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RM-5228-NASA
SEPTEMBER 1967

THE RAND SYNC-SAT CALCULATOR

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

In RAND's studies of advanced communication satellite technology, sponsored by the National Aeronautics and Space Administration, the potential of highly directive satellite antennas for greatly increasing system capabilities has been under investigation. An important element in the application of these high-gain antennas is an attitude control system to provide the desired accuracy and life expectancy. In the investigation of potentially attractive attitude and orbit control techniques, it became necessary to establish the control requirements for possible application of high-gain antennas in synchronous orbits. This Memorandum contributes to this important component of the investigation and should be useful to others concerned with the design of such systems, as well as those concerned with planning for operational application of such systems. A slide chart calculator, called the RAND Sync-Sat Calculator, is provided to facilitate planning studies involving geometric-geographic relations.

SUMMARY

Advancing technology will permit the exploitation of high-gain, narrow antenna beams to greatly increase the capability of communication or broadcast satellites in synchronous orbits. The RAND Sync-Sat Calculator was developed to facilitate the calculation of some of the geometric relations between a synchronous, near-equatorial satellite and one or more earth stations. This Memorandum supplements the instructional information printed on the slide chart, provides some examples of its use, and documents the mathematical background of the slide chart.

CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Az	= azimuth of satellite viewed from earth station. A horizontal angle measured clockwise from the northern direction of the earth station meridian
a_1	= amplitude of the first-order pitch steering rule
B	= azimuth of earth station from subsatellite point
b_1	= amplitude of the first-order roll steering rule
b_2	= amplitude of the second-order roll steering rule
i	= satellite orbit plane inclination relative to the earth's equatorial plane
H	= satellite angular elevation above the horizon when viewed from an earth point
K	= parameter describing contours of first-order roll steering
ℓ	= dimension (length, st mi) of the intersection of a conical beam with the earth's surface. It lies along the great circle, R
N	= number of consecutive contour intervals required to span a set of points
n	= the order of a steering rule (first, second, etc.)
$\vec{P}$	= arbitrary point on the earth's surface described by latitude, φ, and relative longitude, λ, or by Cartesian coordinates x, y, z
p	= geocentric radius of a synchronous satellite, 6.6166 earth radii
R	= great-circle distance (angular measure) from the subsatellite point to point $\vec{P}$
s	= distance from satellite to point $\vec{P}$
T	= tilt, the angle between a vector from the satellite to the subsatellite point and a vector from the satellite to $\vec{P}$
t	= time, days

u, v, w u', v', w' = direction cosines of $\vec{P}$ in the satellite body axes. Unprimed values refer to a reference state, primed values to a displaced state. These direction cosines are used in Appendix B.

w = dimension (width, st mi) of the intersection of a conical beam with the earth's surface. w is perpendicular to R and hence to ℓ. This w appears in the body of the Memorandum and in Appendix A.

X, Y, Z = satellite body axes defined in Appendix B following Eq. (B-3)

x, y, z = a rotating, earth-centered Cartesian frame, with z coincident with north, y directed toward the synchronous satellite's equatorial crossing point, and x orthogonal in a right-handed sense. x, y, and z designate an earth point in this frame; x_s, y_s, and z_s refer to the satellite in the same frame.

α = satellite pitch, a rotation about body axis Z. When unsubscripted, refers to the total of the various pitch terms, α_n

β = satellite roll, a rotation about body axis X. When unsubscripted, refers to the total of the various roll terms, β_n

λ = longitude of an earth station relative to the subsatellite point (geostationary satellite) or to the equatorial crossing point for a nonstationary, synchronous satellite

σ = apex angle of a conical beam

Φ = instantaneous latitude of a nonstationary, synchronous satellite

φ = latitude of an earth station

ω = angular rate of the earth and of a synchronous satellite, 2π rad per day

I. INTRODUCTION

The RAND Sync-Sat Calculator, enclosed in a pocket on the inside of the back cover of this Memorandum, may be used to solve problems involving the geometric relations between a synchronous, near-equatorial, communication satellite and earth stations visible to the satellite. The calculator is divided into four parts, each requiring independent settings. Parts 1, 2, and 3 pertain to an equatorial satellite or a satellite at the instant of equatorial crossing. Part 4 relates to satellites in inclined orbits and to the special problem of directing multiple, narrow beams (beamwidth $< 1.2^{\circ}$) to corresponding earth stations.

Part 3 of the calculator (map side) is useful for determining the field of view of a satellite placed at various longitudes and for estimating the reduction in the field of view if the minimum allowable elevation of the satellite above the horizon is prescribed. The distorted rectangular boxes represent $1^{\circ} \times 1^{\circ}$ solid angles projected from the satellite onto the earth's surface. These 1° dimensions can be used to estimate beam dimensions required to cover large areas.

Parts 1 and 2 of the calculator provide parameters relevant to narrow beams and small areas. The azimuth and elevation of a satellite (at known or assumed longitude) can be found for an earth station located at latitude φ and longitude λ relative to the satellite. The tilt, T, is the angle subtended at the satellite by the subsatellite point and an earth station. The principal use of this parameter is to define the proximity of a beam to the horizon, since, if T plus beam half-width exceeds 8.69°, some portion of the beam will pass over the horizon. The spot size, length (ℓ), and width (w) are the dimensions of a beam outline on the earth's surface. The numbers actually represent the elliptical intersection of a conical beam with an earth tangent plane at φ, λ. The scales read directly for a beamwidth of 1°, and these scale values can be multiplied by actual beamwidth to determine the ellipse axes for other beamwidths.

Part 4 of the calculator concerns the problem of individually tracking several earth points from a synchronous, but nonequatorial, satellite. For example, a single satellite might link a number of metropolitan areas via a corresponding number of narrow beams, with each beam directed toward a particular earth station. Accurate pointing is necessary to insure adequate signal level at the earth stations and at the satellite. In an inclined orbit, and for beam directions fixed relative to the satellite body axes and steered collectively by controlling satellite attitude, there are residual geometric errors (angular deviations) which depend on the distribution of earth points and the orbit plane inclination. For small orbital inclination ($i < 8^{\circ}$), the angular deviations at all visible points on the earth's surface are within $\pm$ 0.08 i°. For beamwidths of the order of 0.16 i°, angular deviations can degrade system performance for some earth station distributions. The contours of Part 4 can be used to determine angular deviations for arbitrary orbit plane inclination ($< 8^{\circ}$) and earth station distributions.

Angular deviations can be decreased by restricting the orbital inclination to small values. The combined effects of the sun, moon, and earth produce an orbital regression(1,2) and, therefore, a time-varying orbit plane inclination. These effects can be countered by thrusts applied to the satellite if a small inclination is to be maintained over an extended period. However, it has been shown(2) that a plane, invariant to these effects, exists at an inclination of $7^{\circ}20'$ to the equatorial plane; no control thrust is necessary to maintain this inclination. Most near-equatorial, synchronous satellites need not be designed for inclinations greater than $7^{\circ}20'$, and angular deviations greater than 0.6° are unlikely. For beams narrower than a few degrees, there will be some trade-off between the desirability of the narrow beam and the cost of maintaining a given small inclination. For the case of synchronous satellites with inclinations greater than 8°, the angular deviations can be calculated by the methods described in Appendix B.

II. CALCULATOR APPLICATIONS

LARGE AREA PROBLEMS (CALCULATOR PART 3)

Definition of the field of view of the satellite, the best longitude at which to place a cross-continental or intercontinental communications satellite, or the beamwidth required to cover a continent are large area problems involving a substantial fraction of the earth's surface. Part 3 of the Sync-Sat Calculator is designed to examine such questions.

Field of View

The outer dotted curve represents the horizon seen by a synchronous satellite located at zero degrees latitude. Points outside this contour cannot be seen by the satellite. Points on this contour are located at a great-circle distance of 81.31° from the subsatellite point.

Earth stations on the next inner contour will view the satellite at an elevation of 5° above the horizon; the contour is 76.4° from the subsatellite point, about 340 st mi inside the outer contour. The tips of the radial arms are at a distance of 69.5° from the subsatellite point. Earth stations at this distance will see the satellite approximately 12° above the horizon.

The inner dotted contours are contours of constant "tilt angle." The tilt angle is defined at the satellite as the angle subtended by an earth station and the subsatellite point. These contours of constant tilt angle are also lines of constant satellite elevation above the horizon and constant distance from earth station to subsatellite point.

The closed contours of constant elevation or constant tilt angle may be used to estimate the usable field of view for various assumed constraints upon the minimum satellite elevation above the horizon.

Determination of Best Satellite Longitude

If the fuel requirements of station keeping are ignored and only the geometric factors are considered, then a logical position to locate a synchronous satellite is the longitude at which the maximum tilt angle (over all earth stations) is a minimum. Thus, if a collection of earth stations is noted on the map, the overlay can be shifted in longitude to find the location at which the collection can be encompassed by the smallest possible tilt angle. If there are only two earth stations in the collection, they will both have the same tilt angle and will be equidistant from the subsatellite point.

The above remarks apply to satellites which are fixed at zero latitude. Apparent motions in latitude will occur if the satellite orbit plane does not coincide with the earth's equatorial plane. Compensating satellite attitude motions are required to keep antennas pointing toward earth stations. This subject is discussed in detail later; here it is only necessary to understand that perfect compensation is not possible, and that angular deviations due to the changing aspect of the earth are intrinsic in the geometry. The calculator figure labeled "Angular Deviations" provides contours along which deviations can be minimized by appropriate compensation. These contours are not of constant distance from the sub-satellite point because the errors stem from the preferential satellite motion in latitude. For satellites having multiple narrow beams and operating in an orbit of small inclinations, the best location might be that longitude which minimizes the maximum angular deviations. The procedure for locating this longitude is to move the calculator in longitude to bring a collection of earth points as closely as possible to a single contour. The maximum angular deviations will occur at those points furthest from the contour.

Beam Dimensions for Large-Area Coverage

The radial arms of calculator Part 3 are centered on great circles which leave the subsatellite point at azimuths of 0°, 30°, 60°, 90°, and 120°, etc. The width of these arms corresponds to a beam 1° wide, and the partitions along the arms provide a 1° measure in the orthogonal dimension. Thus, each partitioned region represents the projection of a $1^\circ \times 1^\circ$ solid angle upon the surface of the earth. These $1^\circ \times 1^\circ$ squares may be used to estimate the number of square degrees of beam required to cover a large area.

SMALL-REGION PROBLEMS (CALCULATOR PARTS 1 AND 2)

Calculator Parts 1 and 2 provide a means of calculating the size, shape, and orientation of the figure obtained by projecting a beam of circular cross section upon the surface of the earth. It is assumed that a satellite position has been fixed by the previous considerations so that the location of an earth station can be specified relative to the subsatellite point. A procedure for plotting the beam outline in the vicinity of this earth station consists of the following steps:

1. On Part 1, set the earth station relative longitude to the earth station latitude and read the dimensions ℓ and w of the 1° solid-angle intersection. Multiply these by the actual beams' dimensions to obtain the actual projected dimensions in statute miles.

2. On Part 2, set earth station relative longitude to earth station latitude and read azimuth of satellite. The azimuth scale reads from 98° to 172° and unless the satellite lies east and south of the earth station, it is necessary to transform scale values into the proper quadrant in order to represent satellite azimuth as an angle measured clockwise from north through 360°. The required transformations are

Satellite Direction from Earth Station	Transformation
East and south	None
West and south	Subtract scale value from 360°
East and north	Subtract scale value from 180°
West and north	Add 180° to scale value

The calculator fails when latitude φ is less than 1° or longitude λ is less than 8°. For these cases, azimuth can be computed according to:

$$\tan Az = - \tan \lambda / \sin \varphi$$

3. On a map or chart of the region containing the earth station, draw a line through the earth station in the direction of the subsatellite point. This line crosses the local meridian at Az and is a portion of the great-circle arc which passes through the subsatellite point. Using the map scale in statute miles, sketch the length dimension along the great-circle arc and sketch the width dimension perpendicular to the great-circle arc. The pattern for a circular beam will be roughly elliptical with major and minor axes given by the length and width dimensions, respectively.

4. The earth station antenna direction is prescribed by the satellite azimuth Az and its apparent elevation H above the horizon. This elevation angle can be found on Part 1 of the calculator.

In many problems, the earth station location is not specified in advance, but is to be located in the vicinity of some known point (for example, a city). In this case, the satellite azimuth Az and the pattern dimensions may be determined using the known point; these parameters will be approximately correct in the vicinity of the known point. The pattern can be shifted (at constant azimuth) over a map of the area to determine the best location of the earth station. In such problems, it may be useful to construct a properly scaled overlay rather than to draw directly on the map. The overlay can then be shifted (at constant orientation) to examine reasonable earth station locations when geographical features and interference with other earth stations constrain the possible choices.

Arc Distance to Subsatellite Point

Having set earth station latitude and relative longitude in Part 1, all other number pairs φ, λ, which can be read at the same setting, are points equidistant from the subsatellite point. In particular, λ at $\varphi = 0$ (or φ at $\lambda = 0$) is the arc distance R from the earth station to the subsatellite point. This arc must always equal 90°-T-H.

Special Techniques Near the Horizon

For earth stations located near the perimeter of the visible earth, some portion of a finite beam may pass over the horizon. If a portion of a narrow circular beam approaches the horizon, the previously described elliptical intersection becomes distorted, and more accurate location of the beam extremities can be found by the following procedure.

For an earth station at φ, λ, find the tilt angle T and the arc length R as indicated in the preceding paragraph. For a beamwidth of σ°, set $T + 0.5\sigma$ on the tilt scale and read R(max) at $\varphi = 0$. Repeat with $T - 0.5\sigma$ to obtain R(min). Then

$$\ell_1 = 69.3[R(\max) - R] \quad \text{st mi}$$

$$\ell_2 = 69.3[R - R(\min)] \quad \text{st mi}$$

ℓ_1 and ℓ_2 are respectively the distances from the beam axis to its down-range and uprange extremities. The value of $\ell_1 + \ell_2$ can differ from the ℓ calculated on Part 1 since the tangent plane approximation is not used when ℓ_1 and ℓ_2 are calculated from arc lengths.

Beam Patterns for Large Beams

The procedure described above permits the plotting of beam outlines on flat maps having approximately uniform mileage scales over the region to be plotted. This assumes that the geographical region is also parallel projected on a tangent plane with satisfactory accuracy. If the largest dimension of an area can be spanned by a great-circle arc of R deg, then scale errors on the tangent plane projection will not exceed

$$E = (R/4.1)^3 \qquad \text{st mi}$$

When this error is excessive, beam outline patterns may be estimated with the graphic aids given in Ref. 3.

Example of a Small Region Problem

By using a number of spatially distinct beams, enough isolation may be provided between adjacent beams to enable the repeated use of the same spectrum, thus increasing the total communication capacity possible to a large metropolitan region. A system of n beams will require n spatially distinct earth stations which could be linked by surface micro-wave relay or land line to a single population center. The question then arises: How many spatially distinct earth stations can be placed in the vicinity of a particular population center? The answer depends upon the assumed beamwidth, as well as the distribution of population centers. The example treated here, using the RAND Sync-Sat Calculator, examines the question of earth station locations in the vicinity of Los Angeles for the hypothetical case of satellite linkage between Los Angeles and New York City.

Figure 1 illustrates seven beams at maximum density. If the minimum angular separation between beams is specified by the angle σ, then the hexagonal arrangement implies that the six outer beams are uniformly spaced about a circle of diameter 2σ. This circle will be projected upon the ground as a near ellipse, representing the minimum perimeter over which the six peripheral earth stations might be disposed. The analysis is outlined in the following steps for a beam separation σ of 0.16°, which would probably require satellite antenna gains in excess of 60 db to provide the isolation needed.

1. With Part 4 of the calculator, it is found that a contour (interpolated) of zero first-order angular deviation can be passed through the New York City and Los Angeles areas if the satellite is placed at approximately 87°W.

2. Taking the actual coordinates of the Los Angeles region as 118°W and 34°N, the coordinates of Los Angeles relative to the subsatellite point are 31°W and 34°N.

3. Part 2 of the calculator indicates that the satellite azimuth is about 133°. A line is sketched on the map in Fig. 2 in the direction 133°E of the 118° meridian. From Part 1 of the calculator, the length and width of the ground pattern are read as 650 and 410 st mi per degree of beamwidth, respectively. The ellipse over which the six peripheral earth stations are to be distributed has a semimajor axis of 104 mi and a semiminor axis of 66 mi.

4. An overlay pattern of these dimensions is constructed at the map scale of Fig. 2. A useful method of constructing these overlays is to establish local coordinates centered on the beam axis.

$$x = (w/2) \cos \theta$$

$$y = (\ell/2) \sin \theta$$

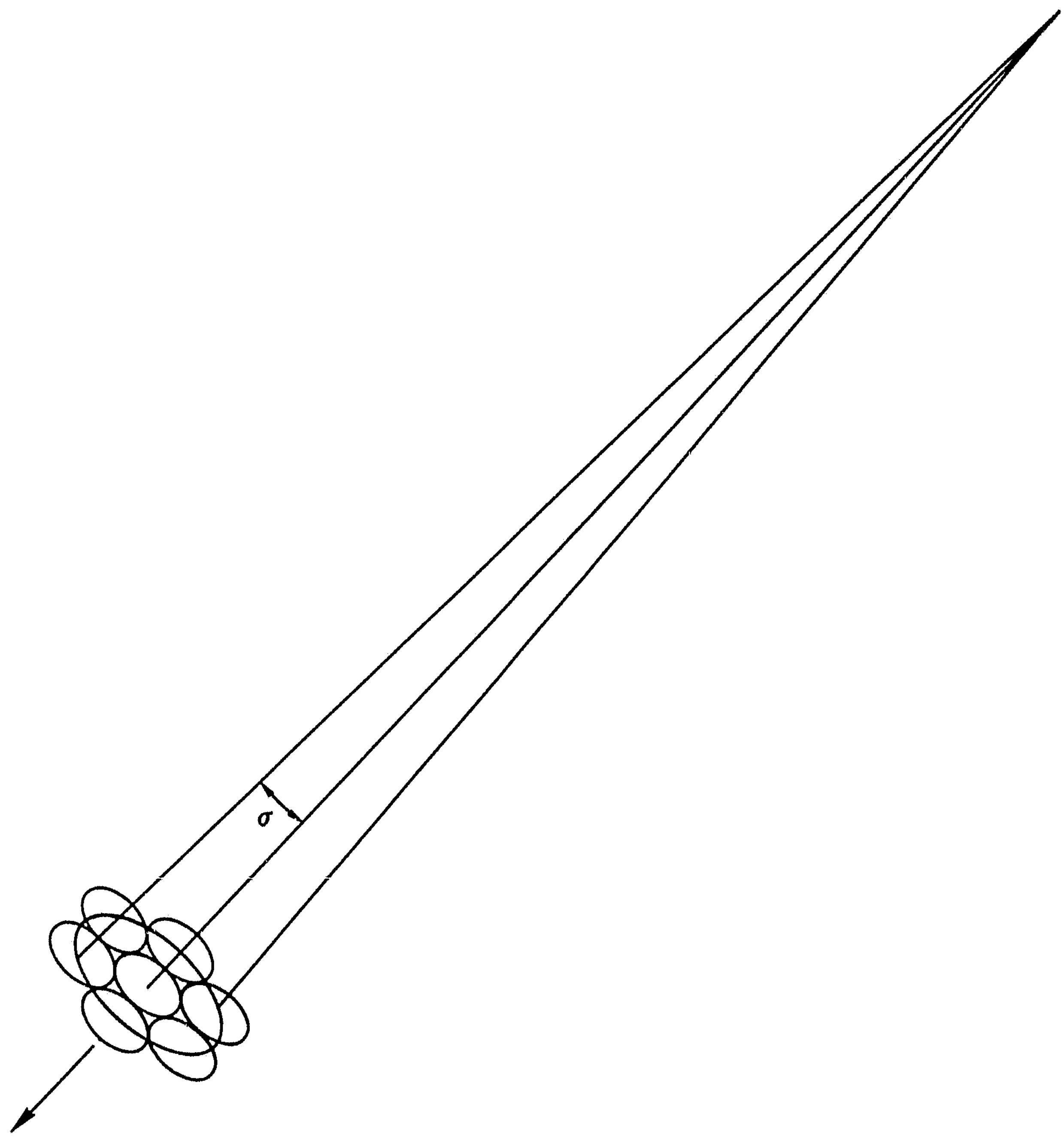

Fig. 1—A seven-beam cluster of maximum density

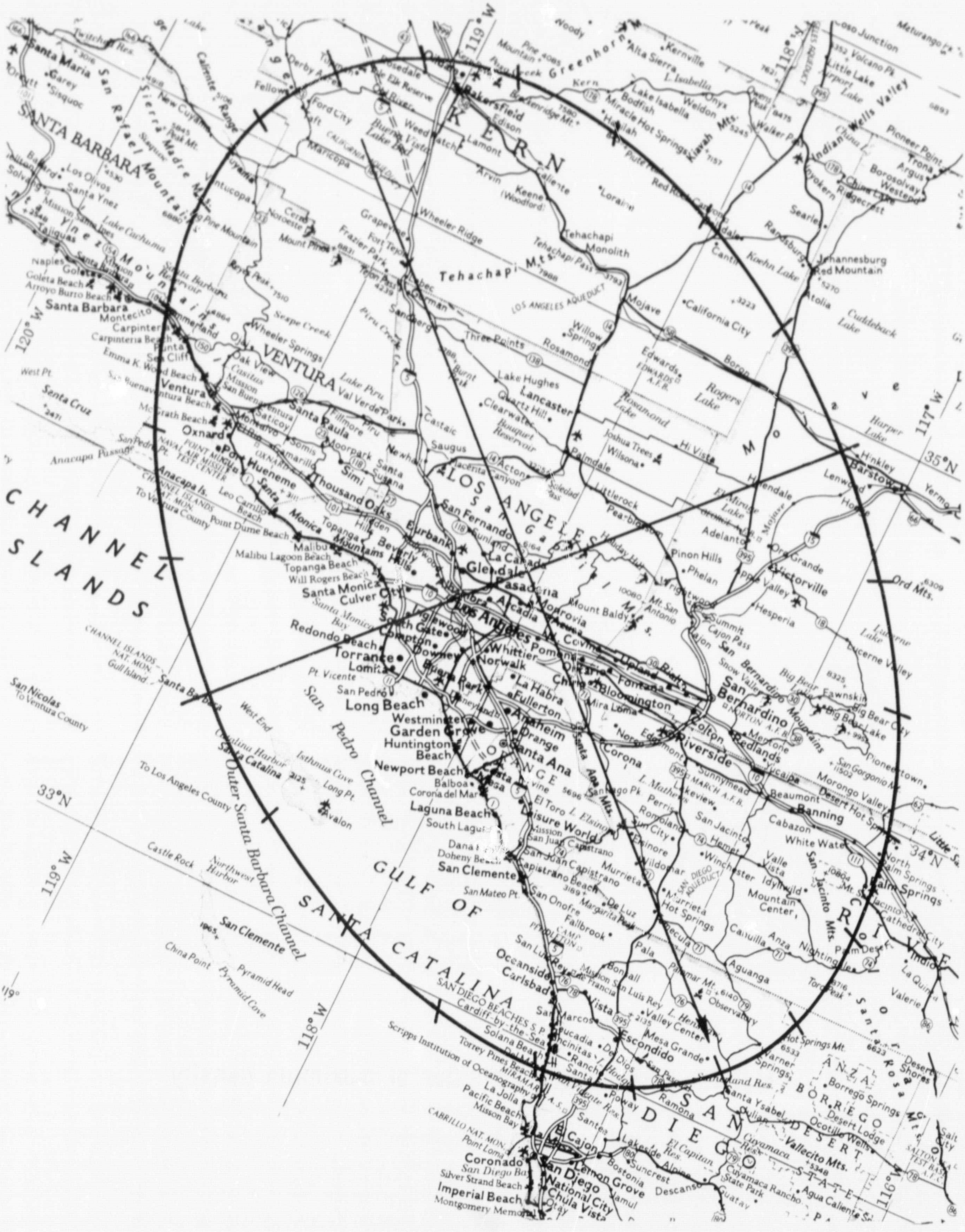

Fig. 2—Earth-station locations (example)

Here $\ell/2$ and w/2 are the semimajor and semiminor axes, respectively. Points are plotted at 15° increments of θ, and these points are useful for referencing the original 60° spacing of the hexagonal beam arrangement, which would otherwise be lost in the projection.

5. Preliminary studies with the overlay indicated that maximum earth station density might be achieved in the vicinity of Los Angeles by placing the center of the pattern near the city center and using one of the Channel Islands (possibly Santa Catalina, San Clemente, or Santa Barbara) for one peripheral station. Mount Wilson is probably a good choice for a central station and the ellipse centered there is shown in Fig. 2. It appears that three of the seven possible earth stations might more aptly serve outlying communities (San Diego, Santa Barbara, Bakersfield), so that four stations remain to be surface-linked to the Los Angeles metropolitan area.

ANGULAR DEVIATIONS (CALCULATOR PART 4)

A synchronous satellite with multiple, narrow beams is conceived as a device for relaying communications between widely separated earth stations. Ideally, each beam remains directed toward a specific earth station, and for a geostationary satellite, the pointing accuracy is limited only by attitude control accuracy. The beams are assumed to have fixed directions in the satellite body axes, and so have fixed relative orientation. They are steered collectively by controlling satellite attitude, and for the geostationary satellite, the steering rule requires that the satellite yaw axis remain directed toward the center of the earth and that the yaw be zero.

If the preceding steering rule is applied to a nonstationary satellite, then earth points viewed from the satellite will exhibit apparent angular motions which vary over the field of view. Earth points do not appear to move collectively, and this prevents the exact, simultaneous tracking by beams of fixed relative orientation. Angular deviations are defined here as the geometric pointing error due to satellite displacement under the assumption of some specific attitude steering rule. Other possible sources of beam pointing error, such as attitude control errors, flexure of the satellite structure, beam-direction bias errors, and atmospheric refraction are excluded from this study.

The principal cause of satellite displacement from a geostationary position is noncoincidence of the satellite's orbit plane and the earth's equatorial plane. If these two planes are inclined at a small angle i, the synchronous satellite will execute sinusoidal oscillations in latitude (rotating, earth-centered coordinates). The amplitude of the oscillation is i, and the frequency ω is 2π rad per day. The attendant longitudinal oscillations are small and will be ignored.

Under the above type of motion, it is possible to define the satellite attitude steering rule necessary to track a specific earth point. For small amplitudes i, the schedules in pitch and roll become separable and independently describable as a collection of sinusoids of frequency $n\omega$ ($n=1,2,3,\cdots$). The amplitude of these sinusoids decreases very rapidly as the order n increases. The different sinusoids are called "terms;" there is a set of terms for the pitch motion (α_1, α_2, $\cdots$, α_n, $\cdots$), and another set (β_1, β_2, $\cdots$, β_n, $\cdots$) for the roll motion. For any single earth point, the amplitude and phase of each term is determined by the orbit plane inclination and the coordinates of the point. The only possible phases are 0°, 90°, 180°, 270°, and the concept of phase can be ignored by specifying whether a given term is a sine or a cosine and whether the amplitude is positive or negative.

The β_1 term is unique; at any point, the amplitude of that term exceeds the amplitudes of all of the other terms. Further, this same amplitude and phase is demanded by a large number of points. All points on a single contour of the overlay (calculator Part 4) require the same amplitude and phase. For these points, a simple harmonic roll motion, $\beta_1 = Ki \sin \omega t$, will permit the first-order tracking of these points. The parameter K characterizes a contour, and every earth point visible to the satellite belongs to some contour. If the first-order roll amplitude is set at a specific value, $K'i$, then <u>primary</u> angular deviations will vanish on contour K'. A point, or points, characterized by another K value, say K'', will experience maximum primary angular deviations $|K' - K''| \cdot i$.

The contours of the overlay differ in K by increments of 1/40, and first-order roll amplitudes differ by i/40 deg per contour interval (i in deg). If a set of points is confined to an annular region of N consecutive contour intervals, then the extreme points (innermost and outermost) require first-order steering rules which differ in amplitude by Ni/40 deg. By choosing the mean amplitude (arithmetic average of extremes), the primary angular deviations reduce to $\pm$ Ni/80 on the bounding contours. Part 4 of the calculator was designed for this situation, and its use should be limited to cases in which $N > 1$ and $i \leq 7^{\circ}20'$.

The limitation, $N > 1$, arises because <u>secondary</u> angular deviations will limit tracking accuracy in regions of vanishing primary deviations. That is, secondary deviations exist almost everywhere and can be made to vanish only at a single point, but for many problems they are negligible with respect to the primary angular deviations. However, if attention is focused on regions of vanishing primary deviation, the secondary deviations become dominant.

As with primary angular deviations, the secondary deviations depend on the choice of steering rules. In this latter case, the secondary steering rules cannot be generalized and secondary deviations can never be made to vanish at more than a single point.

The tracking rules for precisely following a single point are described in Appendix B, Eqs. (B-1), (B-7), and (B-10); however, the present study does not single out any preferred points and describes the secondary angular deviations which arise when second-order steering rules are ignored. The results are then upper limits which might be reduced at specific points by invoking further specialization of steering rules. These results are shown in Figs. 3 and 4 for orbit plane inclinations of 4° and $7^{\circ}20'$, respectively. The maximum secondary angular deviations never exceed i/80 deg at any point of either figure. These figures are primarily useful for determining whether specialization of steering rules beyond first-order is worth considering in any specific situation.

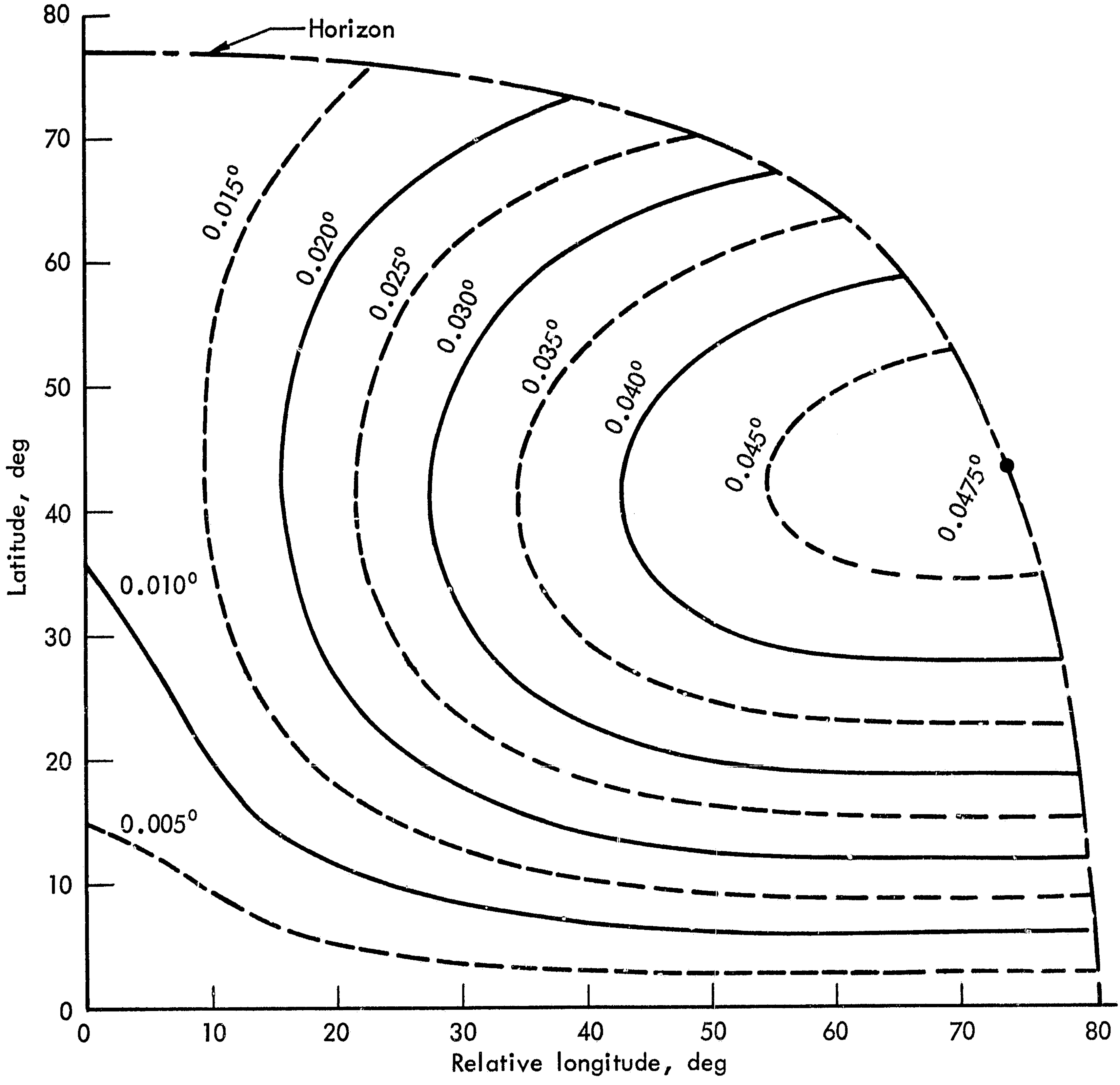

Fig. 3—Secondary angular deviations at i = 4°

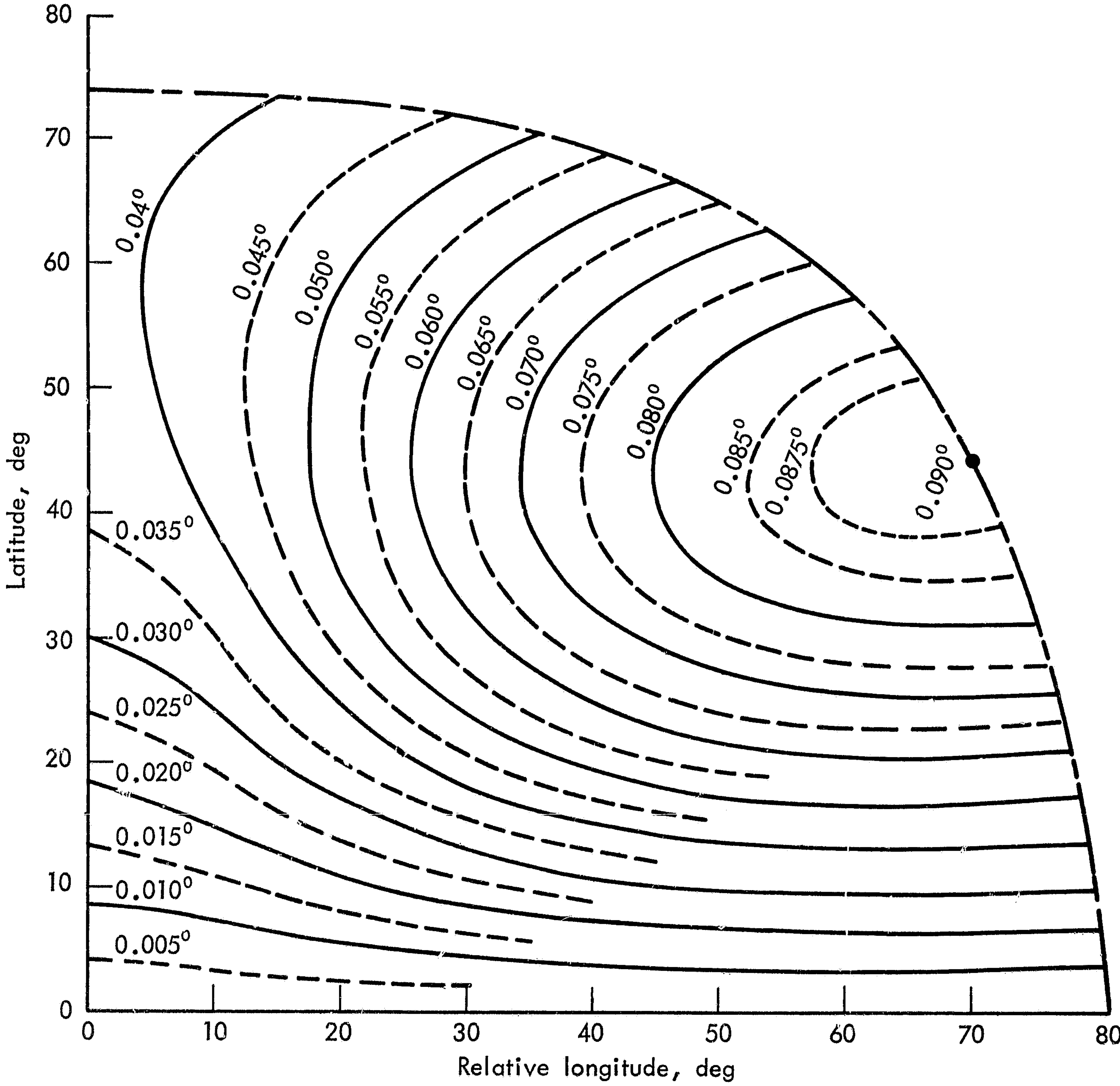

Fig. 4—Secondary angular deviations at i = 7°20'

Appendix A

COORDINATE CONVERSIONS AND DERIVATION OF GRAPHICAL SOLUTIONS

In Fig. 5, a point $\vec{P}$ is located at latitude φ and longitude λ relative to the subsatellite point SS. The arc R can be found by

$$\cos R = \cos \lambda \cdot \cos \varphi \tag{A-1}$$

The azimuth of the earth station as viewed from the subsatellite point can be found by

$$\tan B = \sin \lambda \cdot \operatorname{ctn} \varphi \tag{A-2}$$

Thus, the earth station position λ,φ may equally well be represented by the arc length R and the azimuth B. If a polar map of the visible earth were constructed with the subsatellite point as the pole, then R and B would be the natural coordinates for the map.

Earth pattern orientation is most conveniently described in terms of local azimuth Az, the angle which the great-circle arc R makes with the local meridian of the earth station. This local azimuth is

$$\tan Az = - \tan \lambda / \sin \varphi \tag{A-3}$$

The tilt angle T is defined at the satellite as the angle between the geocentric radius vector and a ray to the earth station at $\vec{P}$. Thus, it is the angle subtended at the satellite by arc R, and by examination of Fig. 6, it is related to R by

$$\left.\begin{aligned} \tan T &= \sin R/(p - \cos R) \\ p &= 6.6166 \text{ earth radii} \end{aligned}\right\} \tag{A-4}$$

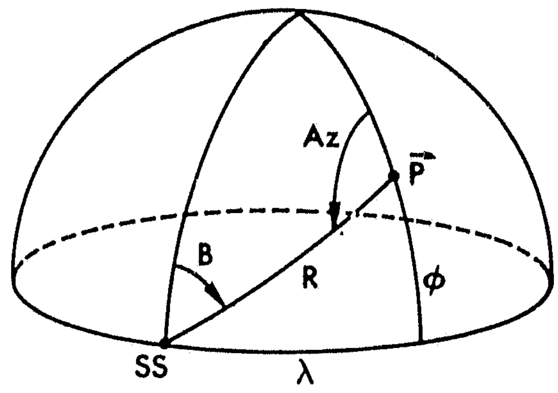

Fig. 5—Coordinate conversions

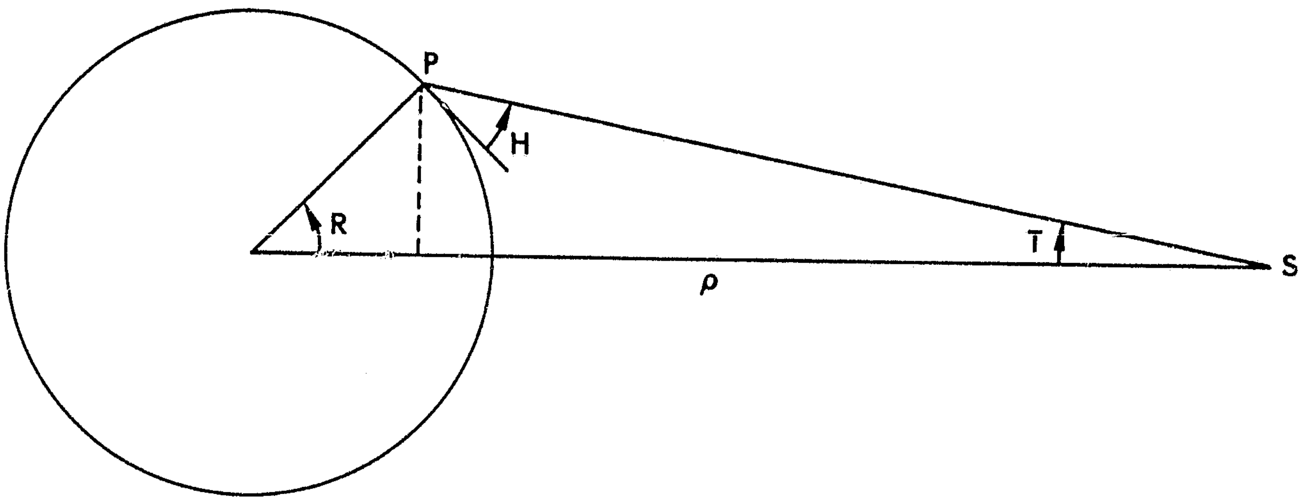

Fig. 6—Tilt-angle definition

The angular elevation of the satellite above the earth station horizon is denoted H and is related to T by

$$\cos H = p \cdot \sin T \qquad \text{(A-5)}$$

For an earth station at the perimeter of satellite vision, H = 0, and the maximum tilt angle is arc sin (1/p) = 8.692°. From Fig. 6, it can be seen that

$$R = 90 - H - T \qquad \text{(A-6)}$$

and the great-circle distance from the subsatellite point to the perimeter of vision is 81.308°.

A narrow communication beam of diameter σ deg is represented by a cone with its apex at the satellite. The cone is described by the rays which generate its surface, and these have tilt angles differing from that of the beam axis. The maximum tilt angle of any ray is $T + 0.5\sigma$, and if this exceeds 8.692°, then some portion of the beam will pass over the horizon. When beams approach the horizon, the extreme rays at $T + 0.5\sigma$ and $T - 0.5\sigma$ provide better estimates of the beam-length dimension upon the earth's surface than does the tangent plane approximation described below.

Figure 7 illustrates the projection of a beam of diameter σ deg upon the surface of the earth. The projection distance from satellite to earth station is given by

$$s^2 = p^2 - 2p \cdot \cos R + 1 \qquad \text{(A-7)}$$

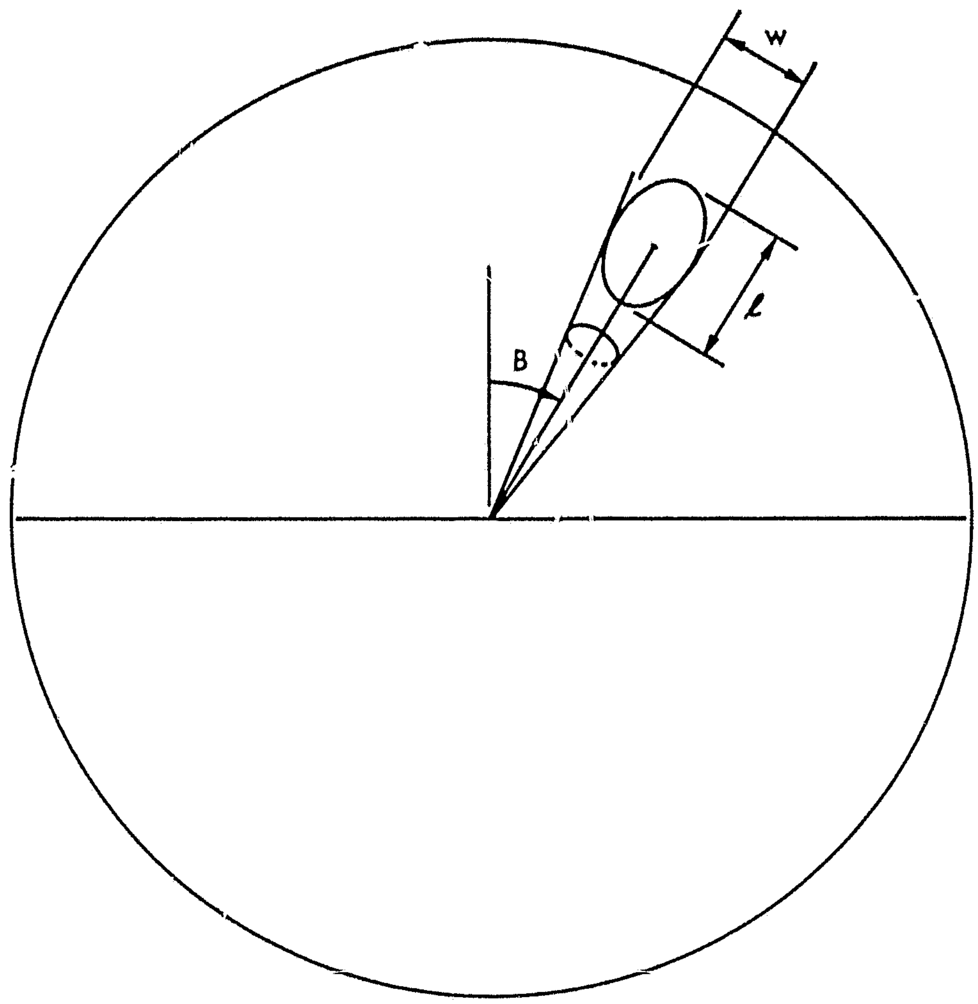

Fig. 7—Projection of beam of circular cross section upon the earth surface

(view from far behind the satellite)

The width of the ground pattern is approximately

$$w = 69.1\ s\sigma\ \text{st mi} \tag{A-8}$$

The length is approximated with $\ell = 68.2\sigma \cdot dR/dT$

$$\ell = 68.2\ s^2\sigma/(p \cos R-1)\ \text{st mi} \tag{A-9}$$

Appendix B

ANGULAR DEVIATIONS

Figure 8 illustrates the geometry of the earth point P and the satellite S. The location of earth points and satellite is defined in a Cartesian coordinate frame (x, y, z). This frame rotates (in inertial space) once per day about its z axis and so retains the synchronous satellite perpetually in the y, z plane. With the earth radius as the unit of length, the Cartesian coordinates of an earth point located at latitude φ and longitude λ (relative to the satellite meridian plane) are

$$\left.\begin{aligned} x &= \cos\varphi \sin\lambda \\ y &= \cos\varphi \cos\lambda \\ z &= \sin\lambda \end{aligned}\right\} \qquad \text{(B-1)}$$

The synchronous satellite is in an orbit plane inclined to the earth's equatorial plane by a small angle i ($i < 7^{o}\ 20'$). The longitudinal oscillations are ignored and for an orbital frequency ω, equal to 2 π rad per day, the satellite latitude versus time is

$$\Phi(t) = i \sin \omega t \qquad \text{(B-2)}$$

Using subscript s to designate satellite position coordinates, the instantaneous position of the satellite is given by

$$\left.\begin{aligned} x_s &= 0 \\ y_s &= p \cdot (1 - \cos\Phi) \\ z_s &= p \cdot \sin\Phi \end{aligned}\right\} \qquad \text{(B-3)}$$

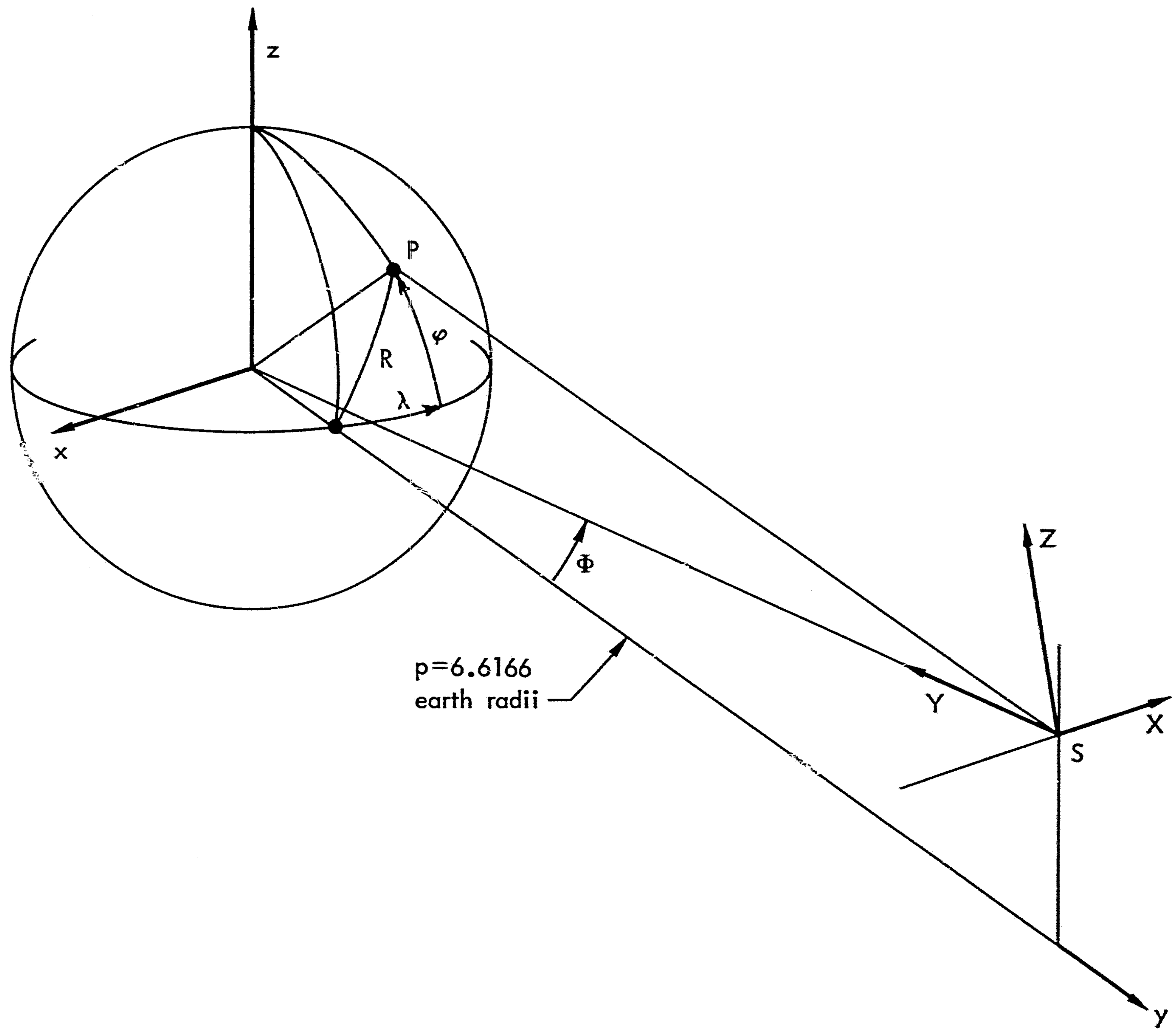

Fig. 8—Geometry of satellite displacement

Satellite orientation is described by Eulerian rotations of its body axes with respect to an initial reference orientation. The satellite body axes X, Y, Z are referred to as the roll, yaw, and pitch axes, respectively. The direction cosines of earth points are specified in this satellite body axes system and will vary as the satellite orientation changes. The initial reference orientation is specified as X in the direction -x, Y in the direction -y, and Z in the direction +z. The reference state is comprised of this reference orientation and $\Phi = 0$. The reference state coincides with that of a stationary satellite. In this reference state, the direction cosines of an earth point P (φ, λ) are given by

$$\left.\begin{aligned} u &= -x/s \\ v &= (p - y)/s \\ w &= z/s \end{aligned}\right\} \qquad \text{(B-4)}$$

The distance, $s = \overline{PS}$, from the satellite to the earth point will disappear in the subsequent algebra and need not be explicitly evaluated.

The displacement motion described by Eqs. (B-3) must be complemented by the addition of a satellite rotation of $-\Phi$ about the satellite X axis. This maintains the satellite Y axis directed toward the center of the earth and is actually an inertial orientation requiring no steering torque. Its explicit introduction is required to complete the description of the satellite inertial motion as projected into the y, z plane of the rotating coordinate frame.

For the satellite displaced to Φ and rotated by $-\Phi$ about x, the direction cosines of point P become

$$\left.\begin{aligned} u' &= -x/s' \\ v' &= (p - y \cos \Phi - z \sin \Phi)/s' \\ w' &= (-y \sin \Phi + z \cos \Phi)/s' \end{aligned}\right\} \qquad \text{(B-5)}$$

This set of direction cosines will differ from those observed in the reference state, but for any single point P, the satellite orientation can be altered to bring the two sets into coincidence. Any two of the three degrees of rotational freedom might be chosen, but pitch and roll are natural choices, since for small changes they create nearly orthogonal components of angular change. The required changes will be small if satellite displacement, Φ, is small. The satellite displacement is given by $\Phi = i \sin \omega t$, and for $i \leq 7^{\circ} 20'$, complete orthogonality will be assumed.

THE ROLL STEERING RULE

A roll rotation, β, about X leaves u' unchanged but can bring v', w' into coincidence with v, w.

$$\begin{aligned} v' \cos \beta + w' \sin \beta &= v \\ -v' \sin \beta + w' \cos \beta &= w \end{aligned}$$

or

$$\tan \beta = \frac{vw' - wv'}{vv' + ww'} \qquad \text{(B-6)}$$

Assuming that β is small and inserting the direction cosines from Eqs. (B-4) and (B-5)

$$\beta = - \frac{(py - y^2 - z^2) \sin \Phi + pz (1 - \cos \Phi)}{(p - y)(p - y \cos \Phi) + z^2 \cos \Phi - pz \sin \Phi} \qquad \text{(B-7)}$$

with $\Phi = i \cdot \sin \omega t$. This equation prescribes the roll, β, required to track one component of the angular deviation at P (λ, φ). With Φ small, $\sin \Phi \simeq \Phi$, and $\cos \Phi \simeq 1 - \Phi^2/2$, Eq. (B-7) becomes

$$\beta = - \frac{y(p - y) - z^2}{(p - y)^2 + z^2} \Phi - \frac{1}{2} \frac{(p + y) pz}{(p - y)^3} \Phi^2 \tag{B-8}$$

The first and second terms of this equation are respectively referred to as the first- and second-order roll steering rules for the point P(φ, λ).

THE PITCH STEERING RULE

A pitch rotation α about Z leaves w' unchanged, but can bring $u'v'$ into coincidence with u,v.

$$\begin{aligned} u' \cos \alpha + v' \sin \alpha &= u \\ -u' \sin \alpha + v' \cos \alpha &= v \end{aligned}$$

or

$$\tan \alpha = \frac{uv' - vu'}{uu' + vv'} \tag{B-9}$$

For α small

$$\alpha = \frac{[z \sin \Phi - y(1 - \cos \Phi)]\, x}{(p - y)(p - y \cos \Phi - z \sin \Phi) + x^2} \tag{B-10}$$

with Φ small, the first-order approximation to this equation is

$$\alpha = \frac{zx}{(p - y)^2 + x^2} \Phi \tag{B-11}$$

The magnitude of this term is comparable to the magnitude of the second-order roll term, and higher order terms are very small.

The First-Order Steering Rule and Primary Angular Deviations

The first-order roll term reduces to

$$\beta_1 = K\,\Phi = -\,K\,i\,\sin\omega t \qquad \text{(B-12)}$$

for all points satisfying

$$\frac{y(p-y) - z^2}{(p-y)^2 + z^2} = K \qquad \text{(B-13)}$$

For a given value of K, a large number of points can satisfy this equation and simultaneously satisfy $1 = x^2 + y^2 + z^2$ (i.e., they lie on the surface of the earth). The contours of the overlay (calculator Part 4) illustrate sets of points satisfying Eq. (B-13) with K = constant.

From Eq. (B-12), it is evident that the first-order steering rule prescribes a sinusoidal roll of frequency ω and amplitude Ki. If a contour is specified by a certain value of K, say K_o, then at any point P (φ, λ) not on K_o, there will be a primary angular deviation given by

$$\left.\begin{aligned} \delta\,\beta_1 &= b_1 \sin\omega t \\ b_1 &= -\left\{\frac{y(p-y) - z^2}{(p-y)^2 + z^2} - K_o\right\}\cdot i\,, \end{aligned}\right\} \qquad \text{(B-14)}$$

An arbitrary set of points, confined to an annular region characterized by an innermost contour K_n and an outermost contour K_m is spanned by N = m - n contour intervals. The required steering rule amplitudes differ by $(K_m - K_n) \cdot i$ deg over the set of points. The contours on Part 4 of the calculator are separated by intervals of i/40 deg so the range of angular deviations is Ni/40 deg across the set. If the steering rule is chosen to bisect the extreme values (K_m and K_n), the maximum deviations are Ni/80 deg.

The Second-Order Steering Rule and Secondary Angular Deviations

The second-order roll rule is

$$\beta_2 = - \frac{(p + y)\ pz}{2(p - y)^3}\ \Phi^2 \qquad \text{(B-15)}$$

The pitch rule is simply

$$\alpha_1 = \frac{zx}{(p - y)^2 + x^2}\ \Phi \qquad \text{(B-16)}$$

The coefficients of Φ and Φ^2 do not describe contours as did the first-order roll coefficient. Taken together, they intersect the earth's surface at only one common point, and so represent a unique steering requirement for each point on the earth's surface. If these secondary steering requirements are ignored, then Eqs. (B-14) and (B-15) describe secondary ngular deviations.

The β_2 term can be represented by

$$\left.\begin{aligned} \beta_2 &= b_2\ (1 - \cos 2\ \omega\ t) \\ b_2 &= - \frac{(p + y)\ pz}{4(p - y)^3}\ i^2 \end{aligned}\right\} \qquad \text{(B-17)}$$

If this is treated as an error, the constant part (at any point y, z) is simply a bias which can be removed by redefining the beam direction. Bias removal is assumed, and the angular deviation at $P(\lambda, \varphi)$ due to neglecting the β_2 steering rule is

$$\delta \beta_2 = + b_2 \cdot \cos 2 \omega t \tag{B-18}$$

Neglect of the pitch steering rule yields an angular deviation (from Eq. (B-15)):

$$\left.\begin{aligned} \delta \alpha_1 &= a_1 \sin \omega t \\ a_1 &= \frac{xz}{(p - y)^2 + x^2} \end{aligned}\right\} \tag{B-19}$$

The quantities $\delta \beta_2$ and $\delta \alpha_1$ are orthogonal, and the magnitude of the vector sum has a maximum value

$$E_s = \sqrt{b_2^2 + a_1^2} \tag{B-20}$$

Values of E are shown in Figs. 3 and 4 for orbit plane inclinations of $i = 4^\circ$ and 7° 20', respectively. These represent the magnitude of the residual angular deviation at any point $P(\varphi, \lambda)$ when the first-order steering rule is applied at that point.

Total Angular Deviation

For a set of points distributed over a relatively large area and spanned by several contour intervals, the primary angular deviations are likely to be dominant with respect to the secondary angular deviations given by Eq. (B-19). On a single contour, the primary deviations can be made to vanish by choice of the first-order steering rule, and the secondary angular deviations shown in Figs. 3 and 4 limit the tracking accuracy.

For a set of points near, but not on, a single contour, both primary and secondary deviations may be important, and the total angular deviation is given by

$$E_{Tot} = \sqrt{\left(|b_1| + |b_2|\right)^2 + a_1^2} \tag{B-21}$$

The individual components of this equation are the amplitudes defined in Eqs. (B-14), (B-17), and (B-19).

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