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SPECIAL
A ROD
SYSTEM
STUDIES

FIFTH QUARTERLY REPORT

VOLUME 2

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Special AROD System Studies

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VOLUME 2

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ROCKVILLE, MARYLAND

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Appendix A

RANGE EQUATION ANALYSIS

Fundamental to the analysis and design of the AROD modulation system, is a determination of the ratio of the signal power to thermal noise power density, $\left(\frac{C}{N_o}\right)_{RS}$, at the spacecraft receiver. In this appendix, the equations for calculating this ratio are presented and a quantitative estimate of this ratio and the corresponding ratio for the ground transponder receiver are made.

The version of the Range Equation which best describes the relationship among the AROD signal parameters is:

$$\left(\frac{C}{N_o}\right)_{RS} = \left(\frac{P_{RG} A_G G_{TG} L_{SU}^{-1} L_{TG}^{-1} G_{RS}}{\left(kT_{RS}\right) + \left(kT_{RG} A_G G_{TG} L_{SU}^{-1} \; L_{TG}^{-1} G_{RS}\right)}\right) \qquad \text{(A-1)}$$

where

P_{RG} = signal power received by the ground transponder

$$= P_{TS} G_{TS} L_{SD}^{-1} G_{RG} L_{TS}^{-1} \qquad \text{(A-2)}$$

where

P_{TS} = spacecraft transmitter output power

G_{TS} = spacecraft transmitting antenna gain

L_{SD} = space loss for the "down link"

G_{RG} = ground transponder receiving antenna gain

L_{TS} = transmitting loss at the spacecraft

A_G = net electronic amplification of the ground transponder

G_{TG} = ground transponder transmitting antenna gain

L_{SU} = space loss for the "up link"

L_{TG} = transmitting loss at the ground transponder

G_{RS} = spacecraft receiving antenna gain

k = Boltzmann's constant = $1.38 \cdot 10^{-23}$

T_{RS} = spacecraft receiver effective noise temperature

T_{RG} = ground receiver effective noise temperature

Both space losses can be calculated from:

$$L_S = \left(\frac{\lambda}{4\pi R}\right)^{-2}$$

Based upon information provided in the RFP for the AROD System Test Model Hardware,* and in conferences with NASA's Contracting Officer's Technical Representative, the following values were assumed for the AROD parameters at maximum range:

$$\text{transmitted frequency} = \begin{cases} 2276 \text{ MHz down} \\ 1800 \text{ MHz up} \end{cases}$$

G_{TS} = 0 db = G_{RS}

R = 4000 naut miles

G_{RG} = 12 db = G_{TG}

L_{TS} = 5 db = L_{TG}

T_{RS} = 870°K (6 db receiver)

T_{RG} = 290°K (3 db receiver)

*RFP No. 1-4-40-01283 from NASA's Marshall Space Flight Center, Huntsville, Ala.

The space loss calculations then yield:

$$L_{SD}^{-1} = -177 \text{ db}$$

$$L_{SU}^{-1} = -175 \text{ db}$$

With these values, $P_{RG} = -170$ dbw for each watt of spacecraft transmitter power. The value of A_G for maximum range* is determined by calculating the difference between the maximum ground transmitter power (50 watts) and the P_{RG} obtained if the maximum spacecraft transmitter power (10 watts) is employed: $A_G = 17 \text{ dbw} - (-160 \text{ dbw}) = 177$ db.

For each watt of spacecraft transmitter power we then obtain:

$$\left(\frac{C}{N_o}\right)_{RS} = 2 \cdot 10^3 \text{ cps.}$$

At the ground transponder receiver, the ratio of signal power to thermal noise power density is given by:

$$\left(\frac{C}{N_o}\right)_{RG} = \frac{P_{RG}}{kT_{RG}} = 2.6 \cdot 10^3 \text{ cps}$$

for each watt of spacecraft transmitter power.

*A_G will be varied by AGC; these calculations assume that the ground transponder output SNR is sufficiently high to put 50 watts into the signal power.

Appendix B

PHASE-LOCK LOOP ANALYSIS

B.1 TRACKING

The linear equivalent representation of phase for the phase-lock loop in the tracking mode can be represented by the linear second order servo loop as shown in Figure B-1. The gain of the loop, K, includes the amplitude of the input signal, A. Thus:

$$K = A K_1 \tag{B-1}$$

where

K = open loop gain

A = signal rms amplitude

K_1 = loop gain exclusing the signal amplitude

Then K_1 is given by:

$$K_1 = K_{VCO} K_M K_A \tag{B-2}$$

K_{VCO} = VCO gain constant rad/sec/volt

K_M = phase detector gain constant volts/radian

K_A = loop amplifier gain

The open loop transfer function of the loop is given by

$$H_o(s) = \frac{KF(s)}{s} \tag{B-3}$$

The closed loop transfer function is

$$H(s) = \frac{\theta_o(s)}{\theta_1(s)} = \frac{H_o(s)}{1+H_o(s)} = \frac{KF(s)}{s+KF(s)} \tag{B-4}$$

Consider the loop filter as being a lag filter as shown in Figure B-2. The transfer function of this filter is:

$$F(s) = \frac{1+sCR_2}{1+sC(R_1+R_2)} = \frac{1+T_2 s}{1+T_1 s} \tag{B-5}$$

The closed loop transfer function is then found to be:

$$H(s) = \frac{1 + T_2(s)}{1 + \left(\frac{1}{K} + T_2\right) s + \frac{T_1}{K} s^2} \tag{B-6}$$

Comparing the characteristic equation of the closed loop transfer function with that of a second order loop obtains:

$$1 + \left(\frac{1}{K} + T_2\right) s + \frac{T_1}{K} s^2 = 1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2} \tag{B-7}$$

The loop natural frequency is given by

$$\omega_n = \sqrt{\frac{K}{T_1}} \quad ; \quad T_1 = \frac{K}{\omega_n^2} \tag{B-8}$$

and the loop damping factor and natural frequencies are related by

$$\frac{2\zeta}{\omega_n} = \frac{1}{K} + T_2 \quad ; \quad T_2 = \frac{2\zeta}{\omega_n} - \frac{1}{K} \tag{B-9}$$

$$T_2 = 2\zeta \sqrt{\frac{T_1}{K}} - \frac{1}{K} \tag{B-10}$$

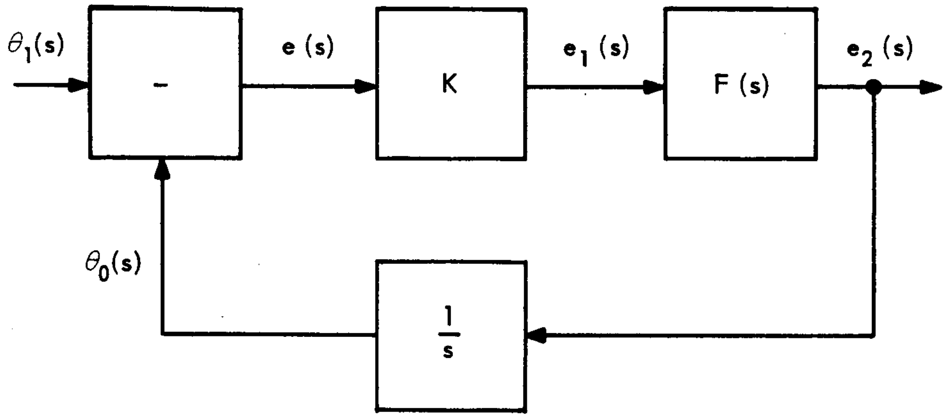

Figure B–1. Linear Equivalent of Phase-lock Loop

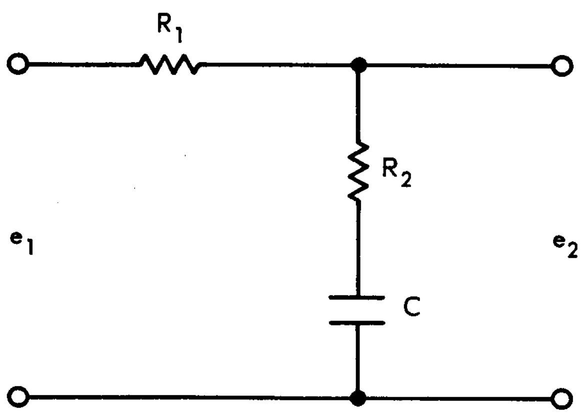

Figure B–2. Lag Filter

If $K >> \omega_n$ then $T_1 >>> 1$

and

$$T_2 \simeq 2\zeta \sqrt{\frac{T_1}{K}} = \frac{2\zeta}{\omega_n} \qquad \text{(B-11)}$$

Then $T_1 >> T_2$.

The closed loop transfer function is then given approximately by:

$$H(s) = \frac{1 + T_2 s}{1 + T_2 s + \frac{T_1}{K} s^2} \qquad \text{(B-12)}$$

The closed loop transfer function is the same as that which would be obtained if a proportional plus integral control filter were used with the transfer function given by

$$F(s) = \frac{1 + T_2 s}{T_1 s} \qquad \text{(B-13)}$$

The loop filter parameters are chosen for a particular signal level which is denoted as the design point or match point. The rms amplitude of the signal at the design point is denoted by A_o. The loop parameters at the match point will be denoted with a subscript $_o$, thus

ζ_o = loop damping factor at match point

ω_{n_o} = loop natural frequency at match point.

The filter time constants for a particular signal level are given by:

$$T_1 = \frac{K}{\omega_{n_o}^2} = \frac{A_o K_1}{\omega_{n_o}^2} \qquad \text{(B-14)}$$

$$T_2 = \frac{2\zeta_o}{\omega_{n_o}} \qquad \text{(B-15)}$$

Substituting these values for T_1 and T_2 in the closed loop transfer function (B-12), we find that the transfer function at other signal levels is given by:

$$H(s) = \frac{1 + \frac{2\zeta_o}{\omega_{n_o}} s}{1 + \frac{2\zeta_o}{\omega_{n_o}} s + \frac{A_o}{A} \frac{s^2}{\omega_{n_o}^2}} \tag{B-16}$$

Equating the characteristic equation with that of the second order loop, the loop natural frequency is given by

$$\omega_n = \sqrt{\frac{A}{A_o}} \quad \omega_{n_o} \tag{B-17}$$

The loop damping factor is given by

$$\zeta = \sqrt{\frac{A}{A_o}} \quad \zeta_o \tag{B-18}$$

Consider the case of the phase-lock loop preceded by a bandpass limiter. The limiter is considered to be an ideal hard limiter, that is, the output is +1 for inputs greater than 0, and -1 for inputs less than 0. The output power is assumed to be constant for all input powers. Thus, if the output power level is L^2, then:

$$L^2 = A^2 + B^2 = S + N \tag{B-19}$$

where

A = output signal rms value

B = output noise rms value

S = signal output power

N = noise output power

The rms signal output is found to be:

$$A = \frac{L}{\sqrt{1 + \frac{B^2}{A^2}}} = \frac{L}{\sqrt{1 + \left(\frac{N}{S}\right)_{out}}} \tag{B-20}$$

The suppression factor is defined as

$$\alpha = \frac{1}{\sqrt{1 + \left(\frac{N}{S}\right)_{out}}} \qquad \text{(B-21)}$$

The loop gain of the phase-lock loop is then given by

$$K = \alpha\ K_1$$

where K_1 includes the limiter constant L, or it could be assumed that $L = 1$. The noise power to signal power ratio at the output of the limiter has been found by Davenport[1] to be related to the noise power to signal power at the input of the limiter by the factor $\frac{4}{\pi}$ for low signal-to-noise ratios at the limiter input.

$$\left(\frac{N}{S}\right)_{out} = \frac{4}{\pi}\left(\frac{N}{S}\right)_{2B_N} \qquad \text{(B-22)}$$

$2B_N$ = Limiter bandwidth

$\left(\frac{N}{S}\right)_{2B_N}$ = Noise power to signal power ratio in bandwidth at limiter input

The suppression factor is then given by

$$\alpha = \frac{1}{\sqrt{1 + \frac{4}{\pi}\left(\frac{N}{S}\right)_{2B_N}}} \qquad \text{(B-23)}$$

The loop parameters are again chosen for a particular signal power and noise power at the design point. The suppression factor at the design point is denoted by α_o.

The loop filter parameters are found from:

$$T_1 = \frac{\alpha_o\ K_1}{\omega_{n_o}{}^2} \ ; \qquad T_2 = \frac{2\ \zeta_o}{\omega_{n_o}} \qquad \text{(B-24)}$$

---

[1] Davenport, W.B., "Signal to Noise Ratios in Bandpass Limiters," Journal of Applied Physics, Vol 24, pp 720-727, June 1963.

The loop natural frequency and loop damping factor will vary at different signal levels and can be found to be given by

$$\omega_n = \sqrt{\frac{\alpha}{\alpha_o}} \quad \omega_{n_o}$$

$$\zeta = \sqrt{\frac{\alpha}{\alpha_o}} \quad \zeta_o \tag{B-25}$$

Assume that the noise input to the loop results in the phase input being a wide sense stationary process with a power spectral density. The power spectral density of the phase at the output of the VCO is then given by

$$S_{\theta_o}(f) = |H(j\,2\,\pi\,f)|^2\,\Phi \tag{B-26}$$

The variance of the phase output, $\sigma_P{}^2$, is given by

$$\sigma_P{}^2 = \int_{-\infty}^{\infty} |H(j\,2\,\pi\,f)|^2\,\Phi\,df \tag{B-27}$$

If the power spectral density Φ is white, the evaluation of the integral of equation (B-27), for the transfer function given in (B-12), results in the variance being given by

$$\sigma_P{}^2 = \Phi\left[\frac{K\,T_2}{2\,T_1} + \frac{1}{2\,T_2}\right] \tag{B-28}$$

Choosing the loop filter time constants, T_1 and T_2, at the design point and substituting their values in equation (B-28), it is found that for the phase-lock loop without the limiter the phase variance is

$$\sigma_P{}^2 = \frac{\omega_{n_o}}{2\,\zeta_o}\,\Phi\left[\frac{A}{A_o}\,2\,\zeta_o^{\,2} + \frac{1}{2}\right] \tag{B-29}$$

For the phase-lock loop with the limiter preceding the phase detector, the phase variance is

$$\sigma_P^2 = \frac{\omega_{n_o}}{2\zeta_o} \; \Phi \left[\frac{\alpha}{\alpha_o} \; 2\zeta_o^2 + \frac{1}{2} \right] \tag{B-30}$$

The phase variance as given by equations (B-29) and (B-30) could be considered as that obtained from an equivalent "square cutoff" filter with a two sided bandwidth of 2 B_{LN}, in which case

$$\sigma_P^2 = 2\, B_{LN}\, \Phi \tag{B-31}$$

Thus, the two sided equivalent noise bandwidth for the phase-lock loop without a limiter is given by

$$2\, B_{LN} = \frac{\omega_{n_o}}{2\zeta_o} \left[\frac{A}{A_o} \; 2\zeta_o^2 + \frac{1}{2} \right] \tag{B-32}$$

The two sided equivalent noise bandwidth for the phase-lock loop with a limiter is given by

$$2\, B_{LN} = \frac{\omega_{n_o}}{2\zeta_o} \left[\frac{\alpha}{\alpha_o} \; 2\zeta_o^2 + \frac{1}{2} \right] \tag{B-33}$$

The phase spectral density for a sine wave signal plus narrow band gaussian noise, for high signal-to-noise noise ratios is given by:

$$\Phi = \frac{N_o}{2\,S}$$

where

N_o = noise power spectral density

S = signal power

For low signal-to-noise ratios, the more pessimistic value of phase spectral density of N_o/S is assumed.

B.2 ACQUISITION

The acquisition time of a phase-lock loop, with a proportional plus integral control filter, for a signal which is offset in frequency from the VCO center frequency by Ω was found by Viterbi[2] to be

$$t_{p_i} = \frac{\Omega^2}{2\zeta\ \omega_n^3} \tag{B-34}$$

This was derived by Viterbi from the non-linear equation of the phase-lock loop and gives the approximate pull in time when the offset is large and when there is no noise. At the design point the pull in time is given by

$$t_{p_i} = \frac{\Omega^2}{2\zeta_o\ \omega_{n_o}^{\ 3}} \tag{B-35}$$

At signal levels other than the design point, the pull in time for the phase-lock loop without a limiter will be

$$t_{p_i} = \frac{\Omega^2}{2\zeta_o\ \omega_{n_o}^{\ 3}\ \left(\frac{A}{A_o}\right)^2} \tag{B-36}$$

In the case of a phase-lock loop preceded by a limiter, the pull in time for signal levels other than the design point is approximately

$$t_{p_i} = \frac{\Omega^2}{2\zeta_o\ \omega_{n_o}^{\ 3}\ \left(\frac{\alpha}{\alpha_o}\right)^2} \tag{B-37}$$

It should be pointed out than Viterbi's formula applies for a noise free input and may not be accurate for a signal plus noise. However, the solution of the pull in

---

[2] Viterbi, A.J., "Acquisition and Tracking Behavior of Phase-Locked Loops," Jet Propulsion Laboratory External Publication 673, AD-234 166.

time for a signal plus noise does not appear amenable to solution, except by simulation techniques. According to Martin[3], the acquisition of the signal in a noisy environment, without a priori knowledge of the received frequency, has been satisfactory for SNR of 6 db in the loop noise bandwidth. The signal to noise ratios for the carrier loop and fine range tracking loop, (using the loop natural frequencies chosen in Volume 1, Sub-section 1.3.2.5), are 21 db and 31.7 db respectively, which are well above the 6 db figure given by Martin.

In Volume 1, it was found that by reducing the doppler uncertainty by 1/500 the acquisition time for the carrier loop and fine range loop were 1.5 seconds and 1.73 seconds, respectively. This time could be reduced by sweeping the VCO through the doppler uncertainty window. Frazier and Page[4] have postulated a formula for the sweep rate for 90 per cent probability of acquisition which holds for loop damping factors equal to or greater than 0.5. The Frazier and Page formula is:

$$R_{90}\ (\text{cps/sec}) = \frac{\left(\frac{\pi}{2} - 2.2\,\sigma_o\right)\left(0.9\,\frac{\alpha}{\alpha_o}\right)\omega_{n_o}{}^2}{2\pi\ (1 + \delta)} \qquad \text{(B-38)}$$

where

σ_o = calculated rms phase jitter

δ = overshoot

α = suppression factor

α_o = suppression factor at design point

ω_{n_o} = loop natural frequency at design point

[3] Martin, B.D., "The Pioneer IV Lunar Probe: A Minimum-Power FM/PM System Design," Jet Propulsion Laboratory Technical Report No. 32-215.

[4] Frazier, J.P. and Page, J., "Phase-lock Loop Frequency Acquisition Study," IRE Transactions on Space Electronics and Telemetry, Sept. 1962, pp 210-227.

For a loop damping factor of $\zeta = 0.707$, we obtain from Figure 24 of Frazier and Page, and $\delta = 0.4$. Thus, solving for R, in (B-38), at the design point it is found that for the carrier tracking loop with $\omega_{n_o} = 80$ rad/sec

$$R_{90}\ (\text{cps/sec}) = 1350\ \text{cps/sec}$$

It is seen, however, that a sweep rate of 1350 cps/sec at the design point, with $\omega_{n_o} = 80$ rad/sec, will result in a steady state phase error of $\pi/2.37$. The loop will not be capable of tracking this sweep rate, even in a noiseless environment. A great deal of caution must be exercised in the use of the Frazier and Page data and analysis. The data was obtained for a particular phase-lock loop with specific loop parameters. The design point or match point was chosen as the input signal to noise ratio which resulted in a signal to noise ratio of 8 db in the loop two-sided bandwidth. In order to choose a sweep rate for a 90 percent probability of acquisition, the same or similar simulation program as run by Frazier and Page will have to be run for the specific loop parameters and design point chosen.

B.3 TRANSIENT ANALYSIS AND MODULATION ERROR

B.3.1 Transient Error

Consider the loop with a closed loop transfer function as given in equation (B-12). The error response is given by

$$\frac{E(s)}{\theta_1(s)} = 1 - H(s) = \frac{\frac{T_i}{K} s^2}{1 + T_2 s + \frac{T_1}{K} s^2} \qquad \text{(B-39)}$$

If the incoming signal has a doppler rate D on the signal then

$$\theta_1(s) = D/s^3$$

and

$$E(s) = \frac{D}{s^3} \frac{T_1/K \ s^2}{1 + T_2 s + \frac{T_1}{K} s^2} \tag{B-40}$$

The steady state value of the error signal can be found by applying the final value theorem for Laplace transforms. We find the steady state error due to the frequency ramp is given by

$$e_{ss}(t) = \lim_{s \to o} s\, E(s) = \frac{\frac{D}{K}}{T_1} \tag{B-41}$$

For the phase-lock loop without the limiter, we find that substituting for T_1 from equation (B-14), that the steady state phase error is given by

$$e_{ss}(t) = \frac{A_o}{A} \frac{D}{\omega_{n_o}{}^2} \tag{B-42}$$

For the phase-lock loop with the limiter, we find that the steady state phase error is

$$e_{ss}(t) = \frac{\alpha_o}{\alpha} \frac{D}{\omega_{n_o}{}^2} \tag{B-43}$$

B.3.2 Modulation Error

The signal into the carrier tracking loop is the received signal with the ranging modulation on it. The input signal will be given by

$$x(t) = \sqrt{2}\, A \cos\left(\omega_o t + \theta(t)\right) \tag{B-44}$$

where

$\theta(t)$ is the modulation.

The modulation will cause an rms error in the output of the VCO, θ_o. Consider the modulation to be angle modulation by a tone, that is

$$\Theta(t) = \Delta\phi \sin \omega_r t$$

The power spectral density of the modulation is:

$$S_\Theta (f) = \frac{\Delta\phi^2}{4} \left[\delta (f + f_r) + \delta (f - f_r)\right] \tag{B-45}$$

The VCO output phase variance due to this modulation is given by

$$\sigma_P^2 = \int_{-\infty}^{\infty} \left|H (j\, 2\pi f)\right|^2 S_\Theta (f)\, df$$

$$= \int_{-\infty}^{\infty} \frac{\Delta\phi^2}{4} \left[\delta (f + f_r) + \delta (f - f_r)\right] \left| \frac{1 + T_2\, j\, 2\pi f}{1 + T_2\, j\, 2\pi f - \frac{T_1}{K} (2\pi f)^2} \right|^2 df$$

$$= \frac{\Delta\phi^2}{2} \cdot \frac{1 + (2\pi f_r)^2 \quad T_2{}^2}{\left(1 - \frac{T_1}{K} (2\pi f_r)^2\right)^2 + (2\pi f_r)^2 \; T_2{}^2} \tag{B-46}$$

$$\sigma_P{}^2 = \frac{\Delta\phi^2}{2} \cdot \frac{1}{1 + \dfrac{\left(\frac{T_1}{K}\, 2\pi f_r\right)^2 \left[(2\pi f_r)^2 - \frac{2K}{T_1}\right]}{1 + (2\pi f_r T_2)^2}} \tag{B-47}$$

Equation (B-47) gives the phase variance at the VCO output for a single tone phase modulating the carrier. If the phase modulation were a number of discrete tones, then the modulation can be represented by

$$\Theta(t) = \sum_{i=1}^{n} \Delta\, \phi_i \sin 2\pi\, f_i\, (t)$$

The total phase variance is then given by

$$\sigma_P^2 = \sum_{i=1}^{n} \frac{\Delta \phi_i^2}{2} \cdot \frac{1}{\dfrac{1 + \left(\dfrac{T_1}{K}\, 2\pi f_i\right)^2 \left[\left(2\pi f_i\right)^2 - \dfrac{2K}{T_1}\right]}{1 + (2\pi f_i\, T_2)^2}} \qquad \text{(B-48)}$$

B.4 INTERFERENCE ERROR

The results of the previous sub-section can be used to find the phase variance in the VCO output due to an interfering sine wave of small amplitude. Assume that the carrier phase-lock loop is tracking a carrier which is given by

$$x(t) = A \cos\left(\omega_c t + \theta(t)\right) \qquad \text{(B-49)}$$

An interfering sine wave is assumed to occur at frequency $\omega_c + \omega_i$. The interfering sine wave can be written as

$$y(t) = B \cos\left[\left(\omega_c + \omega_i\right) t + \psi\right] \qquad \text{(B-50)}$$

The interfering sine wave can be expressed as the sum of a pair of symmetrical sidebands and a pair of anti-symmetrical sidebands (see Goldman[5] p 172), that is

$$y(t) = y_s(t) + y_a(t) \qquad \text{(B-51)}$$

The symmetrical sidebands are

$$y_s(t) = \frac{B}{2} \cos\left[\left(\omega_c + \omega_i\right) t + \psi\right] + \frac{B}{2} \cos\left[\left(\omega_c - \omega_i\right) t - \psi\right] \qquad \text{(B-52)}$$

[5] Goldman, S., "Frequency Analysis Modulation and Noise," McGraw Hill Book Company.

The anti-symmetrical components are

$$y_a(t) = \frac{B}{2} \cos\left[\left(\omega_c + \omega_i\right) t + \psi\right] - \frac{B}{2} \cos\left[\left(\omega_c - \omega_i\right) t - \psi\right] \qquad \text{(B-53)}$$

The superposition of the symmetrical components and the carrier gives rise to pure amplitude modulation and, for the amplitude B much less than A, is negligible. The superposition of the anti-symmetrical components and the carrier gives rise to an approximately pure angle-modulated carrier.

That is:

$$x(t) + y_a(t) = A \cos\left[\omega_c t + \theta(t) + \frac{2B}{A} \sin(\omega_i t + \psi)\right] \qquad \text{(B-54)}$$

The phase variance at the VCO output due to the interfering sine wave can be found from equation (B-47) and is:

$$\sigma_P^2 = 2\left(\frac{B}{A}\right)^2 \frac{1}{\dfrac{1 + \left(\dfrac{T_1}{K} 2\pi f_i\right)^2 \left[\left(2\pi f_i\right)^2 - \dfrac{2K}{T_1}\right]}{1 + (2\pi f_i T_2)^2}} \qquad \text{(B-55)}$$

Appendix C

SPECTRUM ANALYSIS

C.1 SIDETONE SPECTRUM

In a sidetone ranging system in which a number of discrete tones are used to phase modulate the carrier, the modulation index of each tone must be chosen so as to place the desired power in the rf spectral components. If a receiver as shown in Figure 1-2 of Volume 1 is used to coherently track the carrier, then Figure C-1 is an equivalent of this receiver.

The input signal to the carrier-tracking phase-lock loop is given by:

$$x(t) = \sin\left(\omega_c t + \psi(t)\right) \tag{C-1}$$

where $\psi(t)$ = sidetone modulation

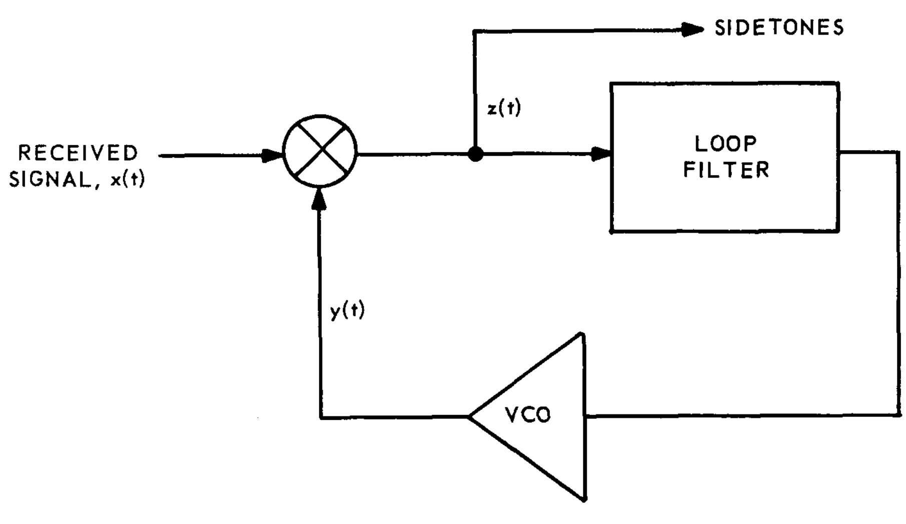

Figure C–1. Equivalent of Sidetone Carrier Tracking Loop

$$\psi(t) = \sum_{i=1}^{k} \Delta \phi_i \sin \omega_i t \tag{C-2}$$

the output of the VCO is assumed to be

$$y(t) = \cos \omega_c t \tag{C-3}$$

The VCO output phase error due to the ranging modulation can be found from equation (B-48) of Appendix B. From this equation, it is seen that if the sidetone modulation frequency is much larger than the loop natural frequency, then at the design point, the phase variance is given by the following approximation

$$\sigma_p^2 \approx \sum_{i=1}^{k} \frac{1}{2} \left(\frac{\Delta \phi_i}{\omega_i} \right)^2 \tag{C-4}$$

For small modulation indices, the total phase variance will be negligible and the phase-lock loop will track the carrier with negligible error.

The received spectrum for the k sidetones phase modulating a carrier has been derived by Giacolleto[6] and is given by

$$x(t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_k=-\infty}^{\infty} J_{n_1}(\Delta \phi_1) J_{n_2}(\Delta \phi_2) \cdots J_{n_k}(\Delta \phi_k) \tag{C-5}$$

From equation (C-5), we see that the carrier component at the phase-lock loop is given by

$$\text{carrier} = J_0(\Delta \phi_1) J_0(\Delta \phi_2) \cdots J_0(\Delta \phi_k) \sin \omega_c t \tag{C-6}$$

The power in the carrier component divided by the total power is:

[6] Giacolleto, L. J., "Generalized Theory of Multitone Amplitude and Frequency Modulation," Proc. of IRE, Vol 35, Jul 1947, pp 680-693.

$$\frac{P_C}{P_T} = \prod_{i=1}^{k} \left[J_0 \; (\Delta \phi_i) \right]^2 \tag{C-7}$$

The output of the phase detector is given by

$$z\,(t) = x(t)\; y\,(t) = \frac{1}{2} \left[\sin \left(2\, \omega_c\, t + \psi\, (t) \right) + \sin \psi\, (t) \right] \tag{C-8}$$

Neglecting the $2\omega_c$ term, which is well outside the band of interest, the following expression for the output is obtained:

$$z\,(t) = \frac{1}{2} \left[\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_k=-\infty}^{\infty} J_{n_1} (\Delta \phi_1)\, J_{n_2} (\Delta \phi_2) \cdots J_{n_k} (\Delta \phi_k)\; \sin \left(\sum_{i=1}^{k} n_i\, \omega_i\; t \right) \right] \tag{C-9}$$

Thus, the j-th sidetone will be equal to:

$$= \frac{1}{2}\; J_0\; (\Delta \phi_1)\; J_0\; (\Delta \phi_2) \cdots J_1 (\Delta \phi_j) \cdots J_0 (\Delta \phi_k) \sin \omega_j t \tag{C-10}$$

The useful power in the j-th sidetone, which can be used for tracking, divided by the total power is then given by

$$\frac{P_{\omega_j}}{P_T} = \frac{1}{2} \left[J_1\; (\Delta \phi_j) \prod_{\substack{i=1 \\ i \neq j}}^{k} J_0\; (\Delta \phi_i) \right]^2 \tag{C-11}$$

All other modulation products will be wasted power. It may be possible to track higher harmonics of the sidetone; however, it can be shown that the spectrum of (C-9) will only contain modulation products which satisfy the relation.

$$\sum_{i=1}^{k} n_i \quad \text{is odd} \tag{C-12}$$

Thus, the second harmonics of the sidetone will be absent. The third harmonic of the j-th tone will be:

$$\frac{1}{2} J_0 (\Delta \phi_1) J_0 (\Delta\phi_2) - - - J_3 (\Delta\phi_j) - - - J_0 (\Delta\phi_k) \sin 3 \omega_j t$$

The power in this term compared to the first harmonic will be the ratio:

$$\frac{P_{3\omega j}}{P_{\omega_j}} = \left[\frac{J_3 (\Delta\phi_j)}{J_1 (\Delta\phi_j)}\right]^2 \tag{C-13}$$

For low modulation indices, the power in the higher tone will be insufficient for tracking.

C.2 PN SPECTRAL ANALYSIS

When a maximal length sequence is used as the ranging code to modulate the carrier, the vehicle receiver must go through an acquisition stage before commencing tracking operations. An examination of the signal spectrum at each phase of system operation is required. Some of the spectra of more interest in the pure and hybrid PN systems discussed are as follows:

1. Event marker spectrum
2. Clock pulse spectrum
3. RF spectrum of clock pulse combined with PN

C.2.1 Event Marker Spectrum

Suppose the event marker is transmitted as n bits of binary information, which phase modulates a carrier. When the modulation is bi-phase in the sense that 0 and π are used as phase shifts, a DSBSC receiver can be used to coherently demodulate the event marker. If this is not the case and phase modulation is used, a phase-lock loop receiver may be considered for use in demodulating the event marker. The power density spectrum of an n bit binary word is of interest to determine the effect of the modulation on the carrier tracking capability of the phase-lock loop receiver.

First, the autocorrelation function of n-binary bits will be obtained with the assumption that:

1. Each binary bit is independent of the other bits
2. The probability is 1/2 that the bit is 0 or 1.

It is to be noted here that the event marker is considered to be performing a more general function than considered in Sub-section 1.3.3.2.1 of Volume 1. Here, the event marker specifies more than one of the possible states at which the vehicle receiver slave PNG ambiguity resolution process may commence.

The method of derivation is that of deriving the autocorrelation function of the n-bit word by considering a shifted version of the word by an amount τ with respect to the original n bit word. If t_o denotes the bit duration, the autocorrelation function,

$$\varphi(\tau) = \frac{t_o - \tau}{2} + (n-1)\left(\frac{\tau}{4} + \frac{t_o - \tau}{2}\right),$$

when $\quad 0 < \tau < t_o$

Similarly, when $t_o < \tau < 2\,t_o$,

$$\varphi(\tau) = \frac{1}{4}\left\{(2\,t_o - \tau) + (n-2)\,t_o\right\}$$

It also follows that, when $2\,t_o < \tau < 3\,t_o$,

$$\varphi(\tau) = \frac{1}{4}\left\{(3\,t_o - \tau) + (n-3)\,t_o\right\}, \text{ etc.}$$

As an example for the case n = 2,

$$\varphi\ (\tau = 0) = t_o$$

$$\varphi(\tau = t_o) = \frac{t_o}{4}$$

and

$$\varphi(\tau = 2\ t_o) = 0.$$

From the Wiener-Khintchine Theorem, the power density spectrum is obtained, for this case, by taking the Fourier transform of $\varphi\ (\tau)$.

$$\Phi\ (\omega) = \int_{-2\ t_o}^{2\ t_o} \varphi(\tau)\ e^{j\ \omega\ \tau}\ d\tau$$

$$= \frac{-\cos^2 \omega t_o\ - \cos\ \omega t_o + 2}{\omega^2} \tag{C-14}$$

$$\Phi\ (\omega = o) = \frac{3}{2}\ t_o^{\ 2}$$

$$\Phi \left(\omega = \frac{2\ \pi k}{t_o}\right) = 0 \quad \text{k integer } (\neq 0)$$

$$\Phi \left(\omega\ = \frac{(2\ k + 1)}{t_o}\right) = \frac{2\ t_o^{\ 2}}{(2\ k + 1)^2\ \pi^2}, \text{ k integer}$$

The effect of the event marker on the phase-lock loop's tracking performance during acquisition can be determined by computing the variance of the loop phase error with $\Phi\ (\omega)$ as the input power density spectrum.

C.2.2 Clock Pulse Spectrum

The clock pulse, $c(t)$, of interest in this discussion is a periodic square wave with amplitude ± 1, and period $T = 2a$. If $\omega_o = \frac{2\pi}{T} = \frac{\pi}{a}$ the Fourier coefficient of the clock pulse is

$$c_n = \frac{1}{T} \int_0^T c(t)\, e^{-jn\omega_o t}\, dt$$

$$= 0 \text{ if n is even}$$

$$= \frac{2}{jn\pi} \text{ if n is odd}$$

Hence,

$$c(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin n\omega_o t$$

The autocorrelation function of the clock pulse $c(t)$ is given by,

$$\varphi(\tau) = \frac{a - 2\tau}{a} \quad \text{for } 0 < \tau < a$$

and

$$\varphi(\tau) = \frac{2\tau - 3a}{a} \quad \text{for } a < \tau < 2a$$

Taking the Fourier transform of $\varphi(\tau)$ the coefficients of the power spectral density $s(f)$ can be obtained, and

$$s(f) = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{4}{n^2\pi^2}\, \delta\,(f - n\, f_o)$$

where

$$f_o = \frac{1}{T}$$

The clock pulse does not have a DC component, and the total effect on the phase-lock loop tracking performance can be evaluated by examining the variance of the loop phase error with s(f) as the input spectral density.

The spectral components of a carrier which is phase modulated by the clock pulse can be readily obtained. Let the deviation be denoted by Φ then the phase modulated signal is,

$$M(t) = A_c \cos\left(\omega_c t + \Phi \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin \frac{2\pi n t}{T}\right)$$

$$= A_c \operatorname{Re}\left(e^{j\omega_c t} \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} e^{j\Phi \frac{4}{n\pi} \sin \frac{2\pi n t}{T}}\right)$$

Using the identity,

$$e^{jX\sin\theta} = \sum_{m=-\infty}^{\infty} J_m(X)\, e^{jm\theta}$$

$$M(t) = A_c \operatorname{Re}\left(e^{j\omega_c t} \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\sum_{m=-\infty}^{\infty} J_m\left(\Phi \frac{4}{n\pi}\right) e^{jm\frac{2\pi n t}{T}}\right)\right), \tag{C-15}$$

where $J_m(X)$ are Bessel functions of the first kind with integer suffix. The expression above is useful in examining the frequency components of a carrier which is phase modulated by a clock pulse.

C.2.3 RF Spectrum of a Clock Pulse Combined with PN.

The problem of interest here is that of a transmission system in which the basic waveforms are,

$$h_1(t) = \cos \omega_c t$$

and

$$h_2(t) = -\cos \omega_c t$$

One or the other of the two waveforms is selected by a maximal sequence generator every $t_o = \frac{2\pi n}{\omega_c}$ seconds, where n is an integer. The power density spectrum for this case was given in Sub-section 1.3.2.2.2 of Volume 1.

The transmission system described above is slightly modified by combining a clock pulse with the maximal length sequence. In the modified system, the combined sequence will select one or the other of the waveforms for transmission. Under consideration here is the relation between the clock pulse period, T, and the PN bit duration, t_o, and its effect on the power density spectrum.

First, the case to be considered is when $T = 2t_o$. In this case, since the period of the maximal length sequence, p, is odd, the period of the combined sequence will be 2p. Furthermore, the joint probability distribution of the PN states separated by a given number of PN bits will also be modified in the combined sequence. These factors will influence the fine line structure of the power density spectrum, but the envelope of the power density spectrum in this case will be proportional to the magnitude square of,

$$
\begin{aligned}
H_1(f) &= \int_0^{t_o} \cos \omega_c t \, e^{j\omega t} dt \\
&= \frac{\omega(-1 + e^{j\omega t_o})}{j(\omega^2 - \omega_c^2)}
\end{aligned}
\tag{C-16}
$$

and application of L'Hospital's Rule, one finds,

$$
H_1(f) \Big|_{\omega = \omega_c} = \frac{t_o}{2}
$$

When the PN bit duration, t_o, is equal to the clock pulse period, T, the envelope of the power density spectrum will be proportional to the magnitude square of,

$$H_1(f) = \int_0^{t_o/2} \cos \omega_c t \, e^{j\omega t} \, dt - \int_{t_o/2}^{t_o} \cos \omega_c t \, e^{j\omega t} \, dt \qquad \text{(C-17)}$$

$$= \frac{1}{4j} \frac{(\omega - \omega_c) \left(e^{j(\omega+\omega_c) t_o/2} - 1\right)^2 + (\omega + \omega_c) \left(e^{j(\omega-\omega_c) t_o/2} - 1\right)^2}{\omega^2 - \omega_c^2}$$

In this case, it is found by application of L'Hospital's rule again, that the power density function vanishes at the carrier frequency.

Equations C-16 and C-17 show that when the clock pulse period, $T = 2\ t_o$, the component at the carrier is present but then $T - t_o$, the carrier component, vanishes.

C.3 SIDETONE HYBRID SPECTRAL ANALYSIS

In the sidetone hybrid systema pseudo-noise sequence is added to a discrete tone and the total signal then phase modulates the carrier. The received signal is then:

$$x(t) = \cos(\omega_c t + \Delta\theta \Phi(t) + \Delta\phi \sin 2\pi f_r t) \qquad \text{(C-18)}$$

In order to derive the spectrum of the received signal, consider first a carrier which is bi-phase modulated by a random sequence of +1, -1. The sequence $\Phi(t)$ may be the sequence of +1, -1 levels or the same sequence modulo two added to a clock. In either case the power spectral density of the sequency $\Phi(t)$ will be denoted by $S_\phi(f)$.

The carrier bi-phase modulated by the sequence $\Phi(t)$ is then

$$x(t) = \cos(\omega_c t + \Delta\theta\, \Phi(t) + \psi) \qquad \text{(C-19)}$$

where ψ is a random variable uniformly distributed between 0 and 2π. Assume that $\Phi(t)$ is independent of the carrier which will be the case unless the sequence is generated coherently with the carrier.

The autocorrelation function of the random process x (t) is

$$R_X(\tau) = E\left\{X(t)\, X(t+\tau)\right\}$$

$$= E\left\{\cos(\omega_c t + \Delta\theta\, \Phi(t) + \psi)\cos(\omega_c(t+\tau) + \Delta\theta\, \Phi(t+\tau) + \psi)\right\}$$

$$= E\left\{\frac{1}{2}\cos\left(\omega_c(2t+\tau) + \Delta\theta\, \Phi(t) + \Delta\theta\, \Phi(t+\tau) + 2\psi\right)\right\}$$

$$+ E\left\{\frac{1}{2}\cos\left(\omega_c\tau + \Delta\theta\, [\Phi(t) - \Phi(t+\tau)]\right)\right\}$$

The first term above is zero, the second term is

$$= E\left\{\frac{1}{2}\cos\omega_c\tau\, \cos\left(\Delta\theta\, [\Phi(t) - \Phi(t+\tau)]\right)\right.$$

$$\left. - \frac{1}{2}\sin\omega_c\tau\, \sin\left(\Delta\theta\, [\Phi(t) - \Phi(t+\tau)]\right)\right\}$$

$$R_X(\tau) = E\left\{\frac{1}{2}\cos\omega_c\tau\, \cos\Delta\theta\,\Phi(t)\, \cos\Delta\theta\,\Phi(t+\tau)\right.$$

$$+ \frac{1}{2}\cos\omega_c\tau\, \sin\Delta\theta\,\Phi(t)\, \sin\Delta\theta\,\Phi(t+\tau)$$

$$- \frac{1}{2}\sin\omega_c\tau\, \sin\Delta\theta\,\Phi(t)\, \cos\Delta\theta\,\Phi(t+\tau)$$

$$\left. - \frac{1}{2}\sin\omega_c\tau\, \cos\Delta\theta\,\Phi(t)\, \sin\Delta\theta\,\Phi(t+\tau)\right\}$$

If bi phase modulation is used then

$$\begin{aligned} \cos\Delta\theta\,\Phi(t) &= \cos\Delta\theta \\ \sin\Delta\theta\,\Phi(t) &= \Phi(t)\sin\theta \end{aligned} \qquad \text{(C-20)}$$

We find then the autocorrelation function is:

$$R_X(t) = \frac{\cos^2\Delta\theta}{2}\cos\omega_c\tau + \frac{1}{2}(\sin^2\Delta\theta\, \cos\omega_c\tau)\, R_\phi(\tau) \qquad \text{(C-21)}$$

where $R_\phi(\tau)$ = autocorrelation function of the sequence.

The power spectral density can be found from the Wiener-Khintchine relation

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

The solving the integral we obtain

$$S_x(f) = \frac{\cos^2 \Delta\theta}{2}\left[\frac{\delta(f - f_c)}{2} + \frac{\delta(f + f_c)}{2}\right. \qquad \text{(C-22)}$$
$$+ \frac{\sin^2\theta}{2}\left[\frac{\delta(f - f_c)}{2} + \frac{\delta(f + f_c)}{2}\right] * S_\phi(f)$$

where * denotes convolution.

In Figure C-2 a typical spectrum of a random sequence is shown along with the spectrum of the carrier bi phase modulated by the sequence. From equation (C-22) we see that if the modulation index $\Delta\theta$ is $\frac{\pi}{2}$, then there is no discrete carrier component. If the modulation index is less than $\frac{\pi}{2}$ then there will be a discrete carrier component. The percentage of total power in the carrier is 100 $\cos^2 \Delta\theta$ and the percentage of total power in the spread spectrum is 100 $\sin^2 \Delta\theta$. From the above, we see that if the local signal fed to the mixer of the receiver is bi-phase modulated by a random sequence with a modulation index of $\frac{\pi}{2}$, then an interfering sine wave will be spread out.

If we consider the random sequence as the modulo-two addition of a random sequence of pulses of amplitude +1 or −1 which are equally likely and of length b and the clock to have a frequency of $\frac{1}{b}$ (that is, the period of the clock is equal to the random sequence bit length). Then the power spectral density of this combination has been found by Springett[7] to be

$$S_\phi(f) = 2b\,\frac{\sin^4\left(\frac{\pi f b}{2}\right)}{\left(\frac{\pi f b}{2}\right)^2} \qquad \text{(C-23)}$$

This spectrum is the one which is shown in Figure C-2a.

[7] Springett, J.C., "Pseudo-Random coding for bit and Word Synchronization of PSK Data Transmissions Systems," International Telemetering Conference, London, Sept 1963, Vol 1 of the proceedings, pp 410-422

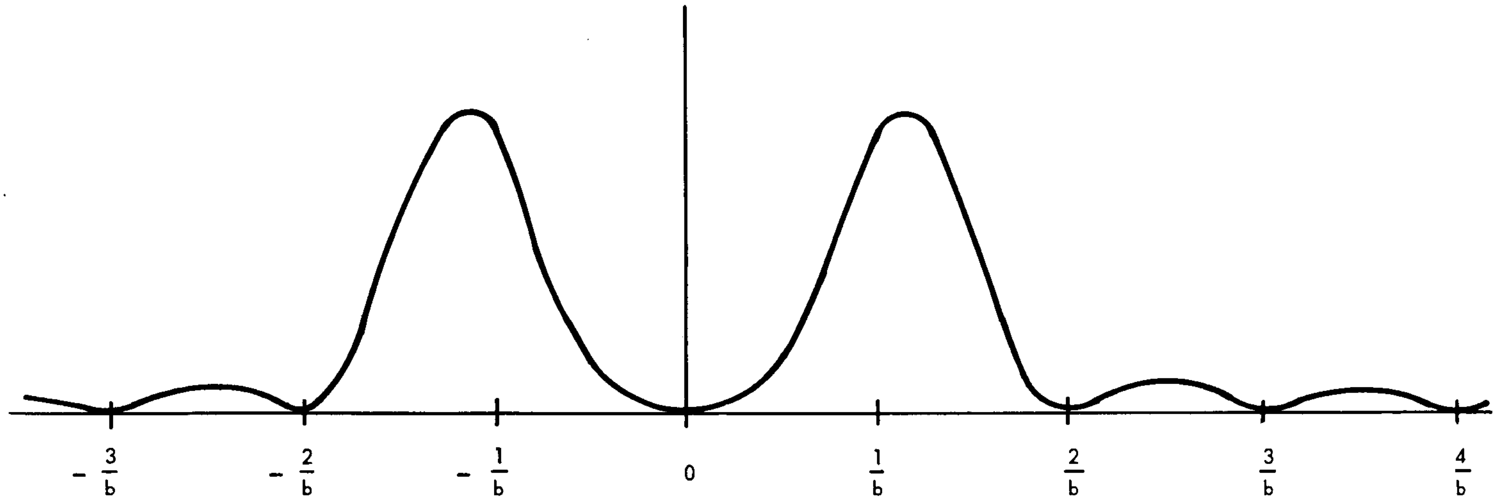

Figure C–2a. PN Spectrum

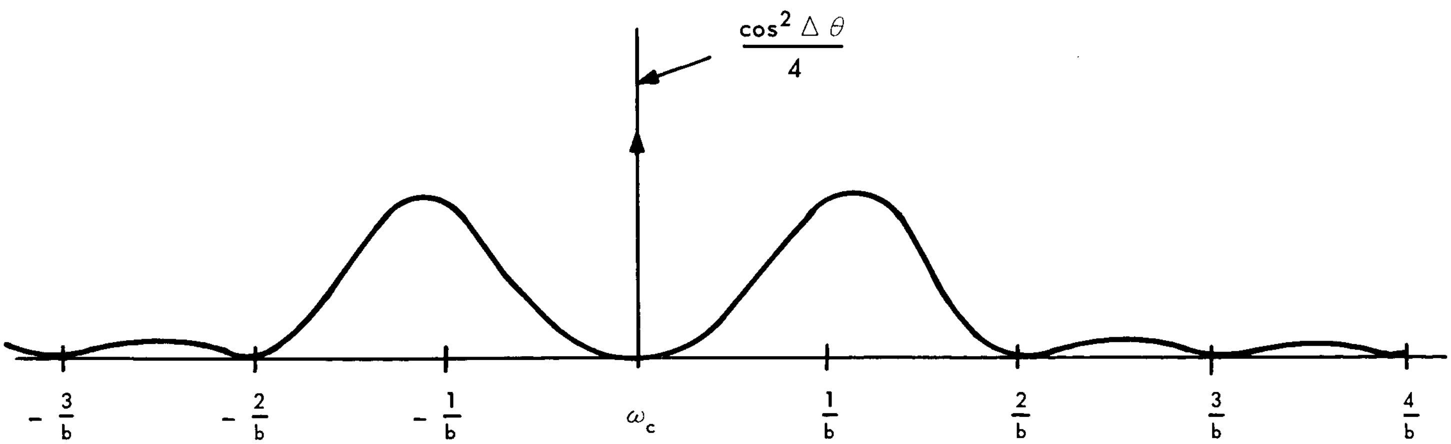

Figure C–2b. RF Spectrum

Figure C–2. Spectral Density

We now consider the spectrum of the signal as given in equation (C-18), where the random sequence plus a discrete tone angle modulates the carrier. We assume that the carrier, random sequence, and tone are independent. Equation (C-18) can be rewritten as:

$$x(t) = \cos(\omega_c t + \Delta\theta\,\Phi(t) + \Delta\phi \sin(2\pi f_r t + \beta) + \psi) \qquad \text{(C-24)}$$

where β and ψ are independent and uniformly distributed from 0 to 2π. We write (C-24) as:

$$x(t) = \cos(\omega_c t + \Delta\theta\,\Phi(t) + \psi)\cos(\Delta\phi \sin(2\pi f_r t + \beta))$$

$$+ \sin(\omega_c t + \Delta\theta\,\Phi(t) + \psi)\sin(\Delta\phi \sin(2\pi f_r t + \beta))$$

If the random sequence is bi phase modulation then using the relations (C-20) we find

$$x(t) = \left[\cos\Delta\theta\cos(\omega_c t + \psi) + \Phi(t)\sin\Delta\theta\sin(\omega_c t + \psi)\right]\cos(\Delta\phi\sin(2\pi f_r t + \beta))$$

$$+ \left[\cos\Delta\theta\sin(\omega_c t + \psi) + \Phi(t)\sin\Delta\theta\cos(\omega_c t + \psi)\right]\sin(\Delta\phi\sin(2\pi f_r t + \beta))$$

Let

$$A(t) = \cos\Delta\theta\cos(\omega_c t + \psi) + \Phi(t)\sin\Delta\theta\sin(\omega_c t + \psi)$$

$$B(t) = \cos\Delta\theta\sin(\omega_c t + \psi) + \Phi(t)\sin\Delta\theta\cos(\omega_c t + \psi)$$

Then

$$\begin{aligned} R_A(\tau) &= R_B(\tau) = \frac{\cos^2\Delta\theta}{2}\cos\omega_c\tau + \frac{1}{2}(\sin^2\Delta\theta\cos\omega_c\tau)\,R_\phi(\tau) \\ R_{AB}(\tau) &= R_{BA}(\tau) = \frac{\cos^2\Delta\theta}{2}\sin\omega_c\tau + \frac{1}{2}(\sin^2\Delta\theta\sin\omega_c\tau)\,R_\phi(\tau) \end{aligned} \qquad \text{(C-25)}$$

Making use of the identities

$$\cos(\Delta\phi\sin(2\pi f_r t + \beta)) = J_0(\Delta\phi) + 2\sum_{n=1}^{\infty} J_{2n}(\Delta\phi)\cos\left[2n(2\pi f_r t + \beta)\right]$$

$$\sin(\Delta\phi\sin(2\pi f_r t + \beta)) = 2\sum_{n=1}^{\infty} J_{2n-1}(\Delta\phi)\sin\left[(2n-1)(2\pi f_r t + \beta)\right]$$

The signal can then be written as

$$x(t) = A(t) \left\{ J_0(\Delta\phi) + 2 \sum_{n=1}^{\infty} J_{2n}(\Delta\phi) \cos \left[2n(2\pi f_r t + \beta) \right] \right\}$$
$$+ B(t) \left\{ 2 \sum_{n=1}^{\infty} J_{2n-1}(\Delta\phi) \sin \left[(2n-1)(2\pi f_r t + \beta) \right] \right\} \tag{C-26}$$

The autocorrelation function of x (t) is

$$R_x(\tau) = E \left\{ X(t) \, X(t+\tau) \right\}$$

$$R_x(\tau) = R_A(\tau) \left\{ J_0^2(\Delta\phi) + 4 \sum_{n=1}^{\infty} J_{2n}^2(\Delta\phi) \frac{\cos 2n \omega_r \tau}{2} \right\}$$
$$+ R_B(\tau) \left\{ 4 \sum_{n=1}^{\infty} J_{2n-1}^2(\Delta\phi) \frac{\cos(2n-1)\omega_r \tau}{2} \right\} \tag{C-27}$$

From this autocorrelation function, we find the power spectral density is given by:

$$S_x(f) = J_0^2(\Delta\phi) \left[\frac{\cos^2\Delta\theta}{2} + \frac{\sin^2\Delta\theta}{2} S_\phi(f) \right] * \left[\frac{\delta(f-f_c)}{2} + \frac{\delta(f+f_c)}{2} \right]$$
$$+ 2 \left[\frac{\cos^2\Delta\theta}{2} + \frac{\sin^2\Delta\theta}{2} S_\phi(f) \right] * \left[\frac{\delta(f-f_c)}{2} + \frac{\delta(f+f_c)}{2} \right] * \left[\sum_{n=1}^{\infty} J_n^2(\Delta\phi) \right.$$
$$\left. \left\{ \frac{\delta(f-nf_r)}{2} + \frac{\delta(f+nf_r)}{2} \right\} \right] \tag{C-28}$$

The ratio of power in each component to the total power in the spectrum can be found from equation C-28, and are

Component	P_c/P_T
carrier	$\cos^2 \Delta\theta \, J_0^2 (\Delta\phi)$
f_r	$2 \cos^2 \Delta\theta \, J_1^2 (\Delta\phi)$
$2 \ f_r$	$2 \cos^2 \Delta\theta \, J_2^2 (\Delta\phi)$
.	.
.	.
.	.
$k \ f_r$	$2 \cos^2 \Delta\theta \, J_k^2 (\Delta\phi)$
.	.
.	.
.	.
PN spectrum	$\sin^2 \theta$

Thus, by proper selection of the modulation indices $\Delta\theta$ and $\Delta\phi$, the desired power can be obtained in the carrier component and fine ranging tone.

Appendix D

PSEUDO-NOISE ANALYSIS

D.1 ANALYSIS OF DSBSC RECEIVER

The receiver under consideration is shown as part of the ranging receiver in Figure 1-4 (Volume 1), and separately in Figure D-1. Under consideration, is the derivation of the input to the loop filter in terms of the input DSBSC signal. Sub-section D. 1. 1 deals with signal only and Sub-section D. 1. 2 deals with an analysis of the case of narrow band noise in the presence of a DSBSC input signal.

D. 1. 1 Analysis Without Noise

The input DSBSC signal is denoted by $f(t) \cos \omega_c t$, and the input to the in-phase arm of the DSBSC receiver is,

$$f(t) \cos \omega_c t \cos (\omega_o t + \theta) = \frac{f(t)}{2} \left[\cos \left((\omega_c + \omega_o) t + \theta\right) + \cos \left((\omega_c - \omega_o) t - \theta\right)\right]$$

where θ is a phase angle.

Low pass filtering in the in-phase arm gives,

$$\frac{f(t)}{2} \cos \left((\omega_c - \omega_o) t - \theta\right) \tag{D-1}$$

In the quadrature arm, the same operation gives a low pass filtered output,

$$\frac{f(t)}{2} \sin \left((\omega_c - \omega_o) t - \theta\right) \tag{D-2}$$

The input to the loop filter is given by the product of equations (D-1) and (D-2),

$$\frac{f^2(t)}{4} \quad \sin \left((\omega_c - \omega_o) \ t - \theta \right) \cos \left((\omega_c - \omega_o) \ t - \theta \right)$$

$$= \frac{f^2(t)}{8} \quad \sin \left(2 \ (\omega_c - \omega_o) \ t - 2 \ \theta \right) \qquad \text{(D-3)}$$

When the loop is locked

$$\omega_c = \omega_o$$

so the input to the loop filter in this case will be,

$$- \frac{f^2(t)}{8} \ \sin \ 2 \ \theta \qquad \text{(D-4)}$$

In the case of interest, f(t) will be a bi-phase modulation, so

$$f(t) = (-1)^{r(t)}$$

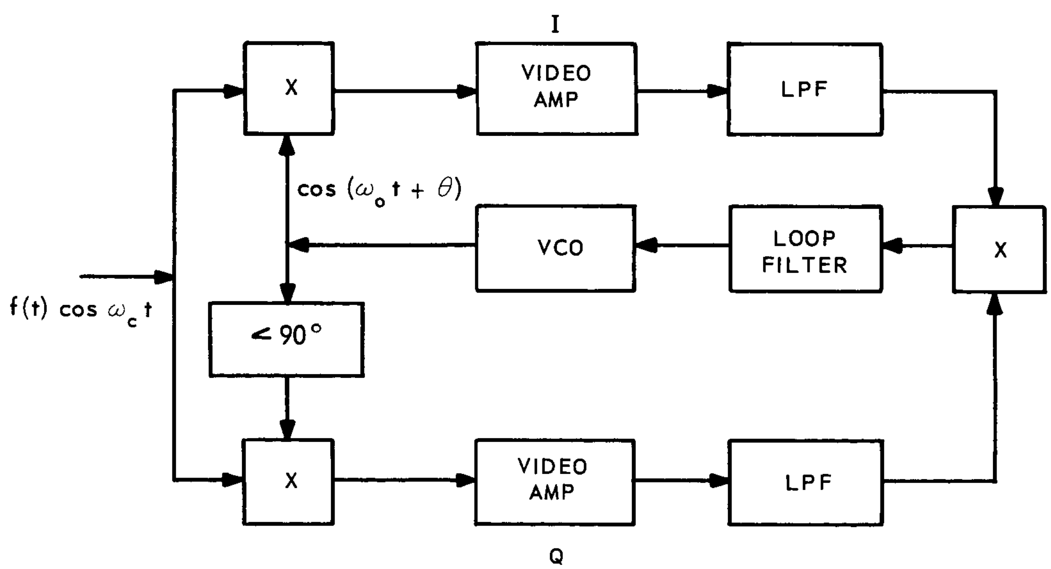

Figure D–1. DSBSC Receiver

Where r(t) takes values 0 and 1 and, in the case of the system shown in Figure 1-4 (Volume 1) will be the event marker or the command transmission from the ground station. In the case of command transmission during ranging operations, the ground station transmits this by combining r(t) with the maximal length ranging sequence. In this case, equation (D-4) becomes,

$$- \frac{\sin 2\theta}{8} \tag{D-5}$$

Comparing the result obtained with the case of the phase-lock loop, the phase error signal into the loop filter is a quantity proportional to the sine of twice the phase error. This may be a fundamental disadvantage of the DSBSC receiver of Figure D-1 in comparison to a phase-lock loop for the following reason. The DSBSC tracking loop will cease to track, in the sense that the phase error will be $\frac{\pi}{2}$, at half the phase error of the phase-lock loop.

D. 1. 2 Input with Narrow Band Noise

In the presence of narrow band gaussian noise, the input signal to the DSBSC receiver is,

$$f(t) \cos \omega_c t + x(t) \cos \omega_c t + y(t) \sin \omega_c t$$

where x(t) and y(t) are independent random variables with gaussian distributions, and

$$\overline{x}(t) = \overline{y}(t) = o$$

$$\overline{x^2(t)} = \overline{y^2(t)} = \sigma^2$$

The input to the in-phase arm is,

$$\left\{ \left(f(t) + x(t)\right) \cos \omega_c t + y(t) \sin \omega_c t \right\} \cos \left(\omega_o t + \theta\right)$$

$$= \frac{f(t) + x(t)}{2} \left[\cos \left((\omega_c + \omega_o) t + \theta\right) + \cos \left((\omega_c - \omega_o) t - \theta\right)\right]$$

$$+ \frac{y(t)}{2} \left[\sin \left((\omega_c + \omega_o) t + \theta\right) + \sin \left((\omega_c - \omega_o) t - \theta\right)\right]$$

The low pass filter removes the high frequency components, and becomes,

$$\frac{f(t) + x(t)}{2} \cos \Big((\omega_c - \omega_o) t - \theta \Big) + \frac{y(t)}{2} \sin \Big((\omega_c - \omega_o) t - \theta \Big) \qquad \text{(D-6)}$$

Similarly, in the quadrature arm, the corresponding quantity is,

$$- \Big\{ \big(f(t) + x(t) \big) \cos \omega_c t + y(t) \sin \omega_c t \Big\} \sin (\omega_o t + \theta)$$

and when low pass filtered,

$$- \frac{f(t) + x(t)}{2} \sin \Big((\omega_o - \omega_c) t + \theta \Big) - \frac{y(t)}{2} \cos \Big((\omega_o - \omega_c) t + \theta \Big) \qquad \text{(D-7)}$$

The input to the loop filter is the product of equations (D-6) and (D-7),

$$\left(\frac{f(t) + x(t)}{2} \right)^2 \sin \Big((\omega_c - \omega_o) t - \theta \Big) \cos \Big((\omega_c - \omega_o) t - \theta \Big)$$

$$+ \frac{(f(t) + x(t)}{2} \cdot \frac{y(t)}{2} \sin^2 \Big((\omega_c - \omega_o) t - \theta \Big) - \frac{f(t) + x(t)}{2} \frac{y(t)}{2} \cos^2 \Big((\omega_c - \omega_o) t - \theta \Big)$$

$$- \left(\frac{y(t)}{2} \right)^2 \sin \Big((\omega_c - \omega_o) t - \theta \Big) \cos \Big((\omega_c - \omega_o) t - \theta \Big) \qquad \text{(D-8)}$$

and it has an average value,

$$\frac{f^2(t)}{8} \sin \Big(2 (\omega_c - \omega_o) t - 2\theta \Big)$$

D.1.3 Performance of DSBSC Receiver in the Presence of a PM Signal

When the input signal to the DSBSC receiver is given by,

$$\cos \Big((\omega_c t + \alpha(t) \Big) ,$$

where $\alpha(t)$ is a phase modulation, the input signal cannot be put in the form of the DSBSC signal considered in Sub-section D.1.1. In this case, it will be shown that the DSBSC receiver will not yield the sine of the phase error term derived previously in Sub-section D.1.1.

The input to the in-phase arm of the receiver is

$$\cos\ (\omega_c t + \alpha(t)\ \ \cos\ \ (\omega_o t + \theta)$$

and the output of the low pass filter is,

$$\frac{1}{2}\cos\ \Big((\omega_c - \omega_o)\, t + \alpha(t) - \theta\Big) \tag{D-9}$$

Similarly, the input to the quadrature arm is

$$-\cos\ (\omega_c t + \alpha(t))\ \sin(\omega_o t + \theta)$$

and low pass filtering gives,

$$-\frac{1}{2}\ \sin\ \Big((\omega_o - \omega_c)\ t + \theta - \alpha(t)\Big) \tag{D-10}$$

The input to the loop filter is given by the product of equations (D-9) and (D-10).

$$\frac{1}{4}\sin\ \Big((\omega_c - \omega_o)\, t + \alpha(t) - \theta\Big)\ \cos\ \Big((\omega_c - \omega_o)\, t + \alpha(t) - \theta\Big)$$

$$= \frac{1}{8}\sin\ \Big(2\,(\omega_c - \omega_o)\ t + 2\,(\alpha(t) - \theta)\Big) \tag{D-11}$$

When $\omega_c = \omega_o$, equation (D-11) becomes

$$\frac{1}{8}\sin\ \Big(2\ (\alpha(t) - \theta)\Big)$$

$$= \frac{1}{8}\ \sin 2\ \alpha(t)\cos\theta - \frac{1}{8}\cos 2\,\alpha(t)\sin 2\,\theta \tag{D-12}$$

From equation (D-12), it is seen that when $\alpha(t) = o, \pi$, which is the case of bi-phase modulation, (D-12) reduces to (D-4). And when this is not the case, there is no point in using the type of receiver shown in Figure D-1, for the quadrature arm only provides the necessary phase error as shown in equation (D-10).

D.2 EVENT MARKER ERROR RATE ANALYSIS

With respect to the system shown in Figure 1-4 (Volume 1), a binary pulse of 0.2 milliseconds is the basic bit. The ground station bi-phase modulates the doppler wiped-out carrier and the DSBSC receiver coherently demodulates the event marker, which will consist of four bits. The satisfactory detection of the four-bit word will be considered to be the event marker.

At maximum range, with 10 watts at the transmitter and with the receiver considered for use, the root mean square received signal power-to-noise power density has been computed to be $\frac{P_s}{N_o} = 20,000$ cps. If E is the total energy per bit of duration 0.2 millisecond,

$$\frac{E}{N_o} = 6 \text{ db cps}$$

For ideal coherent demodulation, the per-digit error at the receiver[1] is 2×10^{-3}. If this per digit error is set equal to $q = 2 \times 10^{-3}$, from the binomial theorem the probability of at least one error occurring is

$$q^4 + \binom{4}{1} p q^3 + \binom{4}{2} p^2 q^2 + \binom{4}{3} p^3 q$$

and from the Ordnance Corps Tables,[2] this value is found to be 0.0080.

D.3 SLAVE PNG AMBIGUITY RESOLUTION WITH THE AID OF THE EVENT MARKER.

The estimate of searching over 50 ambiguous pseudo-noise bits by the slave PNG is an upper bound, and its derivation is given together with a more realistic estimate. In this approach, the event marker is transmitted from the ground preceding some pre-selected reference state in the pseudo-noise sequence. The ground transponder has acquired the S-band transmission from the vehicle, so it has a slave PNG generating the PN sequence synchronously with the incoming PN modulation. The event marker is generated and transmitted from the ground when it is recognized that the reference state will be generated by the synchronized slave PNG

[1] Lawton, J. G., "Comparison of Binary Data Transmission Systems," Conference Proceedings of 2nd National Convention on Military Electronics, p 54

[2] United States Army Ordnance Corps. Tables of the Cumulative Binomial Probabilities, GRDP 20-1, Sept 1952, p 578

in a given number of shift pulses. In the vehicle, the event marker is received, the clock pulses into the spacecraft slave PNG are started at some time after the reference time, and an acknowledgement via VHF is sent to the ground.

When the ground station receives the acknowledgement for receipt of the event marker from the vehicle, the ground inserts the PN modulation. By this time the PN modulation from the ground will not be in synchronism with the spacecraft slave PNG due to the spacecraft velocity.

It has been estimated that, upon arrival of the PN modulation from the ground transponder, the vehicle slave PNG will be within 50 bits of the incoming PN state. This estimate was made by assuming a time delay of 0.2 seconds from the time the spacecraft slave PNG was started and the PN modulation is received from the ground, and maximum vehicle velocity was also assumed.

The method just described is overly complicated. The event marker should precede the PN modulation by a fixed duration of time, and this separation in time should only exist for the purpose of preparing to operate the slave PNG at the vehicle receiver. If the slave PNG always starts at a specified reference state, there is no need to allow more than a PN bit duration between the event marker and the PN modulation.

As in the hybrid PN system in which a clock pulse is modulating the doppler wiped-out carrier, the separation in time between the event marker and the beginning of the PN modulation can be specified in terms of the clock pulse periods.

It is to be noted, however, that with this method, the ground proceeds to put the PN modulation on the doppler wiped-out carrier before the vehicle acknowledges receipt of the event marker. This, however, is not considered to be catastrophic, for a procedure can be set up in which when no acknowledgement is received from the vehicle after a given amount of time, the ground will revert automatically to the retransmission of the event marker.

D.4 INTERFERENCE REJECTION PROPERTIES OF PN MODULATION

D.4.1 Sinusoidal Interference

Suppose ranging functions are being performed by a system such as the one shown in Figure 1-4 of Volume I, and a sinusoidal interfering signal,

$$A \cos \left((\omega_c + \Delta \omega) t + \theta \right)$$

falls in the channel. Here $\Delta \omega$ is the displacement in frequency of the interfering signal from the center frequency of the ranging signal, θ is an arbitrary phase angle, and A is the amplitude. In the bi-phase case the ranging signal

$$(-1)^{p(t)} \cos \omega_c t = \cos \left(\omega_c t + \alpha (t) \right)$$

where p(t) is 0 or 1, and $\alpha(t)$ is 0 or π, as determined by the pseudo-noise generator (PNG). Hence, the total input signal to the receiver is

$$\cos \left(\omega_c t + \alpha (t) \right) + A \cos \left((\omega_c + \Delta \omega) \; t + \theta \right) \tag{D-13}$$

At the multiplier, the input signal is mixed with the local replica, which is given by,

$$\cos \left(\omega_c t + \omega_{IF} \; t + \alpha_e (t) \right) \tag{D-14}$$

The low frequency component of the product of (D-13) and (D-14) is

$$\frac{1}{2} \cos \left(\omega_{IF} t + \alpha_e (t) - \alpha (t) \right) + \frac{A}{2} \cos \left((\omega_{IF} - \Delta \omega) \; t + \alpha_e (t) - \theta \right)$$

When the slave PNG is synchronized with the incoming PN modulation, the above expression reduces to,

$$\frac{1}{2} \cos \omega_{IF} t + \frac{A}{2} \cos \left((\omega_{IF} - \Delta \omega) \; t + \alpha (t) - \theta \right) \tag{D-15}$$

The effect of the interfering sinusoid is given by the second term of equation (D-15) and can be rewritten as,

$$\frac{A}{2} \left[\cos \theta \cos \left((\omega_{IF} - \Delta \omega) t + \alpha (t) \right) + \sin \theta \sin \left((\omega_{IF} - \Delta \omega) t + \alpha (t) \right) \right] \tag{D-16}$$

Equation (D-16) consists of two quadrature signals at frequency $\omega_{IF} - \Delta\omega$, and both are bi-phase modulated by the PN sequence generated by the slave PNG. The in-phase component is weighted by the cosine of the arbitrary phase angle between the ranging signal and the interfering sinusoid, and the quadrature component is multiplied by the sine of the same phase angle, θ. Thus, when $\theta = 0$, the effect of sinusoidal interference is,

$$\frac{A}{2} \cos \left((\omega_{IF} - \Delta\omega)\ t + \alpha(t) \right)$$

There are two more important facts to observe in equation (D-16). The first is that the effects of sinusoidal interference is dependent upon $\Delta\omega$, and is more severe when $\Delta\omega = 0$. The second is that the quadrature signals at frequency $\omega_{IF} - \Delta\omega$ are modulated by the PN, and each will have spectra given by the expression in Subsection 1.3.2.2 of Volume I. The portion of the spectra that falls in the IF frequency bandwidth determines the extent of interference.

D.4.2 Adjacent Channel Interference

The extension of the considerations in the previous subsection, concerning the interference problem due to a sinusoid, can be extended to the problem of interest here. Figure D-2 illustrates the channel arrangement and shows the envelope of the power density spectrum in channel 1. The exact expression for the power density spectrum is given in Subsection 1.3.2.2.2 of Volume I, and only the envelope is shown in Figure D-2. The zeros of the envelope occur at $f_1 \pm$ (1 Mc), $f_1 \pm$ (2 Mc), and so on, for the PN sequence is generated at a one mega bit rate, so the second zero falls in channel 2. The quantity which is proportional to the power in channel 2 shall be estimated by integrating the area under the envelope of the power density spectrum beyond f_1 + (1.6 Mc). This is obtained approximately from,

$$\frac{\pi}{2} - \int_{0}^{5.05\ \text{rad}} \frac{\sin^2 x}{x^2}\, dx = 0.1179$$

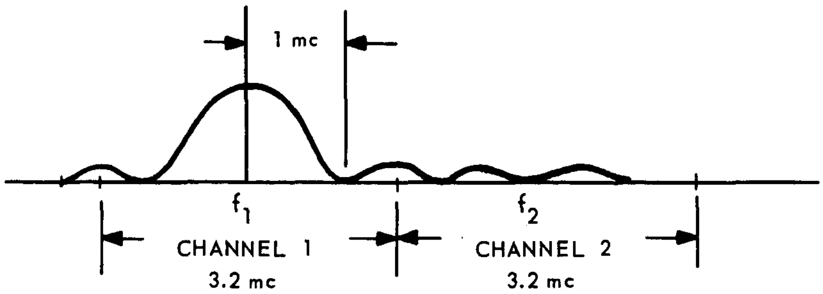

Figure D-2. Adjacent Channel Interference

and this corresponds to 7.5 percent of the total power associated with the ranging signal in channel 1.

In terms of the actual power density spectra of Subsection 1.3.2.2.2, the line spectra that falls in channel 2 are given by the n's in the first summation determined by

$$4.8 \text{ Mc} \geq \frac{n}{pt_o} \geq 1.6 \text{ Mc}$$

and the n's in the second summation is given by

$$4.8 \text{ Mc} \geq \frac{n}{t_o} \geq 1.6 \text{ Mc}.$$

Corresponding to each line spectra, one can consider the existence of a sinusoidal interfering signal. In this manner the adjacent channel interference is reduced to the case of sinusoidal interference previously discussed. Further discussion of this subject, together with a pessimistic estimate of the adjacent channel interference problem, is discussed in Subsection 1.3.2.4, Volume I.

D.5 THE COMBINING OF CLOCK PULSE AND PN FOR PHASE MODULATION

In item d of Subsection 1.3.3.2.1, Volume I, a brief discussion was given concerning the combining of the clock pulse and the PN. The combined clock pulse and PN are received during the vehicle receiver slave PNG ambiguity resolution process. Only the PN is removed from the combined input sequence and the IF signal with clock pulse modulation alone becomes the input to the acquisition and tracking phase lock loop.

The combining logic is shown in Figure D-3. Here, both the clock pulse c(t) and PN p(t) take on the two states $\pm \frac{\pi}{8}$, and are combined according to the table in Figure D-3. If the input signal is represented as

$$\cos\left(\omega_c t + \theta(t)\right)$$

then the $\theta(t)$ will be determined by the clock pulse and PN as specified by the table.

The local replica in this case is

$$\cos\left(\omega_c t + \omega_{IF} t + \theta_e(t)\right)$$

where $\theta_e(t)$ is determined by the slave PNG and takes on values $\pm \frac{\pi}{8}$. The low frequency component of the multiplier output is

$$\cos\left(\omega_{IF} t + \theta_e(t) - \theta(t)\right)$$

When the vehicle receiver slave PNG output is synchronized to the PN component of the incoming combined clock pulse and PN, it can be readily seen that $\theta_e(t) - \theta(t)$ is the clock pulse with values $\pm \frac{\pi}{8}$. Similarly, it can be shown that when $\theta_e(t)$ is not synchronized with the PN component of the received combined clock pulse and PN, $\theta_e(t) - \theta(t)$ has values $\pm \frac{3\pi}{8}$ and $\pm \frac{\pi}{8}$.

p(t) \ c(t)	$\frac{\pi}{8}$	$-\frac{\pi}{8}$
$\frac{\pi}{8}$	$\frac{\pi}{4}$	0
$-\frac{\pi}{8}$	0	$-\frac{\pi}{4}$

Figure D-3. Combining Logic for Clock Pulse and Pseudo Noise

ERRATA SHEETS FOR VOLUME 1

1. Page 1-10 and 1-11. Interchange figure captions
2. Page 1-12. Reference at bottom of page should be 1.3.1.3
3. Page 1-13, 8th line from bottom of page. Change 2000 GC to 2000 MHz
4. Page 1-34, 10 lines from bottom of page. APC instead of PAC
5. Page 1-35, 11 lines from bottom of page. The probability of at least one error occurring is 0.0080 instead of 0.006
6. Page 1-35, 10th and 9th lines from bottom of page. The probability of receiving all four bits correctly is 0.992 instead of the probability of receiving all four bits correctly or with a single error is 0.994.
7. Page 1-36, 6th line from top of page. 32 milliseconds instead of 128 milliseconds
8. Page 1-36, 14th line from top of page (first equation). Replace all zetas with deltas
9. Page 1-36, 6th line from bottom of page should be:

$$H_1(f) = \frac{\sin(\omega + \omega_c)\frac{t_o}{2}}{\omega + \omega_c} + \frac{\sin(\omega - \omega_c)\frac{t_o}{2}}{\omega - \omega_c}$$

10. Page 1-39, 8th line from top of page. Channel 2 refers to Figure D-2 of Appendix D.
11. Page 1-49, 4th line from top of page. Insert missing bracket on right-hand side of equation
12. Page 1-49, 6th line from top of page. Replace the equation by

$$A_c \operatorname{Re}\left(\ell^{j\omega_c t} \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\sum_{m=-\infty}^{\infty} J_m \left(\Phi \frac{4}{n\pi} \right) \ell^{jm \frac{2\pi nt}{T}} \right)\right).$$

13. Page 1-56, 6th line from top of page. Should be:

$$\frac{\dot{\omega}}{\omega_{n_o}^2} = 2.64 \text{ rad.}$$

14. Page 1-56, 6th line from bottom of page. Should be enters not centers

15. Page 1-65, 9th line from bottom of page. Add bracket; thus,

If $\cos(\omega_c t + \theta(t))$ is the modulated signal from the ...

16. Figure 2-1, page 2-4. Change all t to T