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STABLE PERIODIC ORBITS ABOUT THE SUN PERTURBED EARTH-MOON TRIANGULAR POINTS

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ABSTRACT

A method of general perturbations utilizing trigonometric series is used to investigate motion in the vicinity of the triangular equilibrium point of the earth-moon system. The model used is that of the restricted problem of four bodies for the earth-moon-sun system. In this model the three principle bodies are periodic, coplanar, and obey the equations of motion. A stable, periodic, coplanar orbit is calculated. In the synodic system it appears elliptical in shape, having a semimajor axis of 90,000 miles and an eccentricity of 0.5. The minor axis is parallel to the earth triangular point line. The mean motion of the particle describing this orbit is synchronized with that of the sun such that their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse. This orbit, although one and one-half times larger, tends to confirm a conclusion by Schechter predicting that such an orbit exists.

(Paper 67-566, AIAA Guidance, Control and Flight Dynamics Conf. 1967)

A second orbit is also calculated. This stable orbit is similar in size and shape, but 180 degrees out of phase with the first orbit. The calculation of a small unstable coplanar orbit near the triangular point is reported. This result would agree with Schechter's second conclusion that small coplanar motions near the triangular points will grow large.

STABLE PERIODIC ORBITS ABOUT THE SUN PERTURBED EARTH-MOON TRIANGULAR POINTS

I. INTRODUCTION

In a recent paper[1] Schechter concluded a stable, periodic, coplanar orbit can exist about the sun perturbed earth-moon triangular point. The model used for the moon's motion was that of the variational orbit using de Pontécoulant's notation for arguments. The present paper confirms the conclusion by presenting a numerical solution of a somewhat larger orbit, having the same essential features of the orbit predicted by Schechter. In addition, a second similar orbit having a phase difference of 180 degrees is calculated. A linear stability analysis showed both of these orbits to be stable. The model of the earth-moon-sun system used in the calculation of the orbits is one in which these three principle bodies are periodic, coplanar, and obey the equations of motion. The theory of the moon which is used in this paper corresponds to the part of the Hill-Brown classical lunar theory containing variational and purely parallactic terms. Both kinds of terms depend only upon the variational argument, the mean angular distance between the sun and the moon. The numerical procedure used to obtain the model yields both types of terms simultaneously whereas they are obtained separately, using two completely different procedures, in the classical theory. The models of Ref. 1 and this paper are thus seen to be essentially in agreement. A second conclusion of Ref. 1 is that small coplanar motions near the triangular points will grow large. The present paper agrees with this conclusion by reporting the calculation of an unstable periodic orbit near the triangular points.

II. ANALYSIS

The equations of motion used to obtain the periodic orbits of the particle are:

$$\frac{d^2\bar{r}}{dt^2} = -\mu^2 \frac{\bar{r}}{r^3} + \mu^2 \frac{m_3}{m_2 + m}\left(\frac{\bar{\rho}_3}{\rho_3^3} - \frac{\bar{r}_3}{r_3^3}\right) + \mu^2 \frac{m_1}{m_2 + m}\left(\frac{\bar{\rho}_1}{\rho_1^3} - \frac{\bar{r}_1}{r_1^3}\right) \tag{1}$$

All position vectors are referred to the earth and m is the mass of the particle, put equal to zero in the computation, but kept in the equations for completeness.

$\bar{r}$ is the position vector of the particle

$\bar{r}_3$ is the position vector of the moon

$\bar{r}_1$ is the position vector of the sun

$\bar{\rho}_3 = \bar{r}_3 - \bar{r}$

$\bar{\rho}_1 = \bar{r}_1 - \bar{r}$

m_1 is the mass of the sun

m_2 is the mass of the earth

m_3 is the mass of the moon

$\mu^2 = k^2 (m_2 + m)$ where k is the Gaussian constant.

t is the time.

A digital computer is used in obtaining solutions for these equations. No approximations are made in the equations of motion. The masses and mean motions are given numerical values. The values of the constants used are:

mean motion of the sun, $n_1 = 129597742''.38$ per Julian Century

mean motion of the moon, $n_3 = 1732559353''.56$ per Julian Century

geocentric gravitational constant, $Gm_2 = 398603 \times 10^9 m^3 sec^{-2}$

the measure of 1 A.U., $a_1 = 149600 \times 10^6 m$

semimajor axis of the moon, $a_3 = 3.847487965 \times 10^8 m$

$$m_3/m_2 = 1/81.30$$

$$m_1/m_2 = 332958.087932061$$

The procedure used in obtaining a solution for the earth-moon-sun model as well as for the periodic triangular point orbits is based on Musen's[2] method with the perturbations represented in trigonometric series with numerical coefficients. A linear stability analysis was made by considering the variational equation and the stability was determined in the usual way from the characteristic roots.[3] The solutions of the equations of motion are given in the following form (See Figure 1)

$$\bar{r} = (1 + \alpha)\, \bar{r}_o + \beta \bar{w} \tag{2}$$

where α and β are the components of the perturbations, $\bar{r}_o$ is the position vector in a fixed reference ellipse and

$$\bar{w} = \frac{1}{n} \frac{d\bar{r}_o}{dt} \tag{3}$$

Kepler's law is $n^2 a^3 = \mu^2$ where n is the mean motion and a is the semimajor axis of the reference ellipse. Since the only reference ellipses used will have zero eccentricity, $r_o \equiv a$. The functions α and β are represented by the trigonometric series

$$\alpha = \sum_{k=0}^{\infty} \left(\alpha_k^{(c)} \cos k\theta + \alpha_k^{(s)} \sin k\theta \right) \tag{4}$$

$$\beta = \sum_{k=0}^{\infty} \left(\beta_k^{(c)} \cos k\theta + \beta_k^{(s)} \sin k\theta \right) \tag{5}$$

where $\theta = (n_3 - n_1)t$.

III. SOLUTIONS

Solutions for the motions of the moon and sun are found first. Together these solutions define the model of the earth-moon-sun system that will be used. Equations of motion for the moon and sun are obtained by interchanging the symbols in Equation (1). These equations are solved in the following manner. Starting with the sun constrained to move in a circular, coplanar, Keplerian orbit with respect to the earth the equations of motion for the moon are solved. Trigonometric coefficients, α and β, describing the moon's perturbed orbit are thus obtained. The role of the bodies is reversed, the moon's motion is constrained to move in the perturbed orbit defined by α and β and the equations of motion for the sun are solved. The α and β coefficients describing the sun's perturbed orbit are thus obtained. The roles of the bodies are reversed again and again each time using the latest acquired α and β coefficients to define the motion of their respective body. Ultimately the values of the α and β coefficients for each of the bodies do not change from one reversal to the next. The problem is then solved since both bodies, when not constrained, satisfy the equations of motion and move in orbits defined by their respective α and β

coefficients. This is a particular solution of the three body problem in which the bodies move in periodic orbits with respect to each other. The period of the motion is the synodic period of the bodies, P, given by the equation

$$P = 2\pi/(n_3 - n_1) \tag{6}$$

The coefficients for the sun solution are given in Table 1. Since the form of the sun solution is similar to that of the particle, Equations (2), (3), (4) and (5) represent its position if $r_o \equiv a = a_1$ and $n = n_1$. For the number of decimal places given in Table 1, the relative geocentric position of the sun can be obtained to twelve significant figures. In this paper the same accuracy will be given for all orbits. A plot of the position of the sun is shown in Figure 2. Table 2 contains the coefficients for the moon solution. Again Equations (2), (3), (4), and (5) represent the position of the moon if $r_o \equiv a = a_3$ and $n = n_3$. A plot of the position of the moon is shown in Figure 3. This curve corresponds to the Hill-Brown theory containing variational and purely parallactic terms. The sun and moon solutions define the model that will be used in finding periodic orbits about the earth-moon triangular point. Coefficients describing a periodic orbit about a triangular point are given in Table 3. This orbit will be referred to as Orbit I. Linear stability analysis showed this orbit to be stable. Coefficients for a second orbit, Orbit II, are given in Table 4. This orbit was also found to be stable. Equations (2), (3), (4) and (5) describe the position of these orbits if $n = n_3$ and a is found from the equations

$$n_3^2 a^3 = Gm_2 \tag{7}$$

For the constants of this paper $r_o \equiv a = 3.831841237 \times 10^8$m. The two periodic orbits are plotted in Figure 4 in the α, β coordinate system. The origin of this coordinate system is located (See Figure 1) at $\varphi = 60°$, $\bar{r}_o = 3.831841237 \times 10^8$m. These orbits, therefore, go around the triangular point which is in advance of the moon's position. This triangular point is analogous to the L_4 Lagrange equilibrium point of the restricted three body problem. In the restricted three body problem the triangular points (L_4 and L_5) for the earth-moon system are located by the two points making equilateral triangles with the earth-moon positions forming one side of the triangle. If this definition were used in the present case the triangular points would describe orbits identical to the moon's orbit shown in Figure 3. The above method for describing triangular points will not be used in this paper. It has been found to be more convenient to define reference points that are fixed in the synodic system. These reference points, R_4 and R_5, are the two points making equilateral triangles with the center of the earth and the mean geometrical position of the moon $\bar{r}_M$, defined by the equation

$$\bar{r}_M = \frac{1}{2\pi}\int_0^{2\pi} \bar{r}\, d\theta = \frac{a_3}{2\pi}\int_0^{2\pi} \sqrt{(1+\alpha)^2 + \beta^2}\, d\theta \qquad (8)$$

The coefficients in Table 2 were used in this calculation which yielded $\bar{r}_M$ = 3.844099188×10^8m. The location of R_4 on Figure 4 is seen to be $\alpha = 3198.970432 \times 10^{-6}$, $\beta = 0.0$. With this definition of triangular point the α, β coefficients for the orbit implied in Reference 1 are given in Table 5.

IV. DISCUSSION

The orbit described in Reference 1 is plotted in Figure 4 using the coefficients in Table 5. It appears as an ellipse centered at the triangular point, R_4, with its major axis perpendicular to a line joining the earth and R_4 (i.e. parallel to the β axis). This ellipse has an eccentricity of 0.5 and its semimajor axis is 58,128 miles. The scale used on Figure 4 is one in which the value of α or β equivalent to one corresponds to 2.380990996×10^5 miles. The motion of the particle describing this orbit is synchronized with that of the sun such that their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse. At epoch (θ=0) the sun is on the positive α axis as is the particle. The period of the orbit is the synodic period of the earth-moon-sun system given by Equation (6). Orbit I calculated in this paper is elliptical in shape and has its semimajor axis approximately parallel to the β axis. As seen in Figure 4 its center is not located at the R_4 triangular point. In this respect it is similar to the periodic orbits about triangular points that are obtained in the restricted three body problem. The semimajor axis is approximately 90,000 miles and the semiminor axis is approximately 44,000 miles yielding an eccentricity close to 0.5. The mean motion of the particle describing this orbit is synchronized with that of the sun such that their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse.

Orbit II, although having a phase difference of 180 degrees and slightly smaller in size, is very similar to Orbit I. Its semimajor axis is approximately 88,000 miles and semiminor axis is approximately 43,000 miles giving it an eccentricity close to 0.5. At epoch the particle for Orbit II is on the opposite side of the ellipse from the particle for Orbit I. Orbit II is thus seen to be synchronized with the sun so their angular positions almost coincide when the particle crosses one of the axes of the ellipse; however, it is 180 degrees out of phase with Orbit I.

To date these are the only stable periodic orbits for this particular model of the earth-moon-sun system that have been calculated. Orbit I has enough similarities to the orbit predicted in Reference 1 that it may indeed be the same orbit.

An unstable periodic orbit has been calculated about R_4 for the same model of the earth-moon-sun system used in this paper. It is very close to the triangular point remaining within 3100 miles of the point during its orbit. Making two loops about the R_4 point per synodic period, it is similar in geometry to a previous orbit shown by Kolenkiewicz and Carpenter[4]. This orbit agrees with the conclusion, also made in Reference 1, that small coplanar motions near the triangular points will grow large.

REFERENCES

[1]Schechter, H. B., "Three-Dimensional Nonlinear Stability Analysis of the Sun-Perturbed Earth-Moon Equilateral Points," Paper 67-566, AIAA Guidance, Control and Flight Dynamics Conference, Huntsville, Alabama, August 14-16, 1967.

[2]Musen, P. and Carpenter, L., "On the General Planetary Perturbations in Rectangular Coordinates," J. Geophys. Res. 68, 2727-2734 (1963).

[3]Hartman, P., Ordinary Differential Equations (John Wiley and Sons, Inc., New York, 1966).

[4]Kolenkiewicz, R. and Carpenter, L., "Periodic Motion Around the Triangular Libration Point in the Restricted Problem of Four Bodies," Astron, J. 70, 180-183 (1966).

TABLE 1

Three body sun solution, $\varphi = 0°$

k	$\alpha_k^{(c)} \times 10^6$	$\beta_k^{(s)} \times 10^6$
0	0.045105	0.000000
1	30.949868	31.492851
2	-0.004947	-0.005012
3	0.047329	0.047319
4	-0.000005	-0.000005
5	0.000184	0.000183
6	0.000000	0.000000
7	0.000001	0.000001

TABLE 2

Three body moon solution, $\varphi = 0°$

k	$\alpha_k^{(c)} \times 10^6$	$\beta_k^{(s)} \times 10^6$
0	-906.915740	0.000000
1	287.606767	-609.076345
2	-7173.506863	10202.254541
3	-7.507078	7.212259
4	6.028443	5.719334
5	-0.003392	0.005816
6	0.032454	0.027566
7	0.000011	0.000025
8	0.000187	0.000163
9	0.000000	0.000000
10	0.000001	0.000001

TABLE 3

Periodic Orbit I, φ = 60°

k	$\alpha_k^{(c)} \times 10^6$	$\alpha_k^{(s)} \times 10^6$	$\beta_k^{(c)} \times 10^6$	$\beta_k^{(s)} \times 10^6$
0	-19171.568123	0.000000	74753.542768	0.000000
1	187801.135978	17178.314916	-13120.769748	-377986.165218
2	11131.030603	-3722.872058	2352.545921	18027.013465
3	-2874.472418	737.568028	-637.564775	-2521.769547
4	582.134781	-176.988257	173.574850	518.179570
5	-123.327707	47.337357	-46.872029	-110.607813
6	26.570848	-12.692022	12.244230	24.010789
7	-5.830085	3.271408	-3.092692	-5.355760
8	1.327555	-0.827212	0.781757	1.245872
9	-0.314593	0.214693	-0.205945	-0.299764
10	0.075554	-0.058253	0.056669	0.072270
11	-0.017829	0.016064	-0.015680	-0.017002
12	0.004114	-0.004335	0.004214	0.003918
13	-0.000955	0.001136	-0.001100	-0.000914
14	0.000230	-0.000296	0.000288	0.000222
15	-0.000057	0.000080	-0.000078	-0.000055
16	0.000014	-0.000022	0.000022	0.000013
17	-0.000003	0.000006	-0.000006	-0.000003
18	0.000001	-0.000002	0.000002	0.000001

TABLE 4

Periodic Orbit II, φ = 60°

k	$\alpha_k^{(c)} \times 10^6$	$\alpha_k^{(s)} \times 10^6$	$\beta_k^{(c)} \times 10^6$	$\beta_k^{(s)} \times 10^6$
0	-18160.912624	0.000000	72212.688988	0.000000
1	-183627.357659	-16818.088205	11392.917179	370250.263893
2	10460.900708	-3371.790401	2116.289582	17696.741894
3	2715.813738	-648.117571	563.594410	2418.999900
4	544.831763	-152.630065	151.103055	487.952665
5	114.117818	-40.353414	40.270023	102.700732
6	24.309814	-10.697742	10.365541	22.014409
7	5.274431	-2.716444	2.571492	4.851307
8	1.187793	-0.674485	0.637277	1.115552
9	0.278619	-0.171971	0.164972	0.265686
10	0.066360	-0.046021	0.044817	0.063541
11	0.015564	-0.012554	0.012271	0.014861
12	0.003576	-0.003350	0.003259	0.003409
13	0.000826	-0.000865	0.000838	0.000792
14	0.000198	-0.000222	0.000215	0.000192
15	0.000049	-0.000059	0.000057	0.000048
16	0.000012	-0.000016	0.000016	0.000012
17	0.000003	-0.000004	0.000004	0.000003
18	0.000001	-0.000001	0.000001	0.000001

TABLE 5

Triangular point periodic orbit from Reference 1, $\varphi = 60°$

k	$\alpha_k^{(c)} \times 10^6$	$\alpha_k^{(s)} \times 10^6$	$\beta_k^{(c)} \times 10^6$	$\beta_k^{(s)} \times 10^6$
0	3198.970432	0.000000	0.000000	0.000000
1	122066.820297	0.000000	0.000000	-244133.640594

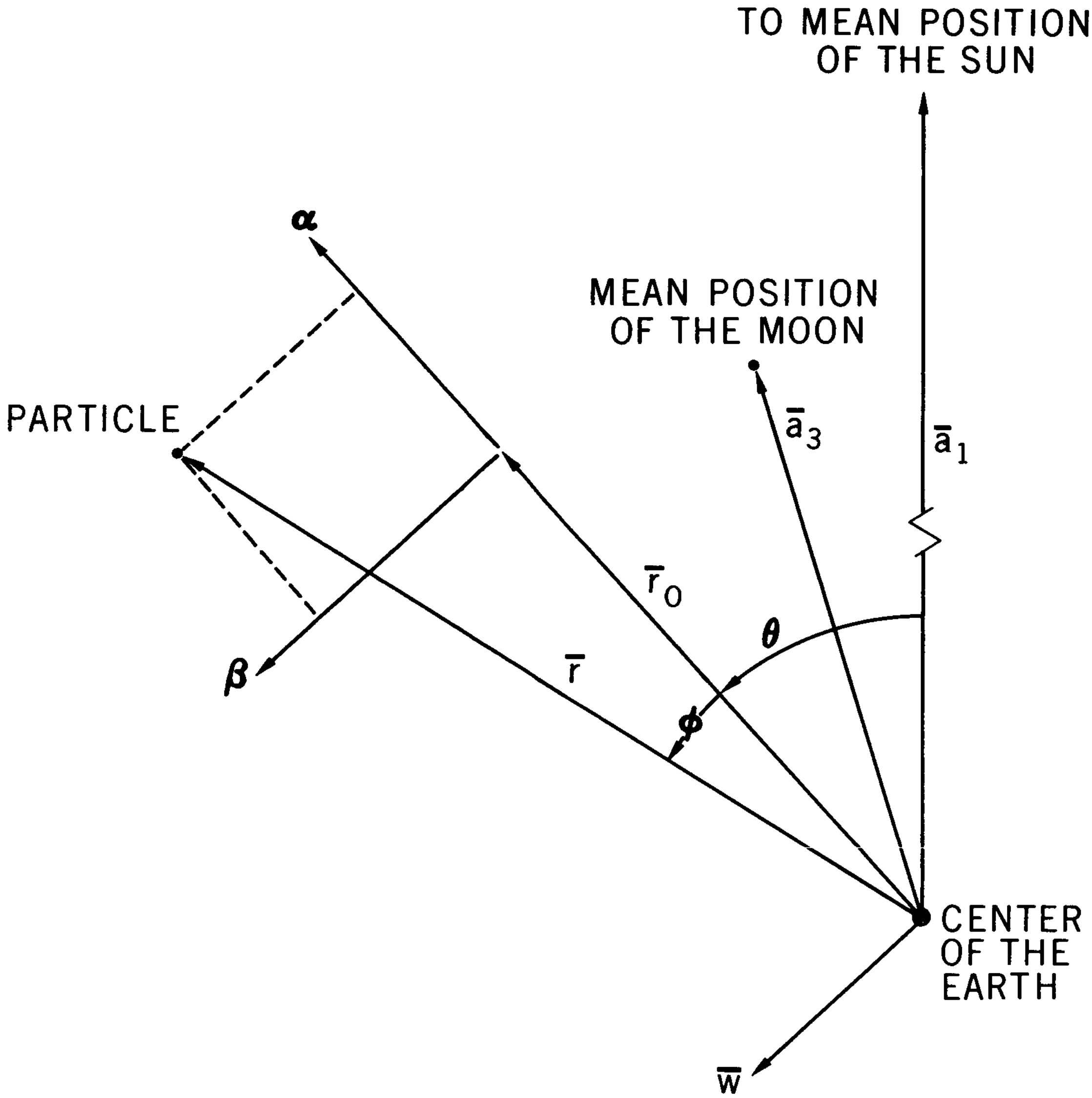

Figure 1. Coordinate System

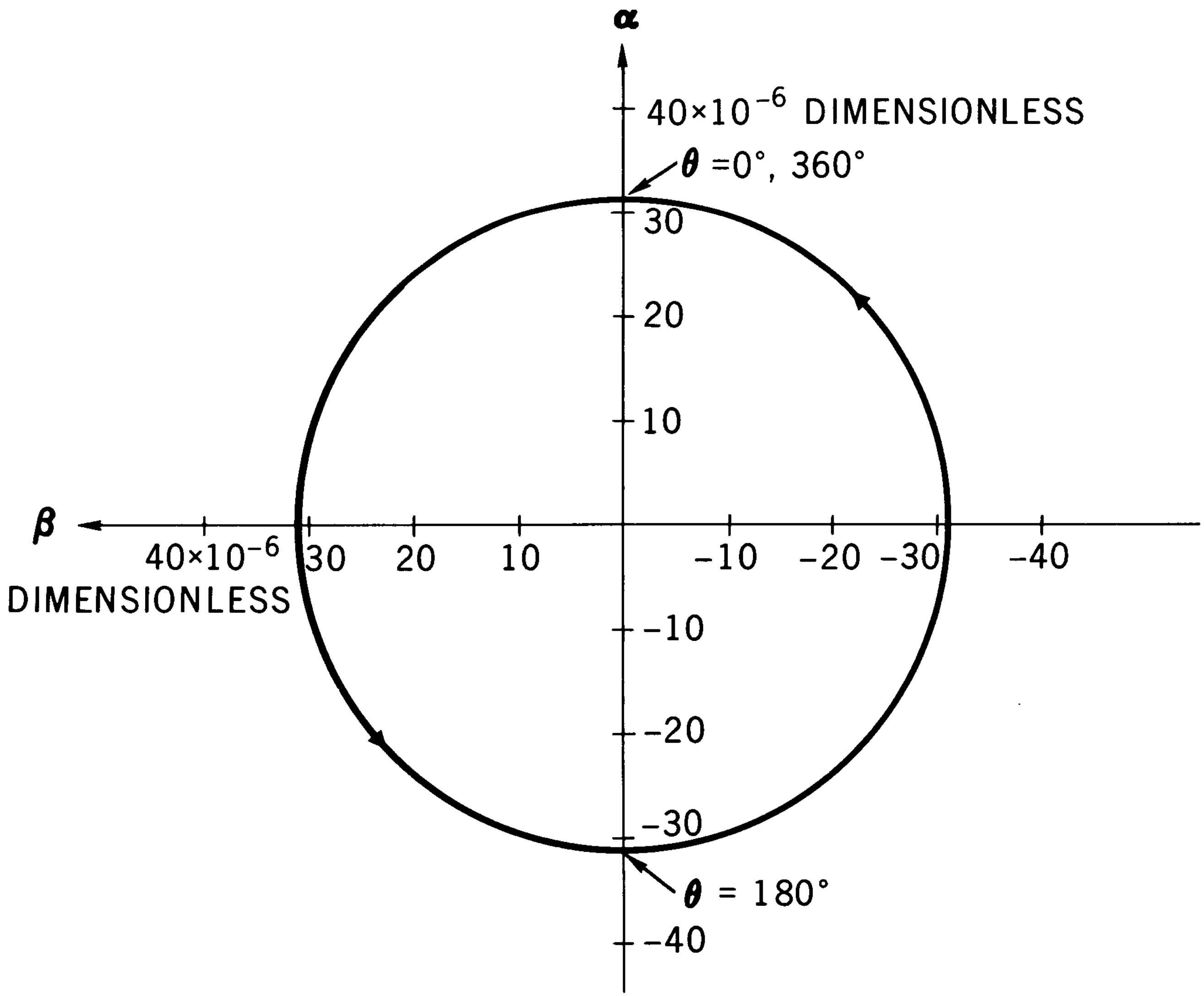

Figure 2. Three Body Sun Solution

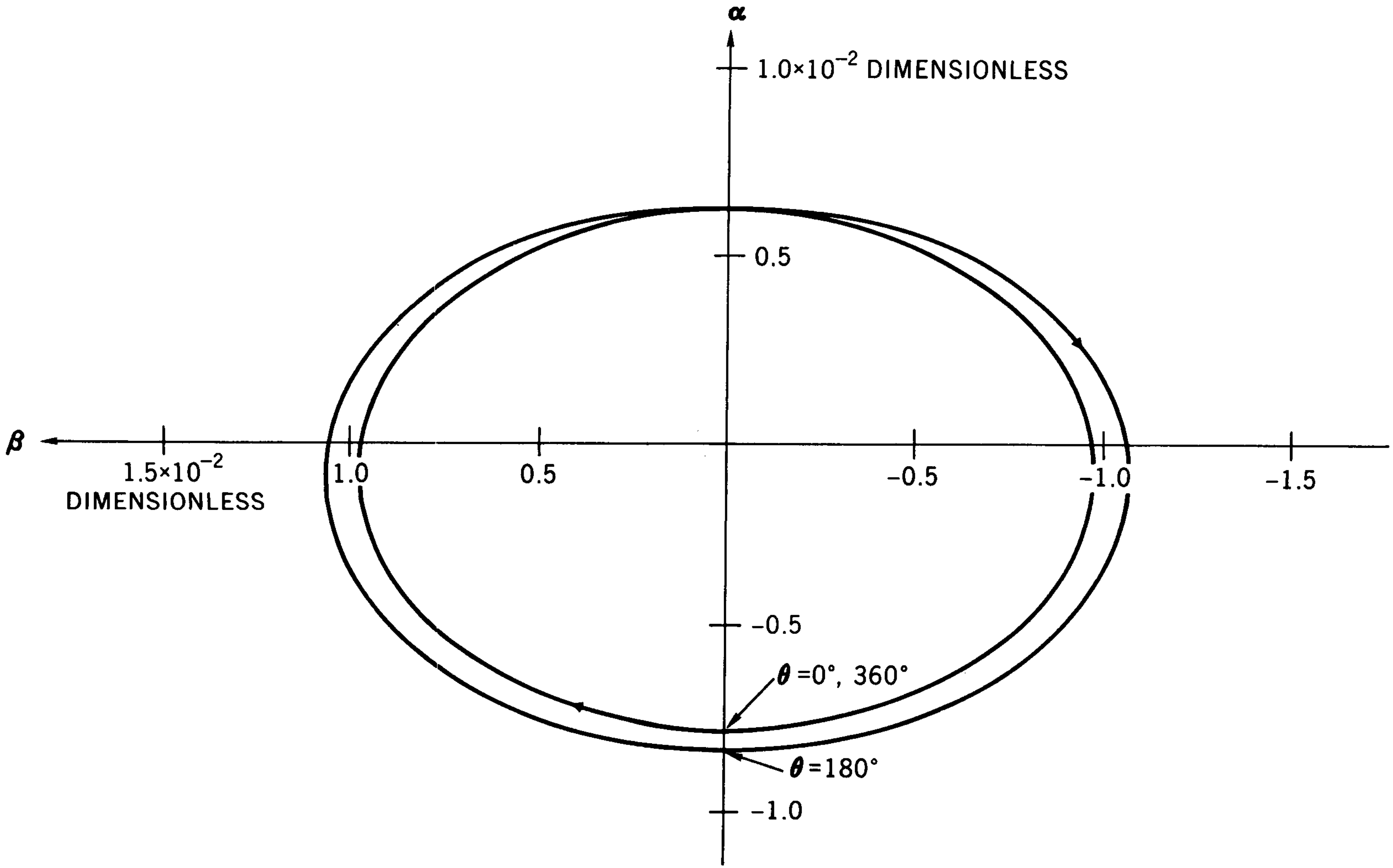

Figure 3. Three Body Moon Solution

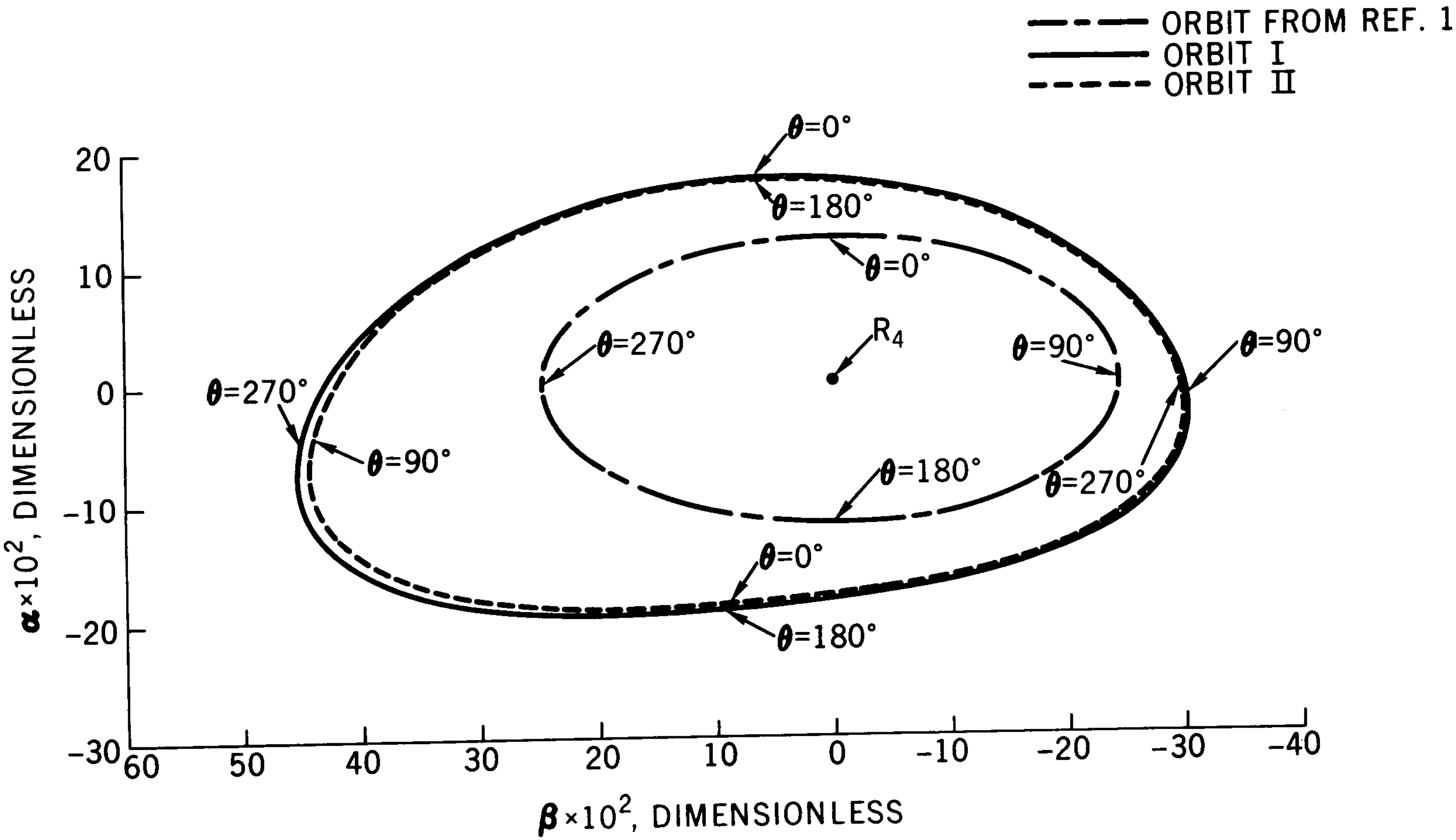

Figure 4. Comparison of Stable Triangular Point Orbits