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DETERMINATION OF STRESSES IN ELASTIC SOLIDS USING THREE STRESS FUNCTIONS AND THREE EQUATIONS

by Robert E. Reed, Jr.

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Moffett Field, Calif.

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SUMMARY

The classical equations of elasticity in terms of the stress components are reduced to a set of three equations in terms of the three Maxwell stress functions and arbitrary functions of integration. The sufficiency of these equations is due to the interrelationships between the six compatibility equations. The difficulty in choosing the arbitrary functions to fit a particular problem accents the advantages of using the inverse method to find solutions. A family of exact solutions is presented for stresses in a rectangular solid with certain applied stresses on two opposite faces, workless boundary conditions on two opposite faces, and the remaining two faces free of applied stress.

INTRODUCTION

The differential equations for determining the stresses and displacements in a three-dimensional elastic solid are well known (e.g., refs. 1-3), but relatively few exact solutions have been found. Primarily, two methods can be used to find these solutions. One method is to choose displacements that satisfy the equilibrium equations and boundary conditions and the other method is to choose stress components that satisfy the equilibrium equations, boundary conditions, and compatibility equations. The choice between these two methods will depend on the quantities desired and the type of boundary conditions present. However, this report will deal only with the method of choosing the stress components; and then it attempts to derive a simplified set of equations from which exact solutions can be obtained. Basically, the complete system of equations for the stress components consists of 12 unknowns and 15 equations. The unknowns are the 6 stress components and the 6 strain components. The equations consist of 3 equilibrium, 6 stress-strain, and the 6 compatibility equations in terms of the 6 strain components which are necessary and sufficient to ensure the continuity of the displacements (for simply connected regions) of any solid undergoing deformation. This system can be reduced to 6 equations and 3 unknowns by defining 3 stress functions that identically satisfy the equilibrium equations and then, with the stress-strain relations, by introducing these stress functions into the compatibility equations. However, it is pointed out in some references (e.g., refs. 1-3) that the 6 compatibility equations are interrelated and that, at most, 3 of the 6 are unrelated equations. That is, certain groups containing 3 compatibility equations will have no equation related to the other 2, but any group of 4 or more equations will have 1 or more equations related to the other equations in the group (see appendix A for a discussion of the compatibility equations). These interrelationships, and the assumption that the material is homogeneous, isotropic, and follows Hooke's law, make it possible to derive a set of 3 equations in terms of 3 unknown stress functions and some arbitrary functions of integration. These equations, along with the boundary conditions, are necessary and sufficient for determining exact solutions, but the arbitrary functions of integration must be chosen to suit each particular problem. Exact solutions of these equations are often difficult where a particular problem is specified. However, use of the "inverse method," where the arbitrary functions of integration are chosen such that solutions can be found, will produce exact solutions to problems which may be of immediate interest in terms of specific problems or can be used to check the validity of approximate solutions which are based on formulations that can be used to solve a wider range of problems.

In the following, the set of three equations is derived and an example solution is presented and discussed. This solution represents a rectangular solid with certain combinations of shear and normal stress applied to opposite faces while two other faces have workless boundary conditions and the remaining opposite faces are free of applied stress.

SYMBOLS

A_i	constants of integration
a	dimension in x direction
B_i	constants of integration
b	dimension in y direction
E	elastic modulus
F_x, F_y, F_z	resultant applied forces
$\left.\begin{matrix} f,g,h, \\ F,G,H \end{matrix}\right\}$	functions of integration
i	$\sqrt{-1}$
L_o	$\frac{a}{b}$
M_x, M_y, M_z	resultant applied moments
P,Q,R	functions of body forces
P_i, Q_i, T_i, U_i	functions of ξ
u,v,w	displacements in x,y,z directions

$\bar{u},\bar{v},\bar{w}$	dimensionless displacements; $\frac{u}{a}$, $\frac{v}{b}$, and $\frac{w}{a}$
X,Y,Z	body forces
x,y,z	rectangular coordinates
β,γ,λ	constant parameters, $\lambda^2 = \gamma^2 - \beta^2 L_o^2$
$\gamma_{xy},\gamma_{xz},\gamma_{yz}$	shear strains
$\epsilon_x,\epsilon_y,\epsilon_z$	normal strains
$\theta_1,\theta_2,\theta_3$	functions of ψ_1,ψ_2,ψ_3,P,Q, and R
ν	Poisson's ratio
ξ,η,ζ	dimensionless coordinates; $\frac{x}{a}$, $\frac{y}{b}$, and $\frac{z}{a}$
$\sigma_x,\sigma_y,\sigma_z$	normal stress components
$\tau_{xy},\tau_{xz},\tau_{yz}$	shear stress components
ψ_1,ψ_2,ψ_3	Maxwell stress functions

Subscripts

I	imaginary part of a complex quantity
R	real part of a complex quantity

HOMOGENEOUS, ISOTROPIC, LINEARLY ELASTIC MATERIALS

The equilibrium equations including the body forces are

$$\left.\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + X = 0 \\
\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z} + Y = 0 \\
\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + Z = 0
\end{aligned}\right\} \tag{1}$$

Hooke's law is

$$
\left.
\begin{aligned}
\epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\
\epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \\
\epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}, \quad \gamma_{xz} = \frac{2(1+\nu)}{E}\tau_{xz}, \quad \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}
\end{aligned}
\right\} \tag{2}
$$

and the compatibility equations are

$$
\left.
\begin{aligned}
&\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = 0 \qquad 2\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 0 \\
&\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} = 0 \qquad 2\frac{\partial^2 \epsilon_y}{\partial x\, \partial z} - \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 0 \\
&\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} = 0 \qquad 2\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} - \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 0
\end{aligned}
\right\} \tag{3}
$$

The equilibrium equations can be satisfied by the Maxwell stress functions modified to account for the body forces. Denoting $\partial^2(\cdot\cdot)/\partial x^2$ by $(\cdot\cdot)_{,xx}$, etc., one obtains

$$
\left.
\begin{aligned}
\sigma_x &= \psi_{2,zz} + \psi_{3,yy} - \int X\,dx, \qquad \tau_{xy} = -\psi_{3,xy} \\
\sigma_y &= \psi_{1,zz} + \psi_{3,xx} - \int Y\,dy, \qquad \tau_{xz} = -\psi_{2,xz} \\
\sigma_z &= \psi_{1,yy} + \psi_{2,xx} - \int Z\,dz, \qquad \tau_{yz} = -\psi_{1,yz}
\end{aligned}
\right\} \tag{4}
$$

Equations (4) can be substituted into equations (2) to give the strains in terms of the stress functions. Equations (2) can then be substituted into the compatibility equations to give six equations in terms of the three unknowns ψ_1, ψ_2, and ψ_3. For convenience, the following definitions are made:

$$\left.\begin{aligned}
\theta_1 &= \nu\nabla^2\psi_1 - \nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + P \\
\theta_2 &= -\nabla^2\psi_1 + \nu\nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + Q \\
\theta_3 &= -\nabla^2\psi_1 - \nabla^2\psi_2 + \nu\nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + R
\end{aligned}\right\} \tag{5}$$

where

$$P = \int X\,dx - \nu\left(\int Y\,dy + \int Z\,dz\right)$$

$$Q = \int Y\,dy - \nu\left(\int X\,dx + \int Z\,dz\right)$$

$$R = \int Z\,dz - \nu\left(\int X\,dx + \int Y\,dy\right)$$

and

$$\nabla^2(\cdot\cdot) = \frac{\partial^2}{\partial x^2}(\cdot\cdot) + \frac{\partial^2}{\partial y^2}(\cdot\cdot) + \frac{\partial^2}{\partial z^2}(\cdot\cdot)$$

Equations (3) then become, respectively (see appendix B for details of the derivation),

$$\left.\begin{aligned}
\theta_{1,yy} + \theta_{2,xx} &= 0 \\
\theta_{1,zz} + \theta_{3,xx} &= 0 \\
\theta_{2,zz} + \theta_{3,yy} &= 0
\end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned}
\theta_{1,yz} &= 0 \\
\theta_{2,xz} &= 0 \\
\theta_{3,xy} &= 0
\end{aligned}\right\} \tag{7}$$

Equations (7) can be integrated to find θ_1, θ_2, and θ_3, but these solutions must be restricted to satisfy equations (6). Therefore, equations (7) yield

$$\left.\begin{aligned} \theta_1 &= f_1(x,y) + g_1(x,z) \\ \theta_2 &= f_2(x,y) + h_2(y,z) \\ \theta_3 &= g_3(x,z) + h_3(y,z) \end{aligned}\right\} \tag{8}$$

and equations (6) require that

$$\left.\begin{aligned} f_{1,yy} + f_{2,xx} &= 0 \\ g_{1,zz} + g_{3,xx} &= 0 \\ h_{2,zz} + h_{3,yy} &= 0 \end{aligned}\right\} \tag{9}$$

Equations (9) can be integrated to give

$$\left.\begin{aligned} f_1(x,y) &= -\iint f_2(x,y)_{,xx}\, dy\, dy + yf_3(x) + F(x) \\ g_3(x,z) &= -\iint g_1(x,z)_{,zz}\, dx\, dx + xg_4(z) + H(z) \\ h_2(y,z) &= -\iint h_3(y,z)_{,yy}\, dz\, dz + zh_4(y) + G(y) \end{aligned}\right\} \tag{10}$$

Substituting equations (10) into equations (8) gives

$$\left.\begin{aligned} \theta_1 &= -\iint f_2(x,y)_{,xx}\, dy\, dy + yf_3(x) + g_1(x,z) + F(x) \\ \theta_2 &= -\iint h_3(y,z)_{,yy}\, dz\, dz + zh_4(y) + f_2(x,y) + G(y) \\ \theta_3 &= -\iint g_1(x,z)_{,zz}\, dx\, dx + xg_4(z) + h_3(y,z) + H(z) \end{aligned}\right\} \tag{11}$$

Equations (5) can be used to write equations (11) in terms of the ψ values. This gives the three equations which, along with the boundary conditions, form a necessary and sufficient set for determining elasticity solutions. These equations are

$$\left.\begin{aligned}
&\nu\nabla^2\psi_1 - \nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + P \\
&\qquad = -\iint f_2(x,y)_{,xx}\, dy\, dy + yf_3(x) + g_1(x,z) + F(x) \\
&-\nabla^2\psi_1 + \nu\nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + Q \\
&\qquad = -\iint h_3(y,z)_{,yy}\, dz\, dz + zh_4(y) + f_2(x,y) + G(y) \\
&-\nabla^2\psi_1 - \nabla^2\psi_2 + \nu\nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) + R \\
&\qquad = -\iint g_1(x,z)_{,zz}\, dx\, dx + xg_4(z) + h_3(y,z) + H(z)
\end{aligned}\right\} \quad (12)$$

where the right-hand sides are arbitrary functions of integration. Note that there are no arbitrary functions of the form $f(x,y,z)$.

Particular solutions can be found for $F(x)$, $G(y)$, $H(z)$, and $yf_3(x)$, $zh_4(y)$, $xg_4(z)$; they have the form:

$$\psi_{1p} = \frac{1}{1 + \nu} \iint [F(x) + yf_3(x)]dx\, dx$$

$$\psi_{2p} = \frac{1}{1 + \nu} \iint [G(y) + zh_4(y)]dy\, dy$$

$$\psi_{3p} = \frac{1}{1 + \nu} \iint [H(z) + xg_4(z)]dz\, dz$$

The stresses resulting from these solutions are zero so these functions can be disregarded. However, the solutions for $f_2(x,y)$, $g_1(x,z)$, and $h_3(y,z)$ will yield nonzero stresses which may be necessary to satisfy the boundary conditions of a specified problem.

COMMENTS ON POSSIBLE SOLUTIONS

General Comments

The relevant arbitrary functions (f_2, g_1, h_3) in equations (12) can be expanded into appropriate double Fourier series, and particular solutions can be found in terms of the constants of the series. These constants would be determined to fit a particular problem. However, these functions and the body forces will be taken to be zero and only the homogeneous equations will be considered. These are

$$\left.\begin{aligned} \nu\nabla^2\psi_1 - \nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) &= 0 \\ -\nabla^2\psi_1 + \nu\nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) &= 0 \\ -\nabla^2\psi_1 - \nabla^2\psi_2 + \nu\nabla^2\psi_3 + (\psi_{1,xx} + \psi_{2,yy} + \psi_{3,zz}) &= 0 \end{aligned}\right\} \tag{13}$$

Subtracting equations (13) in pairs gives

$$\left.\begin{aligned} \nabla^2\psi_1 &= \nabla^2\psi_2 \\ \nabla^2\psi_1 &= \nabla^2\psi_3 \end{aligned}\right\} \tag{14}$$

Therefore, any solution of equations (13) will satisfy equations (14). This gives some indication of the type of solutions one might choose. Another possibility is to let

$$\psi_1 = \psi_2 = \psi_3 \equiv \psi$$

Equations (13) then reduce to a single Laplace equation:

$$\nabla^2\psi = 0 \tag{15}$$

From equations (4), it is seen that solutions to equation (15) represent the case when the first stress invariant $(\sigma_x + \sigma_y + \sigma_z)$ is zero. One obvious form of solution to equation (15) is $\psi = A \cos \alpha x \cos \beta y e^{\gamma z}$, which is similar to a solution discussed in reference 4.

The solution to equations (13) discussed here gives the stress distribution in a rectangular solid having two opposite sides free of applied stress, two opposite sides constrained by workless boundary conditions, and the remaining two sides subjected to some combination of applied stress.

Solution to equations (13).- The following dimensionless coordinates will be used:

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{b}, \qquad \zeta = \frac{z}{a}$$

The term $\nabla^2(")$ is then given by

$$a^2\nabla^2(\cdot\cdot) = (\cdot\cdot)_{,\xi\xi} + L_o^2(\cdot\cdot)_{,\eta\eta} + (\cdot\cdot)_{,\zeta\zeta}$$

The solution is chosen in the form

$$\left.\begin{aligned} \psi_1 &= f(\xi)Y(\eta)Z(\zeta) \\ \psi_2 &= g(\xi)Y(\eta)Z(\zeta) \\ \psi_3 &= h(\xi)Y(\eta)Z(\zeta) \end{aligned}\right\} \tag{16}$$

where it is further assumed that

$$Y_{,\eta\eta} = -\beta^2 Y \ , \qquad Z_{,\zeta\zeta} = \gamma^2 Z \tag{17}$$

Letting $\lambda^2 = \gamma^2 - \beta^2 L_o^2$, one can separate the variables in equations (13) by substituting equations (16). The resulting ordinary differential equations are

$$(1 + \nu)f'' + \nu\lambda^2 f - (g'' + \gamma^2 g) - (h'' - \beta^2 L_o^2 h) = 0 \tag{18}$$

$$-\lambda^2 f + [\nu g'' - (1 + \nu)\beta^2 L_o^2 g + \nu\gamma^2 g] - (h'' - \beta^2 L_o^2 h) = 0 \tag{19}$$

$$-\lambda^2 f - (g'' + \gamma^2 g) + [\nu(h'' - \beta^2 L_o^2 h) + (1 + \nu)\gamma^2 h] = 0 \tag{20}$$

where $f'' = (d^2 f/d\xi^2)$. Subtracting equations (19) and (20) gives

$$g'' + \lambda^2 g = h'' + \lambda^2 h$$

so

$$g = h + A_1 \sin \lambda\xi + A_2 \cos \lambda\xi \tag{21}$$

In the following, A_1 and A_2 are assumed to be zero (which would be required later to satisfy the boundary conditions). Substituting equation (21) into equation (19) or (20) gives f in terms of h. This relation and equation (21) can be substituted into equation (18) to give the following equation for $h(\xi)$:

$$h'''' + 2\lambda^2 h'' + \lambda^4 h = 0 \tag{22}$$

Equation (22) has the solution

$$h(\xi) = B_1 e^{i\lambda\xi} + B_2 e^{-i\lambda\xi} + B_3 i\lambda\xi e^{i\lambda\xi} + B_4 i\lambda\xi e^{-i\lambda\xi} \tag{23}$$

The functions $f(\xi)$ and $g(\xi)$ can now be found and the stresses can be determined from equations (4).

The desired boundary conditions of two opposite surfaces that are free of applied stress are given by:

$\underline{\xi = 0}$

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0$$

$\underline{\xi = 1}$

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0$$

Since τ_{xy} and τ_{xz} are both zero when $h'(\xi) = 0$, these six conditions give the following four homogeneous equations:

$$\left.\begin{aligned} h(0) &= 0 \\ h'(0) &= 0 \\ h(1) &= 0 \\ h'(1) &= 0 \end{aligned}\right\} \tag{24}$$

For the B values to be nonzero, λ must have such a value that the determinant of the coefficients in equations (24) is zero. The B values can then be found in terms of one arbitrary constant from equations (24) and substituted into equation (23) to give

$$h(\xi) = a^2 B_o \lambda[(1 - \xi)\sin \lambda \sin \lambda\xi - \lambda\xi \sin \lambda(1 - \xi)] \tag{25}$$

In order for the determinant to vanish, λ must satisfy one of the two equations

$$\sin \lambda = \pm\lambda \tag{26}$$

The nonzero roots of equations (26) are complex and the lowest five roots of each of equations (26) are given in table I (see, also, ref. 5).

TABLE I.- ROOTS OF $\sin \lambda = \pm\lambda$

$\sin \lambda = -\lambda$		$\sin \lambda = \lambda$	
λ_R	λ_I	λ_R	λ_I
4.21239	2.25072	7.49767	2.76867
10.71253	3.10314	13.89995	3.35220
17.07336	3.55108	20.23851	3.71676
23.39835	3.85880	26.55454	3.98314
29.70811	4.09370	32.85974	4.19325

If equation (25) is used, but not equations (26), which would require the ± sign, the stresses obtained will be complex quantities given by

$$
\left.
\begin{aligned}
\sigma_x^* &= \lambda^3 B_o[(1-\xi)\sin\lambda\ \sin\lambda\xi - \lambda\xi\ \sin\lambda(1-\xi)]Y(\eta)Z(\zeta) \\
\sigma_y^* &= B_o\Big\{\lambda\beta^2 L_o^2[(1-\xi)\sin\lambda\ \sin\lambda\xi - \lambda\xi\ \sin\lambda(1-\xi)] \\
&\quad +2(\beta^2 L_o^2 - \nu\gamma^2)[\sin\lambda\ \cos\lambda\xi - \lambda\ \cos\lambda(1-\xi)]\Big\}Y(\eta)Z(\zeta) \\
\sigma_z^* &= B_o\Big\{\gamma^2\lambda[-(1-\xi)\sin\lambda\ \sin\lambda\xi + \lambda\xi\ \sin\lambda(1-\xi)] \\
&\quad +2(\gamma^2 - \nu\beta^2 L_o^2)[-\sin\lambda\ \cos\lambda\xi + \lambda\ \cos\lambda(1-\xi)]\Big\}Y(\eta)Z(\zeta) \\
\tau_{xy}^* &= \lambda L_o B_o[\sin\lambda\ \sin\lambda\xi + \lambda\ \sin\lambda(1-\xi) - \lambda(1-\xi)\sin\lambda\ \cos\lambda\xi \\
&\quad -\lambda^2\xi\ \cos\lambda(1-\xi)]Y'(\eta)Z(\zeta) \\
\tau_{xz}^* &= \lambda B_o[\sin\lambda\ \sin\lambda\xi + \lambda\ \sin\lambda(1-\xi) - \lambda(1-\xi)\sin\lambda\ \cos\lambda\xi \\
&\quad -\lambda^2\xi\ \cos\lambda(1-\xi)]Y(\eta)Z'(\zeta) \\
\tau_{yz}^* &= L_o B_o\Big\{-\lambda(1-\xi)\sin\lambda\ \sin\lambda\xi + \lambda^2\xi\ \sin\lambda(1-\xi) \\
&\quad -2(1-\nu)[\sin\lambda\ \cos\lambda\xi - \lambda\ \cos\lambda(1-\xi)]\Big\}Y'(\eta)Z'(\zeta)
\end{aligned}
\right\}
\quad (27)
$$

Both the real and imaginary parts of equations (27) are solutions to the equilibrium equations, boundary conditions, etc., so the actual stresses can be taken as the real parts of equations (27). When equations (26) and the notation

$$\gamma = \gamma_R + i\gamma_I$$

$$B_o = B_R + iB_I$$

$$Z(\zeta) = e^{\gamma\zeta} = e^{\gamma_R\zeta}(\cos\gamma_I\zeta + i\ \sin\gamma_I\zeta)$$

are used, the real stresses are

$$\left.\begin{aligned}
\sigma_x &= -Y(\eta)e^{\gamma_R\zeta}\Big\{B_R[P_1(\xi)\cos\gamma_I\zeta - P_2(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[P_2(\xi)\cos\gamma_I\zeta + P_1(\xi)\sin\gamma_I\zeta]\Big\} \\
\sigma_y &= -Y(\eta)e^{\gamma_R\zeta}\Big\{B_R[P_3(\xi)\cos\gamma_I\zeta - P_4(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[P_4(\xi)\cos\gamma_I\zeta + P_3(\xi)\sin\gamma_I\zeta]\Big\} \\
\sigma_z &= Y(\eta)e^{\gamma_R\zeta}\Big\{B_R[P_5(\xi)\cos\gamma_I\zeta - P_6(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[P_6(\xi)\cos\gamma_I\zeta + P_5(\xi)\sin\gamma_I\zeta]\Big\} \\
\tau_{xy} &= L_oY'(\eta)e^{\gamma_R\zeta}\Big\{B_R[Q_1(\xi)\cos\gamma_I\zeta - Q_2(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[Q_2(\xi)\cos\gamma_I\zeta + Q_1(\xi)\sin\gamma_I\zeta]\Big\} \\
\tau_{xz} &= Y(\eta)e^{\gamma_R\zeta}\Big\{B_R[Q_3(\xi)\cos\gamma_I\zeta - Q_4(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[Q_4(\xi)\cos\gamma_I\zeta + Q_3(\xi)\sin\gamma_I\zeta]\Big\} \\
\tau_{yz} &= L_oY'(\eta)e^{\gamma_R\zeta}\Big\{B_R[Q_5(\xi)\cos\gamma_I\zeta - Q_6(\xi)\sin\gamma_I\zeta] \\
&\qquad -B_I[Q_6(\xi)\cos\gamma_I\zeta + Q_5(\xi)\sin\gamma_I\zeta]\Big\}
\end{aligned}\right\} \qquad (28)$$

where $P_1(\xi), \ldots, Q_6(\xi)$ (defined under Definition of Functions) contain the ± sign of equations (26).

The displacements can be found by integrating the strain-displacement equations:

$$\epsilon_x = \frac{\partial\bar{u}}{\partial\xi}\ , \qquad \epsilon_y = \frac{\partial\bar{v}}{\partial\eta}\ , \qquad \epsilon_z = \frac{\partial\bar{w}}{\partial\zeta}$$

The arbitrary functions resulting from the integration are taken to be zero to satisfy the shear strain-displacement equations. The dimensionless displacements are given by

. .

$$
\left.
\begin{aligned}
\bar{u} &= -\frac{1+\nu}{E}\, Y(\eta) e^{\gamma_R \zeta} \Big\{ B_R[U_1(\xi)\cos\gamma_I\zeta - U_2(\xi)\sin\gamma_I\zeta] \\
&\qquad - B_I[U_2(\xi)\cos\gamma_I\zeta + U_1(\xi)\sin\gamma_I\zeta] \Big\} \\
\bar{v} &= -\frac{1+\nu}{E}\, \beta^2 L_o^2 \left[\int Y\, d\eta\right] e^{\gamma_R \zeta} \Big\{ B_R[U_3(\xi)\cos\gamma_I\zeta - U_4(\xi)\sin\gamma_I\zeta] \\
&\qquad - B_I[U_4(\xi)\cos\gamma_I\zeta + U_3(\xi)\sin\gamma_I\zeta] \Big\} \\
\bar{w} &= \frac{1+\nu}{E}\, Y(\eta) e^{\gamma_R \zeta} \Big\{ B_R[Q_5(\xi)\cos\gamma_I\zeta - Q_6(\xi)\sin\gamma_I\zeta] \\
&\qquad - B_I[Q_6(\xi)\cos\gamma_I\zeta + Q_5(\xi)\sin\gamma_I\zeta] \Big\}
\end{aligned}
\right\} \tag{29}
$$

($U_1(\xi)$, . . . , $U_4(\xi)$ are defined in the next section).

The functions $P_1(\xi)$, . . . , $U_4(\xi)$ depend on which of equations (26) is chosen and, depending on the choice, the resulting stresses and displacements will have different symmetry characteristics. For

$\sin\lambda = \lambda$:

$\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \bar{v}, \bar{w}$ are odd about $\xi = \frac{1}{2}$, that is, $\sigma_x(\xi,\eta,\zeta) = -\sigma_x(1-\xi,\eta,\zeta)$

$\tau_{xz}, \tau_{xy}, \bar{u}$ are even about $\xi = \frac{1}{2}$, that is, $\tau_{xz}(\xi,\eta,\zeta) = \tau_{xz}(1-\xi,\eta,\zeta)$

$\sin\lambda = -\lambda$:

$\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \bar{v}, \bar{w}$ are even about $\xi = \frac{1}{2}$

$\tau_{xz}, \tau_{xy}, \bar{u}$ are odd about $\xi = \frac{1}{2}$

Since λ is determined from one of equations (26) and the relation $\lambda^2 = \gamma^2 - \beta^2 L_o^2$ must be satisfied, only one of the three parameters γ, β, and λ remains arbitrary. In the following, the parameter β and $Y(\eta)$ are chosen to satisfy boundary conditions at $\eta = 0, 1$. Two obvious choices are

$$Y = \cos\beta\eta \qquad \beta = n\pi\ , \qquad n = 0,1,2,\ .\ .\ .$$

$$\bar{v} = \tau_{xy} = \tau_{yz} = 0 \qquad \text{at} \quad \eta = 0,1$$

and

$$Y = \sin \beta\eta \qquad \beta = n\pi , \qquad n = 1,2, . . .$$

$$\sigma_y = \bar{u} = \bar{w} = 0 \qquad \text{at} \quad \eta = 0,1$$

Since each of equations (26) contains infinitely many roots and n can have any integer value, the stresses and displacements can be summed to give, for example,

$$\sigma_x = - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \cos \beta_n \eta e^{\gamma_{Rmn}\zeta} \Big\{ B_{Rmn}[P_{1m}(\xi)\cos \gamma_{Imn}\zeta - P_{2m}(\xi)\sin \gamma_{Imn}\zeta]$$

$$-B_{Imn}[P_{2m}(\xi)\cos \gamma_{Imn}\zeta - P_{1m}(\xi)\sin \gamma_{Imn}\zeta]\Big\}$$

Two sets of arbitrary constants are available to satisfy boundary conditions on the surface ζ = constant, so one could specify two stress components on $\zeta = \zeta_o$ and would have to accept whatever the solution gave for the other stress applied to the ζ = constant surface. The functions $\cos \beta_n \eta$ are orthogonal so a function can be easily represented in the η direction; but the functions $P_{1m}(\xi)$, etc., are not orthogonal so it could be difficult to approximate a specified stress in the ξ direction. To avoid this difficulty in the present report, the numerical work is confined to single-term solutions given by equations (28).

Definition of functions.- The functions that comprise equations (28) and (29) are defined in this section. First, the constant parameters are

$$\gamma_I^2 = \frac{1}{2}\left[\sqrt{(\beta^2 L_o^2 + \lambda_1)^2 + \lambda_2^2} - (\beta^2 L_o^2 + \lambda_1)\right]$$

$$\gamma_R = \frac{\lambda_R \lambda_I}{\gamma_I} , \qquad \lambda_1 = \lambda_R^2 - \lambda_I^2 , \qquad \lambda_2 = 2\lambda_R \lambda_I$$

The functions that depend on the choice of equations (26) are

$$P_R(\xi) = \mp(1 - \xi)\sin \lambda_R\xi \cosh \lambda_I\xi + \xi \sin \lambda_R(1 - \xi)\cosh \lambda_I(1 - \xi)$$

$$P_I(\xi) = \mp(1 - \xi)\cos \lambda_R\xi \sinh \lambda_I\xi + \xi \cos \lambda_R(1 - \xi)\sinh \lambda_I(1 - \xi)$$

$$Q_R(\xi) = \mp\cos \lambda_R\xi \cosh \lambda_I\xi + \cos \lambda_R(1 - \xi)\cosh \lambda_I(1 - \xi)$$

$$Q_I(\xi) = \pm\sin \lambda_R\xi \sinh \lambda_I\xi - \sin \lambda_R(1 - \xi)\sinh \lambda_I(1 - \xi)$$

$$T_R(\xi) = \pm\sin \lambda_R\xi \cosh \lambda_I\xi + \sin \lambda_R(1 - \xi)\cosh \lambda_I(1 - \xi)$$

$$T_I(\xi) = \pm\cos \lambda_R\xi \sinh \lambda_I\xi + \cos \lambda_R(1 - \xi)\sinh \lambda_I(1 - \xi)$$

$$U_R(\xi) = \mp(1 - \xi)\cos \lambda_R\xi \cosh \lambda_I\xi - \xi \cos \lambda_R(1 - \xi)\cosh \lambda_I(1 - \xi)$$

$$U_I(\xi) = \pm(1 - \xi)\sin \lambda_R\xi \sinh \lambda_I\xi + \xi \sin \lambda_R(1 - \xi)\sinh \lambda_I(1 - \xi)$$

where the sign corresponding to that of equations (26) must be used (i.e., $\mp \Leftrightarrow \pm\lambda = \sin \lambda$). The functions appearing in equations (28) and (29) are

$$P_1(\xi) = (\lambda_1^2 - \lambda_2^2)P_R(\xi) - 2\lambda_1\lambda_2P_I(\xi)$$

$$P_2(\xi) = 2\lambda_1\lambda_2P_R(\xi) + (\lambda_1^2 - \lambda_2^2)P_I(\xi)$$

$$P_3(\xi) = \beta^2L_o^2[\lambda_1P_R(\xi) - \lambda_2P_I(\xi)] + 2\left\{\lambda_R[(1 - \nu)\beta^2L_o^2 - \nu\lambda_1] + \nu\lambda_I\lambda_2\right\}Q_R(\xi)$$
$$-2\left\{\lambda_I[(1 - \nu)\beta^2L_o^2 - \nu\lambda_1] - \nu\lambda_R\lambda_2\right\}Q_I(\xi)$$

$$P_4(\xi) = \beta^2L_o^2[\lambda_2P_R(\xi) + \lambda_1P_I(\xi)] + 2\left\{\lambda_I[(1 - \nu)\beta^2L_o^2 - \nu\lambda_1] - \nu\lambda_R\lambda_2\right\}Q_R(\xi)$$
$$+2\left\{\lambda_R[(1 - \nu)\beta^2L_o^2 - \nu\lambda_1] + \nu\lambda_I\lambda_2\right\}Q_I(\xi)$$

$$P_5(\xi) = (\lambda_1^2 + \lambda_1L_o^2\beta^2 - \lambda_2^2)P_R(\xi) - \lambda_2(\beta^2L_o^2 + 2\lambda_1)P_I(\xi)$$
$$+2\left\{\lambda_R[\lambda_1 + (1 - \nu)\beta^2L_o^2] - \lambda_I\lambda_2\right\}Q_R(\xi)$$
$$-2\left\{\lambda_I[\lambda_1 + (1 - \nu)\beta^2L_o^2] + \lambda_R\lambda_2\right\}Q_I(\xi)$$

$$P_6(\xi) = \lambda_2(\beta^2L_o^2 + 2\lambda_1)P_R(\xi) + (\lambda_1^2 + \lambda_1\beta^2L_o^2 - \lambda_2^2)P_I(\xi)$$
$$+2\left\{\lambda_I[\lambda_1 + (1 - \nu)\beta^2L_o^2] + \lambda_R\lambda_2\right\}Q_R(\xi)$$
$$+2\left\{\lambda_R[\lambda_1 + (1 - \nu)\beta^2L_o^2] - \lambda_I\lambda_2\right\}Q_I(\xi)$$

$$Q_1(\xi) = \lambda_1T_R(\xi) - \lambda_2T_I(\xi) + (\lambda_R\lambda_1 - \lambda_I\lambda_2)U_R(\xi) - (\lambda_1\lambda_I + \lambda_R\lambda_2)U_I(\xi)$$

$$Q_2(\xi) = \lambda_2T_R(\xi) + \lambda_1T_I(\xi) + (\lambda_I\lambda_1 + \lambda_R\lambda_2)U_R(\xi) + (\lambda_1\lambda_R - \lambda_2\lambda_I)U_I(\xi)$$

$$Q_3(\xi) = (\gamma_R\lambda_1 - \gamma_I\lambda_2)T_R(\xi) - (\gamma_I\lambda_1 + \gamma_R\lambda_2)T_I(\xi)$$
$$+[\lambda_1(\gamma_R\lambda_R - \gamma_I\lambda_I) - \lambda_2(\lambda_R\gamma_I + \lambda_I\gamma_R)]U_R(\xi)$$
$$-[\lambda_1(\gamma_R\lambda_I + \gamma_I\lambda_R) + \lambda_2(\lambda_R\gamma_R - \lambda_I\gamma_I)]U_I(\xi)$$

$$Q_4(\xi) = (\gamma_I\lambda_1 + \gamma_R\lambda_2)T_R(\xi) + (\gamma_R\lambda_1 - \gamma_I\lambda_2)T_I(\xi)$$
$$+[\lambda_1(\gamma_I\lambda_R + \gamma_R\lambda_I) + \lambda_2(\gamma_R\lambda_R - \gamma_I\lambda_I)]U_R(\xi)$$
$$+[\lambda_1(\gamma_R\lambda_R - \gamma_I\lambda_I) - \lambda_2(\gamma_I\lambda_R + \gamma_R\lambda_I)]U_I(\xi)$$

$$Q_5(\xi) = (\gamma_R\lambda_1 - \gamma_I\lambda_2)P_R(\xi) - (\gamma_R\lambda_2 + \gamma_I\lambda_1)P_I(\xi) + 2(1 - \nu)(\lambda_R\gamma_R - \lambda_I\gamma_I)Q_R(\xi)$$
$$-2(1 - \nu)(\lambda_R\gamma_I + \lambda_I\gamma_R)Q_I(\xi)$$

$$Q_6(\xi) = (\gamma_R\lambda_2 + \gamma_I\lambda_1)P_R(\xi) + (\gamma_R\lambda_1 - \gamma_I\lambda_2)P_I(\xi) + 2(1 - \nu)(\gamma_I\lambda_R + \gamma_R\lambda_I)Q_R(\xi)$$
$$+2(1 - \nu)(\gamma_R\lambda_R - \gamma_I\lambda_I)Q_I(\xi)$$

$$U_1(\xi) = (1 - 2\nu)[\lambda_1T_R(\xi) - \lambda_2T_I(\xi)] - (\lambda_1\lambda_R - \lambda_2\lambda_I)U_R(\xi) + (\lambda_1\lambda_I + \lambda_2\lambda_R)U_I(\xi)$$

$$U_2(\xi) = (1 - 2\nu)[\lambda_2T_R(\xi) + \lambda_1T_I(\xi)] - (\lambda_1\lambda_I + \lambda_2\lambda_R)U_R(\xi) - (\lambda_1\lambda_R - \lambda_2\lambda_I)U_I(\xi)$$

$$U_3(\xi) = \lambda_1P_R(\xi) - \lambda_2P_I(\xi) + 2(1 - \nu)[\lambda_RQ_R(\xi) - \lambda_IQ_I(\xi)]$$

$$U_4(\xi) = \lambda_2P_R(\xi) + \lambda_1P_I(\xi) + 2(1 - \nu)[\lambda_IQ_R(\xi) + \lambda_RQ_I(\xi)]$$

NUMERICAL EXAMPLES

The numerical evaluation of any of the solutions given by equations (28) is straightforward and is easily carried out, but only the solution for the lowest value of λ and $\beta = 0,\pi$ is presented here. The boundary conditions satisfied by the solutions considered are

$\underline{\xi = 0,1}$:

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0$$

$\underline{\eta = 0,1}$:

$$v = \tau_{xy} = \tau_{yz} = 0$$

$\underline{\zeta = 0}$:

$$\sigma_z = \cos\,\beta\eta[B_RP_5(\xi) - B_IP_6(\xi)]$$
$$\tau_{xz} = \cos\,\beta\eta[B_RQ_3(\xi) - B_IQ_4(\xi)]$$
$$\tau_{yz} = -\beta L_o\,\sin\,\beta\eta[B_RQ_5(\xi) - B_IQ_6(\xi)]$$

$\underline{\zeta \to \infty}$:

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Since the condition of decreasing stress with increasing z is imposed, γ_I is chosen to be negative. The other parameters are

$$\sin \lambda = -\lambda$$

$$\beta = 0, \pi \ (\beta = 0 \text{ corresponds to plane strain})$$

$$Y(\eta) = \cos \beta\eta$$

$$L_o = 0.01,\ 0.5,\ 1.0$$

Combinations of these parameters are calculated for the following two cases:

Case 1: $B_R \neq 0$, $B_I = 0$ $(\sigma_{x1}, \sigma_{y1}, \text{etc.})$

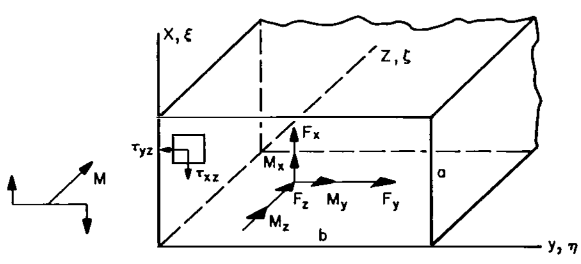
Case 2: $B_R = 0$, $B_I \neq 0$ $(\sigma_{x2}, \sigma_{y2}, \text{etc.})$

For both cases, the applied stresses on the surface $\zeta = 0$ can produce resultant forces and moments that can be found from the following relations:

$$F_\xi = \int_0^1 \int_0^1 -\tau_{xz}\Big|_{\zeta=0} d\xi\, d\eta \qquad M_\xi = -\int_0^1 \int_0^1 \left(\eta - \frac{1}{2}\right) \sigma_z\Big|_{\zeta=0} d\xi\, d\eta$$

$$F_\eta = \int_0^1 \int_0^1 -\tau_{yz}\Big|_{\zeta=0} d\xi\, d\eta \qquad M_\eta = \int_0^1 \int_0^1 \left(\xi - \frac{1}{2}\right) \sigma_z\Big|_{\zeta=0} d\xi\, d\eta$$

$$F_\zeta = \int_0^1 \int_0^1 -\sigma_z\Big|_{\zeta=0} d\xi\, d\eta \qquad M_\zeta = \int_0^1 \int_0^1 \left(\eta - \frac{1}{2}\right)\tau_{xz}\Big|_{\zeta=0} - L_o\left(\xi - \frac{1}{2}\right)\tau_{yz}\Big|_{\zeta=0} d\xi\, d\eta$$



Sketch (a)

where

$$F_\xi,\ F_\eta,\ F_\zeta = \frac{F_x}{ab},\ \frac{F_y}{ab},\ \frac{F_z}{ab}$$

$$M_\xi,\ M_\eta,\ M_\zeta = \frac{M_x}{ab^2},\ \frac{M_y}{a^2b},\ \frac{M_z}{ab^2}$$

and the notation is shown in sketch (a). For the given choice of parameters, the resultants are

$$F_\xi = F_\zeta = 0, \qquad M_\eta = M_\zeta = 0$$

$$F_\eta = -4\nu L_o[(-1)^n - 1][B_R(\gamma_R\lambda_R - \gamma_I\lambda_I) - B_I(\gamma_I\lambda_R + \gamma_R\lambda_I)]$$

$$M_\xi = -4\nu[(-1)^n - 1](B_R\lambda_R - B_I\lambda_I)$$

For plane strain ($n = 0$), all resultant forces and moments are zero.

The calculated stresses for cases 1 and 2 are shown in figures 1 through 12 and several linear combinations of the two cases are shown in figures 13(a) to (d) and 14. Although some discussion of the limited numerical work is given here, a much closer examination of the entire family of solutions presented here would undoubtedly yield much useful information.

Discussion of Numerical Results

Some general comments should first be made about the figures. All the stresses were calculated at the points $\xi = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ so the graphs are accurate at those points and the stresses are either even or odd extensions for $0.5 \leq \xi \leq 1$ as indicated on the graphs. Also, all the graphs are independent of η since this coordinate is contained in the abscissas.

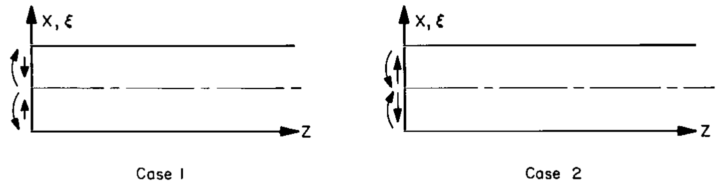
For case 1, the applied stresses are shown by the $\zeta = 0$ curves in figures 1, 3, and 5. The remaining three stress components are shown in figures 7, 9, and 11.

For each stress component the same curve applies for $\beta = 0$ and $\beta = \pi$ when $L_o = 0.01$ ($\tau_{xy} = \tau_{xz} = 0$ as $L_o \rightarrow 0$ or $\beta = 0$) which means that the stresses for $\beta = \pi$ and $L_o = 0.01$ can be represented by $\sigma_{ij} = \bar{\sigma}_{ij} \cos \beta\eta$ where $\bar{\sigma}_{ij}$ are the stresses for the plane strain problem given by $\beta = 0$. In fact, very little difference is apparent up to $L_o \sim 0.1$ (the accuracy is determined by the smallness of βL_o rather than just L_o). This is useful to know since the plane strain solution to this and similar problems can be obtained from two-dimensional theory.

The graphs show the rapid decay of all the stress components as ζ increases. However, σ_{x1} requires a longer distance to reduce its value to a negligible amount because it has its maximum value at about $\zeta = 0.2$ whereas the other components have their maximum values at $\zeta = 0$. At $\zeta = 1$ (a distance equal to the thickness a), σ_{x1} is about 5 percent of its maximum while the other components are about 1 to 2 percent of their maximum value.

A comparison of the curves for $L_o = 1.0$ and $L_o = 0.01$ for $\beta = \pi$ shows that the applied stresses are similar in shape but the magnitude of each component is larger for $L_o = 1.0$. Two of the remaining stresses, σ_{x1} and τ_{xy1}, have similar shapes but σ_{y1} changes both its shape and magnitude. This shows that despite similar applied stress distributions, the internal stress distribution may vary considerably with the thickness ratio.

The applied stresses for case 2 are shown in figures 2, 4, and 6, and the remaining stresses are shown in figures 8, 10, and 12. The main difference between the applied stresses of cases 1 and 2 (figs. 5 and 6) is the distribution of τ_{yz1} and τ_{yz2}. The same comments as for case 1 can be made regarding the similarity between the $\beta = 0$ and $\beta = \pi$ solutions and the decay of stress with increasing ζ except for the behavior of σ_{x2} compared to that of σ_{x1}. In figure 8, one sees that σ_{x2} has its maximum value at $\zeta = 0$, and the maximum value is an order of magnitude larger than that of σ_{x1}. This is true for the solution $\beta = 0$ for which $\tau_{yz1} = \tau_{yz2} = 0$ so the difference in distribution of τ_{yz} does not explain the large maximum value of σ_{x2}. Linear combinations of cases 1 and 2 can be formed to give a better understanding of what governs the behavior of σ_x. Considering only the solutions for $\beta = 0$, one sees that cases 1 and 2 have the resultant forces shown in sketch (b). The shear forces are given by the area $(0 \leq \xi \leq 0.5)$ under the



Sketch (b)

curves for τ_{xz1} and τ_{xz2} for $\zeta = 0$ (figs. 3 and 4). The moments are caused by σ_z, as shown in figures 1 and 2. The relative value of the force and moment can be varied by superposing cases 1 and 2; four examples are shown in figures 13(a) to (h). The values indicated in the figures for the force and moment are not precise since they were determined graphically. In figures 13(a) to (c) the force is kept constant but the moment decreases from a dimensionless value of about 50 in figure 13(a) to zero in figure 13(c). The maximum value of σ_x increases approximately linearly as the moment decreases. Figure 13(d) shows the case for which the force is zero and the moment nonzero, so figures 13(c) and (d) show the uncoupled effect of a force or moment on the stress components regardless of the value of L_o. This points out that it is possible even in thin plates to have large transverse stresses near the edge of the plate, but these stresses, in agreement with Saint Venant's principle, are contained within an edge region whose width about equals the plate thickness. This is clearly seen in figure 14 where σ_x at $\xi = 0.5$ is plotted versus ζ for each example given by figures 13(a) to (d). All the stresses have decreased to small fractions of their maximum values within the distance $\zeta = 1$.

CONCLUDING REMARKS

The set of six fourth-order compatibility equations in terms of the three Maxwell stress functions has been written in the form of three second-order equations with arbitrary functions of integration to simplify the task of finding exact solutions by the inverse method. The solutions studied to date are for body forces and arbitrary functions of integration being zero. The most interesting one found is presented in the form of an infinite family of solutions which gives the stresses in a rectangular solid subjected to applied stresses on two opposite surfaces with two other opposite faces constrained by workless boundary conditions. The remaining two surfaces are free of applied stress. Numerical work is presented to show the characteristics of the solutions and it is worthwhile to note that the stresses are smooth functions throughout the solid which includes the edges and corners. The solutions are not general enough to satisfy an arbitrary distribution of applied stress. It is recommended that further study be devoted to investigating sums of the solutions obtained which may describe additional useful cases.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., 94035, May 29, 1967
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APPENDIX A

DISCUSSION OF COMPATIBILITY EQUATIONS

Since the linear compatibility equations govern the small deformations of any solid, it is worthwhile to see how the relationships between the equations can be used without restricting the strains to any particular stress-strain law. Reference 6 shows that if all six compatibility equations are satisfied on the boundary of a simply connected solid, then satisfying three unrelated compatibility equations throughout the interior of the solid is sufficient to guarantee continuous displacements. In this section, conditions under which three compatibility equations alone are sufficient will be investigated. From the following definitions,

$$\varphi_1 = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{A1}$$

$$\varphi_2 = \frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} \tag{A2}$$

$$\varphi_3 = \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} \tag{A3}$$

$$\varphi_4 = 2\,\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{A4}$$

$$\varphi_5 = 2\,\frac{\partial^2 \epsilon_y}{\partial x\, \partial z} - \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{A5}$$

$$\varphi_6 = 2\,\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} - \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{A6}$$

the six compatibility equations are then

$$\varphi_1 = 0\,, \qquad \varphi_2 = 0\,, \qquad \varphi_3 = 0 \tag{A7}$$

$$\varphi_4 = 0\,, \qquad \varphi_5 = 0\,, \qquad \varphi_6 = 0 \tag{A8}$$

The identities discussed in references 1 to 3 that interrelate equations (A1) to (A6) are

$$\frac{\partial \varphi_4}{\partial y} + \frac{\partial \varphi_5}{\partial x} = 2\,\frac{\partial \varphi_1}{\partial z} \tag{A9}$$

$$\frac{\partial\varphi_4}{\partial z} + \frac{\partial\varphi_6}{\partial x} = 2\,\frac{\partial\varphi_2}{\partial y} \tag{A10}$$

$$\frac{\partial\varphi_5}{\partial z} + \frac{\partial\varphi_6}{\partial y} = 2\,\frac{\partial\varphi_3}{\partial x} \tag{A11}$$

or an alternative form is

$$\frac{\partial^2\varphi_1}{\partial z^2} + \frac{\partial^2\varphi_2}{\partial y^2} - \frac{\partial^2\varphi_3}{\partial x^2} = \frac{\partial^2\varphi_4}{\partial y\,\partial z} \tag{A12}$$

$$\frac{\partial^2\varphi_1}{\partial z^2} - \frac{\partial^2\varphi_2}{\partial y^2} + \frac{\partial^2\varphi_3}{\partial x^2} = \frac{\partial^2\varphi_5}{\partial x\,\partial z} \tag{A13}$$

$$-\frac{\partial^2\varphi_1}{\partial z^2} + \frac{\partial^2\varphi_2}{\partial y^2} + \frac{\partial^2\varphi_3}{\partial x^2} = \frac{\partial^2\varphi_6}{\partial x\,\partial y} \tag{A14}$$

Since φ_4, φ_5, and φ_6 are so closely related to φ_1, φ_2, and φ_3, it seems possible that under certain conditions, strain components that satisfy three compatibility equations will automatically satisfy the other three equations. To explore this, assume that the strain components satisfy the three compatibility equations

$$\varphi_4 = 0\ , \qquad \varphi_5 = 0\ , \qquad \varphi_6 = 0 \tag{A15}$$

Substituting equations (A15) into equations (A9) to (A11) gives

$$\frac{\partial\varphi_1}{\partial z} = 0\ , \qquad \frac{\partial\varphi_2}{\partial y} = 0\ , \qquad \frac{\partial\varphi_3}{\partial x} = 0 \tag{A16}$$

Integrating equations (A16) gives

$$\left.\begin{aligned}
\varphi_1 &= \frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} = f(x,y) \\
\varphi_2 &= \frac{\partial^2\epsilon_x}{\partial z^2} + \frac{\partial^2\epsilon_z}{\partial x^2} - \frac{\partial^2\gamma_{xz}}{\partial x\,\partial z} = g(x,z) \\
\varphi_3 &= \frac{\partial^2\epsilon_y}{\partial z^2} + \frac{\partial^2\epsilon_z}{\partial y^2} - \frac{\partial^2\gamma_{yz}}{\partial y\,\partial z} = h(y,z)
\end{aligned}\right\} \tag{A17}$$

Equations (A17) show that the strains which satisfy equations (A8) may not satisfy equations (A7), but the residues, at most, are the functions f, g, and h. Different forms of the strains can be studied to see what forms can or cannot contribute to the residue. It follows from equations (A17) that some forms that could contribute are

$$\left.\begin{aligned}
\epsilon_x &= f_1(x,y) + g_1(x,z) \\
\epsilon_y &= f_2(x,y) + g_2(y,z) \\
\epsilon_z &= f_3(x,y) + g_3(y,z) \\
\gamma_{xy} &= f_4(x,y) \;, \qquad \gamma_{xz} = f_5(x,z) \;, \qquad \gamma_{yz} = f_6(y,z)
\end{aligned}\right\} \tag{A18}$$

In order to find forms that cannot contribute to the residue and, therefore, will satisfy equations (A7), consider those strains that can be represented by a power series of the form

$$\left.\begin{aligned}
\epsilon_x &= \sum_{k,m,n=o}^{\infty} a_{kmn}x^k y^m z^n & \qquad \gamma_{xy} &= \sum_{k,m,n=o}^{\infty} \alpha_{kmn}x^k y^m z^n \\
\epsilon_y &= \sum_{k,m,n=o}^{\infty} b_{kmn}x^k y^m z^n & \qquad \gamma_{xz} &= \sum_{k,m,n=o}^{\infty} \beta_{kmn}x^k y^m z^n \\
\epsilon_z &= \sum_{k,m,n=o}^{\infty} c_{kmn}x^k y^m z^n & \qquad \gamma_{yz} &= \sum_{k,m,n=o}^{\infty} \gamma_{kmn}x^k y^m z^n
\end{aligned}\right\} \tag{A19}$$

where

$$\sum_{k,m,n=o}^{\infty} = \sum_{k=o}^{\infty} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty}$$

Equations (A1) to (A3) become

$$\left.\begin{aligned}
\varphi_1 &= \sum_{k,m,n=o}^{\infty} \varphi_{1_{kmn}} = \sum_{k,m,n=o}^{\infty} [a_{k,m+2,n}(m+1)(m+2) + b_{k+2,m,n}(k+1)(k+2) \\
&\quad -\alpha_{k+1,m+1,n}(k+1)(m+1)]x^k y^m z^n \\
\varphi_2 &= \sum_{k,m,n=o}^{\infty} \varphi_{2_{kmn}} = \sum_{k,m,n=o}^{\infty} [a_{k,m,n+2}(n+1)(n+2) + c_{k+2,m,n}(k+1)(k+2) \\
&\quad -\beta_{k+1,m,n+1}(k+1)(n+1)]x^k y^m z^n \\
\varphi_3 &= \sum_{k,m,n=o}^{\infty} \varphi_{3_{kmn}} = \sum_{k,m,n=o}^{\infty} [b_{k,m,n+2}(n+1)(n+2) + c_{k,m+2,n}(m+1)(m+2) \\
&\quad -\gamma_{k,m+1,n+1}(m+1)(n+1)]x^k y^m z^n
\end{aligned}\right\} \tag{A20}$$

where the indices have been changed so that terms of like exponents can be grouped. Since the strains given by equations (A19) are assumed to satisfy equations (A8), equations (A16) are satisfied and can be written as

$$\left.\begin{aligned}
nz^{-1}\varphi_{1_{kmn}} &= 0 \\
my^{-1}\varphi_{2_{kmn}} &= 0 \\
kx^{-1}\varphi_{3_{kmn}} &= 0
\end{aligned}\right\} \tag{A21}$$

Equations (A21) must be satisfied for all values of x, y, and z so that

$$\left.\begin{aligned}
\varphi_{1_{kmn}} &= 0 \qquad k = 0,1,2, \ldots; \qquad m = 0,1,2, \ldots; \qquad n = 1,2, \ldots \\
\varphi_{2_{kmn}} &= 0 \qquad k = 0,1,2, \ldots; \qquad m = 1,2, \ldots; \qquad n = 0,1,2, \ldots \\
\varphi_{3_{kmn}} &= 0 \qquad k = 1,2, \ldots; \qquad m = 0,1,2, \ldots; \qquad n = 0,1,2, \ldots
\end{aligned}\right\}$$

(A22)

For equations (A7) to be satisfied, it must also be true that

$$\varphi_{1_{kmo}} = 0 \ , \qquad \varphi_{2_{kon}} = 0 \ , \qquad \varphi_{3_{omn}} = 0 \tag{A23}$$

From equations (A20), it is seen that equations (A23) can be satisfied by requiring that, for $k,m,n = 0,1,2, \ . \ . \ .,$

$$\left.\begin{array}{l} a_{k,m+2,o}(m+1)(m+2) + b_{k+2,m,o}(k+1)(k+2) - \alpha_{k+1,m+1,o}(k+1)(m+1) = 0 \\ \\ a_{k,o,n+2}(n+1)(n+2) + c_{k+2,o,n}(k+1)(k+2) - \beta_{k+1,o,n+1}(k+1)(n+1) = 0 \\ \\ b_{o,m,n+2}(n+1)(n+2) + c_{o,m+2,n}(m+1)(m+2) - \gamma_{o,m+1,n+1}(m+1)(n+1) = 0 \end{array}\right\} \tag{A24}$$

Equations (A24) can be satisfied if the coefficients are combined so that the sum is zero or by each coefficient $a, b, \ . \ . \ ., \gamma$ being zero. An example of a strain that satisfies this latter condition would be if the strain ϵ_x were of the form

$$\epsilon_x = A \cos x \sin y \sin z$$

which has a series expansion where $a_{k,m,o} = a_{k,o,n} = 0$.

The conclusion to be drawn from this discussion is that if the strains are chosen so that their power series expansions exist and the coefficients satisfy equations (A24), then equations (A8) are sufficient to guarantee that all six compatibility equations are satisfied.

One could have assumed first that the strains satisfied equations (A7) and then have looked for forms that would also satisfy equations (A8). By the same procedure, equations analogous to equations (A24) can easily be found; but since equations (A12) to (A14) are of second order, there are six equations between the coefficients $a, b, \ . \ . \ ., \gamma$.

APPENDIX B

DERIVATION OF EQUATIONS

Since some manipulation is required to write the compatibility equations in the form given by equations (6) and (7), it seems appropriate to show a detailed derivation.

By Hooke's law the strains can be found in terms of the stress functions. Substituting equations (4) into equations (2) gives

$$\epsilon_x = \frac{1}{E}\left[\psi_{2,zz} + \psi_{3,yy} - \int X\,dx - \nu\left(\psi_{1,zz} + \psi_{3,xx} - \int Y\,dy + \psi_{1,yy} + \psi_{2,xx} - \int Z\,dz\right)\right] \tag{B1}$$

$$\gamma_{xy} = -\frac{2(1+\nu)}{E}\psi_{3,xy} \tag{B2}$$

The other strains are symmetric with equations (B1) and (B2) and can be obtained by permutation of the coordinates. The strains can be substituted into the first compatibility equation given by

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = 0 \tag{B3}$$

which gives

$$\frac{\partial^2}{\partial y^2}\left[\psi_{2,zz} + \psi_{3,yy} + \psi_{3,xx} - \nu(\nabla^2\psi_1) - \int X\,dx + \nu\left(\int Y\,dy + \int Z\,dz\right)\right]$$

$$+ \frac{\partial^2}{\partial x^2}\left[\psi_{1,zz} + \psi_{3,xx} + \psi_{3,yy} - \nu(\nabla^2\psi_2) - \int Y\,dy + \nu\left(\int X\,dx + \int Z\,dz\right)\right] = 0 \tag{B4}$$

By definition,

$$\nabla^2\psi_i = \psi_{i,xx} + \psi_{i,yy} + \psi_{i,zz} \tag{B5}$$

Using equation (B5), one can write equation (B4) as

$$\frac{\partial^2}{\partial y^2}\left[\nu\nabla^2\psi_1 - \nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{2,xx} + \psi_{2,yy} + \psi_{3,zz}) + \int X\,dx - \nu\left(\int Y\,dy + \int Z\,dz\right)\right] + \frac{\partial^2}{\partial x^2}\left[-\nabla^2\psi_1 + \nu\nabla^2\psi_2 - \nabla^2\psi_3 + (\psi_{1,xx} + \psi_{1,yy} + \psi_{3,zz}) + \int Y\,dy - \nu\left(\int X\,dx + \int Z\,dz\right)\right] = 0 \qquad \text{(B6)}$$

Since $(\partial^2/\partial y^2)(\psi_{2,xx}) = (\partial^2/\partial x^2)(\psi_{2,yy})$, etc., these terms can be interchanged to give the form, which is the first of equations (6). The next two compatibility equations can be obtained by permutation of the coordinates. The fourth equation is

$$2\,\frac{\partial^2\epsilon_x}{\partial y\,\partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{xz}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) = 0$$

Substituting the strains gives

$$2\left[\psi_{2,yzzz} + \psi_{3,yyyz} - \frac{\partial^2}{\partial y\,\partial z}\int X\,dx - \nu\left(\psi_{1,yzzz} + \psi_{3,xxyz} - \frac{\partial^2}{\partial y\,\partial z}\int Y\,dy + \psi_{1,yyyz} + \psi_{2,xxyz} - \frac{\partial^2}{\partial y\,\partial z}\int Z\,dz\right)\right] + 2(1+\nu)(-\psi_{1,xxyz} + \psi_{2,xxyz} + \psi_{3,xxyz}) = 0$$

or

$$2\,\frac{\partial^2}{\partial y\,\partial z}\left[\psi_{2,zz} + \psi_{3,yy} - \psi_{1,xx} + \psi_{2,xx} + \psi_{3,xx} - \int X\,dx - \nu\left(\nabla^2\psi_1 - \int Y\,dy - \int Z\,dz\right)\right] = 0 \qquad \text{(B7)}$$

Equation (B5) can be used to reduce equation (B7) to the first of equations (6) and the remaining equations can be obtained by permutation of the coordinates.

REFERENCES

1. Long, R. R.: Mechanics of Solids and Fluids. Prentice-Hall, 1961. p. 83.

2. Flügge, W., ed.: Handbook of Engineering Mechanics. First ed., McGraw-Hill Book Company, 1962, ch. 33.

3. Pearson, C. E.: Theoretical Elasticity. Harvard University Press, 1959, p. 80.

4. Westergaard, H. M.: Theory of Elasticity and Plasticity. Dover Pub., 1952, p. 131.

5. Fadle, Von J.: Die Selbstspannungs - Eigenwertfunktionen der Quadratischer Scheibe. Ingenieur Archiv., vol. 11, 1941, pp. 125-149.

6. Washizu, K.: A Note on the Conditions of Compatibility. J. Math. Phys., vol. 36, no. 4, April 1957, pp. 306-312.

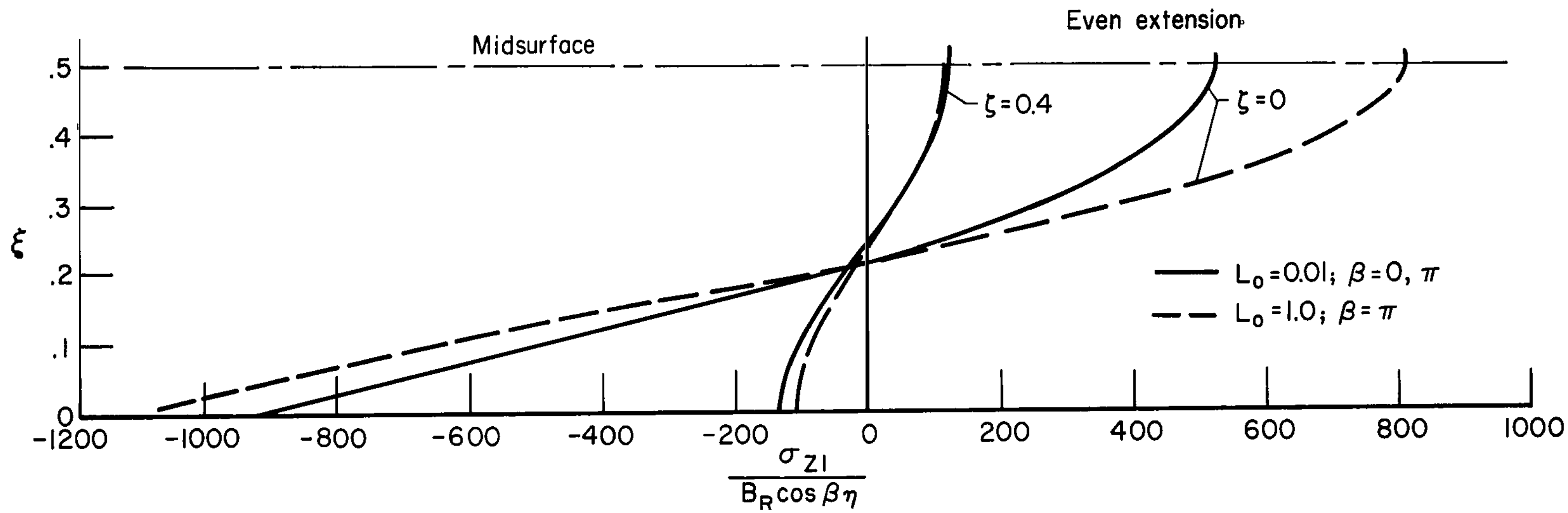

Figure 1.- Variation of σ_{z1} with ξ.

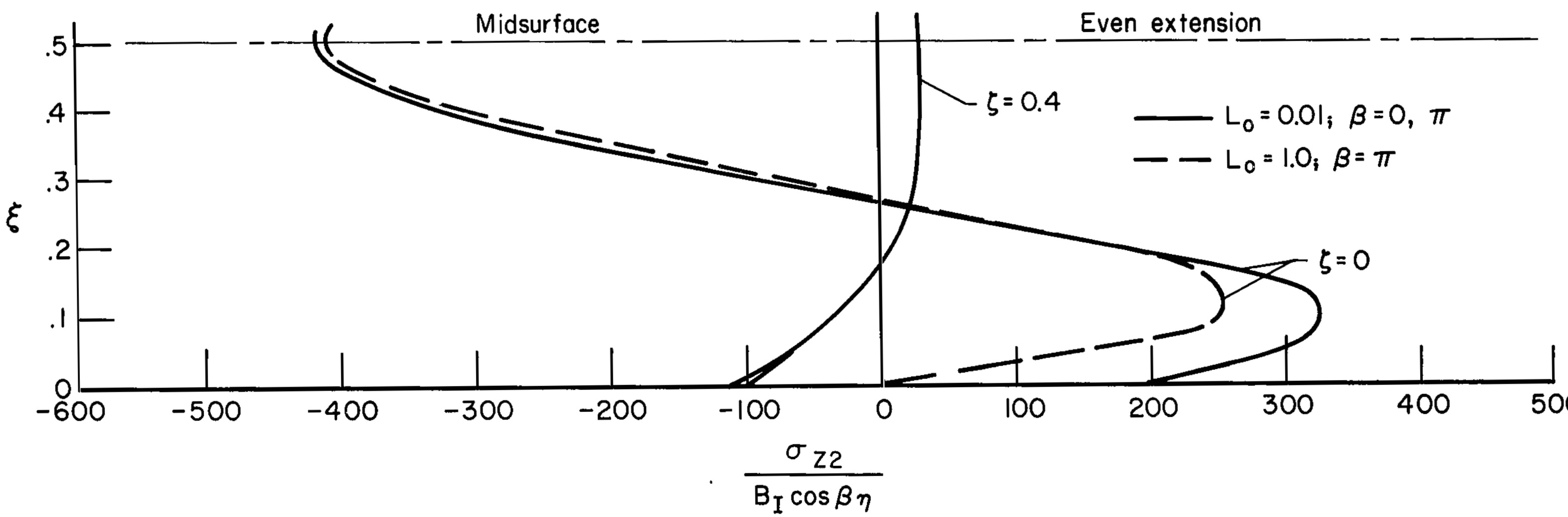

Figure 2.- Variation of σ_{Z2} with ξ.

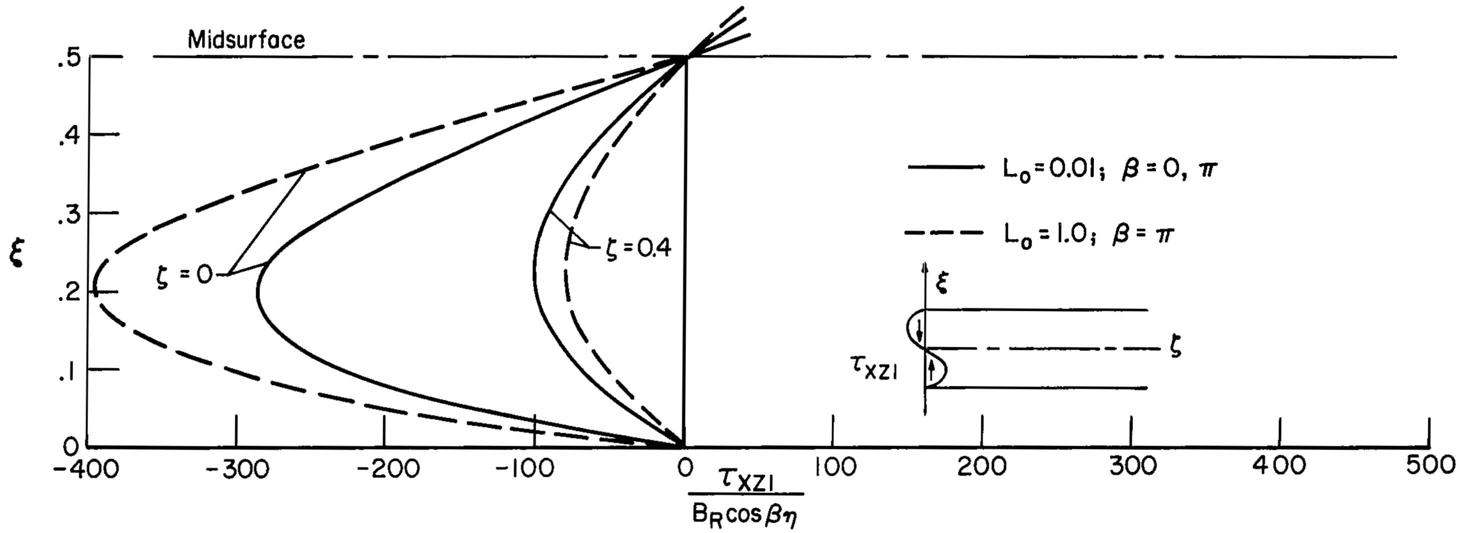

Figure 3.- Variation of τ_{xz1} with ξ.

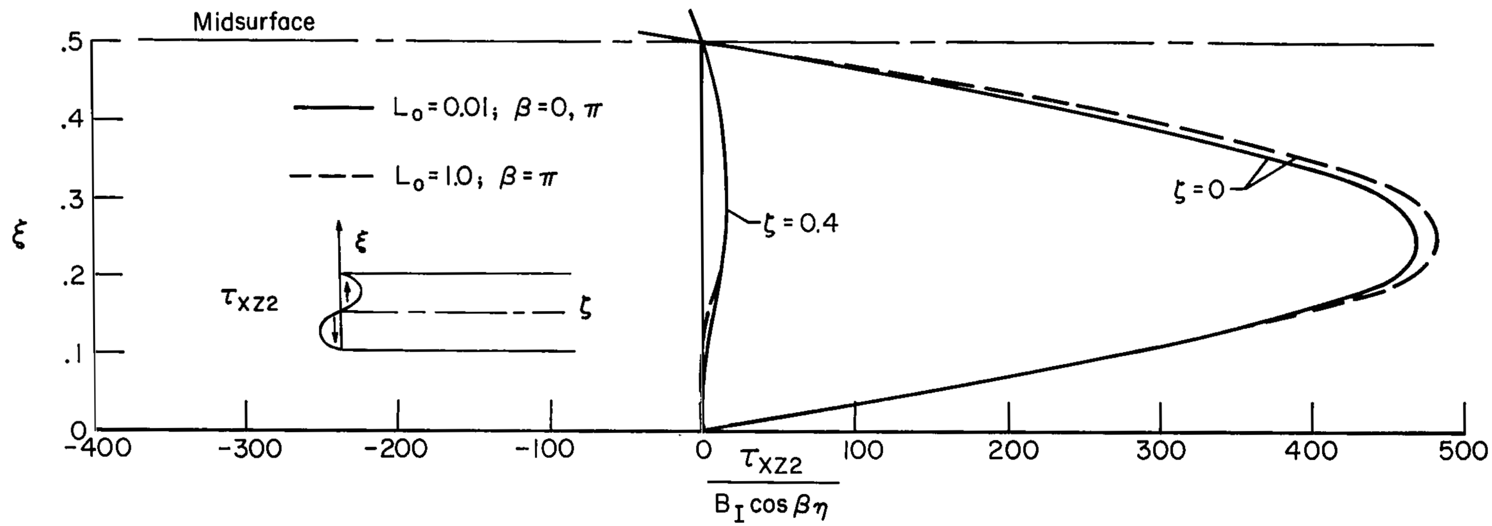

Figure 4.- Variation of τ_{xz2} with ξ.

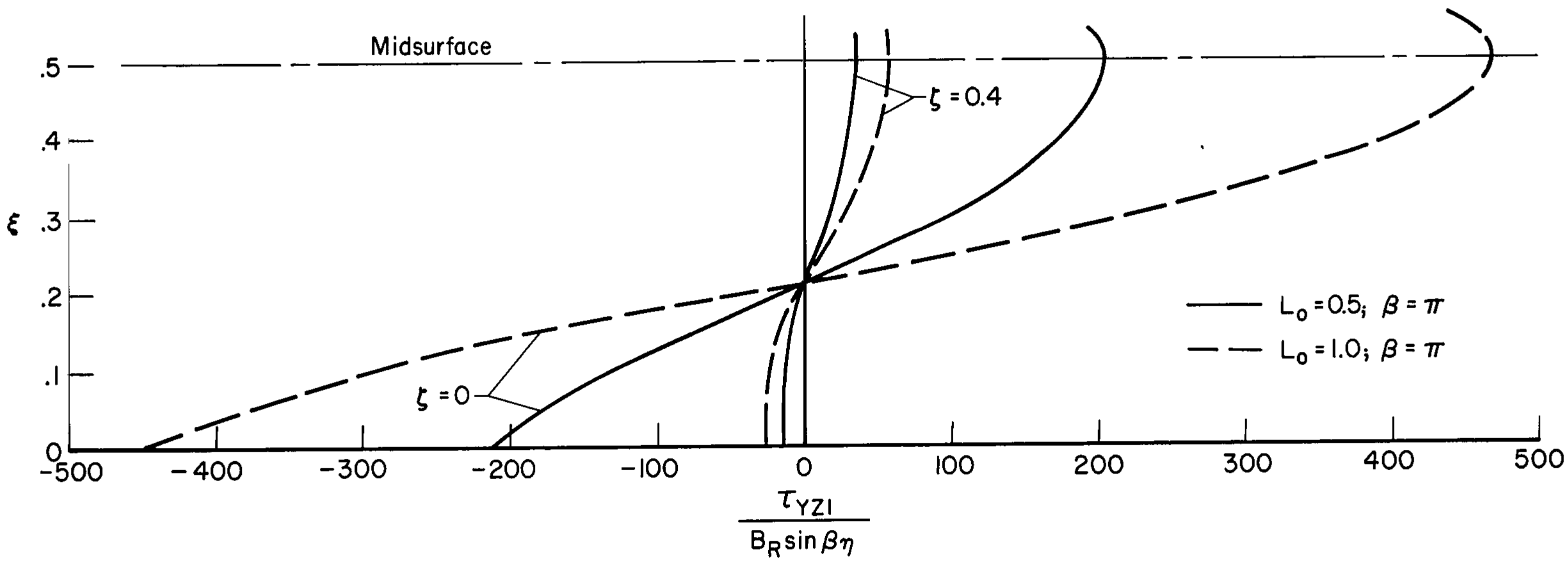

Figure 5.- Variation of τ_{yz1} with ξ.

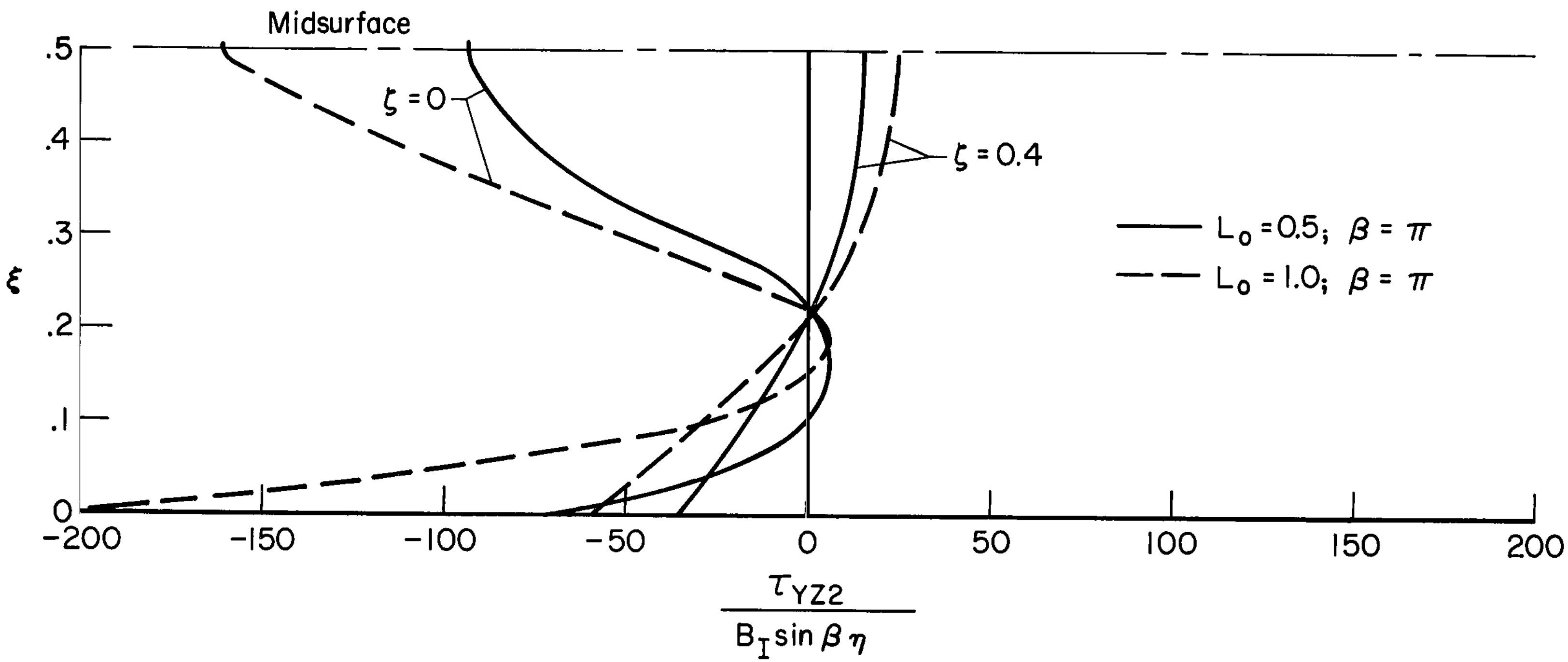

Figure 6.- Variation of τ_{yz2} with ξ.

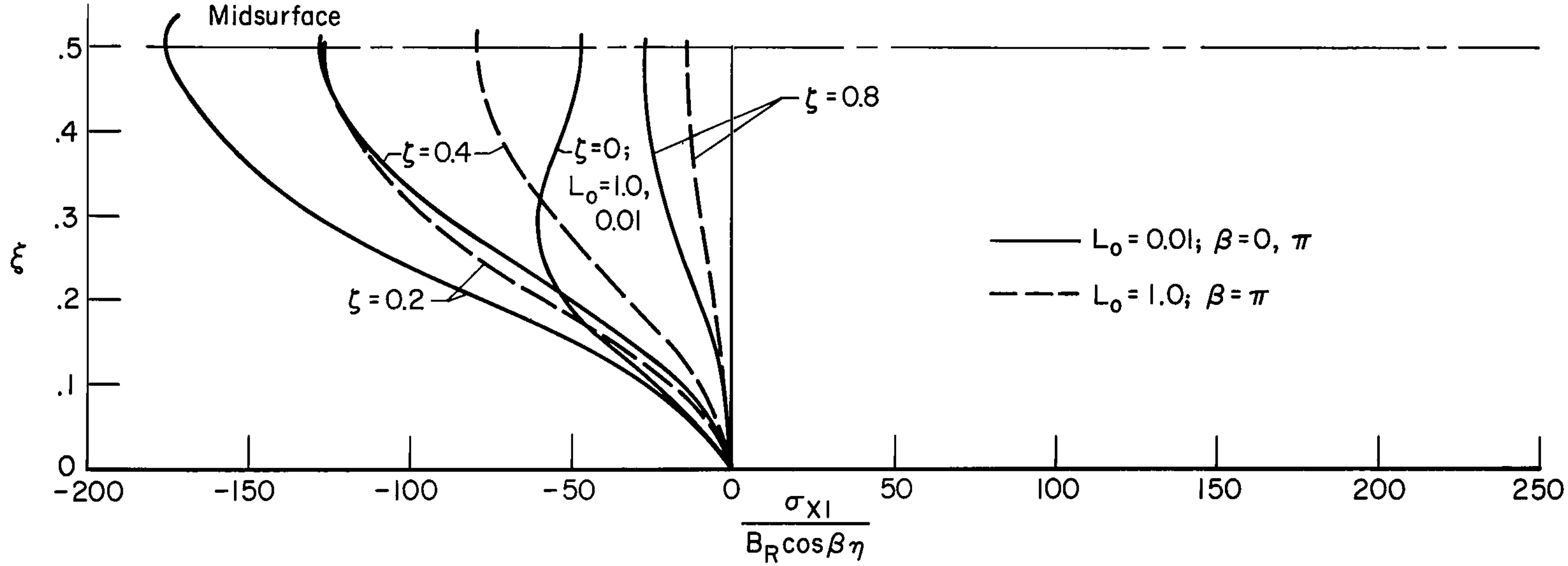

Figure 7.- Variation of σ_{X1} with ξ.

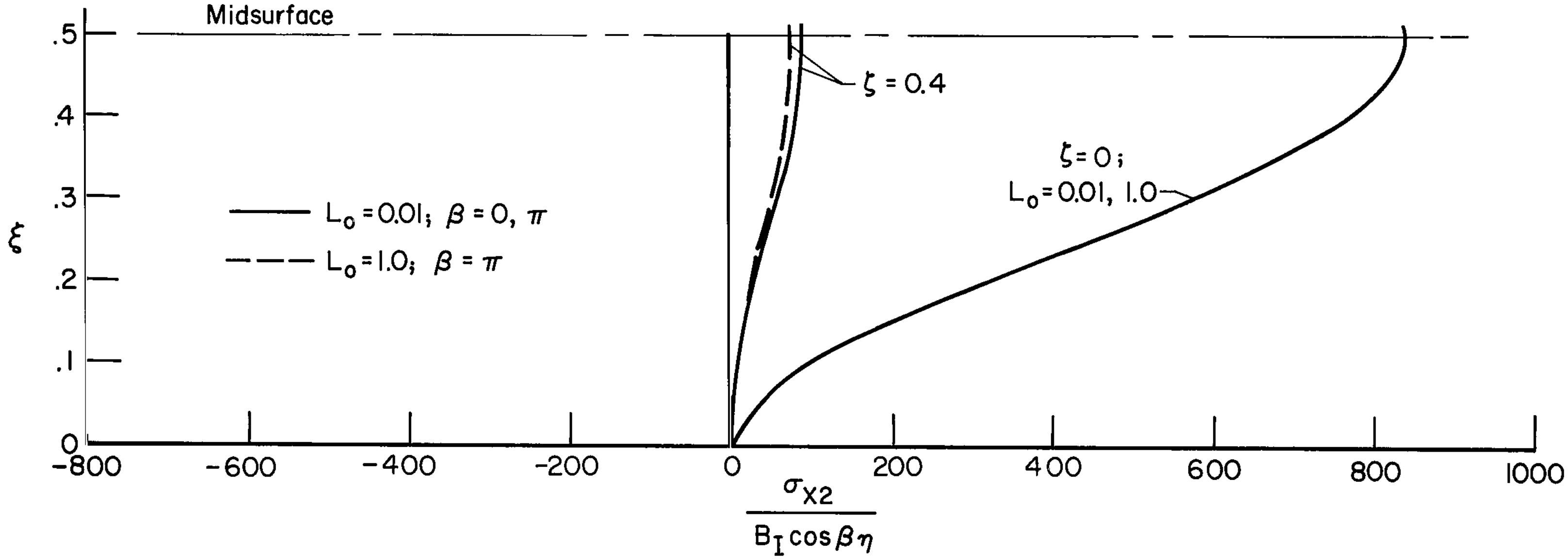

Figure 8.- Variation of σ_{X2} with ξ.

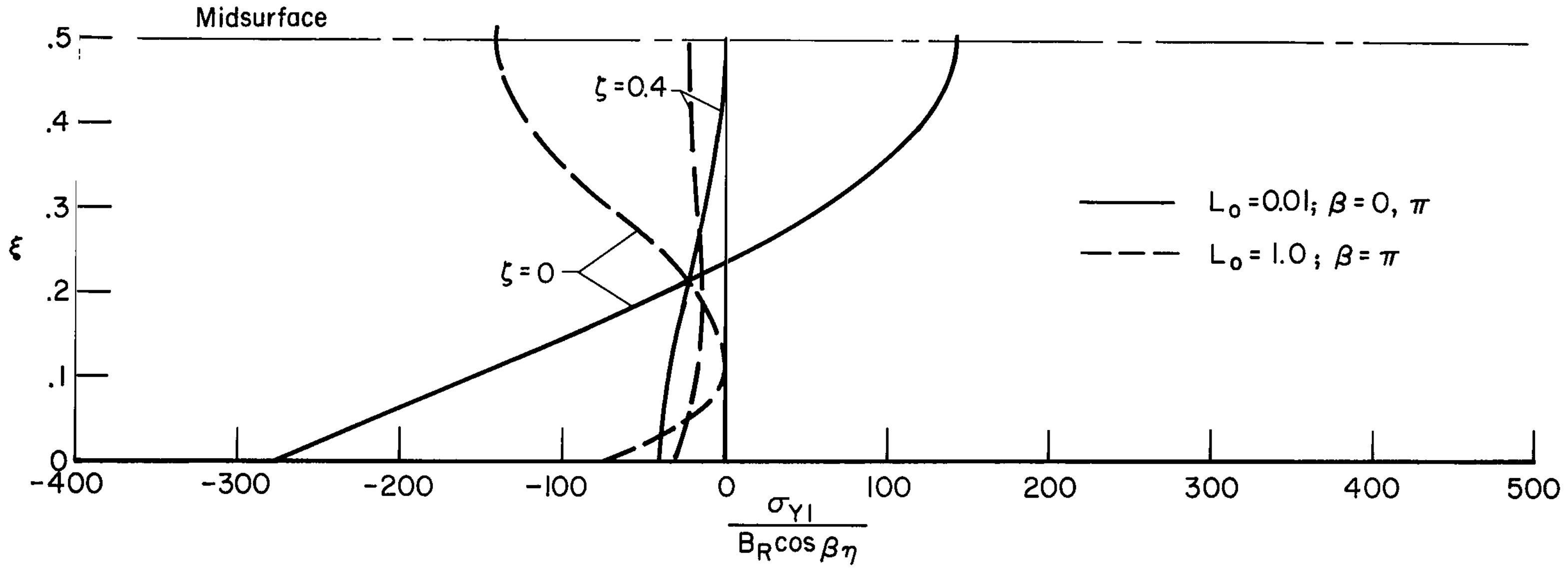

Figure 9.- Variation of σ_{y1} with ξ.

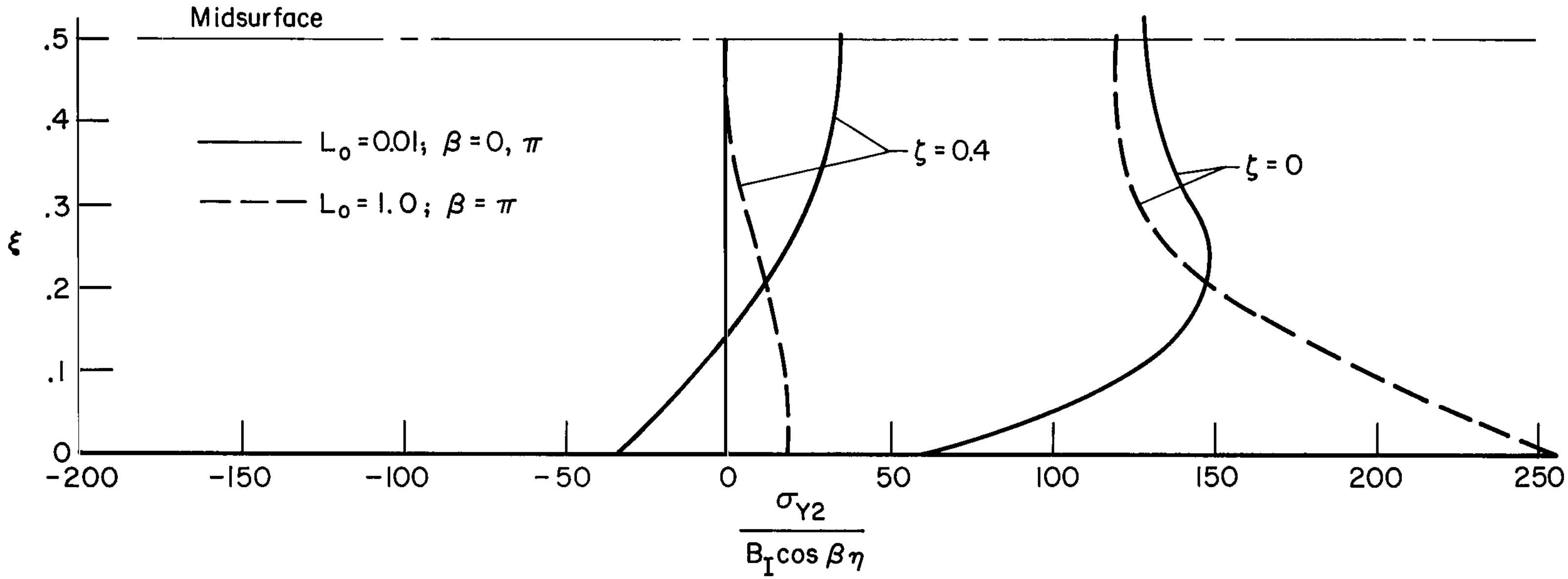

Figure 10.- Variation of σ_{y2} with ξ.

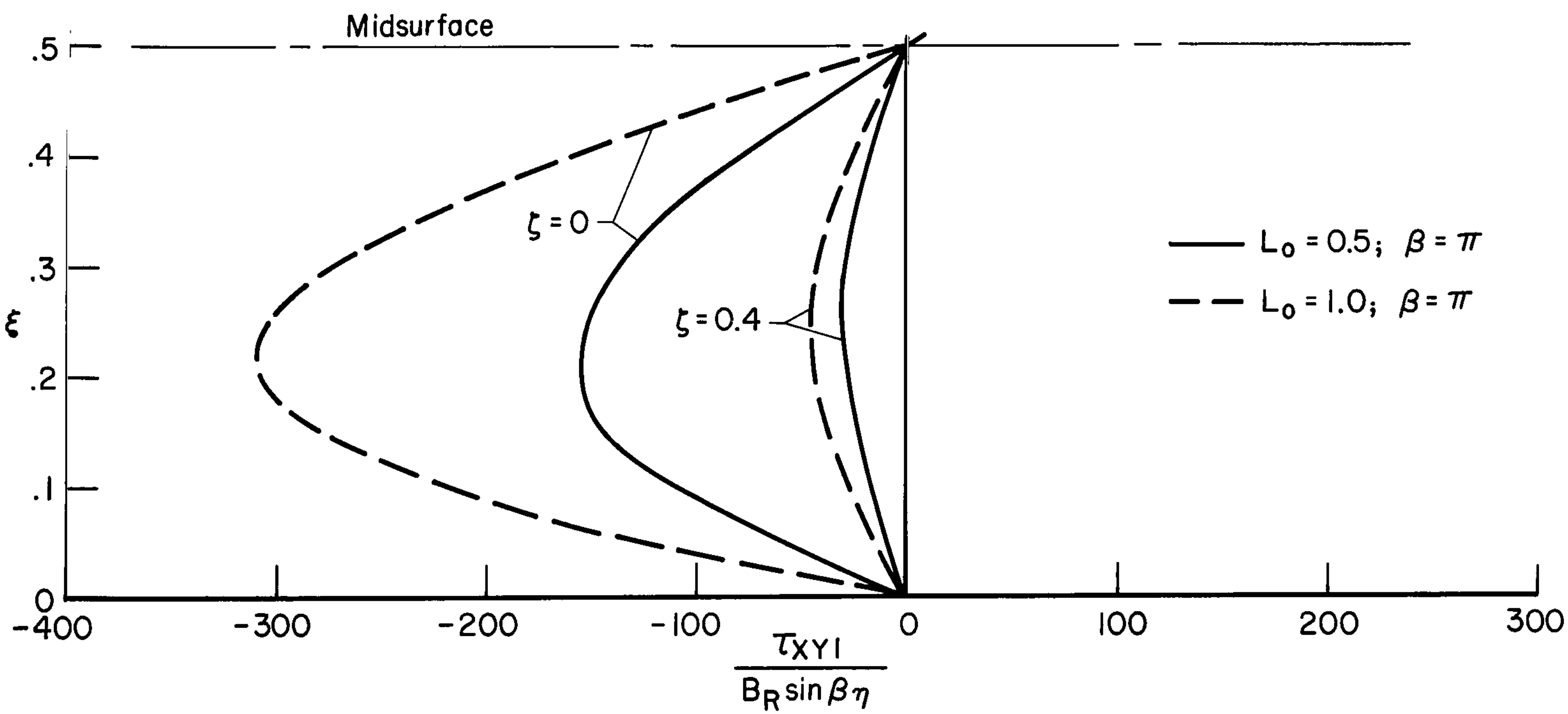

Figure 11.- Variation of τ_{xy_1} with ξ.

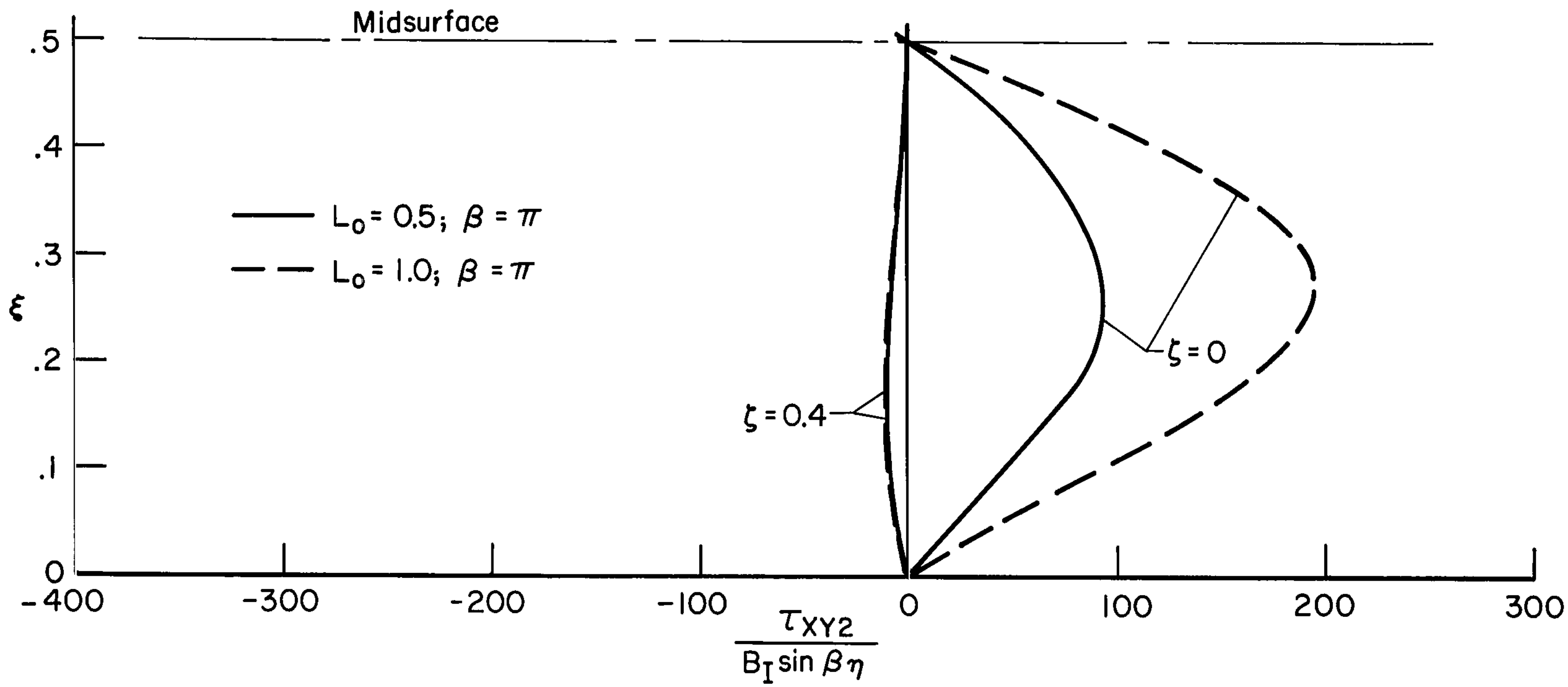

Figure 12.- Variation of τ_{xy2} with ξ.

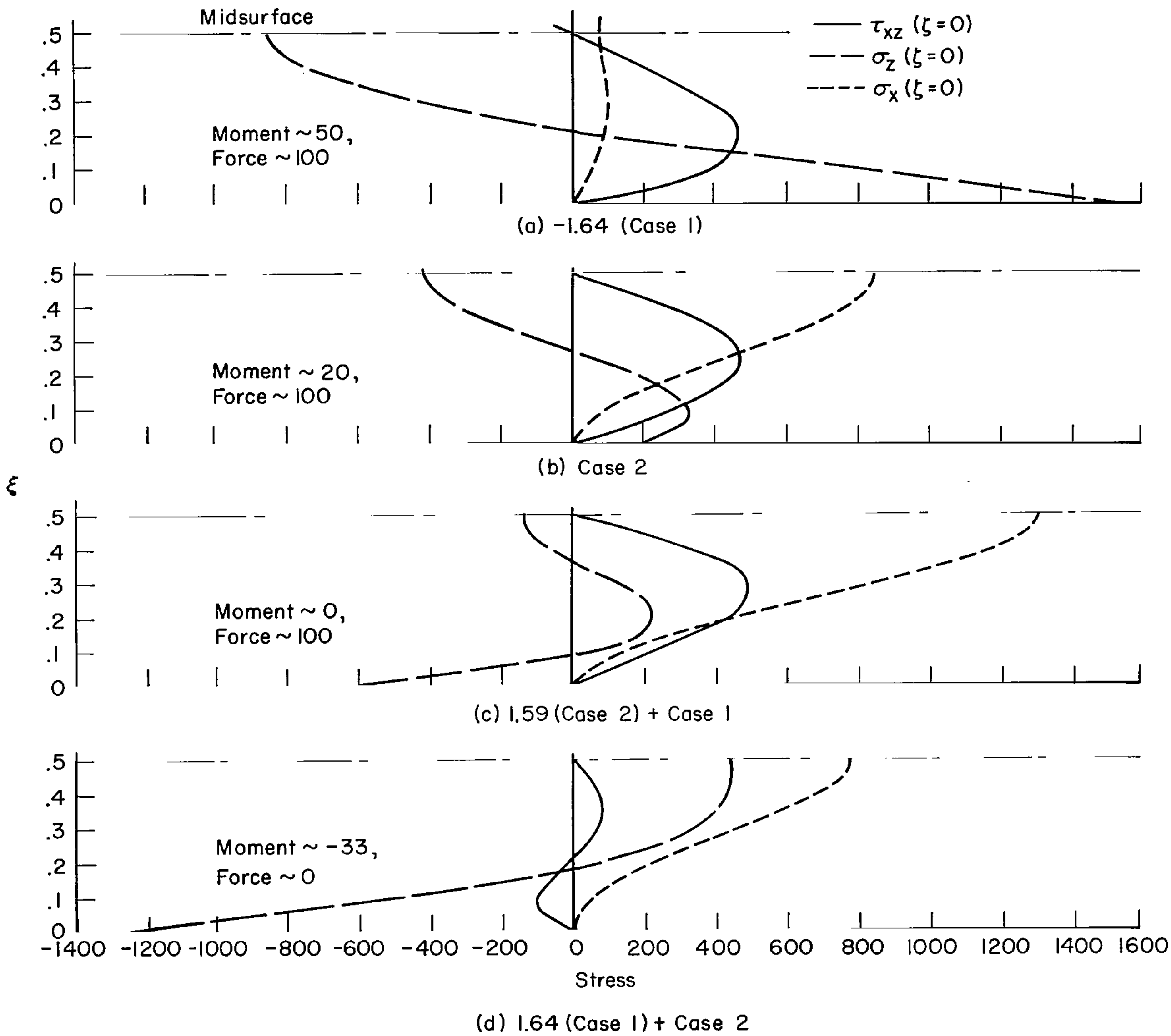

Figure 13.- Variation of τ_{xz}, σ_z, σ_x with ξ at $\zeta = 0$.

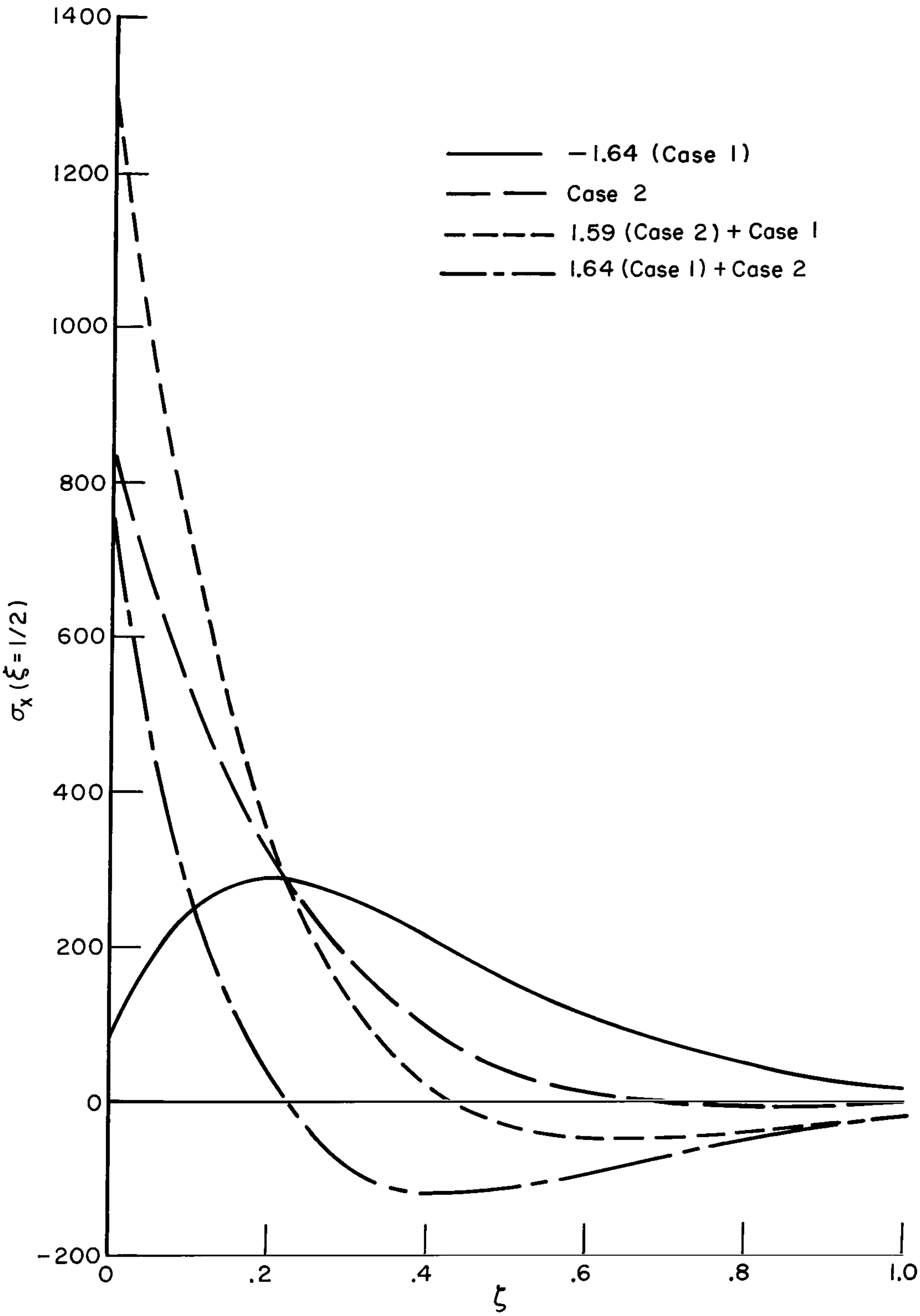

Figure 14.- Variation of σ_x with ζ at $\xi = 1/2$.

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