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ON THE DIFFERENTIAL EQUATIONS OF THE AXISYMMETRIC VIBRATION OF PARABOLOIDAL SHELLS OF REVOLUTION

by James Ting-shun Wang and Chi-wen Lin

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SYMBOLS

a	Geometry parameter
$e_{\varphi\varphi}$, $e_{\theta\theta}$	Linear strains in meridional and circumferential directions
E	Modulus of elasticity
h	Shell thickness
i	$\sqrt{-1}$
$M_{\varphi\varphi}$, $M_{\theta\theta}$	Meridional and circumferential stress couples
$N_{\varphi\varphi}$, $N_{\theta\theta}$	Meridional and circumferential stress resultants
p_1, p_n	Meridional and normal loading components
Q	Transverse shear
r	Radius of a parallel circle
R_1, R_2	Principal radii of curvature
t	Time
v, w	Meridional and normal displacements
ρ	Mass density of shell material
ν	Poisson's ratio

INTRODUCTION

The dynamic behavior of shells of revolution becomes increasingly important in the space vehicle designs. Most of the investigations have been focused on shells having constant curvature. Very limited analytical work can be found in the literature concerning the shells having non-constant curvature. In many cases, such shells with variable curvature may be more desirable from both structural and aerodynamic view points. The simplest surface of non-constant curvature will be the paraboloid of revolution. Hoppmann, Cohen and Kunukkasseril [1] indicated the procedure for formulating the problem. The resulting equations includes two coupled partial differential equations with non-constant coefficients. The authors indicated that the non-constant coefficients are extremely complicated and no detailed expressions were given.

Johnson and Reissner [2] has provided a formula for the frequency of paraboloidal shells of revolution, however, the result is restricted to shallow paraboloid. Lin and Lee [3] discussed the free vibration of paraboloidal shells of revolution based on an inextensional theory. Kalnins [4] presented a numerical procedure similar to Goldberg, Bogdanoff and Marcus [5,6] for the free vibration of rotationally symmetric shells.

The present report provides the derivation of the equations of axi-symmetric motion for paraboloidal shells of revolution according to the linear theory of shells. A method of procedure according to the finite difference technique for the investigation of the free vibration of the shell is presented. The results presented in the report should be useful for the future interested investigators.

GENERAL THEORY

The meridional lines and parallel circles which coincide with the lines of principal curvature are used as coordinate system. The geometry and some symbols are shown in Figure 1. The surface of a paraboloid of revolution is represented by the equation

$$r = \frac{a}{2} \sec \varphi \tag{1}$$

The radii of curvature in the φ and θ directions respectively are

$$R_1 = \frac{a}{2} \sec^3 \varphi \tag{2}$$

$$R_2 = \frac{a}{2} \sec \varphi \tag{3}$$

where a is a constant. The surface becomes flatter as the value of a increases. According to Kirchkoff-Love assumptions, the equations of axisymmetric motion based on bending theory of shells are

$$\frac{\partial}{\partial\varphi} (N_{\varphi\varphi} \tan\varphi) - N_{\theta\theta} \sec^2\varphi - Q \tan\varphi + \frac{a}{2} \sec^3\varphi\tan\varphi\left(P_\varphi - \rho h \frac{\partial^2 v}{\partial t^2}\right) = 0 \tag{4}$$

$$N_{\varphi\varphi} \tan\varphi + N_{\theta\theta} \sec^3\varphi \sin\varphi + \frac{\partial}{\partial\varphi}(Q \tan\varphi)$$
$$+ \frac{a}{2} \sec^3\varphi \tan\varphi\left(P_n - \rho h \frac{\partial^2 v}{\partial t^2}\right) = 0 \tag{5}$$

$$\frac{\partial}{\partial \varphi} (M_{\varphi\varphi} \tan\varphi) - M_{\theta\theta} \sec^2\varphi - \frac{a}{2} Q \sec^3\varphi \tan\varphi = 0 \tag{6}$$

where $N_{\varphi\varphi}$ and $N_{\theta\theta}$ are stress resultants, $M_{\varphi\varphi}$ and $M_{\theta\theta}$ are stress couples, Q is the transverse shear, p_φ and p_n are loading components, v and w are displacement components, ρ is the material density, and h is the thickness of the shell.

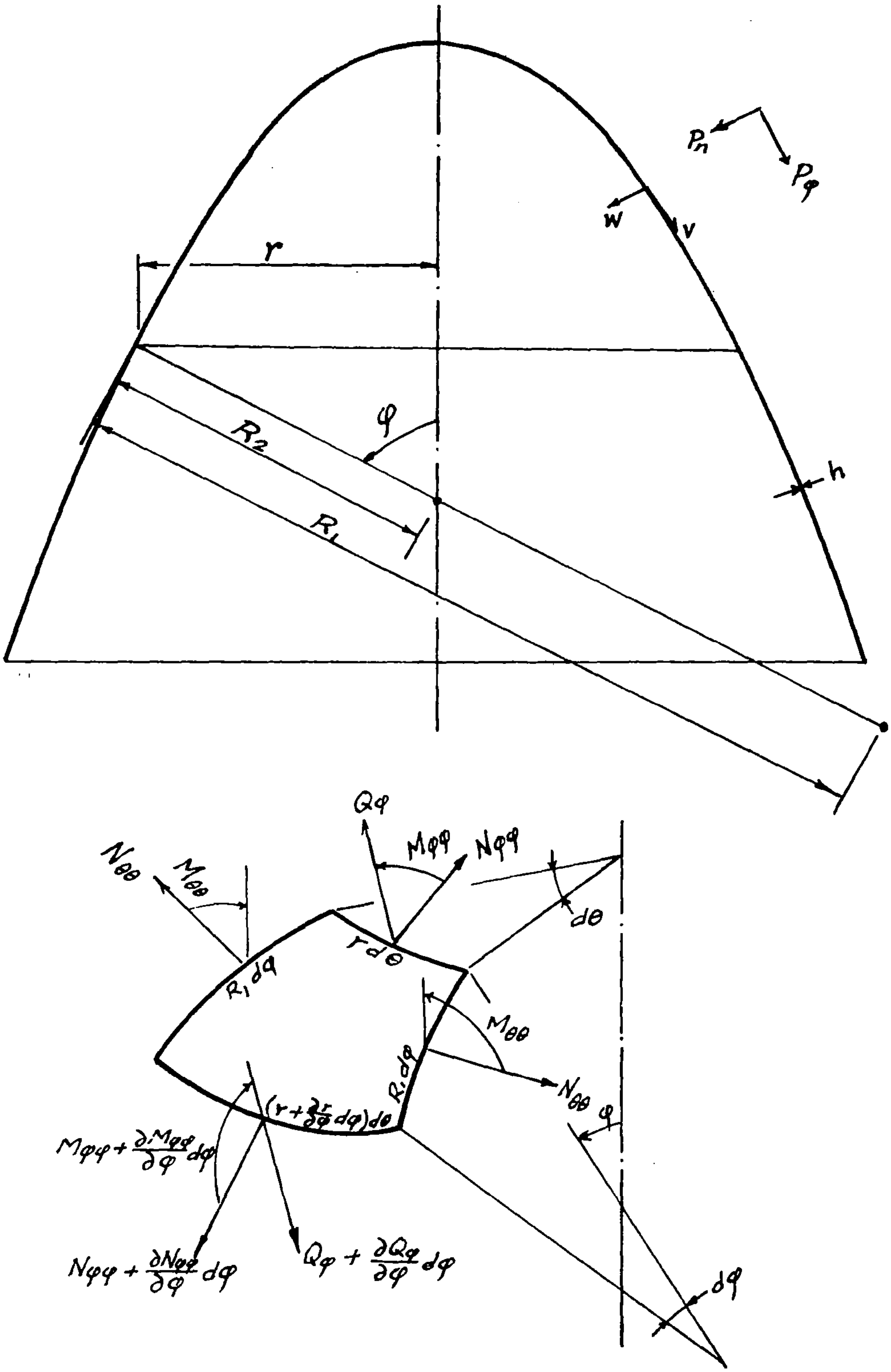

Fig. (1) Geometry

The stress-strain and moment-curvature relationships are

$$N_{\varphi\varphi} = \frac{Eh}{1-\nu^2}(e_{\varphi\varphi} + \nu e_{\theta\theta}) \tag{7a}$$

$$N_{\theta\theta} = \frac{Eh}{1-\nu^2}(e_{\theta\theta} + \nu e_{\varphi\varphi}) \tag{7b}$$

$$M_{\varphi\varphi} = D(\varkappa_{\varphi\varphi} + \nu\varkappa_{\theta\theta}) \tag{8a}$$

$$M_{\theta\theta} = D(\varkappa_{\theta\theta} + \nu\varkappa_{\varphi\varphi}) \tag{8b}$$

where E is the modulus of elasticity, ν is the Poisson's ratio, and D is the plate rigidity. The strain displacement relationships are

$$e_{\varphi\varphi} = \frac{2}{a}\cos^3\varphi\left(\frac{\partial v}{\partial\varphi} - w\right) \tag{9a}$$

$$e_{\theta\theta} = \frac{2}{a}\cos\varphi\ (\cot\varphi\ v - w) \tag{9b}$$

and the changes of curvature and displacement relationships are

$$\varkappa_{\varphi\varphi} = -\left(\frac{2}{a}\right)^2\cos^5\varphi\left[\cos\varphi\left(\frac{\partial v}{\partial\varphi} + \frac{\partial^2 w}{\partial\varphi^2}\right) - 3\sin\varphi\left(v + \frac{\partial w}{\partial\varphi}\right)\right] \tag{10a}$$

$$\varkappa_{\theta\theta} = -\left(\frac{2}{a}\right)^2\cos^4\varphi\ \cot\varphi\left(v + \frac{\partial w}{\partial\varphi}\right) \tag{10b}$$

By substituting Equations (9) and (10) into Equations (7) and (8), one obtains the following relationships

$$N_{\varphi\varphi} = \frac{2Eh}{a(1-\nu^2)}\left[\cos^3\varphi\frac{\partial u}{\partial\varphi} + \nu\cos\varphi\cot\varphi\ u - (\cos^3\varphi + \nu\cos\varphi)w\right] \tag{11a}$$

$$N_{\theta\theta} = \frac{2Eh}{a(1-\nu^2)}\left[\cos\varphi\cot\varphi\ u + \nu\cos^3\varphi\frac{\partial u}{\partial\varphi} - (\cos\varphi + \nu\cos^3\varphi)w\right] \tag{11b}$$

$$M_{\varphi\varphi} = - D\left(\frac{2}{a}\right)^2 \cos^5\varphi \left[\cos\varphi \left(\frac{\partial u}{\partial\varphi} + \frac{\partial^2 w}{\partial\varphi^2}\right) - \left(3 \sin\varphi - \frac{\nu}{\sin\varphi}\right)\left(u + \frac{\partial w}{\partial\varphi}\right)\right] \tag{12a}$$

$$M_{\theta\theta} = - D\left(\frac{2}{a}\right)^2 \cos^5\varphi \left[\nu \cos\varphi \left(\frac{\partial u}{\partial\varphi} + \frac{\partial^2 w}{\partial\varphi^2}\right) - \left(3\nu \sin\varphi - \frac{1}{\sin\varphi}\right)\left(u + \frac{\partial w}{\partial\varphi}\right)\right] \tag{12b}$$

By eliminating the transverse shear, Q, between Equations (4) and (6), and between Equations (5) and (6), and then substituting Equations (11) and (12) into the resulting equations, one obtains the following two partial differential equations governing the displacements v and w:

$$B_1 w''' + B_2 w'' + B_3 w' + B_4 w + B_5 v'' + B_6 v' + B_7 v + B_8 \frac{\partial^2 v}{\partial t^2} = 0 \tag{13}$$

$$C_1 w^{iv} + C_2 w''' + C_3 w'' + C_4 w' + C_5 w + C_6 v''' + C_7 v'' + C_8 v' + C_9 v + C_{10} \frac{\partial^2 w}{\partial t^2} = 0 \tag{14}$$

where

$$B_1 = \frac{8D}{a^3} \cos^9\varphi \sin^2\varphi$$

$$B_2 = \frac{8D}{a^3} \cos^8\varphi \sin\varphi \ (1-5 \sin^2\varphi)]$$

$$B_3 = \frac{8D}{a^3} \cos^7\varphi \ (18 \sin^4\varphi - 6 \sin^2\varphi + 9\nu \sin^2\varphi - 1) - \frac{2Eh}{a(1-\nu^2)} \cos\varphi \sin^2\varphi \cdot (\cos^2\varphi + \nu)$$

$$B_4 = \frac{2Eh}{a(1-\nu^2)} \sin^3\varphi \ (1 + 3 \cos^2\varphi)$$

$$B_5 = \frac{8D}{a^3} \cos^{12}\varphi \sin^2\varphi + \frac{2Eh}{a(1-\nu^2)} \cos^3\varphi \sin^2\varphi$$

$$B_6 = \frac{8D}{a^3} \cos^8\varphi \sin\varphi \ (2-\nu-9 \sin^2\varphi) + \frac{2Eh}{a(1-\nu^2)} \cos^2\varphi \sin\varphi \ (1-3 \sin^2\varphi)$$

$$B_7 = \frac{8D}{a^3} \cos^7\varphi \ (18 \sin^4\varphi - 6 \sin^2\varphi + 9\mu \sin^2\varphi - 1)$$

$$- \frac{2Eh}{a(1-\nu^2)} \cos\varphi \ (1 + \nu \sin^2\varphi)$$

$$B_8 = - \rho h \frac{a \sin^2\varphi}{2 \cos^3\varphi}$$

$$C_1 = - \frac{24D}{a^3} \cos^{12}\varphi$$

$$C_2 = \frac{8D}{a^3} \cos^{10}\varphi \cos\varphi \ (21 \sin^2\varphi - 3 \cos^2\varphi - \cos\varphi \sin\varphi + \nu)$$

$$C_3 = \frac{8D}{a^3} \cos^2\varphi \cot^2\varphi \ (-47 \cos^6\varphi \sin^4\varphi + 25 \cos^8\varphi \sin^2\varphi$$

$$+ 4\nu \cos^6\varphi \sin^2\varphi + 8 \cos^7\varphi \sin^3\varphi - \cos^9\varphi \sin\varphi - 7\nu \cos^6\varphi \sin\varphi$$

$$+ \cos^6\varphi - 3\nu \sin^3\varphi)$$

$$C_4 = \frac{8D}{a^3} \cos^6\varphi \cot^3\varphi \ (72 \sin^6\varphi - 84 \cos^2\varphi \sin^4\varphi + 6 \cos^4\varphi \sin^2\varphi$$

$$- 6\nu \sin^4\varphi + \nu \cos^2\varphi \sin^2\varphi - 21 \cos\varphi \sin^5\varphi + 6 \cos^3\varphi \sin^3\varphi$$

$$+ 7\nu \cos\varphi \sin^3\varphi - 6 \sin^2\varphi - \cos^2\varphi)$$

$$C_5 = -\frac{2Eh}{a(1-\nu^2)} \cos^2\varphi \, (\cos^4\varphi + 2\nu \cos^2\varphi + 1)$$

$$C_6 = C_1 = -\frac{24D}{a^3} \cos^{12}\varphi$$

$$C_7 = C_2 = \frac{8D}{a^3} \cos^{10}\varphi \cot\varphi \, (21 \sin^2\varphi - 3 \cos^2\varphi - \cos\varphi \sin\varphi + \nu)$$

$$C_8 = C_3 + \frac{2Eh}{a(1-\nu^2)} \cos^4\varphi \, (\cos^2\varphi + \nu)$$

$$C_9 = C_4 + \frac{2Eh}{a(1-\nu^2)} \cos^2\varphi \cot\varphi \, (\nu \cos^2\varphi + 1)$$

$$C_{10} = -\rho h \frac{a}{2} \tag{15}$$

The primes represent partial derivatives with respect to φ and dots denote time derivatives.

For very thin shells, the bending effect becomes negligibly small. The equations of axisymmetric motion of a paraboloidal shell of revolution based on membrane theory can be reduced immediately from Equation (13) and (14) by taking the plate rigidity D = 0. The resulting equations are

$$a_1 w' + a_2 w + b_1 v'' + b_2 v' + b_3 v - A\ddot{v} = p_1(\varphi, t) \frac{a^2(1-\nu^2)}{4Eh} \tag{16}$$

$$a_3 w + b_4 v' + b_5 v - A\ddot{w} = p_n(\varphi, t) \frac{a^2(1-\nu^2)}{4Eh} \tag{17}$$

where

$$a_1 = -\cos^4\varphi \, (\cos^2\varphi + \nu)$$

$$a_2 = \cos^3\varphi \sin\varphi \, (1 + 3 \cos^2\varphi)$$

$$a_3 = -\cos^2\varphi\ (\cos^4\varphi + 2\nu\cos^2\varphi + 1)$$

$$b_1 = \cos^6\varphi$$

$$b_2 = \cot\varphi\ \cos^4\varphi\ (1 - 3\sin^2\varphi)$$

$$b_3 = -\cot^2\varphi\ \cos^2\varphi\ (1 + \nu\sin^2\varphi)$$

$$b_4 = \cos^4\varphi\ (\cos^2\varphi + \nu)$$

$$b_5 = \cot\varphi\ \cos^2\varphi\ (1 + \nu\cos^2\varphi)$$

$$A = \rho a^2(1-\nu^2)/4E \tag{18}$$

FREE VIBRATION

The axisymmetric motion corresponding to the free vibration of the shell is considered to be harmonic. The displacements take the following form

$$v = V(\varphi)e^{i\omega t} \tag{19}$$

$$w = W(\varphi)e^{i\omega t} \tag{20}$$

where ω represents the circular frequencies of the shell.

I. Bending Theory

By substituting Equations (19) and (20) into Equations (13) and (14) with $p_1 = p_n = 0$, one obtains the following simultaneous ordinary differential equations:

$$B_1W''' + B_2W'' + B_3W' + B_4W + B_5V'' + B_6V' + (B_7-B_8\omega^2)V = 0 \tag{21}$$

$$C_1W^{IV} + C_2W''' + C_3W'' + C_4W' + C_5W + C_6V''' + C_7V''$$
$$+ C_9V - C_{10}\omega^2W = 0 \tag{22}$$

Elimination of V''' term between Equations (21) and (22) yields the following equation

$$A_1W^{IV} + A_2W''' + A_3W'' + A_4W' + A_5W + A_6V'' + A_7V' + A_8V = 0 \tag{23}$$

where

$$A_1 = C_6B_1 - B_5C_1$$

$$A_2 = C_6(B_1' + B_2) - B_5C_2$$

$$A_3 = C_6(B_2' + B_3) - B_5C_3$$

$$A_4 = C_6(B_3' + B_4) - B_5C_4$$

$$A_5 = C_6B_4' - B_5(C_5 - C_{10}\omega^2)$$

$$A_6 = C_6(B_5' + B_6) - B_5C_7$$

$$A_7 = C_6(B_6' + B_7 - B_8\omega^2) - B_5C_8$$

$$A_8 = C_6(B_7' - B_8'\omega^2) - B_5C_9 \tag{24}$$

By introducing the new variable

$$U = W'' \tag{25}$$

Equations (21) and (23) become

$$B_1U' + B_2U + B_3W' + B_4W + B_5V'' + B_6V' + (B_7-B_8\omega^2)V = 0 \tag{26}$$

$$A_1U'' + A_2U' + A_3U + A_4W' + A_5W + A_6V'' + A_7V' + A_8V = 0 \tag{27}$$

The difference equations equivalent to Equations (25), (26) and (27) according to the following approximations for the first and second derivatives:

$$F_j' = \frac{1}{2\lambda}(F_{j+1} - F_{j-1}) \tag{28}$$

$$F_j'' = \frac{1}{\lambda^2}(F_{j+1} - 2F_j + F_{j-1}) \tag{29}$$

become

$$[\alpha_j]\{X_{j+1}\} + [\beta_j]\{X_j\} + [\gamma_j]\{X_{j-1}\} = 0 \tag{30}$$

where

$$\{X_j\} = \begin{Bmatrix} W_j \\ V_j \\ U_j \end{Bmatrix} \tag{31a}$$

$$\{X_j\} = \{X(j\lambda)\} \quad j = 0,1,\cdots N \tag{31b}$$

and $\lambda = \Delta\varphi$ is the mesh size.

The coefficient matrices, α_j, β_j, and γ_j, in Equation (30) are

$$\alpha_j = \begin{bmatrix} -1 & 0 & 0 \\ B_{3j} & \frac{B_{5j}}{\lambda^2} + \frac{B_{6j}}{2\lambda} & \frac{B_{1j}}{2\lambda} \\ \frac{A_{4j}}{2\lambda} & \frac{A_{6j}}{\lambda^2} + \frac{A_{7j}}{2\lambda} & \frac{A_{1j}}{\lambda^2} + \frac{A_{2j}}{2\lambda} \end{bmatrix} \tag{32}$$

$$\beta_j = \begin{bmatrix} 2 & 0 & \lambda^2 \\ B_{4j} & \left(-\frac{2B_{5j}}{\lambda^2} + B_{7j} - B_{8j}\omega^2\right) & B_{2j} \\ A_{5j} & \left(-\frac{2A_{6j}}{\lambda^2} + A_{8j}\right) & \left(-\frac{2A_{1j}}{\lambda^2} + A_{3j}\right) \end{bmatrix} \tag{33}$$

$$\gamma_j = \begin{bmatrix} -1 & 0 & 0 \\ -B_{3j} & \left(\frac{B_{5j}}{\lambda^2} - \frac{B_{6j}}{2\lambda}\right) & -\frac{B_{1j}}{2\lambda} \\ -\frac{A_{4j}}{2\lambda} & \frac{A_{6j}}{\lambda^2} - \frac{A_{7j}}{2\lambda} & \left(\frac{A_{1j}}{\lambda^2} - \frac{A_{2j}}{2\lambda}\right) \end{bmatrix} \tag{34}$$

When the shell is closed at the apex, it is obvious that $v = 0$ and $\frac{\partial w}{\partial \varphi} = 0$ at $\varphi = 0$ because of symmetry. From the condition that

$$M_{\varphi\varphi} = M_{\theta\theta} \tag{35a}$$

and

$$N_{\varphi\varphi} = N_{\theta\theta} \tag{35b}$$

at the apex of the shell, the following conditions are obtained in conjunction with Equations (11) and (12):

$$\frac{\partial u}{\partial \varphi} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial \varphi^2} = 0 \quad \text{at} \quad \varphi = 0 \tag{36}$$

Therefore, one obtains

$$\begin{aligned} W_0 &= W_1 \\ V_0 &= 0 \\ U_0 &= 0 \end{aligned} \tag{37}$$

or

$$X_0 = R_0 X_1 \tag{38}$$

where $R_0 = [1 \quad 0 \quad 0]$. If the shell is not closed at the apex, R_0 will be different from different supporting conditions. The recurrence relationship for other interior points may be established as follows according to Equation (30) in conjunction with Equation (38):

$$X_j = R_j X_{j+1} \tag{39}$$

where

$$R_j = -(\beta_j + \gamma_j R_{j-1})^{-1} \alpha_j \qquad j=1,2\cdots N \tag{40}$$

and

$$R_0 = [1 \quad 0 \quad 0] \tag{41}$$

The boundary conditions, in finite difference form, along the open end may be written in the following form:

$$\zeta_N X_{N+1} + \eta_N X_N + \xi_N X_{N-1} = 0 \tag{42}$$

where N+1 represents the imaginary point off the edge of the shell. The form of the coefficient matrices, ζ_N, η_N, and ξ_N depend on the supporting conditions. It may be shown that $\zeta_N = -\xi_N$ for

(a) simply supported boundary conditions

$$\begin{aligned} w &= 0 \\ v &= 0 \\ M_{\varphi\varphi} &= 0 \end{aligned} \tag{43}$$

(b) Fixed edge boundary conditions

$$
\begin{aligned}
w &= 0 \\
v &= 0 \\
\frac{\partial w}{\partial \varphi} &= 0
\end{aligned}
\tag{44}
$$

By substituting Equation (39) into Equation (42), one obtains

$$[\zeta_N + (\eta_N + \xi_N R_{N-1}) R_N] X_{N+1} = 0 \tag{45}$$

Equation (45) represents a system of three simultaneous homogeneous algebraic equations. The elements of the coefficient matrix contain the natural circular frequency, ω, of the system. For non-trivial solution of Equation (45), the determinant of the coefficient matrix must vanish. Hence the frequency equation of the shell according to bending theory becomes

$$|\zeta_N + (\eta_N + \xi_N R_{N-1}) R_N| = 0 \tag{46}$$

An iterative procedure similar to the one used in [7] may be used for determining the frequencies of the shell.

II. Membrane Theory

In a similar procedure as presented in I, the following equations are obtained after the substitution of Equations (19) and (20) into Equations (16) and (17) with $p_1 = p_n = 0$:

$$a_1 W' + a_2 W + b_1 V'' + b_2 V' + b_3 V + A\omega^2 V = 0 \tag{47}$$

$$a_3W + b_4V' + b_5V + A\omega^2W = 0 \tag{48}$$

From Equation (48), one obtains

$$W = gV' + fV \tag{49}$$

where

$$g = -b_4/(a_3+A\omega^2) \quad \text{and} \quad f = -b_5/(a_3+A\omega^2)$$

Substitution of Equation (49) into Equation (47) gives

$$G_1V'' + G_2V' + G_3V = 0 \tag{50}$$

where

$$G_1 = a_1g + b_1$$

$$G_2 = a_1g' + a_1f + a_2g + b_2$$

and

$$G_3 = a_1f' + a_2f + b_3 + A\omega^2 \tag{51}$$

According to Equations (28) and (29), the finite difference equations equivalent to Equation (50) becomes

$$\alpha_j{}^*V_{j+1} + \beta_j{}^*V_j + \gamma_j{}^*V_{j-1} = 0 \tag{52}$$

where

$$\alpha_j{}^* = \frac{G_{1j}}{\lambda^2} + \frac{G_{2j}}{2\lambda}$$

$$\beta_j^* = G_{3j} - \frac{2G_{1j}}{\lambda^2}$$

$$\gamma_j^* = \frac{G_{1j}}{\lambda^2} - \frac{G_{2j}}{2\lambda} \tag{53}$$

when the shell is closed at the apex, v=0 or $V_0=0$. The recurrence formulas for j = 1,2$\cdots$N-1 becomes

$$V_j = R_j^* V_{j+1} \tag{54}$$

where

$$R_j^* = -(\beta_j^* + \gamma_j^* R_{j-1}^*)^{-1} \alpha_j^* \tag{55}$$

and

$$R_0^* = 0 \tag{56}$$

In order to allow the membrane state of stress to exist, only two types of boundary conditions are possible, namely

(a) Simply supported:

$$v = 0 \quad \text{or} \quad V_N = 0 \tag{57}$$

Equation (52) corresponding to the station N-1 is

$$\alpha_{N-1}^* V_N + \beta_{N-1}^* V_{N-1} + \gamma_{N-1}^* V_{N-2} = 0 \tag{58}$$

Substituting the condition (57) into Equation (58) in conjunction with Equation (54), one obtains

$$(\beta^*_{N-1} + \gamma^*_{N-1} R^*_{N-2}) V_{N-1} = 0 \tag{59}$$

The coefficient of V_{N-1} in Equation (59) contains the frequency, ω, of the shell. Since the quantity V_{N-1}, in general, does not equal to zero, hence the frequency equation becomes

$$\beta^*_{N-1} + \gamma^*_{N-1} R^*_{N-2} = 0 \tag{60}$$

(b) Free edge:

$$N_{\varphi\varphi} = 0 \quad \text{at} \quad \varphi = \varphi_0 \quad \text{or}$$

$$\zeta^*_N V_{N+1} + \eta^*_N V_N - \zeta^*_N V_{N-1} = 0 \tag{61}$$

where

$$\zeta^*_N = \frac{1}{2\lambda} [\cos^3 \varphi_0 - (\cos^3 \varphi_0 + \nu \cos \varphi_0) g] \tag{62}$$

$$\eta^*_N = \nu \cos \varphi_0 \cot \varphi_0 - (\cos^3 \varphi_0 + \nu \cos \varphi_0) f \tag{63}$$

Substituting Equation (54) into Equation (61), one obtains

$$[\zeta^*_N (1 - R^*_{N-1} R^*_N) + \eta^*_N R^*_N] V_{N+1} = 0 \tag{64}$$

The frequency equation, therefore, becomes

$$\zeta^*_N (1 - R^*_{N-1} R_N) + \eta^*_N R^*_N = 0 \tag{65}$$

An iterative procedure may be used for the determination of the natural frequencies from Equations (60) or (65) whenever applicable.

DISCUSSIONS

The differential equations governing the axisymmetric motion of paraboloidal shells of revolution according to linear bending theory as well as membrane theory are presented. A finite difference technique proposed for the determination of the natural frequency of the shell is believed to be feasible for practical application.

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