REPRESENTATION THEOREMS AND WEAK COMPACTNESS

by

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Let S be a compact Hausdorff space, and C(S,E) the Banach space of continuous E-valued functions defined on S, where E is a Banach space. The main problem of this paper is to establish an integral representation for the bounded linear transformations of C(S,E) into F, F also a Banach space. In the general situation, Foiaş and Singer have developed a Gowurin integral representation for these transformations through a Borel measure μ having values in L(E,F**). Our paper shows that in case S is a compact metric space and F is weakly complete, μ takes its values in L(E,F); and the representation is strong.

As an application of this result, we answer the question as to the weak compactness of bounded linear transformations of the space C(S,E) into F, where S is a compact Hausdorff space and E, F are Banach spaces, posed by Batt in 1960, with our Theorem: If E is reflexive and F is weakly complete, then every bounded linear transformation $T:C(S,E) \to F$ is weakly compact.

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REPRESENTATION THEOREMS AND WEAK COMPACTNESS

1. Introduction. The purpose of this paper is to study bounded linear transformations of the space C(S,E) into F where S is a compact metric space, and E, F are Banach spaces. One aim of our investigation will be a generalization of the famous theorem due to F. Riesz, according to which every bounded linear functional on the space C[0,1] of continuous, real-valued functions defined on the unit interval [0,1] can be represented by a Stieltjes integral. I. Gelfand [8], in 1938, first approached the problem of representing such transformations by letting S be a closed and bounded interval, and taking E to be the real field. He found that these transformations have a Gowurin integral representation. The result of Gelfand has been successfully generalized, notably by R. G. Bartle, N. Dunford and J. Schwartz, by J. Batt and H. König, by N. Dinculeanu, and by C. Foiaş and I. Singer. Shortly after the paper of Batt and König [3] appeared in 1959, Foiaş and Singer [7] announced the result that a bounded linear transformation $T:C(S,E)\to F$, S a comapct Hausdorff space, could be represented by the Gowurin integral; the representation is through a Borel measure μ having values in $L(E,F^{**})$. The sequel shows that by making specific assumptions on S and F, valuable information

can be obtained from the measure μ. We found that if S is a compact metric space and F is weakly complete, the measure μ takes its values in L(E,F). Moreover, the representation of Foiaş and Singer does not give the values of T but rather their images under functionals $y^* \in F^*$; in contrast, our representation is strong.

In section 3, we fix the necessary groundwork by observing that the adjoint space $C(S,E)^*$ is isometrically isomorphic to a space of regular E^*-valued Borel measures. This fact was first noticed by Singer [10]; however, our approach is constructive and lays bare the origin of the measures.

We show that a linear bounded transformation of C(S,E) can be assigned a unique operator-valued Borel measure, which holds much information as to the weak compactness of the transformation. In 1954, Bartle, Dunford, and Schwartz [1] proved that every bounded linear transformation $T:C(S,E)\rightarrow F$ is weakly compact provided S is a compact Hausdorff space, E is the field of real numbers, and F is weakly complete. A similar result was established by Batt [2]; he was interested in the case where S is a bounded and closed interval, E is a reflexive Banach space, and F is weakly complete. The present paper gives a generalization of both these results with our main contribution,

<u>Theorem</u>: Let S be a compact Hausdorff space. If E is reflexive and F is weakly complete, then every bounded linear transformation $T:C(S,E)\to F$ is weakly compact. We note that the conclusion of this theorem is no longer valid if we drop the reflexivity assumption on E, for Foiaş and Singer [7] have given a counter-example for this relaxed situation.

2. <u>Notations and Preliminary Remarks</u>. Throughout, (S,ρ) will be a compact metric space; mention of the metric ρ usually will be suppressed. A useful notion is the *rim* of a set $A \subset S$ with respect to a positive real constant λ, denoted $R(A,\lambda)$, which is defined to be

$$R(A,\lambda) = \{t \mid 0<\rho(A,t)<\lambda\}.$$

We agree that $R(\emptyset,\lambda) = \emptyset$ for all λ. The *concentric disc* about a set $A \subset S$, again with respect to a positive real number λ, is denoted $D(A,\lambda)$; and this set is defined by

$$D(A,\lambda) = \overline{A} \cup R(A,\lambda)$$

where $\overline{A}$ is the closure of A.

By a *ring* of subsets of S, we will mean a collection $\mathcal{R}$ of subsets of S which is closed under complementation and finite union, and to which S belongs. It is clear that a given collection of subsets of S, say $\mathcal{T}$, gives rise to a smallest ring $\mathcal{R} \supset \mathcal{T}$; $\mathcal{R}$ is said to be the ring generated by T. We will denote by Σ the ring generated by the open sets in S.

We define a *partition* of S to be a finite collection $\{A_j\}_{j=1}^{k}$ of non-empty, disjoint sets <u>in Σ</u> such that $S = \bigcup_{j=1}^{k} A_j$. To each partition π of S, we assign the positive real number $|\pi|$ by the rule

$$|\pi| = \max_{A_j \in \pi} \sup \{\rho(x,y) \mid x,y \in A_j\}.$$

The supremums exist because every compact metric space is bounded.

We will restrict our attention to real and complex Banach spaces, denoting the scalar field by K. If E is a Banach space, the vector space C(S,E) of all continuous E-valued functions defined on S becomes a Banach space under the norm

$$||f|| = \max_{t \in S} ||f(t)||, \quad f \in C(S,E).$$

If F is another Banach space, the set of all bounded linear transformations T of E into F will be denoted by L(E,F); this is a Banach space under the norm

$$||T|| = \sup_{||x|| \le 1} ||Tx||.$$

In particular, L(E,K) is the adjoint space of E; and we will use the notation E* for this space. If $\gamma \in E^*$ and $x \in E$, we emphasize that the value of γ at x is a scalar by the notation $\langle x,\gamma\rangle$.

Suppose μ is a set function defined on Σ with values in $L(E,F)$. The *semi-variation* of μ on a set $A \in \Sigma$ is a real number or $+\infty$ defined by

$$\text{Semi-var}\,\mu(A) \quad = \quad \sup \left\| \sum_{j=1}^{k} \mu(C_j)x_j \right\|$$

where the supremum is taken over all finite sequences of disjoint sets $\{C_j\}_{j=1}^{k} \subset \Sigma$ with $C_j \subset A$ $(j=1,\ldots,k)$ and all finite sequences of vectors $\{x_j\}_{j=1}^{k}$ in the closed unit sphere of E. Let $\{x_j\}_{j=1}^{k}$ be a collection of arbitrary vectors in E, set $\alpha = \max\limits_{j} \|x_j\|$, then $\left\|\frac{x_j}{\alpha}\right\| \leq 1$ for all j. Hence,

$$\left\| \sum_{j=1}^{k} \mu(C_j)\, \frac{x_j}{\alpha} \right\| \leq \text{Semi-var}\ \mu(A),$$

and we have that

$$\left\| \sum_{j=1}^{k} \mu(C_j)\ x_j \right\| \leq \max \|x_j\| \cdot \text{Semi-var}\ \mu(A), \tag{1}$$

where $\{C_j\}_{j=1}^{k}$ is a collection of disjoint sets in Σ, $C_j \subset A$ $(j=1,\ldots,k)$.

Consider a function $f:S\to E$ and let μ be a set function defined on Σ with values in $L(E,F)$. We wish to define an integral of f with respect to μ. Let $\pi = \{A_j\}_{j=1}^{k}$ be a partition of S and let $\tau:\tau_j \in A_j$ $(j=1,\ldots,k)$ be a set of intermediate points. We form an approximating sum

$$S_{\pi}(\tau) = \sum_{j=1}^{k} \mu(A_j)f(\tau_j).$$

If $\lim_{|\pi|\to 0} S_{\pi}(\tau)$ exists independently of the choice of τ, in the norm of F, we will say that the limit is the integral of f with respect to μ, and write $\int_S d\mu(t)f(t)$. In other words, this integral exists if and only if for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\|S_{\pi'}(\tau') - S_{\pi''}(\tau'')\| < \varepsilon$$

whenever π' and π'' are partitions of S with $|\pi'|,|\pi''| < \delta$.

We will prove the following sufficient condition for the existence of the integral.

Theorem 1: If $f \in C(S,E)$ and μ is additive on Σ and of bounded semi-variation on S, then $\int_S d\mu(t)f(t)$ exists.

Proof: We introduce the modulus of continuity $\sigma(\delta)$ of a function $f:S\to E$,

$$\sigma(\delta) = \sup_{\rho(t,s)<\delta} \|f(t) - f(s)\|, \quad \delta>0.$$

If f is continuous on S, f is uniformly continuous, and $\sigma(\delta)\to 0$ with δ. Let $\pi' = \{A_j'\}_{j=1}^{k'}$ and $\pi'' = \{A_j''\}_{j=1}^{k''}$ be partitions of S, and let $\tau':\tau_i' \in A_i'$ $(i=1,\ldots,k')$ and $\tau'':\tau_j'' \in A_j''$ $(j=1,\ldots,k'')$ be sets of representative points.

Let $A_{ij} = A_i' \cap A_j''$ $(i=1,\ldots,k';\ j=1,\ldots,k'')$. Then,

$$S_{\pi'}(\tau') - S_{\pi''}(\tau'') = \sum_{i=1}^{k'} \mu\Big(\bigcup_{j=1}^{k''} A_{ij}\Big) f(\tau_i') - \sum_{j=1}^{k''} \mu\Big(\bigcup_{i=1}^{k'} A_{ij}\Big) f(\tau_j'')$$

$$= \sum_{i=1}^{k'} \sum_{j=1}^{k''} \mu(A_{ij}) f(\tau_i') - \sum_{j=1}^{k''} \sum_{i=1}^{k'} \mu(A_{ij}) f(\tau_j'')$$

by the additivity of μ. Thus,

$$S_{\pi'}(\tau') - S_{\pi''}(\tau'') = \sum_{i=1}^{k'} \sum_{j=1}^{k''} \mu(A_{ij})[f(\tau_i') - f(\tau_j'')].$$

The terms of this sum are zero unless $A_i' \cap A_j''$ is non-empty in which case $\max_{i,j} \|f(\tau_i') - f(\tau_j'')\| \leq \sigma(|\tau'| + |\tau''|)$. By (1),

$$\|S_{\pi'}(\tau') - S_{\pi''}(\tau'')\| \leq \sigma(|\pi'| + |\pi''|)\cdot\text{Semi-var}\mu(S). \blacksquare$$

From the approximating sum for the integral and from (1), we see that

$$\|S_\pi(\tau)\| \leq \|f\|\cdot\text{Semi-var}\mu(S).$$

Letting $|\pi| \to 0$, we obtain the useful inequality,

$$\left\|\int_S d\mu(t)f(t)\right\| \leq \|f\|\cdot\text{Semi-var}\mu(S). \tag{2}$$

Let us recall that the *variation* of a set function μ with values in $L(E,F)$, denoted $v(\mu,\cdot)$, is an extended real valued set function defined by

$$v(\mu,A) = \sup \sum_{j=1}^{k} \|\mu(C_j)\|, \ A\varepsilon\Sigma$$

where the supremum is taken over all finite sequences of disjoint sets $\{C_j\}_{j=1}^{k} \subset \Sigma$ with $C_j \subset A$ $(j=1,\ldots,k)$ and $\|\cdot\|$ is the norm of $L(E,F)$. The concepts of variation and semi-variation coincide in case F is finite dimensional [3, p.277]. If the set function μ has values in a Banach space, say F, then μ can be intrepreted as a function with values in $L(K,F)$ and one has for the variation

$$v(\mu,A) = \sup \sum_{j=1}^{k} \|\mu(C_j)\|, \ A\varepsilon\Sigma,$$

where $\|\cdot\|$ is the norm of F.

We will need the concept of regularity of vector-valued set function. A set function μ is said to be *regular* if for every set A in the domain of μ and for every $\varepsilon>0$, there exist sets in the domain of μ: H whose closure is contained in A, and G whose interior contains A such that $v(\mu,G-H)<\varepsilon$.

To complete our list of preliminary definitions, we include, if f is a function $f:S\to E$, we will mean by the *support* of f that subset of S on which f is different from zero. Symbolically,

$$\text{Supp } f = \{t \mid f(t)\neq 0\}.$$

3. <u>Functionals on C(S,E)</u>. Our development of a representation for bounded linear functionals on $C(S,E)$ rests essentially on Lemma 1 and Lemma 3 below.

Lemma 1: Let $\{G_n\}$ be a sequence of open sets of S with $G_n \supset G_{n+1}$ $(n=1,2,\ldots)$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Suppose that $\{f_n\}$ is a bounded sequence in $C(S,K)$ with Supp $f_n \subset G_n$ for all n. Then, for each $x \varepsilon E$ and each $\gamma \varepsilon C(S,E)^*$,

$$\lim_{n\to\infty} \langle xf_n, \gamma\rangle = 0.$$

Proof: Let $C_x(S,K)$ denote the subspace of $C(S,E)$ consisting of all functions of the form xf where $x \varepsilon E$ and $f \varepsilon C(S,K)$. Consider $\gamma \varepsilon C(S,E)^*$ and let $\hat{\gamma}$ be the restriction $\hat{\gamma} = \gamma | C_x(S,K)$. $\hat{\gamma}$ gives rise to a linear function η_x on $C(S,K)$ by the definition

$$\langle xf, \hat{\gamma}\rangle = \langle f, \eta_x\rangle.$$

η_x is bounded, for

$$|\langle f, \eta_x\rangle| = |\langle xf, \hat{\gamma}\rangle| \leq \|\gamma\| \, \|x\| \, \|f\|;$$

and hence, $\|\eta_x\| \leq \|\gamma\| \, \|x\|$. According to the hypothesis, the sequence $\{f_n\}$ is bounded and point-wise convergent to zero, so that

$$\lim_{n\to\infty} \langle f, \eta_x\rangle = 0$$

[6, p. 265]. The proposition follows. ∎

Lemma 2: Let A be a subset of S, and let λ and ε be positive real numbers. If for $f \varepsilon C(S,E)$ with Supp $f \subset R(A,\lambda)$ and for $\gamma \varepsilon C(S,E)^*$ it is true that

$$|\langle f, \gamma\rangle| > \varepsilon,$$

then there exists a real number $\zeta : 0<\zeta<\lambda$ and a function $g\in C(S,E)$ with Supp $g \subset \{t \mid \zeta<\rho(A,t)<\zeta\}$ such that

$$|\langle f,\gamma\rangle|>\varepsilon$$

Proof: By the continuity of γ, there exists $\delta>0$ such that $|\langle h,\gamma\rangle|>\varepsilon$ for all $h\in C(S,E)$ with $\|h-f\|\leq\delta$. Since $f(A)=0$ and f is continuous on S, there exists $\zeta>0$ such that if $t\in R(A,2\zeta)$, then $\|f(t)\|<\delta$. Let $H = \overline{D(A,\zeta)}$ and let $C = S-D(A,2\zeta)$. Then, $C\cap H = \emptyset$, define $u:S\to[0,1]$ by

$$u(t) = \frac{\rho(H,t)}{\rho(H,t)+\rho(C,t)}$$

Clearly, u is continuous and we set $g(t)=u(t)f(t), t\in S$. In fact,

$$\|g-f\|=\|f(u-1)\|\leq \sup_{t\in R(A,2\zeta)} \|f(t)\|\leq\delta;$$

so that,

$$|\langle g,\gamma\rangle|>\varepsilon$$

and Supp $g \subset \{t \mid 0<\zeta<\rho(A,t)<\lambda\}$. ∎

<u>Lemma 3</u>: Let A be a subset of S, and let

$$\Lambda(A,\lambda,M) = \{f\in C(S,E) \mid \text{Supp } f\subset R(A,\lambda),\ \|f\|<M\}$$

where M is a positive real constant. If $\gamma\in C(S,E)^*$, then

$$\lim_{\lambda\to 0} \sup_{f\in\Lambda(A,\lambda,M)} |\langle f,\gamma\rangle| = 0.$$

Proof: Were the proposition false, one could find $\varepsilon_o>0$, a sequence $\{\lambda_n\}$ of positive real numbers, $\lambda_n\downarrow 0$, and a sequence $\{f_n\}$ with $f_n \,\varepsilon\Lambda(A,\lambda_n,M)$ for all n, such that

$$|\langle f_n,\gamma\rangle|>\varepsilon_o, \text{ for all n.}$$

Choose $N > \frac{M\|\gamma\|}{\varepsilon_o}$, then

$$\sum_{n=1}^{N}\|f_n\|\|\gamma\|> N\varepsilon_o>\|\gamma\|.$$

But, by Lemma 2, we may assume the f_n have disjoint support; hence,

$$\sum_{n=1}^{N}\|f_n\|\|\gamma\| = \|\sum_{n=1}^{N}\|\|\gamma\|\leq M\|\gamma\|$$

which is a contradiction. ∎

Let H be a closed subset of S, we wish to obtain a sequence of continuous functions converging point-wise to the characteristic function of H, χ_H. In order to make our work concise, we adopt the convention $\rho(\emptyset,t)=\infty$ for all $t\varepsilon S$. Let $\{\lambda_n\}$ be a positive real sequence, $\lambda_n\downarrow 0$. For each n, let C_n be the complement of the concentric disc $D(H,\lambda_n)$. $C_n\cap H = \emptyset$, for each n. Define the functions $f_n^H:S\to[0,1]$ by

$$f_n^H(t) = \frac{\rho(C_n,t)}{\rho(C_n,t)+\rho(H,t)};\ n=1,2,\ldots. \qquad (\chi)$$

We note that if $H\neq\emptyset$, $f_n^H(H)=1$, for all n; and if $C_n\neq\emptyset$, for some n, then $f_n^H(C_n)=0$; moreover, for every closed $H\subset S$, f_n^H is a continuous function for each n. It is clear that

$$\lim_{n\to\infty} f_n^H(t) = \chi_H(t),\ t\varepsilon S.$$

Henceforth, we will refer to sequences of this type as being of *type* χ. Let $x\varepsilon E$, and form the sequence $\{xf_n^H\}$ where $\{f_n^H\}$ is type χ. Let $\gamma\varepsilon C(S,E)^*$, we assert that $\{<xf_n^H,\gamma>\}$ is a Cauchy sequence in K. Indeed, let m and n be natural numbers with say $m>n$. Then, the support of $f_m^H-f_n^H$ is contained in the open set $R(H,\lambda_n)$. We see that

$$\lim_{n\to\infty} |<xf_m^H,\gamma>-<xf_n^H,\gamma>|=\lim_{n\to\infty} |<x(f_m^H-f_n^H),\gamma>|=0$$

by an application of Lemma 1.

Now, with the bounded linear functional γ on $C(S,E)$ we associate the E^*-valued set function μ on Σ which is defined in the following three steps:

i) if H is a closed subset of S,

$$<x,\mu(H)> = \lim_{n\to\infty} <xf_n^H,\gamma>, x\varepsilon E,$$

where $\{f_n^H\}$ is of type χ. Let $H\neq\emptyset$ be a closed, proper subset of S. If $\{\tilde{f}_n\}$ is a bounded sequence in $C(S,K)$ with $\tilde{f}_n(H)=1$ and $\tilde{f}_n(C_n)=0$ for all n, where C_n is the non-empty complement of $D(H,\tilde{\lambda}_n)$, $\{\tilde{\lambda}_n\}$ a real sequence, $\tilde{\lambda}_n\downarrow 0$, we will show that

$$\lim_{n\to\infty} <xf_n^H,\gamma> = \lim_{n\to\infty} <x\tilde{f}_n,\gamma>, \ x\varepsilon E$$

The function $\tilde{f}_n-f_n^H$ has support in the rim $R(H,\omega_n)$, $\omega_n=\max(\tilde{\lambda}_n,\lambda_n)$; it follows from Lemma 1 that

$$\lim_{n\to\infty} <x(\tilde{f}_n-f_n^H),\gamma> = 0, \ x\varepsilon E.$$

According to the convention $\mu(\emptyset)=0$, for the functions $f_n^\emptyset$ are identically zero.

ii) If G is an open subset of S, we define

$$\mu(G) = \mu(S)-\mu(S-G)$$

It may happen that G is both open and closed, in which case the functions χ_G and χ_{S-G} are both continuous. To see this, one needs to consider four cases, viz., open sets of the closed unit interval which contain: a) 1, b) 0, c) both 1 and 0, and d) neither 1 nor 0. The inverse images of all of these sets under χ_G or χ_{S-G} are open in S. Thus, we may write

$$<x,\mu(G)+\mu(S-G)> = <x(\chi_G+\chi_{S-G}),\gamma> = <x,\mu(S)>$$

and no inconsistencies arise.

iii) If A is an arbitrary set in Σ, A can be written as the disjoint union of an open set G and a closed set H. Of course, A is the union of open and closed sets, say

$$A = \bigcup_{i=1}^{k} G_i \cup \bigcup_{j=1}^{\ell} H_j$$

where each G_i is open and each H_j is closed. Set $G = \bigcup_{i=1}^{k} G_i$ and let $H = (S-G) \cap \bigcup_{j=1}^{\ell} H_j$. Clearly, G is open and H is closed, these sets are disjoint, and

$$A = G \cup H.$$

We define μ at A by the equation

$$\mu(A) = \mu(G)+\mu(H). \qquad (3)$$

We have to show that the definition (3) does not depend on the particular representation of A as the disjoint union of an open set and a closed set. To this end, the following identity will be useful. If H_1 and H_2 are arbitrary closed subsets of S, then

$$\mu(H_1)+\mu(H_2) = \mu(H_1 \cup H_2)+\mu(H_1 \cap H_2). \tag{4}$$

Consider the sequence

$$\{xf_n^{H_1}+xf_n^{H_2}-xf_n^{H_1 \cup H_2}-xf_n^{H_1 \cap H_2}\} = \{xg_n\}$$

where $x \varepsilon E$, and $\{f_n^{H_1}\}$, $\{f_n^{H_2}\}$, $\{f_n^{H_1 \cup H_2}\}$ and $\{f_n^{H_1 \cap H_2}\}$ are all sequences of type χ with respect to the positive real sequence $\{\lambda_n\}$, $\lambda_n \downarrow 0$. $\{g_n\}$ is a bounded sequence in C(S,K) and the support of g_n is contained in the open set $R(H_1,\lambda_n) \cup R(H_2,\lambda_n)$ (n=1,2,...). Thus, the hypothesis of Lemma 1 is satisfied, and we have that $\langle xg_n,\gamma\rangle$ tends to zero as $n \to \infty$. By this observation,

$$\lim_{n\to\infty} \langle xf_n^{H_1},\gamma\rangle + \lim_{n\to\infty} \langle xf_n^{H_2},\gamma\rangle =$$
$$\lim_{n\to\infty} \langle xf_n^{H_1 \cup H_2},\gamma\rangle + \lim_{n\to\infty} \langle xf_n^{H_1 \cap H_2},\gamma\rangle \tag{5}$$

from which (4) follows.

Suppose that for an $A\varepsilon\Sigma$, it is true that

$$A = G_1 \cup H_1 = G_2 \cup H_2$$

where G_i is open, H_i is closed and $G_i \cap H_i = \emptyset$ (i=1,2).

We claim that

$$H_1 \cup (S-G_2) = H_2 \cup (S-G_1)$$

and

$$H_1 \cap (S-G_2) = H_2 \cap (S-G_1).$$

Indeed, if $x \varepsilon H_1 \cup (S-G_2)$ we distinguish two cases:
1) $x \varepsilon H_1$, which means $x \varepsilon S-G_1$ because H_1 and G_1 are disjoint, and thus $x \varepsilon H_2 \cup (S-G_1)$; 2) if $x \varepsilon S-G_2$ two situations arise: first, $x \varepsilon H_2 \subset H_2 \cup (S-G_1)$; and secondly, $x \varepsilon (S-G_2) \cap (S-H_2) = (S-G_1) \cap (S-H_1)$ which means $x \varepsilon S-G_1$ and consequently $x \varepsilon H_2 \cup (S-G_1)$. Hence,

$$H_1 \cup (S-G_2) \subset H_2 \cup (S-G_1).$$

The reverse containment follows from a similar argument. Analoguously, one can prove that

$$G_1 \cup (S-H_2) = G_2 \cup (S-H_1),$$

which implies that

$$H_1 \cap (S-G_2) = H_2 \cap (S-G_1).$$

Now, we wish to prove that

$$\mu(G_1)+\mu(H_1) = \mu(G_2)+\mu(H_2).$$

By the preceding discussion,

$$\mu[H_1 \cup (S-G_2)]+\mu[H_1 \cap (S-G_2)] = \mu[H_2 \cup (S-G_1)]+\mu[H_2 \cap (S-G_1)]$$

and applying (4),

$$\mu(H_1)+\mu(S-G_2) = \mu(H_2)+\mu(S-G_1).$$

Thus, we have that

$$\mu(S)-\mu(S-G_1)+\mu(H_1) = \mu(S)-\mu(S-G_2)+\mu(H_2)$$

or

$$\mu(G_1)+\mu(H_1) = \mu(G_2)+\mu(H_2).$$

We conclude that $\mu:\Sigma\to E^*$ is well defined.

To show that μ is additive on Σ. Consider disjoint sets A_1, A_2 in Σ. Then, $A_1=G_1\cup H_1$ and $A_2=G_2\cup H_2$ where G_i is open, H_i is closed and $G_i\cap H_i=\emptyset$ $(i=1,2)$; and $G_1\cap G_2=H_1\cap H_2=\emptyset$ because A_1 and A_2 are disjoint. A representation for $A_1\cup A_2$ as the disjoint union of an open set and a closed set is

$$A_1\cup A_2 = (G_1\cup G_2) \quad (H_1\cup H_2).$$

Therefore,

$$\begin{aligned}\mu(A_1\cup A_2) &= \mu(G_1\cup G_2)+\mu(H_1\cup H_2)\\ &= \mu(S)-\mu[(S-G_1)\cap(S-G_2)]+\mu(H_1\cup H_2)\end{aligned}$$

We may apply (4),

$$\begin{aligned}\mu(A_1\cup A_2) &= \mu(S)-\mu(S-G_1)-\mu(S-G_2)+\mu[(S-G_1)\cup(S-G_2)]\\ &\qquad +\mu(H_1)+\mu(H_2)\\ &= \mu(G_1)-\mu(S-G_2)+\mu[S-(G_1\cap G_2)]+\mu(H_1)+\mu(H_2)\\ &= \mu(G_1)+\mu(H_1)-\mu(S-G_2)+\mu(S)+\mu(H_2)\end{aligned}$$

$$= \mu(G_1)+\mu(H_1)+\mu(G_2)+\mu(H_2)$$
$$= \mu(A_1)+\mu(A_2).$$

Using the following lemma, we can show that μ is of bounded variation on S.

Lemma 4: If $A\varepsilon\Sigma$ and $\varepsilon>0$, there exists a closed set $H\subset A$ such that $\|\mu(H)-\mu(A)\|<\varepsilon$.

Proof: It is sufficient to prove the proposition for A an open subset of S. Let $\varepsilon>0$ be given. By lemma 3, we are able to choose δ so small that $|<f,\gamma>|<\varepsilon$, for all $f\varepsilon\Lambda(S-A,\delta,1)$. Set $H=S-D(S-A,\delta)$. Clearly, $H\subset A$ and H is closed. We select a positive real sequence $\{\lambda_n\}$ $\lambda_n\downarrow 0$, with $\lambda_1<\delta$; and, we construct sequences $\{f_n^H\}$ and $\{f_n^{S-A}\}$ of type χ, using the sequence $\{\lambda_n\}$ for both. Then, $f_n^H+f_n^{S-A}-1$ has support in $R(S-A,\delta)$ and $\|xf_n^H+xf_n^{S-A}-x\|<1$, for all n and for all x in the closed unit sphere σ of E; so that

$$|<xf_n^H+xf_n^{S-a}-x,\gamma>|<\varepsilon, \text{ all } n,\ x\varepsilon\sigma.$$

By the additivity of γ,

$$|<xf_n^H,\gamma>+<xf_n^{S-A},\gamma>-<x,\gamma>|<\varepsilon, \text{ all } n,\ x\varepsilon\sigma.$$

Passing to the limit,

$$|<x,\mu(H)>+<x,\mu(S-A)>-<x,\mu(S)>|\leq\varepsilon,\ x\varepsilon\sigma;$$

and hence

$$|<x,\mu(H)>-<x,\mu(A)>|\leq \varepsilon, \text{ for all } x\varepsilon\sigma,$$

Thus,

$$\sup_{x\varepsilon\sigma} |\langle x,\mu(H)-\mu(A)\rangle| \leq \varepsilon,$$

and the lemma is proved. ∎

Our task is to prove that $v(\mu,S)$ is finite. The previous lemma allows us to calculate the variation of μ using only finite sequences of disjoint closed sets in S. Let $\{C_j\}_{j=1}^{k}$ be a finite collection of closed subsets of S. If $\varepsilon>0$, there exist $x_j \varepsilon E$ with $\|x_j\|<1$ such that

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \sum_{j=1}^{k} |\langle x_j,\mu(C_j)\rangle| + \frac{\varepsilon}{2}.$$

There are sequences of type χ, say $\{f_n^{C_j}\}$, with $f_1^{C_j}$ $(j=1,\ldots,k)$ having disjoint support; and we may select N so large that

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \sum_{j=1}^{k} |\langle x_j f_n^{C_j},\gamma\rangle| + \varepsilon, \text{ for all } n\geq N.$$

We choose scalars α_j^n with $|\alpha_j^n| = 1$ $(j=1,\ldots,k;\ n\geq N)$; so that

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \sum_{j=1}^{k} \langle \alpha_j^n x_j f_n^{C_j},\gamma\rangle + \varepsilon,\ n\geq N.$$

Thus,

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \|\gamma\| \ \Big\|\sum_{j=1}^{k} \alpha_j^n x_j f_n^{C_j}\Big\| + \varepsilon,\ n\geq N.$$

Since the functions $f_n^{C_j}$ $(j=1,\ldots,k,\ n$ fixed) have disjoint support,

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \|\gamma\| + \varepsilon;$$

and since $\varepsilon>0$ is arbitrary, we have that

$$\sum_{j=1}^{k} \|\mu(C_j)\| < \|\gamma\|.$$

It follows that

$$v(\mu,S) \leq \|\gamma\|. \tag{6}$$

Let $\ell(S,\Sigma,E)$ be the linear space of all functions h of the form

$$h = x_1\chi_{A_1}+\ldots+x_k\chi_{A_k}$$

where $\{A_j\}_{j=1}^{k}$ is a partition of S and the coefficients $\{x_j\}_{j=1}^{k}$ are elements of E. $\ell(S,\Sigma,E)$ is provided with a norm by defining

$$\|h\| = \max_j \|x_j\|$$

The set function μ induces a linear functional γ' on $\ell(S,\Sigma,E)$ by

$$\langle h,\gamma'\rangle = \langle x_1,\mu(A_1)\rangle+\ldots+\langle x_k,\mu(A_k)\rangle,$$

and obviously, $\|\gamma'\| \leq v(\mu,S)$.

Let $\hat{C}(S,\Sigma,E)$ denote the linear space of all sums f+h where $f\varepsilon C(S,E)$ and $h\varepsilon\ell(S,\Sigma,E)$; we define a norm for this space by

$$\|f+h\| = \sup_{t\varepsilon S} \|f(t)+h(t)\|.$$

γ and γ' give rise to a linear functional $\hat{\gamma}$ on $C(S,\Sigma,E)$

through the rule

$$\langle f+h,\hat{\gamma}\rangle = \langle f,\gamma\rangle+\langle h,\gamma'\rangle.$$

We must show that $\hat{\gamma}$ is well defined. Suppose that $f_1+h_1 = f_2+h_2$, $f_i \varepsilon C(S,E)$ and $h_i \varepsilon \ell(S,\Sigma,E)$ $(i=1,2)$. The equation

$$f_1-f_2 = h_2-h_1$$

implies that the element $h_2-h_1 \varepsilon \ell(S,\Sigma,E)$ is continuous. Let $h_2-h_1 = x_1\chi_{A_1}+\ldots+x_k\chi_{A_k}$ where the x_j $(j=1,\ldots,k)$ are distinct. We assert that h_2-h_1 is continuous only if each A_j is both open and closed. Since there are a finite number of the x_j, for fixed j_o: $1\leq j_o \leq k$, there exists a sphere S_{j_o} about x_{j_o} whose closure contains no x_j, $j \neq j_o$. The inverse image of S_{j_o} under h_2-h_1 is A_{j_o} and hence A_{j_o} is open because h_2-h_1 is continuous. On the other hand, the inverse image of $E-\overline{S}_{j_o}$ is the complement of A_{j_o} and must also be open. We conclude, since A_j $(j=1,\ldots,k)$ is both open and closed, that χ_{A_j} is continuous. Hence, we may write

$$\langle f_1-f_2,\gamma\rangle = \langle h_2-h_1,\gamma\rangle = \langle x_1\chi_{A_k},\gamma\rangle +\ldots+\langle x_k\chi_{A_k},\gamma\rangle$$
$$= \langle x_1,\mu(A_1)\rangle +\ldots+\langle x_k,\mu(A_k)\rangle = \langle h_2-h_1,\gamma'\rangle$$

which means

$$\langle f_1,\gamma\rangle + \langle h_1,\gamma'\rangle = \langle f_2,\gamma\rangle + \langle h_2,\gamma'\rangle$$

or

$$<f_1+h_1,\hat{\gamma}> = <f_2+h_2,\hat{\gamma}>.$$

To show that $\hat{\gamma}$ is bounded, let $f+h\varepsilon\hat{C}(S,\Sigma,E)$ where $h = x_1\chi_{A_1}+\ldots+x_k\chi_{A_k}$. Each A_j can be written as the disjoint union of an open set G_j and a closed set H_j, i.e.,

$$A_j = G_j \cup H_j \quad (j=1,\ldots,k).$$

By the definitions of γ' and μ,

$$<h,\gamma'> = \sum_{j=1}^{k} (<x_j,\mu(G_j)> + <x_j,\mu(H_j)>).$$

Since $\mu(G_j) = \mu(S)-\mu(S-G_j)$, for each j,

$$<x_j,\mu(G_j)> = <x_j,\mu(S)>- \lim_{n\to\infty} <x_j f_n^{S-G_j},\gamma>$$
$$= <x_j,\gamma>- \lim_{n\to\infty} <x_j f_n^{S-G_j},\gamma> = \lim_{n\to\infty} <x_j(1-f_n^{S-G_j}),\gamma>$$

where $\{f_n^{S-G_j}\}$ is a sequence of type χ. Let $\{\lambda_n\}$ be a positive real sequence with $\lambda_n\downarrow 0$ and $\lambda_1< \min_{i\neq j} \rho(H_i,H_j)$. This choice is possible because the distance between two closed, disjoint subsets of a compact metric space is positive. Let $\{f_n^{H_j}\}$ and $\{f_n^{S-G_j}\}$ be sequences of type χ with respect to $\{\lambda_n\}$. Then,

$$<h,\gamma'> = \sum_{j=1}^{k} [\lim_{n\to\infty} <x_j(1-f_n^{S-G_j}),\gamma>+ \lim_{n\to\infty} <x_j f_n^{H_j},\gamma>]$$
$$= \lim_{n\to\infty} <\sum_{j=1}^{k} x_j(1-f_n^{S-G_j}+f_n^{H_j}),\gamma>. \qquad (7)$$

Set $f_n = \sum_{j=1}^{k} x_j(1-f_n^{S-G_j}+f_n^{H_j})$. We will show that

$$\overline{\lim_{n\to\infty}} \|f + f_n\| \leq \|f + h\|.$$

In fact, $f+f_n$ agrees with $f+h$ on $\bigcup_{j=1}^{k} H_j$ and on $\bigcup_{j=1}^{k}[S-D(S-G_j,\lambda_n)]$, for each n; so the assertion will be proved if we can show that

$$\overline{\lim_{n\to\infty}} \sup_{t\varepsilon R(S-G_j,\lambda_n)} \|f(t)+f_n(t)\| \leq \|f+h\|$$

for each $j=1,\ldots,k$. Assume the contrary, then for some j_o,

$$\overline{\lim_{n\to\infty}} \sup_{t\varepsilon R(S-G_{j_o},\lambda_n)} \|f(t)+f_n(t)\| > \|f+h\| \geq \sup_{t\varepsilon\partial G_{j_o}} \|f(t)+h(t)\|,$$

where ∂G_{j_o} denotes the boundary of G_{j_o}. $\partial G_{j_o} \neq \emptyset$ for if it were, G_{j_o} would be both open and closed; we could select n_o so large that $R(S-G_{j_o},\lambda_n)=\emptyset$ for all $n \geq n_o$; and the proposition would be vacuously fulfilled. There is a subsequence $\{n_m\}$ of the natural numbers such that

$$\sup_{t\varepsilon R(S-G_{j_o},\lambda_{n_m})} \|f(t)+f_{n_m}(t)\| > \sup_{t\varepsilon\partial G_{j_o}} \|f(t)+h(t)\|$$

for all $m=1,2,\ldots$. The above inequality holds if the sets $R(S-G_{j_o},\lambda_{n_m})$ are replaced by their closures; and consequently, there exists $t_o \varepsilon \bigcap_{m=1}^{\infty} \overline{R(S-G_{j_o},\lambda_{n_m})}$ such that

$$\|f(t_o)+h(t_o)\| > \sup_{t\varepsilon\partial G_{j_o}} \|f(t)+h(t)\|.$$

But, $t_o \varepsilon \bigcap_{m=1}^{\infty} \overline{R(S-G_{j_o},\lambda_{n_m})}$ implies that $\rho(t_o,S-G_{j_o})=0$ which means $t_o \varepsilon \overline{S-G_{j_o}}$, and since $\overline{R(S-G_{j_o},\lambda_{n_m})} \subset \overline{G}_{j_o}$ for all m, $t_o \varepsilon G_{j_o}$; so that $t_o \varepsilon \partial G_{j_o}$ which is a contradiction. Now, according to (7),

$$|\langle f+h,\hat{\gamma}\rangle| = |\langle f,\gamma\rangle + \lim_{n\to\infty} \langle f_n,\gamma\rangle| = \lim_{n\to\infty} |\langle f+f_n,\gamma\rangle|;$$

Thus,

$$|\langle f+h,\hat{\gamma}\rangle| \leq \|\gamma\| \overline{\lim_{n\to\infty}} \|f+f_n\| \leq \|\gamma\| \; \|f+h\|$$

which proves that $\hat{\gamma}$ is bounded and $\|\hat{\gamma}\|\leq\|\gamma\|$.

The preceding work and the following lemma imply that

$$\langle f,\gamma\rangle = \int_S d\mu(t)f(t), \text{ for all } f\varepsilon C(S,E). \qquad (8)$$

Lemma 5: Every function $f\varepsilon C(S,E)$ can be approximated uniformly by a function $f_\pi \varepsilon \ell(S,\Sigma,E)$.

Proof: Let $\varepsilon>0$ and $f\varepsilon C(S,E)$ be given. Since f is uniformly continuous, there exists $\delta>0$ such that $\|f(t)-f(s)\|<\varepsilon$ whenever $\rho(t,s)<\delta$. Let $\pi=\{A_j\}_{j=1}^k$ be a partition of S with diameter less than δ. Select intermediate points $\tau_j \varepsilon A_j$ $(j=1,\ldots,k)$. Then, for the function

$$f_\pi = \sum_{j=1}^k f(\tau_j)\chi_{A_j}$$

and for arbitrary $t\varepsilon S$, we have that

$$\|f(t)-f_\pi(t)\|=\|f(t)-f_\pi(\tau_j)\|<\varepsilon$$

because $\rho(t,\tau_j)<|\pi|<\delta$. Thus, $\|f-f_\pi\|<\varepsilon$. ∎

To prove (8), let $\pi = \{A_j\}_{j=1}^k$ be a partition of S and let $f\varepsilon C(S,E)$ be given, then

$$|\langle f,\gamma\rangle - \sum_{j=1}^k \langle f(\tau_j),\mu(A_j)\rangle| = |\langle f,\gamma\rangle-\langle f_\pi,\gamma'\rangle|$$
$$= |\langle f-f_\pi \; \hat{\gamma}\rangle| \leq \|\hat{\gamma}\| \; \|f-f_\pi\|.$$

The last expression tends to zero with $|\pi|$. Furthermore, if $\|f\|<1$, we have

$$|\langle f,\gamma\rangle| \leq \text{Semi-var } \mu(S) \leq v(\mu,S)$$

as a consequence of the inequality (2). Combining this observation with (6), we see that

$$\|\gamma\| = v(\mu,S).$$

A more detailed study of the set function μ associated with γ is justified. We are in a position to prove that μ is regular on Σ. First, we will treat the closed sets. Let H be a closed subset of S. We know by Lemma 3 that for every $\varepsilon>0$ there corresponds a $\delta>0$ such that $|\langle f,\gamma\rangle|<\varepsilon$ for all $f\varepsilon\Lambda(H,\delta,1)$. If $\{C_j\}_{j=1}^{k}$ is a finite collection of disjoint closed sets, each contained in $R(H,\delta)$, there is, for each C_j, a sequence $\{f_n^{C_j}\}$ of type χ where the $f_1^{C_j}$ $(j=1,\ldots,k)$ have disjoint support. Let $\{x_j\}_{j=1}^{k}$ be a collection of elements in the closed unit sphere of E. Choose scalars α_j^n $(j=1,\ldots,k;\ n=1,2,\ldots)$ with $|\alpha_j^n|=1$ for all j and n; so that,

$$\sum_{j=1}^{k} |\langle x_j f_n^{C_j},\gamma\rangle| = \sum_{j=1}^{k}\langle \alpha_j^n x_j f_n^{C_j},\gamma\rangle, \text{ for all n.}$$

Since $\|\sum_{j=1}^{k} \alpha_j^n x_j f_n^{C_j}\| \leq 1$ $(n=1,2,\ldots)$, we have that

$$\sum_{j=1}^{k} |\langle x_j f_n^{C_j},\gamma\rangle| = |\langle \sum_{j=1}^{k} \alpha_j^n x_j f_n^{C_j},\gamma\rangle|<\varepsilon,$$

for all n and for arbitrary x_j in the closed unit sphere of E.

Passing to the limit,

$$\sum_{j=1}^{k} |\langle x_j, \mu(C_j)\rangle| \leq \varepsilon$$

for arbitrary $x_j \varepsilon E$ with $\|x_j\| \leq 1$. We conclude

$$\sum_{j=1}^{k} \|\mu(C_j)\| \leq \varepsilon,$$

and it follows that

$$v[\mu, R(H,\delta)] \leq \varepsilon.$$

Thus, H is a closed set contained in itself and the concentric disc $D(H,\delta)$ is an open set containing H, and we have shown

$$v[\mu, D(H,\delta)-H] \leq \varepsilon.$$

For an open set G, the above shows that for a preassigned $\varepsilon>0$, there exists $\delta>0$ such that

$$v[\mu, R(S-G,\delta)] < \varepsilon.$$

Thus, G is an open set containing itself and the complement of $D(S-G,\delta)$ is a closed set contained in G, and

$$v[\mu, G-D(S-G,\delta)] < \varepsilon.$$

Now, if A is an arbitrary set in Σ, we have seen that A can be written as the disjoint union of an open set G and a closed set H. We may select a closed set $H' \subset G$ and an open set $G' \supset H$ such that $v(\mu, G-H') < \frac{\varepsilon}{2}$ and $v(\mu, G'-H) < \frac{\varepsilon}{2}$.

Then, $G \cup G'$ is an open set containing $G \cup H$ and $H \cup H'$ is a closed set contained in $G \cup H$; and

$$v[\mu,(G \cup G')-(H \cup H')] \leq v(\mu,G-H')+v(\mu,G'-H)<\varepsilon$$

because $(G \cup G')-(H \cup H') \subset (G-H') \cup (G'-H)$. We conclude μ is regular on Σ.

The question of uniqueness arises.

Uniqueness Lemma: If $\mu:\Sigma \to E^*$ is regular and additive and if $\int_S d\mu(t)f(t)=0$ for all $f \varepsilon C(S,E)$, then $\mu=0$.

Proof: We first show $\mu(S)=0$. Consider the constant function which sends all of S onto the vector $x \varepsilon E$. Then, by the additivity of μ;

$$\sum_{j=1}^{k} \langle x,\mu(A_j)\rangle = \langle x,\mu(S)\rangle$$

for every partition $\{A_j\}_{j=1}^{k}$ of S; and

$$\langle x,\mu(S)\rangle = \langle x,\gamma\rangle = \int_S d\mu(t)x=0,$$

and consequently, $\mu(S)=0$. Since μ is additive, $\mu(A)=\mu(S-A)$, $A \varepsilon \Sigma$; and it is sufficient to prove $\mu(H)=0$ for H closed in S. Let $\{\lambda_n\}$ be a positive real sequence with $\lambda_n \downarrow 0$; and construct a sequence $\{f_n^H\}$ of type χ. Let n be fixed. Any partition π of S gives rise to partitions of H, $R(H,\lambda_n)$ and $S-D(H,\lambda_n)$, the union of these partitions is again a partition of S, say π'; and clearly $|\pi'| \leq |\pi|$. Thus, for all x in the closed unit sphere of E,

$$\int_H d\mu(t)xf_n^H(t) + \int_{R(H,\lambda_n)} d\mu(t)xf_n^H(t) = 0.$$

We will show that it is possible to select n so large that the contribution of the second term is negligible. Let $\{A_j^R\}$ denote the partition of the rim $R(H,\lambda_n)$ generated by the partition π of S. An approximating sum to the integral over $R(H,\lambda_n)$ is given by

$$\sum_j \langle xf_n^H(\tau_j),\mu(A_j^R)\rangle$$

Since $\|x\|\leq 1$,

$$\left|\sum_j \langle xf_n^H(\tau_j),\mu(A_j^R)\rangle\right| \leq v[\mu,R(H,\lambda_n)]$$

But, μ being regular, $v[\mu,R(H,\lambda_n)]$ tends to zero with $\lambda_n \downarrow 0$. Hence,

$$\langle x,\mu(H)\rangle = \int_H d\mu(t)xf_n^H(t) = 0,$$

for all x in the closed unit sphere of E. The proposition follows. ▮

Let $\mathcal{B}$ be the family of Borel subsets of S. It is well known [6, p. 138] that $v(\mu,\cdot)$ has a unique extension $\hat{v}(\mu,\cdot)$ to a regular Borel measure. We assert,

<u>Lemma 6</u>: The set function μ associated with the functional $\gamma\varepsilon C(S,E)^*$ by the relation (8) has a unique extension to a

regular Borel measure $\hat{\mu}$ with values in E^*, and

$$v(\hat{\mu},B) = \hat{v}(\mu,B), \text{ for all } B\varepsilon\mathcal{B}.$$

Proof: Let $B\varepsilon\mathcal{B}$ be arbitrary. Since $\hat{v}(\mu,\cdot)$ is regular, for every $\varepsilon>0$, there exists G_ε open with $G_\varepsilon \supset B$ and H_ε closed with $H_\varepsilon \subset B$, such that $\hat{v}(\mu,G_\varepsilon-H_\varepsilon)<\varepsilon$. We extend $\hat{\mu}$ to $\mathcal{B}$ by

$$\hat{\mu}(B) = \lim_{\varepsilon\to 0} \mu(G_\varepsilon) = \lim_{\varepsilon\to 0} \mu(H_\varepsilon).$$

The limits exist in the norm of E^*. For arbitrary $\varepsilon,\varepsilon'>0$, one has that

$$G_\varepsilon-H_{\varepsilon'} \subset (G_\varepsilon-H_\varepsilon) \cup (G_{\varepsilon'}-H_{\varepsilon'}).$$

It follows that

$$\begin{aligned}\|\mu(G_\varepsilon)-\mu(H_{\varepsilon'})\| = \|\mu(G_\varepsilon-H_{\varepsilon'})\| &\leq v(\mu,G_\varepsilon-H_{\varepsilon'})\\ &\leq v(\mu; G_\varepsilon-H_\varepsilon) +v(\mu,G_{\varepsilon'}-H_{\varepsilon'})\end{aligned}$$

and this last expression tends to zero as $\varepsilon,\varepsilon'\to 0$. Thus,

$$\|\hat{\mu}(B)\| < \hat{v}(\mu,B), \ B\varepsilon\mathcal{B} \tag{9}$$

From the definition of $\hat{\mu}$, we conclude $\hat{\mu}$ is finitely additive on $\mathcal{B}$. The inequality (9) assures that $\hat{\mu}$ with $\hat{v}(\mu,\cdot)$ is σ-additive on $\mathcal{B}$. That $\hat{\mu}$ is unique is easily seen, if $\hat{\nu}$ is another regular extension of μ and $B\varepsilon\mathcal{B}$, let G_ε be an open set with $G_\varepsilon \supset B$ and let H_ε be a closed set with $H_\varepsilon \subset B$ such that $\|\hat{\mu}(B)-\mu(H_\varepsilon)\|<\varepsilon$, then

$$\|\hat{\nu}(B)-\hat{\mu}(B)\| < \|\hat{\nu}(B)-\mu(H_\varepsilon)\|+\varepsilon \leq v(\hat{\nu},B-H_\varepsilon)+\varepsilon \leq v(\hat{\nu},G_\varepsilon-H_\varepsilon)+\varepsilon$$

which tends to zero with ε. Let $v_1(\hat{\mu},\cdot) = v(\hat{\mu},\cdot)|\Sigma$. From (9) it follows that

$$v_1(\hat{\mu},A) \leq v(\mu,A),\ A\varepsilon\Sigma.$$

The reverse inequality is trivial. Thus, $v(\hat{\mu},\cdot) = v(\mu,\cdot)$ on Σ. Because of the uniqueness of the extension $\hat{v}(\mu,\cdot)$, we have that $v(\hat{\mu},B) = \hat{v}(\mu,B)$, $B\varepsilon\mathcal{B}$. ▮

Let *rcabv*$(\mathcal{B},E^*)$ denote the space of all regular countably additive set functions of bounded variation on S defined on $\mathcal{B}$ with values in E^*, the norm of this space is

$$\|\hat{\mu}\| = v(\hat{\mu},S),\ \mu\varepsilon\ rcabv(\mathcal{B},E^*).$$

It is clear that $\hat{\mu}\varepsilon$ *rcabv*$(\mathcal{B},E^*)$ give rise to a linear bounded functional γ on $C(S,E)^*$ through (8).

In summary, we have shown

Theorem 2: There exists an isometric isomorphism between $C(S,E)^*$ and *rcabv*$(\mathcal{B},E^*)$, the map is defined by

$$\langle f,\gamma\rangle = \int_S d\hat{\mu}(t)f(t),\ \text{for all } f\varepsilon C(S,E)^*,$$

and $\|\gamma\| = v(\hat{\mu},S)$.

4. A Strong Representation Theorem. Let $\{y_n\}$ be a sequence of vectors in the Banach space F. If it is true that $\{\langle y_n,y^*\rangle\}$ is a Cauchy sequence of scalars for each functional $y^*\varepsilon F^*$, then $\{y_n\}$ is said to be a *weak Cauchy*

sequence. The space F is called *weakly complete*, if for every weak Cauchy sequence $\{y_n\}$, there exists $y\varepsilon F$ such that $\langle y_n,y^*\rangle \to \langle y,y^*\rangle$, for all $y^*\varepsilon F^*$.

Suppose that T is a bounded linear transformation of C(S,E) into a weakly complete Banach space F. Our goal is to obtain a representation for this operator T through the integral of Section 1. We will denote the adjoint of T by T^*. Let H be a closed subset of S, and let $\{f_n^H\}$ be a sequence of type χ and let $x\varepsilon E$. We have observed that $\{xf_n^H\}$ is a weak Cauchy sequence in C(S,E). But,

$$\langle xf_n^H,T^*y^*\rangle = \langle Txf_n^H,y^*\rangle, \text{ for all } n,$$

and consequently, $\{Txf_n^H\}$ is a weakly Cauchy sequence in F. Now, F is weakly complete; so that, there exists $y\varepsilon F$ such that

$$\lim_{n\to\infty} \langle Txf_n^H,y^*\rangle = \langle y,y^*\rangle, \quad y^*\varepsilon F^*$$

We associate with T the set function μ on the closed subsets of S and taking values in L(E,F) defined by

$$\langle \mu(H)x,y^*\rangle = \lim_{n\to\infty} \langle Txf_n^H,y^*\rangle, \; y^*\varepsilon F^*, x\varepsilon E. \tag{10}$$

We must show that μ does not depend on the particular sequence of type χ. Indeed, if $H\neq\emptyset$ is a proper subset of S, and if $\{\tilde{f}_n^H\}$ is a bounded sequence in C(S,K) with $\tilde{f}_n^H(H)=1$ and $\tilde{f}_n^H(C_n)=0$ for all n, where C_n is the non-empty complement of $D(H,\tilde{\lambda}_n)$, $\{\tilde{\lambda}_n\}$ a real sequence $\tilde{\lambda}_n \downarrow 0$, we have shown (p. 14, par. i) that

$$\lim_{n\to\infty} \langle xf_n^H, T^*y^*\rangle = \lim_{n\to\infty} \langle x\tilde{f}_n^H, T^*y^*\rangle, \quad y^*\varepsilon F^*, x\varepsilon E.$$

Thus,

$$\lim_{n\to\infty} \langle Txf_n^H, y^*\rangle = \lim_{n\to\infty} \langle Tx\tilde{f}_n^H, y^*\rangle, \; y^*\varepsilon F^*, x\varepsilon E.$$

μ may be extended to Σ by

i) if G is open, then we define

$$\mu(G)=\mu(S)-\mu(S-G);$$

and ii) if $A\varepsilon\Sigma$ is arbitrary, let $A=G\cup H$ where G is open and H is closed and these sets are disjoint, then we set

$$\mu(A)=\mu(G)+\mu(H).$$

As before, we must show that this definition does not depend on the particular representation of A as the disjoint union of an open and closed set. The argument of pages 16-7 can be used here if we can prove the identity: if H_1 and H_2 are arbitrary closed subsets of S, then

$$\mu(H_1)+\mu(H_2)=\mu(H_1\cup H_2)+\mu(H_1\cap H_2).$$

By equation (5), we have for arbitrary $y^*\varepsilon F^*$ and arbitrary $x\varepsilon E$,

$$\lim_{n\to\infty} \langle xf_n^{H_1}, T^*y^*\rangle + \lim_{n\to\infty} \langle xf_n^{H_2}, T^*y^*\rangle =$$

$$= \lim_{n\to\infty} \langle xf_n^{H_1\cup H_2}, T^*y^*\rangle + \lim_{n\to\infty} \langle xf_n^{H_1\cap H_2}, T^*y^*\rangle$$

where $\{f_n^{H_1}\}$, $\{f_n^{H_2}\}$ $\{f_n^{H_1 \cup H_2}\}$ and $\{f_n^{H_1 \cap H_2}\}$ are sequences of type χ. It follows that

$$\lim_{n\to\infty} \langle Txf_n^{H_1}, y^*\rangle + \lim_{n\to\infty} \langle Txf_n^{H_2}, y^*\rangle = \\ = \lim_{n\to\infty} \langle Txf_n^{H_1 \cup H_2}, y^*\rangle + \lim_{n\to\infty} \langle Txf_n^{H_1 \cap H_2}, y^*\rangle,$$

and hence,

$$\langle \mu(H_1)x, y^*\rangle + \langle \mu(H_2)x, y^*\rangle = \langle \mu(H_1 \cup H_2)x, y^*\rangle + \langle \mu(H_1 \cap H_2)x, y^*\rangle.$$

Since the last equation holds for all $y^* \varepsilon F^*$, we must have

$$\mu(H_1)x + \mu(H_2)x = \mu(H_1 \cap H_2)x + \mu(H_1 \cup H_2)x, \text{ for all } x \varepsilon E,$$

which means

$$\mu(H_1) + \mu(H_2) = \mu(H_1 \cup H_2) + \mu(H_1 \cap H_2).$$

For each $y^* \varepsilon F^*$, μ induces a set function μ^{y^*} on Σ with values in E^* by

$$\langle x, \mu^{y^*}(A)\rangle = \langle \mu(A)x, y^*\rangle, \quad x \varepsilon E, \ A \varepsilon \Sigma.$$

But, μ^{y^*} is the set function associated with the functional $T^*y^* \varepsilon C(S,E)^*$ according to (10). As in the proof of Lemma 6, if $\hat{\mu}^{y^*}$ is the extension of μ^{y^*} to $\mathcal{B}$ and $B \varepsilon \mathcal{B}$, $\hat{\mu}^{y^*}(B)$ can be thought of as the strong limit of a sequence $\{\mu^{y^*}(G_n)\}$ where the G_n $(n=1,2,\ldots)$ are open sets such that $B \subset G_n$ for all n and $v(\hat{\mu}^{y^*}, G_n - B) \to 0$ as n increases without bound. Since

$$\langle x, \mu^{y^*}(G_n)\rangle = \langle \mu(G_n)x, y^*\rangle, \text{ for all } n, \ x \varepsilon E, \ y^* \varepsilon F^*,$$

$\{\mu(G_n)x\}$ is a weak Cauchy sequence in F. F being weakly complete, we may define $\hat{\mu}(B)$ by

$$<\hat{\mu}(B)x,y^*> = \lim_{n\to\infty} <\mu(G_n)x,y^*> = <x,\mu^{y^*}(B)> \tag{11}$$

where $x\varepsilon E$ and $y^*\varepsilon F^*$. This convention must be justified. Let G_n' be another sequence of open sets containing B with $\mu^{y^*}(G_n')\to\mu^{y^*}(B)$. Then,

$$|<\mu(G_n')x,y^*>-<\mu(G_n)x,y^*>| = |<x,\mu^{y^*}(G_n')>-<x,\mu^{y^*}(G_n)>|$$
$$=|<x,\mu^{y^*}(G_n')-\mu^{y^*}(G_n)>| \leq \|\mu^{y^*}(G_n')-\mu^{y^*}(G_n)\| \, ||x||$$

and the last expression tends to zero with $n\to\infty$ for arbitrary $x\varepsilon E$ and $y^*\varepsilon F^*$.

We wish to show now that $\hat{\mu}$, which was defined on $\mathcal{B}$ above, is a Borel measure of finite semi-variation on S. Let us agree that σ^* will denote the closed unit sphere of F^*.

Lemma 7: Let T be a bounded linear transformation of C(S,E) into a weakly complete Banach space F, and let $\hat{\mu}^{y^*}$ be the measure corresponding to T^*y^* ($y^*\varepsilon F^*$) in the sense of Theorem 2. Then, the σ-additivity of the measures $v(\hat{\mu}^{y^*},\cdot)$ is uniform in $y^*\varepsilon\sigma^*$.

Proof: Let $\{B_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{B}$. We must show

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} v(\hat{\mu}^{y^*},B_n) = 0$$

uniformly in $y^*\varepsilon\sigma^*$. Were the proposition false, one could find $\varepsilon_o>0$, a sequence of natural numbers $\{N_k\}$ with $N_k>K$, and elements $y_k^*\varepsilon\sigma^*$, such that elements $y_k^* \epsilon \sigma^*$, such that

$$\sum_{n=N_k}^{\infty} v(\hat{\mu}^{y_k^*}, B_n) > \varepsilon_o,$$

for all $k=1,2,\ldots.$ Hence for each k, one must have $M_k > N_k$ such that

$$\sum_{n=N_k}^{M_k} v(\hat{\mu}^{y_k^*}, B_n) > \varepsilon_o.$$

By choice of a subsequence, we may take $N_1 < M_1 < N_2 < M_2 \ldots.$ Because of the regularity of $v(\hat{\mu}^{y_k^*}, \cdot)$ there is a sequence of disjoint closed sets $\{H_{n,j}\}$, and a sequence of vectors $\{x_{n,j}\}$ in the closed unit sphere of E ($j=1,\ldots,r_n$ and $n=1,2,\ldots$) such that

$$\sum_{n=N_k}^{M_k} \sum_{j=1}^{r_n} |\langle x_{n,j}, \hat{\mu}^{y_k^*}(H_{n,j})\rangle| = \sum_{n=N_k}^{M_k} \sum_{j=1}^{r_n} |\langle \hat{\mu}(H_{n,j}) x_{n,j}, y_k^* \rangle| > \varepsilon_o; \quad j=1,2,\ldots. \tag{12}$$

On the other hand, the series

$$\sum_{k=1}^{\infty} \sum_{n=N_k}^{M_k} \sum_{j=1}^{r_n} \hat{\mu}(H_{n,j}) x_{n,j}$$

is weakly unconditionally convergent, for if $y^* \epsilon F^*$ and $\sum_{i=1}^{\infty} a_{k_i}$ is a subseries of (13), the weak subseries is dominated by

$$\sum_{i=1}^{\infty} v(\hat{\mu}^{y^*}, B_{k_i})$$

where $B_{k_i} = \bigcup_{n=N_{k_i}}^{M_{k_i}} B_n$; and the weak completeness of F implies the subseries converges. By the lemma of Orlicz and Pettis [8, p.61],

$$\lim_{i\to\infty} \sum_{k=i}^{\infty} \sum_{n=N_k}^{M_k} \sum_{j=1}^{r_n} |<\hat{\mu}(H_{n,j})x_{n,j},y^*>| = 0$$

uniformly in $y^*\varepsilon\sigma^*$. But, this contradicts (12), and the uniform σ-additivity of the $v(\hat{\mu}y^*,\cdot)$; $y^*\varepsilon\sigma^*$ is proved. ∎

Now, let $\{B_n\}$ be a sequence of pairwise disjoint Borel subsets of S, say $\bigcup_{n=1}^{\infty} B_n=B$. We must show $\sum_{n=1}^{N} \hat{\mu}(B_n)\to\hat{\mu}(B)$ with $N\to\infty$ in the norm of L(E,F). The inequality

$$|<[\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)]x,y^*>| = |<x,\hat{\mu}^{y^*}(B)-\sum_{n=1}^{N}\hat{\mu}^{y^*}(B_n)>|$$

$$\leq \|x\| \, \|\hat{\mu}^{y^*}(B)-\sum_{n=1}^{N}\hat{\mu}^{y^*}(B_n)\| \leq \|x\| v(\hat{\mu}^{y^*},B-\bigcup_{n=1}^{N}B_n)$$

shows that the left hand member tends to zero as $N\to\infty$ uniformly for $x\varepsilon E$ with $\|x\|\leq 1$ and $y^*\varepsilon\sigma^*$. Since

$$\sup_{\|y^*\|\leq 1} |<[\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)]x,y^*>| = \|[\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)]x\|,$$

and since

$$\sup_{\|x\|<1} \|[\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)]x\| = \|\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)\|,$$

we see that $\|\hat{\mu}(B)-\sum_{n=1}^{N}\hat{\mu}(B_n)\|$ also tends to zero as N increases without bound.

<u>Lemma 8</u>: The L(E,F)-valued set function $\hat{\mu}$ on $\mathcal{B}$ is of bounded semi-variation if and only if the induced E*-valued set function $\hat{\mu}^{y^*}$ on $\mathcal{B}$, defined by

$$<\hat{\mu}(B)x,y^*> = <x,\hat{\mu}^{y^*}(B)>, \; x\varepsilon E, \; y^*\varepsilon F^*,$$

is of bounded variation for all $y^*\varepsilon F^*$. Furthermore,

$$\text{Semi-var } \hat{\mu}(S) = \sup_{\|y^*\|\leq 1} v(\mu y^*, S).$$

Proof: Let $y^*\varepsilon F^*$. To every $\varepsilon>0$ and to every finite sequence of disjoint sets $\{B_j\}_{j=1}^{k} \subset \mathcal{B}$ there is a k-tuple of vectors $\{x_j\}_{j=1}^{k}$ in the closed unit sphere of E such that

$$\sum_{j=1}^{k} \|\hat{\mu}y^*(B_j)\| < \sum_{j=1}^{k} \langle x_j, \hat{\mu}y^*(B_j)\rangle + \varepsilon.$$

Thus,

$$\sum_{j=1}^{k} \|\hat{\mu}^{y^*}(B_j)\| < \langle \sum_{j=1}^{k} \hat{\mu}(B_j)x_j, y^*\rangle + \varepsilon \leq \|y^*\| \, \|\sum_{j=1}^{k} \hat{\mu}(B_j)x_j\| + \varepsilon.$$

It follows that if $\hat{\mu}$ is of bounded semi-variation, then

$$v(\hat{\mu}y^*, S) \leq \|y^*\| \text{ Semi-var } \hat{\mu}(S) + \varepsilon$$

for every $\varepsilon>0$, hence

$$v(\hat{\mu}y^*, S) \leq \|y^*\| \text{ Semi-var } \hat{\mu}(S) \tag{14}$$

which shows that $\hat{\mu}y^*$ is of bounded variation for all $y^*\varepsilon F^*$. On the other hand, for every finite collection of disjoint Borel sets $\{B_j\}_{j=1}^{k}$ and every k-tuple $\{x_j\}_{j=1}^{k}$ with $\|x_j\| \leq 1$ $(j=1,\ldots,k)$,

$$|\langle \hat{\mu}(B_j)x_j, y^*\rangle| = |\langle x_j, \hat{\mu}y^*(B_j)\rangle| \leq \|\hat{\mu}y^*(B_j)\| \, \|x_j\|;$$

so that,

$$|\langle \sum_{j=1}^{k} \hat{\mu}(B_j)x_j, y^*\rangle| \leq \sum_{j=1}^{k} \|\hat{\mu}y^*(B_j)\| \leq v(\hat{\mu}y^*, S). \tag{15}$$

If $\hat{\mu}_{y^*}$ is of bounded variation for every $y^* \varepsilon F^*$, we may apply the Principle of Uniform Boundedness [6, p.66] to conclude the set $\{\sum_{j=1}^{k} \|\hat{\mu}(B_j)x_j\|\}_k$ is bounded. Since

$$\|\sum_{j=1}^{k} \hat{\mu}(B_j)x_j\| \leq \sum_{j=1}^{k} \|\hat{\mu}(B_j)x_j\|,$$

$\hat{\mu}$ is of bounded semi-variation. From (15),

$$\|\sum_{j=1}^{k} \hat{\mu}(B_j)x_j\| \leq \sup_{\|y^*\| \leq 1} v(\hat{\mu}_{y^*},S),$$

and it follows that Semi-var $\hat{\mu}(S) \leq \sup_{\|y^*\| \leq 1} v(\hat{\mu}_{y^*},S)$. From (14), we see that $\sup_{\|y^*\| < 1} v(\hat{\mu}_{y^*},S) \leq$ Semi-var $\hat{\mu}(S)$. The proposition follows. ∎

We can now prove

<u>Theorem 3</u>: To every bounded linear transformation of $C(S,E)$ into a weakly complete Banach space F there corresponds a unique operator-valued Borel measure $\hat{\mu}$ having values in $L(E,F)$ such that

$$Tf = \int_S d\hat{\mu}(t)f(t), \quad f \varepsilon C(S,E)$$

and

$$\|T\| = \text{Semi-var } \hat{\mu}(S).$$

Proof: Let $\hat{\mu}_{y^*}$ be the regular Borel measure which represents T^*y^* in the sense of Theorem 2, $y^* \varepsilon F^*$. The measure $\hat{\mu}$ is uniquely determined if we require (11) for $\hat{\mu}^{y^*}$ associated with the functional T^*y^* is unique. Since $\hat{\mu}^{y^*}$ is of bounded variation for all $y^* \epsilon F^*$, $\hat{\mu}$ is of bounded semi-variation. Let $\pi = \{A_j\}_{j=1}^{k}$ be a partition of S and $\tau : \tau_j \epsilon A_j$ $(j=1,\ldots,k)$ be a set of intermediate points, we have that

$$\sum_{j=1}^{k} \langle f(\tau_j), \hat{\mu}^{y^*}(A_j) \rangle = \sum_{j=1}^{k} \langle \hat{\mu}(A_j) f(\tau_j), y^* \rangle = \langle \sum_{j=1}^{k} \hat{\mu}(A_j) f(\tau_j), y^* \rangle$$

for each $f \epsilon C(S,E)$. Letting $|\pi| \to 0$, we obtain

$$\langle f, T^*y^* \rangle = \langle Tf, y^* \rangle = \langle \int_S d\hat{\mu}(t) f(t), y^* \rangle$$

by Theorem 2. Since y^* is arbitrary, it follows that

$$Tf = \int_S d\hat{\mu}(t) f(t), \quad f \epsilon C(S,E).$$

To prove the final assertion of the theorem,

$$\text{Semi-var } \hat{\mu}(S) = \sup_{\|y^*\| \leq 1} v(\hat{\mu}^{y^*}, S) = \sup_{\|y^*\| \leq 1} \|T^*y^*\| =$$

$$\|T^*\| = \|T\|. \blacksquare$$

5. Weak Compactness. A subset Q of a Banach space F is said to be *weakly sequentially compact* if every sequence $\{y_n\} \subset Q$ has a subsequence which converges weakly to a vector in F, that is, there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ and a vector $y \epsilon F$ such that $\langle y_{n_k}, y^* \rangle \to \langle y, y^* \rangle$ as $k \to \infty$ for all $y^* \epsilon F^*$. A transformation is *weakly compact* if and only if it maps bounded sets into weakly sequentially compact sets. In this section, we show that if E is reflexive and F is weakly com-

plete, then every bounded linear transformation of C(S,E) into F is weakly compact.

Lemma 9: If E is reflexive, then the subset $Q \subset rcabv(\mathcal{B},E)$ is weakly sequentially compact if the set $\{v(\hat{\mu},\cdot)|\hat{\mu}\varepsilon Q\}$ in $rca(\mathcal{B},R)$ is weakly sequentially compact.

Proof: Assume $\{v(\hat{\mu},\cdot)|\hat{\mu}\varepsilon Q\}$ is weakly sequentially compact in $rca(\mathcal{B},R)$, then this set is bounded and there exists a non-negative measure $\lambda \varepsilon rca(\mathcal{B},R)$ such that all $v(\hat{\mu},\cdot)$ are uniformly λ-continuous with respect to $\hat{\mu}\varepsilon Q$ which means $\lim_{\lambda(B)\to 0} v(\hat{\mu},B)=0$ uniformly in $\hat{\mu}\varepsilon Q$ [6, p.341]. Let $\{\hat{\mu}_k\}$ be a sequence in Q. All $\hat{\mu}_k$ take their values in a separable subspace $X\subset E$. Indeed, the compact metric space S is separable; let D be a countable dense subset of S. From the set $\mathcal{D}$ of all spheres with rational radii centered on points of D we form the ring of subsets of S generated by $\mathcal{D}$, say $\tilde{\Sigma}$ $\tilde{\Sigma}$ is countable [6, p.167]. If k is fixed, we assert $\hat{\mu}_k(\tilde{\Sigma})$ is dense in $\hat{\mu}_k(\mathcal{B})$. Let $B\varepsilon\mathcal{B}$ be given. By the regularity of $\hat{\mu}_k$, we have for every $\varepsilon>0$, an open set $G\supset B$ such that

$$\|\hat{\mu}_k(B)-\hat{\mu}_k(G)\| = \|\hat{\mu}_k(G-B)\| \leq v(\hat{\mu}_k,G-B) <\varepsilon.$$

$\mathcal{D}$, in fact, is a base for the topology on S and hence, $G = \bigcup_{n=1}^{\infty} A_n$ where $A_n\varepsilon\mathcal{D}$. Also, $G = \bigcup_{n=1}^{\infty} \tilde{A}_n$ where the $\tilde{A}_n$ are disjoint sets in $\tilde{\Sigma}$. We have that

$$\hat{\mu}_k(G) = \lim_{n\to\infty} \sum_{n=1}^{N} \hat{\mu}_k(\tilde{A}_n)$$

in the norm of E. Therefore, $\hat{\mu}_k(\tilde{\Sigma})$ is dense in $\hat{\mu}_k(\mathcal{B})$. Let X be the subspace of E generated by $\{\hat{\mu}_k(B) \mid B \varepsilon \mathcal{B},\ k=1,2,\ldots\}$. It is clear that X is separable. The sequence $\{\hat{\mu}_k\}$ is contained in the subspace $rca(\mathcal{B},X,\lambda)$ of λ-continuous measures in $rca(\mathcal{B},X)$. With E, X is also reflexive and X* is separable because its adjoint is such. Thus, X is the adjoint of a separable Banach space, and we may apply the generalization of the Radon-Nikodým theorem by Dunford and Pettis [5, p.339]: there exists an isometric isomorphism between $rca(\mathcal{B},X,\lambda)$ and $L^1(S,X,\lambda)$, the X-valued, λ-integrable (Bochner) functions defined on S, by virtue of

$$\hat{\mu}(B) = \int_B f(t)\lambda(dt),\ B\varepsilon\mathcal{B},$$

and

$$v(\hat{\mu},B) = \int_B \|f(t)\|\lambda(dt),\ B\varepsilon\mathcal{B}.$$

Since E is reflexive, a theorem of S.D. Chatterji, [4, p.14] asserts that a set $Q_1 \subset L^1(S,X,\lambda)$ is weakly sequentially compact if and only if Q_1 is bounded and

$$\lim_{\lambda(B)\to 0} \int_B \|f(t)\|\lambda(dt) = 0,\ B\varepsilon\mathcal{B}$$

uniformly for all $f\varepsilon Q_1$. It follows that $Q \subset rcabv(\mathcal{B},E)$ is weakly sequentially compact. ∎

<u>Theorem 4</u>: If E is reflexive and F is weakly complete, then every bounded linear transformation $T:C(S,E)\to F$ is weakly compact.

Proof: According to Theorem 3, the image of σ^* under T^* is the set $\{\hat{\mu}^{y^*} | y^* \varepsilon \sigma^*\}$. But, we have seen that the σ-additivity of the $v(\hat{\mu}^{y^*},\cdot)$ is uniform in $y^* \varepsilon \sigma^*$, and the norms of these measures are bounded by $\|T\|$. Thus, $\{v(\hat{\mu}^{y^*},\cdot) | y^* \varepsilon \sigma^*\}$ is weakly sequentially compact in *rca*($\mathcal{B}$,R) [6, p.305]. From the previous lemma, we conclude $\{\hat{\mu}^{y^*} | y^* \varepsilon \sigma^*\}$ is weakly sequentially compact in *rcabv*($\mathcal{B}$,E*), noting that E* is reflexive if and only if E is such. Hence, T* is weakly compact; and by the theorem of Gantmacker [6, p.485] T is weakly compact. $\blacksquare$

The result of Theorem 4 is easily generalized; the proof which we present is essentially the method used by Dunford and Schwartz for the case E=K.

<u>Theorem 5</u>: Let S be a compact Hausdorff space. If E is reflexive and F is weakly complete, then every bounded linear transformation $T:C(S,E)\to F$ is weakly compact.

Proof: Let $\{f_n\}$ be a sequence in the closed unit sphere of $C(S,E)$. We partition S into a set of equivalence classes S_o by the relation: $s \sim s'$ if and only if $f_n(s)=f_n(s')$, $n=1,2,\ldots.$ We make S_o into a metric space by

$$\rho_o([s],[t]) = \sum_{n=1}^{\infty} 2^{-n}\|f\ (s)-f\ (t)\|; [s],[t] \varepsilon S_o.$$

Let η be the canonical map of S onto S_o. We see from the continuity of f_n that η is continuous. Thus, S_o is compact metric space. Consider the space $C(S_o,E)$, if $\Phi\varepsilon C(S_o,E)$, then the function f defined by $f(s)=\Phi[\eta(s)]$ is in $C(S,E)$. Define $T_o:C(S_o,E)\to F$ by $T_o = Tf$. T_o is a linear map, and $\|T_o\| \leq \|T\|$. Each f_n gives a well defined function $\Phi_n \varepsilon C(S_o,E)$ such that $f_n(s) = \Phi_n[\eta(s)]$ because of the relation $\sim$. Now, by theorem 4, T_o is weakly compact, so there is a subsequence $\{\Phi_{n_k}\}$ such that $\{T_o\Phi_{n_k}\}$ converges weakly in F. Since $Tf_n = T_o\Phi_{n_k}$, $\{Tf_{n_k}\}$ converges weakly; so that, T is weakly compact. ∎

Works Consulted

1. R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), 289-305.

2. J. Batt, *Darstellung linear Transformationen durch vektorwertige Stieltjes-Integrale und schwache Kompaktheit*, to appear in Mathematische Annalen.

3. J. Batt and H. König, *Darstellung linear Transformationen durch vektorwertige Riemann-Stieltjes-Integrale*, Arch. Math. X (1959), 273-287.

4. S. D. Chatterji, *Weak convergence in certain special Banach spaces*, MRC Technical Summary Report #443, The University of Wisconsin (1963).

5. N. Dunford and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. 47 (1940), 323-392.

6. N. Dunford and J. Schwartz, *Linear Operators*, Part I, Interscience Publishers, New York, 1966.

7. C. Foiaş and I. Singer, *Some remarks on the representation of linear operators in spaces of vector-valued continuous functions*, Rev. Math. pures appl. 5 (1960), 729-752.

8. I. Gelfand, *Abstrakte Functionen und lineare Operatoren*, Mat. Sbornik (46) 4 (1938), 235-286.

9. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31 (1957).

10. I. Singer, *Linear functionals on the space of continuous mappings of a compact Hausdorff space into a Banach space.* Rev. Math. pures appl. 2 (1957), 301-315. (Russian)