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GENERALIZED DECOMPOSITION THEORY OF FINITE SEQUENTIAL MACHINES

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Lewis Research Center
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SUMMARY

As a direct outgrowth of a study of the decomposition structure of finite sequential machines, an extended decomposition theory of finite sequential machines is developed and presented in this report. From the theory it can be determined how any finite sequential machine can be realized from a set of smaller, concurrently operating machines. This ability assumes practical significance in problems vital to the further advancement of space science, such as the design of control systems and of data-gathering data-analyzing systems.

INTRODUCTION

The use of control systems and of data-gathering data-analyzing systems has been and will undoubtedly continue to be instrumental in advancing man's knowledge of space. Among the most exciting and interesting research aspects of this area are the design of control systems which must be devoid of human control for environmental reasons and the design of compact space-borne computers with much greater capabilities.

It has been suggested that solutions to such problems will increasingly depend on formal automata theory to the extent that it concerns itself with the finite and with considerations imposed by dynamic environments demanding the completion of computation within a fixed time (ref. 1). Of the recent studies in this direction, the ones dealing with the theory of decomposition of finite sequential machines are among those which offer the most promise of direct application (refs. 2 to 5).

In the present report, extensions of the theories and results of the aforementioned studies are developed. In particular, the basic result of the research described herein is the formulation of a generalized decomposition theory of finite sequential machines. Unlike previous investigations dealing with the theory of the decomposition of sequential machines, the admissible class of realizations treated is not limited to series, parallel, or combinations of series and parallel inertconnections - nor is it limited to specialized realizations containing feedback loops. But, it does include additionally those realizations having a general class of feedback loop interconnections.

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*Subsequent to the writing of this report in February 1966, the author learned Mr. Arthur T. Pu had submitted a doctoral thesis to the University of Illinois entitled "Generalized Decomposition of Incomplete Finite Automata," December 1965. Some of the results reported in Mr. Pu's thesis are comparable to some of those in this report, but the approaches used were significantly different.

PRELIMINARIES

Before an explanation of the formulations of this report are given, fundamental concepts and terminology are presented as background material.

Definition 1

A finite sequential machine M is a quintuple $\langle I, S, 0, \Delta, \Lambda \rangle$ where $I = \{I_1, I_2, \ldots, I_l\}$ is a set of inputs, $S = \{S_1, S_2, \ldots, S_m\}$ is a set of internal states[1], $0 = \{0_1, 0_2, \ldots, 0_n\}$ is a set of outputs, Δ is the next state function that maps the set of pairs $\langle I_j, S_k \rangle$ into S, and Λ is the output function that maps the set of pairs $\langle I_j, S_k \rangle$ onto 0.

		Inputs							
		I_1	I_2	. . .	I_l	I_1	I_2	. . .	I_l
	S_1	S_{11}	S_{21}	. . .	S_{l1}	0_{11}	0_{21}	. . .	0_{l1}
	S_2	S_{12}	S_{22}	. . .	S_{l2}	0_{12}	0_{22}	. . .	0_{l2}
Present states	.	.	.	. . .	.	.	.	. . .	.
	.	.	.	. . .	.	.	.	. . .	.
	.	.	.	. . .	.	.	.	. . .	.
	S_m	S_{1m}	S_{2m}	. . .	S_{lm}	0_{1m}	0_{2m}	. . .	0_{lm}
		Next states				Outputs			

Figure 1. - Flow representation of sequential machine M.

In figure 1 the behavior of M is described by a flow table. The flow table shows, for instance, that if the present state of M is S_2 and input I_l is applied, then M goes into a new state S_{l2} and supplies the output 0_{l2}. In terms of the definition, this observation is expressed by saying that the next state function Δ maps the pair $\langle I_l, S_2 \rangle$ into S_{l2} a member of S and that the output function Λ maps the pair $\langle I_l, S_2 \rangle$ onto 0_{l2}, a member of 0. If a next state or an output is unspecified, this is denoted in the flow table by means of a dash for the associated entry. If all the next states and outputs of M are specified, then the flow table contains no dashed entries, and M is said to be complete; otherwise, M is incomplete.

Algebraic techniques have been introduced as convenient means of studying sequential machines (refs. 5 to 7). Elements of an algebraic theory of sequential machines and certain results obtained in previous studies using algebraic techniques will be introduced

[1]The states are also represented by positive integers in cases where it is the more convenient notation.

	1	2	1	2
1	4	2	0_1	--
2	1	5	0_2	0_1
3	5	2	0_2	0_1
4	3	1	0_1	0_2
5	--	4	0_2	0_1

Figure 2. - Sequential machine A.

as they are needed. With the aid of the example machine A of figure 2, some of the elements of the algebraic theory are now presented.

Definition 2

The Cartesian product $P \times Q$ of two sets P and Q is the set of all ordered pairs $\langle p, q \rangle$ with p in P and q in Q - that is, with $p \in P$ and $q \in Q$.

A concrete illustration of the Cartesian product is provided by considering the sets $I = \{1, 2\}$ and $S = \{1, 2, 3, 4, 5\}$ of machine A. The Cartesian product $I \times S$ is the set $\{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle\}$. This set is precisely the one that Δ and Λ of machine A map into S and onto 0, respectively.

Definition 3

A binary relation between two sets P and Q is a set ρ of ordered pairs $\langle p, q \rangle$ with $p \in P$ and $q \in Q$. Usually $\langle p, q \rangle \in \rho$ is expressed as $p\rho q$. A binary relation between the set P and itself is called a binary relation on P.

Associated with the output function Λ of machine A are two binary relations between the sets $S = \{1, 2, 3, 4, 5\}$ and $0 = \{0_1, 0_2\}$. Corresponding to inputs 1 and 2 are the binary relations $\Lambda(1)$ and $\Lambda(2)$, respectively; $\Lambda(1)$ and $\Lambda(2)$ are given as follows:

$$\Lambda(1) = \{\langle 1, 0_1\rangle, \langle 2, 0_2\rangle, \langle 3, 0_2\rangle, \langle 4, 0_1\rangle, \langle 5, 0_2\rangle\}$$

$$\Lambda(2) = \{\langle 2, 0_1\rangle, \langle 3, 0_1\rangle, \langle 4, 0_2\rangle, \langle 5, 0_1\rangle\}$$

It is worthwhile noting that there is no pair in $\Lambda(2)$ involving state 1. The reason for this is that the output is unspecified when the input 2 is applied to machine A in state 1.

Each pair in these relations is essential to a complete description of the behavior of A. For instance, $\langle 3, 0_2 \rangle \in \Lambda(1)$ shows the relation between state 3 and output 0_2 when input 1 is applied to A.

There are associated with the next state function Δ of machine A two binary relations between the set S and itself:

$$\Delta(1) = \{\langle 1,4\rangle,\ \langle 2,1\rangle,\ \langle 3,5\rangle,\ \langle 4,3\rangle\}$$

$$\Delta(2) = \{\langle 1,2\rangle,\ \langle 2,5\rangle,\ \langle 3,2\rangle,\ \langle 4,1\rangle,\ \langle 5,4\rangle\}$$

The absence of a pair involving a present state 5 in $\Delta(1)$ indicates that the relation between present state 5 and any next state is unspecified for input 1. Each pair in these relations is also essential to a complete description of the behavior of A. In fact, the flow table representation of A can be readily derived from the four binary relations $\Delta(1)$, $\Delta(2)$, $\Lambda(1)$, $\Lambda(2)$, and vice versa. The aforementioned relations, besides providing a description of the behavior of A, are useful in algebraic manipulations. This discussion serves as an introduction to the first two of the next three definitions.

Definition 4

The next state function Δ of a sequential machine M is a mapping that associates with every input $i \in I$, a next state relation $\Delta(i)$ on S.

Definition 5

The output function Λ of a sequential machine M is a mapping that associates with every input $i \in I$, an output relation $\Lambda(i)$ between S and 0.

Definition 6

For two sets P and Q, P is a subset of Q, written $P \subseteq Q$, if each element of P is an element of Q. P is a proper subset of Q if P is a subset of Q and there is an element of Q which is not in P.

In the case of machine A, $\Delta(1)$ and $\Delta(2)$ are proper subsets of $S \times S$. Similarly, $\Lambda(1)$ and $\Lambda(2)$ are proper subsets of $S \times 0$. Also, $\{1, 2, 5\}$ is a proper subset of S. The set S as a subset of itself is an example of a subset that is not proper.

The next two definitions are conveniently illustrated together.

Definition 7

If P is a set, the set N of all those elements in P which have a property R is expressed as

$$N = \{p \in P \mid p \text{ has the property } R\}$$

Definition 8

If ρ is a binary relation between P and Q, the domain $D(\rho)$ of ρ is given by

$$D(\rho) = \{p \in P \mid p\rho q \text{ for some } q \text{ in } Q\}$$

With respect to machine A,

$$N = \{s \in S \mid s\Lambda(2)o \text{ for all } o \text{ in } 0\} = \{2, 3, 4, 5\}$$

since the elements of S which have the property $s\Lambda(2)o$ for 0_1 and 0_2 are 2, 3, 4, and 5. Furthermore, N is the domain $D(\Lambda(2))$ of the output relation $\Lambda(2)$. The fact that $D(\Lambda(2))$ is a proper subset of S shows that machine A is not complete. In general, for all $i \in I$, both $D(\Delta(i))$ and $D(\Lambda(i))$ must be S for a sequential machine M to be complete.

Definition 9

If $N \subseteq P$, then

$$N\rho = \{q \in Q \mid p\rho q \text{ for some } p \text{ in } N\}$$

A specific illustration of this definition is provided by considering a set $N = \{1, 2, 5\}$, a proper subset of S of machine A, and the next state relation $\Delta(1)$:

$$N\,\Delta(1) = \{1, 2, 5\}\{\langle 1, 4\rangle, \langle 2, 1\rangle, \langle 3, 5\rangle, \langle 4, 3\rangle\}$$

$$= \{4, 1\}$$

Definition 10

If $\rho \subseteq P \times Q$ and $\sigma \subseteq Q \times R$, then

$$\rho\sigma = \{\langle p, r\rangle \in P \times R \mid p\rho q \text{ and } q\sigma r \text{ for some } q \text{ in } Q\}$$

For machine A, $\Delta(1) \subseteq S \times S$ and $\Lambda(2) \subseteq S \times 0$; thus,

$$\Delta(1)\Lambda(2) = \{\langle 1,4\rangle, \langle 2,1\rangle, \langle 3,5\rangle, \langle 4,3\rangle\}\{\langle 2,0_1\rangle, \langle 3,0_1\rangle, \langle 4,0_2\rangle, \langle 5,0_1\rangle\}$$

$$= \{\langle 1,0_2\rangle, \langle 3,0_1\rangle, \langle 4,0_1\rangle\}$$

The binary relation $\Delta(1)\Lambda(2)$ relates the present states of A to the outputs supplied after the successive application of inputs 1 and 2.

Definition 11

The set intersection of a family $P_1, P_2, \ldots, P_n$ of sets, written $P_1 \cap P_2 \cap \ldots \cap P_n$, is the set of elements which are in all of the sets of the family. Two sets P and Q are disjoint if $P \cap Q = \varphi$, where φ, called the empty set, is the set with no elements in it.

In the case of machine A, $P_1 = \{1,2,5\}$, $P_2 = \{1,4\}$, $P_3 = \{1,4,5\}$ is a family of the subsets of S. The intersection of these subsets is

$$P_1 \cap P_2 \cap P_3 = \{1,2,5\} \cap \{1,4\} \cap \{1,4,5\} = \{1\}$$

Two other subsets, $P_4 = \{1,3\}$ and $P_5 = \{2,4\}$, of S are disjoint since there is no element common to both P_4 and P_5 so that

$$P_4 \cap P_5 = \varphi$$

Definition 12

The set union of a family $P_1, P_2, \ldots, P_n$ of sets, written $P_1 \cup P_2 \cup \ldots \cup P_n$, is the set of elements in at least one set of the family.

The union of the subsets P_1, P_2, and P_3 of the previous example is

$$P_1 \cup P_2 \cup P_3 = \{1,2,5\} \cup \{1,4\} \cap \{1,4,5\} = \{1,2,4,5\}$$

Definition 13

A partition π of a set S is a family $P_1, P_2, \ldots, P_n$ of nonempty pairwise disjoint sets such that $P_1 \cup P_2 \cup \ldots \cup P_n = S$. The sets $P_1, P_2, \ldots, P_n$ are called the blocks of π.

The subsets $P_1 = \{1,5\}$, $P_2 = \{2,4\}$, and $P_3 = \{3\}$ of S of machine A are pairwise disjoint. Their union is $P_1 \cup P_2 \cup P_3 = S$ and the partition π is given by

$$\pi = \{P_1, P_2, P_3\}$$

Definition 14

The canonical relation π^* between S and π is given by

$$\pi^* = \{\langle s, P\rangle \mid s \in P \in \pi\}$$

In particular, the relation π^* between S of machine A and the partition π of the previous example is

$$\pi^* = \{\langle 1, P_1\rangle, \langle 5, P_1\rangle, \langle 2, P_2\rangle, \langle 4, P_2\rangle, \langle 3, P_3\rangle\}$$

Then

$$S\pi^* = \{1,2,3,4,5\}\{\langle 1, P_1\rangle, \langle 5, P_1\rangle, \langle 2, P_2\rangle, \langle 4, P_2\rangle, \langle 3, P_3\rangle\}$$

$$= \{P_1, P_2, P_3\} = \pi$$

Definition 15

The inverse of a binary relation ρ is

$$\rho^{-1} = \{\langle q, p\rangle \mid p\rho q,\ p \in P \text{ and } q \in Q\}$$

The inverse of the relation π^* of the previous example is given, for instance, by

$$(\pi^*)^{-1} = \{\langle P_1, 1\rangle,\ \langle P_1, 5\rangle,\ \langle P_2, 2\rangle,\ \langle P_2, 4\rangle,\ \langle P_3, 3\rangle\}$$

Thus,

$$\pi(\pi^*)^{-1} = \{P_1, P_2, P_3\}\{\langle P_1, 1\rangle,\ \langle P_1, 5\rangle,\ \langle P_2, 2\rangle,\ \langle P_2, 4\rangle,\ \langle P_3, 3\rangle\}$$

$$= \{1, 5, 2, 4, 3\} = S$$

Furthermore, $S\pi^*(\pi^*)^{-1} = S$ and $\pi(\pi^*)^{-1}\pi^* = \pi$.

Definition 16

The greatest lower bound (g.l.b.) of two partitions π_α and π_β is given by

$$\pi_\alpha \cdot \pi_\beta = \{P \cap Q \mid P \in \pi_\alpha \text{ and } Q \in \pi_\beta\}.$$

There are two trivial partitions of a set S. The partition I contains one element S. The partition 0 is the partition whose blocks are precisely the elements of the set S. These partitions have the properties

$$I \cdot \pi = \pi$$

$$0 \cdot \pi = 0$$

The use of the symbols I and 0 for these partitions as well as for input and output sets is consistent with conventional usage. Whether these symbols refer to the trivial partitions or the aforementioned sets in subsequent discussions should be clear from the context.

For machine A, the definition is illustrated by considering partitions $\pi_1 = \{P_1 = \{1, 2, 3\}, P_2 = \{4, 5\}\}$, $\pi_2 = \{Q_1 = \{1, 3, 4\}, Q_2 = \{2, 5\}\}$, $I = \{S = \{1, 2, 3, 4, 5\}\}$, and $0 = \{1, 2, 3, 4, 5\}$:

$$\pi_1 \cdot \pi_2 = \{P_1 \cap Q_1 = \{1,3\},\ P_1 \cap Q_2 = \{2\},\ P_2 \cap Q_1 = \{4\},\ P_2 \cap Q_2 = \{5\}\}$$

$$I \cdot \pi_1 = \{S \cap P_1 = P_1, S \cap P_2 = P_2\} = \pi_1$$

$$0 \cdot \pi_1 = \{1 \cap P_1 = 1, 2 \cap P_1 = 2, 3 \cap P_1 = 3, 4 \cap P_2 = 4, 5 \cap P_2 = 5\} = 0$$

Definition 17

That a partition π_α of S is the refinement of a partition π_β of S is expressed by

$$\pi_\alpha \leq \pi_\beta = \{P \subseteq Q \mid \text{ for every } P \in \pi_\alpha \text{ and some } Q \in \pi_\beta\}$$

Illustrative of this definition are the partitions $\pi_1 \cdot \pi_2$ and π_2 of the previous example:

$$\pi_1 \cdot \pi_2 \leq \pi_1$$

since

$$\{1,3\} \subseteq \{1,2,3\},\ \{2\} \subseteq \{1,2,3\},\ \{4\} \subseteq \{4,5\},\ \{5\} \subseteq \{4,5\}$$

Definition 18

Partitions π and π' of S of a sequential machine M form a partition pair (π, π') if and only if for every $P \in \pi$ and every $i \in I$ there exists a $Q \in \pi'$ such that $P\ \Delta(i) \subseteq Q$.

With respect to machine A,

$$(\pi = \{P_1 = \{1,2\},\ P_2 = \{3,5\},\ P_3 = \{4\}\},\ \pi' = \{Q_1 = \{1,2,4,5\},\ Q_2 = \{3\}\})$$

is a partition pair since

$$P_1\ \Delta(1) = \{1,2\}\{\langle 1,4\rangle,\ \langle 2,1\rangle,\ \langle 3,5\rangle,\ \langle 4,3\rangle\} = \{4,1\} \subseteq \{1,2,4,5\} = Q_1$$

$$P_2\ \Delta(1) = \{5\} \subseteq Q_1$$

$$P_3\ \Delta(1) = \{3\} \subseteq Q_2$$

$$P_1\ \Delta(2) = \{1,2\}\{\langle 1,2\rangle,\ \langle 2,5\rangle,\ \langle 3,2\rangle,\ \langle 4,1\rangle,\ \langle 5,4\rangle\} = \{2,5\} \subseteq Q_1$$

$$P_2\ \Delta(2) = \{2,4\} \subseteq Q_1$$

$$P_3\ \Delta(2) = \{1\} \subseteq Q_1$$

DECOMPOSITION OF SEQUENTIAL MACHINES

In previous studies valuable insight into the structure of sequential machines has been gained by investigating the decompositional properties of these machines (refs. 2 to 5). The problem of sequential machine decomposition is that of determining how a sequential machine can be realized from interconnected sets of smaller machines. The aforementioned studies have been concentrated either on specialized classes of feedback loop realizations of machines or else on loop-free connections of machines, that is, machines connected in series, parallel, or series-parallel combinations. The present study is a presentation of a unified decomposition theory that encompasses both loop-free decompositions and a general class of feedback loop decompositions.

In the loop-free decomposition theory all smaller machines in any given realization are assumed to be Moore type machines (ref. 4). A Moore machine is a sequential machine for which the output mapping function Λ is independent of the input set I (refs. 4 and 7). When all component machines in a realization are of the Moore type, the output of each machine is used as an input to the other machines. Since the outputs are determined by the present state of the machines, all component machines can compute their next states simultaneously after an external input has been applied. Hence, the advantage of a realization composed of an interconnection of Moore machines is that it does not slow down the machine because all smaller machines operate concurrently and do not require waits for carry computations. This property of concurrency of operation will be exploited further in the development of the extended theory of decomposition of sequential machines.

Definition 19

The set $\{M_1, M_2, \ldots, M_j\}$ is a set of Moore machines. The set of inputs, states, and outputs of machine M_k, $1 \leq k \leq j$, are $I^{(k)}$, $S^{(k)}$, and $0^{(k)}$, respectively. Furthermore, the input set $I^{(k)}$ is given by

$$I^{(k)} \subseteq I \times C^{(k)}$$

unless either I or $C^{(k)}$ contains one and only one element, in which case

$$I^{(k)} \subseteq C^{(k)} \text{ or } I^{(k)} \subseteq I$$

respectively. The set $C^{(k)}$ is the set of internal inputs or carry inputs, which are derived from the outputs of other machines in a realization of a sequential machine M as an interconnection of machines $M_1, M_2, \ldots, M_j$. The set I is the set of external inputs applied to M.

Definition 20 (ref. 4)

The set $\{M_1, M_2, \ldots, M_j\}$ of interconnected Moore machines, in which the outputs of any $M_k (1 \leq k \leq j)$ may be used as inputs to other machines, is concurrently operating if the next state of each machine M_k depends on the present state of M_k, the present outputs of the machines to which it is connected, and the present external input. The ordered j-tuple of the present states of the machines $M_1, M_2, \ldots, M_j$ is referred to as the state of the interconnected machine.

Definition 21 (ref. 4)

The state behavior of the sequential machine M, the behavior of M exclusive of its outputs, is realized by a concurrently operating interconnection of the machines M_1, $M_2, \ldots, M_j$ with the sets of states $S^{(1)}, S^{(2)}, \ldots, S^{(j)}$, respectively, if the following obtains:

(1) The input I of M is a subset of the set of possible inputs of the machine realized from the machines $M_1, M_2, \ldots, M_j$.

(2) There is a one-to-one mapping Φ between the set of states S and M and a subset R of the Cartesian product of the sets of states of $M_1, M_2, \ldots, M_j$

$$S^{(1)} \times S^{(2)} \times \ldots \times S^{(j)}$$

which is preserved by the operations of M and the machine realized from M_1, $M_2, \ldots, M_j$. In other words, if the two machines are initially in corresponding states, they will again be in corresponding states after any sequence of inputs from I.

Central to the decomposition theory developed in this report is the concept of an $S^{(k)}$-image of a finite sequential machine M. This concept is embodied in the next definition.

Definition 22

Given sequential machines M and M_k, M_k is an $S^{(k)}$-image of M if and only if for every $s^{(k)} \in S^{(k)}$, $i \in I$, and $i^{(k)} = \langle i, c^{(k)} \rangle \in I^{(k)}$

(1) Every $s \in S$ belongs to some $s^{(k)} \in S^{(k)}$ and to some $c^{(k)} \in C^{(k)}$

(2) $(c^{(k)} \cap s^{(k)})\Delta(i) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$

For the case in which $I^{(k)} \subseteq I$, the quantity $c^{(k)} \cap s^{(k)}$ of part (2) reduces to $s^{(k)}$. Also, when $I^{(k)} \subseteq I$, the concept of an $S^{(k)}$-image reduces to a concept which has found some previous usage (ref. 7).

In the development of the theory of this section it is convenient to consider the subsets $s^{(k)}$ of S to be disjoint and also the subsets $c^{(k)}$ of S to be disjoint. In a later section the case for which these subsets may not necessarily be disjoint is considered.

An example of the concept of an $S^{(k)}$-image is furnished by the consideration of machine A of figure 2 and machine A_1 of figure 3. The elements of the sets, $S^{(1)} = \{S_1^{(1)} = \{1,3,5\}, S_2^{(1)} = \{2,4\}\}$ and $C^{(1)} = \{C_1^{(1)} = \{1,2,4,5\}, C_2^{(1)} = \{3\}\}$, of

	$\langle 1, C_1^{(1)} \rangle$	$\langle 1, C_2^{(1)} \rangle$	$\langle 2, C_1^{(1)} \rangle$	$\langle 2, C_2^{(1)} \rangle$	$O^{(1)}$
$S_1^{(1)}$	$S_2^{(1)}$	$S_1^{(1)}$	$S_2^{(1)}$	$S_2^{(1)}$	$O_1^{(1)}$
$S_2^{(1)}$	$S_1^{(1)}$	---	$S_1^{(1)}$	---	$O_2^{(1)}$

Figure 3. - Machine A_1.

machine A_1 satisfy condition (1):

$$1,3,5 \in S_1^{(1)} \qquad 2,4 \in S_2^{(1)} \qquad S_1^{(1)} \cup S_2^{(1)} = S$$

$$1,2,4,5 \in C_1^{(1)} \qquad 3 \in C_2^{(1)} \qquad C_1^{(1)} \cup C_2^{(1)} = S$$

Condition (2) is also satisfied:

$$(C_2^{(1)} \cap S_2^{(1)})\Delta(1) = (\{3\} \cap \{2,4\})\Delta(1) = \varphi\ \Delta(1) = \varphi = S_2^{(1)}\Delta^{(1)}(\langle 1, C_2^{(1)} \rangle)$$

$$(C_2^{(1)} \cap S_2^{(1)})\Delta(2) = \varphi\ \Delta(2) = \varphi = S_2^{(1)}\Delta^{(1)}(\langle 2, C_2^{(1)} \rangle)$$

$$(C_1^{(1)} \cap S_1^{(1)})\Delta(1) = \{1,5\}\{\langle 1,4\rangle, \langle 2,1\rangle, \langle 3,5\rangle, \langle 4,3\rangle\}$$

$$= \{4\} \subseteq S_2^{(1)} = S_1^{(1)}\Delta^{(1)}(\langle 1, C_1^{(1)}\rangle)$$

$$(C_1^{(1)} \cap S_2^{(1)})\Delta(1) = \{2,4\}\Delta(1) = \{1,3\} \subseteq S_1^{(1)} = S_2^{(1)}\Delta^{(1)}(\langle 1, C_1^{(1)}\rangle)$$

$$(C_2^{(1)} \cap S_1^{(1)})\Delta(1) = \{3\}\Delta(1) = \{5\} \subseteq S_1^{(1)} = S_1^{(1)}\Delta^{(1)}(\langle 1, C_2^{(1)}\rangle)$$

$$(C_1^{(1)} \cap S_1^{(1)})\Delta(2) = \{1,5\}\{\langle 1,2\rangle, \langle 2,5\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 5,4\rangle\}$$

$$= \{2,4\} \subseteq S_2^{(1)} = S_1^{(1)}\Delta^{(1)}(\langle 2, C_1^{(1)}\rangle)$$

$$(C_1^{(1)} \cap S_2^{(1)})\Delta(2) = \{2,4\}\Delta(2) = \{5,1\} \subseteq S_1^{(1)} = S_2^{(1)}\Delta^{(1)}(\langle 2, C_1^{(1)}\rangle)$$

$$(C_2^{(1)} \cap S_1^{(1)})\Delta(2) = \{3\}\Delta(2) = \{2\} \subseteq S_2^{(1)} = S_1^{(1)}\Delta^{(1)}(\langle 2, C_2^{(1)}\rangle)$$

Therefore, machine A_1 is an $S^{(1)}$-image of machine A.

The following theorem relates the concept of an $S^{(k)}$-image of a machine M to a partition pair on the states of S of M. The partition pair as in previous investigations into the decomposition structure of sequential machines (refs. 3 and 4) is a valuable tool in the development of the theory. Adding to the importance of this relation between the concept of an $S^{(k)}$-image and a partition pair is the fact that one of the aforementioned partitions corresponds to the set of states $S^{(k)}$ of M_k and the other to both $S^{(k)}$ and the set of carry inputs $C^{(k)}$ to M_k.

Theorem 1:

Given a sequential machine M, there exists a sequential machine M_k, which is an $S^{(k)}$-image of M, if and only if there exists a partition pair (π_k, π_k') on the states of S of M such that $\pi_k = C^{(k)} \cdot \pi_k'$, where $C^{(k)}$ and π_k' are partitions whose blocks are the elements of the carry input set $C^{(k)}$ to M_k and the set of states $S^{(k)}$ of M_k, respectively.

Proof:

According to the hypothesis of the "only if" half of the theorem, M_k is an $S^{(k)}$-image of M, and $C^{(k)}$ and π_k' are partitions whose blocks are the elements of the carry input set $C^{(k)}$ and of the state set $S^{(k)}$ of M_k, respectively. All pairs of intersections $P = c^{(k)} \cap s^{(k)}$ are disjoint because of the disjointness of the pairs of the $c^{(k)}$ and of the $s^{(k)}$. The union of all nonempty P defines a partition π_k of the states of S since every s is in some $c^{(k)}$ and some $s^{(k)}$ according to (1) of definition 22. The partition π_k, in accordance with definition 16, is the g.l.b. $C^{(k)} \cdot \pi_k'$ of the partitions $C^{(k)}$ and π_k'. According to definition 22, $P\,\Delta(i) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)}) \subseteq Q$, where $Q \in \pi_k'$ and $P \in \pi_k$. This is precisely the definition of the partition pair

$$(\pi_k, \pi_k') = (C^{(k)} \cdot \pi_k', \pi_k').$$

Now for the "if" half of the theorem, it can be assumed that there exists a partition pair (π_k, π_k') on the states of S of M such that $\pi_k = C^{(k)} \cdot \pi_k'$, where $C^{(k)}$ is a partition of the states of S. With the use of this information a sequential machine $M_k = \langle I^{(k)}, S^{(k)}, 0^{(k)}, \Delta^{(k)}, \Lambda^{(k)} \rangle$ can be defined as follows: According to the definition of a partition pair $(C^{(k)} \cdot \pi_k', \pi_k')$, there exists, for every $P = c^{(k)} \cap s^{(k)} \in \pi_k$ and every $i \in I$, a $Q \in \pi_k'$ such that $P\,\Delta(i) \subseteq Q$. Let the blocks of π_k' and $C^{(k)}$ be the elements of the sets $S^{(k)}$ and $C^{(k)}$ of M_k. Because π_k' and $C^{(k)}$ are partitions corresponding to the sets $S^{(k)}$ and $C^{(k)}$, respectively, every $s \in S$ necessarily belongs to some $s^{(k)} \in S^{(k)}$ and to some $c^{(k)} \in C^{(k)}$; that is, condition (1) of definition 22 is satisfied. Let $I^{(k)}$ be defined by $i^{(k)} = \langle i, c^{(k)} \rangle \in I^{(k)}$. Moreover, the definition of $\Delta^{(k)}$ is established by letting $P\,\Delta(i) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)}) \subseteq Q$. Thus, condition (2) of definition 22 is satisfied. Furthermore, $0^{(k)}$ and $\Lambda^{(k)}$ can be defined by letting $s^{(k)}\Lambda^{(k)}(i^{(k)}) = o^{(k)} \in 0^{(k)}$ for all $i \in I$. In the process of defining M_k it was shown that definition 22 was satisfied. Therefore, machine M_k is indeed an $S^{(k)}$-image of M.

In the previous example it was shown that machine A_1 is an $S^{(1)}$-image of machine A. The partition pair (π_1, π_1') which is implied as a consequence of the theorem is

$$(\pi_1 = \{P_1 = \{1,5\},\ P_2 = \{2,4\},\ P_3 = \{3\}\},\ \pi_1' = S^{(1)} = \{S_1^{(1)} = \{1,3,5\},\ S_2^{(1)} = \{2,4\}\})$$

where $\pi_1 = C^{(1)} \cdot \pi_1'$ and $C^{(1)} = \{C_1^{(1)} = \{1,2,4,5\},\ C_2^{(1)} = \{3\}\}$. Furthermore, given machine A and the partition pair (π_1, π_1'), one can construct machine A_1 as an $S^{(1)}$-image of machine A. Illustrative of this construction is the following: Let

$$I^{(1)} = \{\langle 1, C_1^{(1)} \rangle,\ \langle 1, C_2^{(1)} \rangle,\ \langle 2, C_1^{(1)} \rangle,\ \langle 2, C_2^{(1)} \rangle\}$$

As a consequence of

$$(C_1^{(1)} \cap S_1^{(1)})\Delta(1) = \{4\} \subseteq S_2^{(1)}$$

$$(C_1^{(1)} \cap S_2^{(1)})\Delta(1) = \{1, 3\} \subseteq S_1^{(1)}$$

let

$$S_2^{(1)} = S_1^{(1)}\Delta^{(1)}(\langle 1, C_1^{(1)}\rangle)$$

$$S_1^{(1)} = S_2^{(1)}\Delta^{(1)}(\langle 1, C_1^{(1)}\rangle)$$

so that

$$\Delta^{(1)}(\langle 1, C_1^{(1)}\rangle) = \{\langle S_1^{(1)}, S_2^{(1)}\rangle, \langle S_2^{(1)}, S_1^{(1)}\rangle\}$$

Similar constructions yield

$$\Delta^{(1)}(\langle 1, C_2^{(1)}\rangle) = \{\langle S_1^{(1)}, S_1^{(1)}\rangle\}$$

$$\Delta^{(1)}(\langle 2, C_1^{(1)}\rangle) = \{\langle S_1^{(1)}, S_2^{(1)}\rangle, \langle S_2^{(1)}, S_1^{(1)}\rangle\}$$

$$\Delta^{(1)}(\langle 2, C_2^{(1)}\rangle) = \{\langle S_1^{(1)}, S_2^{(1)}\rangle\}$$

$$\Lambda^{(1)}(\langle 1, C_1^{(1)}\rangle) = \Lambda^{(1)}(\langle 1, C_2^{(1)}\rangle) = \Lambda^{(1)}(\langle 2, C_1^{(1)}\rangle)$$

$$= \Lambda^{(1)}(\langle 2, C_2^{(1)}\rangle) = \{\langle S_1^{(1)}, 0_1^{(1)}\rangle, \langle S_2^{(1)}, 0_2^{(1)}\rangle\}$$

Lemma 1:

If the state behavior of a sequential machine M is realized by a concurrently operating interconnection of two machines M_1 and M_2, then there exist partition pairs (π_1, π_1') and (π_2, π_2') on the states of S of machine M such that

$$\pi_1' \cdot \pi_2' = 0$$

$$\pi_1 = C^{(1)} \cdot \pi_1'$$

$$\pi_2 = C^{(2)} \cdot \pi_2'$$

where

$$\pi_2' \leq C^{(1)}$$

$$\pi_1' \leq C^{(2)}$$

Proof:

By the hypothesis the state behavior of a sequential machine M is realized by a concurrently operating interconnection of two machines M_1 and M_2. According to definition 20, the state of the interconnection is the ordered pair of present states of M_1 and M_2, that is, $\langle s^{(1)}, s^{(2)}\rangle$ where $s^{(1)} \in S^{(1)}$ and $s^{(2)} \in S^{(2)}$. Also, by definition 21 there is a one-to-one mapping Φ between the states of S of M and members of a subset R of $S^{(1)} \times S^{(2)}$. Thus, R is the set of states of the interconnection of M_1 and M_2. In accordance with the mapping Φ, each element $S_j^{(1)} \in S^{(1)}$ is given by

$$S_j^{(1)} = \{u \mid u \in S \text{ such that } u \text{ corresponds to an ordered pair } \langle s^{(1)}, s^{(2)}\rangle \in R \text{ for which } s^{(1)} \text{ is } S_j^{(1)}\}$$

Similarly,

$$S_k^{(2)} = \{v \mid v \in S \text{ such that } v \text{ corresponds to an ordered pair } \langle s^{(1)}, s^{(2)}\rangle \in R \text{ for which } s^{(2)} \text{ is } S^{(2)}\}$$

No two elements of $S^{(1)}$ can have any states $s \in S$ in common if the mapping Φ is to be preserved. A similar statement applies to the elements of $S^{(2)}$. Also, the mapping Φ requires that every $s \in S$ belongs to some $s^{(1)} \in S^{(1)}$ and to some $s^{(2)} \in S^{(2)}$. Each element $C_l^{(2)}$ of the carry input set $C^{(2)}$ to M_2 is obtained from the output set $0^{(1)}$ of M_1 according to

$$C_l^{(2)} = \{S_j^{(1)} \mid S_j^{(1)}\Lambda^{(1)}(i^{(1)}) = 0_l^{(1)}$$

where

$$S_j^{(1)} \in S^{(1)} \text{ and } 0_l^{(1)} \in 0^{(1)}\}$$

If the output set $O^{(1)}$ contains one and only one element $O_l^{(1)}$, then the only element of $C_l^{(2)}$ is $C_l^{(2)} = S^{(1)}$ in which case $I^{(2)} \subseteq I$; otherwise, $I^{(2)} \subseteq I \times C^{(2)}$. It should be noted that $\Lambda^{(1)}(i^{(1)})$ is the same for all $i \in I$ because M_1 is a Moore machine. Each element $C_l^{(2)}$ of $C^{(2)}$ is the union of the subsets $S_j^{(1)}$ of the set of states S and hence can also be expressed as a subset of S. As such, no two elements of $C^{(2)}$ can have any state $s \in S$ in common and also every $s \in S$ belongs to some $c^{(2)} \in C^{(2)}$. A similar consideration of $C^{(1)}$ shows that a statement analogous to the previous one applies to $C^{(1)}$ and its elements, Hence, thus far it has been shown that M_1 and M_2 satisfy condition (1) of definition 22. To prove that these two machines also satisfy condition (2) of definition 22 and consequently are $S^{(1)}$- and $S^{(2)}$-images, respectively, of M, the following analysis is made: According to the mapping Φ,

$S_j^{(1)} \cap S_k^{(2)} = \{u \cap v \,|\, u \in S$ and $v \in S$ such that u corresponds to an ordered pair

$\langle s^{(1)}, s^{(2)}\rangle \in R$ for which $s^{(1)}$ is $S_j^{(1)}$, and v corresponds to an ordered

pair $\langle s^{(1)}, s^{(2)}\rangle \in R$ for which $s^{(2)}$ is $S_k^{(2)}\}$

Thus, either $S_j^{(1)} \cap S_k^{(2)} = \varphi$ when there exists no ordered pair $\langle s^{(1)}, s^{(2)}\rangle \in R$ for which both $s^{(1)}$ is $S_j^{(1)}$ and $s^{(2)}$ is $S_k^{(2)}$, or else $S_j^{(1)} \cap S_k^{(2)}$ is the state $s \in S$ corresponding to $\langle S_j^{(1)}, S_k^{(2)}\rangle$. Since $C_l^{(2)}$ is the union of subsets $S_j^{(1)}$, the quantity $C_l^{(2)} \cap S_k^{(2)}$ is either φ or else a subset of S. In the latter case, the subset corresponds to a set of ordered pairs $\langle s^{(1)}, s^{(2)}\rangle \in R$ whose first elements are the subsets $S_j^{(1)}$ and whose second elements are all the same, the subset $S_k^{(2)}$. The aforementioned set of ordered pairs is mapped by $\Delta^{(1)}(i^{(1)})$ and $\Delta^{(2)}(i^{(2)})$ into another not necessarily dintinct set of ordered pairs. In particular, $\Delta^{(1)}(i^{(1)})$ is used to map the first elements of the given set of ordered pairs into the first elements of the resulting set of ordered pairs. The second elements of the resulting ordered pairs are obtained from the mapping $\Delta^{(2)}(i^{(2)})$ in an analogous manner. If all the ordered pairs resulting from the mappings correspond to $s^{(1)} \cap s^{(2)} = \varphi$, then the resulting states of the interconnection of M_1 and M_2 are unspecified. In such a case, to preserve the mapping Φ

$$(C_l^{(2)} \cap S_k^{(2)})\Delta(i) = \varphi \subseteq S_k^{(2)}\Delta^{(2)}(i^{(2)})$$

If one or more of the resulting ordered pairs do belong to R, they must correspond to the set of states resulting from the mapping $\Delta(i)$ of $C_l^{(2)} \cap S_k^{(2)}$ to preserve the mapping Φ. Since each of the resulting ordered pairs contains the same second element $s^{(2)} \in S^{(2)}$, the union of intersections $s^{(1)} \cap s^{(2)}$ corresponding to this resulting set or ordered pairs must be a subset of the element $s^{(2)} \in S^{(2)}$ resulting from the mapping $\Delta^{(2)}(i^{(2)})$ of $S_k^{(2)}$. Consequently,

$$(C_l^{(2)} \cap S_k^{(2)})\Delta(i) \subseteq S_k^{(2)}\Delta^{(2)}(i^{(2)})$$

Therefore, machine M_2 satisfies condition (2) of definition 22 and is an $S^{(2)}$-image of M. A similar analysis shows M_1 to be an $S^{(1)}$-image of M. By Theorem 1, then there exist partition pairs $(\pi_1, \pi_1') = (C^{(1)} \cdot \pi_1', \pi_1')$ and $(\pi_2, \pi_2') = (C^{(2)}) \cdot \pi_2', \pi_2')$. Because π_1' and π_2' are partitions whose blocks are the elements of $S^{(1)}$ and $S^{(2)}$, respectively, and because each intersection $s^{(1)} \cap s^{(2)}$ is either φ or a single element $s \in S$, it follows that $\pi_1' \cdot \pi_2' = 0$. Furthermore, since the partition $C^{(2)}$ is the partition whose blocks are the elements of the carry input set $C^{(2)}$ and the elements of the latter are composed of subsets of the elements $s^{(1)} \in S^{(1)}$, it follows that $\pi_1' \leq C^{(2)}$, in the case where the carry input set $C^{(2)}$ is composed of a single element, then the partition $C^{(2)}$ is the trivial partition I and it is still true that $\pi_1' \leq C^{(2)}$. Similarly, $\pi_2' \leq C^{(1)}$.

The machines B, B_1, and B_2 of figure 4 are used to exemplify Lemma 1. Machines B_1 and B_2 comprise the concurrently operating interconnection realizing the

B

	1	2	
1	4	4	O_1
2	4	5	O_1
3	5	6	O_2
4	1	1	O_2
5	6	2	O_1
6	6	3	O_2

B_1

	$\langle 1, 11\rangle$	$\langle 1, 12\rangle$	$\langle 2, 11\rangle$	$\langle 2, 12\rangle$	
7	9	--	9	--	$O_1^{(1)}$
8	9	9	9	10	$O_2^{(1)}$
9	7	10	7	8	$O_1^{(1)}$
10	--	10	--	8	$O_2^{(1)}$

B_2

	$\langle 1, \{7, 9\}\rangle$	$\langle 1, \{8, 10\}\rangle$	$\langle 2, \{7, 9\}\rangle$	$\langle 2, \{8, 10\}\rangle$	
11	11	11	11	12	$O_1^{(2)}$
12	12	12	11	12	$O_2^{(2)}$

Figure 4. - Representations of machines B, B_1, and B_2.

state behavior of machine B. Under the mapping Φ the following correspondences obtain:

$$1 \sim \langle 7, 11 \rangle$$

$$2 \sim \langle 8, 11 \rangle$$

$$3 \sim \langle 8, 12 \rangle$$

$$4 \sim \langle 9, 11 \rangle$$

$$5 \sim \langle 9, 12 \rangle$$

$$6 \sim \langle 10, 12 \rangle$$

The state 7 of machine B_1 appears in only one ordered pair, and that pair corresponds to state 1 of machine B; therefore, $7 = \{1\}$. The state 8 appears in two ordered pairs corresponding to states 2 and 3; hence, $8 = \{2, 3\}$. Similarly, $9 = \{4, 5\}$ and $10 = \{6\}$. The states of machine B_2 are found to be $11 = \{1, 2, 4\}$ and $12 = \{3, 5, 6\}$. Thus, there exists the partition pair (π_1, π_1') on S of machine B, where $\pi_1' = \{7 = \{1\}, 8 = \{2, 3\}, 9 = \{4, 5\}, 10 = \{6\}\}$, $C^{(1)} = \{11 = \{1, 2, 4\}, 12 = \{3, 5, 6\}\}$, and $\pi_1 = C^{(1)} \cdot \pi_1' = 0$. Moreover, there exists (π_2, π_2'), where $\pi_2' = \{11 = \{1, 2, 4\}, 12 = \{3, 5, 6\}\}$, $C^{(2)} = \{\{7, 9\} = \{1, 4, 5\}, \{8, 10\} = \{2, 3, 6\}\}$, and $\pi_2 = C^{(2)} \cdot \pi_2' = \{\{1, 4\}, \{2\}, \{3, 6\}, \{5\}\}$. Finally, it is observed that $\pi_2' = C^{(1)}$, $\pi_1' < C^{(2)}$, and $\pi_1' \cdot \pi_2' = 0$.

Theorem 2:

The state behavior of a sequential machine M is realized by a concurrently operating interconnection of two machines M_1 and M_2 if and only if there exist partition pairs (π_1, π_1') and (π_2, π_2') on the states of S of M such that

$$\pi_1' \cdot \pi_2' = 0$$

$$\pi_1 = C^{(1)} \cdot \pi_1'$$

$$\pi_2 = C^{(2)} \cdot \pi_2'$$

Proof:

The "only if" half of the theorem follows directly from Lemma 1.

For the "if" half of the theorem it can be assumed that there exist partition pairs (π_1, π_1') and (π_2, π_2') on S of M such that $\pi_1' \cdot \pi_2' = 0$, $\pi_1 = C^{(1)} \cdot \pi_1'$, and

$\pi_2 = C^{(2)} \cdot \pi_2'$. Moreover, there exist partition pairs (π_1, π_1') and (π_2, π_2') on S of M with the additional properties that $\pi_2' \leq C^{(1)}$ and $\pi_1' \leq C^{(2)}$. That this last assertion is true is verified by the following arguments: Suppose that $\pi_1 = 0$; then $C^{(1)}$ can be taken to be π_2' since $\pi_1 = C^{(1)} \cdot \pi_1' = \pi_2' \cdot \pi_1' = 0$. Next, suppose that $\pi_1 \neq 0$ and $C^{(1)} < \pi_2'$; then $C^{(1)} \cdot \pi_2' = C^{(1)}$ so that $\pi_1 = C^{(1)} \cdot \pi_1' = C^{(1)} \cdot \pi_2' \cdot \pi_1' = 0$, which is a contradiction. Now, suppose that the first of the partition pairs assumed to exist is represented by $(\underline{\pi}_1 = \underline{C}^{(1)} \cdot \pi_1', \pi_1')$ such that neither $\underline{C}^{(1)}$ nor π_2' is a refinement of the other. Hence, $\underline{C}^{(1)} \cdot \pi_2' = \tau$, where $\tau < \underline{C}^{(1)}$ and $\tau < \pi_2'$. Then $\tau \cdot \pi_1' \leq \underline{C}^{(1)} \cdot \pi_1' = \underline{\pi}_1$. Because of the existence of $(\underline{\pi}_1, \pi_1')$ on S of M, the partition pair $(\tau \cdot \pi_1', \pi_1')$ also exists on S of M (ref. 3). Therefore, let the partition pair $(\tau \cdot \pi_1', \pi_1')$ replace $(\underline{\pi}_1, \pi_1')$ in all further considerations; also let $\tau = C^{(1)}$ and $\pi_1 = \tau \cdot \pi_1'$ so that $C^{(1)} < \pi_2'$. For this case, it has already been shown that π_1 must be 0, and hence $C^{(1)}$ can be taken to be π_2'. Thus, it has been shown that there exists a partition pair (π_1, π_1') on S of M such that $\pi_2' \leq C^{(1)}$. Similar arguments show that there also exists a partition pair (π_2, π_2') on S of M such that $\pi_1' \leq C^{(2)}$. According to Theorem 1, there exist machines M_1 and M_2 which are $S^{(1)}$- and $S^{(2)}$-images, respectively, of machine M. Because π_1' π'_2 are partitions whose blocks are elements of $S^{(1)}$ and $S^{(2)}$, respectively, and because $\pi_1' \cdot \pi_2' = 0$, it follows that each intersection $s^{(1)} \cap s^{(2)}$ is either φ or a single $s \in S$. The nonempty intersections $s^{(1)} \cap s^{(2)}$ can be put into a one-to-one correspondence with a subset R of $S^{(1)} \times S^{(2)}$. With the aid of definition 22 and a reversal of the arguments presented in Lemma 1, it follows that there exists a mapping Φ which is a one-to-one correspondence between the states of S of M and the subset R such that M_1 and M_2 are the component machines of a concurrently operating interconnection realizing the state behavior of machine M. This completes the proof of the theorem.

The reader should be warned that the partition pairs given in the "if" half of the theorem are not necessarily those directly associated with M_1 and M_2 as $S^{(1)}$- and $S^{(2)}$-images, respectively, of M. As is brought out in the proof, if the given partition pairs do not have the property that $\pi_2' \leq C^{(1)}$ and $\pi_1' \leq C^{(2)}$, then they imply the existence of partition pairs that do possess this property. The implied partitions pairs only differ from the given ones in that their first partitions are refinements of the corresponding partitions of the given partition pairs. Thus, the implied partition pairs are the ones that are directly associated with M_1 and M_2.

A consideration of machine B again along with the partition pairs derived fn the previous example afford a concrete illustration of Theorem 2. Machines B_1 and B_2 can be derived as $S^{(1)}$- and $S^{(2)}$-images, respectively, of B in the manner illustrated in the example following Theorem 1. Typical of a specific demonstration that B_1 and

B_2 are interconnected to operate concurrently and realize the state behavior of B is the following: Suppose that machine B is in state 2 when the input 1 is applied; then, the next state of B is 4 according to $\{2\}\Delta(1) = \{4\}$. Since $8 \cap 11 = \{2\}$ where $8 \in S^{(1)}$ and $11 \in S^{(2)}$, let it be supposed that machines B_1 and B_2 are in states 8 and 11, respectively, when B is in state 2: that is, let $2 \sim \langle 8, 11 \rangle$. Because for B_1

$$(11 \cap 8)\Delta(1) = \{2\}\Delta(1) = \{4\} \subseteq 8\Delta^{(1)}(\langle 1, 11 \rangle) = 9$$

for B_2

$$(\{8, 10\} \cap 11)\Delta(1) = \{2\}\Delta(1) = \{4\} \subseteq 11\, \Delta^{(2)}(\langle 1, \{8, 10\} \rangle) = 11$$

and

$$9 \cap 11 = \{4\}$$

the combined next state of B_1 and B_2 after the application of the input 1 is 4; that is, $4 \sim \langle 9, 11 \rangle$.

The fundamental theorem of the decomposition theory developed here is a generalization of Theorem 2 and is as follows:

Theorem 3:

The state behavior of a sequential machine M is realized by a concurrently operating interconnection of j machines $M_1, M_2, \ldots, M_j$ if and only if there exist partition pairs (π_1, π_1'), (π_2, π_2'), $\ldots$, (π_j, π_j') on the states of S of M such that

$$\pi_1' \cdot \pi_2' \cdots \pi_j' = 0$$

$$\pi_1 = C^{(1)} \cdot \pi_1'$$

$$\pi_2 = C^{(2)} \cdot \pi_2'$$

$$\cdots$$

$$\pi_j = C^{(j)} \cdot \pi_j'$$

Proof:

For the "only if" half of the theorem, it can be assumed that the state behavior of M is realized by a concurrently operating interconnection of j machines M_1, $M_2, \ldots, M_j$. That part of the given interconnection consisting of all machines M_1, $M_2, \ldots, M_{k-1}, M_{k+1}, \ldots, M_j$ but M_k is a Moore machine which will be called $M_{\alpha+k-1}$. The carry inputs to $M_{\alpha+k-1}$ come from the outputs of machine M_k. The outputs from $M_{\alpha+k-1}$ serve as the carry inputs to M_k. Thus, the state behavior of M is realized by a concurrently operating interconnection of machines $M_{\alpha+k-1}$ and M_k. By Lemma 1 there exist partition pairs $(C^{(\alpha+k-1)} \cdot \pi'_{\alpha+k-1}, \pi'_{\alpha+k-1})$ and $(C^{(k)} \cdot \pi'_k, \pi'_k)$ on the states of S of M such that $\pi'_{\alpha+k-1} \cdot \pi'_k = 0$ where $\pi'_k \leq C^{(\alpha+k-1)}$ and $\pi'_{\alpha+k-1} \leq C^{(k)}$. There are j such pairs of machines $M_{\alpha+k-1}$ and M_k corresponding to $k = 1, 2, \ldots, j$. Their existence implies the existence of j partition pairs on the states of S of M, $(\pi_k = C^{(k)} \cdot \pi'_k, \pi'_k)$, where $k = 1, 2, \ldots, j$. Also implied is that $\pi'_{\alpha+k-1} = \pi'_1 \cdot \pi'_2 \cdot \ldots \cdot \pi'_{k-1} \cdot \pi'_{k+1} \cdot \ldots \cdot \pi'_j$. Hence, $\pi'_1 \cdot \pi'_2 \cdot \ldots \cdot \pi'_j = 0$.

For the "if" half of the theorem, it can be assumed that there exist j partition pairs $(\pi_k = C^{(k)} \cdot \pi'_k, \pi'_k)$, $k = 1, 2, \ldots, j$, on the states of S of M such that $\pi'_1 \cdot \pi'_2 \cdot \ldots \cdot \pi'_j = 0$. As a notational convenience the given partition pairs are represented by $(\underline{\pi}_k = \underline{C}^{(k)} \cdot \pi'_k, \pi'_k)$. There exists a partition pair $(\underline{\pi}_\beta, \pi'_\beta) = (\underline{\pi}_1 \cdot \underline{\pi}_2 \cdot \ldots \cdot \underline{\pi}_{j-1}, \pi'_1 \cdot \pi'_2 \cdot \ldots \cdot \pi'_{j-1})$ on the states of S of M (ref. 3). Since $\underline{\pi}_\beta = \underline{C}^{(1)} \cdot \underline{C}^{(2)} \cdot \ldots \cdot \underline{C}^{(j-1)} \cdot \pi'_\beta$, there certainly exists a partition $\underline{C}^{(\beta)}$ such that $\underline{\pi}_\beta = \underline{C}^{(\beta)} \cdot \pi'_\beta$. Therefore, according to Theorem 2, the state behavior of machine M is realized by a concurrently operating interconnection of two machines M_β and M_j. The proof of Theorem 2 also indicates how the partition pairs (π_β, π'_β) and (π_j, π'_j), which are directly associated with machines M_β and M_j as $S^{(\beta)}$- and $S^{(j)}$-images of M, are derived from $(\underline{\pi}_\beta, \pi'_\beta)$ and $(\underline{\pi}_j, \pi'_j)$. A partition, which will be called $\tilde{\pi}'_k$, on the states of $S^{(\beta)}$ of M_β can be derived from π'_k on the states of S of M as follows: In accordance with definition 14, there exist canonical relations

$$\pi'^*_\beta = \{ \langle s, s^{(\beta)} \rangle \mid s \in s^{(\beta)} \in S^{(\beta)} \}$$

$$\pi'^*_k = \{ \langle s, s^{(k)} \rangle \mid s \in s^{(k)} \in S^{(k)} \}$$

By definition 15, the inverse of π'^*_β is

$$(\pi'^*_\beta)^{-1} = \{ \langle s^{(\beta)}, s \rangle \mid s \pi'^*_\beta s^{(\beta)}, \ s \in S \text{ and } s^{(\beta)} \in S^{(\beta)} \}$$

By definition 10,

$$\tilde{\pi}'^*_k = (\pi'^*_\beta)^{-1} \pi'^*_k = \{ \langle s^{(\beta)}, s^{(k)} \rangle \mid s^{(\beta)} (\pi'^*_\beta)^{-1} s \text{ and } s \pi'^*_k s^{(k)} \text{ for some } s \in S \}$$

Thus,

$$S\pi'_\beta{}^* = S^{(\beta)}, \ S\pi'^*_k = S^{(k)} \text{ so that } S^{(\beta)} (\pi'^*_\beta)^{-1} = S$$

and

$$S\pi_k'^* = S^{(\beta)}(\pi_\beta'^*)^{-1}\pi_k'^* = S^{(\beta)}\tilde{\pi}_k'^* = S^{(k)}$$

Then, $\tilde{\pi}_k'$, is the partition whose blocks are the elements of $S^{(k)}$. That the blocks of $\tilde{\pi}_k'$ as defined by the previous mapping are indeed pairwise disjoint is guaranteed by the fact that $\pi_\beta' \leq \pi_k'$. A partition pair $(\underline{\tilde{\pi}}_k, \tilde{\pi}_k')$ on the states of $S^{(\beta)}$ of M_β can be derived by an application of definition 18 (ref. 3). Since $(0, \tilde{\pi}_k')$ always exists, it follows that there exists a partition pair of the form $(\underline{\tilde{\pi}}_k = \underline{C}^{(k)} \cdot \tilde{\pi}_k', \tilde{\pi}_k')$ on the states of $S^{(\beta)}$ of M_β. Hence, there exist $j - 1$ partition pairs $(\underline{\tilde{\pi}}_k = \underline{C}^{(k)} \cdot \tilde{\pi}_k', \tilde{\pi}_k')$, $k = 1, 2, \ldots, j - 1$, on the states of $S^{(\beta)}$ of M_β such that $\tilde{\pi}_1' \cdot \tilde{\pi}_\beta' \cdot \ldots \cdot \tilde{\pi}_{j-1}' = 0$ where 0 on $S^{(\beta)}$ of M_β corresponds to π_β' on S of M. The process given thus far in this half of the proof can now be iterated to show that the state behavior of machine $M_{\beta+k-1}$ is realized by a concurrently operating interconnection of machines $M_{\beta+k}$ and M_{j-k} for $k = 1, 2, \ldots, j - 3$. The process terminates when it is established that the state behavior of machine $M_{\beta+j-3}$ is realized by a concurrently operating interconnection of machines M_1 and M_2. Hence, the state behavior of M is realized by a concurrently operating interconnection of machines $M_1, M_2, \ldots, M_j$, which establishes the theorem.

Machine B of figure 4 provides a specific example of the "if" half of Theorem 3. Associated with machine B are partition pairs $(\underline{\pi}_2 = \underline{C}^{(2)} \cdot \pi_2', \pi_2')$, $(\underline{\pi}_3 = \underline{C}^{(3)} \cdot \pi_3', \pi_3')$, and $(\underline{\pi}_4 = \underline{C}^{(4)} \cdot \pi_4', \pi_4')$, where

$$\pi_2' = \{11 = \{1,2,4\},\ 12 = \{3,5,6\}\} \qquad \underline{C}^{(2)} = \{13 = \{1,4,5\},\ 14 = \{2,3,6\}\}$$

$$\pi_3' = \{13 = \{1,4,5\},\ 14 = \{2,3,6\}\} \qquad \underline{C}^{(3)} = \{17 = \{1,2,4\},\ 18 = \{3,5\},\ 19 = \{6\}\}$$

$$\pi_4' = \{15 = \{1,2,3\},\ 16 = \{4,5,6\}\} \qquad \underline{C}^{(4)} = \{20 = \{1,2,3,5,6\},\ 21 = \{4\}\}$$

such that $\pi_2' \cdot \pi_3' \cdot \pi_4' = 0$. According to the theorem, this implies that the state behavior of machine B can be realized by a concurrently operating interconnection of three machines - B_2, B_3, and B_4. This interconnection can be derived by the process introduced in the proof: The partition pairs $(\underline{\pi}_2, \pi_2')$ and $(\underline{\pi}_\beta, \pi_\beta') = (\underline{\pi}_3 \cdot \underline{\pi}_4, \pi_3' \cdot \pi_4') = 0, \{7 = \{1\}, 8 = \{2,3\}, 9 = \{4,5\}, 10 = \{6\}\})$ are considered. The partition π_β' is precisely the partition π_1' given in the previous example. From that example it is also seen that there exists a partition $\pi_\beta = \pi_1 = \underline{\pi}_\beta$ such that $\pi_2' \leq C^{(1)}$ and that there exists a partition $\pi_2 = \underline{\pi}_2$ such that $\pi'_1 \le \underline{C}^{(2)} = C^{(2)}$. The construction of flow tables for machines B_1 and B_2 would ordinarily comprise the next step of the process, but these tables have already been presented in the previous example and are shown in figure 4. Next, partitions $\tilde{\pi}'_3$ and $\tilde{\pi}'_4$ on the states of $S^{(1)}$ of B_1 are derived as

$$\tilde{\pi}'_3 = \{13 = \{7,9\},\ 14 = \{8,10\}\}$$

$$\tilde{\pi}'_4 = \{15 = \{7,8\},\ 16 = \{9,\ 10\}\}$$

With the use of definition 18, the partition pairs

$$(\underline{\tilde{\pi}}_3,\ \tilde{\pi}'_3) = (\{15 = \{7,8\},\ 16 = \{9,10\}\},\ \{13 = \{7,9\},\ 14 = \{8,10\}\})$$

and

$$(\underline{\tilde{\pi}}_4,\ \tilde{\pi}'_4) = (\{15 = \{7,8\},\ 16 = \{9,10\}\},\ \{15 = \{7,8\},\ 16 = \{9,10\}\})$$

are generated. An inspection of partitions $\underline{\tilde{\pi}}_3$ and $\underline{\tilde{\pi}}_4$ reveals the existence of partitions $\tilde{\pi}_3 = C^{(3)} \cdot \tilde{\pi}'_3 \le \underline{\tilde{\pi}}_3$ and $\tilde{\pi}_4 = C^{(4)} \cdot \tilde{\pi}_4 \le \underline{\tilde{\pi}}_4$, where $C^{(3)} = \tilde{\pi}'_4$ and $C^{(4)} = I$. Hence, $\tilde{\pi}'_4 \le C^{(3)}$, $\tilde{\pi}'_3 \le C^{(4)}$, and $\tilde{\pi}'_3 \cdot \tilde{\pi}'_4 = 0$ on $S^{(1)}$ so that the state behavior of machine B_1 is realized by a concurrently operating interconnection of machines B_3 and B_4. These machines constructed as $S^{(3)}$- and $S^{(4)}$-images of machine B_1 are shown as the flow tables of figure 5.

It should be noted in the flow table of machine B_3 that the unspecified next stages can be selected in such a way that, for every input $i^{(3)} \in I^{(3)}$, $13\ \Delta^{(3)}(i^{(3)}) = 14\ \Delta^{(3)}(i^{(3)})$. With such a specification machine B_3 can be simplified to the extent that its next states are independent of its present states. Such a machine is said to be

B_3

	⟨⟨1, 11⟩, 15⟩	⟨⟨1, 11⟩, 16⟩	⟨⟨1, 12⟩, 15⟩	⟨⟨1, 12⟩, 16⟩	⟨⟨2, 11⟩, 15⟩	⟨⟨2, 11⟩, 16⟩	⟨⟨2, 12⟩, 15⟩	⟨⟨2, 12⟩, 16⟩	
13	13	13	--	14	13	13	--	14	$O_1^{(3)}$
14	13	--	13	14	13	--	14	14	$O_2^{(3)}$

B_4

	⟨1, 11⟩	⟨1, 12⟩	⟨2, 11⟩	⟨2, 12⟩	
15	16	16	16	16	$O_1^{(4)}$
16	15	16	15	15	$O_2^{(4)}$

Figure 5. - Machines B_3 and B_4.

feedback free. It is worthwhile noting that the existence of a feedback-free machine B_3 could have been anticipated. On the states S of machine B there exists a partition pair $(C^{(3)} \cdot I, \pi_3')$, which implies for every $i \in I$, $i^{(3)} \in I^{(3)}$, $c^{(3)} \in C^{(3)}$, and $s^{(3)} \in S^{(3)}$ that $c^{(3)}\ \Delta(i) \subseteq s^{(3)}$ or equivalently $\Delta^{(3)}(i^{(3)}) \subseteq s^{(3)}$. The latter relation implies the existence of a machine B_3 independent of its present states.

ADDITIONAL BACKGROUND

Additional basic background material (ref. 2) is introduced in this section to facilitate the concluding theoretical investigations of this report. The presentation of a logical design problem serves a dual purpose as a convenient vehicle for illustrating the concepts comprising the additional background material and as an example of how the newly developed decomposition theory might be used in practice.

The design problem is that of deriving a two-layer "and/or" diode gate realization of a machine C represented by the flow table of figure 6. The solution of this problem is facilitated by the introduction of four definitions.

	1	2	
1	5	3	0_2
2	3	4	0_1
3	7	2	0_1
4	2	4	0_2
5	1	6	0_2
6	5	5	0_2
7	3	2	0_1

Figure 6. - Tabular representation of machine C.

Definition 23

A partition π on the set of states S of a sequential machine M is said to be output consistent, if for every block P of π, all the states contained in the block have the same output. That is, for every $P \in \pi$ and every $i \in I$, $P\Lambda(i) = 0$, where $o \in 0$ is not necessarily the same for any two inputs i.

One such partition on the set of states S of machine C is

$$\pi = \{P_1 = \{1,6\},\ P_2 = \{2,3\},\ P_3 = \{7\},\ P_4 = \{4,5\}\}$$

since

$$P_1\Lambda(i) = 0_2$$

$$P_2\Lambda(i) = 0_1$$

$$P_3\Lambda(i) = 0_1$$

$$P_4\Lambda(i) = 0_2$$

where

$$\Lambda(i) = \Lambda(1) = \Lambda(2) = \{\langle 1, 0_2\rangle,\ \langle 2, 0_1\rangle,\ \langle 3, 0_1\rangle,\ \langle 4, 0_2\rangle,\ \langle 5, 0_2\rangle,\ \langle 6, 0_2\rangle,\ \langle 7, 0_1\rangle\}$$

Definition 24

Given sequential machines M and M_r, M_r is a reduction of M if and only if the following conditions are satisfied:

(1) There exists a partition pair (π_r, π_r') on the states of S of M such that $\pi_r = \pi_r'$.

(2) π_r' is output consistent.

(3) The output set $0^{(r)}$ of M_r is a subset of the output set 0 of M and has elements

$$o = s^{(r)}\Lambda(i) = s^{(r)}\Lambda^{(r)}(i^{(r)})$$

where

$$o \in 0,\ s^{(r)} \in S^{(r)},\ i \in I, \text{ and } i^{(r)} \in I^{(r)}.$$

From figure 6 it is observed that there exists a partition pair (π_r, π_r') on the states S of machine C such that

$$\pi_r = \pi_r' = \{8 = \{1\},\ 9 = \{2\},\ 10 = \{3,7\},\ 11 = \{4\},\ 12 = \{5\},\ 13 = \{6\}\}$$

	1	2	
8	12	10	0_2
9	10	11	0_1
10	10	9	0_1
11	9	11	0_2
12	8	13	0_2
13	12	12	0_2

Figure 7. - Tabular representation of reduction C_r.

The calculation

$$8\Lambda(i) = 0_2 = 8\Lambda^{(r)}(i^{(r)})$$

$$9\Lambda(i) = 0_1 = 9\Lambda^{(r)}(i^{(r)})$$

$$10\Lambda(i) = 0_1 = 10\Lambda^{(r)}(i^{(r)})$$

$$11\Lambda(i) = 0_2 = 11\Lambda^{(r)}(i^{(r)})$$

$$12\Lambda(i) = 0_2 = 12\Lambda^{(r)}(i^{(r)})$$

$$13\Lambda(i) = 0_2 = 13\Lambda^{(r)}(i^{(r)})$$

where $i = i^{(r)}$, serves both to show that π_r' is output consistent and to establish the output set of a reduction C_r of machine C. That such a machine exists is guaranteed by Theorem 1 which implies that machine C_r is an $S^{(r)}$-image of C. The flow table representation of machine C_r is given in figure 7.

Definition 25

Two sequential machines are said to be equivalent if they have the same input-output relations. A sequential machine M and any reduction M_r of M are equivalent.

Definition 26

The state behavior of a sequential machine M is said to be realized by a concurrently operating interconnection of the machines $M_1, M_2, \ldots, M_j$ if the state behavior of a sequential machine M_e equivalent to M is realized by the aforementioned interconnection.

To obtain the desired realization of machine C the decomposition theory is now used to derive a concurrently operating interconnection of three machines C_1, C_2, and C_3 which realize the state behavior of machine C_r. An examination of the flow table of machine C_r given in figure 7 shows that there exist on the states of $S^{(r)}$ the partition pairs $(\pi_1 = C^{(1)} \cdot \pi_1', \pi_1')$, $(\pi_2 = C^{(2)} \cdot \pi_2', \pi_2')$, and $(\pi_3 = C^{(3)} \cdot \pi_3', \pi_3')$, where

$$\pi_1' = \{14 = \{8,12\},\ 15 = \{9,10,11,13\}\} \qquad C^{(1)} = \{20 = \{8,13\},\ 21 = \{9,10,11,12\}\}$$

$$\pi_2' = \{16 = \{8,10,13\},\ 17 = \{9,11,12\}\},\ C^{(2)} = \{22 = \{8\}, 23 = \{12\},\ 24 = \{11,13\}, 25 = \{9,10\}\}$$

$$\pi_3' = \{18 = \{8,11,13\},\ 19 = \{9,10,12\}\},\ C^{(3)} = \{26 = \{8\},\ 27 = \{12\}, 28 = \{10,13\},\ 29 = \{9,11\}\}$$

and $\pi_1' \cdot \pi_2' \cdot \pi_3' = 0$. Therefore, Theorem 3 is satisfied. Furthermore, because

$$\pi_2' \cdot \pi_3' = \pi_4' < C^{(1)} \text{ and } \pi_1' = C^{(4)} \text{ for } (\pi_1, \pi_1') \text{ and } (\pi_4, \pi_4')$$

$$\pi_1' \cdot \pi_3' = \pi_5' = C^{(2)} \text{ and } \pi_2' = C^{(5)} \text{ for } (\pi_2, \pi_2') \text{ and } (\pi_5, \pi_5')$$

$$\pi_1' \cdot \pi_2' = \pi_6' = C^{(3)} \text{ and } \pi_3' = C^{(6)} \text{ for } (\pi_3, \pi_3') \text{ and } (\pi_6, \pi_6')$$

it follows as indicated in the proof of Theorem 3 that the three given partition pairs are those directly associated with the machines C_1, C_2, and C_3 as $S^{(1)}$-, $S^{(2)}$-, and $S^{(3)}$-images, respectively, of C_r. The flow table representations of these machines are shown in figure 8.

A representation of this decomposition in terms of physical switching elements is afforded by making the following associations: Associate with the external inputs 1 and 2 of the three machines the binary variables $\bar{x}$ and x, respectively; $\bar{x}$ denotes the complement of x. Likewise, with the present states 14, 15, 16, 17, 18, 19 associate the binary variables $y_1, \bar{y}_1, y_2, \bar{y}_2, \bar{y}_3, y_3$, respectively. Also, let the next states 14, 15, 16, 17, 18, 19 be associated with the binary variables $Y_1, \bar{Y}_1, Y_2, \bar{Y}_2, \bar{Y}_3, Y_3$, respectively. Since $20 = \{8,13\} = 16 \cap 18$, associate with the element 20 of the carry input set $C^{(1)}$ to C_1 the "and" function $y_2\bar{y}_3$ of y_2 and $\bar{y}_3$. Similarly, associate

C_1

	⟨1, 20⟩	⟨1, 21⟩	⟨2, 20⟩	⟨2, 21⟩	
14	14	14	15	15	$O_1^{(1)}$
15	14	15	14	15	$O_2^{(1)}$

C_2

	⟨1, 22⟩	⟨1, 23⟩	⟨1, 24⟩	⟨1, 25⟩	⟨2, 22⟩	⟨2, 23⟩	⟨2, 24⟩	⟨2, 25⟩	
16	17	--	17	16	16	--	17	17	$O_1^{(2)}$
17	--	16	17	16	--	16	17	17	$O_2^{(2)}$

C_3

	⟨1, 26⟩	⟨1, 27⟩	⟨1, 28⟩	⟨1, 29⟩	⟨2, 26⟩	⟨2, 27⟩	⟨2, 28⟩	⟨2, 29⟩	
18	19	--	19	19	19	--	19	18	$O_1^{(3)}$
19	--	18	19	19	--	18	19	18	$O_2^{(3)}$

Figure 8. - Machines C_1, C_2, and C_3.

$Y_1 = \bar{x}y_1 + \bar{y}_1y_2\bar{y}_3$

	$\bar{x}y_2\bar{y}_3$	$\bar{x}(\bar{y}_2 + y_3)$	$xy_2\bar{y}_3$	$x(\bar{y}_2 + y_3)$	
y_1	Y_1	Y_1	$\bar{Y}_1$	$\bar{Y}_1$	$O_1^{(1)}$
$\bar{y}_1$	Y_1	$\bar{Y}_1$	Y_1	$\bar{Y}_1$	$O_2^{(1)}$

$Y_2 = \bar{x}y_3 + xy_1$

	$\bar{x}y_1\bar{y}_3$	$\bar{x}y_1y_3$	$\bar{x}\bar{y}_1\bar{y}_3$	$\bar{x}\bar{y}_1y_3$	$xy_1\bar{y}_3$	xy_1y_3	$x\bar{y}_1\bar{y}_3$	$x\bar{y}_1y_3$	
y_2	$\bar{Y}_2$	Y_2	$\bar{Y}_2$	Y_2	Y_2	Y_2	$\bar{Y}_2$	$\bar{Y}_2$	$O_1^{(2)}$
$\bar{y}_2$	$\bar{Y}_2$	Y_2	$\bar{Y}_2$	Y_2	Y_2	Y_2	$\bar{Y}_2$	$\bar{Y}_2$	$O_2^{(2)}$

$Y_3 = \bar{x}\bar{y}_1 + y_2$

	$\bar{x}y_1y_2$	$\bar{x}y_1\bar{y}_2$	$\bar{x}\bar{y}_1y_2$	$\bar{x}\bar{y}_1\bar{y}_2$	xy_1y_2	$xy_1\bar{y}_2$	$x\bar{y}_1y_2$	$x\bar{y}_1\bar{y}_2$	
$\bar{y}_3$	Y_3	$\bar{Y}_3$	Y_3	Y_3	Y_3	$\bar{Y}_3$	Y_3	$\bar{Y}_3$	$O_1^{(3)}$
y_3	Y_3	$\bar{Y}_3$	Y_3	Y_3	Y_3	$\bar{Y}_3$	Y_3	$\bar{Y}_3$	$O_2^{(3)}$

Figure 9. - Truth tables of the switching functions Y_1, Y_2, Y_3.

with 21 the "or" function $\bar{y}_2 + y_3$. In an analogous manner, associate with the carry input elements 22, 23, 24, 25 or $C^{(2)}$ to C_2 $y_1\bar{y}_3$, y_1y_3, $\bar{y}_1\bar{y}_3$, $\bar{y}_1y_3$, respectively. Finally, associate with the carry input elements 26, 27, 28, 29 of $C^{(3)}$ to C_3 y_1y_2, $y_1\bar{y}_2$, $\bar{y}_1y_2$, $\bar{y}_1\bar{y}_2$, respectively. Under these associations the flow tables of machines C_1, C_2, and C_3 become truth tables for the logical switching functions Y_1, Y_2, and Y_3, respectively (see fig. 9). Moreover, the part of the flow table of machine C_r specifying the input-output relations is transformed into a truth table for the output function of the decomposition. This output switching function, designated by Z, is

$Z = \bar{y}_1y_3$

8	$y_1y_2\bar{y}_3$	$\bar{Z}$
9	$\bar{y}_1\bar{y}_2y_3$	Z
10	$\bar{y}_1y_2y_3$	Z
11	$\bar{y}_1\bar{y}_2\bar{y}_3$	$\bar{Z}$
12	$y_1\bar{y}_2y_3$	$\bar{Z}$
13	$\bar{y}_1y_2\bar{y}_3$	$\bar{Z}$

Figure 10. - Truth table of the switching function Z.

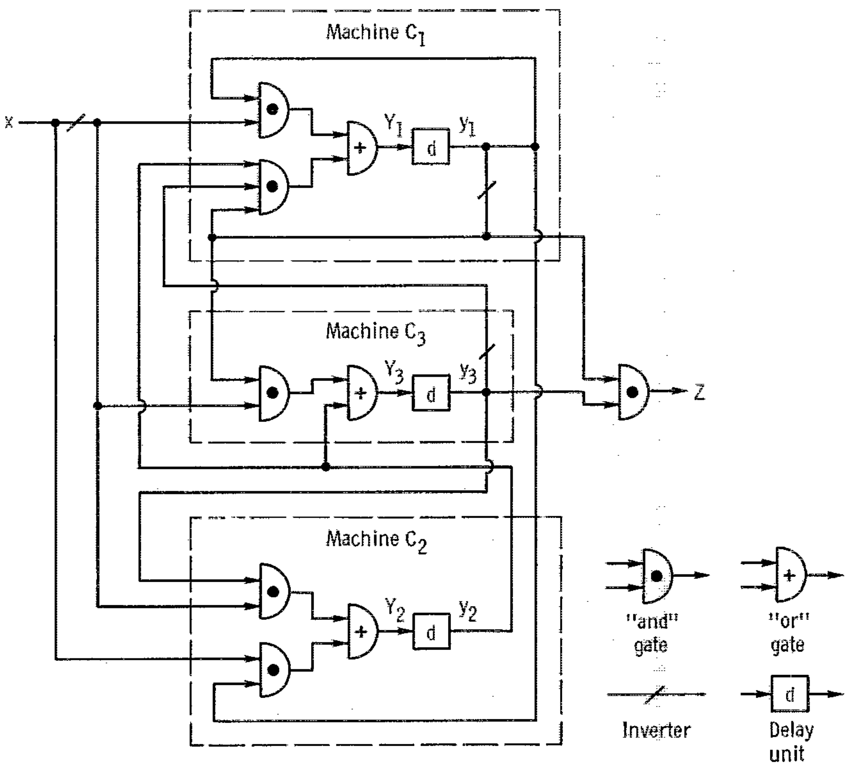

Figure 11. - Realization of machine C.

associated with 0_1; whereas its complement $\overline{Z}$ is associated with 0_2. The truth table for Z is given in figure 10.

Corresponding to each of the switching functions Y_1, Y_2, Y_3, and Z are logical switching circuits made up of transistor inverters, diode "and/or" gates, and delay units. These circuits, shown in figure 11, comprise the sought after realization of machine C.

STATE SPLITTING

An important concern of the previous section was the realization of the state behavior of a given machine M as that interconnection of concurrently operating machines realizing the state behavior of a reduction M_r of M. A reciprocal concern of equal importance is the realization of the state behavior of a given machine M_r as that interconnection of concurrently operating machines realizing the state behavior of a machine M of which M_r is a reduction (refs. 5 and 9). It is clear that the latter realization can be obtained merely by applying the newly developed decomposition theory to machine M. There is a problem, however, of deriving the machine M from M_r. The derivation of such a machine M from a given machine M_r is achieved by means of a process called state splitting (refs. 5 and 10).

Definition 27

A state $S_j^{(r)} \in S^{(r)}$ of a sequential machine M_r is said to be split if it is replaced by two or more states $S_j^{(r)1}$, $S_j^{(r)2}$, . . ., $S_j^{(r)n}$. A sequential machine M is said to be a state split version of M_r if it is derived from M_r as follows:

(1) If a state $S_j^{(r)}$ of M_r is split, then the states $S_j^{(r)1}$, $S_j^{(r)2}$, . . ., $S_j^{(r)n}$ are taken as states of S of M.

(2) If a state $S_k^{(r)}$ of M_r is not split, then the state $S_k^{(r)}$ is taken as a state of S of M.

(3) If $S_k^{(r)}\ \Delta^{(r)}(i^{(r)}) = S_j^{(r)}$ for M_r, then for M $S_k^{(r)}\ \Delta(i) = S_j^{(r)l}$, where $i^{(r)} = i \in I$ and l is an integer such that $1 \leq l \leq n$.

(4) If $S_j^{(r)}\Delta^{(r)}(i^{(r)}) = S_k^{(r)}$ for M_r, then for M $S_j^{(r)l}\ \Delta(i) = S_k^{(r)}$, where l assumes all integer values such that $1 \leq l \leq n$.

(5) If $S_j^{(r)}\Delta^{(r)}(i^{(r)}) = S_j^{(r)}$ for M_r, then for M $S_j^{(r)l}\Delta(i) = S_j^{(r)m}$, where l assumes all integer values such that $1 \leq l \leq n$ and m takes on some integer values such that $1 \leq m \leq n$.

(6) If $S_k^{(r)}\Delta^{(r)}(i^{(r)}) = S_l^{(r)}$, where both $S_k^{(r)}$ and $S_l^{(r)}$ are not split, for M_r, then for M $S_k^{(r)}\Delta(i) = S_l^{(r)}$.

A concrete illustration of the definition is provided by considering machine D_r of figure 12. If only state 4 of machine D_r is split into states 4^1, and 4^2, then according

D_r

	1	2	1	2
1	5	2	$0_1^{(r)}$	$0_1^{(r)}$
2	4	1	$0_1^{(r)}$	$0_2^{(r)}$
3	6	4	$0_1^{(r)}$	$0_1^{(r)}$
4	7	4	$0_2^{(r)}$	$0_1^{(r)}$
5	6	3	$0_2^{(r)}$	$0_2^{(r)}$
6	4	6	$0_2^{(r)}$	$0_1^{(r)}$
7	5	3	$0_1^{(r)}$	$0_2^{(r)}$

Figure 12. - Machine D_r.

to parts (1) and (2) of the definition, the set S of states of machine D, which is a state split version of D_r, is $\{1, 2, 3, 4^1, 4^2, 5, 6, 7\}$. By part (3) of the definition, because it is true that

$$2\Delta^{(r)}(1) = 4$$

$$6\Delta^{(r)}(1) = 4$$

$$3\Delta^{(r)}(2) = 4$$

for machine D_r, then for machine D

$$2\Delta(1) = 4^1$$

$$6\Delta(1) = 4^1$$

$$3\Delta(2) = 4^1$$

where the choice of 4^1 in each case is arbitrary. By part (4) of the definition

$$4\Delta^{(r)}(1) = 7$$

for D_r implies that

$$4^1\Delta(1) = 7 \text{ and } 4^2\Delta(1) = 7$$

for machine D. By part (5) of the definition

$$4\Delta^{(r)}(2) = 4$$

for D_r implies that

$$4^1\Delta(2) = 4^2 \text{ and } 4^2\Delta(2) = 4^1$$

for D, where the choice of next states 4^2 and 4^1, respectively, is arbitrary. Typical of part (6) of the definition is

$$3\Delta^{(r)}(1) = 6$$

for D_r implies that

$$3\Delta(1) = 6$$

for machine D. The flow table representation of machine D is shown in figure 13.

It is clearly impractical to determine at random what states of a given machine should be split and to what degree they should be split. Bases for systematic determination procedures must be developed if state splitting is to be useful. Information about the decomposition structure of a given sequential machine should most properly be the basis for state splitting when the primary objective is to design the machine as a con-

D

	1	2	1	2
1	5	2	0_1	0_1
2	4^1	1	0_1	0_2
3	6	4^1	0_1	0_1
4^1	7	4^2	0_2	0_1
4^2	7	4^1	0_2	0_1
5	6	3	0_2	0_2
6	4^1	6	0_2	0_1
7	5	3	0_1	0_2

Figure 13. - Machine D.

currently operating interconnection of smaller machines. A step toward the development of such a systematic state splitting determination procedure has been made by introducing concepts among which are generalizations of partitions and partition pairs (ref. 5). These generalizations carry information about the loop-free decomposition structure of sequential machines. These concepts must, however, be extended further to convey sufficient information about the general decomposition structure of sequential machines. Obvious extensions of the aforementioned concepts are now introduced.

Definition 28

A nondisjoint partition θ of a set $S^{(r)}$ is a family $P_1, P_2, \ldots, P_n$ of nonempty sets such that $P_1 \cup P_2 \cup \ldots \cup P_n = S^{(r)}$. The sets $P_1, P_2, \ldots, P_n$ are called the blocks of θ. The blocks of θ are not necessarily pairwise disjoint.

With respect to machine D_r, a nondisjoint partition θ_1 is given by

$$\theta_1 = \{P_1 = \{1,2\},\ P_2 = \{3,4\},\ P_3 = \{4,5\},\ P_4 = \{6,7\}\}$$

where the union of the blocks P_1, P_2, P_3, P_4 is

$$\{1,2\} \cup \{3,4\} \cup \{4,5\} \cup \{6,7\} = \{1,2,3,4,5,6,7\} = S^{(r)}$$

and blocks P_2 and P_3 are nondisjoint.

Definition 29

A canonical relation θ^* between $S^{(r)}$ and θ is given by

$$\theta^* = \{\langle s^{(r)}, P\rangle \mid s^{(r)} \in P \in \theta\}$$

In particular, the relation θ_1^* between $S^{(r)}$ of machine D_r and θ_1 of the previous example is

$$\theta_1^* = \{\langle 1,P_1\rangle,\ \langle 2,P_1\rangle,\ \langle 3,P_2\rangle,\ \langle 4,P_2\rangle,\ \langle 4,P_3\rangle,\ \langle 5,P_3\rangle,\ \langle 6,P_4\rangle,\ \langle 7,P_4\rangle\}$$

Definition 30

Nondisjoint partitions θ and θ' of $S^{(r)}$ of a sequential machine M_r form a nondisjoint partition pair (θ, θ'), if and only if for every $P \in \theta$ and every $i^{(r)} \in I^{(r)}$ there exists a $Q \in \theta'$ such that $P \Delta^{(r)}(i^{(r)}) \subseteq Q$.

Illustrative of this definition is the nondisjoint partition pair

$$(\theta_1, \theta_1') = (\{P_1 = \{1,2\},\ P_2 = \{3,4\},\ P_3 = \{4,5\},\ P_4 = \{6,7\}\},$$

$$\{Q_1 = \{1,2,4,5\},\ Q_2 = \{3,4,6,7\}\})$$

of the machine D_r:

$$\{1,2\}\Delta^{(r)}(1) = \{5,4\} \subseteq Q_1$$

$$\{3,4\}\Delta^{(r)}(1) = \{6,7\} \subseteq Q_2$$

$$\{4,5\}\Delta^{(r)}(1) = \{7,6\} \subseteq Q_2$$

$$\{6,7\}\Delta^{(r)}(1) = \{4,5\} \subseteq Q_1$$

$$\{1,2\}\Delta^{(r)}(2) = \{2,1\} \subseteq Q_1$$

$$\{3,4\}\Delta^{(r)}(2) = \{4\} \subseteq Q_1 \text{ or } Q_2$$

$$\{4,5\}\Delta^{(r)}(2) = \{4,3\} \subseteq Q_2$$

$$\{6,7\}\Delta^{(r)}(2) = \{6,3\} \subseteq Q_2$$

Definition 31

A nondisjoint partition θ_α of $S^{(r)}$ is said to be a refinement of a nondisjoint partition θ_β of $S^{(r)}$, written $\theta_\alpha \leq \theta_\beta$, if and only if the following conditions are satisfied:

(1) Every $P \in \theta_\alpha$ is contained in some $Q \in \theta_\beta$.

(2) The union of all P contained in the same Q equals that Q.

The nondisjoint partitions θ_1 and θ_1' of machine D_r afford a concrete example of the definition. Since $P_1 \subseteq Q_1$ and $P_3 \subseteq Q_1$ such that $P_1 \cup P_3 = Q_1$ and $P_2 \subseteq Q_2$ and $P_4 \subseteq Q_2$ such that $P_2 \cup P_4 = Q_2$, $\theta_1 \leq \theta_1'$.

Definition 32

The multiplicity of the element $S_j^{(r)}$ in θ_α of $S^{(r)}$ is one less than the number of blocks of θ_α that contain $S_j^{(r)}$. The multiplicity of $S_j^{(r)}$ in θ_α is denoted by $m_{\alpha; S_j^{(r)}}$

In particular, the multiplicity $m_{1;4}$ of state 4 in θ_1 of machine D_r is 1; whereas, the multiplicity of all other states of $S^{(r)}$ in θ_1 is 0.

Definition 33

The multiplicity-dependent intersection of two blocks $P \in \theta_\alpha$ and $Q \in \theta_\beta$, written as $P \text{ ⓜ } Q$, is any subset of $P \cap Q$ such that the following are true:

(1) Every $S_j^{(r)}$ contained in $P \cap Q$, for which either or both $m_{\alpha;S_j^{(r)}}$ and $m_{\beta;S_j^{(r)}}$ is 0, also belongs to $P \text{ ⓜ } Q$.

(2) Any $S_j^{(r)}$ contained in $P \cap Q$, for which $m_{\alpha;S_j^{(r)}} > 0$ and $m_{\beta;S_j^{(r)}} > 0$, may be chosen to be a member of $P \text{ ⓜ } Q$, but is not necessarily so.

A specific illustration of the multiplicity-dependent intersection of two blocks is provided by considering the nondisjoint partitions

$$\theta_1' = \{8 = \{1,2,4,5\}, 9 = \{3,4,6,7\}\} \text{ and } \theta_2' = \{10 = \{1,2,3,4\},\ 11 = \{4,5,6,7\}\}$$

of machine D_r. By part (1) of definition 33

$$\{1,2\} \subseteq 8 \text{ ⓜ } 10$$

and by part (2) either

$$\{1,2\} = 8 \text{ ⓜ } 10$$

or

$$\{1,2,4\} = 8 \text{ ⓜ } 10$$

Definition 34

The multiplicity-dependent greatest lower bound of two nondisjoint partitions θ_α and θ_β, written as $\theta_\alpha \dot{m}\, \theta_\beta$, is a nondisjoint partition

$$\theta_\gamma = \{P ⓜ Q \mid P \in \theta_\alpha, Q \in \theta_\beta, \text{ and } m_{\gamma; S_k^{(r)}} = m_{\alpha; S_j^{(r)}} \text{ if } m_{\alpha; S_j^{(r)}} \geq m_{\beta; S_j^{(r)}};$$

$$\text{otherwise, } m_{\gamma; S_j^{(r)}} = m_{\beta; S_j^{(r)}}\}$$

such that

$$\theta_\gamma \leq \theta_\alpha$$

$$\theta_\gamma \leq \theta_\beta$$

Specifically, the multiplicity-dependent greatest lower bound of θ_1' and θ_2' of the previous example is

$$\theta_3 = \{8 ⓜ 10,\ 8 ⓜ 11,\ 9 ⓜ 10,\ 9 ⓜ 11\}$$

where

$$8 ⓜ 10 = \{1, 2\} \text{ or } \{1, 2, 4\}$$

$$8 ⓜ 11 = \{5\} \text{ or } \{4, 5\}$$

$$9 ⓜ 10 = \{3\} \text{ or } \{3, 4\}$$

$$9 ⓜ 11 = \{6, 7\} \text{ or } \{4, 6, 7\}$$

such that precisely two blocks of θ_3 contain the state 4 and

$$(8 ⓜ 10) \cup (8 ⓜ 11) = 8$$

$$(9 ⓜ 10) \cup (9 ⓜ 11) = 9$$

$$(8 \textcircled{m} 10) \cup (9 \textcircled{m} 10) = 10$$

$$(8 \textcircled{m} 11) \cup (9 \textcircled{m} 11) = 11$$

Thus, θ_3 can be chosen to be either

$$\{8 \textcircled{m} 10 = \{1,2\},\ 8 \textcircled{m} 11 = \{4,5\},\ 9 \textcircled{m} 10 = \{3,4\},\ 9 \textcircled{m} 11 = \{6,7\}\}$$

or

$$\{8 \textcircled{m} 10 = \{1,2,4\},\ 8 \textcircled{m} 11 = \{5\},\ 9 \textcircled{m} 10 = \{3\},\ 9 \textcircled{m} 11 = \{4,6,7\}\}$$

Definition 35

Given sequential machines M_r and M_k, M_k is a generalized $S^{(k)}$-image of M_r if and only if for every $s^{(k)} \in S^{(k)}$, $i^{(r)} \in I^{(r)}$, and $i^{(k)} \in I^{(k)}$ the following conditions are satisfied:

(1) Every $s^{(r)} \in S^{(r)}$ belongs to some $s^{(k)} \in S^{(k)}$ and to some $c^{(k)} \in C^{(k)}$.

(2) $(c^{(k)} \textcircled{m} s^{(k)})\Delta^{(r)}(i^{(r)}) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$.

The machine D_1, shown in figure 14, is a generalized $S^{(1)}$-image of D_r and affords a specific example of the previous definition. The carry input set to D_1 is

$$C^{(1)} = \{10 = \{1,2,3,4\},\ 11 = \{4,5,6,7\}\}$$

and the state set is

$$S^{(1)} = \{8 = \{1,2,4,5\},\ 9 = \{3,4,6,7\}\}$$

From calculations carried out in examples 30, 31, 33, and 34, it is clear that machine D_1 satisfies the conditions of definition 35 to be a generalized $S^{(1)}$-image of D_r.

D_1

	$\langle 1,10\rangle$	$\langle 1,11\rangle$	$\langle 2,10\rangle$	$\langle 2,11\rangle$	
8	8	9	8	9	$0_1^{(1)}$
9	9	8	--	9	$0_2^{(1)}$

Figure 14. - Representation of machine D_1 as a generalized $S^{(1)}$-image of machine D_r.

Theorem 4:

Implied by a generalized $S^{(k)}$-image M_k of a given sequential machine M_r is a family of state split versions each of which is an $S^{(k)}$-image of some state split version M of M_r. Reciprocally, implied by the $S^{(k)}$-image of any state split version M of M_r is a generalized $S^{(k)}$-image M_k of M_r.

Proof:

For the first half of the theorem, it can be assumed that there is given the sequential machine M_k which is a generalized $S^{(k)}$-image of a given machine M_r. Also, according to part (1) of definition 35, corresponding to the sets $S^{(k)}$ and $C^{(k)}$ of machine M_k are nondisjoint partitions θ'_k and $C^{(k)}$. Each state $S_j^{(r)}$ whose multiplicity $m_{k;S_j^{(r)}}$ is greater than zero for either θ'_k or $C^{(k)}$ can be split as follows:

(1) If the multiplicity of $S_j^{(r)}$ in θ'_k is greater than that of $S_j^{(r)}$ in $C^{(k)}$ and is $m_{k;S_j^{(r)}} = n - 1 > 0$, replace $S_j^{(r)}$ in each block of θ'_k that it appears by one of the states $S_j^{(r)1}, S_j^{(r)2}, \ldots, S_j^{(r)n}$. The replacement can be made such that $S_j^{(r)1}$ is assigned to the first block of the arbitrarily ordered blocks of θ'_k containing $S_j^{(r)}$, $S_j^{(r)2}$ to the second such block, etc. Corresponding to every multiplicity-dependent block intersection $c^{(k)} ⓜ s^{(k)}$ originally containing $S_j^{(r)}$, the state $S_j^{(r)}$ for the sake of compatibility must be replaced in $C^{(k)}$ by the state $S_j^{(r)l}$ just assigned to the associated block of θ'_k.

(2) If the multiplicity of $S_j^{(r)}$ in $C^{(k)}$ is greater than that of $S_j^{(r)}$ in θ'_k and is $m_{k;S_j^{(r)}} = n - 1 > 0$, replace $S_j^{(r)}$ in each block of $C^{(k)}$ that it appears by one of the states $S_j^{(r)1}, S_j^{(r)2}, \ldots, S_j^{(r)n}$. The replacement can be made such that $S_j^{(r)1}$ is assigned to the first block of the arbitrarily ordered blocks of $C^{(k)}$ containing $S_j^{(r)}$, $S_j^{(r)2}$ to the second such block, etc. If the state $S_j^{(r)}$ is contained in the multiplicity-dependent block intersection $c^{(k)} ⓜ s^{(k)}$, for the sake of compatibility replace $S_j^{(r)}$ in the block $S^{(k)}$ of θ'_k by the state $S_j^{(r)l}$ just assigned to the block $c^{(k)}$ of $C^{(k)}$. Some blocks of θ'_k will now contain two or more replacements for the original state $S_j^{(r)}$ under this procedure.

(3) Once the present states of M_r have been split as described, the next states of M_r must be split. Each next state of M_k, before state splitting, corresponded to a block of θ'_k. Let the next states of M_r be split in such a way that each next state of M_k corresponds to the associated block resulting from the procedure of (1) and (2). Each next state which is left unsplit after this process can be arbitrarily replaced by any member of its replacement set of states.

In the state splitting process (1) to (3) just given, the parts (1) and (2) of definition 27 are obviously satisfied. In accordance with step (3) of the state splitting process, to every state $S_j^{(r)} \in S^{(r)}$ in some $s^{(k)} \in S^{(k)}$ implied by a state $s^{(r)} \in c^{(k)} ⋒ s^{(k)}$ for an input $i^{(r)}$ there corresponds a state $S_j^{(r)l}$, where l is an integer such that $1 \leq l \leq n$. Thus, part (3) of definition 27 is also satisfied. Moreover, similar reasoning shows that parts (4), (5), and (6) of definition 27 are satisfied. Therefore, the state splitting process presented herein serves to derive a state split version M of M_r. The derived machine M is a member of a family of state split versions of M_r. The number of members in this family is 2^a, where a is the number of replacements which can be made arbitrarily in step (3) of the above state splitting process. The state splitting has been accomplished in such a way that the relation

$$(c^{(k)} ⋒ s^{(k)})\Delta^{(r)}(i^{(r)}) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$$

where $c^{(k)}$ and $s^{(k)}$ are elements of the given sets, corresponds to

$$(c^{(k)} \cap s^{(k)})\Delta(i) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$$

where $c^{(k)}$ and $s^{(k)}$ are blocks of the newly derived partitions $C^{(k)}$ and π_k', respectively, Hence, the partition pair $(C^{(k)} \cdot \pi_k', \pi_k')$ on S of M implies by Theorem 1 the existence of an $S^{(k)}$-image of M.

For the second half of the theorem, it can be assumed that machine M and an $S^{(k)}$-image of M are given. By Theorem 1 there exists the partition pair $(C^{(k)} \cdot \pi_k', \pi_k')$ on the states of S of M. Machine M_r can be derived directly from M as a reduction of M. The states of $S^{(r)}$ of M_r are the blocks of an output-consistent partition pair (π_r', π_r') on the states of S of M. Canonical relations $\pi_k'^*$, $C^{(k)*}$, $(\pi_r'^*)^{-1}$ exist such that $(\pi_r'^*)^{-1}\pi_k'^*$ maps the partition π_k' into a nondisjoint partition θ_k'; likewise, $(\pi_r'^*)^{-1}C^{(k)*}$ maps the partition $C^{(k)}$ into a nondisjoint partition $C^{(k)}$ on the states of $S^{(r)}$ of M_r. Because $i^{(r)} = i$, the relation

$$(c^{(k)} \cap s^{(k)})\Delta(i) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$$

under the aforementioned mappings goes into

$$(c^{(k)} ⋒ s^{(k)})\Delta^{(r)}(i^{(r)}) \subseteq s^{(k)}\Delta^{(k)}(i^{(k)})$$

where the latter elements $c^{(k)}$ and $s^{(k)}$ are those belonging to new sets $C^{(k)}$ and $S^{(k)}$ corresponding to the derived nondisjoint partitions $C^{(k)}$ and θ_k', respectively. Therefore, in accordance with definition 35 there does exist a machine M_k which is a generalized $S^{(k)}$-image of M_r. This completes the proof of the theorem.

A consideration of machines D_r and D_1 provides a concrete illustration of the first half of the theorem. Corresponding to the sets $S^{(1)}$ and $C^{(1)}$ of D_1, a generalized $S^{(1)}$-image of machine D_r, are the nondisjoint partitions

$$\theta_1' = \{8 = \{1,2,4,5\},\ 9 = \{3,4,6,7\}\} \text{ and } C^{(1)} = \{10 = \{1,2,3,4\},\ 11 = \{4,5,6,7\}\}.$$

The only state whose multiplicity is greater than 0 in either θ_1' or $C^{(1)}$ is state 4. In accordance with step (1) of the proof, state 4 is split such that θ_1' becomes

$$\pi_1' = \{8 = \{1,2,4^1,5\},\ 9 = \{3,4^2,6,7\}\}$$

Since $9 \overset{\frown}{m}\ 10 = \{3,4\}$, state 4 in 10 of $C^{(1)}$ is replaced by 4^2. Likewise, since $8 \overset{\frown}{m}\ 11 = \{4,5\}$, state 4 in 11 of $C^{(1)}$ is replaced by 4^1. Thus, $C^{(1)}$ becomes

$$C^{(1)} = \{10 = \{1,2,3,4^2\},\ 11 = \{4^1,5,6,7\}\}$$

By step (3) of the proof, the next state implied by state 2 when the input $\langle 1,10\rangle$ is applied is 4^1 since

$$(10 \cap 8)\Delta^{(r)}(1) = \{1,2\}\Delta^{(r)}(1) = \{5,4^1\} \subseteq 8$$

Similarly, the next states implied by present states 6 and 4^1 for inputs $\langle 1,11\rangle$ and $\langle 2,11\rangle$, respectively, are 4^1 and 4^2, respectively, since

$$(11 \cap 9)\Delta^{(r)}(1) = \{6,7\}\Delta^{(r)}(1) = \{4^1,5\} \subseteq 8$$

$$(11 \cap 8)\Delta^{(r)}(2) = \{4^1,5\}\Delta^{(r)}(2) = \{4^2,3\} \subseteq 9$$

The next states implied by present states 3 and 4^2 must be the same because

$$(10 \cap 9)\Delta^{(r)}(2) = \{3,4^2\}\Delta^{(r)}(2) = \{4^1\} \text{ or } \{4^2\}$$

Let these next states be arbitrarily chosen to be 4^1. With this choice, state 4 of machine D_r will have been split precisely as it was in the illustration of definition 27. Hence, machine D of figure 13 is an implied state split version of D_r. Furthermore, from the nondisjoint partitions θ_1' and $C^{(1)}$ comes the partition pair

$$(\pi_1 = \{10 \cap 8 = \{1,2\},\ 10 \cap 9 = \{3,4^2\},\ 11 \cap 8 = \{4^1,5\},\ 11 \cap 9 = \{6,7\}\}$$

$$\pi_1' = \{8 = \{1,2,4^1,5\},\ 9 = \{3,4^2,6,7\}\})$$

which implies the existence of an $S^{(1)}$-image of machine D whose flow table representation is like that of machine D_1 of figure 14. The two representations differ only in that the next state implied by state 9 for input $\langle 2, 10\rangle$ is unspecified for machine D_1 but is state 8 for the $S^{(1)}$-image of machine D.

For an illustration of the second half of the theorem, machine D and its $S^{(1)}$-image are considered. An examination of the flow table of machine D shown in figure 13 reveals in addition to (π_1, π_1') the existence of the output-consistent partition pair (π_r', π_r'), where $\pi_r' = \{1,2,3,4 = \{4^1, 4^2\},5,6,7\}$. From (π_r', π_r') the reduction D_r of machine D is directly derivable and is seen to have the representation given in figure 12. Additionally,

$$\begin{aligned}(\pi_r'^*)^{-1}\,\pi_1'^* &= \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4^1\rangle, \langle 4,4^2\rangle, \langle 5,5\rangle, \langle 6,6\rangle, \langle 7,7\rangle\} \\ &\quad \{\langle 1,8\rangle, \langle 2,8\rangle, \langle 4^1,8\rangle, \langle 5,8\rangle, \langle 3,9\rangle, \langle 4^2,9\rangle, \langle 6,9\rangle, \langle 7,9\rangle\} \\ &= \{\langle 1,8\rangle, \langle 2,8\rangle, \langle 4,8\rangle, \langle 5,8\rangle, \langle 3,9\rangle, \langle 4,9\rangle, \langle 6,9\rangle, \langle 7,9\rangle\} \\ &= \theta_1'^*\end{aligned}$$

from which

$$\theta_1' = \{8 = \{1,2,4,5\},\ 9 = \{3,4,6,7\}\}$$

on the states of $S^{(r)}$ of D_r. Similarly,

$$\begin{aligned}(\pi_r'^*)^{-1}C^{(1)*} &= (\pi_r'^*)^{-1}\{\langle 1,10\rangle, \langle 2,10\rangle, \langle 3,10\rangle, \langle 4^2,10\rangle, \langle 4^1,11\rangle, \langle 5,11\rangle, \langle 6,11\rangle, \langle 7,11\rangle\} \\ &= \{\langle 1,10\rangle, \langle 2,10\rangle, \langle 3,10\rangle, \langle 4,10\rangle, \langle 4,11\rangle, \langle 5,11\rangle, \langle 6,11\rangle, \langle 7,11\rangle\}\end{aligned}$$

from which

$$C^{(1)} = \{10 = \{1,2,3,4\},\ 11 = \{4,5,6,7\}\}$$

on the states of $S^{(r)}$ of D_r. The two thusly derived partitions define the nondisjoint partition pair ($\theta_1 = C^{(1)} \dot{m}\theta_1', \theta_1'$) considered previously and shown to imply the existence of machine D_1, a generalized $S^{(1)}$-image of D_r.

Theorem 5:

Given a sequential machine M_r, then there exists an $S^{(k)}$-image M_k of a state split version M of M_r if and only if there exists a nondisjoint partition pair (θ_k, θ_k') on the states of $S^{(r)}$ of M_r such that $\theta_k = C^{(k)} \dot{m}\theta_k'$ and such that when the states are split, the blocks of $C^{(k)}$ and θ_k' become the elements of the carry input set $C^{(k)}$ to M_k and of the set of states $S^{(k)}$ of M_k, respectively.

Proof:

The proof follows immediately from Theorems 1 and 4.

Theorem 5 shows that the nondisjoint partition pairs of a sequential machine convey vital information about its decomposition structure. Such information is of the type sought for as a basis for the development of a systematic state splitting determination procedure. Included in the proof of Theorem 4 is a model of such a procedure.

The fact that Theorem 5 is the state split generalization of Theorem 1 furnishes some promise that the fundamental theorem, Theorem 3, also has a state split generalization. The following embodies the fulfillment of this promise:

Theorem 6:

The state behavior of a sequential machine M_r can be realized by a concurrently operating interconnection of j machines $M_1, M_2, \ldots, M_j$ if and only if there exist nondisjoint partition pairs $(\theta_1, \theta_1'), (\theta_2, \theta_2'), \ldots, (\theta_j, \theta_j')$ on the states of $S^{(r)}$ of M_r such that $\theta_1'\dot{m}\theta_2'\dot{m} \ldots \dot{m}\theta_j' = 0$, $\theta_1 = C^{(1)}\dot{m}\theta_1'$, $\theta_2 = C^{(1)} \dot{m}\theta_2'$, $\ldots$, $\theta_j = C^{(j)}\dot{m}\theta_j'$, and furthermore the machines $M_1, M_2, \ldots, M_j$ are $S^{(1)}$-, $S^{(2)}$-, $\ldots$, $S^{(j)}$-images, respectively, of the same split version M of M_r.

Proof:

With the use of definitions 31 and 34, the proof follows directly from Theorems 3, 4, and 5.

The next example shows the significance of the requirement in the theorem that the machines $M_1, M_2, \ldots, M_j$ are $S^{(1)}$-, $S^{(2)}$-, $\ldots$, $S^{(j)}$-images, respectively, of the same state split version M of M_r. Previously, it was shown that an $S^{(1)}$-image of a state split version of machine D_r existed corresponding to the nondisjoint partition pair $(C^{(1)}\dot{m}\theta_1', \theta_1')$. With the aid of Theorem 5, it is found that additionally there exist $S^{(2)}$- and $S^{(3)}$-images of state split versions of D_r corresponding to $(C^{(2)}\dot{m}\theta_2', \theta_2')$ and $(C^{(3)}\dot{m}\theta_3', \theta_3')$, respectively, where

$$\theta_2' = C^{(1)}, \ C^{(2)} = \{12 = \{1,3,4,6\},\ 13 = \{2,4,5,7\}\}$$

$$\theta_2 = \{12 \textcircled{m} 10 = \{1,3\},\ 12 \textcircled{m} 11 = \{4,6\},\ 13 \textcircled{m} 10 = \{2,4\},\ 13 \textcircled{m} 11 = \{5,7\}\}$$

$$\theta_3' = C^{(2)}, \ C^{(3)} = \theta_1'$$

$$\theta_3 = \{8 \textcircled{m} 12 = \{1,4\},\ 8 \textcircled{m} 13 = \{2,5\},\ 9 \textcircled{m} 12 = \{3,6\},\ 9 \textcircled{m} 13 = \{4,7\}\}$$

The nondisjoint partitions θ_1', θ_2', and θ_3' are such that

$$\theta_1'\dot{m}\theta_2'\dot{m}\theta_3' = 0$$

Nevertheless, there exists no state split version D of machine D_r whose state behavior is realized by a concurrently operating interconnection of the aforementioned $S^{(1)}$-, $S^{(2)}$-, and $S^{(3)}$-images. The reason for this is made evident by deriving from $(C^{(2)}\dot{m}\theta_2', \theta_2')$ a state split version of machine D_r. The only such state split version, which can be derived so that the transformation of the nondisjoint partition $\theta_2' = C^{(1)}$ to the partition π_2' is compatible with transforming $C^{(1)}$ on $S^{(r)}$ of D_r to $C^{(1)}$ on S of D, is the machine represented in figure 15. This machine is not the same state split

	1	2	1	2
1	5	2	0_1	0_1
2	4^1	1	0_1	0_2
3	6	4^2	0_1	0_1
4^1	7	4^1	0_2	0_1
4^2	7	4^2	0_2	0_1
5	6	3	0_2	0_2
6	4^1	6	0_2	0_1
7	5	3	0_1	0_2

Figure 15. - State split version of D_r derived from (θ_2, θ_2').

version of D_r derived previously and shown in figure 13. Moreover it is not the same as any member of the family of state split versions derivable from $(C^{(1)}\dot{m}\theta_1', \theta_1')$. Therefore Theorem 6 must indeed include a compatibility condition requiring in essence that all nondisjoint partition pairs concerned imply a common state split version of the given machine.

It is suggested that the interested reader illustrate to himself the further workings of Theorem 6 by deriving a concurrently operating interconnection of three machines realizing the state behavior of D_r as follows: Replace the nondisjoint partition $C^{(2)}$ already given in the example by $C^{(2)} = \theta_3$ such that $C^{(2)}\dot{m}\theta_2' = 0$. Then by using Theorem 4 proceed to verify that now the three nondisjoint partition pairs do imply for machine D_r a common state split version D, shown in figure 13. During the verification process, partition pairs (π_1, π_1'), (π_2, π_2'), and (π_3, π_3') on the states of S of D such that $\pi_1' \cdot \pi_2' \cdot \pi_3' = 0$ will have been derived. As a final step, with the derived partition pairs apply Theorem 3, as illustrated in previous examples, to construct the machines D_1, D_2, and D_3 of the required interconnection.

Theorem 6 is of twofold importance. First, it is a fundamental statement of the generalized decomposition theory; as such Theorem 6 also represents the attainment of the primary objective of this report. Second, it provides a basis for an adequate state splitting determination procedure insofar as the design of machines from their decompositional properties is concerned.

CONCLUDING REMARKS

The basic result of the research reported herein is the development of a generalized decomposition theory of finite sequential machines.

It is hoped that the newly developed theory will lay the foundation for the successful completion of many other research projects. Such projects might, for instance, involve the following:

1. The formulation of algorithms for the derivation of the "best" decompositions of sequential machines under varying criteria
2. The founding of similar algorithms for determining the best decomposition of a combination of two machines given one machine already realized (the realized machine might be a computer and the other, an addition to it)
3. The establishment of criteria under which the specification of sequential machines can be changed to improve their decomposition structure
4. The gaining of a more practical understanding of iterative arrays of logical circuits (ref. 11)

5. The development of methods for the synthesis of sequential machines with given classes of modular building blocks (refs. 12 and 13)

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REFERENCES

1. Pedelty, M. J.: A Review of the Field of Artificial Intelligence and Its Possible Applications to NASA Objectives. American University (NASA CR-62346), Feb. 1965.
2. Hartmanis, J.: On the State Assignment Problem for Sequential Machines. I. IRE Trans. Electronic Computers, vol. EC-10, no. 2, June 1961, pp. 157-165.
3. Stearns, R. E.; and Hartmanis, J.: On the State Assignment Problem for Sequential Machines. II. IRE Trans. Electronic Computers, vol. EC-10, no. 4, Dec. 1961, pp. 593-603.
4. Hartmanis, J.: Loop-Free Structure of Sequential Machines. Information and Control, vol. 5, no. 1, Mar. 1962, pp. 25-43.
5. Hartmanis, J.; and Stearns, R. E.: Pair Algebra and Its Application to Automata Theory. Information and Control, vol. 7, no. 4, Dec. 1964, pp. 485-507.
6. Ginsburg, Seymour: An Introduction to Mathematical Machine Theory. Addison-Wesley, Publ. Co., 1962.
7. Yoeli, Michael: Generalized Cascade Decompositons of Automata. J. Assoc. Computing Mach., vol. 12, no. 3, July 1965, pp. 411-422.
8. Moore, E. F.: Gedanken-Experiments on Sequential Machines. Automata Studies. Princeton University Press, 1956.
9. Hartmanis, J.; and Stearns, R. E.: Some Dangers in State Reduction of Sequential Machines. Information and Control, vol. 5, no. 3, Sept. 1962, pp. 252-260.
10. Kohavi, Zvi: Secondary State Assignment for Sequential Machines. IEEE Trans. Electronic Computers, vol. EC-13, no. 3, June 1964, pp. 193-203.
11. Hennie, Frederick C.: Iterative Arrays of Logical Circuits. John Wiley and Sons, Inc., 1961.

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