GENERALIZED STOCHASTIC EQUATIONS

AND THEIR APPLICATIONS TO PLASMAS

by

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<div align="center">ERRATA</div>

	ERRORS	CORRECTIONS

P. 3 Eq.(2.2)

$$-\epsilon^2 \frac{\partial^2}{\partial x^2} \int_{-t}^{0} K \ldots \qquad -\epsilon^2 \frac{\partial}{\partial x} \int_{-t}^{0} K \ldots$$

P. 4 Line 11   Stratonivch   Stratonovich

P. 10 Eq.(2.30)

$$\epsilon^2 (-\frac{\partial}{\partial x}) \int_{0}^{t} <F\frac{\partial F(x,t)}{\partial x} \ldots \qquad \epsilon^2 (-\frac{\partial}{\partial x}) \int_{0}^{t} < \frac{\partial F(x,t)}{\partial x} \ldots$$

P.14 Line 18 $(\psi_1(t,t_0,x_0),..\psi_N(t,t_0,x))$ $(\psi_1(t,t_0,x_0),...\psi_N(t,t_0,x_0))$

P.24 Eq.(3.40)

$$\frac{\partial \omega_x(x,t)}{\partial \tau} \qquad \frac{\partial \omega_x(x,t)}{\partial t}$$

P.25 Line 11

$$\epsilon^2 \frac{\partial^2}{\partial y_i \partial y_j} \int_{-t}^{0} d\ K[\ldots \qquad \epsilon^2 \frac{\partial^2}{\partial y_i \partial y_j} \int_{-t}^{0} d\sigma K[\ldots$$

P. 26 Line 3

$$\int_{-\tau}^{0} d\sigma\ <(\frac{\partial \lambda_j^{-1}}{\partial x\ell})F_{\ell\sigma}/Y(0)=y>\omega_y \qquad \int_{-\tau}^{0} d\sigma<(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} F_\ell)_\sigma/Y(0)=y>\omega_y$$

P. 37 Eq.(4.32)

$$\frac{\partial \omega}{\partial t} = (\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial y^2})D(t)\omega \qquad \frac{\partial \omega}{\partial t} = (\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2})D(t)\omega$$

P.40 Eq.(4.45)

$$\ldots \int_{-t}^{0} K(\sigma)\cos\theta(t,t+\sigma)d\sigma \qquad \ldots \int_{-t}^{0} K(\sigma)\cos\theta(t,t+\sigma)d\sigma\omega$$

P.41 Eq.(4.46)

$$D(t) = \int_{-t}^{0} K(\sigma)\ldots \qquad D(t) = (\frac{q}{m})^2 \int_{-t}^{0} K(\sigma)\ldots$$

P.43 Line 18 $z = z_0 + v_{z0}$ $z = z_0 + v_{z0}t$

P.55 Eq.(5.24)

$$= \sum_{k\neq j}^{N} \frac{\partial\phi(\vec{x}_j,\vec{x}_k(t_j))}{\partial x_{j\partial}} \qquad = \sum_{k\neq j}^{N} \frac{\partial\phi(\vec{x}_j-\vec{x}_k(t_j))}{\partial x_{j\alpha}}$$

ABSTRACT

The Fokker-Planck equation has often been used to describe the distribution of a particle in a random field. When such a technique is applied to a plasma in order to obtain a kinetic equation, certain difficulties arise. In the first place, the electric field must be determined consistently and the coefficients cannot be found explicitly without further assumptions which may not be consistent. Previously various attempts have been made to find the coefficients including the use of the collision concept by means of the Boltzmann equation and the BBGKY theory. In certain cases, this amounts to a manipulation of a kinetic equation into the form of a Fokker-Planck equation. Moreover, the coefficients are given as averages of functionals of the random field. A naive interpretation of these averages, however, gives incorrect result.

In view of these difficulties, a new formulation of the Fokker-Planck method is established. It is observed that the coefficients actually involve conditional averages of functionals of the random field. Furthermore, just as a Fokker-Planck equation can be used to describe the lowest distribution, similar equations can also be used to describe higher distributions. Such a set of equations is called generalized stochastic (or generalized Fokker-Planck) equations following Stratonovich. These equations are applicable to systems perturbed by small random forces provided sufficient knowledge regarding the unperturbed system is available. When the random forces are specified statistically, the generalized stochastic equations give immediate results. In this respect, of particular interest in plasma application, is the heating of electrons

by random electric fields. For the Coulomb potential problem in which the electric field has to be determined consistently, it is possible to decouple the set of generalized equations by using a cluster expansion. The coefficients can then be found explicitly resulting in a kinetic equation. The results indicate that the use of a Vlasov equation in finding the polarization effect is essentially justified. The polarization effect is seen to be the consequence of the condition imposed on the average in evaluating the coefficient. Unlike the test particle theory, however, no test particle is artificially introduced. Finally, a similarity is observed between the set of generalized stochastic equations and the BBGKY hierarchy. Closer examination reveals that the BBGKY hierarchy may be regarded, in a sense, as a special mode of generalized stochastic equations.

ACKNOWLEDGMENT

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I. INTRODUCTION

The Fokker-Planck equation has often been used to describe the distribution of a particle in a random field. If the distribution is spatially homogeneous, the Fokker-Planck equation has the following form[1]

$$\frac{\partial f(v,t)}{\partial t} = - \frac{\partial}{\partial v} < \frac{\Delta v}{\Delta t} > f + \frac{1}{2} \frac{\partial^2}{\partial v^2} < \frac{(\Delta v)^2}{\Delta t} > f$$

where v is the velocity of the particle, Δv is the increment of velocity in an increment of time Δt, and f is the velocity distribution. The coefficients $< \frac{\Delta v}{\Delta t} >$, $< \frac{(\Delta v)^2}{\Delta t} >$ may be found in terms of the random field.[1] When applied to the case of a spatially homogeneous plasma, certain difficulties arise. In the first place, the electric field must be determined consistently and the coefficients cannot be found explicitly without further assumptions. Moreover, the polarization effect[2] cannot be obtained if the average operator < > is interpreted in a naive manner. To find the coefficients Gasiorowicz, et al.,[2] use the Vlasov equation for the polarization effect and a Holtsmark distribution for the fluctuation of the electric field. It is not clear how this can be done since this may lead to inconsistency.

We observe that the coefficients are actually average values of functionals of the random field under certain conditions. Furthermore just as a Fokker-Planck equation can be used to describe the distribution of the particle, similar equations can also be used to describe higher distributions. Such a set of equations is called generalized stochastic (or generalized Fokker-Planck) equations. When applied to a plasma,

these equations replacing the BBGKY hierarchy can be decoupled by using a cluster expansion to determine the coefficients.

In Chapter II, the generalized stochastic equations for a system with only small random forces are derived by modifying Stratonovich's[3] results. For systems having additional forces which are not small, reduction is possible through a suitable transformation provided sufficient knowledge regarding the unperturbed systems is available. This is considered in Chapter III. Some examples in which the random forces are assumed to be specified are given in Chapter IV. In Chapter V the Coulomb potential problem is considered with generalized stochastic equations replacing the BBGKY hierarchy. Using a cluster expansion to decouple the equations and freezing the lowest distribution in the evaluation of higher distributions, the Fokker-Planck or kinetic equations for a plasma are obtained. Although the mathematics involved in this case is close to that in the BBGKY theory, the starting point is different. It can be seen, however, that the BBGKY hierarchy may be regarded as a special form of generalized stochastic equations.

II. DERIVATION OF GENERALIZED STOCHASTIC EQUATIONS

It is known that the fluctuation equation

$$\dot{x} = \epsilon F(x,t) \tag{2.1}$$

leads to a Fokker-Planck type equation

$$\frac{\partial \omega(x,t)}{\partial t} = - \epsilon \frac{\partial}{\partial x} <F> \omega - \epsilon^2 \frac{\partial^2}{\partial x^2} \int_{-t}^{0} K[\frac{\partial F}{\partial x}, F_\tau] d\tau \; \omega + \epsilon^2 \frac{\partial^2}{\partial x^2} \int_{-t}^{0} K[F,F_\tau] d\tau \; \omega + O(\epsilon^3)$$

$$\tag{2.2}$$

where $F = F(x,t)$, $F_\tau = F(x,t+\tau)$, and K stands for the correlation

function. The force F as well as the initial value of x are random, and

ω represents the probability density of x at t. Stratonovich[3] gives a

detailed derivation for a system of equations in the form of (2.1). It

seems that, however, he has not taken into account of the correlation

between the force and the initial distribution of x. To account for

this, we generalize his results and obtain equations similar to (2.2)

which we shall refer to as the generalized stochastic equations following

Stratonovich.

Before we indicate how Stratonovich's results can be modified and

generalized, a few remarks concerning notations are in order. In the

mathematical literature, a random variable is often denoted by a capital

letter. Thus, $X(t)$ or X_t represents a family of random variables

indexed by t, a parameter, whereas lower cases are used to denote real

numbers. Unfortunately, in the engineering literature, usually no such

distinctions are made. Thus, the expected value of $F(X(0),t)$ given

$X(0) = x$, where $F(\cdot,t)$ is a random function, can be unambiguously

expressed as $< F(X(0),t) / X(0) = x> \square <F(x,t) / X(0) = x>$. It would be extremely confusing if not impossible to express such a functional without using the capital letters for random variable convention. To compromise, we shall follow the engineering literature except in situations where it is essential to distinguish a random variable from a real number. Thus, to emphasize that $x(t)$ is a random function, we may write $X(t)$ instead, and $X(t) = x$ means that the random variable $X(t)$ assumes the value of x.

2.1 Stratonovich's Derivation

We now review Stratonovich's derivation briefly. Consult Stratonivch[3] for details. The equation of interest is

$$\dot{x} = \epsilon F(x,t) \qquad (2.3)$$

with the initial condition

$$x(0) = x_0 \qquad (2.4)$$

Write the increment by

$$x(t) - x_0 = \epsilon Z_1 + \epsilon^2 Z_2 + \ldots = Z \qquad (2.5)$$

and expand $F(x,t)$ as follows

$$\epsilon F(x,t) = \epsilon F(x_0,t) + \epsilon \frac{\partial F}{\partial x}(\epsilon Z_1 + \ldots) + \ldots \qquad (2.6)$$

One then finds

$$\dot{Z}_1 = F(x_0,t), \quad \text{or} \quad Z_1(x_0,t) = \int_0^t F(x_0,t_1)dt_1$$

$$\dot{Z}_2 = \frac{\partial F}{\partial x}Z^1, \quad \text{or} \quad Z_2(x_0,t) = \int_0^t dt_2 \frac{\partial F}{\partial x}(x_0,t_2) \int_0^{t_2} F(x_0,t_1)dt_1 \quad \text{etc.} \quad (2.7)$$

The characteristic function of the random increment

$$z(t) = x(t) - x_0 \tag{2.8}$$

is

$$\phi_{Z(t)}(u) = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} <Z^n> \tag{2.9}$$

Inverting

$$\omega_{Z(t)}(z/X(0)=x_0) = \frac{1}{2\pi} \int e^{-iuz}[1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} <Z^n>]du$$

$$= [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial z})^n <Z^n>]\delta(z) \tag{2.10}$$

where $\omega_{Z(t)}(z/X(0)=x_0)$ is the probability density of $Z(t)$, given $X(0) = x_0$, and we shall write as $\omega_{Z(t)}(z/x_0)$ when confusion is not likely to arise. Now

$$\omega_{X(t)}(x/x_0) = \omega_{Z(t)}(x-x_0/x_0)$$

$$= [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial x})^n <Z^n(x_0,t)>]\delta(x-x_0) \tag{2.11}$$

Averaging over x_0

$$\omega_{X(t)}(x) = [1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{\partial}{\partial x})^n <Z^n(x,t)>]\omega_{X(0)}(x) = [1+L]\omega_{X(0)}(x) \tag{2.12}$$

where L is defined by (2.12). Using the more familiar notations

$$\omega(x,t) = \omega_{X(t)}(x)$$

$$\omega(x,0) = \omega_{X(0)}(x)$$

we have

$$\omega(x,t) = (1+L) \, \omega(x,0) \qquad (2.13)$$

Taking partial derivative with respect to time in (2.13) and substituting $\omega(x,0)$ by taking inverse of (2.13), we obtain

$$\frac{\partial \omega(x,t)}{\partial t} = \dot{L}(1+L)^{-1} \, \omega(x,t) \qquad (2.14)$$

Expanding $\dot{L}(1+L)^{-1}$, Equation (2.2) is obtained. Details are given by Stratonovich.[3]

2.2 Need for Conditional Averages

It appears that Stratonovich has assumed that F is not affected by the initial specification of X^{*}. Otherwise the averages in the characteristic function and hence $\dot{L}(1+L)^{-1}$ should be conditional averages. This modification is extremely important in the Coulomb potential problem as we shall see in Chapter V. In order to show how the modification comes about we first introduce the conditional characteristic functions.

Let Y,Z be random variables. Define the conditional characteristic function $\phi_Z(u/Y=y)$ as the conditional average of e^{iuZ} given Y = y. That is

$$\phi_Z(u/Y=y) = <e^{iuZ}/Y=y>$$

$$= \int e^{iuz} \omega(z/Y=y) \, dz \qquad (2.15)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} <Z^n/Y=y> \qquad (2.16)$$

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*If F does not depend on t explicitly, the results are correct.

where $< /Y=y>$ denotes the expected value given $Y = y$. Obviously, y enters only as a parameter so that the inversion is

$$\omega_Z(z/Y=y) = \frac{1}{2\pi} \int e^{-iuz} \phi_Z(u/Y=y) du$$

$$= \frac{1}{2\pi} \int e^{-iuz} [1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} <Z^n/Y=y>] du$$

$$= [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (- \frac{\partial}{\partial z})^n <Z^n/Y=y>] \delta(z) \qquad (2.17)$$

Similarly, if Y' is another random variable, then

$$\omega_Z(z/Y=y, Y'=y') = [1 + \sum_{n=1}^{\infty} \frac{1}{n!}(- \frac{\partial}{\partial z})^n <Z^n/Y=y, Y'=y'>] \delta(z) \qquad (2.18)$$

This shows that it is sufficient to replace the averages in the L operator in (2.14) with the proper conditional averages. In details, suppose we have random variables X, Y, Z such that

$$X - Y = Z \qquad (2.19)$$

Then $\omega_X(x/Y=y) = \omega_Z(x-y/Y=y)$

$$= [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (- \frac{\partial}{\partial x})^n <Z^n/Y=y>] \delta(x-y) \qquad (2.20)$$

Averaging over Y,

$$\omega_X(x) = [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (- \frac{\partial}{\partial x})^n <Z^n/Y=x>] \omega_Y(x) \qquad (2.21)$$

If we put $X = X(t)$, $Y = X(0)$, with $\omega_X(x) = \omega(x,t)$, $\omega_Y(y) = \omega(y,0)$, we have

$$\omega(x,t) = [1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial x})^n <Z^n/X(0)=x>] \, \omega(x,0) \qquad (2.22)$$

which is Stratonovich's result with the conditional average $<Z^n/X(0)=x>$ instead of just $<Z^n>$.

Comparing (2.22) with (2.13), we see that the operator L in (2.13) should be modified to $\bar{L}$ with $<Z^n/X(0)=x>$ replacing $<Z^n>$. In order to bring out the formal differences due to this modification, let us write out the equation corresponding to (2.14) by using the modified operator $\bar{L}$. We have

$$\frac{\partial \omega(x,t)}{\partial t} = \dot{\bar{L}}(1 + \bar{L})^{-1} \, \omega(x,t) \qquad (2.23)$$

Here

$$(1 + \bar{L}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-\frac{\partial}{\partial x})^n <Z^n/X(0)=x> \qquad (2.24)$$

where

$$Z = \epsilon Z_1 + \epsilon^2 Z_2 + \ldots \qquad (2.25)$$

Hence,

$$\dot{\bar{L}} = - \epsilon \frac{\partial}{\partial x} <\dot{Z}_1/X(0)=x> - \epsilon^2 \frac{\partial}{\partial x} <\dot{Z}_2/X(0)=x>$$

$$+ \epsilon^2 (-\frac{\partial}{\partial x})^2 <\dot{Z}_1 Z_1/X(0)=x> + O(\epsilon^3) \qquad (2.26)$$

and

$$(1 + \bar{L})^{-1} = 1 - \bar{L} + 0(\epsilon^2)$$

$$= 1 - \epsilon(-\frac{\partial}{\partial x}) <Z_1/X(0)=x> \ + 0(\epsilon^2) \qquad (2.27)$$

Combining (2.26) and (2.27)

$$\dot{\bar{L}}(1 + \bar{L})^{-1} = - \epsilon \frac{\partial}{\partial x} <\dot{Z}_1/X(0)=x> \ - \epsilon^2 \frac{\partial}{\partial x} <\dot{Z}_2/X(0)=x>$$

$$+ \epsilon^2(-\frac{\partial}{\partial x})^2 <\dot{Z}_1 Z_1/X(0)=x>$$

$$- \epsilon^2(-\frac{\partial}{\partial x}) <\dot{Z}_1/X(0)=x> \ (-\frac{\partial}{\partial x}) <Z_1/X(0)=x>$$

$$+ 0(\epsilon^3) \qquad (2.28)$$

$$= (-\frac{\partial}{\partial x}) <\epsilon\dot{Z}_1 + \epsilon^2\dot{Z}_2/X(0)=x>$$

$$+ (-\frac{\partial}{\partial x})^2 <\epsilon^2\dot{Z}_1 Z_1/X(0)=x>$$

$$- (-\frac{\partial}{\partial x})(-\frac{\partial}{\partial x}) <\epsilon\dot{Z}_1/X(0)=x> \ <\epsilon Z_1/X(0)=x>$$

$$+ (-\frac{\partial}{\partial x}) [(-\frac{\partial}{\partial x}) <\epsilon\dot{Z}_1/X(0)=x>] <\epsilon Z_1/X(0)=x>$$

$$+ 0(\epsilon^3)$$

$$= \epsilon(-\frac{\partial}{\partial x}) \ <\dot{Z}_1/X(0)=x>$$

$$+ \epsilon^2(-\frac{\partial}{\partial x})^2 K[\dot{Z}_1, Z_1/X(0)=x]$$

$$+ \varepsilon^2 (- \frac{\partial}{\partial x}) \quad <\dot{Z}^2/X(0)=x>$$

$$+ \varepsilon^2 (- \frac{\partial}{\partial x}) [(- \frac{\partial}{\partial x}) \quad <\dot{Z}_1/X(0)=x>] <\dot{Z}_1/X(0)=x>$$

$$+ O(\varepsilon^3) \tag{2.29}$$

where $K[A,B/X(0)=x] = <AB/X(0)=x> - <A/X(0)=x> <B/X(0)=x>$. From (2.7) we substitute the functions Z_1, Z_2, and find that

$$\dot{\bar{L}}(1 + \bar{L})^{-1} = \varepsilon(- \frac{\partial}{\partial x}) <F(x,t)/X(0) = x>$$

$$+ \varepsilon^2 (- \frac{\partial}{\partial x})^2 \int_0^t K[F(x,t), F(x,t_1)/X(0) = x] dt_1$$

$$+ \varepsilon^2 (- \frac{\partial}{\partial x}) \int_0^t <F \frac{\partial F(x,t)}{\partial x} F(x,t_1)/X(0) = x> dt_1$$

$$+ \varepsilon^2 (- \frac{\partial}{\partial x}) [(- \frac{\partial}{\partial x}) <F(x,t)/X(0) = x>] \int_0^t <F(x,t_1)/X(0)=x> dt_1$$

$$+ O(\varepsilon^3) \tag{2.30}$$

If F is not affected by the specification of X(0), the third and fourth terms on the right hand side of (2.30) combine to give

$$\varepsilon^2 (- \frac{\partial}{\partial x}) \int_0^t K[\frac{\partial F(x,t)}{\partial x}, F(x,t_1)] dt_1$$

as given by Stratonovich. However, we cannot do so in general. Thus, to second order, (2.23) gives the desired equation

$$\frac{\partial \omega(x,t)}{\partial t} = \left\{ -\varepsilon \frac{\partial}{\partial x} \left[<F(x,t)/X(0)=x> + \varepsilon \int_0^t < \frac{\partial F(x,t)}{\partial x} F(x,t_1)/X(0)=x> dt_1 \right] \right.$$

$$+ \varepsilon^2 \frac{\partial^2}{\partial x^2} \int_0^t K[F(x,t), F(x,t_1)/X(0)=x]dt_1$$

$$\left. + \varepsilon^2 \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} <F(x,t)/X(0)=x> \right] \int_0^t <F(x,t_1)/X(0)=x> dt_1 \right\} \omega(x,t)$$

$$(2.31)$$

We remark here that in the cases we consider later, the last term

contributes to terms of order higher than second and hence can be

ignored.

2.3 Generalization to Higher Density Functions and to Higher Dimensions

Above results can be easily generalized to higher density functions

and to higher dimensions.

Let $\omega_{X(t)}(x/X'(t') = x') = \omega(x,t/x',t')$ be the probability density

of $X(t) = x$ given $X'(t') = x'$. Here $X'(t')$ is another random function.

Equation (2.31) can be applied to the present case, provided we impose

the condition $X'(t') = x'$ on all averages. This can be seen from

Equation (2.18), with $Y = X(0)$, $Y' = X'(t')$ where Z is again defined by

the system (2.3) and

$$X(t) - X(0) = Z$$

Thus,

$$\omega_{X(t)}(x/X(0) = x_0, X'(t') = x')$$

$$= \omega_Z(x-x_0/X(0) = x_0, X'(t') = x')$$

$$= [1 + \sum_{n=1}^{\infty} (-\frac{\partial}{\partial x})^n <Z^n/X(0) = x_0, X'(t')=x>] \delta(x-x_0) \qquad (2.32)$$

Multiplying by $\omega_{X(0)}(x_0/X'(t') = x')$ and integrating over x_0,

$$\omega_{X(t)}(x/X'(t') = x') = \omega(x,t/x',t')$$

$$= [1 + \sum_{n=1}^{\infty} (-\frac{\partial}{\partial x})^n <Z^n/X(0)=x, X'(t') = x'>] \omega_{X(0)}(x/X'(t') = x')$$

$$= [1 + \sum_{n=1}^{\infty} (-\frac{\partial}{\partial x})^n <Z^n/X(0)=x, X'(t') = x'>] \omega(x,0/x',t')$$

$$(2.33)$$

Comparing (2.33) with (2.22), we see that the results are formally the same provided we impose the condition $X'(t') = x'$. Thus, (2.33) can be rewritten as

$$\omega(X,t/x',t') = (1 + \bar{L}_{x'}) \omega(X,0/x',t') \qquad (2.34)$$

Since the condition $X'(t') = x'$ enters as only a parameter, it does not have any effect on the manipulation of $\dot{\bar{L}}_{x'}(1 + \bar{L}_{x'})^{-1}$. Hence, (2.31) with the extra condition $X'(t') = x'$ is the equation for $\omega(x,t/x',t')$.

The results can also be generalized in a straight forward manner to multi-dimensional cases.[3] Thus, if instead of (2.3), we have

$$\dot{x}_j = F_j(x,t) \qquad j = 1, 2, \ldots, N \qquad (2.3a)$$

where

$$x = (x_1, x_2, \ldots, x_N)$$

The generalization of (2.31) is

$$\frac{\partial \omega(x,t)}{\partial t} = \left\{ - \epsilon \frac{\partial}{\partial x_j}[<F_j/X(0)=x> + \epsilon \int_{-t}^{0} < \frac{\partial F_j}{\partial x_k} F_{k\tau}/X(0)=x> d\tau] \right.$$

$$+ \epsilon^2 \frac{\partial^2}{\partial x_j \partial x_k} \int_{-t}^{0} K[F_j, F_{k\tau}/X(0)=x] d\tau$$

$$\left. + \epsilon^2 \frac{\partial}{\partial x_j} [\frac{\partial}{\partial x_k} <F_j/X(0)=x>] \int_{-t}^{0} <F_{k\tau}/X(0)=x> d\tau \right\} \omega \qquad (2.31a)$$

Here $F_j = F_j(x,t)$, $F_{k\tau} = F_{k\tau}(x, t + \tau)$, $j,k = 1,2, \ldots, N$, and the convention of summation index has been used.

To summarize, we have shown in this chapter how Stratonovich's results can be generalized so that the small random forces may not be independent of the initial distribution. This allows us to consider self-consistent forces other than applied external forces. Furthermore, higher density functions also satisfy similar equations. Thus, the process can be studied by using such a series of equations which we may call generalized stochastic (or Fokker-Planck) equations. They are given by (2.31) and the appropriate generalizations.

III. GENERALIZED STOCHASTIC EQUATIONS FOR SYSTEMS PERTURBED
BY RANDOM FORCES

In the previous chapter, we deal with a system of differential

equations of the form:

$$\dot{x} = \epsilon F$$

In other words, the "random force" is small for such systems. In this

chapter, we wish to study the effect of additional deterministic forces

which are not small, while assuming we have sufficient knowledge regarding

the system without the small random forces.

3.1 General Theorems in Differential Equations

We begin by collecting some relevant results in differential

equations.[4],[5] In this section we will use the Einstein summation

convention. Furthermore, an explicit notation for vectors will not be

used since the vector property does not play a significant roll except

in dot products where we use the summation convention. Thus, if we are

dealing with an N-dimensional space, by x we shall mean the N-tuple

$(x_1, \ldots, x_N)$, and by $\psi(t,t_0,x_0)$, we shall mean the N-tuple

$(\psi_1(t,t_0,x_0), \ldots, \psi_N(t,t_0,x))$. Here t is the scaler time. Thus,

Equation (2.3a) under this convention becomes

$$\dot{x} = \epsilon F(x,t)$$

which has the same appearance as (2.3).

Consider the system

$$\dot{x} = F(x,t) \tag{3.1}$$

Under appropriate conditions (such as the Lipschitz condition) on F,

there is a unique solution $x = \psi(t)$ for each pair (t_0,x_0) as initial

conditions. Thus, there is a function $\psi(t,t_0,x_0)$ such that it satisfies

(3.1) and $\psi(t_0,t_0,x_0) = x_0$. Under appropriate conditions (such as existence of $\frac{\partial F_j}{\partial x_k}$, $\frac{\partial F_j}{\partial t}$) the first partial derivatives of $\psi(t,t_0,x_0)$ exist.

Theorem 1

Define $J \equiv \dfrac{\partial(\psi_1, \ldots, \psi_N)}{\partial(x_1^0, \ldots, x_N^0)}$ (3.2)

Then $\dfrac{dJ}{dt} = J \dfrac{\partial F_j}{\partial x_j}$ along each solution and, hence, $J \equiv 1$ if and only if $\dfrac{\partial F_j}{\partial x_j} \equiv 0$.

The proof is available in standard literature[4] and will be omitted. In the cases we will be interested in, this can also be directly verified.

Theorem 2

In case of an autonomous system (i.e., $\dfrac{\partial F}{\partial t} = 0$),

$$\psi(t + \bar{t}, t_0 + \bar{t}, x_0) = \psi(t, t_0, x_0)$$

$$= \psi(t - t_0, 0, x_0)$$

        (3.3)

Proof: Let $V(t)$ be a solution through $(0,x_0)$. Define $U(t) = V(t-t_0)$. Then $U(t)$ is a solution through (t_0,x_0). But by the definition of $\psi(t,t_0,x_0)$, we have $V(t) = \psi(t,0,x_0)$, and $U(t) = \psi(t,t_0,x_0)$. Hence,

$$\psi(t,t_0,x_0) = U(t) = V(t-t_0) = \psi(t-t_0,0,x_0) \qquad (3.4)$$

Replacing t by $t + \bar{t}$, t_0 by $t_0 + \bar{t}$ in (3.4), we have

$$\psi(t + \bar{t}, t_0 + \bar{t}, x_0) = \psi(t - t_0, 0, x_0) \qquad (3.5)$$

Hence

$$\psi(t + \bar{t}, t_0 + \bar{t}, x_0) = \psi(t - t_0, 0, x_0)$$

$$= \psi(t, t_0, x_0)$$

Theorem 3

$$\psi(t, t_0, x_0) = \psi(t, t_1, \psi(t_1, t_0, x_0)) \tag{3.6}$$

Proof: $\psi(t, t_0, x_0)$ is a solution through $(t_1, \psi(t_1, t_0, x_0))$.

$\psi(t, t_1, \psi(t_1, t_0, x_0))$ is a solution through the same point, since

$\psi(t_1, t_1, \psi(t_1, t_0, x_0)) = \psi(t_1, t_0, x_0)$ by the definition of ψ. By uniqueness, these two solutions are identical.

Theorem 4

The following is a transform pair for each t, t_0.

$$\begin{cases} x = \psi(t, t_0, y) \\ \\ y = \psi(t_0, t, x) \end{cases} \tag{3.7}$$

Proof: Given y, x is uniquely determined by $x = \psi(t, t_0, y)$. Suppose there are y_1, y_2 such that for given x

$$x = \psi(t, t_0, y_1)$$

$$x = \psi(t, t_0, y_2)$$

Then $\psi(t, t_0, y_1) \equiv \psi(t, t_0, y_2)$ since both are solution through (t, x). In particular $\psi(t_0, t_0, y_1) = y_1 = \psi(t_0, t_0, y_2) = y_2$. By Theorem 3,

$$x = \psi(t, t_0, y) = \psi(t, t_0, \psi(t_0, t, x))$$

and the proof is therefore complete.

Note that in autonomous case, we may as well choose $t_0 = 0$, in Theorem 4 and write

$$\left\{ \begin{array}{l} x = \psi(t,0,y) = \lambda(t,y) \\[2ex] y = \psi(0,t,x) = \lambda^{-1}(t,x) \end{array} \right. \tag{3.8}$$

Corollary

In case of an autonomous system, then

$$\lambda(t,x) = \lambda^{-1}(-t,x) \tag{3.9}$$

and

$$\lambda(t+\sigma,\lambda^{-1}(t,x)) = \lambda(\sigma,x) \tag{3.10}$$

Proof: (3.9) can be readily shown to be true by applying Theorem 2. To prove (3.10) we apply Theorem 3. By Theorem 3,

$$\psi(t+\sigma,0,\psi(0,t,x)) = \psi(t+\sigma,t,x)$$

$$= \psi(\sigma,0,x)$$

which gives (3.10).

Theorem 3 has the following consequence in partial differentiations. Since $\psi_j(t_0,t+\sigma,x) = \psi_j(t_0,t,\psi(t,t+\sigma,x))$ we have

$$\left.\frac{\partial\psi_j}{\partial x_k}\right|_{t_0,t+\sigma,x} = \left.\frac{\partial\psi_j}{\partial x_\ell}\right|_{t_0,t,\psi(t,t+\sigma,x)} \cdot \left.\frac{\partial\psi_\ell}{\partial x_k}\right|_{t,t+\sigma,x} \tag{3.11}$$

In the autonomous case, if $\lambda^{-1}(t,x)$ is a linear function of x for each t, then

$$\left.\frac{\partial\lambda_j^{-1}}{\partial x_k}\right|_{t+\sigma} = \left.\frac{\partial\lambda_j^{-1}}{\partial x_\ell}\right|_t \cdot \left.\frac{\partial\lambda_\ell^{-1}}{\partial x_k}\right|_\sigma \tag{3.12}$$

where λ^{-1} is defined as in (3.8). Since λ^{-1} is assumed to be linear in x, $\partial\lambda^{-1}/\partial x$ is independent of x and hence the subscript x in (3.12) is omitted.

Theorem 5

$$\frac{\partial\psi_j(t_0,t,x)}{\partial t} + F_k(x,t)\frac{\partial\psi_j(t_0,t,x)}{\partial x_k} = 0 \qquad (3.13)$$

Proof: $\psi_j(t_0,t,\psi(t,t_0,x_0)) = x_{j0}$ by Theorem 3. Hence,

$$0 = \frac{d}{dt}\psi_j(t_0,t,\psi(t,t_0,x_0))$$

$$= \frac{\partial}{\partial t}\psi_j(t_0,t,\psi(t,t_0,x_0))$$

$$+ F_k(\psi(t,t_0,x_0),t)\cdot\frac{\partial\psi_j(t_0,t,\psi(t,t_0,x_0))}{\partial x_k}$$

Since the left hand side is independent of x_0, we may replace $\psi(t,t_0,x_0)$ by x (e.g., by choosing $x_0 = \psi(t_0,t,x)$). (3.13) then follows. More generally $f(\psi(t_0,t,x))$ satisfies (3.13) where f is arbitrary. Thus the partial differential Equation (3.13) with initial condition f(x) has $f(\psi(t_0,t,x))$ as its solution.

Finally we evaluate an expression needed later. Now

$$\frac{\partial\psi(t,t_0,y)}{\partial t} = F(\psi(t,t_0,y),t)$$

so that

$$\left.\frac{\partial \psi}{\partial t}\right|_{t,t_0,\psi(t_0,t,x)}$$

$$= F(\psi(t,t_0,\psi(t_0,t,x)),t)$$

$$= F(\psi(t,t,x),t)$$

$$= F(x,t) \tag{3.14}$$

3.2 Systems Perturbed by Random Forces

We now consider the system (3.1) under the influence of small random forces. Let the perturbed system be described by

$$\dot{x}_j = F_j^0 + \varepsilon F_j(x,t) \qquad j = 1, \ldots, N \tag{3.15}$$

We shall for simplicity, assume F_j^0 does not depend on t explicitly, so that the associated system in the unperturbed state

$$\dot{x}_j = F_j^0(x) \tag{3.16}$$

is autonomous.

Let $\lambda_j(t,t_0,x_0)$ be the solution of (3.16) with $\lambda_j(t_0,t_0,x_0) = x_{0j}$. Since $\lambda_j(t,t_0,x_0) = \lambda_j(t-t_0,0,x_0)$, we shall choose $t_0 = 0$ and just write $\lambda_j(t,x_0)$. Because of Theorem 4, we can define the transformation pair

$$\begin{cases} \tau = t \\ y_j = \lambda_j^{-1}(t,x) \end{cases} \qquad \begin{cases} t = \tau \\ x_j = \lambda_j(\tau,y) \end{cases} \tag{3.17}$$

where $\lambda_j^{-1}(t,x) = \lambda_j(-t,x)$ according to (3.19). Equation(3.14) is transformed to

$$\frac{dy_j}{d\tau} = \frac{\partial \lambda_j^{-1}}{\partial t} \cdot \frac{dt}{d\tau} + \frac{\partial \lambda_j^{-1}}{\partial x_k} \frac{dx_k}{dt} \frac{dt}{d\tau}$$

$$= \frac{\partial \lambda_j^{-1}}{\partial t} \cdot + \frac{\partial \lambda_j^{-1}}{\partial x_k} \frac{dx_k}{dt}$$

$$= \frac{\partial \lambda_j^{-1}}{\partial t} + F_k^0 \frac{\partial \lambda_j^{-1}}{\partial x_k} + \epsilon \frac{\partial \lambda_j^{-1}}{\partial x_k} F_k \qquad (3.19)$$

By Theorem 5,

$$\frac{\partial \lambda_j^{-1}}{dt} + F_k^0 \frac{\partial \lambda_j^{-1}}{\partial x_k} = 0$$

so that

$$\frac{dy_j}{d\tau} = \epsilon \frac{\partial \lambda_j^{-1}}{\partial x_k} F_k \qquad (3.20)$$

Thus system (3.15) goes to the system (3.20) in which the forces are small. In (3.20) we should have x and t in terms of y, τ using (3.17). We shall do this explicitly only for F. Defining

$$\mathcal{F}(y,\tau) = F(\lambda(\tau,y),\tau) \qquad (3.21)$$

Equation (3.20) becomes

$$\frac{dy_\ell}{d\tau} = \epsilon \frac{\partial \lambda_\ell^{-1}}{\partial x_k} \mathcal{F}_k(y,\tau) \qquad (3.22)$$

Therefore the zeroth order non-random force in (3.15) can be transformed away through (3.17). The resulting equation (3.22) has the identical form

as the one studied in the last chapter. We shall now apply these results to our present system.

Let $\omega_Y(y,\tau)$ be the density of y at τ. From Equation (2.31a):

$$\frac{\partial}{\partial \tau}\omega_Y(y,\tau) = -\varepsilon \frac{\partial}{\partial y_i} \left\langle \frac{\partial \lambda_i^{-1}}{\partial x_j} \mathcal{F}_j / Y(0) = y \right\rangle \omega_Y$$

$$- \varepsilon^2 \frac{\partial}{\partial y_i} \int_{-\tau}^0 d\sigma \left\langle \left(\frac{\partial}{\partial y_j} \frac{\partial \lambda_i^{-1}}{\partial x_k} \mathcal{F}_k \right)\left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \mathcal{F}_\ell \right)_\sigma / Y(0) = y \right\rangle \omega_Y$$

$$+ \varepsilon^2 \frac{\partial^2}{\partial y_i \partial y_j} \int_{-\tau}^0 d\sigma \ K\left[\frac{\partial \lambda_i^{-1}}{\partial x_k} \mathcal{F}_k, \left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \mathcal{F}_\ell \right)_\sigma / Y(0) = y \right] \omega_Y$$

$$+ \varepsilon^2 \frac{\partial}{\partial y_i} \left[\frac{\partial}{\partial y_j} \left\langle \frac{\partial \lambda_i^{-1}}{\partial x_k} \mathcal{F}_k / Y(0) = y \right\rangle \right] \int_{-\tau}^0 d\sigma \left\langle \left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \mathcal{F}_\ell \right)_\sigma / Y(0) = y \right\rangle \omega_Y$$

$$(3.23)$$

where by $(\quad)_\sigma$, we mean the function in the parenthesis is to be evaluated at a time interval σ later. In order to transform (3.23) back to variables x and t, we need several relations which we first derive in the following.

Equation (3.23) is general and valid even if system (3.16) is non-autonomous, provided we take the general transformation (3.7). In such case, however, the equation will depend strongly on the initial time. Considerable simplification results in a special but important case, when (3.16) is linear with constant coefficient so that

$$\lambda_i^{-1}(t,x) = \lambda_{ij}^{-1}(t)x_j \qquad i = 1, \ldots, N \qquad (3.24)$$

where the λ_{ij}^{-1} are scalar functions of t, forming a matrix which we shall designate by $\Omega^{-1}(t)$. Thus in matrix notation

$$\Omega^{-1}(t) = (\lambda_{ij}^{-1}(t)) \qquad (3.25)$$

and (3.17) becomes

$$y = \Omega^{-1}(t)x \qquad (3.26)$$

Since $\lambda_j^{-1}(t,x) = \lambda_j(-t,x)$, we can define λ_{ij} and $\Omega(\tau)$ similarly by relations

$$\lambda_i(\tau,y) = \lambda_{ij}(\tau)y_j \qquad (3.27)$$

and

$$x = \Omega(\tau)y \qquad (3.28)$$

where

$$\Omega(\tau) = (\lambda_{ij}(\tau)) \qquad (3.29)$$

Now we present some interesting properties. They are

$$\lambda_{ij}(\tau) = \lambda_{ij}^{-1}(-\tau) \quad \text{or} \quad \Omega^{-1}(\tau) = \Omega(-\tau) \qquad (3.30)$$

$$\lambda_{ij}^{-1} = \frac{\partial \lambda_i^{-1}}{\partial x_j} \;, \qquad \lambda_{ij} = \frac{\partial \lambda_i}{\partial y_j} \qquad (3.31)$$

and

$$\Omega(0) = \Omega^{-1}(0) = \Omega(t)\Omega^{-1}(t) = \Omega^{-1}(t)\Omega(t) = I_N \qquad (3.32)$$

since $\lambda_j(0,x_0) = x_{0j}$. Here I_N is the identity matrix. Theorem 1 then asserts

$$\text{det. } \Omega = 1 = \text{det. } \Omega^{-1} \qquad (3.33)$$

if the hypothesis $\dfrac{\partial F_j^0}{\partial x_j} = 0$ is satisfied. This is always true since the unperturbed system (3.15) is a conservative Hamiltonian system in the cases we are interested in.[6] Equation (3.12) implies that

$$\Omega^{-1}(t + \sigma) = \Omega^{-1}(t)\Omega^{-1}(\sigma) \qquad (3.34)$$

$$= \Omega^{-1}(\sigma)\Omega^{-1}(t)$$

For an arbitrary function $H(y,\tau)$, we have the following chain rules:

$$\frac{\partial H}{\partial \tau} = \frac{\partial h}{\partial t}(x,t) + \frac{\partial \lambda_j}{\partial \tau}\frac{\partial h}{\partial x_j}(x,t) \qquad (3.35)$$

$$\frac{\partial H}{\partial y_j} = \frac{\partial \lambda_i}{\partial y_j}\frac{\partial h(x,t)}{\partial x_i} \qquad (3.36)$$

where $h(x,t) = H(\lambda^{-1}(t,x),t)$.

Using (3.13) and (3.31), (3.35) and (3.36) give

$$\frac{\partial H}{\partial \tau} = \frac{\partial h}{\partial t} + F_j^0(x)\frac{\partial h}{\partial x_j} \qquad (3.37)$$

$$\frac{\partial H}{\partial y_j} = \lambda_{ij}(t)\,\frac{\partial h}{\partial x_i} \tag{3.38}$$

Also, we have

$$\omega_Y(\lambda^{-1}(t,x),t) = \omega_X(x,t) \tag{3.39}$$

where $\omega_X(x,t)$ is the density of $X(t) = x$, because of (3.33). Using (3.37) through (3.39), equation (3.23) can be transformed back to the variables x,t. Thus, the left-hand side of (3.23) becomes:

$$\frac{\partial \omega_Y(y,\tau)}{\partial \tau} = \frac{\partial \omega_X(x,t)}{\partial \tau} + F_j^0(x)\,\frac{\partial \omega_X(x,t)}{\partial x_j} \tag{3.40}$$

The first term on the right side of (3.23):

$$- \varepsilon\,\frac{\partial}{\partial y_i} < \frac{\partial \lambda_i^{-1}}{\partial x_j}\,\mathcal{F}_j \big/ Y(0) = y >\,\omega_Y$$

$$= - \varepsilon\,\lambda_{ki}\frac{\partial}{\partial x_k} < \lambda_{ij}^{-1}\,F_j \big/ X(0) = \lambda^{-1}(t,x) >\,\omega_X(x,t)$$

$$= - \varepsilon\,\frac{\partial}{\partial x_j} < F_j \big/ X(0) = \lambda^{-1}(t,x) >\,\omega_X \tag{3.41}$$

where we have made use of the fact that λ_{ij} is a function of t alone, and that Equation (3.32) implies $\lambda_{ki}\lambda_{ij}^{-1} = \delta_{kj}$, $X(0) = Y(0)$.

For the second term of (3.23):

$$- \varepsilon^2 \frac{\partial}{\partial y_i} \int_{-\tau}^{0} d\sigma < \left(\frac{\partial}{\partial y_j} \frac{\partial \lambda_i^{-1}}{\partial x_k} \mathcal{F}_k \right) \left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \mathcal{F}_\ell \right)_\sigma / Y(0) = y > \omega_Y$$

$$= - \varepsilon^2 \lambda_{mi}(t) \frac{\partial}{\partial x_m} \int_{-\tau}^{0} d\sigma < \lambda_{nj}(t) \frac{\partial}{\partial x_n} \lambda_{ik}^{-1}(t) F_k \lambda_{j\ell}^{-1}(t+\sigma) \mathcal{F}_\ell(\lambda^{-1}(t,x),t+\sigma)/X(0)$$

$$= \lambda^{-1}(t,x) > \omega_X \tag{3.42}$$

Now from (3.21)

$$\mathcal{F}_\ell(\lambda^{-1}(t,x),t+\sigma) = F_\ell(\lambda(t+\sigma,\lambda^{-1}(t,x)), t+\sigma) = F_\ell(\lambda(\sigma,x),t+\sigma) \tag{3.43}$$

by (3.10).

Thus (3.42) simplifies to

$$-\varepsilon^2 \frac{\partial}{\partial x_k} \int_{-t}^{0} d\sigma \frac{\partial F_k}{\partial x_i} \lambda_{i\ell}^{-1}(\sigma) F_\ell(\lambda(\sigma,x),t+\sigma)/X(0) = \lambda^{-1}(t,x) > \omega_X \tag{3.44}$$

in which we have used (3.32) and (3.34).

For the third term in (3.23):

$$\varepsilon^2 \frac{\partial^2}{\partial y_i \partial y_j} \int_{-t}^{0} d \ K \left[\frac{\partial \lambda_i^{-1}}{\partial x_k} \mathcal{F}_k, \left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \mathcal{F}_\ell \right)_\sigma / Y(0) = y \right] \omega_Y$$

$$= \varepsilon^2 \lambda_{mi}(t) \frac{\partial}{\partial x_m} \lambda_{nj}(t) \frac{\partial}{\partial x_n} \int_{-t}^{0} d\sigma \ K[\lambda_{ik}^{-1}(t) F_k, \lambda_{j\ell}^{-1}(t+\sigma) F_\ell (\lambda(\sigma,x),t+\sigma)/X(0)$$

$$= \lambda^{-1}(t,x)] \omega_X$$

$$= \varepsilon^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \int_{-t}^{0} d\sigma \; K[F_k, \lambda_{j\ell}^{-1}(\sigma)F_\ell(\lambda(\sigma,x),t+\sigma)/X(0) = \lambda^{-1}(t,x)]\omega_X \qquad (3.45)$$

For the last term in (3.23):

$$\varepsilon^2 \frac{\partial}{\partial y_i} \left[\frac{\partial}{\partial y_j} < \frac{\partial \lambda_i^{-1}}{\partial x_k} \mathscr{F}_k / Y(0) = y> \right] \int_{-\tau}^{0} d\sigma \; < \left(\frac{\partial \lambda_j^{-1}}{\partial x_\ell} \right) \mathscr{F}_\ell {}_\sigma / Y(0) = y> \; \omega_Y$$

$$= \varepsilon^2 \lambda_{mi} \frac{\partial}{\partial x_m} [\lambda_{nj} \frac{\partial}{\partial x_n} <\lambda_{ik}^{-1} F_k / X(0) = \lambda^{-1}(t,x)>] \int_{-t}^{0} d\sigma \; <\lambda_{j\ell}^{-1}(t+\sigma)F_\ell(\lambda(\sigma,x),t+\sigma)$$

$$/X(0) = \lambda^{-1}(t,x)> \; \omega_X$$

$$= \varepsilon^2 \frac{\partial}{\partial x_k} [\frac{\partial}{\partial x_j} <F_k / X(0) = \lambda^{-1}(t,x)>] \int_{-t}^{0} d\sigma \; <\lambda_{j\ell}^{-1}(\sigma)F_\ell(\lambda(\sigma,x),t+\sigma)/X(0)$$

$$= \lambda^{-1}(t,x)> \quad \omega_X \qquad (3.46)$$

Combining (3.40), (3.41), (3.44), (3.45) and (3.46),

$$\frac{\partial \omega_X}{\partial t} + F_j^0(x,t)\frac{\partial \omega_X}{\partial x_j} = - \varepsilon \frac{\partial}{\partial x_j} <F_j / X(0) = \lambda^{-1}(t,x)> \; \omega_X$$

$$- \varepsilon^2 \frac{\partial}{\partial x_k} \int_{-t}^{0} d\sigma \; \lambda_{i\ell}^{-1}(\sigma) < \frac{\partial F_k}{\partial x_i} F_\ell(\lambda(\sigma,x),t+\sigma)/X(0) = \lambda^{-1}(t,x)> \; \omega_X$$

$$+ \varepsilon^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \int_{-t}^{0} d\sigma \lambda_{j\ell}^{-1}(\sigma) \; K[F_k, F_\ell(\lambda(\sigma,x),t+\sigma)/X(0) = \lambda^{-1}(t,x)]\omega_X$$

$$+ \; \varepsilon^2 \; \frac{\partial}{\partial x_k} [\frac{\partial}{\partial x_j} \; <F_k/X(0) \; = \; \lambda^{-1}(t,x) \; >] \int_{-t}^{0} d\sigma \; \lambda_{j\ell}^{-1} \; < \; F_\ell(\lambda(\sigma,x),t+\sigma)/X(0)$$

$$= \lambda^{-1}(t,x) > \; \omega_X \qquad\qquad\qquad (3.47)$$

In passing we remark that the basic idea is to transform system (3.15) into a system (3.22) with only small forces which will permit us to write down a stochastic equation. Such a transformation can be effected through the integrals of the unperturbed system, and in our case, this amounts to describing the system along its unperturbed orbits. As is pointed out in the derivation, Equation (3.47) applies in cases where (3.16) is linear with constant coefficients and F^0 is a conservative force. If F^0 is not conservative, there is then need of introducing a Jacobian in (3.39) and consequently in (3.47). When (3.16) is non-linear but still autonomous, we then have to go back to (3.23). In the most general case when (3.16) is non-autonomous the λ-transformation in (3.23) should be replaced by the more general transformation (3.7).

IV. EXAMPLES FOR SPECIFIED RANDOM FORCES

In this chapter we consider some examples in which the random forces are assumed to be specified. Of particular interest in plasma application is the heating of electrons by random electric field. Sturrock[1.] has considered the heating of electrons without reference to the mechanism for generation of the random electric field. Puri[7] has performed an experiment in which a collisionless plasma is heated by an electric field of a noisy generator. We obtain similar results by using equations developed earlier. The random electric field may be due to an incident plane wave and this is considered in the last example.

4.1 Example 1. Charged Particle in a Weak Electric Field

Consider a charged particle with charge q, mass m, in an electric field $E(x,t) = - \frac{\partial \phi}{\partial x}$, where ϕ is the potential. Note that the electric field is not a function of velocity. The equation of motion gives

$$\dot{x} = v$$
$$\dot{v} = \frac{q}{m} E$$
$$(4.1)$$

If we normalize the equations by[*]

$$x = \underline{x} \, x_0$$
$$v = \underline{v} \, v_0$$
$$t = \underline{t} \, \frac{x_0}{v_0}$$
$$\phi = \underline{\phi} \, \phi_0$$
$$(4.2)$$

---

[*] We may choose the correlation time of the random field as the characteristic time and the thermal velocity as the characteristic velocity.

where x_0, v_0, ϕ_0 are characteristic quantitites not to be confused with initial values. Equation (4.1) in the normalized quantities is then given by

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = (\frac{q\phi_0}{mv_0^2}) \; (- \frac{\partial \phi}{\partial x}) \tag{4.3}$$

We express "weak field" by considering $\frac{q\phi_0}{mv_0^2} = 0(\epsilon)$. Formally then we can regard $\frac{q}{m} E$ in (4.1) as $0(\epsilon)$. This corresponds to $F_1^0 = v$, $F_2^0 = 0$, $F_1 = 0$, $\epsilon F_2 = \frac{q}{m} E$.

Also

$$\lambda_1^{-1}(t, x, v) = x - vt$$

$$\lambda_2^{-1}(t, x, v) = v \tag{4.4}$$

or

$$\Omega_1^{-1}(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

Using (3.47), we obtain the stochastic equation for the first example as

$$\frac{\partial \omega_x}{\partial t} + v \frac{\partial \omega_x}{\partial x} = - \frac{q}{m} \frac{\partial}{\partial v} < E/X(0) = x - vt, \; V(0) = v > \omega_x$$

$$- (\frac{q}{m})^2 \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma(-\sigma) < \frac{\partial E}{\partial x} E(x + v\sigma, t+\sigma)/X(0) = x - vt,$$

$$V(0) = v > \omega_x$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v} \frac{\partial}{\partial x} \int_{-t}^{0} d\sigma(-\sigma) \; K[E, E(x + v\sigma, t + \sigma)/X(0) =$$

$$x - vt, \; V(0) = v] \; \omega_x$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v} \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma \; K[E, E(x + v\sigma, t + \sigma)/X(0) = x -$$

$$vt, \; V(0) = v] \; \omega_x$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v} \left[\frac{\partial}{\partial x} < E/X(0) = x - vt, \; V(0) = v >\right] \int_{-t}^{0} d\sigma(-\sigma)$$

$$< E(x + v\sigma, t + \sigma)/X(0) = x - vt, \; V(0) = v > \omega_x$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v} \left[\frac{\partial}{\partial v} < E/X(0) = x - vt, \; V(0) = v >\right] \int_{-t}^{0} d\sigma$$

$$< E(x + v\sigma), t + \sigma)/X(0) = x - vt), \; V(0) = v > \omega_x$$

$$(4.6)$$

To compare with Sturrock's results[1] we assume the electric field is external, with

$$< E(x,t) > = 0. \tag{4.7}$$

The electric field is further assumed to be steady and homogeneous in the statistical sense, i.e.

$$< E(x,t) \; E(x + x', t + \sigma) >$$

$$= < E(0,0) \; E(x', \sigma) >$$

$$= K(x', \sigma) \tag{4.8}$$

Let $S(k,\omega)^*$ be the Fourier Transform of the correlation function $K(x',\sigma)$ so that

$$\left\{\begin{array}{l} K(x',\sigma) = \iint dk\,d\omega\, e^{i(kx' - \omega\sigma)} S(k,\omega) \\[2em] S(k,\omega) = (\frac{1}{2\pi})^2 \iint dx'\,d\sigma\, e^{-i(kx' - \omega\sigma)} K(x',\sigma) \end{array}\right. \tag{4.9}$$

Since we have an external field, the conditions on all averages in (4.6) may be dropped. If we are interested in the spatially homogeneous case where

$$\frac{\partial \omega_x}{\partial x} = 0 \tag{4.10}$$

we find Equation (4.6) reduces to

$$\frac{\partial \omega_x}{\partial t} = - (\frac{q}{m})^2 \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma(-\sigma) < \frac{\partial E}{\partial x} E(x + v\sigma, t + \sigma) > \omega_x$$

$$+ (\frac{q}{m})^2 \frac{\partial}{\partial v} \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma < E\, E(x + v\sigma, t + \sigma) > \omega_x \tag{4.11}$$

Now
$$\int_{-t}^{0} d\sigma(-\sigma) < \frac{\partial E}{\partial x} E(x + v\sigma, t + \sigma) > = \int_{-t}^{0} d\sigma(-\sigma) \frac{\partial}{\partial x} < EE(x + v\sigma, t+\sigma)>$$

$$- \int_{-t}^{0} d\sigma(-\sigma) < E \frac{\partial E}{\partial x} (x + v\sigma, t + \sigma) >$$

$$= 0 + \int_{-t}^{0} d\sigma < E\, \sigma \frac{\partial E}{\partial x} (x + v\sigma, t + \sigma) >$$

---

* The ω here stands for angular frequency and is not to be confused with the probability density function.

$$= \int_{-t}^{0} d\sigma < E \frac{\partial E}{\partial v} (x + v\sigma, t + \sigma) >$$

$$= \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma < EE(x + v\sigma, t + \sigma) > \tag{4.12}$$

in which we have used (4.7). The use of (4.12) further reduces the stochastic Equation (4.11) to

$$\frac{\partial \omega_x}{\partial t} = - (\frac{q}{m})^2 \frac{\partial}{\partial v} (\frac{\partial}{\partial v} \int_{-t}^{0} d\sigma < EE(x + v\sigma, t + \sigma) >) \omega_x$$

$$+ (\frac{q}{m})^2 \frac{\partial}{\partial v} \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma < EE(x + v\sigma, t + \sigma) > \omega_x$$

or

$$\frac{\partial \omega_x}{\partial t} = (\frac{q}{m})^2 \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma < EE(x + v\sigma, t + \sigma) > \frac{\partial}{\partial v} \omega_x$$

$$= (\frac{q}{m})^2 \frac{\partial}{\partial v} \int_{-t}^{0} d\sigma \, K(v\sigma, \sigma) \frac{\partial}{\partial v} \omega_x \tag{4.13}$$

Defining

$$D(v,t) = (\frac{q}{m})^2 \int_{-t}^{0} d\sigma \, K(v\sigma, \sigma) \tag{4.14}$$

we obtain finally the desired equation

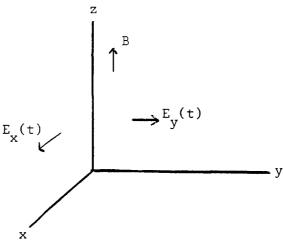$$\frac{\partial \omega_x}{\partial t} = \frac{\partial}{\partial v} D(v,t) \frac{\partial}{\partial v} \omega_x \tag{4.15}$$

Here

$$D(v, t \to \infty) = \pi (\frac{q}{m})^2 \int dk \, S(k, vk) \tag{4.16}$$

and by $t \to \infty$, we mean t much larger than the correlation time of E.
Equation (4.15) is a diffusion equation. It has been obtained also by

Sturrock. The reader is referred to Sturrock for a discussion of some of its properties.

4.2 Example 2. Charged Particle in a Weak Electric Field with a Constant Magnetic Field

Consider a charged particle under the influence of a constant magnetic field perpendicular to its plane of motion. A small random external electric field is applied in the transverse direction as shown in the figure. For simplicity we assume E is only a function of time. The orientation of the magnetic field B and the electric field E(t) is depicted in the figure.

The equations of motion for the present problem are

$$
\begin{cases}
\dot{x} = v_x \\
\dot{v} = \omega_c v_y + \dfrac{q}{m} E_x \\
\dot{y} = v_y \\
\dot{v}_y = -\omega_c v_x + \dfrac{q}{m} E_y
\end{cases}
\tag{4.17}
$$

where $\omega_c = \dfrac{qB}{m}$ (4.18)

Formally we take $\frac{q}{m} E = O(\varepsilon)$

<div align="right">(4.19)</div>

With

$$x_1 = x$$

$$x_2 = y$$

$$x_3 = v_x$$

$$x_4 = v_y$$

<div align="right">(4.20)</div>

(4.17) becomes

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = \omega_c x_4 + \frac{q}{m} E_x$$

$$\dot{x}_4 = -\omega_c x_3 + \frac{q}{m} E_y$$

<div align="right">(4.21)</div>

From the unperturbed system we obtain

$$
\begin{cases}
\lambda_1(t,x_{10},x_{20},x_{30},x_{40}) = x_{10} + x_{30} \dfrac{\sin \omega_c t}{\omega_c} - \dfrac{x_{40}[\cos \omega_c t - 1]}{\omega_c} \\[4mm]
\lambda_2(t,x_{10},x_{20},x_{30},x_{40}) = x_{20} + \dfrac{x_{30}[\cos \omega_c t - 1]}{\omega_c} + \dfrac{x_{40} \sin \omega_c t}{\omega_c} \\[4mm]
\lambda_3(t,x_{10},x_{20},x_{30},x_{40}) = x_{30} \cos \omega_c t + x_{40} \sin \omega_c t \\[4mm]
\lambda_4(t,x_{10},x_{20},x_{30},x_{40}) = -x_{30} \sin \omega_c t + x_{40} \cos \omega_c t
\end{cases}
$$

<div align="right">(4.22)</div>

The Corollary (3.9) and Equation (3.24) can be used to compute Ω^{-1}, yielding

$$
\Omega^{-1}(t) = \begin{pmatrix}
1 & 0 & -\dfrac{\sin \omega_c t}{\omega_c} & -\dfrac{[\cos \omega_c t - 1]}{\omega_c} \\[2ex]
0 & 1 & \dfrac{[\cos \omega_c t - 1]}{\omega_c} & -\dfrac{\sin \omega_c t}{\omega_c} \\[2ex]
0 & 0 & \cos \omega_c t & -\sin \omega_c t \\[2ex]
0 & 0 & \sin \omega_c t & \cos \omega_c t
\end{pmatrix}
\tag{4.23}
$$

Substituting into (3.47),

$$
\frac{\partial \omega}{\partial t} + \vec{v} \cdot \frac{\partial \omega}{\partial \vec{x}} + \vec{v} \times \vec{B} \cdot \frac{\partial \omega}{\partial \vec{v}}
$$

$$
= \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_x} \frac{\partial}{\partial x} \int_{-t}^{0} d\sigma \left\{ K_{xx}(\sigma) \left(\frac{-\sin \omega_c \sigma}{\omega_c}\right) + K_{xy}(\sigma)\left[-\frac{1}{\omega_c}(\cos \omega_c \sigma - 1)\right] \right\} \omega
$$

$$
+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_x} \frac{\partial}{\partial y} \int_{-t}^{0} d\sigma \left\{ K_{xx}(\sigma)\left[\frac{1}{\omega_c}(\cos \omega_c \sigma)\right] + K_{xy}(\sigma)\left(-\frac{\sin \omega_c \sigma}{\omega_c}\right) \right\} \omega
$$

$$
+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_x} \frac{\partial}{\partial v_x} \int_{-t}^{0} d\sigma \left\{ K_{xx}(\sigma)(\cos \omega_c \sigma) + K_{xy}(\sigma)(-\sin \omega_c \sigma) \right\} \omega
$$

$$
+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_x} \frac{\partial}{\partial v_y} \int_{-t}^{0} d\sigma \left\{ K_{xx}(\sigma)(\sin \omega_c \sigma) + K_{xy}(\sigma)(\cos \omega_c \sigma) \right\} \omega
$$

$$
+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_y} \frac{\partial}{\partial x} \int_{-t}^{0} d\sigma \left\{ K_{yx}(\sigma)\left(\frac{-\sin \omega_c \sigma}{\omega}\right) + K_{yy}\left[-\frac{1}{\omega_c}(\cos \omega_c - 1)\right] \right\} \omega
$$

$$
+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_y} \frac{\partial}{\partial y} \int_{-t}^{0} d\sigma \left\{ K_{yx}(\sigma)\left[\frac{1}{\omega_c}(\cos \omega_c \sigma - 1)\right] + K_{yy}(\sigma)\left(\frac{-\sin \omega_c \sigma}{\omega_c}\right) \right\} \omega
$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_y} \frac{\partial}{\partial v_x} \int_{-t}^{0} d\sigma \; K_{yx}(\sigma)[(\cos \omega_c \sigma)] + K_{yy}(\sigma)(-\sin \omega_c \sigma) \quad \omega$$

$$+ \left(\frac{q}{m}\right)^2 \frac{\partial}{\partial v_y} \frac{\partial}{\partial v_y} \int_{-t}^{0} d\sigma \; K_{yx}(\sigma)(\sin \omega_c \sigma) + K_{yy}(\sigma)(\cos \omega_c \sigma) \quad \omega$$

$$(4.24)$$

where

$$K_{xy}(\sigma) = K[E_x, E_{y\sigma}]$$

$$K_{yx}(\sigma) = K[E_y, E_{x\sigma}]$$

$$K_{xx}(\sigma) = K[E_x, E_{x\sigma}]$$

$$K_{yy}(\sigma) = K[E_y E_{y\sigma}] \qquad (4.25)$$

and we have assumed $\quad < E_x > = < E_y > = 0.$ $\qquad (4.26)$

(4.24) will be considerably simplified if we assume

$$K_{xy}(\sigma) = K_{yx}(\sigma) = 0 \qquad (4.27)$$

$$K_{xx}(\sigma) = K_{yy}(\sigma) = K(\sigma) \qquad (4.28)$$

This is equivalent to saying that the electric field along any particular direction has the same statistical property (up to 2nd order). Indeed, (4.25), (4.26) and (4.27) imply that $< E_\theta > = 0$, $< E_\theta E_{\theta\sigma} > = K_{xx}(\sigma)$, where E_θ is the electric field seen in any particular θ-direction. If we restrict to homogeneous and isotropic distributions, i.e.

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial y} = 0$$

$$\omega = \omega(v), \quad v = \sqrt{v_x^2 + v_y^2}$$

so that
$$\frac{\partial \omega}{\partial v_x} = \frac{v_x}{v} \frac{\partial \omega}{\partial v}$$

$$\frac{\partial \omega}{\partial v_y} = \frac{v_y}{v} \frac{\partial \omega}{\partial v}$$

(4.24) simplifies to

$$\frac{\partial \omega}{\partial t} = (\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2})(\frac{q}{m})^2 \left[\int_{-t}^{0} K(\sigma) \cos \omega_c \sigma \, d\sigma \right] \omega \qquad (4.29)$$

or
$$\frac{\partial \omega}{\partial t} = 2 \frac{\partial}{\partial u} u \frac{\partial}{\partial u} \frac{q^2}{m} \left[\int_{-t}^{0} K(\sigma) \cos \omega_c \sigma \, d\sigma \right] \omega \qquad (4.30)$$

where $u = \frac{1}{2} mv^2$ \qquad (4.31)

Equation (4.29) can be rewritten as

$$\frac{\partial \omega}{\partial t} = (\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial_y^2}) D(t)\omega \qquad (4.32)$$

where

$$D(t) = (\frac{q}{m})^2 \int_{-t}^{0} K(\sigma) \cos \omega_c \sigma \, d\sigma$$

We note that

$$D(t \to \infty) = (\frac{q}{m})^2 \int_{-\infty}^{0} K(\sigma) \cos \omega_c \sigma \, d\sigma$$

$$= \frac{1}{2}(\frac{q}{m})^2 2\pi S(\omega_c) \qquad (4.33)$$

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\sigma) \cos \omega\sigma \, d\sigma \qquad (4.34)$$

Aside from the fact that we allow spatial dependence of the electric field in Example 1, the magnetic field is seen to cause a shift in the frequency at which the spectral density is to be evaluated. It is clear that the solution subject to an initial Maxwellian distribution is Maxwellian. In fact if

$$\omega = \frac{1}{2\pi} \frac{m}{\kappa T} e^{\frac{-mv^2}{\kappa T}} \quad , \quad t = 0 \qquad (4.35)$$

then the solution of Equation (4.32) is

$$\omega = \frac{1}{2\pi \frac{\kappa}{m}(T + \Delta T)} e^{\frac{-v^2}{2\frac{\kappa}{m}(T + \Delta T)}} \qquad (4.36)$$

where

$$\Delta T = \frac{2m}{\kappa} \int_{0}^{t} D(\sigma) d\sigma \qquad (4.37)$$

Here κ is the Boltzmann constant, T the temperature. The heating rate is

$$\frac{d\Delta T}{dt} = \frac{2m}{\kappa} D(t) \qquad (4.38)$$

$$\lim_{t \to \infty} \frac{d\Delta T}{dt} = \frac{q^2}{m\kappa} 2\pi S(\omega_c) \qquad (4.39)$$

We have omitted any motion in the z-direction (i.e. along the magnetic field). If the correlation between the longitudinal and transverse

motions is ignored, the E_x, E_y may be evaluated along the rectilinear trajectory in the z-direction. The results agree with Sturrock's.[1] Unlike Equation (4.6), however, Sturrock's diffusion equation is different from (4.32) since he uses the guiding center approach. One of the disadvantages in using the guiding center approach is that the radius of gyration approaches to infinity as the magnetic field decreases towards zero. Hence it is difficult to recover the zero magnetic field limit from the theory. The present approach does not have such difficulties.

If the magnetic field assumes different values over different periods, which are much longer than the correlation time of the electric field, we may expect that the result can be readily obtained by combining the above results over the intervals. Puri[7] has considered the case with a time varying magnetic field. Except for some minor modifications, the approach is similar and straight-forward, so that we will just briefly indicate how this can be done. Instead of a constant B, we let

$$\omega_c(t) = \omega_c[1 + A \cos (pt + \Psi)] \tag{4.40}$$

where ω_c, A, p, Ψ are constants. We have

$$\begin{cases} \dot{v}_x = \omega_c(t) \, v_y + \dfrac{q}{m} E_x \\\\ \dot{v}_y = -\omega_c(t) \, v_x + \dfrac{q}{m} E_y \end{cases} \tag{4.41}$$

Instead of the submatrix

$$
\begin{pmatrix}
\cos \omega_c t & -\sin \omega_c t \\
\\
\sin \omega_c t & \cos \omega_c t
\end{pmatrix}
$$

in (4.23), we have

$$
\begin{pmatrix}
v_{x0} \\
\\
v_{y0}
\end{pmatrix}
=
\begin{pmatrix}
\cos \theta(t_0,t) & -\sin \theta(t_0,t) \\
\\
\sin \theta(t_0,t) & \cos \theta(t_0,t)
\end{pmatrix}
\begin{pmatrix}
v_x \\
\\
v_y
\end{pmatrix}
\tag{4.42}
$$

$$
=
\begin{pmatrix}
\Psi_{jk}(t_0,t)
\end{pmatrix}
\begin{pmatrix}
v_x \\
v_y
\end{pmatrix}
, \quad j,k = 1,2
\tag{4.43}
$$

where

$$
\theta(t_0,t) = \omega_c(t-t_0) + \beta \sin(pt_0+\Psi) - \beta \sin(pt+\Psi)
\tag{4.44}
$$

$$
\beta = \frac{A\omega_c}{p}
$$

Note that we do not have an autonomous system, and the more general relations (3.7) and (3.11) should be used. The only modification, however, is in (3.45) where $\lambda_{j\ell}^{-1}(\sigma)$ should be replaced by $\Psi_{j\ell}(t,t+\sigma)$ so that we need only replace $\cos \omega_c \sigma$ in (4.29) by $\cos \theta(t,t+\sigma)$, resulting in

$$
\frac{\partial \omega}{\partial t} = \left(-\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2}\right)\left(\frac{q}{m}\right)^2 \int_{-t}^{0} K(\sigma) \cos \theta(t,t+\sigma)d\sigma
\tag{4.45}
$$

The result is formally the same with

$$D(t) = \int_{-t}^{0} K(\sigma) \cos \theta(t, t+\sigma) d\sigma \qquad (4.46)$$

If we merely extend the lower limit -t to -∞ in (4.46) D(t) becomes a periodic function with a period $2\pi/p$. Usually the measuring device can not respond to such fast changes. Therefore, it seems reasonable to take the time average of D(t) for the heating rate. We first expand $\cos \theta(t, t+\sigma)$, using

$$\cos \alpha \sin z = J_0(\alpha) + 2 \sum_{n=1}^{\infty} J_{2n}(\alpha) \cos 2nz$$

$$\sin \alpha \sin z = 2 \sum_{n=1}^{\infty} J_{2n+1}(\alpha) \sin(2n+1)z$$

where J_n is the Bessel function of the first kind of order n. We find that

$$\cos \theta(t, t+\sigma) = \cos[\omega_c \sigma + \beta \sin(pt + \Psi) - \beta \sin(pt + p\sigma + \Psi)]$$

$$= \cos \omega_c \sigma \left\{ C_1(pt) \, C_1(pt + p\sigma) + C_2(pt) \, C_2(pt + p\sigma) \right\}$$

$$- \sin \omega_c \sigma \left\{ C_2(pt) \, C_1(pt + p\sigma) - C_1(pt) \, C_2(pt + p\sigma) \right\} \qquad (4.47)$$

where

$$C_1(pt) = J_0(\beta) + 2 \sum_{n=1}^{\infty} J_{2n}(\beta) \cos 2n(pt + \Psi) \qquad (4.48)$$

$$C_2(pt) = 2 \sum_{n=0}^{\infty} J_{2n+1}(\beta) \sin(2n+1)(pt + \Psi) \qquad (4.49)$$

The time average of (4.46) can now be performed by using (4.47). It gives

$$< D(t) >_t = (\frac{q}{m})^2 \int_{-\infty}^{0} K(\sigma) \sum_{n=-\infty}^{\infty} J_{|n|}^2(\beta) \cos(\omega_c + np)\sigma \, d\sigma \qquad (4.50)$$

Here $< >_t$ means time average. Note that averaging on time is formally the same as averaging on Ψ.

Equation (4.50) is same as Puri's result. Several points are worth notice. Puri uses a randomization period technique. It is not clear whether he has taken into account the non-autonomous nature of the system. Because of the non-autonomous nature, we find it necessary to use the time average technique, which is absent in Puri's treatment. Also he has assumed $E_x \equiv 0$. In the limit $B \rightarrow 0$, we see that the particle will be accelerated anisotropically.

Finally, in order to measure the heating rate experimentally, one starts out with an ensemble of non-interacting particles with an initial distribution. It is pointed out by Puri that if all the particles see an identical electric field, the distribution will not be Maxwellian. He then argues that if Ψ represents the phase angle of the magnetic field along its axis, the particle velocity will consist of a sum of 'quasi-independent" modes which enables the use of central limit theorem. It is difficult to see how this is done. More basically, if only a single realization of the random electric field is carried out, it is even more difficult to see the significance. This motivates the following example.

4.3 Example 3. Plasma Heating by a Plane Wave

So far we have not considered the origin of the randomness of the electric field. We now consider a physical situation that may arise.

Consider a plane wave propagating in the direction of the magnetic field. For simplicity, assume the wave is circularly polarized. The equations of motion are taken as

$$\dot{z} = v_z$$

$$\dot{v}_z = 0$$

$$\dot{x} = v_x$$

$$\dot{v}_x = \omega_c v_y + \frac{q}{m} E_x$$

$$\dot{y} = v_y$$

$$\dot{v}_y = - \omega_c v_x + \frac{q}{m} E_y \qquad\qquad (4.51)$$

The circularly polarized electric field is given by

$$E_x = E_0 \sin(\omega_0 t - kz)$$

$$E_y = E_0 \cos(\omega_0 t - kz) \qquad\qquad (4.52)$$

where ω_0 is the frequency and k the wave number of the wave. We have ignored the force associated with the magnetic field of the wave. Substituting $z = z_0 + v_{z0}$ into the equations for the transverse velocities,

$$\dot{v}_x = \omega_c v_y + \frac{q}{m} E_0 \sin((\omega_0 - kv_{z0})t - kz_0)$$

$$\dot{v}_y = -\omega_c v_x + \frac{q}{m} E_0 \cos((\omega_0 - kv_{z0})t - kz_0) \tag{4.53}$$

Suppose we have an ensemble of non-interacting particles uniformly distributed in space. The initial distribution of longitudinal velocity is to be independent of that of transverse velocity. Consider the space generated by $kz_0 = 0$, to $kz_0 = 2\pi$, for example. (This will roughly be a slab if $v_{z0}t$ is small.) Let $\omega(v_x, v_y)$ be the probability density of a particle with (v_x, v_y) in this space. The electric field can be regarded as random with random phase kz_0 and random frequency $\omega_0 - kv_{z0}$. For any preassigned correlation function $K(\sigma)$, there is a distribution of $\omega_0 - kv_{z0}$ or a distribution $S(v_{z0})$ of v_{z0} so that the electric field E_x is a stationary process with this correlation function.[8] In fact

$$< E_x > = < E_y > = 0 \tag{4.54}$$

$$< E_x(t) \, E_x(t + \sigma) > = < E_y(t) \, E_y(t + \sigma) >$$

$$= \frac{1}{2} E_0^2 \int_{-\infty}^{\infty} \cos[(\omega_0 - kv_{z0})\sigma] S(v_{z0}) \, dv_{z0}$$

$$= K(\sigma) \tag{4.55}$$

Here S is seen to be proportional to the spectral density of $E_x(t)$. The cross correlations are not zero but antisymmetric,

$$< E_x(t) \, E_y(t + \sigma) >$$

$$= - < E_y(t) \, E_x(t + \sigma) >$$

$$= \frac{1}{2} E_0^2 \int_{-\infty}^{\infty} \sin(\omega_0 - k v_{z0}) \sigma \, S(v_{z0}) \, dv_{z0} \tag{4.56}$$

Applying (4.24), we have the following diffusion equation.

$$\frac{\partial \omega}{\partial t} = (\frac{q}{m})^2 \left(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \right) \int_{-t}^{0} d\sigma \, \frac{1}{2} E_0^2 \int_{-\infty}^{\infty} \cos(\omega_c + \omega_0 - k v_{z0}) \sigma$$

$$S(v_{z0}) \, dv_{z0} \quad \omega \tag{4.57}$$

As $t \to \infty$, we have

$$\frac{\partial \omega}{\partial t} = (\frac{q}{m})^2 \left(\frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \right) \pi \, \frac{1}{2} E_0^2 \, S\left(\frac{\omega_c + \omega_0}{k} \right) \omega \tag{4.58}$$

and the heating rate is

$$\frac{q^2}{m\kappa} \pi \, E_0^2 \, S\left(\frac{\omega_c + \omega_0}{k} \right) \tag{4.59}$$

Note that the maximum occurs when $\dfrac{\omega_c + \omega_0}{k}$ equals the most probable velocity along the direction of propagation.

4.4 Summary

In this Chapter, we have applied our previous results to several situations, in which a charged particle experiences a random electric field. By considering a noninteracting ensemble of charged particles, it is found that heating is possible with a random electric field,

and the heating rate is in general proportional to the spectral density of the random field. In general. the presence of a magnetic field has the effect of a shift in the frequency at which the spectral density is to be evaluated. This is particularly clear in the last example, in which the origin of the randomness of the field is also considered.

V. GENERALIZED STOCHASTIC EQUATIONS APPLIED
TO THE COULOMB POTENTIAL PROBLEM

In the introduction we have already pointed out that the application of the conventional Fokker-Planck equation to the Coulomb potential problem involves some difficulties. For one reason the coefficients cannot be determined explicitly without the use of some model or other theory. From a theoretical point of view, this may lead to inconsistency. On the other hand, the kinetic equation of a plasma based on the BBGKY theory can be manipulated into the form of a Fokker-Planck equation.[9,10] Aside from the fact that we wish to derive the kinetic equation using the Fokker-Planck equation approach, this does not give a general exact form of the coefficients in terms of the random force.

Early attempts in finding the coefficients introduces the collision concept by means of the Boltzmann equation.[10,12] Gasiorowizc et al.,[2] use the Vlasov equation and a Holtzmark distribution. Following Bogoliubov's method, Tolmachev[13] introduces a chain of linked distributions at different times for the Coulomb potential problem. The results are divergent at extreme distances of interaction. Temko[14] makes use of a ternary correlation to insure convergence at large distances. His results have been further refined by Tchen.[9] These methods are primarily based on the Liouville equation or the BBGKY theory, and are especially formulated for the Coulomb potential problem. As a result, these methods may not be suitable for systems with more general forces.

In this chapter gneralized stochastic equations are used for the lowest and higher distributions. With a cluster expansion, the equations are decoupled and the coefficients are determined.

A diagram relating various approaches is as follows:

```
┌──────────┐            ┌──────────────┐          ┌──────────────┐
│Lionville │            │ Generalized  │          │Markov Model  │
│Equation  │ · · · · · ·│ Stochastic   │          │or Lagrange   │
│          │            │ Equations    │          │Expansion     │
└──────────┘            └──────────────┘          └──────────────┘
     │                                                    │
     │                                                    │
┌──────────┐      ┌──────────────┐  ┌──────────────┐  ┌──────────────┐
│BBGKY     │      │ Higher       │  │ Lowest       │  │Conventional  │
│Hierarchy │      │ Distribution │  │ Distribution │  │Fokker-Planck │
└──────────┘      └──────────────┘  └──────────────┘  │Equation      │
     │                                                 └──────────────┘
     │                                                    │
     │                                            ┌──────────────────┐
     │                                            │BBGKY Theory      │
     │                                            │Boltzmann Equation│
     │                                            │Dressed particle  │
     │                                            │theory, etc.      │
     │                                            └──────────────────┘
     │                                                    │
     └───────────┬──────────────┬─────────────────────────┘
            ┌─────────────────────────────┐
            │      Kinetic Equation        │
            └─────────────────────────────┘
```

The dotted line indicates some relation between the generalized stochastic equations and Lionville equation. This point is discussed in the final chapter.

5.1 Formulation of the Coulomb Potential Problem

We consider a system of N identical electrons in a volume V, with a uniform background of immobile positive charge. Let $\vec{x}_j$ be the spatial coordinate of the jth electron, $\vec{v}_j$ the velocity and $\vec{E}_j$ the electric field seen by the electron. The equations of motion are then

$$\begin{cases} \dot{\vec{x}}_j = \vec{v}_j \\ \\ \dot{\vec{v}}_j = \frac{1}{m}(-e\vec{E}_j) \end{cases} \qquad j = 1, \ldots, N \qquad (5.1)$$

where $-e$ and m are the charge and mass of the electron. Introducing the potential

$$\phi_j = \sum_{k \neq j}^{N} \frac{e^2}{\left| \vec{x}_j - \vec{x}_k \right|} \tag{5.2}$$

we have

$$e\vec{E}_j = \frac{\partial \phi_j}{\partial \vec{x}_j} \tag{5.3}$$

Normalizing Equation (5.1) by

$$\vec{x}_j = \vec{\underline{x}}_j r_0$$

$$\vec{v}_j = \vec{\underline{v}}_j v_0$$

$$\phi_m = \bar{\phi} \, \phi_0$$

$$t = \bar{t} \, \frac{r_0}{v_0} \tag{5.4}$$

Equations (4.1) become

$$\frac{d\vec{\underline{x}}_j}{d\bar{t}} = \vec{\underline{v}}_j$$

$$\frac{d\vec{\underline{v}}_j}{d\bar{t}} = \frac{\phi_0}{mv_0^2} \left(- \frac{\partial \bar{\phi}_j}{\partial \vec{\underline{x}}_j} \right) \tag{5.5}$$

Here we take r_0 an effective range of the force, v_0 as the thermal velocity, with

$$\phi_0 = \frac{e^2}{r_0} \tag{5.6}$$

$$\phi_j = \sum_{k \neq j}^{N} \frac{1}{|\vec{x}_j - \vec{x}_k|} \tag{5.7}$$

The characteristic time is $\frac{r_0}{v_0}$ and should be such that the higher

distributions relax in a time of a few characteristic times, enabling us

to drop the initial conditions and extend the limit of integration t to

∞ as in the later development.* We assign

$$\frac{\phi_0}{mv_0^2} = \varepsilon \tag{5.8}$$

to be a small number. This means the interaction is weak compared to

the thermal energy.

    Let

$$\vec{\xi} = (\vec{x}, \vec{v}) \tag{5.9}$$

be a six dimensional vector. Let $D_{\mu_1; \cdots; \mu_j}(\vec{\xi}_1, t_1; \cdots; \vec{\xi}_j, t_j)$ be

the probability density of $(\vec{x}_{\mu_1}(t_1), \vec{v}_{\mu_1}(t_1)) = \vec{\xi}_1, \cdots, (\vec{x}_{\mu_j}(t_j), \vec{v}_{\mu_j}(t_j))$

$= \vec{\xi}_j^{**}, j = 1, 2, \cdots$. Here μ maps the set $\{1, \cdots, j\}$ into $\{1, \cdots, N\}$.

We consider only symmetrical distributions, i.e., if M maps $\{1, \cdots, N\}$

onto $\{1, \cdots, N\}$ in a 1-1 manner, then

$$D_{M(\mu_1), \cdots, M(\mu_j)}(\vec{\xi}_1, t_1; \cdots; \vec{\xi}_j, t_j) = D_{\mu_1, \cdots, \mu_j}(\vec{\xi}_1, t_1; \cdots; \vec{\xi}_j, t_j)$$
$$\tag{5.10}$$

_____

*For a plasma, r_0 is usually taken as the Debye length and the
characteristic time is taken as the reciprocal of the plasma frequency.[15]
**To keep our notations simple and comparable to those in the literature,
we will not use capital letters as in Chapter I.

Since the system (5.1) is invariant under the transformation M, this is equivalent to requiring $D_{1, \ldots N}(\vec{\xi}_1, 0; \ldots ; \vec{\xi}_N, 0)$ to be symmetrical. In view of the symmetry, we will suppress the subscripts $\mu_1, \ldots \mu_j$ whenever they are distinct. Thus

$$D(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j) \equiv D_{1, \ldots , j}(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j)$$

$$= D_{M(1), \ldots , M(j)}(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j)$$

(5.11)

Let

$$D_{\mu_1, \ldots , \mu_j / \mu_1', \ldots , \mu_k'}(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j / \vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k')$$

$$\equiv \frac{D_{\mu_1, \ldots , \mu_j, \mu_1', \ldots , \mu_k'}(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j; \vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k')}{D_{\mu_1', \ldots , \mu_k'}(\vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k')}$$

(5.12)

be the conditional probability density of

$$(\vec{x}_{\mu_1}(t_1), \vec{v}_{\mu_1}(t_1)) = \vec{\xi}_1, \ldots , (\vec{x}_{\mu_j}(t_j), \vec{v}_{\mu_j}(t_j)) = \vec{\xi}_j$$

given

$$(\vec{x}_{\mu_1}'(t_1'), \vec{v}_{\mu_1}'(t_1')) = \vec{\xi}_1', \ldots , (\vec{x}_{\mu_k}'(t_k'), \vec{v}_{\mu_k}'(t_k')) = \vec{\xi}_k'$$

where

$$j = 1, 2, \ldots \quad \text{and} \quad k = 1, 2, \ldots$$

Again, when the subscripts $\mu_1 \ldots , \mu_j, \mu_1', \ldots , \mu_k'$ are distinct, we will just write as $D(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j / \vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k')$.

Since we have a large volume V, it is convenient (particularly in the case of spatial homogeneity, i.e., $\frac{\partial D(\vec{\xi}, t)}{\partial \vec{x}} = 0$) to write

$$D(\vec{\xi}, t) = \frac{1}{V} f_1(\vec{\xi}, t) \qquad (5.13)$$

More generally, we let

$$D(\vec{\xi}_1, t; \ldots ; \vec{\xi}_j, t) = (\frac{1}{V})^j f_j(\vec{\xi}_1, \ldots \vec{\xi}_j, t), \quad j \leq N \qquad (5.14)$$

Similarly let

$$D(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j / \vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k')$$

$$= (\frac{1}{V})^j g_{j/k}(\vec{\xi}_1, t_1; \ldots ; \vec{\xi}_j, t_j / \vec{\xi}_1', t_1'; \ldots ; \vec{\xi}_k', t_k'), \quad j + k \leq N \qquad (5.15)$$

We now introduce the cluster expansion.[16] We shall write down only the first few relations explicitly. Define P, T, Q as follows:

$$D(\vec{\xi}_1, t_1; \vec{\xi}_2, t_2) = D(\vec{\xi}_1, t_1) D(\vec{\xi}_2, t_2) + P(\vec{\xi}_1, t_1; \vec{\xi}_2, t_2) \qquad (5.16)$$

$$D(\vec{\xi}_1, t_1; \vec{\xi}_2, t_2; \vec{\xi}_3, t_3) = \prod_{i=1}^{3} D(\vec{\xi}_i, t_i) + \sum_{j<k}' D(\vec{\xi}_i, t_i) P(\vec{\xi}_j, t_j; \vec{\xi}_k, t_k)$$

$$+ T(\vec{\xi}_1, t_1; \vec{\xi}_2, t_2; \vec{\xi}_3, t_3) \qquad (5.17)$$

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$$D(\vec{\xi}_1,t_1;\vec{\xi}_2,t_2;\vec{\xi}_3,t_3;\vec{\xi}_4,t_4) = \prod_{i=1}^{4} D(\vec{\xi}_i,t_i)$$

$$+ \sum_{\substack{i<j,k<\ell}}' D(\vec{\xi}_i,t_i)D(\vec{\xi}_j,t_j)P(\vec{\xi}_k,t_k;\vec{\xi}_\ell t_\ell)$$

$$+ \sum_{\substack{i<k<\ell}}' D(\vec{\xi}_i,t_i)T(\vec{\xi}_j,t_j;\vec{\xi}_k,t_k;\vec{\xi}_\ell,t_\ell)$$

$$+ \sum_{\substack{i<j,k<\ell}}' P(\vec{\xi}_i,t_i;\vec{\xi}_j,t_j)P(\vec{\xi}_k,t_k;\vec{\xi}_\ell,t_\ell)$$

$$+ Q(\vec{\xi}_1,t_1;\vec{\xi}_2,t_2;\vec{\xi}_3,t_3;\vec{\xi}_4,t_4)$$

$$(5.18)$$

where Σ' means the summation indices i, j, k, ℓ are to be all distinct.

In the later development, we require P to be of first order, T and Q of higher order than P. To zeroth order, since the trajectories of the particles are independent, this implies that

$$D(\vec{\xi}_1,t_1;\vec{\xi}_2,t_2;\vec{\xi}_3,t_3;\vec{\xi}_4,t_4) = \prod_{i=1}^{4} D(\vec{\xi}_i,t_i) \qquad (5.19)$$

and

$$f_4(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3,\vec{\xi}_4,t) = \prod_{i=1}^{4} f_1(\vec{\xi}_i,t_i) \qquad (5.20)$$

Similarly, we have

$$g_{j/k}(\vec{\xi}_1,t_1; \cdots ;\vec{\xi}_j,t_j/\vec{\xi}_1',t_1'; \cdots ;\vec{\xi}_k',t_k') = v^j \prod_{i=1}^{j} D(\vec{\xi}_i,t_i)$$

$$= \prod_{i=1}^{j} f_1(\vec{\xi}_i,t_i) \quad j+k \leq 4 \qquad (5.21)$$

to the zeroth order. We will not employ special notations for $D_{\mu_1, \cdots, \mu_j}(\vec{\xi}_1,t_1; \cdots ;\vec{\xi}_j,t_j)$ when the μ_j's are not all distinct. It should be noted in such cases, $D_{\mu_1, \cdots, \mu_j}$ may contain δ-functions. In fact, to zeroth order,

$$D_{1,2,3,4,1,2,3,4}(\vec{\xi}_1,t_1;\vec{\xi}_2,t_2;\vec{\xi}_3,t_3;\vec{\xi}_4,t_4;\vec{\xi}_1',t_1';\vec{\xi}_2',t_2';\vec{\xi}_3',t_3';\vec{\xi}_4',t_4')$$

$$= \prod_{i=1}^{4} D(\vec{\xi}_i,t_i)\delta(\vec{x}_i'-\vec{x}_i-\vec{v}_i(t_i'-t_i))\delta(\vec{v}_i'-\vec{v}_i)$$

$$= (\frac{1}{v})^4 \prod_{i=1}^{4} f_1(\vec{\xi}_i,t_i)\delta(\vec{x}_i'-\vec{x}_i-\vec{v}_i(t_i'-t_i))\delta(\vec{v}_i'-\vec{v}_i) \qquad (5.22)$$

where

$$\vec{\xi}_i = (\vec{x}_i,\vec{v}_i)$$

$$\vec{\xi}_i' = (\vec{x}_i',\vec{v}_i')$$

We are now ready to write down some stochastic equations. Let us define the following quantities.

$$eE_{j\alpha} = eE_{j\alpha}(\vec{x}_j,t_j) \qquad (5.23)$$

$$= \sum_{\substack{k \neq j}}^{N} \frac{\partial \phi(\vec{x}_j, \vec{x}_k(t_j))}{\partial x_{j\alpha}} \qquad j = 1, 2, \dots, N \qquad (5.24)$$

$$\phi(\vec{x}) = \frac{e^2}{|\vec{x}|} \qquad (5.25)$$

$$E_{j\beta\sigma} \equiv E_{j\beta}(\vec{x}_j + \vec{v}_j \sigma, t_j + \sigma) \qquad (5.26)$$

$$\vec{\xi}_j(t) = (\vec{x}_j(t), \vec{v}_j(t)) \qquad j = 1, 2, \dots, N \qquad (5.27)$$

$$\vec{\xi}_j = (\vec{x}_j, \vec{v}_j) \qquad j = 1, 2, \dots, N \qquad (5.28)$$

$$\vec{\eta}_j = (\vec{x}_j - \vec{v}_j t_j, \vec{v}_j) \qquad j = 1, 2, \dots, N \qquad (5.29)$$

where the subscripts α, β refer to the αth and βth component of a three dimensional vector.

Taking $j = 1$ in system (5.1), we can write down the stochastic equation for $D(\vec{\xi}_1, t_1)$ or $f_1(\vec{\xi}_1, t_1)$. This is formally the same as in example 1 (note that however the potentials are defined somewhat differently). To allow for a three dimensional space, refer to Equation (2.31a). The result is:

$$\frac{\partial f_1(\vec{\xi}_1, t_1)}{\partial t_1} + v_{1\alpha} \frac{\partial f_1}{\partial x_{1\alpha}} = \frac{e}{m} \frac{\partial}{\partial v_1} < E_{1\alpha} / \vec{\xi}_1(0) = \vec{\eta}_1 > f_1$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^{0} d\sigma(-\sigma) < \frac{\partial E_{1\alpha}}{\partial x_{1\beta}} E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1 > f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^{0} d\sigma K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1] f_1$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial x_{1\beta}} \int_{-\infty}^{0} \sigma d\sigma K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1] f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} [\frac{\partial}{\partial x_{1\beta}} < E_{1\alpha}/\xi_1(0) = \vec{\eta}_1 >] \int_{-\infty}^{0} d\sigma(-\sigma) < E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1 > f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} [\frac{\partial}{\partial v_{1\beta}} < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 >] \int_{-\infty}^{0} d\sigma < E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1 > f_1$$

$$(5.30)$$

where α, β are to be summed from 1 through 3, and we have replaced the lower limit -t, of integration by $-\infty$, since we are interested in time long compared to the correlation time.

According to the results in Chapter I, $D(\vec{\xi}_1, t_1/\vec{\xi}_2, t_2)$ satisfies a similar Equation as $D(\vec{\xi}_1, t_1)$ so that $g_{1/1}(\vec{\xi}_1, t_1/\vec{\xi}_2, t_2)$ satisfies (5.30) provided the extra condition

$$\vec{\xi}_2(t_2) = \vec{\xi}_2 \qquad (5.31)$$

is imposed on all averages, so that,

$$\frac{\partial g_{1/1}(\vec{\xi}_1, t_1 / \vec{\xi}_2, t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial x_{1\alpha}}$$

$$= \frac{e}{m} \frac{\partial}{\partial v_{1\alpha}} <E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2> g_{1/1}$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^{0} d\sigma(-\sigma) <\frac{\partial E_1}{\partial x_{1\beta}}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2> g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^{0} d\sigma \quad K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2]g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial x_{1\beta}} \int_{-\infty}^{0} d\sigma(-\sigma)K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2]g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} [\frac{\partial}{\partial x_{1\beta}} <E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2>] \int_{-\infty}^{0} d\sigma(-\sigma) <E_{1\beta\sigma}/\vec{\xi}_1(0)$$

$$= \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2> g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} [\frac{\partial}{\partial v_{1\beta}} <E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2>] \int_{-\infty}^{0} d\sigma <E_{1\beta\sigma}/\vec{\xi}_1(0)$$

$$= \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2> g_{1/1} \qquad (5.32)$$

Thus, if we know the various conditional averages, we can find f_1 and $g_{1/1}$. The fact is that these conditional averages of E_1 are functions of f_1, $g_{1/1}$ and perhaps higher density functions and must be determined

consistently. This is clear from Equation (5.3) which defines E_1. Indeed, if we make use of the definition of the various density functions, we find that

$$e < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 > = < \sum_{j=2}^{N} \frac{\partial \phi(\vec{x}_1 - \vec{x}_j(t_1))}{\partial x_{1\alpha}} / \vec{\xi}_1(0) = \vec{\eta}_1 >$$

$$= \int \sum_{j=2}^{N} \frac{\partial \phi(\vec{x}_1 - \vec{x}_j)}{\partial x_{1\alpha}} \frac{g_{1/1}}{V} (\vec{\xi}_j, t_1/\vec{\eta}_1, 0) d\vec{\xi}_j$$

$$= \frac{(N-1)}{V} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_2, t_1/\vec{\eta}_1, 0) d\vec{\xi}_2 \tag{5.33}$$

where we have made use of the assumption that the particles are indistinguishable. Similarly,

$$e^2 < E_{1\alpha} E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1 >$$

$$= \int \sum_{\substack{k \neq j \\ k,j > 1}}^{N} \frac{\partial \phi(\vec{x}_1 - \vec{x}_j)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_k')}{\partial x_{1\beta}} \frac{g_{2/1}}{V^2} (\vec{\xi}_j, t_1; \vec{\xi}_k', t_1 + \sigma/\vec{\eta}_1, 0) d\vec{\xi}_j d\vec{\xi}_k'$$

$$+ \int \sum_{k=j \neq 1}^{N} \frac{\partial \phi(\vec{x}_1 - \vec{x}_j)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_k')}{\partial x_{1\beta}} D_{j,k/1}(\vec{\xi}_j, t_1; \vec{\xi}_k', t_1 + \sigma/\vec{\eta}_1, 0) d\vec{\xi}_j d\vec{\xi}_k'$$

$$= \frac{(N-1)(N-2)}{V^2} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_3')}{\partial x_{1\beta}} g_{2/1}(\vec{\xi}_2, t_1; \vec{\xi}_3', t_1 + \sigma/\vec{\eta}_1, 0) d\vec{\xi}_2 d\vec{\xi}_3'$$

$$+ (N-1) \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} D_{2,2/1}(\vec{\xi}_2, t_1; \vec{\xi}_2', t, +\sigma/\vec{\eta}_1, 0) d\vec{\xi}_2 d\vec{\xi}_2'$$

$$(5.34)$$

Using (5.21), (5.22), (5.33) and (5.34), we can estimate the orders of magnitude of the various terms in Equation (5.30). The ratios of these terms are found to be

1st term	2nd	3rd	4th	5th through 8th
1	: 1	: $n_0 r_0^3 (\phi_0/mv_0^2)$	: $n_0 r_0^3 (\phi_0/mv_0^2)^2$	: $(n_0 r_0^3)^2 (\phi_0/mv_0^2)^2$

$$(5.35)$$

Here we have taken N, V to be large, with a finite ratio

$$\frac{N}{V} = n_0 \qquad (5.36)$$

We will consider two cases. In the first case, $n_0 r_0^3 \sim 1$, so that (5.35) is of the form

$$1 \;\; : \;\; 1 \;\; : \;\; \varepsilon \;\; : \;\; \varepsilon^2 \;\; : \;\; \varepsilon^2 \qquad (5.37)$$

Keeping terms up to ε^2, we will show that (5.30) results in the Fokker-Planck equation. In the second case, we consider the limit $n_0 r_0^3 \to \frac{1}{\varepsilon}$, which leads us to the kinetic equation. In both cases, we assume spatial homogeneity, i.e.,

$$\frac{\partial f_1(\vec{\xi}_1, t_1)}{\partial \vec{x}_1} = 0$$

5.2 The Fokker-Planck Equation[17]

In this case we consider

$$n_0 r_0^3 \sim 1 \qquad (5.38)$$

and keep terms of (5.30) to second order. According to (5.33), (5.34) and (5.37), we need $g_{1/1}$ to first order in (5.33), $g_{2/1}$ as well as $D_{2,2/1}$ to zeroth order in (5.34). The last two can be obtained from (5.21) and (5.22). Thus our primary concern is to find $g_{1/1}$ to first order. Note that the ordering in (5.37) applies to (5.32) as well. Thus to first order, $g_{1/1}$ satisfies

$$\frac{\partial g_{1/1}(\vec{\xi}_1, t_1 / \vec{\xi}_2, t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial v_{1\alpha}}$$

$$= \frac{e}{m} \frac{\partial}{\partial v_{1\alpha}} < E_{1\alpha} / \vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > g_{1/1} \qquad (5.39)$$

By symmetry of the density functions as well as that of system (5.1) (under the transformation M),

$$\frac{\partial g_{1/1}}{\partial t_2}(\vec{\xi}_2, t_2 / \vec{\xi}_1, t_1) + v_{2\alpha} \frac{\partial g_{1/1}}{\partial v_{2\alpha}} = \frac{e}{m} \frac{\partial}{\partial v_{2\alpha}} < E_{2\alpha} / \vec{\xi}_2(0) = \vec{\eta}_2, \vec{\xi}_1(t_1) = \vec{\xi}_1 > g_{1/1}$$

$$\qquad (5.40)$$

The result is expected since it is simply Equation (5.32) or Equation (5.39) for the second electron. By first setting $t_1 = 0$, then $\vec{\xi}_1 = \vec{\eta}_1$,

$$\frac{\partial g_{1/1}(\vec{\xi}_2,t_2/\vec{\eta}_1,0)}{\partial t_2} + v_{2\alpha}\frac{\partial}{\partial v_2}g_{1/1} = \frac{e}{m}\frac{\partial}{\partial v_{2\alpha}} <E_{2\alpha}/\vec{\xi}_2(0) = \vec{\eta}_2,\vec{\xi}_1(0)$$

$$= \vec{\eta}_1> g_{1/1} \qquad (5.41)$$

Now $<E_{2\alpha}/\vec{\xi}_2(0) = \vec{\eta}_2,\vec{\xi}_1(0) = \vec{\eta}_1>$ is found in a way similar to (5.33).

$$e <E_{2\alpha}/\vec{\xi}_2(0) = \vec{\eta}_2,\vec{\xi}_1(0) = \vec{\eta}_1>$$

$$= \int \sum_{j=3}^{N} \frac{\partial\phi(\vec{x}_2-\vec{x}_j)}{\partial x_{2\alpha}} \frac{g_{1/2}}{V}(\vec{\xi}_j,t_2/\vec{\eta}_1,0;\vec{\eta}_2,0)d\vec{\xi}_j$$

$$+ \int \frac{\partial\phi(\vec{x}_2-\vec{x}_1')}{\partial x_{2\alpha}} D_{1/1,2}(\vec{\xi}_1',t_2/\vec{\eta}_1,0;\vec{\eta}_2,0)d\vec{\xi}_1'$$

$$= n_0 \int \frac{\partial\phi(\vec{x}_2-\vec{x}_3)}{\partial x_{2\alpha}} g_{1/2}(\vec{\xi}_3,t_2/\vec{\eta}_1,0;\vec{\eta}_2,0)d\vec{\xi}_3$$

$$+ \int \frac{\partial\phi(\vec{x}_2-\vec{x}_1')}{\partial x_{2\alpha}} D_{1/1,2}(\vec{\xi}_1',t_2/\vec{\eta}_1,0;\vec{\eta}_2,0)d\vec{\xi}_1' \qquad (5.42)$$

Since, according to (5.37), these are of first order, we need only $g_{1/2}$, $D_{1/1,2}$ to zeroth order. From (5.21) and (5.22),

$$g_{1/2}(\vec{\xi}_3,t_2/\vec{\eta}_1,0;\vec{\eta}_2,0) = f_1(\vec{\xi}_3,t_2) \qquad (5.43)$$

$$D_{1/1,2}(\vec{\xi}_1',t_2/\vec{n}_1,0;\vec{n}_2,0) = \delta(\vec{x}_1'-[(\vec{x}_1-\vec{v}_1 t)+\vec{v}_1 t_2]) \; \delta(\vec{v}_1'-\vec{v}_1) \qquad (5.44)$$

Similarly, in the last term of (5.41), we need only $g_{1/1}$ to zeroth order, which is f_1 from (5.21). Putting in (5.41)

$$\frac{\partial g_{1/1}(\vec{\xi}_2,t_2/\vec{n}_1,0)}{\partial t_2} + v_{2\alpha} \frac{\partial g_{1/1}}{\partial x_{2\alpha}}$$

$$= \cdot \frac{n_o}{m} \frac{\partial}{\partial v_{2\alpha}} \int \frac{\partial \phi(\vec{x}_2-\vec{x}_3)}{\partial x_{2\alpha}} f_1(\vec{v}_3,t_2) d\vec{\xi}_3 f_1(\vec{v}_2,t_2)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{2\alpha}} \int \frac{\partial \phi(\vec{x}_2-\vec{x}_1')}{\partial x_{2\alpha}} \delta(\vec{x}_1'-[\vec{x}_1-\vec{v}_1 t+\vec{v}_1 t_2])\delta(\vec{v}_1'-\vec{v}_1) d\vec{\xi}_1' f_1(\vec{v}_2,t_2)$$

$$= \frac{1}{m} \frac{\partial}{\partial v_{2\alpha}} \frac{\partial \phi(\vec{x}_2-\vec{x}_1-\vec{v}_1(t_2-t_1))}{\partial x_{2\alpha}} \; f_1(\vec{v}_2,t_2) \qquad (5.45)$$

where we have used $\dfrac{\partial f_1(\vec{\xi},t)}{\partial \vec{x}} = 0$, so that

$$f_1(\vec{\xi},t) = f_1(\vec{v},t) \qquad (5.46)$$

To solve Equation (5.45), we "freeze" f_1.[15] This assumes that $g_{1/1}$ approaches its $t_2 \to \infty$ form in a time short compared with that over which f_1 is varying. The solution is*

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*Formally, (5.45) can be solved by taking the Fourier transform on $\vec{x}_2$ followed by the standard method of variation of parameters.[17] It can also be solved by using Laplace transform.

$$g_{1/1}(\vec{k},\vec{v}_2,t_2/\vec{\eta}_1,0) = \frac{e^2}{2\pi^2 k^2 m} e^{-i\vec{k}\cdot(\vec{v}_2 t_2 + \vec{x}_1 - \vec{v}_1 t_1)} i\vec{k}\cdot\frac{\partial f_1(\vec{v}_2)}{\partial\vec{v}_2}$$

$$\int_0^{t_2} e^{i\vec{k}(\vec{v}_2-\vec{v}_1)\tau} d\tau + e^{-i\vec{k}\cdot\vec{v}_2 t_2} g_{1/1}(\vec{k},\vec{v}_2,0/\vec{\eta}_1,0) \qquad (5.47)$$

where

$$g_{1/1}(\vec{k},\vec{v}_2,t_2/\vec{\eta}_1,0) = \frac{1}{(2\pi)^3} \int e^{-i\vec{k}\cdot\vec{x}_2} g_{1/1}(\vec{\xi}_2,t_2/\vec{\eta}_1,0)d\vec{x}_2 \qquad (5.48)$$

$$k = |k| \qquad (5.49)$$

For simplicity, let us omit the term due to initial value. This amounts to assuming initially there is no correlation. Equation (5.33) can then be evaluated in a straightforward manner, by using (5.47) with $t_2 = t_1 \to \infty$.

$$e <E_\alpha/\vec{\xi}_1(0) = \vec{\eta}_1>$$

$$= n_0 \int \frac{\partial\phi(\vec{x}_1-\vec{x}_2)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_2,t/\vec{\eta}_1,0)d\vec{\xi}_2$$

$$= n_0 e^2 \int\int \frac{ik_\alpha}{2\pi^2 k^2} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} g_{1/1}(\vec{\xi}_2,t_1/\vec{\eta}_1,0)d\vec{k}d\vec{\xi}_2$$

$$= n_0 e^2 \int dk \frac{ik_\alpha}{2\pi^2 k^2} e^{i\vec{k}\cdot\vec{x}_1}(2\pi)^3\int g_{1/1}(\vec{k},\vec{v}_2,t_1/\vec{\eta}_1,0)d\vec{v}_2$$

$$= n_0 e^2 \int dk \frac{ik_\alpha}{2\pi^2 k^2} e^{i\vec{k}\cdot\vec{x}_1} (2\pi)^3 \iint_0^{t_1=\infty} \frac{e^2}{2\pi^2 k^2 m} e^{-i\vec{k}\cdot(\vec{v}_2-\vec{v}_1)t_1 - i\vec{k}\cdot\vec{x}_1}$$

$$i\vec{k} \cdot \frac{\partial f_1(\vec{v}_2)}{\partial \vec{v}_2} e^{i\vec{k}\cdot(\vec{v}_2-\vec{v}_1)\tau} d\tau d\vec{v}_2$$

$$= \frac{-n_0 e^4}{m} \int d\vec{k} \frac{k_\alpha \vec{k}}{k^4} \cdot \frac{(2\pi)^3}{(2\pi^2)^2} \pi \int \frac{\partial f_1(\vec{v}_2)}{\partial \vec{v}_2} \delta(\vec{k}\cdot\vec{v}_2 - \vec{k}\cdot\vec{v}_1) d\vec{v}_2$$

$$= -\frac{2n_0 e^4}{m} \int d\vec{k} \frac{k_\alpha}{k^4} F'\left(\frac{\vec{k}\cdot\vec{v}_1}{k}\right) \tag{5.50}$$

where

$$F\left(\frac{\vec{k}\cdot\vec{v}_1}{k}\right) = \int f_1(\vec{v}_2) \delta\left(\frac{\vec{k}\cdot\vec{v}_1}{k} - \frac{\vec{k}\cdot\vec{v}_2}{k}\right) d\vec{v}_2 \tag{5.51}$$

and F' is the derivative of F with respective to its argument.

We have omitted the initial condition. For the case of non-zero initial correlations, there are conditions on the initial correlations. These conditions have been studied by Sandri and Frieman.[18,17]

To find $e^2 < E_{1\alpha} E_{1\beta\sigma} / \vec{\xi}_1(0) = \vec{\eta}_1 >$, according to (5.34) and (5.37), we need $g_{2/1}$ and $D_{2,2/1}$ to zeroth order, which can be found from (5.21) and (5.22). Because of spatial homogeneity, there is no contribution due to $g_{2/1}$, so that

$$e^2 < E_{1\alpha} E_{1\beta\sigma} / \vec{\xi}_1(0) = \vec{\eta}_1 >$$

$$= (N-1)\int \frac{\partial \phi(\vec{x}_1-\vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1+\vec{v}_1 \sigma-\vec{x}_2')}{\partial x_{1\beta}} D_{2,2/1}(\vec{\xi}_2,t_1;\vec{\xi}_2',t_1+\sigma/\vec{\eta}_1,0)d\vec{\xi}_2 d\vec{\xi}_2'$$

$$= n_0 \int \frac{\partial \phi(\vec{x}_1-\vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_2')}{\partial x_{1\beta}} f_1(\vec{v}_2) \delta(\vec{x}_2'-\vec{x}_2-\vec{v}_2\sigma) \delta(\vec{v}_2'-\vec{v}_2)d\vec{\xi}_2 d\vec{\xi}_2'$$

$$= n_0 \int \frac{\partial \phi(\vec{x}_1-\vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1-\vec{x}_2+(\vec{v}_1-\vec{v}_2)\sigma)}{\partial x_{1\beta}} f_1(\vec{v}_2)d\vec{\xi}_2$$

$$= n_0 e^2 \iint \frac{ik_\alpha}{2\pi^2 k^2} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} \frac{\partial \phi(\vec{x}_1-\vec{x}_2+(\vec{v}_1-\vec{v}_2)\sigma)}{\partial x_{1\beta}} f_1(\vec{v}_2)d\vec{k}d\vec{\xi}_2$$

$$= n_0 e^4 \iint \frac{ik_\alpha(-ik_\beta)}{2\pi^2 k^2} e^{-i\vec{k}\cdot(\vec{v}_1-\vec{v}_2)\sigma} \frac{1}{2\pi^2 k^2} (2\pi)^3 f_1(\vec{v}_2)d\vec{v}_2 d\vec{k}$$

$$= \frac{2n_0 e^4}{\pi} \iint \frac{k_\alpha k_\beta}{k^4} e^{-i\vec{k}\cdot(\vec{v}_1-\vec{v}_2)\sigma} f_1(\vec{v}_2)d\vec{v}_2 d\vec{k} \tag{5.52}$$

Hence

$$\int_{-\infty}^{0} e^2 <E_{1\alpha}E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1> d\sigma$$

$$= \frac{2n_0 e^4}{\pi} \iiint_{-\infty}^{0} \frac{k_\alpha k_\beta}{k^4} e^{-i\vec{k}\cdot(\vec{v}_1-\vec{v}_2)\sigma} f_1(\vec{v}_2)d\sigma d\vec{v}_2 d\vec{k}$$

$$= 2n_0 e^4 \iint \frac{k_\alpha k_\beta}{k^4} \delta(\vec{k}\cdot\vec{v}_1-\vec{k}\cdot\vec{v}_2)f_1(\vec{v}_2)d\vec{v}_2 d\vec{k}$$

$$= 2n_0 e^4 \int \frac{k_\alpha k_\beta}{k^4} F\left(\frac{\vec{k}\cdot\vec{v}_1}{k}\right)d\vec{k} \tag{5.53}$$

Here again we have frozen f_1 and taken the limit $t_1 \to \infty$.

We are ready to write down Equation (5.30) with all the functionals involved explicitly. The following points are observed:

1. Terms involving $<E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1><E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1>$ vanish to the order concerned (here second order). For, according to (5.33) and (5.37), we need $g_{1/1}$ to zeroth order for each average. Because of spatial homogeneity, there is no contribution. It follows that

$$K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1] = <E_{1\alpha} E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1> \qquad (5.54)$$

2. To the order concerned,

$$K[E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1] = <E_{1\alpha} E_{1\beta\sigma}> \qquad (5.55)$$

Since we need $g_{2/1}$ and $D_{2,2/1}$ in (5.34) to zeroth order, which are given by (5.21) and (5.22).

3. $\dfrac{\partial}{\partial \vec{x}_1} < E_{1\alpha} E_{1\beta\sigma}> = 0$ \qquad (5.56)

This is observed in (5.52).

4. $\dfrac{\partial}{\partial v_{1\beta}} <E_{1\alpha} E_{1\beta\sigma}>$

$$= \frac{\partial}{\partial v_{1\beta}} <E_{1\alpha}(\vec{x}_1, t_1) E_{1\beta}(\vec{x}_1 + \vec{v}_1\sigma, t_1 + \sigma)>$$

$$= <E_{1\alpha}(\vec{x}_1, t_1) \frac{\partial E_{1\beta}(\vec{x}_1 + \vec{v}_1\sigma, t_1 + \sigma)}{\partial v_{1\beta}}>$$

$$= \sigma <E_{1\alpha}(\vec{x}_1, t_1) \frac{\partial E_{1\beta}(\vec{x}_1 + \vec{v}_1\sigma, t_1 + \sigma)}{\partial x_{1\beta}}>$$

$$= \sigma \frac{\partial}{\partial x_{1\beta}} < E_{1\alpha}(\vec{x}_1,t_1)E_{1\beta}(\vec{x}_1+\vec{v}_1\,\sigma,t+\sigma) >- \sigma <\frac{\partial E_{1\alpha}}{\partial x_{1\beta}} E_{1\beta\sigma}>$$

$$= - \sigma < \frac{\partial E_{1\alpha}}{\partial x_{1\beta}} E_{1\beta\sigma} > \tag{5.57}$$

because of (5.56). Thus (5.30) can be written as

$$\frac{\partial f_1}{\partial t_1} = \frac{e}{m} \frac{\partial}{\partial v_{1\alpha}} < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 >f_1$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \left(\frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^0 d\sigma <E_{1\alpha}E_{1\beta\sigma}>\right) f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^0 d\sigma < E_{1\alpha}E_{1\beta\sigma}> f_1$$

$$= \frac{e}{m} \frac{\partial}{\partial v_{1\alpha}} < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 > f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^0 d\sigma < E_{1\alpha}E_{1\beta\sigma} > \frac{\partial f_1}{\partial v_{1\beta}} \tag{5.58}$$

where we have dropped terms involving $\frac{\partial f_1}{\partial \vec{x}_1}$ because of spatial homogeneity. Putting (5.50) and (5.53) in (5.58)

$$\frac{\partial f_1}{\partial t_1} = \frac{\partial}{\partial \vec{v}_1} \cdot \left\{ - \frac{2n_0 e^4}{m^2} \int d\vec{k}\, \frac{\vec{k}}{k^4}\, F'\left(\frac{\vec{k}\cdot\vec{v}_1}{k}\right)f_1 + \frac{2n_0 e^4}{m^2} \int d\vec{k}\frac{\vec{k}\vec{k}}{k^5} \cdot F\left(\frac{\vec{k}\cdot\vec{v}}{k}\right)\frac{\partial f_1}{\partial v_1} \right\} \tag{5.59}$$

which is the Fokker-Planck equation.

5.3 The Kinetic Equation[19]

We now investigate the limit by letting $n_0 r_0^3 \to \frac{1}{\varepsilon}$ so that $n_0 r_0^3 (\frac{\phi_0}{mv_0^2})$

$= 0(1)$. (5.35) now gives

$$1 : 1 : 1 : \varepsilon : 1 \tag{5.60}$$

We will therefore keep terms up to first order. In the cluster

expansion, T, and Q are assumed to be of higher order than P, which

is of first order. As before, we assume spatial homogeneity. We must

find the various averages in (5.30).

According to (5.33), we need $g_{1/1}$ to first order for $< E_{1\alpha}/\vec{\xi}_1(0) =$

$\vec{\eta}_1 >$. According to (5.34) and (5.60), we need $g_{2/1}$ to first order but

only the zeroth order of $D_{2,2/1}$ since the term due to $D_{2,2/1}$ is already

of first order.

Consider the term involving $g_{2/1}$ in (5.34). We have

$$n_0^2 \int -\frac{\partial\phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \; \frac{\partial\phi(\vec{x}_1 + \vec{v}_1\sigma - \vec{x}_2)}{\partial x_{1\beta}} \; g_{2/1}(\vec{\xi}_2, t_1; \vec{\xi}_3', t_1 + \sigma/\vec{\eta}_1, 0) d\vec{\xi}_2 d\vec{\xi}_3' \tag{5.61}$$

Using (5.15), (5.16) and (5.17), we find that

$$\frac{1}{V^2} g_{2/1}(\vec{\xi}_2, t_2; \vec{\xi}_3, t_3/\vec{\xi}_1, t_1)$$

$$= \frac{1}{D(\vec{\xi}_1, t_1)} \left\{ -2 \prod_{i=1}^{3} D(\vec{\xi}_i, t_i) + \sum_{j<k}' D(\vec{\xi}_i, t_i) D(\vec{\xi}_j, t_j; \vec{\xi}_k, t_k) \right\} \tag{5.62}$$

Putting (5.62) in (5.61),

$$n_0^2 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} g_{2/1}(\vec{\xi}_2, t_1; \vec{\xi}_3', t_1 + \sigma / \vec{\eta}_1, 0) \, d\vec{\xi}_2 d\vec{\xi}_3$$

$$= n_0^2 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} v^2 D(\vec{\xi}_2, t_1; \vec{\xi}_3', t_1 + \sigma) \, d\vec{\xi}_2 d\vec{\xi}_3'$$

$$= n_0^2 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} f_1(\vec{\xi}_3', t_1 + \sigma) g_{1/1}(\vec{\xi}_2, t_1 / \vec{\xi}_3'. t + \sigma) d\vec{\xi}_2 d\vec{\xi}_3'$$

$$(5.63)$$

Here we have made use of the fact that integrals of the type

$$\int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} f_1(\vec{\xi}_2, t_1) \, d\vec{\xi}_2 = 0 \qquad (5.64)$$

and may be omitted because of spatial homogeneity.

Equation (5.63) together with the fact that we need $D_{2,2/1}$ to zeroth order reveals that

$$< E_{1\alpha} E_{1\beta\sigma} / \vec{\xi}_1(0) = \vec{\eta}_1 > = < E_{1\alpha} E_{1\beta\sigma} > \qquad (5.65)$$

and this average is a functional of $g_{1/1}$. Thus it will be sufficient to determine $g_{1/1}$ to first order. Also, because of spatial homogeneity, terms involving products of the average of E_1 (conditional or not), may be omitted to the order concerned. Hence Equation (5.30) may be rewritten as

$$\frac{\partial f_1}{\partial t_1} = \frac{e}{m} \frac{\partial}{\partial v_{1\alpha}} < E_{1\alpha} / \vec{\xi}_1(0) = \vec{\eta}_1 > f_1 - \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^{0} d\sigma(-\sigma) < \frac{\partial E_{1\alpha}}{\partial x_{1\beta}} E_{1\beta\sigma} > f_1$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^{0} d\sigma < E_{1\alpha} E_{1\beta\sigma} > f_1 + \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial x_{1\beta}} \int_{-\infty}^{0} d\sigma(-\sigma) < E_{1\alpha} E_{1\beta\alpha} > f_1$$

$$(5.66)$$

where

$$e < E_{1\alpha} / \vec{\xi}_1(0) = \vec{\eta}_1 > = n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_2, t_2 / \vec{\eta}_1, 0) \, d\vec{\xi}_2 \qquad (5.67)$$

according to (5.33) and

$$e^2 < E_{1\alpha} E_{1\beta\sigma} > = n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} f_1(\vec{\xi}_2, t_1) \, \delta(\vec{x}_2' - \vec{x}_2 - \vec{v}_2 \sigma)$$

$$\delta(\vec{v}_2' - \vec{v}_2) \, d\xi_2 d\xi_2'$$

$$+ n_0^2 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_2')}{\partial x_{1\beta}} f_1(\vec{\xi}_3', t_1 + \sigma) \, g_{1/1}(\vec{\xi}_2 t_1 / \vec{\xi}_3', t_1 + \sigma) \, d\vec{\xi}_2 d\vec{\xi}_3'$$

$$(5.68)$$

according to (5.34), (5.63) and (5.65). Here we have substituted

$$\frac{1}{V} f_1(\vec{\xi}_2, t_1) \, \delta(\vec{x}_2' - \vec{x}_2 - \vec{v}_2 \sigma) \, \delta(\vec{v}_2' - \vec{v}_2) = D_{2,2/1}(\vec{\xi}_2, t_1; \vec{\xi}_2', t_1 + \sigma / \vec{\eta}_1, 0)$$

to zeroth order.

To find $g_{1/1}$ to first order, let us rewrite Equation (5.32), omitting products of averages because of spatial homogeneity.

$$\frac{\partial g_{1/1}(\vec{\xi}_1, t_1 / \vec{\xi}_2, t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial x_{1\alpha}}$$

$$= \frac{e}{m} \frac{\partial}{\partial v_1} < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > g_{1/1}$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^{0} d\sigma(-\sigma) < \frac{\partial E_1}{\partial x_{1\beta}} E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^{0} d\sigma \quad < E_{1\alpha} E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > g_{1/1}$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial x_{1\beta}} \int_{-\infty}^{0} d\sigma(-\sigma) < E_{1\alpha}, E_{1\beta\sigma}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > g_{1/1}$$

$$(5.69)$$

Now

$$e < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$= < \sum_{j=2}^{N} \frac{\partial\phi(\vec{x}_1 - \vec{x}_j(t_1))}{\partial x_{1\alpha}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$= \int \sum_{j=3}^{N} \frac{\partial\phi(\vec{x}_1 - \vec{x}_j)}{\partial x_{1\alpha}} \frac{g_{1/2}}{V} (\vec{\xi}_j, t_1/\vec{\eta}_1, 0; \vec{\xi}_2, t_2) d\vec{\xi}_j$$

$$+ \int \frac{\partial\phi(\vec{x}_1 - \vec{x}_2')}{\partial x_{1\alpha}} D_{2/1,2}(\vec{\xi}_2', t_1/\vec{\eta}_1, 0; \vec{\xi}_2, t_2) d\vec{\xi}_2$$

$$= n_0 \int \frac{\partial\phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/2}(\vec{\xi}_3, t_1/\vec{\eta}_1, 0; \vec{\xi}_2, t_2) d\vec{\xi}_3$$

$$+ \int \frac{\partial\phi(\vec{x}_1 - \vec{x}_2')}{\partial x_{1\alpha}} \delta(\vec{x}_2' - [\vec{x}_2 + \vec{v}_2(t_1 - t_2)]) \delta(\vec{v}_2' - \vec{v}_2) d\vec{\xi}_2' \qquad (5.70)$$

where we have substituted the zeroth order of $D_{2/1,2}$ since the term is of first order already.

Again we use the cluster expansion to find $g_{1/2}$. Using (5.16), (5.17) and (5.64), we find

$$n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/2}(\vec{\xi}_3, t_1 / \vec{\eta}_1, 0 : \vec{\xi}_2, t_2) \, d\vec{\xi}_3$$

$$= n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1 / \vec{\eta}_1, 0) \, d\vec{\xi}_3$$

$$+ n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1 / \vec{\eta}_2, t_2) \, d\vec{\xi}_3 \qquad (5.71)$$

so that (5.70) becomes

$$e < E_{1\alpha} / \vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$= n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1 / \vec{\eta}_1, 0) \, d\vec{\xi}_3$$

$$+ n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1 / \vec{\eta}_2, t_2) \, d\vec{\xi}_3$$

$$+ \frac{\partial \phi(\vec{x}_1 - \vec{x}_2 - \vec{v}_2(t_1 - t_2))}{\partial x_{1\alpha}} \qquad (5.72)$$

Next, we want to show

$$< E_{1\alpha} E_{1\beta\sigma} / \vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 > = < E_{1\alpha} E_{1\beta\sigma} > \qquad (5.73)$$

For

$$e^2 < E_{1\alpha} E_{1\beta\sigma} / \vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$= < \sum_{\substack{j \neq k \\ j,k > 2}}^{N} \frac{\partial \phi(\vec{x}_1 - \vec{x}_j(t_1))}{\partial x_{1\alpha}} \frac{\partial \phi(\vec{x}_1 + \vec{v}_1 \sigma - \vec{x}_k(t_1 + \sigma))}{\partial x_{1\beta}} / \vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$+ < \sum_{j=k>2}^{N} \frac{\partial\phi(\vec{x}_1-\vec{x}_j(t_1))}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_k(t_1+\sigma))}{\partial x_{1\beta}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$+ < \sum_{k>2}^{N} \frac{\partial\phi(\vec{x}_1-\vec{x}_2(t_1))}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_k(t_1+\sigma))}{\partial x_{1\beta}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$+ < \sum_{j>2}^{N} \frac{\partial\phi(\vec{x}_1-\vec{x}_j(t_1))}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_2(t_1+\sigma))}{\partial x_{1\beta}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$+ < \frac{\partial\phi(\vec{x}_1-\vec{x}_2(t_1))}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_2(t_1+\sigma))}{\partial x_{1\beta}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$(5.74)$$

The last term is of second order. The third and fourth terms are proportional to N and are of first order, so that in evaluation, we need only density functions to zeroth order. To this order the average on $\vec{x}_k(t_1+\sigma)$ or on $\vec{x}_j(t_1)$ (j,k > 2) has no contribution because of (5.64). In the second term, the density function needed is of zeroth order and in fact this term is the same as the first term of (5.64). For the first term, we have

$$< \sum_{\substack{j\neq k \\ j,k>2}}^{N} \frac{\partial\phi(\vec{x}_1-\vec{x}_j(t_1))}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_k(t_1+\sigma))}{\partial x_{1\beta}} /\vec{\xi}_1(0) = \vec{\eta}_1, \vec{\xi}_2(t_2) = \vec{\xi}_2 >$$

$$= n_0^2 \int \frac{\partial\phi(\vec{x}_1-\vec{x}_3)}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_4)}{\partial x_{1\beta}} g_{2/2}(\vec{\xi}_3,t_1;\vec{\xi}_4,t_1+\sigma/\vec{\eta}_1,0;\vec{\xi}_2;t_2) \, d\vec{\xi}_3 d\vec{\xi}_4$$

$$= n_0^2 \int \frac{\partial\phi(\vec{x}_1-\vec{x}_3)}{\partial x_{1\alpha}} \frac{\partial\phi(\vec{x}_1+\vec{v}_1\sigma-\vec{x}_4)}{\partial x_{1\beta}} f_1(\vec{\xi}_4,t_1+\sigma) \, g_{1/1}(\vec{\xi}_3,t_1/\vec{\xi}_4,t_1+\sigma)$$

$$d\vec{\xi}_3 d\vec{\xi}_4 \qquad\qquad (5.75)$$

by using the cluster expansion and the spatial homogeneous property (i.e. (5.64)). Essentially the reasoning is as follows:

$$\frac{1}{v^2} g_{2/2} = \frac{D_{1,2,3,4}}{D_{1,2}}$$

Because of (5.64), we may restrict $D_{1,2,3,4}$ to its first order component, which is $\sum\limits_{i<j,k<\ell}^{4} D_i D_j D_{k,\ell}$, and $D_{1,2}$ to its zeroth order component which is $D_1 D_2$. Again because of (5.64), the only surviving term is when $k = 3$, $\ell = 4$.

Thus (5.73) is valid. Since (5.72) and (5.74) are all of first order, we need $g_{1/1}$ to zeroth order on the right hand side of (5.69), which is f_1. Substituting into (5.69),

$$\frac{\partial g_{1/1}(\vec{\xi}_1,t_1/\vec{\xi}_2,t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial x_{1\alpha}}$$

$$= \frac{n_0}{m} \frac{\partial}{\partial v_{1\alpha}} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3,t_1/\vec{\xi}_2,t_2) \, d\vec{\xi}_3 f_1(\vec{\xi}_1,t_1)$$

$$+ \frac{n_0}{m} \frac{\partial}{\partial v_{1\alpha}} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3,t_1/\vec{n}_1,0) \, d\vec{\xi}_3 f_1(\vec{\xi}_1,t_1)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial \phi(\vec{x}_1 - \vec{x}_2 - \vec{v}_2(t_1-t_2))}{\partial x_{1\alpha}} f_1(\vec{\xi}_1,t_1)$$

$$- \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \int_{-\infty}^{0} d\sigma(-\sigma) < \frac{\partial E_{1\alpha}}{\partial x_{1\beta}} E_{1\beta\sigma} > f_1(\vec{\xi}_1,t_1)$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \int_{-\infty}^{0} d\sigma \qquad < E_{1\alpha} E_{1\beta\sigma} > f_1 (\vec{\xi}_1, t_1)$$

$$+ \frac{e^2}{m^2} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial x_{1\beta}} \int_{-\infty}^{0} d\sigma(-\sigma) < E_{1\alpha} E_{1\beta\sigma} > f_1(\vec{\xi}_1, t_1) \qquad (5.76)$$

Finally, using (5.66)

$$\frac{\partial g_{1/1}(\vec{\xi}_1, t_1/\vec{\xi}_2, t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial x_{1\alpha}}$$

$$= \frac{n_0}{m} \frac{\partial}{\partial v_{1\alpha}} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1/\vec{\xi}_2, t_2) \, d\vec{\xi}_3 f_1(\vec{\xi}_1, t_1)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial \phi(\vec{x}_1 - \vec{x}_2 - \vec{v}_2(t_1 - t_2))}{\partial x_{1\alpha}} f_1(\vec{\xi}_1, t_1) + \frac{\partial f_1(\vec{\xi}_1, t_1)}{\partial t} \qquad (5.77)$$

Since we are interested in the functionals of $g_{1/1}$ such as $< E_{1\alpha}/\vec{\xi}_1(0) = \eta_1 >$ and $< E_{1\alpha} E_{1\beta\sigma} >$, we may omit the term $\frac{\partial f_1}{\partial t}$ in (5.77) which has no contribution because of spatial homogeneity. (This amounts to writing $g_{1/1} = f_1 + \delta g$ and consider only δg). Furthermore, we "freeze" f_1 as before, in finding these functionals, so that

$$\frac{\partial g_{1/1}(\vec{\xi}_1, t_1 \vec{\xi}_2, t_2)}{\partial t_1} + v_{1\alpha} \frac{\partial g_{1/1}}{\partial x_{1\alpha}}$$

$$= \frac{n_0}{m} \frac{\partial}{\partial v_{1\alpha}} \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_3)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_3, t_1/\vec{\xi}_2, t_2) \, d\vec{\xi}_3 f_1(\vec{v}_1)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{1\alpha}} \frac{\partial \phi(\vec{x}_1 - \vec{x}_2 - \vec{v}_2(t_1 - t_2))}{\partial x_{1\alpha}} f_1(\vec{v}_1) \qquad (5.78)$$

By symmetry,

$$\frac{\partial g_{1/1}(\vec{\xi}_2,t_2/\vec{\xi}_1,t_1)}{\partial t_2} + v_{2\alpha} \frac{\partial g_{1/1}}{\partial x_{2\alpha}}$$

$$= \frac{n_0}{m} \frac{\partial}{\partial v_{2\alpha}} \int \frac{\partial \phi(\vec{x}_2-\vec{x}_3)}{\partial x_{2\alpha}} g_{1/1}(\vec{\xi}_3,t_2/\vec{\xi}_1,t_1) \, d\vec{\xi}_3 f_1(\vec{v}_2)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{2\alpha}} \frac{\partial \phi(\vec{x}_2-\vec{x}_1-\vec{v}_1(t_2-t_1))}{\partial x_{2\alpha}} f_1(\vec{v}_2) \qquad (5.79)$$

so that

$$\frac{\partial g_{1/1}(\vec{\xi}_2,t_2/\vec{\eta}_1,0)}{\partial t_2} + v_{2\alpha} \frac{\partial g_{1/1}}{\partial x_{2\alpha}}$$

$$= \frac{n_0}{m} \frac{\partial}{\partial v_{2\alpha}} \int \frac{\partial \phi(\vec{x}_2-\vec{x}_3)}{\partial x_{2\alpha}} g_{1/1}(\vec{\xi}_3,t_2/\vec{\eta}_1,0) \, d\vec{\xi}_3 f_1(\vec{v}_2)$$

$$+ \frac{1}{m} \frac{\partial}{\partial v_{2\alpha}} \frac{\partial \phi(\vec{x}_2-\vec{x}_1-\vec{v}_1(t_2-t_1))}{\partial x_{2\alpha}} f_1(\vec{v}_2) \qquad (5.80)$$

Equation (5.80) may be solved by using Laplace and Fourier transforms.[20]

Let

$$g_{1/1}(\vec{k}_2,\vec{v}_2,p) = \int_0^\infty e^{-pt_2} \, dt_2 \left(\frac{1}{2\pi}\right)^3 \int e^{-i\vec{k}_2\cdot\vec{x}_2} g_{1/1}(\vec{\xi}_2,t_2/\vec{\eta}_1,0)$$

$$d\vec{k}_2 \qquad (5.81)$$

and define the operator

$$V(2) = \frac{1}{p_2 + i\vec{k}_2\cdot\vec{v}_2} \left[1 + \frac{(2\pi)^3 n_0}{m} \frac{\partial f_1(\vec{v}_2)}{\partial \vec{v}_2} \frac{i\vec{k}_2\phi(k_2)}{\varepsilon(\vec{k}_2,p_2)} \int \frac{d\vec{v}_2}{p_2 + i\vec{k}_2\cdot\vec{v}_2}\right]$$

$$(5.82)$$

where
$$\phi(k_2) = \frac{e^2}{2\pi^2 k_2^2} \qquad (5.83)$$

is the transform of $\phi(x)$, and

$$\epsilon(\vec{k}_2, p_2) = 1 - \frac{(2\pi)^3 n_0 \phi(k_2)}{m} i\vec{k}_2 \cdot \int \frac{\partial f_1(\vec{v}_2)}{\partial \vec{v}_2} \frac{d\vec{v}_2}{p_2 + i\vec{k}_2 \cdot \vec{v}_2} \qquad (5.84)$$

Then
$$g_{1/1}(\vec{k}_2, \vec{v}_2, p_2) = V(2) S_1 \qquad (5.85)$$

where

$$S_1 = \frac{1}{m} \frac{\partial f_1(v_2)}{\partial \vec{v}_2} \cdot \frac{i\vec{k}_2 \phi(k_2)}{p_2 + i\vec{k}_2 \cdot \vec{v}_1} e^{-i\vec{k}_2 \cdot (\vec{x}_1 - \vec{v}_1 t_1)} \qquad (5.86)$$

is the transform of $\dfrac{1}{m} \dfrac{\partial}{\partial v_2} \cdot \dfrac{\partial \phi(\vec{x}_2 - \vec{x}_1 - \vec{v}_1(t_2 - t_1))}{\partial x} f_1(\vec{v}_2)$

Here we have ignored the initial condition. We find that

$$\int g_{1/1}(\vec{k}_2, \vec{v}_2, p_2) d\vec{v}_2$$

$$= \frac{1}{\epsilon(\vec{k}_2, p_2)} \int \frac{d\vec{v}_2}{p_2 + i\vec{k}_2 \cdot \vec{v}_2} \frac{1}{m} \frac{\partial f_1(\vec{v}_2)}{\partial \vec{v}_2} \cdot \frac{i\vec{k}_2 \phi(k_2)}{p_2 + i\vec{k}_2 \cdot \vec{v}_1} e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{v}_1 t_1)}$$

$$\qquad (5.87)$$

Taking $t_2 = t_1 \to \infty$, we may evaluate the residue to obtain

$$\int g_{1/1}(\vec{k}_2, \vec{v}_2, t_2 = t_1 \to \infty) d\vec{v}_2$$

$$= \frac{e^{-i\vec{k}_2 \cdot \vec{x}_1}}{\varepsilon(\vec{k}_2, -i\vec{k}_2 \cdot \vec{v}_1)} \frac{1}{(2\pi)^3 n_0} (1 - \varepsilon(\vec{k}_2, -i\vec{k}_2 \cdot \vec{v}_1)) \qquad (5.88)$$

From (5.67)

$$e < E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 >$$

$$= n_0 \int \frac{\partial \phi(\vec{x}_1 - \vec{x}_2)}{\partial x_{1\alpha}} g_{1/1}(\vec{\xi}_2, t_1/\vec{\eta}_1, 0) d\vec{\xi}_2$$

$$= n_0 \int d\vec{k} \, ik_\alpha \phi(k) e^{i\vec{k} \cdot \vec{x}_1} (2\pi)^3 \int g_{1/1}(\vec{k}, \vec{v}_2, t_1) d\vec{v}_2$$

$$= \int d\vec{k} \, ik_\alpha \phi(k) \frac{1}{\varepsilon(\vec{k}, -i\vec{k} \cdot \vec{v}_1)} (1 - \varepsilon(\vec{k}, -i\vec{k} \cdot \vec{v}_1))$$

$$= \int d\vec{k} \, k_\alpha \phi(k) \, \mathrm{Im} \frac{1}{\varepsilon(\vec{k}, -i\vec{k} \cdot \vec{v}_1)}$$

$$= \int d\vec{k} \frac{k_\alpha e^2}{2\pi^2 k^2} \frac{\frac{(2\pi)^3 n_0 e^2}{2\pi^2 m k^2} - \pi \, F'\left(-\frac{\vec{k} \cdot \vec{v}_1}{k}\right)}{|\varepsilon(k, -i\vec{k} \cdot \vec{v}_1)|^2} = -\frac{2n_0 e^4}{m} \int \frac{d\vec{k}}{k^4} \frac{k_\alpha F'\left(\frac{\vec{k} \cdot \vec{v}_1}{k}\right)}{|\varepsilon(\vec{k}, -i\vec{k} \cdot \vec{v}_1)|^2}$$

$$\qquad (5.89)$$

where $\quad F\left(\dfrac{\vec{k} \cdot \vec{v}_1}{k}\right) = \int f(\vec{v}_2) \, \delta\left(\dfrac{\vec{k} \cdot \vec{v}_1}{k} - \dfrac{\vec{k} \cdot \vec{v}_2}{k}\right)$ as before.

We now turn to the correlation term. According to (5.68) we need

$$f_1(\vec{\xi}_3', t_1 + \sigma) \, g_{1/1}(\vec{\xi}_2, t_1/\vec{\xi}_3', t_1 + \sigma) = v^2 D(\vec{\xi}_2, t_1; \vec{\xi}_3', t + \sigma). \text{ In other}$$

words, we need a two-time two-body distribution. Multiplying (5.79) by

$$f_1(\vec{\xi}_1, t_1),$$

$$\frac{\partial V^2 D(\vec{\xi}_2,t_2;\vec{\xi}_1,t_1)}{\partial t_2} + v_{2\alpha}\frac{\partial V^2 D(\vec{\xi}_2,t_2;\vec{\xi}_1,t_1)}{\partial x_{2\alpha}}$$

$$= \frac{n_0}{m}\frac{\partial}{\partial v_{2\alpha}}\int \frac{\partial \phi(\vec{x}_2-\vec{x}_3)}{\partial x_{2\alpha}} V^2 D(\vec{\xi}_3,t_2;\vec{\xi}_1,t_1)\, d\vec{\xi}_3\, f_1(\vec{v}_2)$$

$$+ \frac{1}{m}\frac{\partial}{\partial v_{2\alpha}}\frac{\partial \phi(\vec{x}_2-\vec{x}_1-\vec{v}_1(t_2-t_1))}{\partial x_{1\alpha}} f_1(\vec{v}_2)f_1(\vec{v}_1) \qquad (5.90)$$

Here we have omitted the $f_1(\vec{\xi}_2,t_2)\cdot f_1(\vec{\xi}_1,t_1)$ component of $V^2 D(\vec{\xi}_2,t_2;$ $\vec{\xi}_1,t_1)$ just as in (5.79) since it has no contribution to the correlation.

Equation (5.90) is to be solved subject to the initial condition $V^2 D(\vec{\xi}_2,t_1;\vec{\xi}_1,t_1)$ at $t_2 = t_1$.

To find $V^2 D(\vec{\xi}_2,t_1;\vec{\xi}_1,t_1)$, we add to (5.90) its symmetrical counter part by interchanging the subscripts 1 and 2, setting $t_2 = t_1$. The result is

$$\frac{\partial V^2 D(\vec{\xi}_2,t_1;\vec{\xi}_1,t_1)}{\partial t_1} + \sum_{j=1}^{2} v_{j\alpha}\frac{\partial V^2 D(\vec{\xi}_2,t_1;\vec{\xi}_1,t_1)}{\partial x_{j\alpha}}$$

$$= \frac{n_0}{m}\frac{\partial}{\partial v_{1\alpha}}\int \frac{\partial \phi(\vec{x}_1-\vec{x}_3)}{\partial x_{1\alpha}} V^2 D(\vec{\xi}_3,t_1;\vec{\xi}_2,t_1)\, d\vec{\xi}_3\, f_1(\vec{v}_1)$$

$$+ \frac{n_0}{m}\frac{\partial}{\partial v_{2\alpha}}\int \frac{\partial \phi(\vec{x}_1-\vec{x}_3)}{\partial x_{2\alpha}} V^2 D(\vec{\xi}_3,t_1;\vec{\xi}_1,t_1)\, d\vec{\xi}_3 f_1(\vec{v}_2)$$

$$+ \frac{1}{m}\left[\frac{\partial}{\partial v_{1\alpha}}\frac{\partial \phi(\vec{x}_1-\vec{x}_2)}{\partial x_{1\alpha}} + \frac{\partial}{\partial v_{2\alpha}}\frac{\partial \phi(\vec{x}_2-\vec{x}_1)}{\partial x_{2\alpha}}\right] f_1(\vec{v}_1)f_1(\vec{v}_2) \qquad (5.91)$$

For t_1 sufficiently large so that $v^2 D(\vec{\xi}_2, t_1; \vec{\xi}_1, t_1)$ relaxes to its asymptotic form, (5.91) may be solved omitting the initial condition. This is then used in (5.90) to solve for $D(\vec{\xi}_2, t_1 + \tau; \vec{\xi}_1, t_1)$ which now depends only on τ. Substituting this into (5.68), $e^2 < E_{1\alpha} E_{1\beta\sigma} >$ can be evaluated. This has been done by Rostoker.[21] The result is

$$< E_{1\alpha} E_{1\beta\sigma} >$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega\sigma} \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{v}_1 \sigma} (4\pi)^2 n_0 e^2 \frac{k_\alpha k_\beta}{k^4} \frac{2\pi}{k} \frac{F(-\frac{\omega}{k})}{|\epsilon(\vec{k}, i\omega)|^2}$$

$$(5.92)$$

Integrating over σ,

$$\int_{-\infty}^{0} d\sigma < E_{1\alpha} E_{1\beta\sigma} >$$

$$= \int \frac{d\vec{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \pi \delta(\omega + \vec{k}\cdot\vec{v}_1) (4\pi)^2 n_0 e^2 \frac{k_\alpha k_\beta}{k^4} \frac{2\pi}{k} \frac{F(-\frac{\omega}{k})}{|\epsilon(\vec{k}, i\omega)|^2}$$

$$= 2 n_0 e^2 \int d\vec{k} \frac{k_\alpha k_\beta}{k^5} \frac{F(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(\vec{k}, -i\vec{k}\cdot\vec{v}_1)|^2} \qquad (5.93)$$

Since $< E_{1\alpha} E_{1\beta\sigma} >$ does not depend on $\vec{x}_1$, Equation (5.57) is valid. Furthermore, the last term in (5.66) vanishes. Putting (5.89) and (5.93) in (5.66), we obtain

$$\frac{\partial f_1}{\partial t_1} = - \frac{\partial}{\partial v_{1\alpha}} \frac{2 n_0 e^4}{m^2} \int d\vec{k} \frac{k_\alpha}{k^4} \frac{F'(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(k, -i\vec{k}\cdot\vec{v}_1)|^2} f_1$$

$$- \frac{\partial}{\partial v_{1\alpha}} \left(\frac{\partial}{\partial v_{1\beta}} \frac{2n_0 e^4}{m^2} \int dk \frac{k_\alpha k_\beta}{k^5} \frac{F(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(\vec{k},-i\vec{k}\cdot\vec{v}_1)|^2} \right) f_1$$

$$+ \frac{\partial}{\partial v_{1\alpha}} \frac{\partial}{\partial v_{1\beta}} \frac{2n_0 e^4}{m^2} \int d\vec{k} \frac{k_\alpha k_\beta}{k^5} \frac{F(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(\vec{k},-i\vec{k}\cdot\vec{v}_1)|^2} f_1$$

$$= - \frac{\partial}{\partial \vec{v}_1} \cdot \left\{ \frac{2n_0 e^4}{m^2} \int d\vec{k} \frac{\vec{k}}{k^4} \frac{F'(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(\vec{k},-i\vec{k}\cdot\vec{v}_1)|^2} f_1 \right.$$

$$\left. + \frac{2n_0 e^4}{m^2} \int d\vec{k} \frac{\vec{k}\vec{k}}{k^5} \frac{F(\frac{\vec{k}\cdot\vec{v}_1}{k})}{|\epsilon(\vec{k},-i\vec{k}\cdot\vec{v}_1)|^2} \cdot \frac{\partial f_1}{\partial \vec{v}_1} \right\} \qquad (5.94)$$

Letting $u_1 = \frac{\vec{k}\cdot\vec{v}_1}{k}$, (5.94) can be rewritten as

$$\frac{\partial f_1}{\partial t_1} = - \frac{\partial}{\partial \vec{v}_1} \cdot \left\{ \frac{2n_0 e^4}{m^2} \int d\vec{k} \frac{\vec{k}}{k^4 |\epsilon(\vec{k},-i\vec{k}\cdot\vec{v}_1)|^2} [f_1(\vec{v}_1) \frac{\partial F(u_1)}{\partial u_1} \right.$$

$$\left. - F(u_1) \frac{\partial f_1(\vec{v}_1)}{\partial u_1}] \right\} \qquad (5.95)$$

Equation (5.95) is the kinetic equation.[19]

5.4 Discussion

We have seen in the previous sections, decoupling the set of the generalized stochastic equations by using the cluster expansion and freezing the lowest distribution, result in the Fokker-Planck or kinetic equation. We have assumed that the electric field experienced by a particle is of small order. This of course cannot be proved, and is simply false when particles are close. There is, however, another

problem. Actually, the force experienced by a particle consists of

forces due to a large number of other particles, and we have only assumed

each individual force is small. In other words the electric field

seen by the jth electron is given by

$$\vec{E}_j = \sum_{i \neq j}^{N} \vec{E}_{ij}$$

where

$$\vec{E}_{ij} = 0(\epsilon)$$

We wish to indicate it is reasonable to take

$$\vec{E}_j = 0(\epsilon).$$

With the Fokker-Planck case, where $n_0 r_0^3 = 0(1)$, $\vec{E}_j = 0(\epsilon)$ since the

summation actually only extends to $N \sim n_0 r_0^3$. With the kinetic case,

however, $n_0 r_0^3 = 0(\frac{1}{\epsilon})$. Let us write

$$\vec{E}_j = \sum_{i \neq j}^{N} \frac{\vec{F}_{ij}}{N}$$

where $\vec{F}_{ij} = 0(1)$, $\frac{1}{N} = 0(\epsilon)$. To zeroth order the $\vec{F}_{ij}$'s are identically

distributed and independent, since the trajectories of the particles

are independent and have identical distribution. Thus $\vec{E}_j \to < \vec{F}_{ij} > = 0$,

as $N \to \infty$, which indicates $\vec{E}_j$ is small.

When the cluster expansion was introduced, we required T, Q of

higher order than P. Actually this is not necessary for the case

$n_0 r_0^3 = O(1)$. Also in this case, the omission of the initial correlation
is reasonable in most cases. In fact the contribution of the initial
correlation as given by (5.47) can be written as $g_{1/1}(\vec{x}_2 - \vec{v}_2 t_2, \vec{v}_2, 0/$
$\vec{x}_1 - \vec{v}_1 t_1, \vec{v}_1, 0)$ with $t_2 = t_1 \rightarrow \infty$. This will be small if the difference
between the spatial arguments is large, which is the case with the
exception $\vec{v}_2 = \vec{v}_1$. This is expressed by Sandri as "the principle of
absence of parallel motion."[18]

For the kinetic case, the major problem is in the evaluation of
the correlation of the electric field since this involved the asymptotic
distribution of two particles. The following, however, is observed.
If the equilibrium distribution is used, the correlation is found to be
formally the same as given by (5.92),

$$< E_{1\alpha} E_{1\beta\sigma} >$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega\sigma} \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{v}_1\sigma} (4\pi)^2 \frac{k_\alpha k_\beta}{k^4} \frac{2\pi}{k} \frac{F(-\frac{\omega}{k})}{|\epsilon(\vec{k}, i\omega)|^2}$$

where

$$F(u) = \int f_m(\vec{v}) \delta(u - \frac{\vec{k}\cdot\vec{v}}{k}) d\vec{v}$$

and f_m is the Maxwellian distribution. If the distribution function
f_1 differs from f_m only in first order, then (5.92) may be expected
since only the zeroth order of F is needed.

Another way to obtain the fluctuation of the electric field is to
make use of the concept of the dielectric constant $\epsilon(\vec{k}, i\omega)$[22] or the

concept of dressed particles.[9] While these techniques yield almost immediate result, a mathematical justification, however, involves considerable difficulty.[23]

The term $< E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 >$ gives the polarization effect,[2] also referred to as the field due to quasi-particles by Rostoker.[9] Note that in the kinetic case, this has an extra factor $\dfrac{1}{|\epsilon(\vec{k},-i\vec{k}\cdot\vec{v})|^2}$ compared to that in the Fokker-Planck case and in fact this factor is the only difference between the kinetic and the Fokker-Planck equation. This force has been obtained by Gasiorowicz, et al.[2] and Rostoker[9] by using the linearized Vlasov equation. The use of the Vlasov equation is essentially justified, according to (5.80). However, in Rostoker's calculation, the Vlasov equation is solved with either an external driving force or a particular initial distribution, so that there seems to be some arbitrariness. When an external force is assumed, it is taken to be that due to an electron moving in its rectilinear orbit. The resulting polarization force has an extra factor $e^{-i\vec{k}\cdot\vec{v}_1 t}$, which is not explained. Such a factor does not appear in our calculation and is due to the condition $\vec{\xi}_1(0) = \vec{\eta}_1$ in $< E_{1\alpha}/\vec{\xi}_1(0) = \vec{\eta}_1 >$. Here $\vec{\eta}_1 = (\vec{x}_1 - \vec{v}_1 t, \vec{v}_1)$. If the condition had been $\vec{\xi}_1(0) = \vec{\xi}_1 = (\vec{x}_1, \vec{v}_1)$, the factor $e^{-i\vec{k}\cdot\vec{v}_1 t}$ will appear. Another point to be noted is that while we obtain the polarization effect, no test particle is introduced artificially.

The Fokker-Planck and the kinetic equations (5.59) and (5.95) have been discussed in the literature.[17,19] It is to be emphasized that we do not obtain these equations from the BBGKY hierarchy, although the mathematics turn out to be similar. This similarity has deeper signifi-

cance than it appears, as we will show in the next chapter, and justifies

to some extent the hybrid use of the conventional Fokker-Planck

equation and the BBGKY theory. The generalized stochastic equations,

of course, are applicable to very general situations.

VI. FINAL REMARKS AND CONCLUSION

In the previous chapters we have shown how a set of generalized

stochastic (or generalized Fokker-Planck) equations can be used to

describe the evolution of a system. When applied to the Coulomb

potential problem, a strong similarity between the set of generalized

stochastic equations and the BBGKY hierarchy is observed. In fact, the

BBGKY hierarchy may be regarded as a special mode of generalized

stochastic equations. To see this let us take the limit $t \to 0$ in

Equation (2.31). All terms involving integration vanish, leaving

$$\frac{\partial \omega}{\partial t} \bigg|_{t=0} = - \frac{\partial}{\partial x} < \varepsilon \ F(x,0)/X(0) = x > \omega \bigg|_{t=0}$$

Since we have chosen 0 arbitrarily as the initial time, in general, we

have

$$\frac{\partial \omega}{\partial t} = - \frac{\partial}{\partial x} < \varepsilon \ F(x,t)/X(t) = x > \omega$$

This equation is exact and a set of generalized stochastic equations

obtained in this manner is equivalent to the BBGKY hierarchy.[*] Thus,

in this sense, the BBGKY hierarchy may be regarded as a special mode

of generalized stochastic equations. Note that while it is not possible

to ignore the condition $X(t) = x$ in $< F(x,t)/X(t) = x >$ for all t,

since $X(t)$ depends on F, it is possible to ignore the condition $X(0)$ ▫

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[*] See appendix

x in < F(x,t)/X(0) = x > provided there is no correlation between F(x,t) and X(0).

Usually the Fokker-Planck equation is derived by using a Markov model or using the Lagrange expansion.[2,24] The coefficients are averages of the force and no explicit condition is imposed. We have seen in Chapter V that while we can drop the condition on the correlation of the electric field, the exact condition is important in calculating the polarization effect. In fact for a spatially homogeneous plasma, the unconditional average of the electric field will be zero.

Our results may be summarized as follows. We modify Stratonovich's results to obtain a set of generalized stochastic equations which describe the higher distributions as well as the lowest forming a chain similar to the BBGKY hierarchy. It is possible to obtain the kinetic equation by decoupling this chain with the cluster expansion. This set of generalized stochastic equations has the following characteristics which distinguishes them from the conventional Fokker-Planck equation. Firstly, the coefficients involve conditional averages of the force taking into account the correlation of the force and initial position. Moreover, the equations serve to describe higher distributions as well as the lowest distribution, making it possible to determine the coefficients without introducing certain models or appealing to other theories. There is a similarity between the generalized stochastic equations and the BBGKY hierarchy. In fact the BBGKY hierarchy may be regarded as a special mode of the generalized stochastic equations.

For a plasma Sturrock,[1] Gasiorowicz, et.al.[2] obtained essentially the same Fokker-Planck equation. To determine the coefficients, Gasiorowicz, et.al. use the Vlasov equation and a Holtsmark distribution.

This may be inconsistent although we find the use of Vlasov equation is essentially justified. Also we find that an artificially introduced test particle is unnecessary. In an entirely opposite direction, a kinetic equation on the basis of the BBGKY theory may be manipulated into the form of a Fokker-Planck equation as done by Rostoker[9] and Tchen.[10] Our approach is to regard the set of generalized stochastic equations as basic from which the coefficients in the equation for the lowest distribution are determined resulting in the kinetic equation. In this case either approach involves similar mathematics. However, the generalized stochastic equations are applicable for general situations. We have considered several situations where the random force is assumed to be specified, with some results comparable to those of Sturrock[1] and Puri.[7]

We conclude that the generalized stochastic equations give a general formulation applicable to self-consistent systems as well as to systems with specified random forces.

APPENDIX

We give a heuristic argument to obtain the equation

$$\frac{\partial \omega}{\partial t} = - \frac{\partial}{\partial x} < F(x,t)/X(t) = x > \omega \qquad \text{(A.1)}$$

The notations of Chapter III will be adopted.

Consider the system,

$$\dot{x} = F(x,t) \qquad \text{(A.2)}$$

Let $\omega(x,t)$ be the probability density of $X(t)$. Firstly, we assume $F(x,t)$ is a given function. We have

$$\omega(x_0,0) = \omega(\psi(t,t_0,x_0),t)J \qquad \text{(A.3)}$$

where J is defined by (3.2). Differentiating (A.3) with respect to t, we obtain

$$\frac{d}{dt} \omega(\psi,t)J = 0$$

or

$$J[\frac{\partial \omega(\psi,t)}{\partial t} + F_j(\psi,t) \frac{\partial \omega(\psi,t)}{\partial x_j}] + \omega(\psi,t) \frac{dJ}{dt} = 0 \qquad \text{(A.4)}$$

Now[4]

$$\frac{dJ}{dt} = J \frac{\partial F_j}{\partial x_j} \qquad \text{(A.5)}$$

along the solution. Putting in (A.4), cancelling J, and replacing ψ by x, we have

$$\frac{\partial \omega(x,t)}{\partial t} + \frac{\partial}{\partial x_j} F_j(x,t)\omega(x,t) = 0 \qquad\qquad (A.6)$$

If the force F is random, we can write Equation (A.6) for each sample F; here $\omega(x,t)$ should be replaced by $\omega(x,t/F)$, the conditional density of X(t) given F. Thus,

$$\frac{\partial \omega(x,t/F)}{\partial t} + \frac{\partial}{\partial x_j} F_j(x,t)\omega(x,t/F) = 0 \qquad\qquad (A.7)$$

Averaging over the function space of F,

$$\frac{\partial \omega(x,t)}{\partial t} + \frac{\partial}{\partial x_j} \int F_j(x,t)\omega(x,t/F)dP(F) = 0 \qquad\qquad (A.8)$$

where P(F) is the distribution of the force F. Formally we take

$$\omega(x,t/F)dP(F) = \omega(x,t)dP(F/X(t) = x) \qquad\qquad (A.9)$$

according to the usual rules on conditional probability. Here $P(F/X(t) = x)$ is the distribution of F given X(t) = x. (A.8) then yields

$$\frac{\partial \omega(x,t)}{\partial t} + \frac{\partial}{\partial x_j} \int F_j(x,t)dP(F/X(t) = x)\,\omega(x,t) = 0$$

or

$$\frac{\partial \omega(x,t)}{\partial t} + \frac{\partial}{\partial x_j} < F_j(x,t)/X(t) = x > \omega(x,t) = 0$$

which is (A.1)

Let us apply the result to (5.1) with j = 1. (A.6) gives

$$\frac{\partial f_1(\vec{\xi}_1,t)}{\partial t} + \frac{\partial}{\partial x_1} \cdot < \vec{V}_1(t)/\vec{X}(t) = \vec{x}_1, \vec{V}_1(t) = \vec{v}_1 > f_1$$

$$+ \frac{\partial}{\partial \vec{v}_1} \cdot < \frac{1}{m} \left(-\frac{\partial \phi_1}{\partial \vec{x}_1} \right) / \vec{X}(t) = \vec{x}_1, \vec{V}_1(t) = \vec{v}_1 > f_1 = 0$$

$$(A.10)$$

Now $< \vec{V}_1(t) / \vec{X}_1(t) = \vec{x}_1, \vec{V}_1(t) = \vec{v}_1 > = \vec{v}_1,$

$$< \frac{1}{m} \left(-\frac{\partial \phi_1}{\partial \vec{x}_1} \right) / \vec{X}_1(t) = \vec{x}_1, \vec{V}_1(t) = \vec{v}_1 >$$

$$= < \sum_{j \neq 1} -\frac{1}{m} \frac{\partial \phi}{\partial \vec{x}_1} (\vec{x}_1 - \vec{x}_j(t)) / \vec{X}_1(t) = \vec{x}_1, \vec{V}_1(t) = \vec{v}_1 >$$

$$= \sum_{j \neq 1} -\frac{1}{m} \int \frac{\partial \phi}{\partial \vec{x}_1} (\vec{x}_1 - \vec{x}_j) \frac{\frac{1}{V^2} f_2(\vec{\xi}_1, \vec{\xi}_j, t) d\vec{\xi}_j}{\frac{1}{V} f_1(\vec{\xi}_1,t)}$$

$$= -\frac{1}{m} \frac{N-1}{V} \int \frac{\partial \phi}{\partial \vec{x}_1} (\vec{x}_1 - \vec{x}_2) f_2(\vec{\xi}_1, \vec{\xi}_2, t) d\vec{\xi}_2 \cdot \frac{1}{f_1(\vec{\xi},t)}$$

Putting in (A.10),

$$\frac{\partial f_1(\vec{\xi}_1,t)}{\partial t} + \vec{v}_1 \cdot \frac{\partial f_1}{\partial \vec{x}_1} - \frac{n_0}{m} \frac{2}{\partial \vec{v}_1} \cdot \int \frac{\partial \phi}{\partial \vec{x}_1} (\vec{x}_1 - \vec{x}_2) f_2(\vec{\xi}_1,\vec{\xi}_2,t) d\vec{\xi}_2 = 0$$

$$(A.11)$$

which is the first equation in the BBGKY hierarchy. The entire hierarchy
including the Liouville equation can be obtained in a similar manner.

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13. ABSTRACT

The Fokker-Planck equation hs often been used to describe the distribution of a particle in a random field. When such a technique is applied to a plasma in order to obtain a kinetic equation, certain difficulties arise. In the first place, the electric field must be determined consistently and the coefficients cannot be found explicitly without further assumptions which may not be consistent. Previously various attempts have been made to find the coefficients including the use of the collision concept by means of the Boltzmann equation and the BBGKY theory. In certain cases, this amounts to a manipulation of a kinetic equation into the form of a Fokker-Planck equation. Moreover, the coefficients are given as averages of functionals of the random field. A naive interpretation of these averages, however, gives incorrect result.

In view of these difficulties, a new formulation of the Fokker-Planck method is established. It is observed that the coefficients actually involve conditional averages of functionals of the random field. Furthermore, just as a Fokker-Planck equation can be used to describe the lowest distribution, similar equations can also be used to describe higher distributions. Such a set of equations is called generalized stochastic (or generalized Fokker-Planck) equations following Stratonovich. These equations are applicable to systems perturbed by small random forces provided sufficient knowledge regarding the unperturbed system is available. When the random forces are specified statistically, the generalized stochastic equations give immediate results. In this respect, of particular interest in plasma application, is the heating of electrons

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by random electric fields. For the Coulomb potential problem in which the electric field has to be determined consistently, it is possible to decouple the set of generalized equations by using a cluster expansion. The coefficients can then be found explicitly resulting in a kinetic equation. The results indicate that the use of a Vlasov equation in finding the polarization effect is essentially justified. The polarization effect is seen to be the consequence of the condition imposed on the average in evaluating the coefficient. Unlike the test particle theory, however, no test particle is artificially introduced. Finally, a similarity is observed between the set of generalized stochastic equations and the BBGKY hierarchy. Closer examination reveals that the BBGKY hierarchy may be regarded, in a sense, as a special mode of generalized stochastic equations.