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THEORY OF THE INVESTIGATION OF THE ATMOSPHERE OF BINARY STARS BY USE OF THE ANALOGY BETWEEN GAS DYNAMICS AND SHALLOW WATER FLOW

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SUMMARY

Approximate solutions of the differential equations of continuum isentropic gas dynamics may be obtained for flows in the equatorial plane of a binary star atmosphere if the analogy between gas dynamics and shallow water flows is employed to define pertinent experiments in a rotating water tank. The magnitudes of the local inclination of the tank's bottom and of the resulting local gravitational force at the corresponding point in the binary star atmosphere are proportional. The analogy requires a constant polytropic coefficient of two. This application of the analogy rests on its extension presented here to rotating systems with external force fields. A theoretical evaluation of the analogy, for the case of the earth's atmosphere under hydrostatic conditions with polytropic coefficient equal to two, yields the correct result for the maximum height of the atmosphere where density reaches zero. This study is motivated by an application to the early stages after a hypothetical fission of the earth and the moon.

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LIST OF SYMBOLS

A	characteristic constant defined in (26)
B	characteristic constant defined in (32)
$\bar{c}_{pg}$	specific heat at constant pressure
F_{jw}	surface function defined in connection with (10)
$\mathcal{F}$	FROUDE-number
$\bar{g}$	gravitational acceleration
$\bar{H}_w$	depth of water rotating as a solid body, defined in (13)
$\bar{H}_{max}$	maximum altitude of atmosphere
$\bar{M}$	mass of celestial body
$\bar{p}$	pressure
$\bar{p}_{oo}$	atmospheric pressure at the free surface of water
$\bar{r}$	distance from center of rotation
$\bar{r}_o$	reference length
$\bar{R}_g$	gas constant
$\bar{t}$	time
$\bar{T}$	absolute temperature
$\bar{u},\bar{v},\bar{w}$	velocity components parallel to the coordinate axes
$\bar{x},\bar{y},\bar{z}$	Cartesian coordinates
$\bar{\Gamma}$	universal gravitational constant
ϵ	small dimensionless length, defined in connection with (34)
$\bar{\eta}_w$	disturbance amplitude of water depth
$\varkappa$	polytropic exponent of a gas
$\bar{\lambda}_w$	wave length corresponding to $\bar{\eta}_w$
μ	mass ratio
$\bar{\rho}$	density
Ψ	nondimensional gravitational potential
$\bar{\Psi}$	gravitational potential
$\bar{\omega}$	constant angular velocity

Subscripts

i	gas and water
g	gas
w	water
α,β	first and second celestial body, respectively
o	reference quantities or quantities taken at the surface of the earth
1	bottom of container of water
2	free surface of water
II	free surface of water in case of rotation of water as a solid body

All dimensional quantities are marked by a bar above the letters.

INTRODUCTION

According to observation, a binary star system may possess a common atmosphere which envelops both stars. The motion of a particle in such an atmosphere may be treated by use of the restricted three-body-problem in celestial mechanics. This approach has been employed in several publications, e.g., Kopal (1956, 1957) and Mrs. Gould (1957, 1959). (See part E of this paper.) According to Prendergast (1960), the mean free path in these atmospheres is of the order of magnitude of 1 - 10 km, i.e., this characteristic length is so much smaller than either the separation of the stars or the radius of one of them that collisions between particles in the atmosphere have to be taken into account. Therefore, Prendergast (1960) studied stationary solutions of the differential equations for continuum isentropic gas flows in the equatorial plane of the binary system by neglecting (a) the pressure gradient and (b) the velocity component normal to the "Lagrangian surfaces". Subsequently, Huang (1965) employed a stead-state celestial mechanics approach which neglects the pressure gradient but takes into account collisions between particles by invoking the statistical properties of the Jacobian constants of the colliding particles.

The theory presented in this paper was developed because of a desire to study gaseous motions in the vicinity of earth and moon during the early stages after a fission of their hypothetical parent body. By such motions, mass, energy, and angular momentum would be redistributed and possibly exported from this binary system.

The relatively large mean free path quoted above may be assumed to represent an average value for the entire atmosphere enveloping the stars revolving around each other. A rigorous hydrodynamic treatment of the atmosphere including the pressure gradient, therefore, seems desirable at least for the regions of maximum density within the atmosphere. In addition, the observed eruption phenomena in binary systems, e.g., Huang (1965), render desirable a non-stationary treatment of the atmosphere. If these atmospheric motions always are symmetrical with respect to the equatorial plane, the differential equations of motion and energy for three-dimensional isentropic gas flows reduce regorously to their two-dimensional versions in this plane; in the continuity equation, however, one term has to be neglected in this two-dimensional transient approach. Since Prendergast (1960) points out that the total mass of the atmosphere is concentrated about the equatorial plane, a study of flows in this plane appears to be relatively unimportant a restriction of generality which, in addition, governs the preceding pertinent publications mentioned above. Even in the equatorial plane, the time-dependent nonlinear system of differential equations with three independent variables precludes any analytic solutions. Numerical solutions reveal general characteristics only after a very large number of individual cases have been computed and evaluted. Because of this situation, the analogy between gas dynamics and shallow water motions is employed in the following to furnish the theoretical background for a water-tank-experiment simulating transient flows in the equatorial plane of the binary star atmosphere.* This requires a discussion of shallow water flows (part B) and an extension of the validity of this analogy to flows in rotating systems which are subjected to external force fields (part C). A presentation of an analogy such as in this paper should include some typical and sufficiently general results. Unfortunately, means are not available to the authors to carry out these experiments for the time being. Since this situation will not change in the foreseeable future, it was decided to publish the paper now. As a restricted test of validity and accuracy of the method presented here, the hydrostatic density distribution in the earth's atmosphere is obtained as a special application in part D.

SHALLOW WATER FLOWS - PART B

A container, schematically shown in Figure 1 is filled to a certain height with water. At time $\bar{t}_w = 0$, this water tank is rotating about the $\bar{z}_w$-axis with constant angular velocity $\bar{\omega}_w$. The acceleration of gravity $\bar{g}_w$ is assumed to act in the negative $\bar{z}_w$-direction. A cylinder with radius $\bar{r}_{ow}$ is mounted at the bottom of the container; its axis coincides with the $\bar{z}_w$-axis. Along the wetted surfaces of the cylinder and/or the bottom, sources or sinks may add water to the system or drain it off according to some law, which may vary locally and/or in time. Thus the total amount of water in the contained need not necessarily be constant. The shape of the bottom will be treated later on.

The subscripts w and g denote the water and the gas, respectively; the subscript i stands for both w and g. Dimensional properties are marked by an upper bar. The angular velocities $\bar{\omega}_i$, the lengths $\bar{r}_{oi}$, and the densities $\bar{\rho}_{oi}$ will be used as constant reference quantities. Non-dimensional Cartesian coordinates x_i, y_i, z_i, velocities u_i, v_i, w_i, times t_i, pressures p_i and densities ρ_i are introduced by relations such as the following:

$$x_i = \frac{\bar{x}_i}{\bar{r}_{oi}} \; , \; u_i = \frac{\bar{u}_i}{\bar{r}_{oi}\bar{\omega}_i} \; , \; t_i = \bar{\omega}_i \, \bar{t}_i \; , \; p_i = \frac{\bar{p}_i}{\bar{\rho}_{oi}\bar{\omega}_i^2 \bar{r}_{oi}^2} \; , \tag{1}$$

$$\rho_i = \frac{\bar{\rho}_i}{\bar{\rho}_{oi}} \text{ with } \bar{\rho}_w = \bar{\rho}_{ow} = \text{const. and } \bar{\rho}_{og} = \bar{\rho}_g(\bar{r}_{og}) \; .$$

*For general information about this analogy, e.g., Courant/Friedrichs (1948) and Wehausen/Laitone (1960) are recommended.

$$\mathcal{F}_i = \bar{r}_{oi}\,\bar{\omega}_i^2/\bar{g}_i = \text{const.}, \qquad (2)$$

a FROUDE-number $\mathcal{F}_i$ is introduced as the ratio of the centrifugal acceleration $\bar{r}_{oi}\,\bar{\omega}_i^2$ at the distance $\bar{r}_{oi}$ from the axis fo rotation to the gravitational acceleration $\bar{g}_i$. The symbol D/Dt_w denotes the substantial derivative with respect to time t_w. According to Lamb (1932, p.4 and p.318), the equations of motion of the water flow read as follows in a system rotating at constant angular velocity $\bar{\omega}_w$:

$$\frac{Du_w}{Dt_w} - 2v_w - x_w = -\frac{\partial p_w}{\partial x_w}, \quad \frac{Dv_w}{Dt_w} + 2u_w - y_w = -\frac{\partial p_w}{\partial y_w}, \qquad (3)$$

and

$$\frac{Dw_w}{Dt_w} = -\frac{\partial p_w}{\partial z_w} - \frac{1}{\mathcal{F}_w}. \qquad (4)$$

Here (4) was obtained from

$$\bar{\rho}_w\,\frac{D\bar{w}_w}{D\bar{t}_w} = -\frac{\partial \bar{p}_w}{\partial \bar{z}_w} - \bar{\rho}_w\,\bar{g}_w, \qquad (5)$$

using relations (1). The left hand sides of (3) - (5) express the mass-fixed accelerations in a system at rest provided the quantities of the rotating system are used. According to Goldstein (1960, p. 12), the equation of continuity is independent of the chosen coordinate system, since it simply states the constancy of a scalar quantity. It thus reads

$$\frac{\partial u_w}{\partial x_w} + \frac{\partial v_w}{\partial y_w} + \frac{\partial w_w}{\partial z_w} = 0 \quad . \tag{6}$$

For $\omega_w = 0$, the basic simplifying assumptions of the shallow water theory are, see e.g., Wehausen and Laitone (1960, p. 667):

(I) irrespective of flows in the water, the pressure distribution in a vertical column of water is the same as in hydrostatics and

(II) the horizontal velocity components u_w and v_w are independent of z_w, (considering the no-slip condition at the bottom, this assumption can be satisfied only approximately and for short times.)

In addition to assuming the validity of (I) and (II) for $\omega_w \neq 0$, here also the following assumption has to be made:

(III) the hydrostatic pressure distribution holds true even in the presence of motions relative to the rotating frame of reference.

If the fluid rotates as a solid body, its free surface (subscript II) is given by

$$\bar{z}_{II}(\bar{r}_w) = \text{const.} + \bar{r}_w^{\,2}\bar{\omega}_w^{\,2}/2\bar{g}_w \text{ with } \bar{r}_w^{\,2} = \bar{x}_w^{\,2} + \bar{y}_w^{\,2} \quad . \tag{7}$$

Should any flows relative to the rotating frame of reference ("disturbances") be superimposed to this solid body rotation, the free surface is given by

$$z_w = z_{2w}(x_w,y_w,y_w) = \eta_w\,(x_w,y_w,t_w) + z_{II}\,(x_w,y_w) \quad .$$

If the vertical accelerations Dw_w/Dt_w caused by these disturbances are negligible due to a sufficiently small water depth, then (5) is satisfied by the hydrostatic distribution specified in assumption (III) above,

$$\bar{p}_w = \bar{\rho}_w \bar{g}_w \left[\bar{z}_{II}(\bar{x}_w,\bar{y}_w) + \bar{\eta}_w(\bar{x}_w,\bar{y}_w,\bar{t}_w) - \bar{z}_w\right] + \bar{p}_{oo}, \tag{8}$$

where $\bar{p}_{oo}$ is the atmospheric pressure acting upon the free surface. Eq. (8) has been derived in Lamb (1932, p. 318) for rotating shallow water flows. A nondimensional form of (8) is given by

$$p_w = p_{oo} + \left[(z_{2w} - z_{1w}) + (z_{1w} - z_w)\right] / \mathcal{F}_w \ . \tag{9}$$

The functions $F_{1w} = z_w - z_{1w}(x_w,y_w) = 0$ and $F_{2w} = z_w - z_{2w}(x_w,y_w,t_w) = 0$ represent the wetted bottom and the free water surface, respectively, see Figure 1. Therefore,

$$DF_{jw}/Dt_w = 0 \qquad\qquad j = 1,\ 2\ , \tag{10}$$

according to Lamb (1932, p. 7), since these surfaces are always composed of the same fluid particles and thus

$$w_{jw} = \frac{\partial z_{jw}}{\partial t_w} + u_{jw}\frac{\partial z_{jw}}{\partial x_w} + v_{jw}\frac{\partial z_{jw}}{\partial y_w}\ , \qquad j = 1,\ 2\ . \tag{11}$$

Because of assumption (II) and eq. (11), integration of the continuity equation (6) with respect to z_w in the limits z_{1w} and z_{2w} gives

$$\frac{\partial(z_{2w} - z_{1w})}{\partial t_w} + \frac{\partial}{\partial x_w}\left[(z_{2w} - z_{1w})u_w\right] + \frac{\partial}{\partial y_w}\left[(z_{2w} - z_{1w})v_w\right] = 0. \qquad (12)$$

For $\bar{\omega}_w = 0$, equations (3) and (12) represent the shallow water theory which is a first order approximation by expanding in powers of a parameter ε, defined as the ratio of the squares of the characteristic horizontal and vertical scales of length, Wehausen and Laitone (1960, p. 466 and 667). Subsequently, the wave length $\bar{\lambda}_w$ and the amplitude $\bar{\eta}_w$ of the disturbance are employed for these scales. According to Laitone (1962), a second order approximation yields in case of $\bar{\lambda}_w \rightarrow \infty$ and $\bar{\omega}_w = 0$

$$\frac{\partial \bar{p}_w}{\partial z_w} \propto - \bar{\rho}_w \bar{g}_w \left[1 - \frac{3}{2}\left(\frac{\bar{\eta}_w}{\bar{H}_w}\right)^2\right] \text{ and } \frac{\partial \bar{u}_w}{\partial \bar{z}_w} \propto \sqrt{\bar{g}_w \bar{H}_w} \left(\frac{\bar{\eta}_w}{\bar{H}_w}\right)^2 \qquad (13)$$

$$\text{where } \bar{H}_w = \bar{z}_{II} - \bar{z}_{1w} \quad .$$

Correspondingly, Laitone (1962) shows for $\bar{\lambda}_w \rightarrow 0$ and $\bar{\omega}_w = 0$

$$\frac{\partial \bar{p}_w}{\partial \bar{z}_w} \propto - \rho_w g_w \left[1 - \left(\frac{2\pi \bar{H}_w}{\bar{\lambda}_w}\right)^2 \frac{\bar{\eta}_w}{\bar{H}_w}\right] \text{ and}$$

$$(14)$$

$$\frac{\partial \bar{u}_w}{\partial \bar{z}_w} \propto \sqrt{\bar{g}_w \bar{H}_w} \left(\frac{2\pi \bar{H}_w}{\bar{\lambda}_w}\right)^2 \frac{\bar{\eta}_w}{\bar{H}_w} \quad .$$

Relations (13) and (14) show for $\bar{\omega}_w = 0$ that the following quantities have to be sufficiently small in order to allow an inviscid water flow to be approximated by the shallow water theory:

(a) the hydrostatic depth $\bar{H}_w(\bar{x}_w,\bar{y}_w) = \bar{z}_{II} - \bar{z}_{1w}$ of the water,

(b) the ratio $\bar{\eta}_w/\bar{H}_w$ of the disturbance amplitude $\bar{\eta}_w$ to $\bar{H}_w$, and

(c) the ratio of $\bar{H}_w/\bar{\lambda}_w$.

It is assumed here that conditions (a), (b), and (c) also govern the validity of the shallow water theory in the rotating water tank provided both $\bar{\omega}_w$ and $|\text{grad } z_{1w}(\bar{x}_w,\bar{y}_w)|$ are sufficiently small.

THE ANALOGY BETWEEN SHALLOW WATER AND GAS FLOWS - PART C

By assuming $z_g = w_g = 0$, a two-dimensional transient gas flow (subscript g) will be related to the three-dimensional transient shallow water flow (subscript w) treated in part B. By use of $H_w(1)$ as defined in (13) and taken at station $x_w^2 + y_w^2 = 1$, the following functional relationships now are introduced between the water flow and gas flows:

$$x_g = x_w; \quad y_g = y_w; \quad u_g = u_w; \quad t_g = t_w; \tag{15}$$

$$\rho_g = \left[z_{2w}(x_w,y_w,t_w) - z_{1w}(x_w,y_w)\right] / H_w(1) , \tag{16}$$

and

$$p_g = \left[z_{2w}(x_w,y_w,t_w) - z_{1w}(x_w,y_w)\right]^2 /2\mathcal{F}_w H_w(1) =$$

$$= \frac{\rho_g^{\,2} H_w(1)}{2\mathcal{F}_w} \quad . \tag{17}$$

The scales of length and time of the two flows under consideration are connected by the quotients $\bar{r}_{ow}/\bar{r}_{og}$ and $\bar{t}_w/\bar{t}_g = \bar{\omega}_g/\bar{\omega}_w$ = const., respectively. Since the isentropic flow of a chemically homogeneous gas is considered here, there exists a functional relation $p_g = p_g(\rho_g)$ between pressure and density. The special polytropic relation $p_g \propto \rho_g^{\,\varkappa}$ with $\varkappa = 2$, which follows immediately from (16) and (17), can only approximately hold true, since the kinetic theory of gases shows that $1 < \varkappa \leq 1.67$.

Substitution of (15) - (17) into (3) and (12) yields by use of (9)

$$\rho_g \left[\frac{Du_g}{Dt_g} - 2v_g - x_g\right] = -\frac{\partial p_g}{\partial x_g} - \rho_g \frac{\partial(z_{1w}/\mathcal{F}_w)}{\partial x_g}, \tag{18}$$

$$\rho_g \left[\frac{Dv_g}{Dt_g} + 2u_g - y_g\right] = -\frac{\partial p_g}{\partial y_g} - \rho_g \frac{\partial(z_{1w}/\mathcal{F}_w)}{\partial y_g}, \tag{19}$$

and

$$\frac{\partial \rho_g}{\partial t_g} + \frac{\partial}{\partial x_g}(\rho_g u_g) + \frac{\partial}{\partial y_g}(\rho_g v_g) = 0 . \tag{20}$$

The gas flow defined by (15) - (17) is two-dimensional, since the third coordinate z_w of the corresponding three-dimensional water flow is needed to describe the density ρ_g of the gas. Because of

$$\frac{\partial w_w}{\partial z_w} \neq 0,$$

it does not follow from (6) and (20) that ρ_g is constant even though this seems to be suggested by relations (1) and (15). By the argument following equation (5), the left-hand-sides of (18) and (19) are the mass-fixed accelerations in the rotating system of reference, and (20) is the two-dimensional equation of continuity of a compressible medium in this system. The last terms in the right hand sides of both (18) and (19) may be interpreted as due to a conservative body force, which can be derived from a potential $z_{1W}(x_g, y_g)/\mathcal{F}_W$. Thus (18) - (20) are the equations of motion and continuity (in the continuum regime) of two-dimensional gas flows subjected to conservative body forces in a system rotating at a constant angular velocity.

The first law of thermodynamics yields for any mass element of the supposedly isentropic gas flow the relation $\bar{c}_{pg}\, d\bar{T}_g = d\bar{p}_g / \bar{\rho}_g$ between the specific heat $\bar{c}_{pg}$ at constant pressure, the absolute temperature $\bar{T}_g$, the pressure $\bar{p}_g$, and the density $\bar{\rho}_g$ of the gas. Therefore, the energy following (5)

$$\bar{\rho}_g \, \bar{c}_{pg} \frac{D\bar{T}_g}{D\bar{t}_g} = \frac{D\bar{p}_g}{D\bar{t}_g} . \tag{21}$$

Eq. (21) is satisfied by any two-dimensional transient flow provided a thermally perfect gas and the polytropic relation $\bar{p}_g \propto \bar{\rho}_g{}^{\varkappa}$ with $\varkappa$ = const. are specified. Thus the relation between water and gas flows as given by (15) - (17) is a complete analogy within the limitations of the assumptions mentioned at various placed above.

The generalization of the analogy developed above consists of admitting (a) conservative body forced which determine the bottom shape $z_W = z_{1W}(x_W, y_W)$ of the water tank and (b) rotating systems. Variable bottom configurations have been used in many applications of the analogy to problems without body forces in order to admit arbitrary constant values of $\varkappa$. During the years 1952 to 1962, this issue initiated a discussion in which body forces were not mentioned at all. In agreement with the derivation presented above, Laitone (1952, 1953, 1961, 1962) and Wehausen and Laitone (1960) pointed out the necessity of $z_{1W}(x_W, y_W) \equiv$ const. in case of problems without body forces which were considered solely by them, Loh (1959, 1962) and Bryant (1960, 1962) remarked that $z_{1W}(x_W, y_W) \neq$ const. is admissible in view of the other approximations involved in the analogy, such as constant values of $\varkappa$.

The conditions for the validity of the shallow-water theory mentioned at the end of section B also apply to the analogy. Experiments by Laitone (1952) confirmed that the analogy yields satisfactory agreement between gas dynamics and water flows as long as for tap water with usual surface tension the water depth is smaller than one quarter inch and the wave length $\bar{\lambda}_w$ of the disturbances exceeds 3 inches, see also Laitone (1953, 1961, 1962). Under consideration of the surface tension of water, Gupta's analysis (1965) yields about the same limits for the validity of the analogy carried out in water tanks.

For sufficiently small values of $\bar{\omega}_w$, the wave systems caused by the tank's rotation (e.g., Lamb (1932)) presumably are negligible as compared to the wave systems which have been discussed above for $\omega_w = 0$. Since, in addition, a transient boundary layer growth at the tank bottom is due to the waves disturbing the solid-body rotation of the water, the relation $\bar{t}_g \bar{\omega}_g = \bar{t}_w \bar{\omega}_w$ following from (1) and (15) prohibits atmospheric motions to be studied in the water tank beyond sufficiently short time intervals $\bar{t}_g$ after the beginning of the said waves. The required small values of $\bar{\omega}_w$ demand a sufficiently small maximum inclination of the tank bottom. This may necessitate the addition of a detergent to the water because of the surface tension.

APPLICATION TO AN ASTROPHYSICAL AND TO A GEOPHYSICAL PROBLEM - PART D

The right-hand sides of (18) and (19) were interpreted as representing body forces, which result from a potential $z_{1w}(x_g, y_g)/\mathcal{F}_w$. The gravitational potential of spherically symmetric celestial bodies now will be employed to demonstrate the analogy. Figure 2 shows a "binary" system of two celestial bodies rotating about their common center of gravity S at the constant angular velocity $\bar{\omega}_g$. The masses of the bodies are $\bar{M}_\alpha$ and $\bar{M}_\beta$, $\bar{r}_\alpha$ and $\bar{r}_\beta$ denote the distance of any point $Q(\bar{x},\bar{y},\bar{z})$ from the centers of the two masses, respectively. The gravitational potential in the space outside M_α and M_β then is given by

$$\bar{\Psi} = \frac{\bar{\Gamma}\,\bar{M}_\alpha}{\bar{r}_\alpha} + \frac{\bar{\Gamma}\,\bar{M}_\beta}{\bar{r}_\beta}\,, \tag{22}$$

where $\bar{\Gamma}$ is the universal gravitational constant. In the following, the contribution of the atmospheric matter to the gravitational potentials will be neglected. The mass ratio $\mu = \bar{M}_\beta/\bar{M}_\alpha$ is introduced and the radius of any one of the two celestial bodies is used as a constant reference length $\bar{r}_{og}$, i.e., $\bar{r}_\alpha/\bar{r}_{og}$ and $r_\beta = \bar{r}_\beta/\bar{r}_{og}$. Since $\bar{\Psi}$ has the units (cm^2 sec^{-2}) the expression $\bar{r}_{og}{}^2\,\bar{\omega}_g{}^2 = \text{const.}$ is employed to obtain nondimensional potential

$$\Psi = \frac{1}{\mathcal{F}_\alpha}\left[\frac{1}{r_\alpha} + \frac{\mu}{r_\beta}\right], \tag{23}$$

where the Froude-number

$$\mathcal{F}_\alpha = \frac{\bar{r}_{og}^{\,3}\,\bar{\omega}_g^{\,2}}{\Gamma\,\bar{M}_\alpha}$$

is the ratio of centrifugal to gravitational forces at the surface of the celestial body $\bar{M}_\alpha$. In applications of the analogy, $\mathcal{F} = \mathcal{F}_g$. The components of the gradient of this potential now will be equated to the last terms in equations (18) and (19) to obtain an expression for the bottom shape in the water tank

$$-\ \mathrm{grad}\ \frac{z_{1W}}{\mathcal{F}_W} = \mathrm{grad}\ \Psi\ . \tag{24}$$

With z_o as a constant of integration, it is found that

$$z_o - z_{1W} = \frac{\mathcal{F}_W}{\mathcal{F}_\alpha}\left[\frac{1}{r_\alpha} + \frac{\mu}{r_\beta}\right]\ . \tag{25}$$

The analogy as expressed by (15) - (17) enables one to study two-dimensional atmospheric motions subjected to a gravitational field $\bar{\Psi}(\bar{x}_g,\bar{y}_g,\bar{z}_g)$ by experimental simulation in a water tank with the bottom $z_W = z_{1W}(x_W,y_W)$. These atmospheric flows are restricted to the equatorial plane $z_g = 0$ where the complete differential equations of motion and energy take on the special two-dimensional forms (18) - (19) and (21), respectively, provided the motion is isentropic and symmetrical with respect to the plane $z_g = 0$. Due to this symmetry, the derivatives with respect to z_g of u_g, v_g, p_g, ρ_g, and T_g vanish identically as functions of x_g, y_g and t in the plane $z_g = 0$. Since $\partial w_g/\partial z_g \neq 0$ for $z_g = 0$, the three-dimensional equation of continuity retains this term $\partial w_g/\partial z_g$ which is absent in (20). If

$$(\partial w_g/\partial z_g)\big|_{z_g} = 0$$

is known as a function of x_g, y_g and t_g by use of some information outside of the analogy under discussion, this term can be represented by a suitable distribution of sources and sinks in the bottom $z_w = z_{1w}\ (x_w\, my_w)$ of the water tank. The equations of continuity (6) and (20) then possess corresponding additional source terms.

Since an experimental verification and test of the accuracy of the analogy can not be carried out by the authors for the time being, a theoretical check is desirable. Such a test of accuracy is provided by the application of the analogy to the earth's atmosphere under conditions of a known density distribution, i.e., in case of the hydrostatic atmosphere. In this case, the above equations simplify somewhat since $\mu = 0$ and $\Gamma M_\alpha / r_{og}^{\ 2} = g_o$, where g_o is the gravitational acceleration at the surface of the earth. Then the quotient $_w/\ _\alpha$ reduces to

$$\frac{\ _w}{\ _\alpha} = \frac{r_{ow}\,\omega_w^{\ 2}}{r_{og}\,\omega_g^{\ 2}} = A \ , \tag{26}$$

i.e., the ratio of the centrifugal forces in the container to those at the earth's equator. By realizing that $r_\alpha = r_g = r_w$ because of (15) and by taking $z_{1w} = 0$ at $r_w = 1$, equation (25) yields

$$z_{1w} = A\ (1 - \frac{1}{r_w}) \ . \tag{27}$$

The atmosphere enveloping the earth is assumed to rotate as a solid body, such that in the analogy $z_{2w} = z_{IIw}$ and $\eta_w = 0$. From (7) it is thus found

$$z_{IIw} = \text{const.} + \frac{\ _w}{2}\ r_w^{\ 2} \ . \tag{28}$$

Because of (16) and (17),

$$(29)\quad \frac{\rho_g}{\rho_g} = \frac{1}{2\mathcal{F}_w}\left(z_{IIw} - z_{1w}\right).$$

The equation of state for a thermally perfect gas, $\bar{p}_{og} = \bar{\rho}_{og}\,\bar{R}_g\,\bar{T}_{og}$, requires at the surface of the earth (subscript o)

$$(30)\quad \frac{\rho_{og}}{\rho_{og}} = \frac{\bar{R}_g\,\bar{T}_{og}}{\bar{g}_o\,\bar{r}_{og}}\,\frac{1}{\mathcal{F}_g}\,.$$

Because of $\mathcal{F}_\alpha = \mathcal{F}_g$ and $z_{1w} = 0$ at $r_w = 1$, the expressions (29) and (30) yield

$$(31)\quad z_{IIw}\Big|_{r_w=1} = 2\,A\,B$$

with

$$(32)\quad B = \frac{\bar{R}_g\,\bar{T}_{og}}{\bar{g}_o\,\bar{r}_{og}}\,.$$

From (27), (28), and (31) finally

$$(33)\quad z_{IIw} - z_{1w} = A\left[2B + \frac{\mathcal{F}_g}{2}\left(r_w^2 - 1\right) - \left(1 - \frac{1}{r_w}\right)\right]$$

is obtained to express the nondimensional density distribution in the earth's atmosphere under hydrostatic conditions and for $\varkappa = 2$. From left to right, the terms in the bracket of (33) represent the contributions of the earth's surface temperature, of the centrifugal force due to the assumed rigid-body-rotation, and of the gravitational force.

Because of the small ratio of the maximum extension of the atmosphere to the earth's radius, it is reasonable to write $r_g = r_w = 1 + \epsilon$, where second and higher powers of ϵ are to be neglected. By realizing that $1/(1+\epsilon) = 1 - \epsilon + \epsilon^2 \cdots$, the following relation is obtained to replace (33):

$$(34)\quad z_{IIw} - z_{1w} = A\left[2B - \left(1 - \mathcal{F}_g\right)\epsilon\right].$$

Because of (17), the pressure variation in the earth's atmosphere according to the analogy is

$$\frac{p_g}{p_{o_g}} = \left(\frac{z_{IIw} - z_{Iw}}{(z_{IIw} - z_{Iw})_o}\right)^2 = \left(1 - \frac{1 - \mathcal{F}_g}{2B}\,\epsilon\right)^2 , \tag{35}$$

where the effect of the centrifugal force is negligible even at the earth's equator since $\mathcal{F}_g = 3.32 \times 10^{-3}$. Because of $B = 1.27 \times 10^{-3}$ for a surface temperature $\bar{T}_{og} = 293$ ($^\circ$ K), the analogy predicts that the atmospheric pressure $\bar{p}_g$ vanishes at a maximum altitude as given by

$$\bar{H}_{max} = \bar{r}_{og} \cdot \epsilon_{max} = \bar{r}_{og} \cdot \frac{2B}{1 - \mathcal{F}_g} \approx \frac{1}{394}\,\bar{r}_{og} \approx 16 \text{ km} .$$

It will now be shown that this result for $\bar{H}_{max}$ agrees with the one of the well-known hydrostatic polytropic atmosphere with $\varkappa = 2$, where is defined by (a) neglecting rotation, i.e., $\mathcal{F}_g = 0$ and (b) by assuming a constant acceleration of gravity. Under these conditions Prandtl/Tietjens (p. 33) show that

$$\frac{p_g}{p_{o_g}} = \left[1 - \frac{\varkappa - 1}{\varkappa}\,\frac{\bar{h}}{\bar{h}_o}\right]^{\frac{\varkappa}{\varkappa - 1}} . \tag{36}$$

where in the notation of this paper

$$\bar{h} = \bar{r}_g - \bar{r}_{og} \quad \text{and} \quad \bar{h}_o = B\,\bar{r}_{og} . \tag{37}$$

Because of $\epsilon = h/r_{og}$, Eq. (26) yields in case of $\kappa = 2$ the relationship

$$\frac{p_g}{p_{og}} = (1 - \frac{\epsilon}{2B})^2, \tag{38}$$

which is identical to Eq. (35) for $\mathcal{F}_g = 0$. It should be recalled here that the polytropic hydrostatic atmospheres defined by conditions (a) and (b) mentioned above possess a finite maximum altitude $\bar{H}_{max}$, with the exception of $\kappa = 1$.

OBSERVATIONS AND GENERAL CONSIDERATIONS WITH RESPECT TO THE ATMOSPHERE OF BINARY STARS - PART E

Prominance activity in the sun is a well known phenomenon. Observation of binary stars suggests that the prominence activity of the secondary component is stronger on the hemisphere facing the primary star. Appropriate initial velocity vectors of these prominences could result in a gaseous stream toward the primary or escape from the binary system, Sahade (1960).

Within the context of the restricted three-body-problem of celestial mechanics, several authors have studied trajectories of ejected particles in the equatorial plane. For a particle moving in this plance, the Jacobi integral shows the existence of curves of zero velocity in a frame rotating with the binary system. These closed curves are the "Roche equipotentials" on which the geopotential is constant, i.e., the sum of the gravitational potentials of the component stars and the potential of the centrifugal force due to the rotation of the system. These Roche equipotentials define admissible cross sections of each component star in the equatorial plane. The equipotential enclosing the largest area in this plane and only one component star is called the Roche limit. For a particle having a given value of the Jacobi integral, the corresponding Roche equipotential represents a barrier which the particle, of moving inside its folds, can never penetrate. The trajectories of particles ejected from a component star completely filling its Roche limit take place in an area of the equatorial plane whose Roche equipotential boundary envelopes both component stars. For this case, trajectories of particles have been computed by numerical integrations of the equations of motion of the restricted three-body-problem, Kopal (1959). The results could be summarized as follows: (1) particles fall upon the companion star, (2) particles fall back upon the parent star, and (3) particles escape from the system if they possess sufficiently high initial velocities, generally of the order of several hundred kilometers per second.

According to Sahade (1960), velocities of 700 km/sec have been observed in the binary system V 444 Cygni. Because of the smaller masses, considerably smaller velocities would be needed for the escape from the earth-moon system after fission of its hypothetical parent body.

These results of celestial mechanics referred to above do not account for interactions between particles. Struve (1957), therefore, has pointed out that it would be important to consider the problem at hand also from a hydrodynamical point of view and to allow for the effect of radiation pressure and magnetic forces. The theory presented in Parts B, D, and D of this paper furnishes a background for a water-tank experiment which attempts to simulate gaseous streams in the equatorial plane of a binary star system by accounting for gravitational, inertial, and hydrodynamic pressure forces. These results would pertain to the continuum flow regime which is separated from motions due to applications of celestial mechanics by the rarefied flow region. The transient water-tank experiment, therefore, would furnish a necessary supplement to published results from (a) applications of celestial mechanics, Kopal (1959) and (b) stationary continuum flow theory under rather restrictive assumptions (see part A), Prendergast (1960).

COMMENTS WITH RESPECT TO THE EXPERIMENTAL SIMULATION OF THE FLOW IN THE EQUATORIAL PLANE OF THE BINARY STAR SYSTEM - PART F

In the water tank, the primary and the secondary stars are represented by cylinders mounted parallel to the table's z-axis of rotation, see Fig. 3. Since the primary star by definition possesses the larger mass, the bottom of the tank reaches its smallest z-coordinate at its intersection with the "primary" cylinder. If the surfaces of the cylinders are sufficiently smooth, the motion of the water is very little affected by rotations of the cylinders about their own aces of symmetry. Therefore, the wetted portions of these surfaces may have to be roughened artificially if this eigen-rotation of the cylinders is to influence appreciably the water flow. Even though the gravitational fields extend to infinity in the equatorial plane, it is sufficient to place a board parallel to the axis of rotation of the system on the bottom of this tank, provided the board is located beyond the "synchronous satellite orbit" of the system. A water sink may be placed at the intersection of the board and the bottom.

Under hydrostatic conditions, the water depth at the circumference of one of these cylinders may be prescribed arbitrarily.

Because of (32) and (33), this depth is proportional to the temperature of the atmosphere at the surface of the corresponding component star. This temperature should be sufficiently small to ensure the gaseous state of the atmosphere since (16), (17), and the equation of state of a thermally perfect gas show that the temperature of the atmosphere is proportional to its density in the analogy being employed here. The tank's bottom between the cylinders does not have to be covered by water continuously in time. As far as mass transfer from the secondary to the primary star is concerned, it is sufficient to have intermittently water jets emanate from the "secondary" cylinder. The mass flow from the secondary to the primary cylinder can be visualized by adding dye to the water pouring out of the secondary cylinder.

CONCLUDING REMARKS - PART G

The objective of conducting a water-tank-experiment should be a qualitative simulation of transient astrophysical effects such as the ones mentioned in Part E of this paper. The following details militate against quantitative validity of the results:

(1) the assumption of a constant polytropic coefficient (of 2) which is essential to the analogy in every possible version:

(2) the assumption of isentropic gas flow;

(3) the treatment of flows in the equatorial plane without regard for the interaction with the general three-dimensional flow field outside of this plane; this shortcoming manifests itself in neglecting the term $\partial w/\partial z$ in the three-dimensional continuity equation;

(4) the omission of radiation pressure, turbulence, viscous friction, heat conduction, and other real gas effects;

(5) experimental difficulties, e.g, such as the ones discussed at the ends of part B and part C.

Even though this seems to be a formidable accumulation of shortcomings of the method presented in this paper, the approach may be expected to yield more reliable and realistic results in sufficiently dense atmospheres than the theoretical methods employed by other authors, see Section A.

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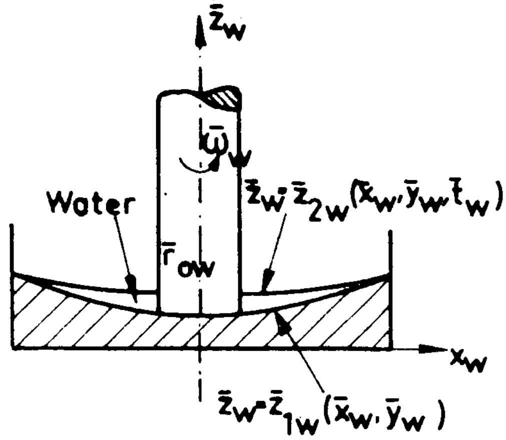

Figure 1: Water Tank.

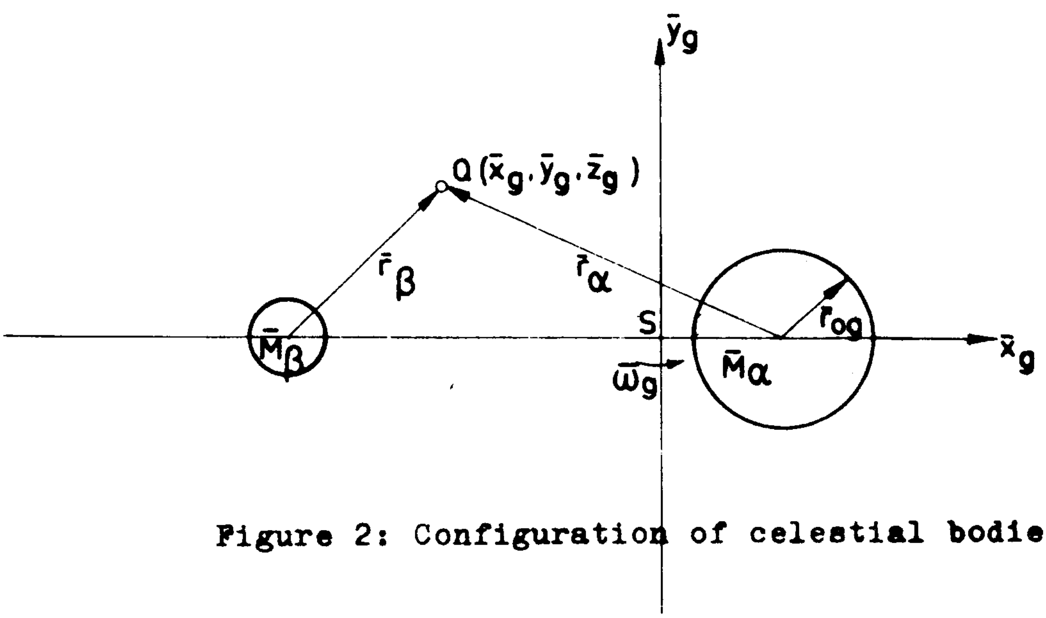

Figure 2: Configuration of celestial bodies.

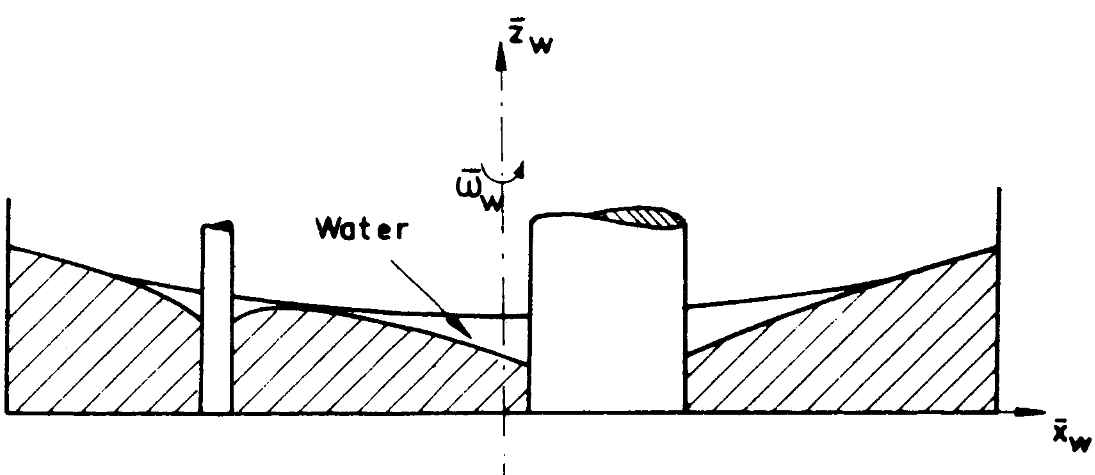

Figure 3: Schematic configuration of water tank suggested to study the atmosphere around the celestial bodies shown in figure 2.