{SB,j,k} = 0$$

BLOCK 15:

$$S^{(0)}_{RB,j,k} = S^{(0)}_{B,j,k}$$

BLOCK 16: If $H^{(0)}_{B,j} \geqq 0$, the foot is to the surface side of the plane; otherwise, the foot is not to the surface side of the plane.

BLOCK 17:

$$\overset{*}{X}^{(0)}_{FB,j}, \overset{*}{Y}^{(0)}_{FB,j}, \overset{*}{Z}^{(0)}_{FB,j} = 0,0,0$$

BLOCK 18: Use block 16 of integration cycle.

BLOCK 19: Use blocks 18 to 22 of the integration cycle and delete from block 18 the calculation of reference length $S^{(n)}_{RB,j,k}$.

BLOCK 20: Use block 23 of integration cycle.

BLOCK 21: Use block 24 of integration cycle.

BLOCK 22: Use block 26 of integration cycle.

BLOCK 23: Use block 28 of integration cycle.

BLOCK 24: Use blocks 31 and 32 of integration cycle.

BLOCK 25: Use block 33 of integration cycle.

BLOCK 26: Use block 34 of integration cycle.

BLOCK 27: Use block 36 of integration cycle.

BLOCK 28: Use blocks 41 and 42 of integration cycle.

BLOCK 29: Use blocks 43 and 44 of integration cycle.

Comments on Output

Printing or plotting quantities generated by a digital computer can involve substantial computer time. Therefore, when the procedure described in this paper is programed, outputting should be made flexible so that the number of output quantities can be kept to a minimum consistent with the objectives of a particular investigation. It is usually not necessary to output after each execution of the integration cycle since the time step required for satisfactory integration is usually much smaller than that required for plotting. Conversion from dimensionless to dimensional output is readily programed by using the inverse forms of the relations given in "Dimensionless Equations."

RELATION BETWEEN PRESENT PAPER AND REFERENCE 1

Reference 1 gives the results obtained by using the procedure to compute landing stability boundaries, that is, to compute bounds on approach velocities and orientations within which a vehicle will not overturn and beyond which it will overturn. The calculations were made for a 1/6-scale dynamic model of a lunar landing vehicle suitable for manned landing. The computed stability boundaries were compared with boundaries obtained by landing tests of the model.

The computed stability boundaries presented in reference 1 fall into three categories termed "elastic shocks," "inelastic shocks," and "rigid body." The first two categories were obtained by using an IBM 7094 digital computer programed to execute a procedure in all essentials equivalent to the procedure of this paper. The boundaries designated "rigid body" were obtained by using a procedure reported in reference 5 which is based on the assumption that the entire vehicle moves as a rigid unit. The difference between the boundaries designated "elastic shocks" and "inelastic shocks" lies in the representation of the shock absorber. In the first instance, there is assumed to be a spring in the shock absorber; in the second, the shock is assumed to provide no elastic restoring force.

For precise cross-referencing, the numerical values of input to describe the model and the landing surface of reference 1 are given in table I on page 66.

SOME NEEDED IMPROVEMENTS IN ANALYSIS

It is extremely important to render analytical procedures much more efficient in regard to consumption of computer time in order to consider the multitude of landing situations which may be encountered. Design of vehicles and planning of missions would be greatly enhanced if the time for calculating an impact history could be reduced to the order of a hundredth of a minute as opposed to present times of 2 minutes or more with fast computers. The possibilities for improvement in this area are largely unexplored.

A systematic method is needed for including the effects of overall elasticity of the vehicle. A way should be devised so that the data describing the elastic characteristics of the system can be obtained either by structural analysis or by feasible tests if the vehicle is available.

TABLE I.- INPUT FOR LANDING STABILITY CALCULATIONS REPORTED IN REFERENCE 1

[g = 32.2 ft/sec^2 (9.8146 m/sec^2); L = 2.26 ft (0.68885 m); Δt_B = 0.000369 sec]

(a) Input for configuration

M	1.662 $\frac{\text{lb-sec}^2}{\text{ft}}$	2.473 $\frac{\text{kg-sec}^2}{\text{m}}$
I_ξ	2.08 ft-lb-sec^2	0.2876 m-kg-sec^2
I_η	2.08 ft-lb-sec^2	0.2876 m-kg-sec^2
I_ζ	1.098 ft-lb-sec^2	0.1518 m-kg-sec^2
$\xi_{H,1,1}$	0.895 ft	0.273 m
$\eta_{H,1,1}$	-0.370 ft	-0.113 m
$\zeta_{H,1,1}$	-1.47 ft	-0.448 m
$\xi_{H,1,2}$	1.4675 ft	0.447 m
$\eta_{H,1,2}$	0	0
$\zeta_{H,1,2}$	-0.647 ft	-0.197 m
$\xi_{H,1,3}$	0.895 ft	0.273 m
$\eta_{H,1,3}$	0.370 ft	0.113 m
$\zeta_{H,1,3}$	-1.47 ft	-0.448 m
$\xi_{H,2,1}$	0.370 ft	0.113 m
$\eta_{H,2,1}$	0.895 ft	0.273 m
$\zeta_{H,2,1}$	-1.47 ft	-0.448 m
$\xi_{H,2,2}$	0	0
$\eta_{H,2,2}$	1.4675 ft	0.447 m
$\zeta_{H,2,2}$	-0.647 ft	-0.197 m
$\xi_{H,2,3}$	-0.370 ft	-0.113 m
$\eta_{H,2,3}$	0.895 ft	0.273 m
$\zeta_{H,2,3}$	-1.47 ft	-0.448 m
$\xi_{H,3,1}$	-0.895 ft	-0.273 m
$\eta_{H,3,1}$	0.370 ft	0.113 m
$\zeta_{H,3,1}$	-1.47 ft	-0.448 m
$\xi_{H,3,2}$	-1.4675 ft	-0.447 m
$\eta_{H,3,2}$	0	0
$\zeta_{H,3,2}$	-0.647 ft	-0.197 m
$\xi_{H,3,3}$	-0.895 ft	-0.273 m
$\eta_{H,3,3}$	-0.370 ft	-0.113 m
$\zeta_{H,3,3}$	-1.47 ft	-0.448 m

$\xi_{H,4,1}$	-0.370 ft	-0.113 m
$\eta_{H,4,1}$	-0.895 ft	-0.273 m
$\zeta_{H,4,1}$	-1.47 ft	-0.448 m
$\xi_{H,4,2}$	0	0
$\eta_{H,4,2}$	-1.4675 ft	-0.447 m
$\zeta_{H,4,2}$	-0.647 ft	-0.197 m
$\xi_{H,4,3}$	0.370 ft	0.113 m
$\eta_{H,4,3}$	-0.895 ft	-0.273 m
$\zeta_{H,4,3}$	-1.47 ft	-0.448 m
$\xi^{(0)}_{F,1}$	2.205 ft	0.672 m
$\eta^{(0)}_{F,1}$	0	0
$\zeta^{(0)}_{F,1}$	-2.26 ft	-0.689 m
$\xi^{(0)}_{F,2}$	0	0
$\eta^{(0)}_{F,2}$	2.205 ft	0.672 m
$\zeta^{(0)}_{F,2}$	-2.26 ft	-0.689 m
$\xi^{(0)}_{F,3}$	-2.205 ft	-0.672 m
$\eta^{(0)}_{F,3}$	0	0
$\zeta^{(0)}_{F,3}$	-2.26 ft	-0.689 m
$\xi^{(0)}_{F,4}$	0	0
$\eta^{(0)}_{F,4}$	-2.205 ft	-0.672 m
$\zeta^{(0)}_{F,4}$	-2.260 ft	-0.689 m

TABLE I.- INPUT FOR LANDING STABILITY CALCULATIONS REPORTED IN REFERENCE 1 – Concluded

(b) Input for landing surface

$A_{X,j}$	0	0
$A_{Y,j}$	0.17364818	0.17364818
$A_{Z,j}$	0.98480775	0.98480775
A_j	0	0
$R_{N,j}$	10^{-11} ft/lb-sec	0.672×10^{-11} m/kg-sec
$R_{T,j}$	10^{-11} ft/lb-sec	0.672×10^{-11} m/kg-sec
$K_{i,j}$	0	0

(c) Input for shock absorber

For elastic shocks:

$P_{1,j,k}$	0	0
$P_{2,j,1}$	53,600 lb/ft	79,765 kg/m
$P_{2,j,2}$	73,900 lb/ft	109,975 kg/m
$P_{2,j,3}$	53,600 lb/ft	79,765 kg/m
$P_{3,j,k}$	0	0
$P_{4,j,k}$	0	0
$S_{E,j,1}$	0.00239 ft	0.000728 m
$S_{E,j,2}$	0.003465 ft	0.00106 m
$S_{E,j,3}$	0.00239 ft	0.000728 m
$F_{EC,j,k}$	5 lb	2.27 kg
$F_{EE,j,k}$	5 lb	2.27 kg
$F_{CC,j,1}$	128 lb	58.1 kg
$F_{CC,j,2}$	256 lb	116 kg
$F_{CC,j,3}$	128 lb	58.1 kg
$F_{CE,j,k}$	5 lb	2.27 kg
$\dot{S}_{EC,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{EE,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{CC,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{CE,j,k}$	0.01 ft/sec	0.003048 m/sec
K_S	73,900 lb/ft	109,975 kg/m

For inelastic shocks:

$P_{i,j,k}$	0	0
$S_{E,j,1}$	0.00203 ft	0.000619 m
$S_{E,j,2}$	0.00350 ft	0.00107 m
$S_{E,j,3}$	0.00203 ft	0.000619 m
$F_{EC,j,k}$	5 lb	2.27 kg
$F_{EE,j,k}$	5 lb	2.27 kg
$F_{CC,j,1}$	128 lb	58.1 kg
$F_{CC,j,2}$	256 lb	116 kg
$F_{CC,j,3}$	128 lb	58.1 kg
$F_{CE,j,k}$	5 lb	2.27 kg
$\dot{S}_{EC,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{EE,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{CC,j,k}$	0.01 ft/sec	0.003048 m/sec
$\dot{S}_{CE,j,k}$	0.01 ft/sec	0.003048 m/sec
K_S	73,900 lb/ft	109,975 kg/m

For the usual vehicle configuration, a rocket nozzle protrudes downward beneath the body when the vehicle is in a landing attitude. As the shock absorbers stroke, the nozzle may impinge on the landing surface and affect the course of the vehicle during the impact and possibly affect the stability against overturning. The subject of nozzle impingement requires study and documentation.

CONCLUDING REMARKS

This paper has given the development of a procedure for computing the motions during impact of a spacecraft with legs representative of presently conceived vehicles for lunar landing.

Idealization of the vehicle and landing surface is discussed in a general way. The vehicle is treated as an arbitrary rigid body to which there are attached up to four legs, each leg consisting of three struts in an inverted tripod arrangement. The struts are connected to the body by universal joints and the junction point of the three struts at the foot of a leg is also a universal joint. There is a shock absorber in each strut. The individual struts may shorten or lengthen because of stroking of the shock absorbers but otherwise do not deform. Locations of the points where the struts attach to the body and the initial positions of the feet relative to the body may be arbitrarily chosen. The legs are considered to have no mass. The shock absorber in a strut is considered to produce forces directed along the axis of the strut. The magnitudes of the shock absorber forces are assumed to depend on the instantaneous length of the strut and the rate of change of this length. The boundary of the landing surface is represented by a set of arbitrarily oriented planes, one plane associated with each foot. If a foot is interacting with the surface material, a force is assumed to act on the foot. This interaction force is assumed to be a function of the position and velocity of the foot.

A derivation of the differential equations which govern the motion of the rigid body part of the idealized vehicle is given. The equations consist basically of Newton's equations of translation of the body and the Euler equations for rotation of the body. Forces and torques from the shock absorbers appear in the equations as specified variables.

Idealization of the shock absorbers is discussed in detail. Emphasis is placed on practical considerations in representing the aluminum honeycomb shock absorbers which were utilized on a model vehicle reported in NASA TN D-4215. These considerations include representing constant force crushing characteristic of honeycomb, representing the overall vehicle elasticity by springs in the shock absorbers, and representing gross system damping to prevent unrealistic bounding of the idealized system. Expressions are derived relating shock absorber forces and torques to the system variables, and thereby completing the equations of motion of the body.

The idealization of the landing surface is developed and equations of motion of the feet are derived. In representing forces generated by the interaction of the feet with the surface, two facts must be considered. First, knowledge of the properties of the lunar surface which would affect landing performance is as yet limited. Second, soil mechanics has not progressed to the point where one can predict with any confidence the forces on an arbitrary body impinging upon or passing through soil even under laboratory conditions. Therefore, cases where the feet move substantially through or along the surface are studied by assuming laws for the force on a foot. The definitions and associated equations allow one to construct a variety of laws for the force on a foot by specialization of constants. The derivation of the equations of motion of the feet rests on the assumptions that a foot not interacting with the landing surface material moves to maintain a static balance between the interaction force and the forces from the shock absorbers bearing on the foot.

The equations of motion of the body and feet and necessary auxiliary relations are converted to equivalent dimensionless forms. Working with the dimensionless equations facilitates application of results to both model and full-scale versions of a vehicle. Also, replacing time by a dimensionless variable allows one to rely to some extent on previous experience in sizing the time step for numerical integration of the equations of motion.

The scheme for numerical integration of the equations of motion is given; the method used is a slight modification of Euler's method.

Explicit instructions are given for programing a digital computer to compute a general impact history.

Some improvement is needed in the analysis of landing vehicles with legs. Analytical procedures should be made much more efficient in regard to consumption of computer time, a systematic method is needed for including overall elasticity of the vehicle, and the effects of impingement of the rocket nozzle on the landing surface require study and documentation.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 8, 1967,
124-08-04-13-23.

REFERENCES

1. Herr, Robert W.; and Leonard, H. Wayne: Dynamic Model Investigation of Touchdown Stability of Lunar Landing Vehicles. NASA TN D-4215, 1967.
2. Anon.: Final Report of Lunar Landing Dynamics Systems Investigation. Rept. No. MM-64-8 (Dept. 870), Bendix Products Aerospace Div., Bendix Corp., Oct. 1964.
3. Anon.: Documentation of Digital Computer Programs for Lunar Landing Dynamics System Investigation. Rept. No. MM-64-9 (Dept. 870), Bendix Products Aerospace Div., Bendix Corp., Nov. 1964.
4. Lavender, Robert E.: On Touchdown Dynamics Analysis for Lunar Landing. AIAA Symposium on Structural Dynamics and Aeroelasticity, American Inst. Aeron. Astronaut., Aug.-Sept. 1965, pp. 239-244.
5. Walton, W. C., Jr.; Herr, R. W.; and Leonard, H. W.: Studies of Touchdown Stability for Lunar Landing Vehicles. J. Spacecraft, vol. 1, no. 5, Sept.-Oct. 1964, pp. 552-556.
6. Walton, William C., Jr.; Herr, Robert W.; and Leonard, H. Wayne: Landing Stability for Lunar-Landing Vehicles. Conference on Langley Research Related to Apollo Mission. NASA SP-101, 1965, pp. 77-85.
7. Mantus, M.; Lerner, E.; and Elkins, W.: Landing Dynamics of the Lunar Excursion Module (Method of Analysis). LED-520-6 (Contract NAS 9-1100), Mar. 6, 1964.
8. Goldstein, Herbert: Classical Mechanics. Addison-Wesley Pub. Co., Inc., c.1950.
9. Levy, H.; and Baggott, E. A.: Numerical Studies in Differential Equations. Watts & Co., 1934.

NASA-Langley, 1967 —— 31 L-4857