THE ELECTRICALLY LONG ANTENNA - CURRENT, ADMITTANCE AND FAR FIELD

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ABSTRACT

An approximate formula is derived for the distribution of current in and the driving point admittance of electrically long and thin dipole antennas. It consists of four terms to represent the real and four terms to represent the imaginary part of the current referred to the driving voltage. These combine simple sines and cosines for the leading parts with linearly and logarithmically tapered sines and cosines to shape the amplitudes. The currents per unit voltage at the driving points give the admittance.

I INTRODUCTION

In recent papers[1,2] representations of the currents in cylindrical dipoles were derived in the form of two and three trigonometric terms with suitable complex coefficients. These were shown to combine simplicity with quantitative accuracy for antennas in the range of electrical half-lengths given by $0 \leq \beta_o h \leq 5\pi/4$ whereas conventional sinusoidal theory is satisfactory only for very thin antennas in the much more restricted range $0 \leq \beta_o h < \pi/2$. An approximate representation specifically for long resonant antennas has been reported[3], but no general extension of the theory to electrically long antennas has been made except in the rigorous analysis by Wu[4] which does not provide a simple trigonometric formula for the current. Such a formula is generally useful in providing physical insight into the behavior of antennas and for many applications, notably those that involve superposition and transient response. The purpose of this paper is to provide a simple, reasonably accurate formula for the distribution of current in a cylindrical antenna that may be many wavelengths long. Experimental studies of such antennas have been made by Iizuka et al[5] and by Altshuler[6]. The analytical procedure for deriving the currents resembles that developed for shorter antennas[1,2], but is necessarily somewhat more involved.

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1. R. W. P. King, "Linear arrays: currents, impedances and fields," IRE Trans. on Antennas and Propagation, vol. AP-7, pp. 5440-5457, Dec., 1959.

2. R. W. P. King and T. T. Wu, "Currents, charges and near fields of cylindrical antennas," Radio Science, vol. 69D, pp. 429-446, March, 1965.

3. R. W. P. King and S. S. Sandler, IEEE Trans. on Antennas and Propagation, vol. AP-14, pp. 639-641, Sept., 1966.

4. T. T. Wu, "Theory of the dipole antenna and the two-wire transmission line," J. Math. Phys., vol. 2, pp. 550-574, July-August, 1961.

5. K. Iizuka, R. W. P. King and S. Prasad, "The admittance of very long cylindrical antennas," Proc. IEE(London), vol. 110, pp. 303-310, Feb., 1963.

6. E. E. Altshuler, "The traveling-wave linear antenna," Ph.D. dissertation, Harvard University, Cambridge, Mass, 1960.

II INTEGRAL EQUATIONS AND APPROXIMATE CURRENTS

The well-known integral equation for the current in a thin center-driven cylindrical antenna with half-length h and radius a is readily expressed in the following form:

$$4\pi\mu_o^{-1}A_z(z) = \int_{-h}^{h} I_z(z')K(z,z')dz' = \frac{j4\pi}{\zeta_o\cos\beta_o h}\left[\frac{1}{2}V_o^e\sin\beta_o(h-|z|) + U\cos\beta_o z\right], \tag{1}$$

where $A_z(z)$ is the vector potential on the surface of the antenna, $\zeta_o \doteq 120\pi$ ohms, and

$$U = -\frac{j}{c}A_z(h) = -\frac{j\zeta_o}{4\pi}\int_{-h}^{h} I_z(z')K(h,z')dz' \tag{2}$$

is proportional to the vector potential at the end $z = h$ of the antenna.

The kernel is $$K(z,z') = \frac{e^{-j\beta_o R}}{R}, \quad R = \sqrt{(z-z')^2 + a^2} . \tag{3}$$

If the real and imaginary parts of the current, kernel, and function U are introduced in the form $I_z(z) = I''_z(z) + jI'_z(z)$, $U = U'_R + jU_I$, $K(z,z') = K_R(z,z') + jK_I(z,z') = R^{-1}\cos\beta_o R - jR^{-1}\sin\beta_o R$, (1) can be separated into two equations as follows:

$$\int_{-h}^{h} I'_z(z')K_R(z,z')dz' = \frac{4\pi}{\zeta_o\cos\beta_o h}\left[\frac{1}{2}V_o^e\sin\beta_o(h-|z|) + U_R\cos\beta_o z\right] - \int_{-h}^{h} I''_z(z')K_I(z,z')dz' \tag{4a}$$

$$\int_{-h}^{h} I''_z(z')K_R(z,z')dz' = -\frac{4\pi U_I}{\zeta_o\cos\beta_o h}\cos\beta_o z + \int_{-h}^{h} I'_z(z')K_I(z,z')dz' , \tag{4b}$$

where

$$U_R = \frac{\zeta_o}{4\pi} \int_{-h}^{h} \left[I_z'(z')K_R(h,z') + I_z''(z)K_I(h,z') \right] dz' \tag{5}$$

$$U_I = \frac{\zeta_o}{4\pi} \int_{-h}^{h} \left[I_z''(z')K_R(h,z') - I_z'(z')K_I(h,z') \right] dz' \ . \tag{6}$$

The functions U_R and U_I are potentials in volts, V_o^e is the driving voltage to which all phases are referred.

Owing to the quite different properties of the real and imaginary parts of the kernel[2] $K(z,z') = K_R(z,z') + jK_I(z,z')$, the integrals that involve the former have leading terms that are well approximated by

$$\int_{-h}^{h} I(z')K_R(z,z')dz' \sim I(z) \quad , \tag{7}$$

whereas those that involve the latter have a dependence on z that is largely independent of I(z'). If use is made of this fact, it follows directly from (4a) and (4b) that $I_z'(z)$ must have a distribution that includes as leading terms the trigonometric functions $\sin\beta_o(h - |z|)$ and $\cos\beta_o z$; $I_z''(z)$ must include $\cos\beta_o z$. Additional terms come from the integrals on the right. However, since $\beta_o K_I(z,z') \leq 1$, whereas $\beta_o K_R(z,z') \leq (\beta_o a)^{-1} >> 1$ these are relatively small and insensitive to the variation of $I_z(z)$ with z. It follows that satisfactory approximations may be obtained with $I_z'(z) \doteq A \sin \beta_o(h - |z'|) + B' \cos \beta_o z + B'' f(z)$. $I_z''(z) \doteq B'' \cos \beta_o z + A\ g(z) + B'\ f(z)$ in these integrals where

$$f(z) = \int_{-h}^{h} \cos\beta_o z'\ K_I(z,z')dz' \doteq - \left[\frac{\pi}{2} \cos\beta_o z + \frac{z}{h} \sin\beta_o z\right] \tag{8a}$$

$$g(z) = \int_{-h}^{h} \sin \beta_o(h-|z'|)K_I(z,z')dz' \doteq -\left[\frac{\pi}{2} \sin \beta_o(h-|z|) + \frac{z}{h} \sin\beta_o z \sin\beta_o h - L_1(z)\cos\beta_o z \cos\beta_o h\right] . \quad (8b)$$

The approximate expressions on the right in (8a,b) are derived in Appendix A. The function

$$L_1(z) = \frac{1}{2}\left[\mathrm{Cin}\ 2\beta_o(h-z) + \mathrm{Cin}\ 2\beta_o(h+z) - 2\mathrm{Cin}\ 2\beta_o z\right]$$

$$\doteq \ell n \sqrt{(h^2/z^2) - 1} + e^{-2\beta_o z} \ell n\ 2\beta_o z\ \gamma\ , \quad 2\beta_o(h-z) > \frac{\pi}{2} \quad (9)$$

is slowly varying except near $z = 0$. Note that $\ell n\ \gamma = 0.5772..$(Euler's constant) and $\mathrm{Cin}\ x = \int_0^x x^{-1}(1 - \cos x)dx$ is the modified cosine integral. If these formulas are used in the evaluation of the integrals on the right in (4a,b) (see Appendix A), relatively simple expressions are obtained that are acceptable approximations for all values of z not too near the ends at $z = h$. The resulting approximate distributions of the real and imaginary parts of the current have the forms

$$I_z'(z) \doteq A \sin\beta_o(h-|z|) + \left[E' + H'L_1(z)\right]\cos\beta_o z + D'(z/h)\sin\beta_o z \quad (10a)$$

$$I_z''(z) \doteq \left[E'' + H''L_1(z)\right]\cos\beta_o z + \left[F + D''(|z|/h)\right]\sin\beta_o|z| \quad . \quad (10b)$$

The coefficients A to F remain to be determined. The formulas (10a) and (10b) are useful approximations when $0 \lesssim \beta_o z \lesssim \beta_o h - \pi/4$.

An approximate simple expression for the current near the ends $z = \pm h$ of a long antenna is readily obtained if note is taken of the following fact - which has been verified both by reference to measurements[4] and to detailed calculations from the analytically correct theory of Wu[4]. The currents in antennas in the range $0 \leq \beta_o z \leq 5\pi/4$ are very well represented in the form[1]

$$I_z'(z) \doteq A \sin \beta_o(h - |z|) + B'(\cos \beta_o z - \cos \beta_o h) \tag{11a}$$

$$I_z''(z) \doteq B''(\cos \beta_o z - \cos \beta_o h) \quad . \tag{11b}$$

If a given antenna with $\beta_o h \leq 5\pi/4$ is increased in length by integral multiples of the wavelength so that the new electrical length is $\beta_o h + n\pi$, $n = 1,2,\ldots$ the distribution of current in the outer pieces of length h is still very well represented in the forms (11a) and (11b) with appropriately modified coefficients A, B', and B". This suggests the following representation for antennas of any length: use (10a) and (10b) in the range $0 \leq \beta_o z \leq \beta_o h' = n\pi$; use (11a) and (11b) in the range $\beta_o h' \leq \beta_o z \leq \beta_o h$, where $\frac{\pi}{4} \lesssim \beta_o(h - h') < \frac{5\pi}{4}$.

In order to provide continuity of current, the two sets of formulas must be equated at $\beta_o z = \beta_o h' = n\pi$. This requires the two coefficients A to be the same, and

$$B' = b\left[E' + H'L_1(h')\right] \; ; \qquad B'' = b\left[E'' + H''L_1(h')\right] \; , \tag{12}$$

where

$$b = \frac{\cos \beta_o h'}{\cos \beta_o h' - \cos \beta_o h} = \frac{1}{1 - \cos \beta_o(h - h')} \tag{13}$$

and $L_1(h') \doteq \ln\sqrt{(h/h')^2 - 1}$. The components that involve $\sin \beta_o z$ vanish at $z = h'$. They can be continued with components that are zero at $z = h$. Suitable expressions are

$$I_z'(z) = A \sin \beta_o(h-|z|) + B'(\cos \beta_o z - \cos \beta_o h) + D'\left[\frac{h'(h-|z|)}{h(h - h')}\right]\sin \beta_o|z| \tag{14a}$$

$$I_z''(z) = B''(\cos \beta_o z - \cos \beta_o h) + \left[F + D''(h'/h)\right]\left[\frac{h - |z|}{h - h'}\right] \sin \beta_o |z| \quad . \tag{14b}$$

Actually, the two added terms are small and usually negligible.

III DETERMINATION OF COEFFICIENTS

The several coefficients in the distribution of current are determined by substituting the currents (10a,b) and (11a,b) in the equations (4a,b) and approximating the integrals as obtained by trigonometric functions. This is carried out in Appendix A; the desired approximations are in (A-30) to (A-33). If these are substituted in (4a,b), the following equations are obtained:

$$\left[A\Psi_R + \frac{F\pi}{2\cos \beta_o h} - \frac{2\pi V_o^e}{\zeta_o \cos \beta_o h}\right] \sin \beta_o(h-|z|) + \left[E'\Psi_{CR} - \frac{4\pi U_R}{\zeta_o \cos \beta_o h}\right.$$

$$\left. - \frac{\pi}{2}\,(E'' + 0.55H'' + F \tan \beta_o h) - D''\right] \cos \beta_o z$$

$$+ \left[H'\Psi_{CR} - F\right] L_1(z) \cos \beta_o z + \left[D'\Psi_{hR} - E'' - 0.55H'' - \frac{\pi}{2}D''\right] \left(\frac{z}{h}\sin\beta_o z\right)$$

$$= 0 \tag{15a}$$

$$(F\Psi_{SR} - A\frac{\pi}{2} \cos \beta_o h)\sin \beta_o|z| + \left[E''\Psi_{CR} + \frac{4\pi U_I}{\zeta_o \cos \beta_o h} + \frac{\pi}{2}(A \sin \beta_o h + E' + 0.55H')\right.$$

$$\left. + D'\right] \cos \beta_o z + \left[H''\Psi_{CR} - A \cos \beta_o h\right] L_1(z) \cos \beta_o z + \left[D''\Psi_{hR} + A \sin \beta_o h\right.$$

$$\left. + E' + 0.55H' + \frac{\pi}{2}\, D')\right] \left(\frac{z}{h} \sin \beta_o z\right) = 0 \quad . \tag{15b}$$

These equations are satisfied if the coefficients of the trigonometric functions are individually equated to zero. In this manner the following simple results are obtained directly:

$$F = \frac{A\pi \cos \beta_o h}{2\Psi_{SR}} = \frac{\pi^2 V_o^e}{\zeta_o \Psi_{R1} \Psi_{SR}} \tag{16a}$$

$$A = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1} \cos \beta_o h} \quad , \quad \Psi_{R1} = \Psi_R + \frac{\pi^2}{4\Psi_{SR}} \tag{16b}$$

$$H' = \frac{F}{\Psi_{CR}} = \frac{A\pi \cos \beta_o h}{2\Psi_{SR} \Psi_{CR}} = \frac{\pi^2 V_o^e}{\zeta_o \Psi_{R1} \Psi_{SR} \Psi_{CR}} \tag{16c}$$

$$H'' = \frac{A \cos \beta_o h}{\Psi_{CR}} = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1} \Psi_{CR}} \quad . \tag{16d}$$

The following equations must also be satisfied:

$$E'\Psi_{CR} - \frac{\pi}{2}(E'' + 0.55H'' + F \tan \beta_o h) - D'' - \frac{4\pi U_R}{\zeta_o \cos \beta_o h} = 0 \tag{17}$$

$$E''\Psi_{CR} + \frac{\pi}{2}(E' + 0.55H' + A \sin \beta_o h) + D' + \frac{4\pi U_I}{\zeta_o \cos \beta_o h} = 0 \tag{18}$$

$$D'\Psi_{hR} - E'' - 0.55H'' - \frac{\pi}{2} D'' = 0 \tag{19}$$

$$D''\Psi_{hR} + E' + 0.55H' + \frac{\pi}{2} D' + A \sin \beta_o h = 0 \quad . \tag{20}$$

It remains to evaluate D', D'', E', and E'' from (17) to (20). This involves the functions U_R and U_I as defined in (5, 6). They are readily evaluated if (1Ca,b) and (11a,b) are substituted in (5, 6). The results are:

$$\frac{4\pi U_R}{\zeta_o} = A\Psi_{VR}(h) + E'\Psi_{CR}(h) + H'\psi_{LR}(h) + D'\Psi_{ZR}(h)$$

$$+ E''\Psi_{CI}(h) + H''\Psi_{LI}(h) + D''\Psi_{ZI}(h) + F\Psi_{SI}(h) \tag{21}$$

$$\frac{4\pi U_I}{\zeta_o} = E''\Psi_{CR}(h) + H''\Psi_{LR}(h) + D''\Psi_{ZR}(h) + F\Psi_{SR}(h)$$

$$- A\Psi_{VI}(h) - E'\Psi_{CI}(h) - H'\Psi_{LI}(h) - D'\Psi_{ZI}(h) \quad , \qquad (22)$$

where the several constants $\Psi(h)$ are defined in Appendix B. Now let

$$A_{1R} = A\Psi_{VR}(h) + H'\Psi_{LR}(h) + H''\Psi_{LI}(h) + F\Psi_{SI}(h) - \frac{\pi}{2}\left[0.55H'' + F \tan \beta_o h\right]\cos \beta_o h \qquad (23a)$$

$$A_{1I} = A\Psi_{VI}(h) + H''\Psi_{LI}(h) - H''\Psi_{LR}(h) - F\Psi_{SR}(h) + \frac{\pi}{2}\left[0.55H' + A \sin \beta_o h\right]\cos \beta_o h \qquad (23b)$$

With (16a-d) these become

$$A_{1R} = A\left[\Psi_{VR}(h) + f_{1R} \cos \beta_o h\right] \quad , \quad A_{1I} = A\left[\Psi_{VI}(h) + f_{1I} \cos \beta_o h\right] \, , \qquad (24a)$$

where

$$f_{1R} = (\pi/2)\Psi_{LR}(h)\Psi_{SR}^{-1}\Psi_{CR}^{-1} + \Psi_{CR}^{-1}\left[\Psi_{LI}(h) - 0.275\pi \cos \beta_o h\right]$$

$$+ (\pi/2)\Psi_{SR}^{-1}\left[\Psi_{SI}(h) - (\pi/2) \sin \beta_o h\right] \qquad (24b)$$

$$f_{1I} = (\pi/2)\left\{\sin \beta_o h + \Psi_{SR}^{-1}\Psi_{CR}^{-1}\left[\Psi_{LI}(h) + 0.275\pi \cos \beta_o h\right] - \Psi_{SR}(h)\Psi_{SR}^{-1}\right\}$$

$$- \Psi_{ZR}(h)\Psi_{CR}^{-1} \, . \qquad (24c)$$

With (21) to (24), the equations (17) and (18) now become

$$D'\Psi_{ZR}(h) + D''\left[\Psi_{ZI}(h) - \cos \beta_o h\right] + E'\left[\Psi_{CR}(h) + \Psi_{CR} \cos \beta_o h\right]$$

$$+ E''\left[\Psi_{CI}(h) - (\pi/2)\cos \beta_o h\right] = -A_{1R} \qquad (25)$$

$$D'\left[\Psi_{ZI}(h) + \cos\beta_o h\right] - D''\Psi_{ZR}(h) + E'\left[\Psi_{CI}(h) + (\pi/2)\cos\beta_o h\right]$$

$$- E''\left[\Psi_{CR}(h) - \Psi_{CR}\cos\beta_o h\right] = -A_{1I}\ . \qquad (26)$$

Equations (19) and (20) may be arranged as follows:

$$D'\Psi_{hR} - D''\frac{\pi}{2} - E'' = A(0.55\Psi_{CR}^{-1}\cos\beta_o h) \qquad (27)$$

$$D'\frac{\pi}{2} + D''\Psi_{hR} + E' = -A(\sin\beta_o h + 0.275\pi\ \Psi_{SR}^{-1}\Psi_{CR}^{-1}\cos\beta_o h)\ . \qquad (28)$$

The simultaneous solution of (25) to (28) for D', D", E', and E" expresses these coefficients in terms of the known constant A. It is carried out in Appendix C. This completes the determination of all of the coefficients in (10a,b) and (11a,b) so that the current in the antenna is completely known.

IV DISTRIBUTIONS OF CURRENT AND ADMITTANCES

The distribution of current in the form $I_z(z) = I''_z(z) + jI'_z(z)$ is well approximated in the range $|\beta_o z| \leq \beta_o h' = n\pi$ by

$$I'_z(z) = \frac{2\pi\ V_o^e}{\zeta_o\Psi_{R1}}\left[\frac{\sin\beta_o(h-|z|)}{\cos\beta_o h} + T'_C\cos\beta_o z + T'_L L_1(z)\cos\beta_o z + T'_z(z/h)\sin\beta_o z\right] \qquad (29a)$$

$$I''_z(z) = \frac{2\pi\ V_o^e}{\zeta_o\Psi_{R1}}\left[T''_C\cos\beta_o z + T''_L L_1(z)\cos\beta_o z + T'_z(z/h)\sin\beta_o z + T_S\sin\beta_o|z|\right], \qquad (29b)$$

where $T'_C = E'/A\cos\beta_o h$, $T''_C = E''/A\cos\beta_o h$, $T'_z = D'/A\cos\beta_o h$, $T''_z = D''/A\cos\beta_o h$, $T'_L = H'/A\cos\beta_o h = \pi/2\ \Psi_{SR}\Psi_{CR}$, $T''_L = H''/A\cos\beta_o h = 1/\Psi_{CR}$, $T_S = F/A\cos\beta_o h = \pi/2\ \Psi_{SR}$. When $\beta_o h' = n\pi \leq \beta_o z \leq \beta_o h$, where $\frac{\pi}{4} < \beta_o(h-h') \leq \frac{5\pi}{4}$, the formulas for the current are

$$I_z'(z) = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \left\{ \frac{\sin \beta_o(h-|z|)}{\cos \beta_o h} + b\left[T_C' + T_L' L_1(h')\right] (\cos \beta_o z - \cos \beta_o h) \right\} \tag{30a}$$

$$I_z''(z) = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \, b\left[T_C'' + T_L'' L_1(h')\right] (\cos \beta_o z - \cos \beta_o h) \, . \tag{30b}$$

The driving-point admittance $Y_o = G_o + jB_o$ is given by

$$B_o = \frac{2\pi}{\zeta_o \Psi_{R1}} \left[\tan \beta_o h + T_C' + T_L' L_1(0)\right] \tag{31a}$$

$$G_o = \frac{2\pi}{\zeta_o \Psi_{R1}} \left[T_C'' + T_L'' L_1(0)\right] \, . \tag{31b}$$

When $\cos \beta_o h = 0$ all of the coefficients T in (29a,b) remain finite except T_C'. However, if the two first terms in (29a) are combined in the form

$$\frac{\sin \beta_o(h-|z|)}{\cos \beta_o h} + T_C' \cos \beta_o z = T_{1C}' \cos \beta_o z - \sin \beta_o |z| , \tag{32}$$

where $\quad T_{1C}' = \tan \beta_o h + T_C' = (A \sin \beta_o h + E')/A \cos \beta_o h \qquad (33)$

the new coefficient T_{1C}' remains finite when $\cos \beta_o h = 0$. This is shown in Appendix C. It may be convenient to use the alternative form

$$I_z'(z) = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \left[- \sin \beta_o |z| + T_{1C}' \cos \beta_o z + T_L' L_1(z) \cos \beta_o z + T_z'\left(\frac{z}{h} \sin \beta_o z\right)\right] \tag{34}$$

in place of (29a) when $\cos \beta_o h$ is very small or zero. In this form the susceptance is

$$B_o = \frac{2\pi}{\zeta_o \Psi_{R1}} \left[T_{1C}' + T_L' \, L_1(0)\right] , \tag{35}$$

where, from (9), $\quad L_1(0) = \text{Cin } 2\beta_o h \quad . \qquad (36)$

The current in the range $\beta_o h' = n\pi \leq \beta_o z \leq \beta_o h$ that corresponds to (34) is obtained from (30a) with the substitution $T'_C = T'_{1C} - \tan \beta_o h$ as given by (33). Since,

$$\frac{\sin \beta_o(h-|z|)}{\cos \beta_o h} = \sin \beta_o h - \sin \beta_o z + \tan \beta_o h(\cos \beta_o z - \cos \beta_o h) \quad (37)$$

$$I'_z(z) = \frac{2\pi V^e_o}{\zeta_o \Psi_{R1}} \left\{ \sin \beta_o h - \sin \beta_o |z| + b\left[T'_{1C} + T'_L L_1(h') - \sin \beta_o h \sec \beta_o h'\right] (\cos \beta_o z - \cos \beta_o h) \right\}, \quad (38)$$

where, as defined in (13), $b = \cos \beta_o h'/(\cos \beta_o h' - \cos \beta_o h)$. Note that (33) reduces to (34) at $z = h'$, since $\beta_o h' = n\pi$.

V THE FIELD PATTERN

The radiation field of an electrically long antenna has been determined by Wu[4] in a useful form. It is readily computed from the well-known formula

$$E''_\Theta = \frac{j\zeta_o I_z(0)}{2\pi} \frac{e^{-j\beta_o R_o}}{R_o} f(\theta), \quad (39a)$$

where the field factor is

$$f(\theta) = \frac{\sin \theta}{2V^e_o Y_o} \int_{-h}^{h} I(z') e^{-j\beta_o z \cos \theta} \beta_o dz'. \quad (39b)$$

In the form given in (34b) the field is referred to the $I_z(0) = V^e_o Y_o$ at the driving point. It may, of course, be referred to the driving voltage instead, but this is not conventional. The current distribution in (34b) is that given in (29a,b) with (30a,b). Alternatively, (34) and (38) may be used instead of (29b) and (30b). In the evaluation of the far field pattern it is adequate to replace the logarithmic term with its value at $z = h'$. Hence, with $I_z(z) = I_z''(z) + jI_z'(z)$,

$$I_z(z) = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \left[j \frac{\sin \beta_o(h-|z|)}{\cos \beta_o h} + T_{CL} \cos \beta_o z + T_Z (z/h) \sin \beta_o z + T_S \sin \beta_o |z| \right] \tag{40a}$$

$$I_z(z) = \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \left[j \frac{\sin \beta_o(h-|z|)}{\cos \beta_o h} + bT_{CL}(\cos \beta_o z - \cos \beta_o h) \right], \tag{40b}$$

where (40a) applies for $0 \leq \beta_o|z| \leq \beta_o h' = n\pi$, (40b) for $\beta_o h' \leq \beta_o|z| \leq \beta_o h$. In (40a,b), $T_{CL} = T_C'' + jT_C' + (T_L'' + jT_L')L_1(h')$, $T_Z = T_Z'' + jT_Z'$. Note that T_S is real. $Y_o = G_o + j B_o$ is obtained from (31a,b).

If (40a,b) are used in (39b), the following integrals are obtained:

$$(1/2)\sin \theta \int_{-h}^{h} \sin \beta_o(h-|z|)\, e^{j\beta_o z \cos \theta} \beta_o dz = F_m(\Theta,\beta_o h)$$

$$= \frac{\cos(\beta_o h \cos \theta) - \cos \beta_o h}{\sin \theta} \tag{41}$$

$$(1/2)\sin \theta \int_{-h'}^{h'} \sin \beta_o|z|\, e^{j\beta_o z \cos \theta} \beta_o dz = F_S(\Theta,n\pi)$$

$$= \frac{1 - (-1)^n \cos(n\pi \cos \theta)}{\sin \theta} \tag{42}$$

$$(1/2)\sin \theta \int_{-h'}^{h'} \cos \beta_o z\, e^{j\beta_o z \cos \theta} \beta_o dz = F_C(\Theta,n\pi)$$

$$= -(-1)^n \cot \theta \sin(n\pi \cos \theta) \tag{43}$$

$$(1/2)\sin \theta \int_{-h'}^{h'} (z/h)\sin \beta_o z\, e^{j\beta_o z \cos \theta} \beta_o dz = F_z(\Theta,n\pi)$$

$$= \frac{-(-1)^n}{\beta_o h \sin^4 \theta} \left[n\pi \sin^2 \theta \cos(n\pi \cos \theta) + 2\cos \theta \sin(n\pi \cos \theta) \right] \tag{44}$$

$$(1/2)\sin\theta \left(\int_{-h}^{h} - \int_{-h'}^{h'} \right) (\cos\beta_o z - \cos\beta_o h)\, e^{j\beta_o z \cos\theta}\, \beta_o dz$$

$$= F_C(\Theta,\beta_o h) - F_C(\Theta,n\pi) - \left[F_E(\Theta,\beta_o h) - F_E(\Theta,n\pi) \right] \cos\beta_o h \quad , \qquad (45)$$

where

$$F_C(\Theta,\beta_o h) = \frac{1}{\sin\Theta}\left[\sin\beta_o h \cos(\beta_o h \cos\Theta) - \cos\beta_o h \sin(\beta_o h \cos\Theta)\cos\Theta\right] \qquad (46)$$

$$F_E(\Theta,\beta_o h) = \tan\Theta \sin(\beta_o h \cos\theta) \quad . \qquad (47)$$

It follows that the field pattern is given by

$$f(\Theta) = \left[T_C + T_L L_1(0) + j \tan\beta_o h \right]^{-1} \left\{ j\, F_m(\Theta,\beta_o h)\sec\beta_o h + T_{CL} F_C(\Theta,n\pi) \right.$$

$$+ T_Z F_Z(\Theta,n\pi) + T_S F_S(\Theta,n\pi) + bT_{CL} \Big| F_C(\Theta,\beta_o h) - F_C(\Theta,n\pi) - E_C(\Theta,\beta_o h)$$

$$\cos\beta_o h + E_C(\Theta,n\pi)\cos\beta_o h \quad . \qquad (48)$$

A similar formula is readily derived for the range of $\beta_o h$ near and at $(2n + 1)(\pi/2)$. Usually $|f(\theta)|$ is of primary interest.

APPENDIX A: INTEGRALS

The following four integrals occur in the equations (4a,b):

$$J_R' = \int_{-h}^{h} I_z'(z')\, K_R(z,z')dz' \quad ; \quad J_R'' = \int_{-h}^{h} I_z''(z')\, K_R(z,z')dz' \tag{A-1}$$

$$J_I' = \int_{-h}^{h} I_z'(z')\, K_I(z,z')dz' \quad ; \quad J_I'' = \int_{-h}^{h} I_z''(z')\, K_I(z,z')dz' \quad . \tag{A-2}$$

In the approximate evaluation the currents are given by

$$I_z'(z) = A \sin \beta_o(h-|z|) + E' \cos \beta_o z + H'L(z) \cos \beta_o z + D'(z/h) \sin \beta_o z \tag{A-3a}$$

$$I_z''(z) = E'' \cos \beta_o z + H''L(z) \cos \beta_o z + D''(z/h) \sin \beta_o z + F \sin \beta_o |z| \, , \tag{A-3b}$$

when $|z| \leq h' = n\pi/\beta_o$, and by

$$I_z'(z) = A \sin \beta_o(h-|z|) + b\left[E' + H'L_1(h')\right](\cos \beta_o z - \cos \beta_o h) \tag{A-4a}$$

$$I_z''(z) = b\left[E'' + H''L_1(h')\right](\cos \beta_o z - \cos \beta_o h) \tag{A-4b}$$

in the range $h' \leq |z| \leq h$ where $\frac{\pi}{4} < \beta_o(h-h') \leq \frac{5\pi}{4}$.

When these distributions are substituted in the several integrals certain well-known integrals are encountered. These are

$$S_a(h,z) = \int_{-h}^{h} \sin \beta_o |z'| \, K(z,z')dz' \tag{A-5}$$

$$C_a(h,z) = \int_{-h}^{h} \cos \beta_o z' \, K(z,z')dz' \tag{A-6}$$

$$E_a(h,z) = \int_{-h}^{h} K(z,z')dz' \quad . \tag{A-7}$$

The properties of the real part of the kernel permits the following approximations:

$$\text{Re } S_a(h,z) \doteq \Psi_{SR}\sin \beta_o|z| \;, \quad \text{Re } C_a(h,z) \doteq \Psi_{CR}\cos \beta_o z \;,$$

$$\text{Re } E_a(h,z) \doteq \Psi_{ER} \;, \tag{A-8}$$

where $\Psi_{SR} = S_a(h,\lambda/4)$, $\Psi_{CR} = C_a(h,0)$, $\Psi_{ER} = E_a(h,0)$.

In the imaginary parts of the integrals (A-5) to (A-7), the approximation $R \doteq |z' - z|$ can be made. It then follows that

$$\text{Im } S_a(h,z) = -\left[L_1(z) \cos \beta_o z + P_1(z) \sin \beta_o|z|\right] \doteq -\left[L_1(z) \cos \beta_o z + \frac{\pi}{2} \sin \beta_o|z|\right], \tag{A-9}$$

where, with $\text{Cin } x = \int_o^x x^{-1}(1 - \cos x)dx$,

$$L_1(z) = \frac{1}{2}\left[\text{Cin } 2\beta_o(h-z) + \text{Cin } 2\beta_o(h+z) - 2 \text{ Cin } 2\beta_o z\right] \tag{A-10}$$

$$\doteq \ln\sqrt{(h^2/z^2) - 1} + \ln(2\beta_o z\, \gamma)e^{-2\beta_o z} \quad \text{when} \quad 2\beta_o(h-z) > \frac{\pi}{2} \;. \tag{A-11}$$

In (A-11) the first term is a well-known approximation for arguments that are not too small. The second term, in which $\ln \gamma = 0.577...$ as Euler's constant, is added to provide a formula that has approximately the correct behavior near and at $z = 0$. In (A-10), with $\text{Si } x = \int_o^x x^{-1} \sin x\, dx$,

$$P_1(z) = \frac{1}{2}\left[\text{Si } 2\beta_o(h-z) - \text{Si } 2\beta_o(h+z) + 2 \text{ Si } 2\beta_o z\right] \;. \tag{A-12a}$$

In the range $2\beta_o(h-z) > \frac{\pi}{2}$,

$$P_1(z) \sin \beta_o|z| \doteq \frac{\pi}{2} \sin \beta_o|z| \;. \tag{A-12b}$$

Actually, near $z = 0$, $P_1(z) \to 2\beta_o z$, so that $P_1(z) \sin \beta_o|z| \to 2\beta_o^2 z^2$, whereas $\frac{\pi}{2} \sin \beta_o|z| \to \frac{\pi}{2} \beta_o z$. However, the precise shape of these small terms in a narrow range near $z = 0$ is not important so long as the value at $z = 0$ is correct. Hence, the simple form (A-12b) is adequate.

Corresponding to (A-10)

$$\text{Im}\, C_a(h,z) = -\left[P_2(z)\cos\beta_o z + L_2(z)\sin\beta_o|z|\right] \doteq -\left[\frac{\pi}{2}\cos\beta_o z + \frac{z}{h}\sin\beta_o z\right] , \qquad \text{(A-13)}$$

where

$$L_2(z) = \frac{1}{2}\left[\text{Cin}\, 2\beta_o(h+z) - \text{Cin}\, 2\beta_o(h-z)\right] \qquad \text{(A-14a)}$$

$$P_2(z) = \frac{1}{2}\left[\text{Si}\, 2\beta_o(h+z) + \text{Si}\, 2\beta_o(h-z)\right] . \qquad \text{(A-14b)}$$

Subject to the condition $2\beta_o(h-z) \gtrsim \pi/2$,

$$L_2(z) \doteq \ln\sqrt{\frac{h+z}{h-z}} \doteq \frac{z}{h} , \quad P_2(z) \doteq \frac{\pi}{2} . \qquad \text{(A-15)}$$

Strictly, the approximation $L_2(z) \doteq \frac{z}{h}$ is valid only when $(z/h) < 1$. However, near $z = h$, $L_2(z)$ rises rapidly to $L_2(h) = \frac{1}{2}\,\text{Cin}\, 4\beta_o h \doteq \frac{1}{2}\left[0.577 + \ln 4\beta_o h\right]$ which is considerably greater than 1. However, in a small correction term, the simple form $L_2(z) \doteq \frac{z}{h}$ should be adequate even though considerably in error near $z = h$.

Finally,

$$\text{Im}\, E_a(h,z) = -\left[\text{Si}\, 2\beta_o(h+z) + \text{Si}\, 2\beta_o(h-z)\right] \doteq -\pi . \qquad \text{(A-16)}$$

The approximation on the right is valid when $2\beta_o(h-z) \gtrsim \pi/2$.

Additional integrals - which occur only in quite small terms - involve distributions of current of the forms $L_1(z)\cos\beta_o z$ and $(z/h)\sin\beta_o z$. The associated integrals are

$$\int_{-h}^{h} L_1(z') \cos \beta_o z' \, K_R(z,z')dz' \doteq \Psi_{CR} \, L_1(z) \cos \beta_o z \quad . \tag{A-17}$$

Since $L_1(z)$ is quite slowly varying, Ψ_{CR} is an appropriate amplitude.

$$\int_{-h}^{h} (z'/h) \sin \beta_o z' \, K_R(z,z')dz' \doteq \Psi_{hR} \, (z/h) \sin \beta_o z + (\sin^2 \beta_o h/\beta_o h) \cos \beta_o z \,, \tag{A-18}$$

where $\Psi_{hR} = \ln(2h/\beta_o a^2) - 1.577$. Note that when the limits are $-h'$ and h', $\sin^2 \beta_o h' = 0$ since $\beta_o h' = n\pi$.

$$\int_{-h}^{h} L_1(z') \cos \beta_o z' \, K_I(z,z')dz' \doteq L_1(h/2) \int_{-h}^{h} \cos \beta_o z' \, K_I(z,z')dz'$$

$$\doteq - 0.55\left[\frac{\pi}{2} \cos \beta_o z + \frac{z}{h} \sin \beta_o z\right]. \tag{A-19}$$

In this small term, the slowly-varying function $L_1(z)$ is replaced by an approximately average value at $z = h/2$, viz. $L_1(h/2) = \frac{1}{2}\left[\mathrm{Cin}\ 3\beta_o h - \mathrm{Cin}\ \beta_o h\right] \doteq (\ln 3)/2$, and the remaining integral is the same as (A-13),

$$\int_{-h}^{h} (z'/h) \sin \beta_o z' \, K_I(z,z')dz' = -\left\{\cos \beta_o z \left[1 - \frac{\sin 2\beta_o h}{2\beta_o h}\right] + \frac{z}{h}\left[P_2(z) \sin \beta_o |z| - L_2(z) \cos \beta_o z\right]\right\}$$

$$\doteq -\left\{\left[1 - \frac{\sin 2\beta_o h}{2\beta_o h} - \frac{z^2}{h^2}\right] \cos \beta_o z + \frac{\pi}{2} \cdot \frac{z}{h} \sin \beta_o z\right\} \quad . \tag{A-20}$$

Note that when the limits are $-h'$ to h' $\sin 2\beta_o h' = 0$ since $\beta_o h' = n\pi$. This integral is evaluated directly with $R \doteq |z - z'|$ in the kernel. The approximate expression is obtained with (A-15). Since the term $(z/h)^2 \cos \beta_o z$ is of higher order in the current distribution than the terms in (10a,b), it will be omitted.

Integrals that actually arise in addition to (A-5) to (A-7) include

$$\int_{-h}^{h} \sin \beta_o(h-|z'|)\, K_R(z,z')dz' = \mathrm{Re}\Big[C_a(h,z) \sin \beta_o h - S_a(h,z) \cos \beta_o h\Big]$$

$$\doteq \Psi_R \sin \beta_o(h-|z|) \quad , \tag{A-21}$$

where

$$\Psi_R = \mathrm{Re}\Big[C_a(h,h-\lambda/4) \sin \beta_o h - S_a(h,h-\lambda/4) \cos \beta_o h\Big] \quad \text{when } \beta_o h > \frac{\pi}{2} \tag{A-22}$$

$$\int_{-h}^{h} \sin \beta_o(h-|z'|)\, K_I(z,z')dz' = \mathrm{Im}\Big[C_a(h,z) \sin \beta_o h - S_a(h,z) \cos \beta_o h\Big]$$

$$\doteq \Big[\frac{\pi}{2} \sin \beta_o(h-|z|) - L_1(z) \cos \beta_o z \cos \beta_o h + \frac{z}{h} \sin \beta_o z \sin \beta_o h\Big] \tag{A-23}$$

$$\int_{-h'}^{h'} \cos \beta_o z'\, K_R(z,z')dz' + b\left(\int_{-h}^{h} - \int_{-h'}^{h'}\right)(\cos \beta_o z' - \cos \beta_o h)\, K_R(z,z')dz'$$

$$\doteq \Psi_{CR} \cos \beta_o z \tag{A-24}$$

in the range $|z| \leq h'$. The constant amplitude Ψ_{CR} is deferred at a maximum of $\cos \beta_o z$ such as at $z = 0$ where

$$\Psi_{CR} = \mathrm{Re}\left[C_a(h',0) + b\left\{C_a(h,0) - C_a(h',0) - \Big[E_a(h,0) - E_a(h',0)\Big]\cos \beta_o h\right\}\right]. \tag{A-25}$$

Since $L_1(z)$ is slowly varying,

$$\int_{-h'}^{h'} L_1(z') \cos \beta_o z'\, K_R(z,z')dz' + b\, L_1(h')\left(\int_{-h}^{h} - \int_{-h'}^{h'}\right)(\cos \beta_o z' - \cos \beta_o h)$$

$$K_I(z,z')dz' \doteq \Psi_{CR}\, L_1(z) \cos \beta_o z \quad . \tag{A-26}$$

Since integrals with the kernel $K_I(z,z')$ are not sensitive to the distribution of current and since this is continuous at $z = h'$, the added slightly different integral for $h' \leq |z| \leq h$ cannot significantly alter the value obtained with an unchanged distribution. Hence, with (A-13),

$$\int_{-h'}^{h'} \cos \beta_o z' \, K_I(z,z')dz' + b\left(\int_{-h}^{h} - \int_{-h'}^{h'}\right)(\cos \beta_o z' - \cos \beta_o h) \, K_I(z,z')dz'$$

$$\doteq -\left[\frac{\pi}{2} \cos \beta_o z + \frac{z}{h} \sin \beta_o z\right] . \quad \text{(A-27)}$$

Similarly, with (A-19)

$$\int_{-h}^{h} L_1(z') \cos \beta_o z' \, K_I(z,z')dz' + b \, L_1(h')\left(\int_{-h}^{h} - \int_{-h'}^{h'}\right)(\cos \beta_o z' - \cos \beta_o h)$$

$$K_I(z,z')dz' \doteq - 0.55\left[\frac{\pi}{2} \cos \beta_o z + \frac{z}{h} \sin \beta_o z\right] . \quad \text{(A-28)}$$

With the several approximate formulas, it is now possible to express the original integrals (A-1) and (A-2) with (A-3a,b) and (A-4a,b) as follows:

$$J_R' \doteq A \, \Psi_R \sin \beta_o(h-|z|) + E'\Psi_{CR} \cos \beta_o z + H'\Psi_{CR} \, L_1(z) \cos \beta_o z + D'\Psi_{hR} \frac{z}{h} \sin \beta_o z$$

$$\text{(A-29)}$$

$$J_R'' \doteq E''\Psi_{CR} \cos \beta_o z + H''\Psi_{CR}L_1(z) \cos \beta_o z + D''\Psi_{hR}(z/h) \sin \beta_o z + F \, \Psi_{SR} \sin \beta_o|z|$$

$$\text{(A-30)}$$

$$J_I' \doteq A \frac{\pi}{2} \cos \beta_o h \sin \beta_o|z| - \left[A \frac{\pi}{2} \sin \beta_o h + E'\frac{\pi}{2} + 0.55H'\frac{\pi}{2} + D'\right]\cos \beta_o z$$

$$+ A \cos \beta_o h \, L_1(z) \cos \beta_o z - \left[A \sin \beta_o h + E' + 0.55H' + D'\frac{\pi}{2}\right]\left(\frac{z}{h} \sin \beta_o z\right)$$

$$\text{(A-31)}$$

$$J_I'' = -\left[E''\frac{\pi}{2} + 0.55H''\frac{\pi}{2} + D'' + \frac{\pi}{2} F \tan \beta_o h\right]\cos \beta_o z - \left[E'' + 0.55H'' + D''\frac{\pi}{2}\right]$$

$$\left(\frac{z}{h} \sin \beta_o z\right) - F \, L_1(z) \cos \beta_o z + \frac{\pi}{2} F \sin \beta_o(h-|z|)/\cos \beta_o h \quad . \quad \text{(A-32)}$$

APPENDIX B: COMPONENTS OF U

The coefficients U_R and U_I are defined by (5, 6). Approximate values are obtained from the substitution of the currents (10a,b) and (11a,b). They may be expressed as in (21) and (22) in terms of a set of numbers denoted by $\Psi(h)$ since they correspond to integrals evaluated at $z = h$, the end of the antenna. The several values are defined below.

$$\Psi_V(h) = \Psi_{VR}(h) + j\Psi_{VI}(h) = \int_{-h}^{h} \sin \beta_o(h-|z'|)\, K(h,z')dz'$$

$$= C_a(h,h) \sin \beta_o h - S_a(h,h) \cos \beta_o h \qquad \text{(B-1)}$$

$$\Psi_C(h) = \Psi_{CR}(h) + j\Psi_{CI}(h) = \int_{-h'}^{h'} \cos \beta_o z'\, K(h,z')dz' + b\left(\int_{-h}^{h} - \int_{-h'}^{h'}\right)(\cos \beta_o z' - \cos \beta_o h)\, K(h,z')dz'$$

$$= C_a(h',h) + b\left[C_a(h,h) - C_a(h',h)\right] + b\left[E_a(h,h) - E_a(h',h)\right] \cos \beta_o h \qquad \text{(B-2)}$$

$$\Psi_L(h) = \Psi_{LR}(h) + j\Psi_{LI}(h) = \int_{-h'}^{h'} L_1(z') \cos \beta_o z'\, K(h,z')dz' + b\, L_1(h') \left(\int_{-h}^{h} - \int_{-h'}^{h'}\right) (\cos \beta_o z - \cos \beta_o h)\, K(h,z')dz' \qquad \text{(B-3a)}$$

$$\doteq L_1(h')\, \Psi_C(h) \qquad \text{(B-3b)}$$

$$\Psi_Z(h) = \Psi_{ZR}(h) + j\Psi_{ZI}(h) = (\beta_o h)^{-1} \int_{-h'}^{h'} \beta_o z' \sin \beta_o z'\, K(h,z')dz' \quad . \qquad \text{(B-4)}$$

Note that

$$K(h,z') = \frac{\cos \beta_o R_h}{R_h} - j\frac{\sin \beta_o R_h}{R_h} \;,$$

where $$R_h = \sqrt{(h - z')^2 + a^2} \;.$$

APPENDIX C: SOLUTION OF EQUATIONS

The four equations (25) to (27) for D', D", E', and E" can be solved quite simply if E' and E" are first eliminated from (25) and (26). Thus, from (27) and (28)

$$E' = -\left[D'\frac{\pi}{2} + D''\Psi_{hR} + A(\sin \beta_o h + 0.275\pi\ \Psi_{SR}^{-1}\Psi_{CR}^{-1} \cos \beta_o h)\right] , \qquad \text{(C-1a)}$$

$$E'' = D'\Psi_{hR} - D''\frac{\pi}{2} - 0.55A\ \Psi_{CR}^{-1} \cos \beta_o h \quad . \qquad \text{(C-1b)}$$

When these expressions are substituted in (25) and (26) these become

$$D'\ \Phi_1' + D''\ \Phi_1'' = A\ \Phi_3 \qquad \text{(C-2a)}$$

$$D'\ \Phi_2' + D''\ \Phi_2'' = A\ \Phi_4 \ , \qquad \text{(C-2b)}$$

where

$$\Phi_1' = \Psi_{ZR}(h) - \Psi_{CR}(h)\pi/2 + \Psi_{hR}\Psi_{CI}(h) - (\Psi_{CR} + \Psi_{hR})(\pi/2) \cos \beta_o h \qquad \text{(C-3)}$$

$$\Phi_1'' = \Psi_{ZI}(h) - \Psi_{CR}(h)\Psi_{hR} - \Psi_{CI}(h)\pi/2 - (1 + \Psi_{CR}\Psi_{hR} - \pi^2/4) \cos \beta_o h \qquad \text{(C-4)}$$

$$\Phi_2' = \Psi_{ZI}(h) - \Psi_{CI}(h)\pi/2 - \Psi_{CR}(h)\Psi_{hR} + (1 + \Psi_{CR}\Psi_{hR} - \pi^2/4) \cos \beta_o h \qquad \text{(C-5)}$$

$$\Phi_2'' = - \Psi_{ZR}(h) - \Psi_{CI}(h)\Psi_{hR} + \Psi_{CR}(h)\pi/2 - (\Psi_{CR} + \Psi_{hR})(\pi/2) \cos \beta_o h \qquad \text{(C-6)}$$

$$\Phi_3 = - \Psi_{VR}(h) + \Psi_{CR}(h) \sin \beta_o h + f_3 \cos \beta_o h \qquad \text{(C-7)}$$

$$\Phi_4 = - \Psi_{VI}(h) + \Psi_{CI}(h) \sin \beta_o h + f_4 \cos \beta_o h \ , \qquad \text{(C-8)}$$

where

$$f_3 = - f_{1R} + 0.55\ \Psi_{CR}^{-1}\left[\Psi_{CI}(h) - (\pi/2) \cos \beta_o h\right] + \Psi_{CR} \sin \beta_o h + 0.275\pi\ \Psi_{SR}^{-1}\Psi_{CR}^{-1} \left[\Psi_{CR}(h) + \Psi_{CR} \cos \beta_o h\right] \qquad \text{(C-9)}$$

$$f_4 = - f_{1I} - 0.55\ \Psi_{CR}^{-1}\left[\Psi_{CR}(h) - \Psi_{CR} \cos \beta_o h\right] + (\pi/2) \sin \beta_o h + 0.275\pi$$

$$\Psi_{SR}^{-1}\Psi_{CR}^{-1}\left[\Psi_{CI}(h) + (\pi/2) \cos \beta_o h\right] \quad . \qquad (C-10)$$

It follows directly that

$$D' = A\ \Delta^{-1}\left[\Phi_3\Phi_2'' - \Phi_4\Phi_1''\right] = A\ \Delta^{-1}\Big\{\Phi_2''\left[- \Psi_{VR}(h) + \Psi_{CR}(h) \sin \beta_o h + f_3 \cos \beta_o h\right]$$

$$- \Phi_1''\left[- \Psi_{VI}(h) + \Psi_{CI}(h) \sin \beta_o h + f_4 \cos \beta_o h\right] \qquad (C-11)$$

$$D'' = A\ \Delta^{-1}\left[\Phi_4\Phi_1' - \Phi_3\Phi_2'\right] = A\ \Delta^{-1}\Big\{\Phi_1'\left[- \Psi_{VI}(h) + \Psi_{CI}(h) \sin \beta_o h + f_4 \cos \beta_o h\right]$$

$$- \Phi_2'\left[- \Psi_{VR}(h) + \Psi_{CR}(h) \sin \beta_o h + f_3 \cos \beta_o h\right]\Big\} \ , \qquad (C-12)$$

where $\Delta = \Phi_1'\ \Phi_2'' - \Phi_1''\ \Phi_2'$,

and $A = - 2\pi\ V_o^e/\zeta_o\ \Psi_{R1} \cos \beta_o h$. Since when $\beta_o h \to (2n + 1)(\pi/2)$, $\Psi_{VR}(h) \to \Psi_{CR}(h)$, $\Psi_{VI}(h) \to \Psi_{CI}(h)$, it is clear that all terms not multiplied by cos $\beta_o h$ in the numerators of (C-11) and (C-12) vanish so that the cos $\beta_o h$ in the denominator of A is cancelled and finite values of D' and D'' remain.

With D' and D'' given in (C-11) and (C-12), E' and E'' are given by (C-1a,b). Thus,

$$E' = - A\Big\{\left[\Phi_3\Phi_2'' - \Phi_4\Phi_1''\right]\Delta^{-1}(\pi/2) + \left[\Phi_4\Phi_1' - \Phi_3\Phi_2'\right]\Delta^{-1}\Psi_{hR} + \sin \beta_o h$$

$$+ 0.275\pi\ \Psi_{SR}^{-1}\Psi_{CR}^{-1} \cos \beta_o h\Big\} \qquad (C-13)$$

$$E'' = - A\Big\{\left[\Phi_3\Phi_2'' - \Phi_4\Phi_1''\right]\Delta^{-1}\Psi_{hR} - \left[\Phi_4\Phi_1' - \Phi_3\Phi_2'\right]\Delta^{-1}(\pi/2) - 0.55\ \Psi_{CR}^{-1} \cos \beta_o h\Big\} \ . \qquad (C-14)$$

Since the coefficients Φ_3 and Φ_4 individually reduce to the form f cos $\beta_o h$ when $\beta_o h \to (2n + 1)(\pi/2)$, it is clear that E'' remains finite when cos $\beta_o h$

$\rightarrow 0$ since $\cos \beta_o h$ in the denominator of A cancels with $\cos \beta_o h$ in the numerator. Exactly the same is true in E' except for the one term, $\sin \beta_o h$. When $\cos \beta_o h$ in the denominator of A is multiplied through this becomes $\tan \beta_o h$.

Hence, $E' \rightarrow \frac{2\pi V_o^e}{\zeta_o \Psi_{R1}} \left\{ (\text{finite term}) + \tan \beta_o h \right\}$. As pointed out in conjunction with (32), the sum of the terms $A \sin \beta_o(h-|z|) + E' \cos \beta_o z$ yields simply $(A \sin \beta_o h + E') \cos \beta_o z - A \cos \beta_o h \cos \beta_o z$ which remains finite when $\beta_o h \rightarrow (2n + 1)(\pi/2)$.