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THE EFFECT OF INSERTION SHIP TRACKING ERRORS ON THE APOLLO GO, NO-GO DECISION

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APRIL 28, 1967

GODDARD SPACE FLIGHT CENTER
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ABSTRACT

After insertion into the near-earth parking orbit a Go, No-Go decision has to be made, based on one minute of ship's tracking data. The deciding factor for the Go, No-Go decision is the perigee height of the parking orbit; a minimum perigee height of 75 n.mi. is required. In this paper, a statistical analysis has been made of the error in the perigee height, as determined from one minute of ship's tracking data. It is shown, that the computed perigee height has to be 0.5 to 17 n.mi. higher than 75 n.mi. in order to insure that the actual perigee height exceeds 75 n.mi. with 99.5% probability. The use of a 17.n.mi. padding is recommended in order to cover worst case conditions.

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SUMMARY

At insertion, the following parameters:

radius	r_o
speed	v_o
flight path angle	γ_o

will be determined from one minute of a ship's tracking data. In reference 2, it is required that these parameters be determined to the following accuracies:

$$3\sigma_{r_o} = 2.4 \text{ n.mi.} \quad (4.44 \text{ km})$$

$$3\sigma_{v_o} = 16 \text{ ft/sec} \quad (4.87 \text{ m/sec})$$

$$3\sigma_{\gamma_o} = 0.16° \quad (2.79 \text{ mrad})$$

A statistical analysis of the perigee height error due to the above insertion parameter uncertainties is presented in this paper. Gaussian distributions are assumed for the errors in the insertion parameters (i.e., measured minus actual) and correlations between the errors are taken into account. The results are applied to the Apollo Go, No-Go decision at insertion.

For a Go decision, an actual perigee height of 75 n.mi. is required with 99.5% probability (reference 1). It is shown that the computed perigee height (from ship's tracking data) has to be 0.5 to 17 n.mi. higher than 75 n.mi. in order to meet this requirement. The required padding is a function of the eccentricity e of the parking orbit, the true anomaly θ at insertion, the errors in the insertion parameters and their correlations.

The results of the analysis may be summed up in the following recommendations:

1. A padding of 17 n.mi. should be used to insure with a 99.5% probability that a Go decision is correct. This value covers the following cases:
   a. All eccentricities in the region $0 \leq e \leq 0.01$.
   b. All values of true anomaly at insertion.
   c. All values of correlation coefficients ρ between errors of the insertion parameters in the region $0 \leq |\rho| \leq 0.9$.
2. The parking orbit should have the smallest possible eccentricity and a true anomaly close to 0° at insertion in order to minimize the perigee height error.

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1. INTRODUCTION

For the Apollo mission a decision to continue or not continue, i.e., the Go, No-Go decision, will have to be made shortly after the spacecraft is inserted into the parking orbit.

Three sources of data are available for the Go, No-Go decision: the insertion ship and two on board inertial guidance systems. One of the on board systems is located in the S-IVB and the other in the Apollo spacecraft (Command Module). Normally, the Go, No-Go decision will be made using the two on board systems. In the event these two systems are in disagreement, the insertion ship will be used as an arbiter (reference 1).

This report is a statistical analysis of the effect of tracking measurement errors by the insertion ship on the error in perigee for near-earth Apollo parking orbits. The results of this analysis are applied to the Apollo Go, No-Go decision at insertion.

Figure 1 shows the geometry at insertion. The critical insertion parameters are the radius r_o, the magnitude of the velocity v_o, and the flight path angle γ_o.

As required in reference 2, these insertion parameters must be determined from the ship's tracking data with the following accuracies:

$$3\sigma_{r_o} = 2.4 \text{ n.mi} \quad (4.44 \text{ km})$$

$$3\sigma_{v_o} = 16 \text{ ft/sec} \quad (4.87 \text{ m/sec})$$

$$3\sigma_{\gamma_o} = 0.16^\circ \quad (2.79 \text{ mrad})$$

It is shown in reference 3, Chapter 3, that these accuracy requirements can be met with one minute of tracking by the ship if the tracking error model from reference 4 is used. Three minutes of tracking is obtainable with the positions of the insertion ship as planned in reference 3. Thus, since only one minute of tracking data is needed, this provides a margin for loss of data due to delayed acquisition, interrupted tracking, etc.

The Go, No-Go decision will be based upon the perigee height. In reference 2 a minimum perigee height of 71 n.mi. (131 km) is required. In order to allow

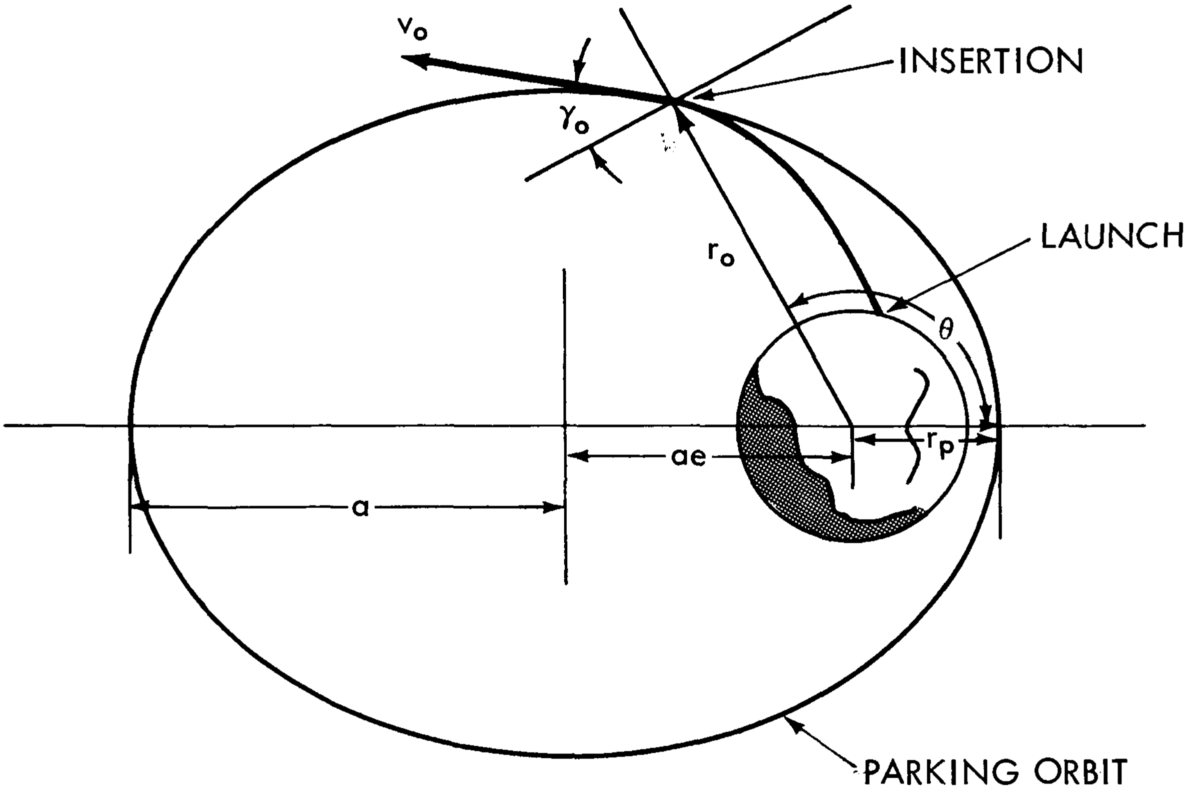

Figure 1. Insertion Geometry.

for two orbits before re-entry, a minimum perigee height of 75 n.mi. (139 km) is suggested in reference 1, and a 99.5% probability is required to achieve this minimum height if a Go decision is made (reference 1).

The perigee height may be calculated from the insertion parameters by means of the Keplerian two-body equations of motion. However, due to errors in the insertion parameters as determined from the ship's tracking data, there will be an error associated with the calculated perigee height. It is the purpose of this report to study the probability distribution of perigee error and apply the results to the Apollo Go, No-Go decision at insertion.

Distribution determination of the perigee error is important from the standpoint of astronaut safety. If the actual perigee height is lower than the calculated perigee height, an incorrect Go decision could be made (i.e., one where the actual perigee height is lower than 75 n.mi.). This would be detrimental to the safety of the astronauts due to the excessive re-entry heat.

As stated in reference 3, a circular parking orbit is planned. However, an elliptical orbit should not be ruled out, in which case the perigee height may be at 85 n.mi. (157 km) and the apogee height at 150 n.mi. (278 km). Both circular and elliptical parking orbits are therefore treated in this analysis.

2. THE CUMULATIVE DISTRIBUTION OF PERIGEE ERROR

2.1 The Perigee Error

The spacecraft has the actual or true insertion parameters r_o, v_o, and γ_o. The actual perigee radius r_p can be determined from the Keplerian two-body equations of motion.

$$v_o^2 = \mu \left(\frac{2}{r_o} - \frac{1}{a} \right) \tag{2.1.1}$$

$$(1 - e^2) = \left(\frac{1}{\mu a} \right) \left(v_o^2 r_o^2 \cos^2 \gamma_o \right) \tag{2.1.2}$$

$$r_p = a(1 - e) \tag{2.1.3}$$

where a = semi-major axis of the parking orbit

e = eccentricity of the parking orbit

μ = gravitational constant of the earth (398603.2 (km)^3/sec^2)

by eliminating a and e as shown in Appendix A. Thus, the perigee radius r_p is a function of r_o, v_o, and γ_o.

Since the insertion parameters are determined from the ship's tracking data, they will be in error by the amounts Δr_o, Δv_o, and $\Delta\gamma_o$. Therefore, the calculated perigee radius $r_{p_{cal}}$ will deviate by Δr_p from the actual perigee radius r_p.

$$\Delta r_p = r_{p_{cal}} - r_p = r_p(r_o + \Delta r_o,\ v_o + \Delta v_o,\ \gamma_o + \Delta\gamma_o) - r_p(r_o,\ v_o,\ \gamma_o) \tag{2.1.4}$$

This equation determines Δr_p as a function of the insertion parameters and their errors. By eliminating r_o, v_o, and γ_o with the aid of equations (2.1.1) through (2.1.3), an alternate expression for Δr_p results (equations B.63, B.64, and B.68 of Appendix B).

$$\Delta r_p = \Delta r_p(a,\ e,\ \theta,\ \Delta r_o,\ \Delta v_o,\ \Delta\gamma_o) \tag{2.1.5}$$

The advantage of using equation (2.1.5) is that Δr_p is almost independent of the semi-major axis a. In the range of a-values of interest, we may write

$$\Delta r_p = \Delta r_p (e, \theta, \Delta r_o, \Delta v_o, \Delta \gamma_o) \tag{2.1.6}$$

In this form Δr_p depends only upon two parameters, the eccentricity e of the actual orbit and the true anomaly θ at insertion. This greatly simplifies the statistical analysis of Δr_p.

2.2 Use of the Cumulative Distribution Function F

An example of a cumulative distribution function

$$F(\Delta r_p) = P(\Delta R_p \leq \Delta r_p)$$

where ΔR_p = the random variable and Δr_p = a particular value of ΔR_p, is shown in Figure 2a. (For derivation of the distribution function see Chapter 2.3.) The use of this function can be demonstrated best by an example.

Assume that the calculated perigee height ($r_{p_{cal}} - r_e$) is 79 n.mi. (146 km). The allowed minimum perigee height ($r_{p_{min}} - r_e$) is 75 n.mi. (139 km), and it appears that a Go decision could be made. However, the condition for making a correct Go decision is

$$r_p - r_e \geq 75 \text{ n.mi.} \tag{2.2.1}$$

Using (2.1.4) and substituting ΔR_p for Δr_p (since we wish to make a statement of probability we will use the random variable ΔR_p) we can write

$$(r_{p_{cal}} - r_e) - \Delta R_p \geq (r_{p_{min}} - r_e)$$

Therefore,

$$79 \text{ n.mi.} - \Delta R_p \geq 75 \text{ n.mi.}$$

or

$$\Delta R_p \leq 4 \text{ n.mi. } (7.4 \text{ km}). \tag{2.2.2}$$

From the cumulative distribution function in Figure 2b, we see that the probability for satisfying (2.2.2), i.e., making a correct Go decision, is:

$$P(\Delta R_p \leq 4 \text{ n.mi.}) = F = 93\%. \tag{2.2.3}$$

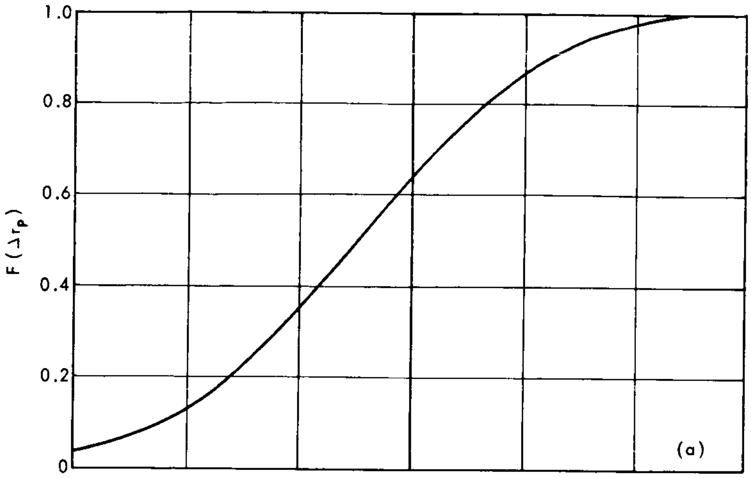

Figure 2a. Cumulative Distribution Function F of the Perigee Error ΔR_p. $F = F(\Delta r_p) = P(\Delta R_p \leq \Delta r_p)$.

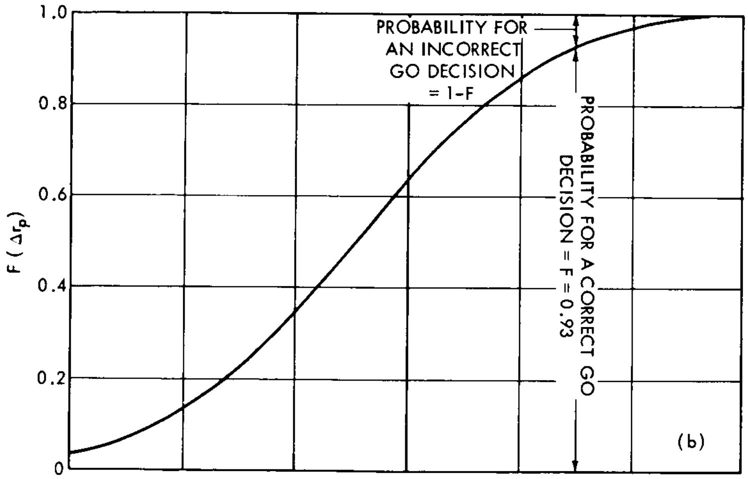

Figure 2b. With a Padding of 4 Nautical Miles, the Probability for a Correct GO Decision is 93%.

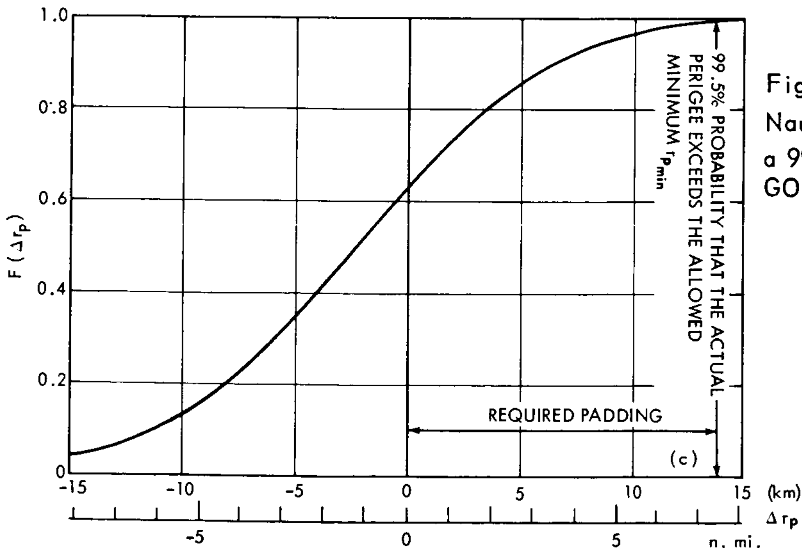

Figure 2c. A Padding of 7.4 Nautical Miles Is Required for a 99.5% Probability of a Correct GO Decision.

Using the same function (see Figure 2c), for a 99.5% probability, the calculated perigee height ($r_{p_{cal}} - r_e$) has to be

$$(r_{p_{cal}} - r_e) - 75 \text{ n.mi.} \geq 7.4 \text{ n.mi.}$$

or

$$(r_{p_{cal}} - r_e) \geq 82.4 \text{ n.mi.} \tag{2.2.4}$$

In other words, a padding of 7.4 n.mi. (13.7 km) is required to insure with a 99.5% probability that the true or actual perigee height exceeds 75 n.mi. (139 km).

The cumulative distribution function F used in the above example is valid for $e = 0.001$, $\theta = 180°$, and no correlation between the random variables ΔR_o, ΔV_o, and $\Delta\Gamma_o$ (1). The required padding for other insertion parameters and different correlations between ΔR_o, ΔV_o, and $\Delta\Gamma_o$ is analyzed in Chapter 3.

2.3 Computation of the Cumulative Distribution Function F.

Normally, the first step in an error analysis is to develop a variational equation expressing Δr_p as a function of Δr_o, Δv_o, and $\Delta\gamma_o$. This is done in Appendix B (Equation B.79), and it is shown that the variational equation is a second order equation in Δr_p of the form:

$$\begin{aligned} &\alpha_1 \Delta r_p + \alpha_2 \Delta r_o \Delta r_p + \alpha_3 \Delta v_o \Delta r_p + \alpha_4 (\Delta r_p)^2 \\ &= \beta_1 \Delta r_o + \beta_2 \Delta v_o + \beta_3 \Delta\gamma_o + \beta_4 \Delta r_o \Delta v_o + \beta_5 \Delta v_o \Delta\gamma_o \\ &+ \beta_6 \Delta r_o \Delta\gamma_o + \beta_7 (\Delta r_o)^2 + \beta_8 (\Delta v_o)^2 + \beta_9 (\Delta\gamma_o)^2 \\ &+ \text{terms of higher order} \end{aligned} \tag{2.3.1}$$

where the coefficients α_i ($i = 1,2,3,4$) and β_i ($i = 1,2, \ldots, 9$) are functions of the insertion parameters r_o, v_o, and γ_o.

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(1) Here ΔR_o, ΔV_o and $\Delta\Gamma_o$ are random variables representing the errors in insertion radius, velocity magnitude, and flight path angle, to be distinguished from Δr_o, Δv_o, and $\Delta\gamma_o$ which are values which ΔR_o, ΔV_o, and $\Delta\Gamma_o$ can assume.

Equation (2.3.1) has simple solutions for two special cases. For a circular orbit, the coefficients of the linear terms vanish and the result is Equation (B.82) of Appendix B:

$$\Delta r_p = 2\left[\Delta r_o + \left(\frac{r_o}{v_o}\right)\Delta v_o\right] - \sqrt{\left[\Delta r_o + 2\left(\frac{r_o}{v_o}\right)\Delta v_o\right]^2 + (r_o \Delta\gamma_o)^2} . \quad \text{(B.82)}$$

For large e-values the second order terms may be neglected, and the result is a simple linear expression in Δr_o, Δv_o, and $\Delta\gamma_o$. In particular, if $0.005 \leq e \leq 0.05$, equation (9) from reference 5 is valid

$$\Delta r_p \approx (2 - \cos\theta)\Delta r_o + 2(1 - \cos\theta)\left(\frac{r_o}{v_o}\right)\Delta v_o - (r_o \sin\theta)\Delta\gamma_o . \quad (2.3.2)$$

The range of e-values of interest in this report is $0 \leq e \leq 0.01$. In the region $0 \leq e \leq 0.005$, the variational equation (2.3.1) confuses rather than aids the analysis. Therefore, the basic definition, equation (2.1.4), has been used.

$$\Delta r_p = r_p(r_o + \Delta r_o, v_o + \Delta v_o, \gamma_o + \Delta\gamma_o) - r_p(r_o, v_o, \gamma_o) . \quad (2.1.4)$$

It should be noted that this latter expression is exact. The value of equations (B.82) and (2.3.2) lies in their simplicity. Consequently, in their region of validity, they have been used as a check against the numerical computations performed using (2.1.4).

If ΔR_o, ΔV_o, and $\Delta\Gamma_o$ are assumed to be normally distributed random variables, then the cumulative distribution function of perigee error[(2)]

$$\Delta R_p = r_p(r_o + \Delta R_o, v_o + \Delta V_o, \gamma_o + \Delta\Gamma_o) - r_p(r_o, v_o, \gamma_o) \quad (2.3.3)$$

can be written in integral form. However, this integral cannot be expressed in terms of known or tabulated functions, but must be evaluated by numercial techniques – either a Monte Carlo approach or numerical integration. The latter approach has been used in this analysis, and is described in Appendix B.

If the error in perigee is expressed in terms of the orbital elements a, e, θ

$$\Delta R_p = \Delta R_p(a, e, \theta, \Delta R_o, \Delta V_o, \Delta\Gamma_o) \quad (2.3.4)$$

(2) Again, since we are dealing with probability distributions we will use random variable notations.

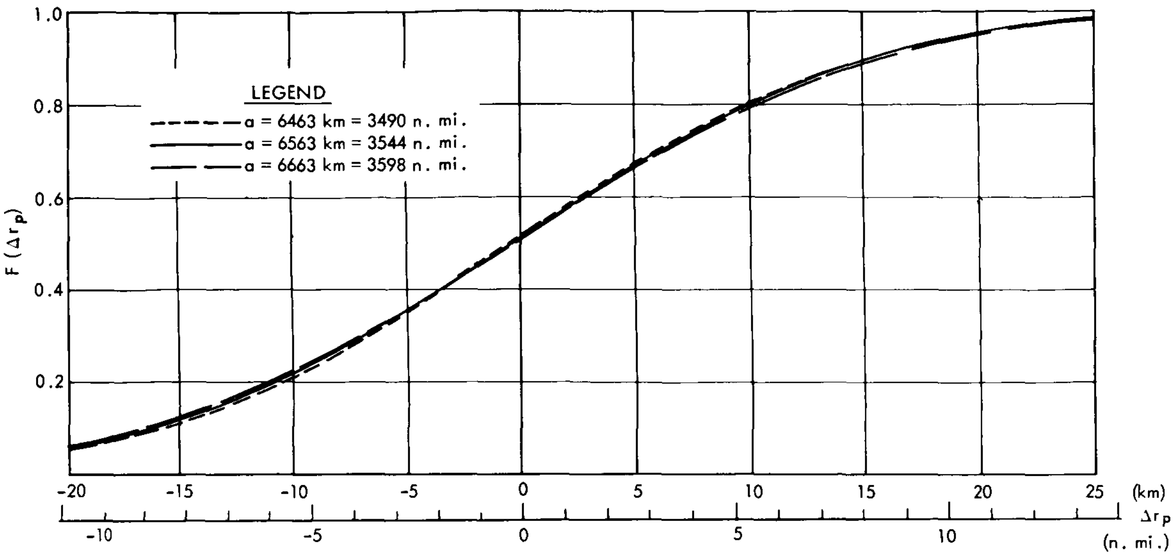


Figure 3. The Cumulative Distribution Function $F(\Delta r_p)$ Varies Insignificantly if the Semi-Major Axis a Is Varied ±54 Nautical Miles (±100 km) Which Is Approximately Six Times the Variation in a for this GO, NO-GO Analysis. Insertion Parameters: Eccentricity e = 0.01, True Anomaly at Insertion θ = 225°.

we see in Figure 3 that ΔR_p is almost independent of the semi-major axis a for a-values of interest in the Go, No-Go analysis. Figure 3 shows the cumulative distribution functions for ΔR_p for the following values of a:

3490 n.mi., 3544 n.mi., and 3598 n.mi.
(6463 km, 6563 km, 6663 km)

Although a varies ± 54 n.mi. (± 100 km), or approximately six times the expected variation in a for this analysis, the cumulative distribution functions vary insignificantly. We may therefore neglect the variation in a, and use an average value of 3544 n.mi. (6563 km). This will greatly simplify the analysis. Hence,

$$\Delta R_p = \Delta R_p (e, \theta, \Delta R_o, \Delta V_o, \Delta\Gamma_o) . \tag{2.3.5}$$

A computed probability density function of perigee error, $f(\Delta r_p)$ for an actual circular parking orbit ($e = 0$) for both correlated and uncorrelated insertion errors ΔR_o, ΔV_o, and $\Delta\Gamma_o$ is shown in Figure 4a. The computed cumulative distribution function

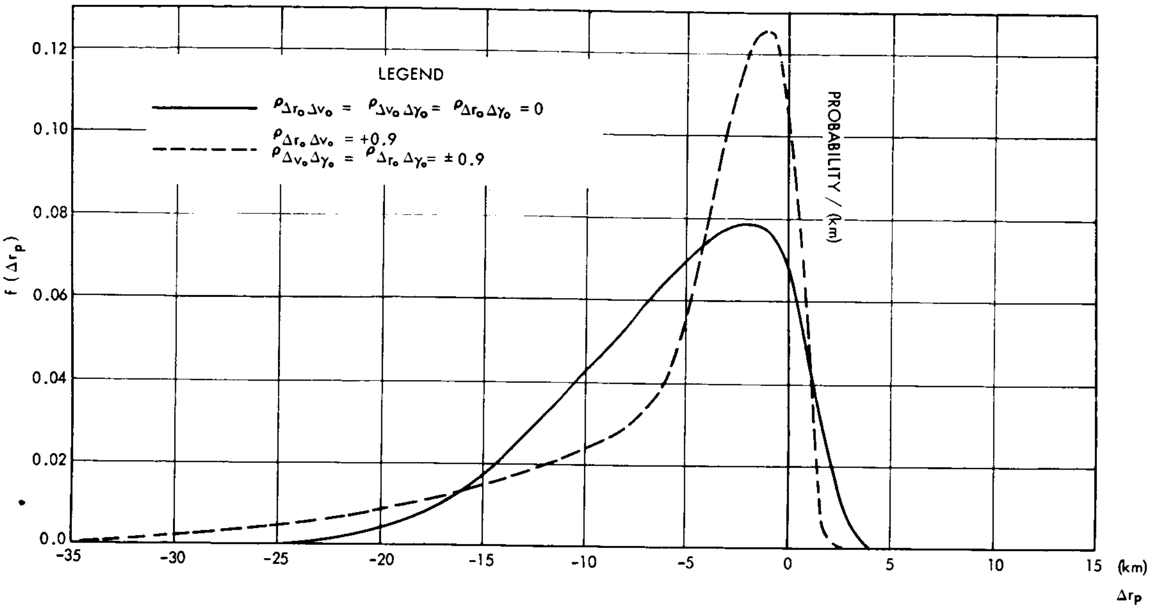


Figure 4a. Probability Density Function f of Perigee Error ΔR_p for a Circular Orbit. Insertion Parameters: Semi-Major Axis a = 3544 n.mi., Eccentricity e = 0.

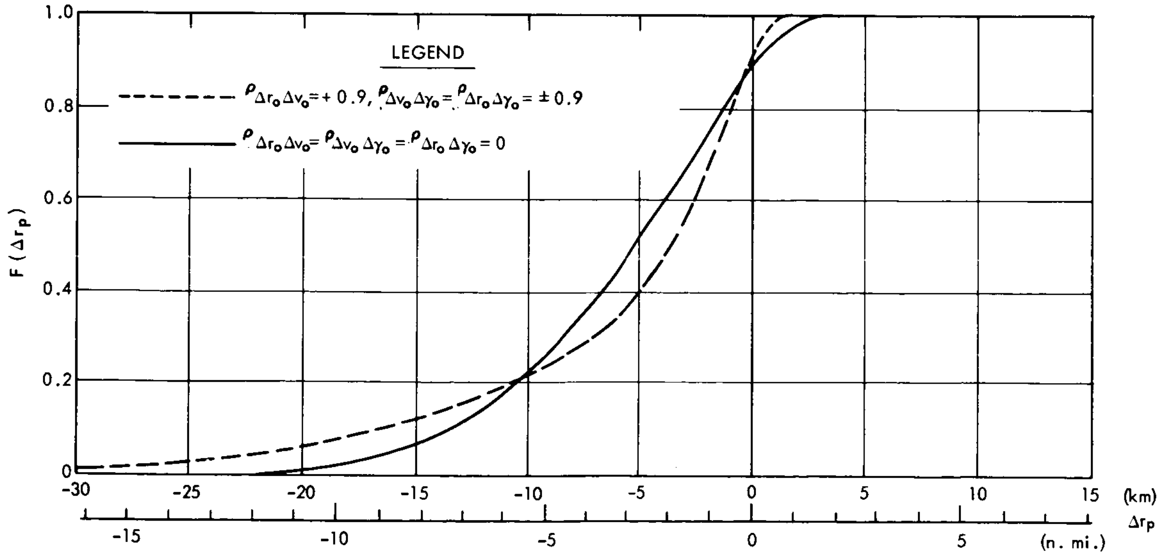


Figure 4b. Cumulative Distribution Function F for the Above Probability Density Function f.

$$F = F(\Delta r_p) = \int_{-\infty}^{\Delta r_p} f(x)\, dx \qquad (2.3.6)$$

is shown for the same conditions in Figure 4b. It is of interest to note that the perigee error does not have a mean of zero and is not normally distributed.

Figures 5 through 8 show cumulative distribution functions for eccentricities of $e = 0.0005$, $e = 0.001$, $e = 0.005$, and $e = 0.01$. In each instance there are curves for both correlated and uncorrelated insertion errors. For uncorrelated errors, curves are plotted for true anomalies at insertion of $\theta = 0°$, $90°$, and $180°$. The F-curves are even functions of θ for uncorrelated errors and hence are also valid for $\theta = -90°$. For correlated errors, the curves are no longer even functions of θ, and have therefore been plotted for $\theta = 0°$ and $\theta = 180° \pm 45°$. In all of these figures, the "3 sigma" limits for the insertion errors have been taken as

$$3\sigma_{\Delta r_o} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{\Delta v_o} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\Delta \gamma_o} = 0.16° \ (2.79 \text{ m rad}).$$

2.4 Verification of the Numerical Integration

It has already been mentioned that in its region of validity equation (2.3.2) is a very good approximation for the perigee error. This can be seen in Figures 9a and 9b where the cumulative normal distribution function using (2.3.2) is compared with the cumulative distribution function of perigee error obtained by numerical integration. In both figures, the semi-major axis was taken to be 3544 n.mi. (6563 km) and the eccentricity of the parking orbit $e = 0.01$. In Figure 9a, $\theta = 225°$ and the coefficients of correlation of the errors at insertion ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are all equal to 0.75. It can be seen that the agreement between the two methods is very good. In Figure 9b, $\theta = 180°$, and the errors are uncorrelated. Again the agreement between the two methods is very good.

In Appendix B, Figures 19 and 20, the required padding for 99.5% probability is compared using both numerical integration and equation (2.3.2). Again, there is good agreement.

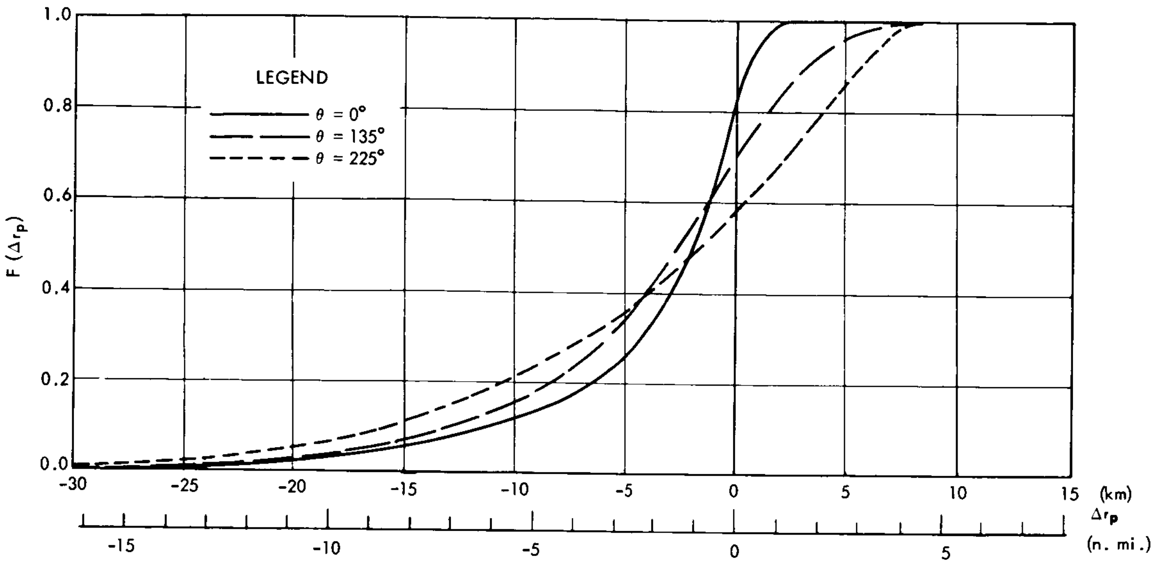


Figure 5a. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.0005, True Anomaly at Insertion θ = 0°, 135°, 225°. Insertion Errors Are Positively Correlated, Coefficients of Correlation = +0.75.

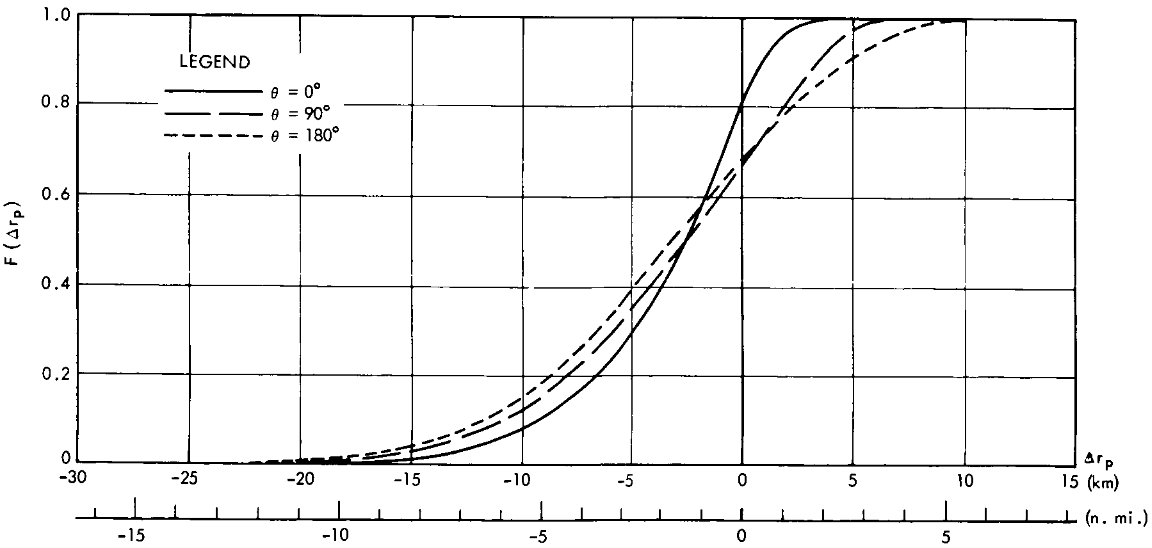


Figure 5b. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.0005, θ = 0°, 90°, 180°. Insertion Errors Are Uncorrelated.

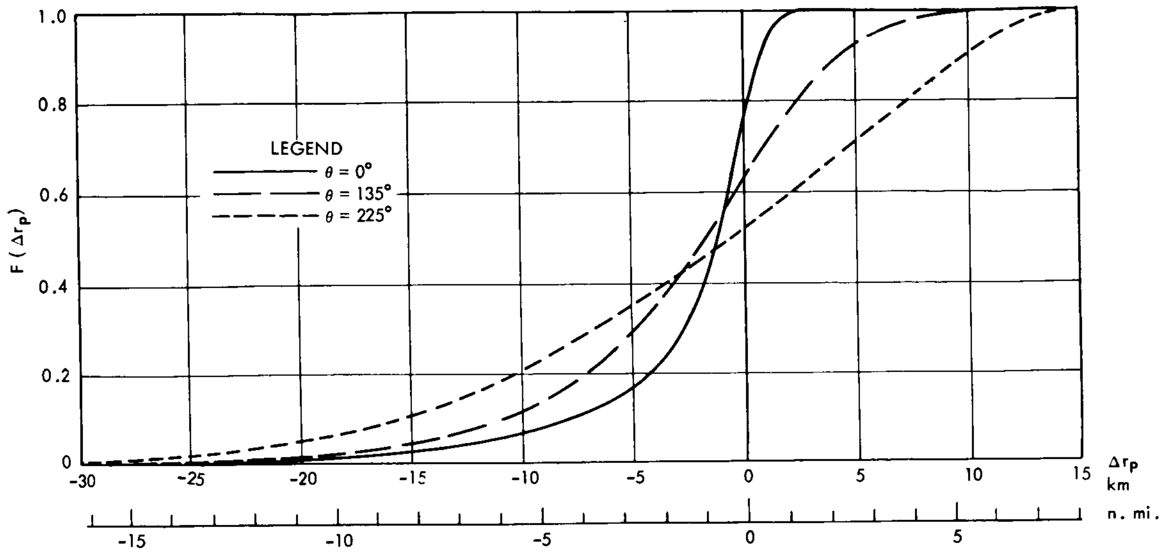


Figure 6a. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001, θ = 0°, 135°, 225°. Insertion Errors Are Positively Correlated, Coefficients of Correlation = +0.75.

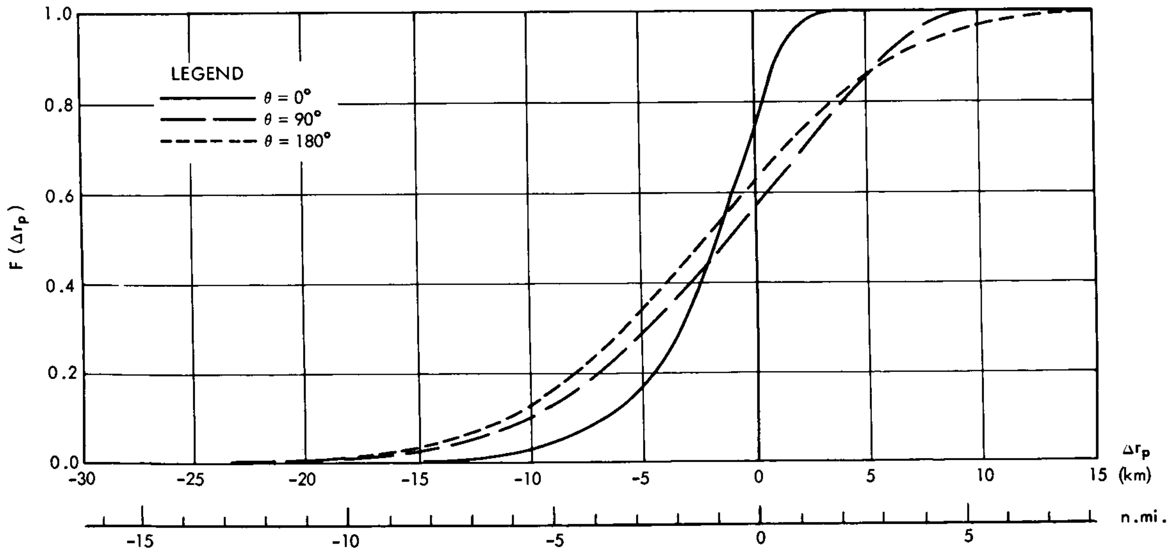


Figure 6b. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001, θ = 0°, 90°, 180°. Insertion Errors Are Uncorrelated.

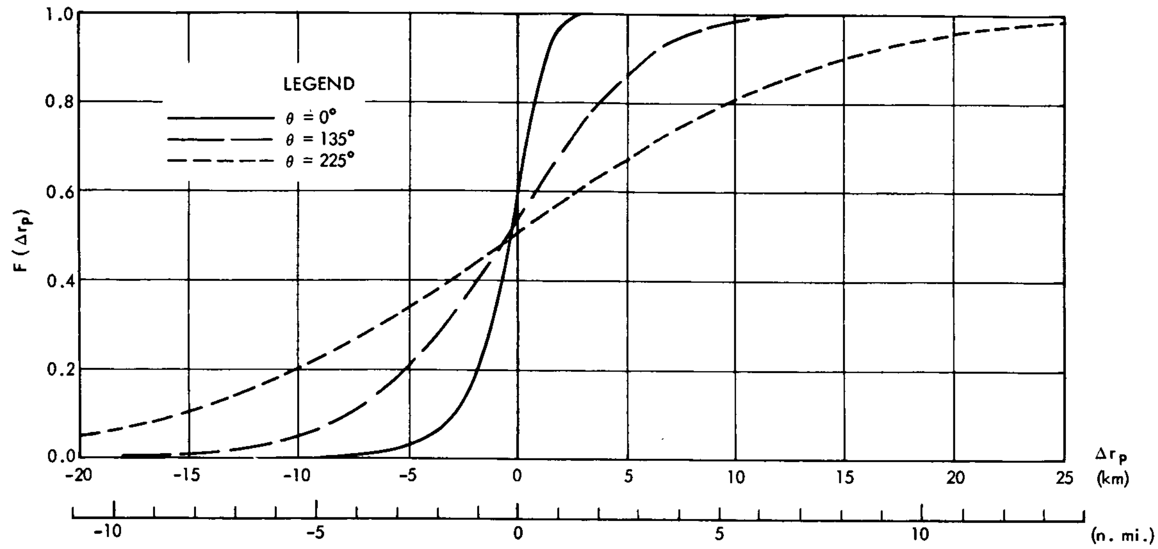


Figure 7a. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005, θ = 0°, 135°, 225°. Insertion Errors Are Positively Correlated, Coefficients of Correlation = +0.75.

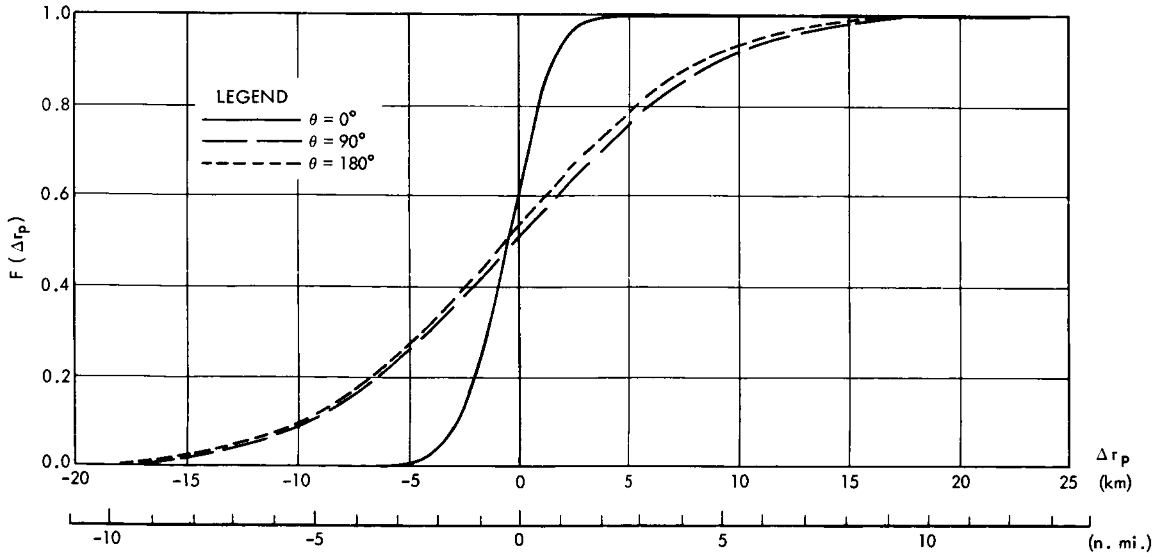


Figure 7b. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005, θ = 0°, 90°, 180°. Insertion Errors Are Uncorrelated.

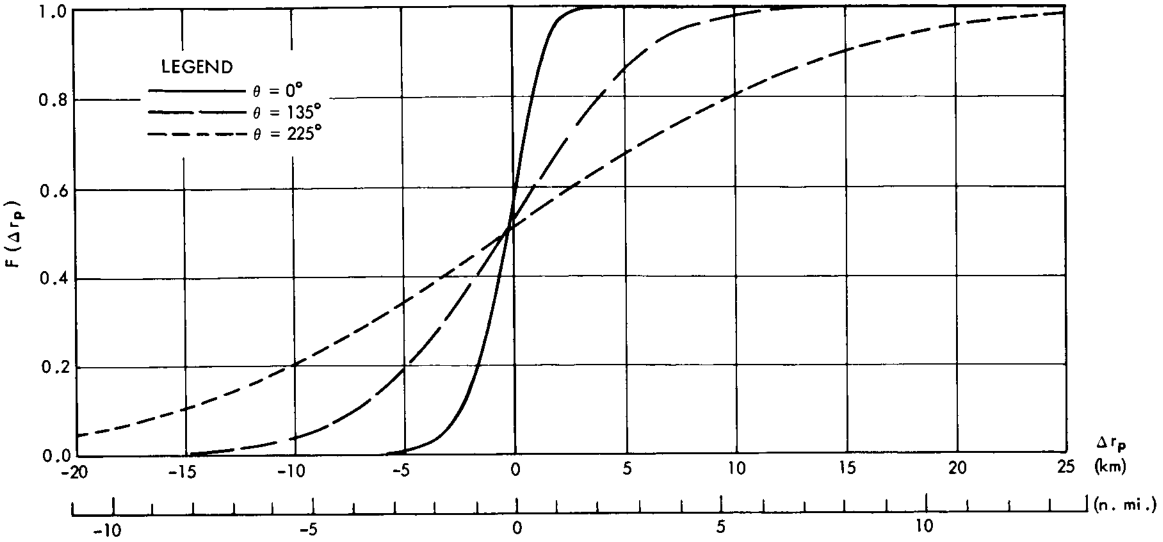


Figure 8a. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01, θ = 0°, 135°, 225°. Insertion Errors Are Positively Correlated, Coefficients of Correlation = +0.75.

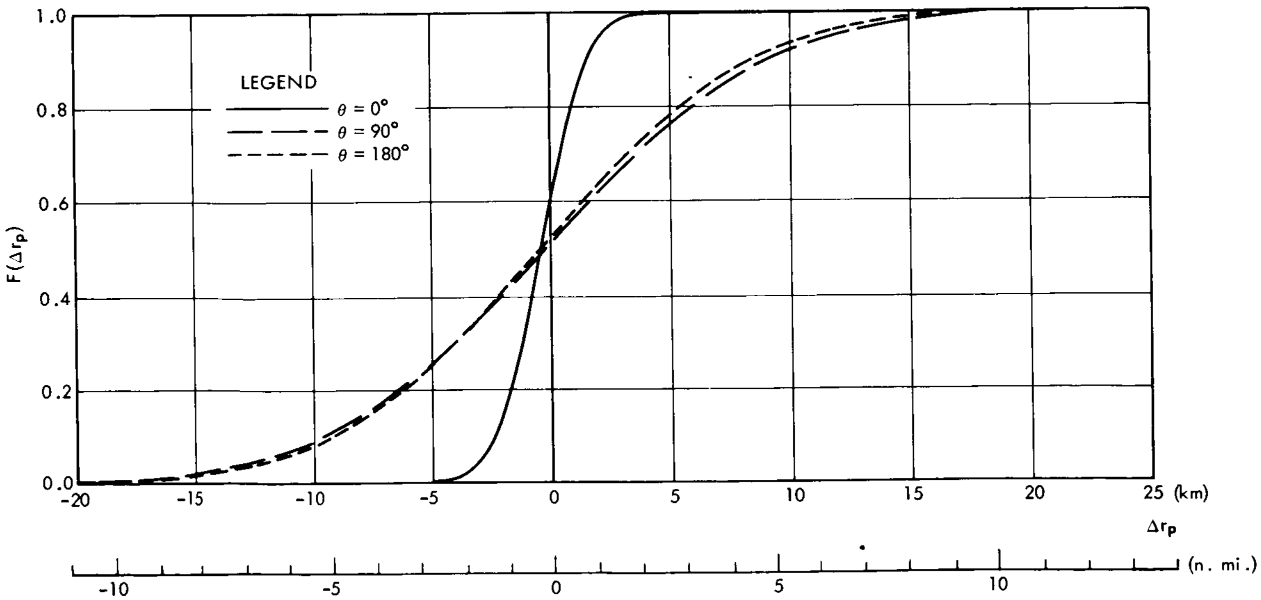


Figure 8b. Cumulative Distribution Function F of Perigee Error ΔR_p. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01, θ = 0°, 90°, 180°. Insertion Errors Are Uncorrelated.

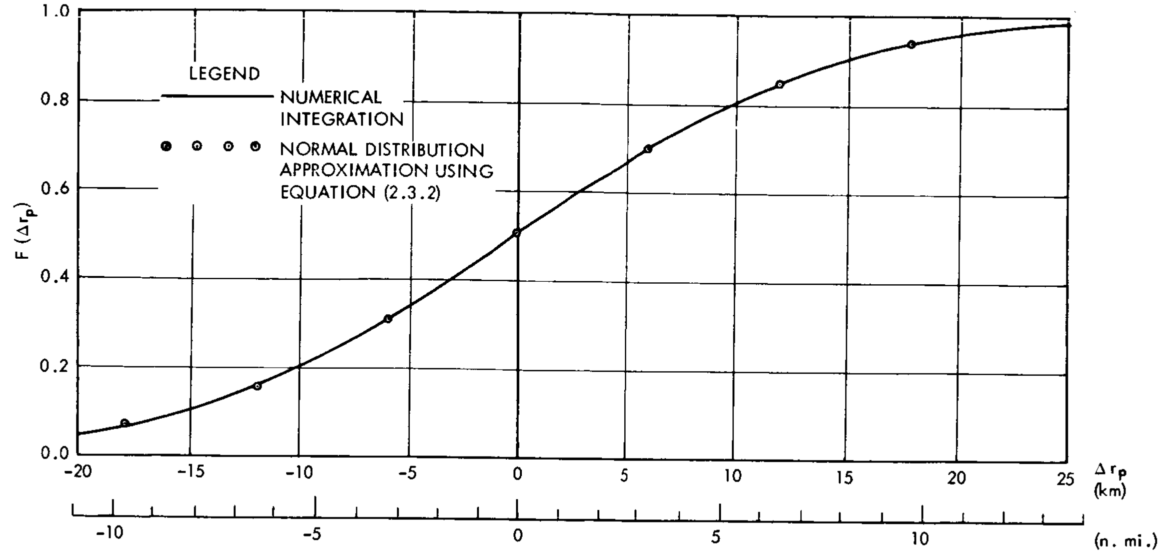


Figure 9a. The Computed Cumulative Distribution Function F of Perigee Error ΔR_p Agrees with the Cumulative Normal Distribution Approximation for Large Eccentricities. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01, θ = 225°. Insertion Errors Are Positively Correlated, Coefficients of Correlation = +0.75.

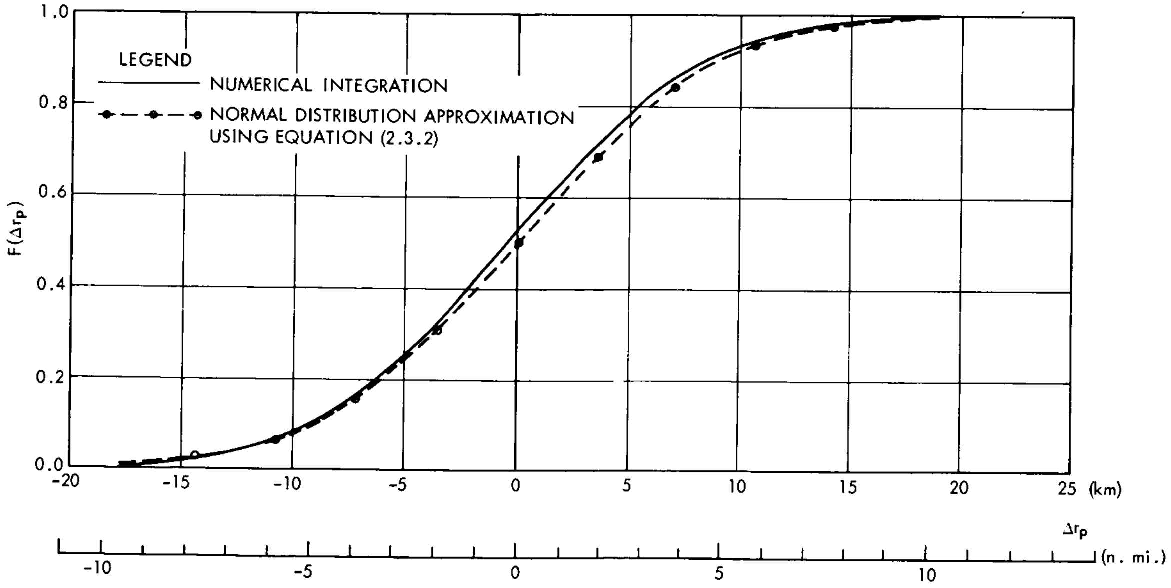


Figure 9b. The Computed Cumulative Distribution Function F of Perigee Error ΔR_p Agrees With the Cumulative Normal Distribution Approximation for Large Eccentricities. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01, θ = 180°. Insertion Errors Are Uncorrelated.

3. THE GO, NO-GO DECISION

An incorrect Go decision (one where the true or actual perigee height is less than 75 n.mi.) could endanger the lives of the astronauts. Therefore, a 99.5% probability is required for a correct Go decision (reference 1). Since there will always be an error Δr_p in the calculation of the perigee height, a padding must be added to the minimum required perigee height of 75 n.mi. (139 km) in order to have a 99.5% probability that the actual perigee height exceeds 75 n.mi. The required padding must be equal to or greater than that value of Δr_p for which $F(\Delta r_p) = 0.995$. Hence, it can be read off the cumulative distribution curves in Chapter 2 at the point where $F = 0.995$.

In this chapter, the required padding is analyzed and is seen to depend significantly on the eccentricity e, the true anomaly at insertion θ, and the correlation coefficients between the insertion errors. As discussed in paragraph 2.1, the required padding depends insignificantly on the semi-major axis a. Since it is practically impossible to determine the actual coefficients of correlation between the insertion errors, the calculations have been carried out for coefficients of correlation of 0, ±0.75, and ±0.9. That is, all possible combinations of signs have been considered with the exception of those for which the covariance matrix becomes singular. Throughout the analysis, it has been assumed that the insertion errors ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are normally distributed random variables having means of zero and "3 sigma" values of:

$$3\sigma_{\Delta r_o} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{\Delta v_o} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\Delta \gamma_o} = 0.16^\circ \ (2.79 \text{ m rad})$$

Table 1 shows the maximum required padding for eccentricities from 0 through 0.01. Also shown are the associated true anomalies at insertion and the coefficients of correlation for the insertion errors.

Figures 10 through 14 show the required padding for a 99.5% probability of a correct Go decision. The figures are for eccentricities of 0, 0.0005, 0.001, 0.005, and 0.01, and the curves are plotted versus true anomaly at insertion θ. In each figure there are two sets of curves — those for correlation coefficients of 0.75 and those for correlation coefficients of 0.9.

It is of interest to note that for eccentricities in the range $e \leq 0.001$, the maximum required padding occurs when the insertion errors are uncorrelated.

Table 1

Maximum Required Padding for a 99.5% Probability for a Correct Go-Decision

Eccentricity	Max. Required Padding (n.mi.)	Max. Required Padding (km)	True anomaly at insertion (degrees)	Coefficients of Correlation $\rho_{\Delta r_o \Delta v_o}$	$\rho_{\Delta v_o \Delta \gamma_o}$	$\rho_{\Delta r_o \Delta \gamma_o}$
0	1.6	2.9	All values	0	0	0
0.0005	4.8	8.9	180	0	0	0
0.001	7.5	13.8	180	0	0	0
0.005	17.1	31.7	135	+0.9	-0.9	-0.9
			225	+0.9	+0.9	+0.9
0.01	17.2	31.8	135	+0.9	-0.9	-0.9
			225	+0.9	+0.9	+0.9

For eccentricities in the range $e \geq 0.005$, high correlation gives the worst case. The maximum required padding does not increase substantially if e is increased above 0.005. This result obtained from numerical integration also agrees with equation (2.3.2) which is valid in the region $0.005 \leq e \leq 0.05$, and is independent of e.

4. CONCLUSIONS

4.1 The Optimum Orbit

From the standpoint of the Go, No-Go decision, a circular parking orbit is most desirable for two reasons:

1. For a given insertion energy, the perigee height is the greatest, giving the largest margin between the actual perigee height and the required minimum.

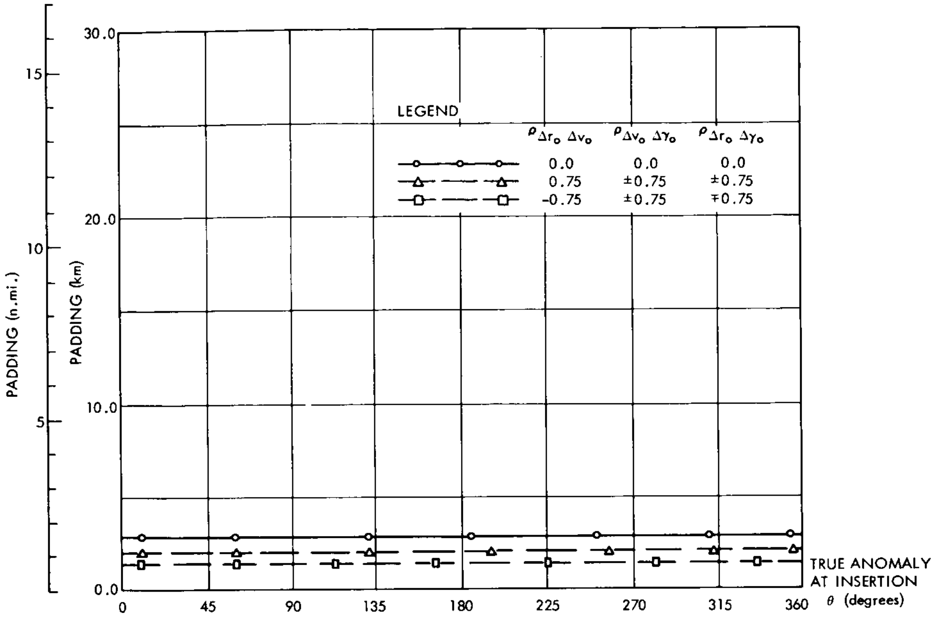


Figure 10a. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0 (Circular Orbit). Correlated (All Possible Combinations of Signed Values of 0.75) and Uncorrelated Insertion Errors.

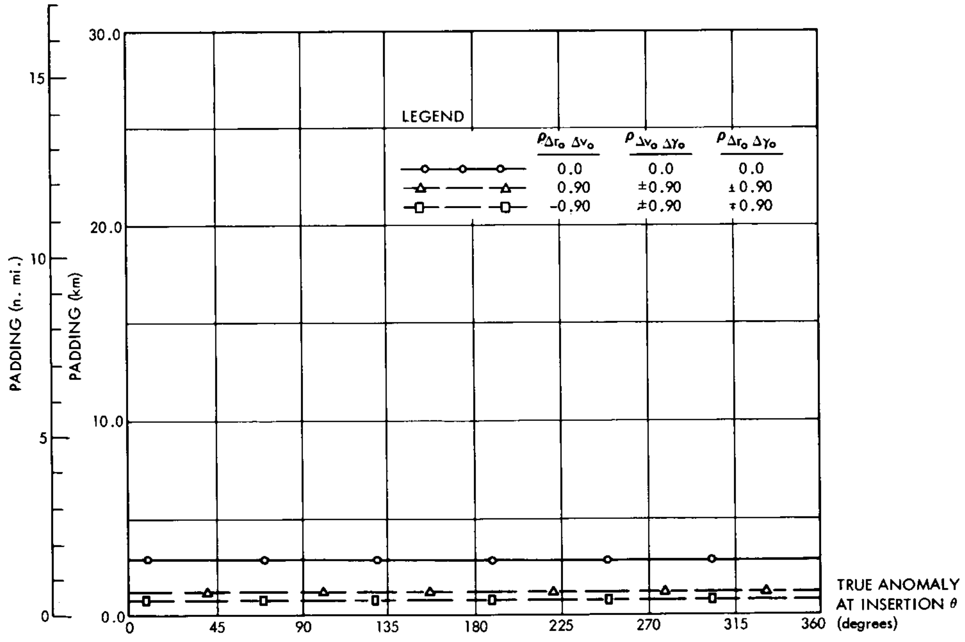


Figure 10b. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0 (Circular Orbit). Correlated (All Possible Combinations of Signed Values of 0.90) and Uncorrelated Insertion Errors.

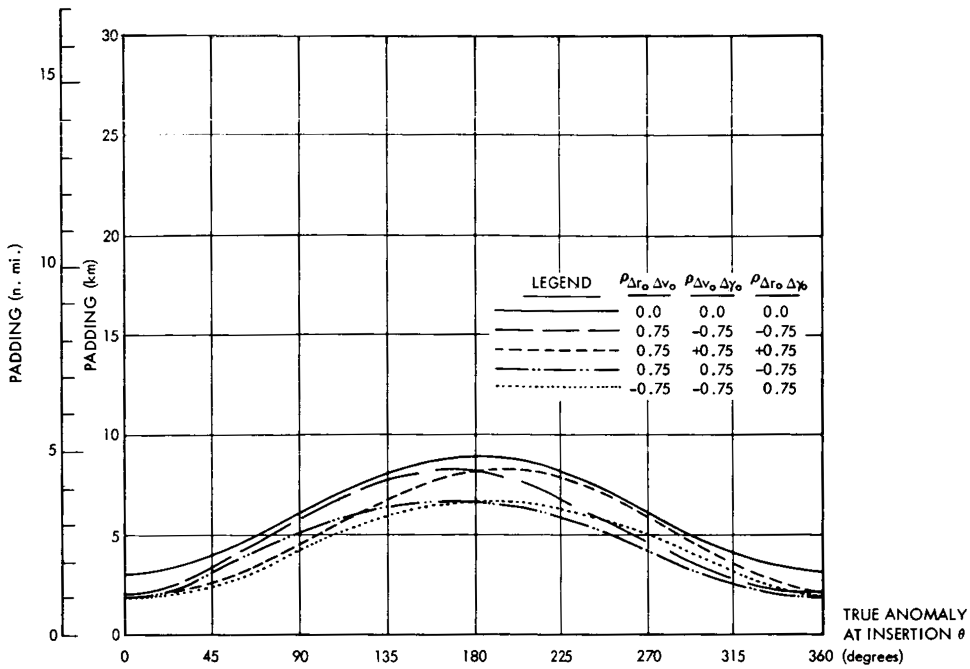


Figure 11a. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.0005. Correlated (All Possible Combinations of Signed Values of 0.75) and Uncorrelated Insertion Errors.

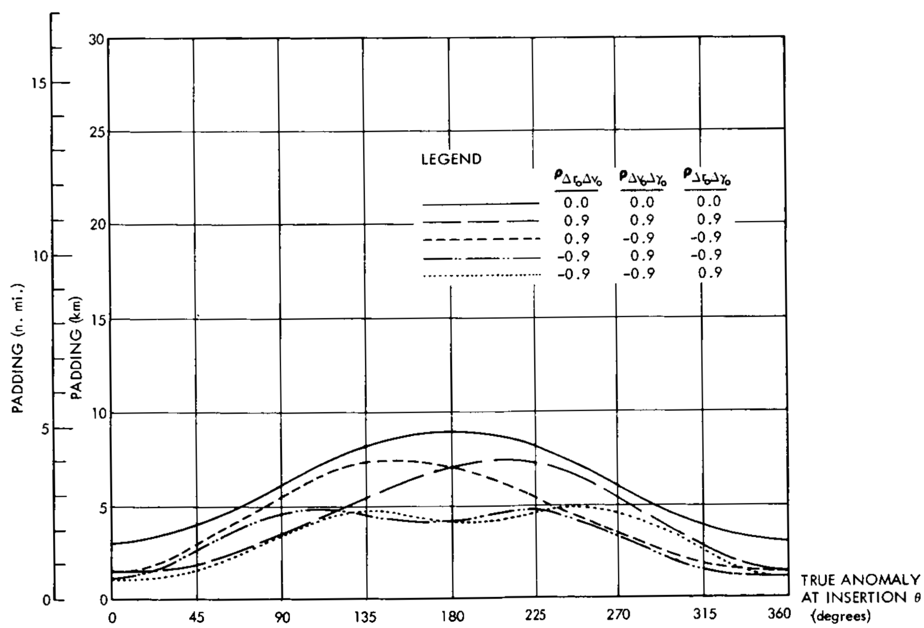


Figure 11b. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.0005. Correlated (All Possible Combinations of Signed Values of 0.90) and Uncorrelated Insertion Errors.

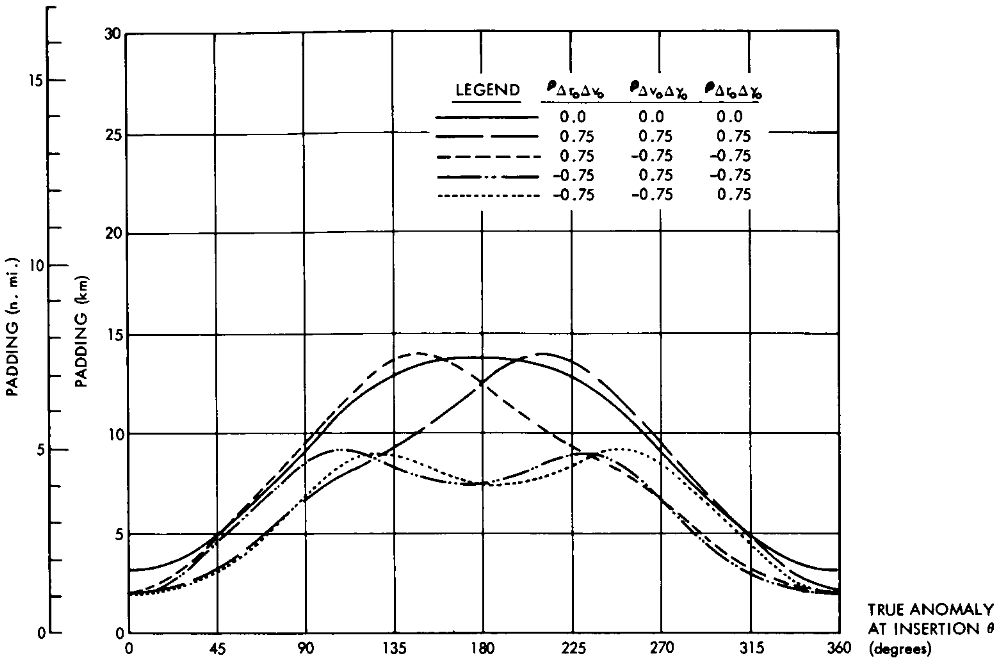


Figure 12a. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{P_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001. Correlated (All Possible Combinations of Signed Values of 0.75) and Uncorrelated Insertion Errors.

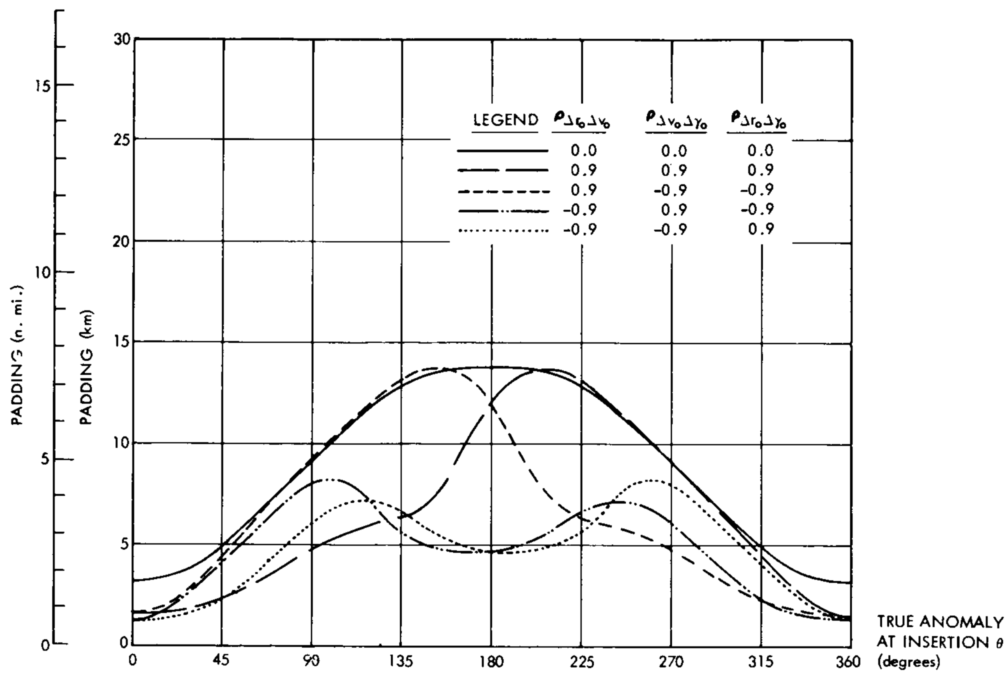


Figure 12b. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{P_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001. Correlated (All Possible Combinations of Signed Values of 0.90) and Uncorrelated Insertion Errors.

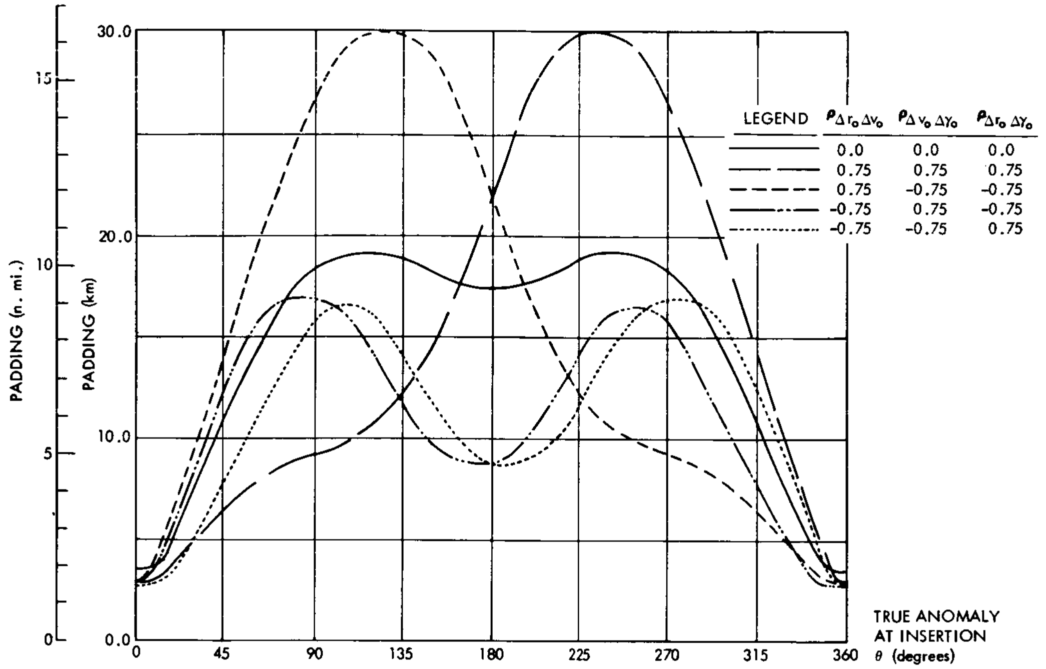


Figure 13a. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{P_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Correlated (All Possible Combinations of Signed Values of 0.75) and Uncorrelated Insertion Errors.

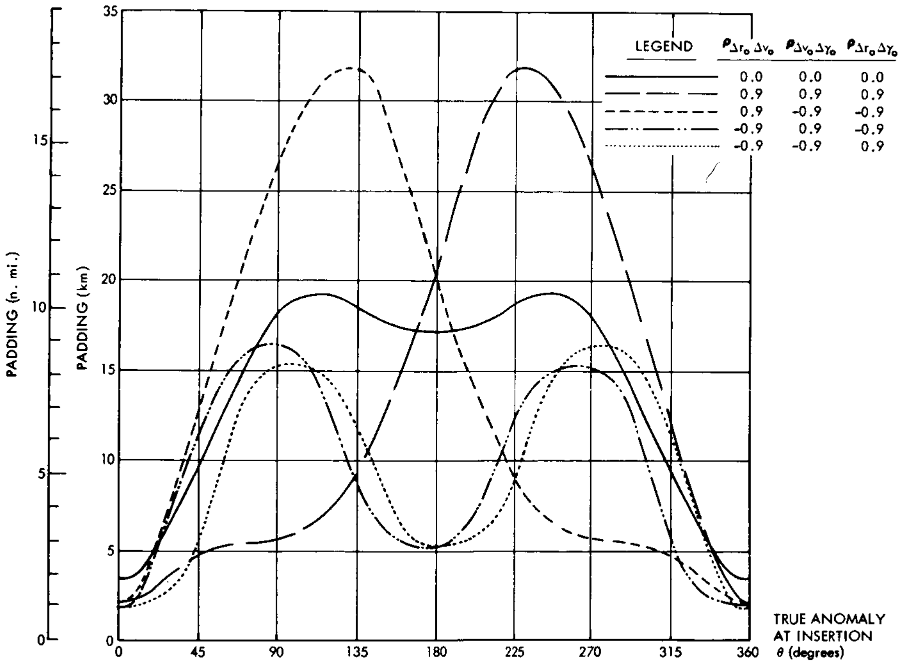
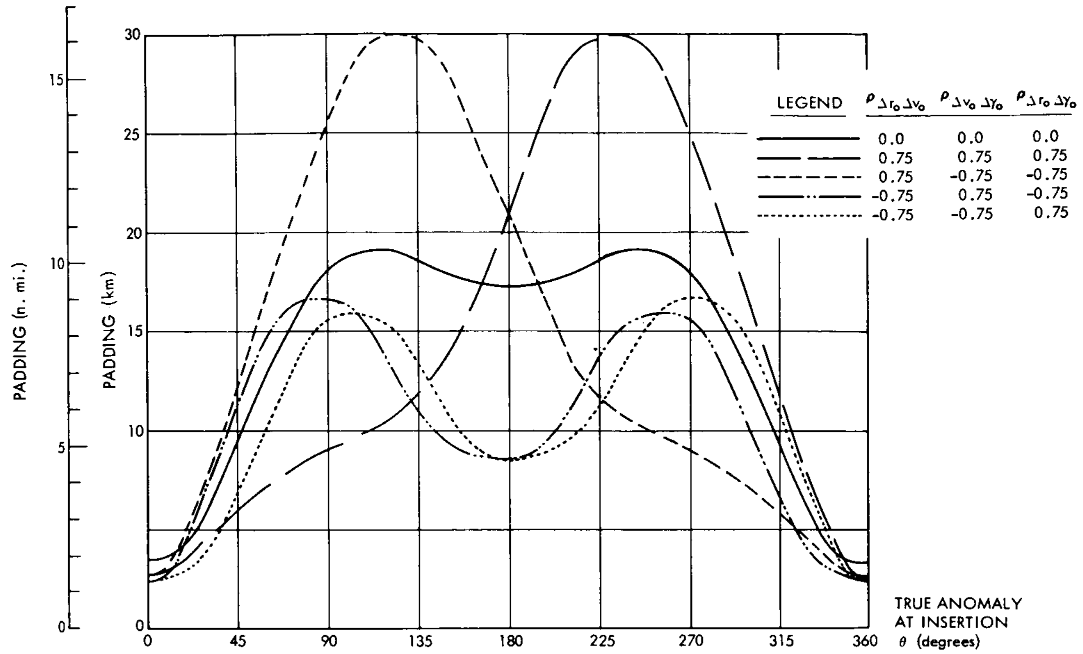
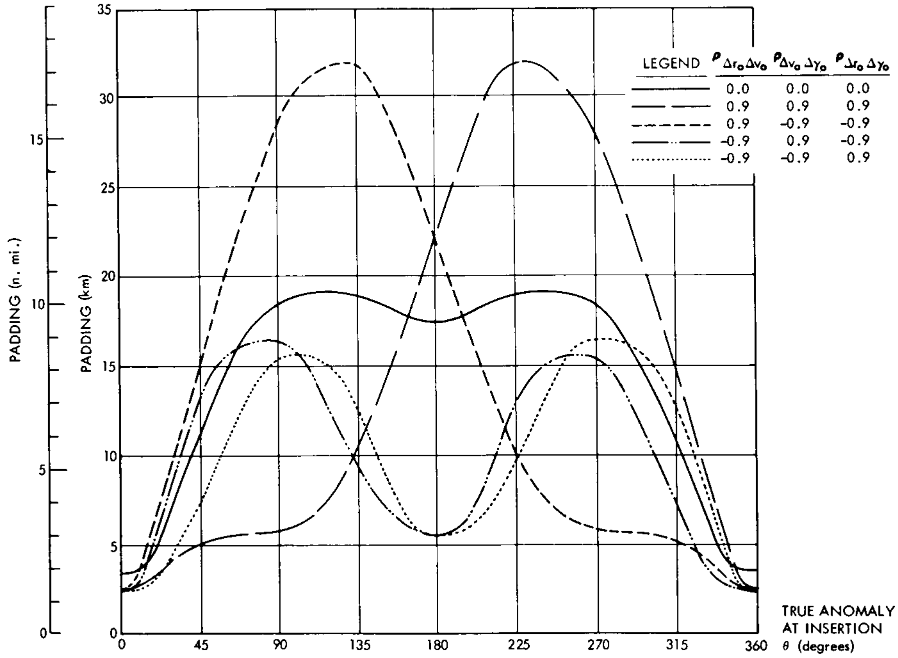


Figure 13b. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{P_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Correlated (All Possible Combinations of Signed Values of 0.90) and Uncorrelated Insertion Errors.



LEGEND	$\rho_{\Delta r_0 \Delta v_0}$	$\rho_{\Delta v_0 \Delta \gamma_0}$	$\rho_{\Delta r_0 \Delta \gamma_0}$
———	0.0	0.0	0.0
— —	0.75	0.75	0.75
- - - -	0.75	-0.75	-0.75
—··—	-0.75	0.75	-0.75
······	-0.75	-0.75	0.75

Figure 14a. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01. Correlated (All Possible Combinations of Signed Values of 0.75) and Uncorrelated Insertion Errors.



LEGEND	$\rho_{\Delta r_0 \Delta v_0}$	$\rho_{\Delta v_0 \Delta \gamma_0}$	$\rho_{\Delta r_0 \Delta \gamma_0}$
———	0.0	0.0	0.0
— —	0.9	0.9	0.9
- - - -	0.9	-0.9	-0.9
—··—	-0.9	0.9	-0.9
······	-0.9	-0.9	0.9

Figure 14b. Required Padding as a Function of True Anomaly at Insertion in Order to Insure with a 99.5% Probability that the Actual Perigee Height Exceeds the Allowed Minimum ($r_{p_{min}} - r_e$) of 75 n.mi. (139 km). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01. Correlated (All Possible Combinations of Signed Values of 0.90) and Uncorrelated Insertion Errors.

2. As shown in Chapter 3, the perigee error is smallest.

Since the circular orbit cannot be achieved in actual flight, the optimum orbit is therefore one which has the smallest eccentricity. In addition, as shown in Chapters 2 and 3, insertion at a true anomaly of $\theta = 0°$ is desirable in order to minimize the error in perigee Δr_p.

4.2 The Required Padding

In Chapter 3 it was shown that the required padding depends significantly on the eccentricity e of the parking orbit, increasing with e in the region $0 \leq e \leq 0.005$, and remaining nearly constant in the region $0.005 \leq e \leq 0.01$ (Figures 10 through 14). In order to use the smaller required padding corresponding to smaller e-values, the insertion ship must have the capability of determining the eccentricity of the orbit with sufficient accuracy. This can be clarified with an example.

The maximum required padding for an eccentricity of $e = 0.001$ is 7.5 n.mi. (14 km) according to Figures 12a and 12b in Chapter 3. In order to use this value, however, the ship must be able to verify that $e \leq 0.001$ for the actual orbit. The calculated eccentricity e_{cal} is a function of the measurement errors. Figures 15a and 15b show the cumulative distribution function $H(\Delta e)$ of the error in eccentricity ΔE for orbits with

$$e = 0.001$$

$$3\sigma_{r_o} = 2.4 \text{ n.mi. } (4.44 \text{ km})$$

$$3\sigma_{v_o} = 16 \text{ ft/sec } (4.87 \text{ m/sec})$$

$$3\sigma_{\gamma_o} = 0.16° \ (2.79 \text{ mrad})$$

For the derivation and calculation of $H(\Delta e)$ see Appendix C.

It is seen from these curves that the insertion ship cannot verify that $e \leq 0.001$. Therefore, the small padding of 7.5 n.mi. (14 km) cannot be used.

The cumulative distribution function $H(\Delta e)$ for an actual orbit with $e = 0.005$ is shown in Figures 16a and 16b. It is evident from these figures that even an eccentricity as large as $e = 0.005$ cannot be determined accurately enough by the tracking ship. Therefore, the smaller required padding corresponding to the lower e-values cannot be used.

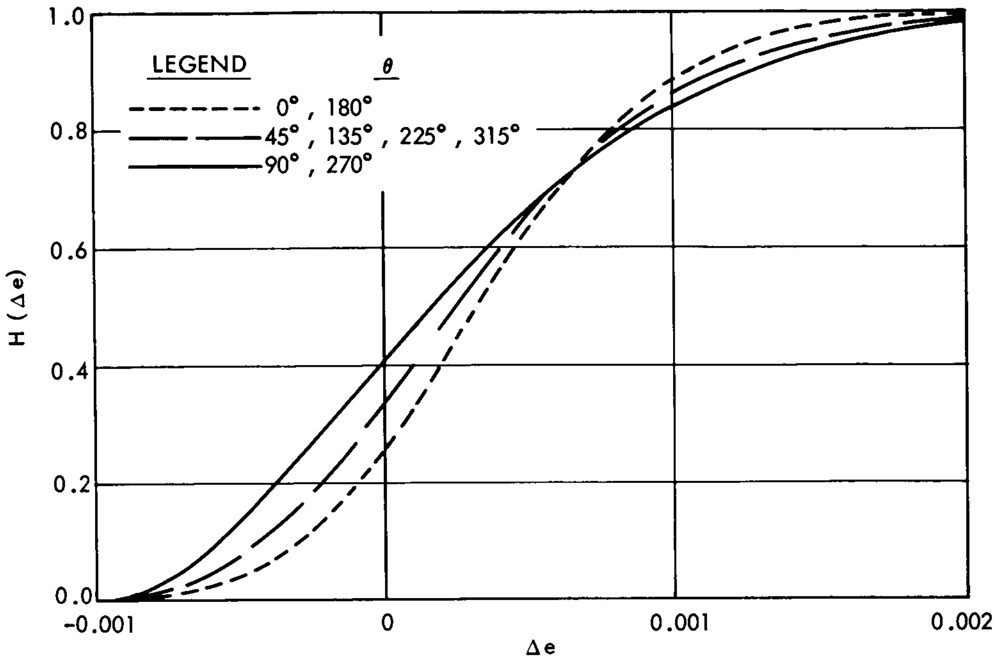


Figure 15a. Cumulative Distribution Functions of the Error in Eccentricity for Various Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001. Uncorrelated Insertion Errors.

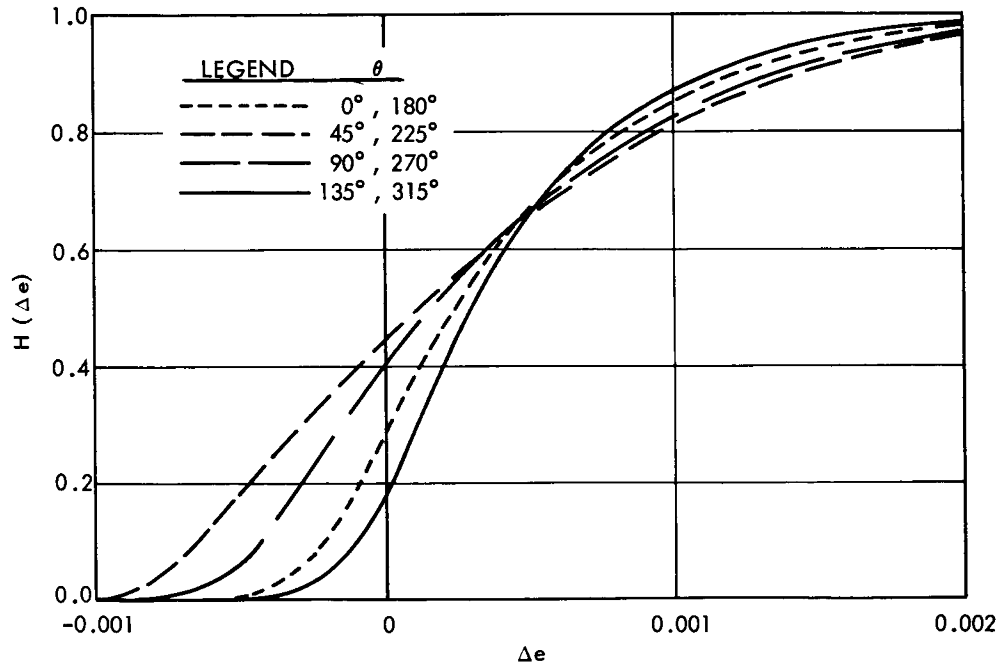


Figure 15b. Cumulative Distribution Functions of the Error in Eccentricity for Various Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001. Correlated Insertion Errors. ($\rho_{\Delta r_o \Delta v_o}$ = 0.9, $\rho_{\Delta v_o \Delta \gamma_o}$ = $\rho_{\Delta r_o \Delta \gamma_o}$ = ± 0.9)

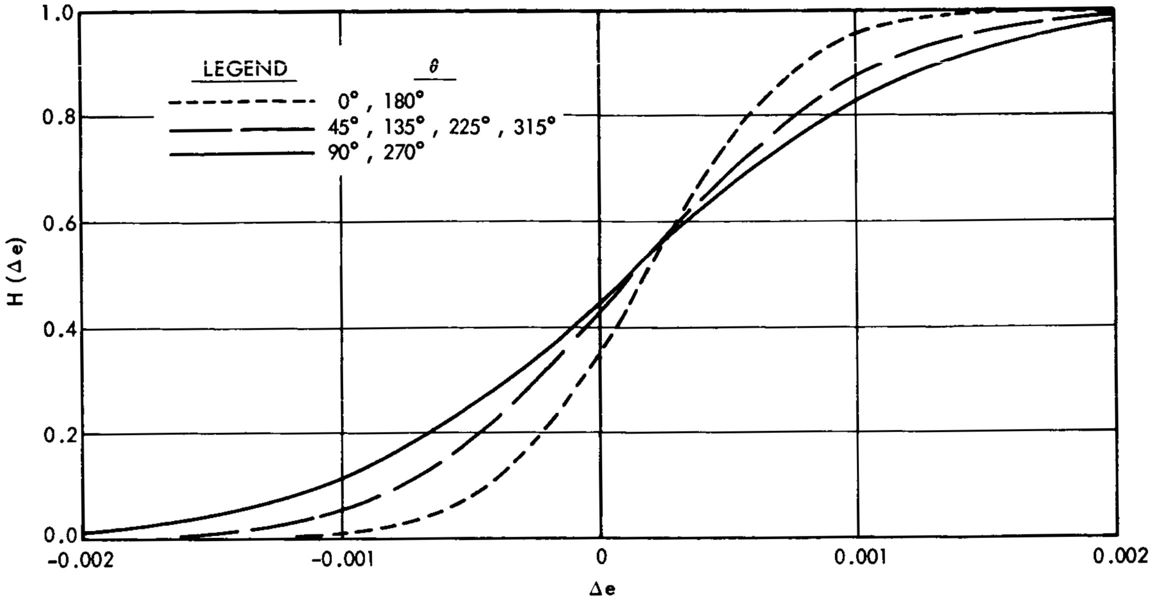


Figure 16a. Cumulative Distribution Functions of the Error in Eccentricity for Various Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Uncorrelated Insertion Errors.

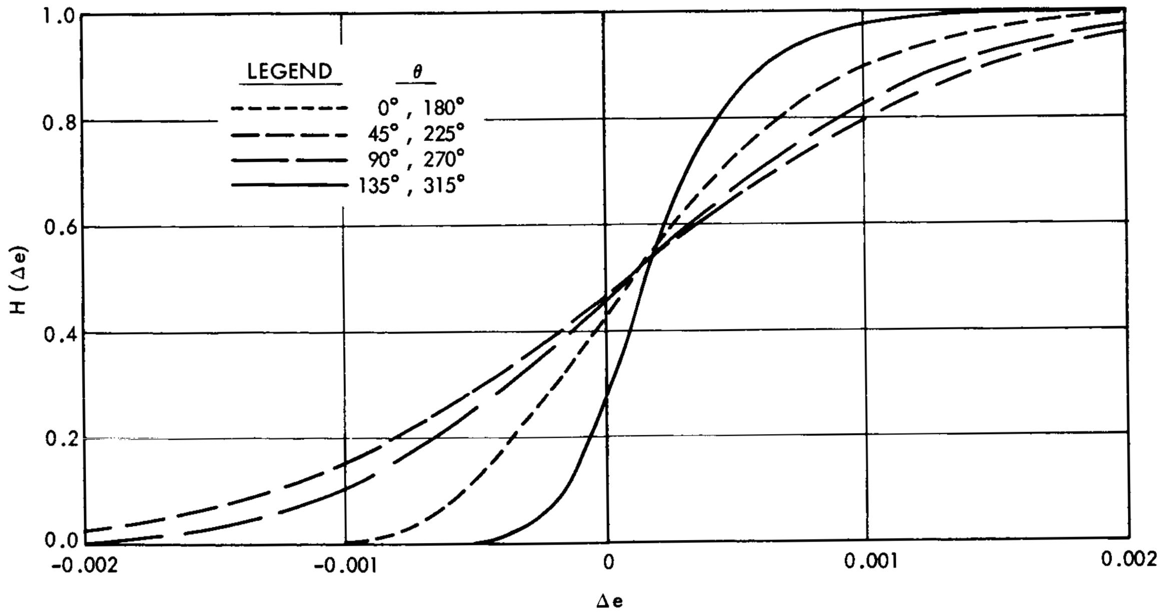


Figure 16b. Cumulative Distribution Functions of the Error in Eccentricity for Various Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Correlated Insertion Errors ($\rho_{\Delta r_o \Delta v_o}$ = +0.9, $\rho_{\Delta v_o \Delta \gamma_o}$ = $\rho_{\Delta r_o \Delta \gamma_o}$ = ±0.9)

As previously mentioned, the required padding does not increase significantly for $e \geq 0.005$. It is therefore recommended that the maximum padding of 17 n.mi. (32 km) for $e \geq 0.005$ be used as a required padding. This value insures with a 99.5% probability that the actual perigee height exceeds the required minimum perigee height for the following cases:

1. All eccentricities in the region $0 \leq e \leq 0.01$.
2. All values of true anomaly at insertion.
3. All values of correlation coefficients ρ between the insertion errors in the region $0 \leq |\rho| \leq 0.9$.

4.3 Recommendations

The results of this analysis may be briefly stated in the form of the following recommendations:

1. A padding (calculated perigee height $\geq$ required minimum perigee height + padding) of 17 n.mi. (32 km) should be used in order to insure with a 99.5% probability that a positive Go decision is correct.

2. The near-earth Apollo parking orbit should have the smallest possible eccentricity in order to achieve the largest possible margin between the actual perigee height and the required minimum perigee height.

3. Insertion of the Apollo spacecraft should occur at a true anomaly as close as possible to $\theta = 0°$ in order to minimize the perigee height error.

ACKNOWLEDGEMENTS

The authors wish to express their gratitude to Mr. R. T. Groves of the Mission and Trajectory Analysis Division for his suggestion to make an orthogonal transformation from the correlated space of the insertion errors to an uncorrelated space in order to more easily calculate the probability distribution of perigee error.

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GLOSSARY AND DEFINITION OF SYMBOLS

Random vector. Let $V_1, V_2, \ldots, V_p$ be p random variables. Then the $(p \times 1)$ vector $\mathbf{V}$,

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_p \end{pmatrix}$$

is a random vector.

Multivariate Normal Distribution. Let the p-dimensional random vector $\mathbf{V}$ have the probability density function

$$|\Omega|^{-1/2} (2\pi)^{-\frac{p}{2}} e^{-1/2 (\mathbf{V}-\eta)^T \Omega^{-1} (\mathbf{V}-\eta)}$$

where η is the vector of constants

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_p \end{pmatrix}$$

and Ω is a $(p \times p)$ positive definite matrix. Then $\mathbf{V}$ has a nonsingular multivariate normal distribution with mean vector $\mathcal{E}(\mathbf{V}) = \eta$ and covariance matrix $\mathcal{E}\left[(\mathbf{V} - \eta)(\mathbf{V} - \eta)^T\right] = \Omega$.

Uncorrelated	The random variables V_1 and V_2 are said to be uncorrelated if the coefficient of correlation $\rho_{v_1 v_2}$ between them is zero.
Error	The measured or calculated value of a quantity minus its actual or true value
a	Semi-major axis of the parking orbit

b_{ij} — Element in the i^{th} row and j^{th} column of the orthogonal matrix **B** relating the normal vector **Y** (components are uncorrelated) to the normal vector **X** (components are correlated) such that $\mathbf{Y} = \mathbf{BX}$

A — The (3×3) diagonal matrix relating the normal vector **W** to the normal vector **X** such that $\mathbf{W} = \mathbf{AX}$. The matrix **A** has the diagonal elements $\sigma_{\Delta r_o}$, $\sigma_{\Delta v_o}$, and $\sigma_{\Delta \gamma_o}$.

B — The (3×3) orthogonal matrix relating the normal vector **Y** (components are uncorrelated) to the normal vector **X** (components are correlated) such that $\mathbf{Y} = \mathbf{BX}$. The matrix **B** is also the transpose of the matrix **C**.

c_{ij} — Element in the i^{th} row and j^{th} column of the (3×3) orthogonal matrix **C** relating the normal vector **Y** (components are uncorrelated) to the normal vector **X** (components are correlated) such that $\mathbf{X} = \mathbf{CY}$. The matrix **C** is also the transpose of the matrix **B**.

C — The (3×3) orthogonal matrix relating the normal vector **Y** (components are uncorrelated) to the normal vector **X** (components are correlated) such that $\mathbf{X} = \mathbf{CY}$. The matrix **C** is also the transpose of the matrix **B**.

$\mathbf{C}_1$ — An orthogonal matrix (3×3) which diagonalizes the correlation matrix $\mathbf{P}_1$, i.e., $\mathbf{C}_1^T\mathbf{P}_1\mathbf{C}_1$ is a diagonal matrix.

$\mathbf{C}_2$ — An orthogonal (3×3) matrix which diagonalizes the correlation matrix $\mathbf{P}_2$, i.e., $\mathbf{C}_2^T\mathbf{P}_2\mathbf{C}_2$ is a diagonal matrix.

$\mathbf{C}_3$ — An orthogonal (3×3) matrix which diagonalizes the correlation matrix $\mathbf{P}_3$, i.e., $\mathbf{C}_3^T\mathbf{P}_3\mathbf{C}_3$ is a diagonal matrix.

D — A (3×3) diagonal matrix having λ_1, λ_2, and λ_3 as diagonal elements. $\mathbf{D} = \mathbf{E}^T\mathbf{PE}$ where **E** is an orthogonal matrix and **P** is the covariance matrix of **X** (correlated components).

e_{cal} — A calculated value of eccentricity.
$e_{cal} = e(r_o + \Delta r_o,\ v_o + \Delta v_o,\ \gamma_o + \Delta \gamma_o)$

e — The actual or true value of eccentricity.

$\mathcal{E}(X_1)$	The expected value of the random variable X_1. $\mathcal{E}$ is the expectation operator.
E	A (3×3) orthogonal matrix which diagonalizes the correlation matrix **P**, i.e., $\mathbf{D} = \mathbf{E}^T\mathbf{P}\mathbf{E}$ is a diagonal matrix having λ_1, λ_2, and λ_3 as diagonal elements.
$f_1(y_1)$	Probability density function of the normally distributed random variable Y_1.
$f_2(y_2)$	Probability density function of the normally distributed random variable Y_2.
$f_3(y_3)$	Probability density function of the normally distributed random variable Y_3.
$f(y_1, y_2, y_3)$	Joint probability density function of Y_1, Y_2, and Y_3 where Y_1, Y_2, and Y_3 are independent normal random variables. $f(y_1, y_2, y_3) = f_1(y_1)\ f_2(y_2)\ f_3(y_3)$.
f	The probability density function of the error in perigee ΔR_p. We have $f = F'(\Delta r_p) = f(\Delta r_p)$
$F_1(\Delta r_o)$	Cumulative distribution function of ΔR_o
F	The cumulative distribution function of the error in perigee ΔR_p where $F = P(\Delta R_p \leq \Delta r_p) = F(\Delta r_p)$
$g(x)$	The probability density function of the standardized normal random variable X.
$g(x_1, x_2, x_3)$	The joint probability density function of X_1, X_2, and X_3 where X_1, X_2, and X_3 are components of the normal vector **X**.
$G(x)$	The cumulative distribution function for the standardized normal random variable X.
$H(\Delta e)$	The cumulative distribution function for the error in eccentricity ΔE.
I	The (3×3) identity matrix.
$J(x_1, x_2, x_3)$	The Jacobian of the transformation between X_1, X_2, X_3 and Y_1, Y_2, Y_3.

$$J(x_1, x_2, x_3) = \text{mod} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

where "mod" means the absolute value of the expression following it.

$K(\Delta\theta)$ — The cumulative distribution function for the error in true anomaly $\Delta\Theta$.

l — A constant given by (see m also).

$$l = \left(\frac{2 r_o}{v_o}\right) \sigma_{\Delta v_o} + 2\sigma_{\Delta r_o} \qquad (m > 1)$$

L — Upper and lower limit (in number of standard deviations) for the approximation to the range of each independent normal random variable Y_i $(i = 1, 2, 3)$.

m — A constant given by (see l also)

$$m = \sqrt{\left[2\left(\frac{r_o}{v_o}\right)\sigma_{\Delta v_o} + \sigma_{\Delta r_o}\right]^2 + \left(r_o \sigma_{\Delta \gamma_o}\right)^2}$$

where $(m > 1)$

mod — An operator which means the absolute value of the expression following it.

$(2N + 1)$ — Total number of subdivisions (extending from $-L\sigma_{y_i}$ to $+L\sigma_{y_i}$) for the approximation to the range of each independent normally distributed random variable Y_i $(i = 1, 2, 3)$.

p_{i_k} — The probability that the discrete approximation Y_i' to the normally distributed random variable Y_i assumes the value y'_{i_k} where y'_{i_k} is the midpoint of the interval $(y'_{i_k} - \Delta y_i, \; y'_{i_k} + \Delta y_i)$. We have

$$P\left(y'_{i_k} - \Delta y_i \leq Y_i < y'_{i_k} + \Delta y_i\right) \approx P\left(Y'_i = y'_{i_k}\right) = p_{i_k}$$

where

$$(i = 1, 2, 3) \text{ and } \left(k = 1, 2, 3, \ldots, (2N + 1)\right)$$

Also

$$p_{i_k} = \frac{q_{i_k}}{S_i} \qquad (\text{see } q_{i_k} \text{ and } S_i)$$

p_{jkl} — The joint probability that the discrete random variable approximations Y'_1, Y'_2, Y'_3 to the independent normal random variables Y_1, Y_2, Y_3 assume the values y'_{1_j} , y'_{2_k}, y'_{3_l} . We have

$$p_{jkl} = P\left(Y'_1 = y'_{1_j}, Y'_2 = y'_{2_k}, Y'_3 = y'_{3_l}\right) = p_{1_j}\ p_{2_k}\ p_{3_l}$$

where $(j, k, l = 1, 2, 3, \ldots, 2N + 1)$

$P\left(Y'_i = y'_{i_k}\right)$ — The probability that Y'_i $(i = 1, 2, 3)$ takes the value y'_{i_k} $(k = 1, 2, 3, \ldots, 2N + 1)$. P denotes "probability" here.

P — Correlation matrix or covariance matrix of the normal vector **X** when $\rho_{x_1x_2} = \rho_{x_2x_3} = \rho_{x_1x_3} = \rho$, $(0 \leq \rho < 1)$.

A real symmetric positive definite square matrix of order 3.

$\mathbf{P}_1$ — Correlation matrix or covariance matrix of the normal vector **X** when $\rho_{x_1x_2} = -\rho$, $\rho_{x_1x_3} = -\rho$, $\rho_{x_2x_3} = \rho$ $(0 \leq \rho < 1)$. A real symmetric positive definite square matrix of order 3.

$\mathbf{P}_2$ — Correlation matrix or covariance matrix of the normal vector **X** when $\rho_{x_1x_2} = -\rho$, $\rho_{x_1x_3} = \rho$, $\rho_{x_2x_3} = -\rho$, $(0 \leq \rho < 1)$. A real symmetric positive definite square matrix of order 3.

$\mathbf{P}_3$ — Correlation matrix or covariance matrix of the normal vector **X** when $\rho_{x_1x_2} = \rho$, $\rho_{x_1x_3} = -\rho$, $\rho_{x_2x_3} = -\rho$, $(0 \leq \rho < 1)$. A real symmetric positive definite square matrix of order 3.

q_{i_k} — Value of the normal probability density function of Y_i $(i = 1, 2, 3)$ at the point y'_{i_k} $(k = 1, 2, 3, \ldots, 2N + 1)$, i.e.,

$$q_{i_k} = \left(2\pi\sigma_{y_i}^2\right)^{-1/2} e^{-\left(y'_{i_k}\right)^2/2\sigma_{y_i}^2}$$

(see also p_{i_k}).

Q The covariance matrix of the normal vector **Y**. We have

$$\mathbf{Q} = \mathcal{E}(\mathbf{YY}^T) = \mathbf{BPB}^T = \mathbf{C}^T\mathbf{PC}.$$

r_e Radius of the earth.

r_p Perigee radius.

$r_{p_{cal}}$ Calculated perigee radius (from the ship's tracking data).

$r_{p_{min}}$ Required minimum perigee radius.

r_o Magnitude of the radius vector at insertion.

S_i A normalizing sum, i.e.,

$$S_i = \sum_{k=1}^{2N+1} q_{i_k} \qquad (i = 1, 2, 3)$$

(see also p_{i_k}).

v_o The magnitude of the velocity vector of insertion.

W Normal random vector having the components ΔR_o, ΔV_o, and $\Delta \Gamma_o$.

x-space Refers to the correlated "space" of the normal random variables X_1, X_2, and X_3.

$\vec{x}_1$ Normalized eigenvector corresponding to the eigenvalue $\lambda_1 = 1 + 2\rho \quad (0 \le \rho < 1)$.

$\vec{x}_2$ Normalized eigenvector corresponding to the eigenvalue $\lambda_2 = 1 - \rho \quad (0 \le \rho < 1)$.

$\vec{x}_1 \cdot \vec{x}_2$ The dot product of the normalized eigenvector $\vec{x}_1$ with the normalized eigenvector $\vec{x}_2$.

$\vec{x}_3$	The normalized eigenvector corresponding to the eigenvalue $\lambda_3 = 1 - \rho \quad (0 \leq \rho < 1)$.
$\mathbf{X}$	Normal random vector having correlated standardized normal random variables X_1, X_2, and X_3 as components.
X_1	Standardized normal random variable. A component of $\mathbf{X}$.
X_2	Standardized normal random variable. A component of $\mathbf{X}$.
X_3	Standardized normal random variable. A component of $\mathbf{X}$.
X'_1, X'_2, X'_3	Discrete random variable approximations to the standardized normal random variables X_1, X_2, and X_3.
X	A standardized normal random variable.
y-space	Refers to the uncorrelated "space" of the uncorrelated normal random variables Y_1, Y_2, and Y_3.
$\mathbf{Y}$	Normal random vector having the uncorrelated normal random variables Y_1, Y_2, and Y_3 as components.
y'_{1_j}, y'_{2_k}, y'_{3_l}	Values that the discrete random variable approximations Y'_1, Y'_2, Y'_3 to the uncorrelated normal random variables Y_1, Y_2, Y_3 can take $(j, k, l = 1, 2, 3, \ldots, 2N + 1)$.
Y_1, Y_2, Y_3	Uncorrelated normal random variables. Components of the normal vector $\mathbf{Y}$.
Y'_1, Y'_2, Y'_3	Discrete random variable approximations to the uncorrelated normal random variables Y_1, Y_2, and Y_3.
$\mathbf{Z}$	Normal random vector having the uncorrelated standardized normal random variables Z_1, Z_2, and Z_3 as components.
Z_1, Z_2, Z_3	Uncorrelated standardized normal random variables. Components of the normal random vector $\mathbf{Z}$.
α_i	Coefficients in variational equation for Δr_p. Functions of r_o, v_o, and γ_o. $(i = 1, 2, 3, 4)$
β_i	Coefficients in variational equation for Δr_p. Functions of r_o, v_o, and γ_o. $(i = 1, 2, \ldots, 9)$.

Δe	A value for the error in eccentricity ΔE due to metric tracking errors by the insertion ship. We have $\Delta e = e_{cai} - e$.
ΔE	A random variable representing the error in eccentricity. ΔE has the cumulative distribution function $H(\Delta e)$.
$\Delta \gamma_o$	A value for the error in flight path angle at insertion $\Delta \Gamma_o$ where $\Delta \Gamma_o$ is normally distributed.
$\Delta \Gamma_o$	A normal random variable representing the error in flight path angle at insertion.
$\Delta \Gamma_o'$	A discrete random variable approximation to the error in flight path angle at insertion $\Delta \Gamma_o$($\Delta \Gamma_o$ is normally distributed).
Δr_p	A value for the error in perigee radius ΔR_p.
ΔR_p	A random variable representing the error in perigee radius. ΔR_p has the cumulative distribution function $F(\Delta r_p)$, and probability density function $f(\Delta r_p)$.
$\Delta R_p'$	A discrete random variable approximation to the error in perigee radius ΔR_p.
Δr_o	A value for the error in insertion radius ΔR_o where ΔR_o is normally distributed.
$\Delta \theta$	The error in true anomaly at insertion. A value $\Delta \Theta$ can assume.
$\Delta \Theta$	A random variable representing the error in true anomaly of the spacecraft at insertion.
ΔR_o	A normal random variable representing the error in insertion radius.
$\Delta R_o'$	A discrete random variable approximation to the normal random variable ΔR_o where ΔR_o represents the error in radius at insertion.
Δv_o	A value for the error in the magnitude of the insertion velocity ΔV_o which is a normal random variable.

ΔV_o — A normal random variable representing the error in the magnitude of the velocity at insertion.

$\Delta V_o'$ — A discrete random variable approximation to the normal random variable ΔV_o which represents the error in insertion velocity magnitude.

Δy_i — A subdivision of the range of the normal random variable approximation for Y_i ($i = 1, 2, 3$). We have

$$\Delta y_i = \left(\frac{L}{N}\right)\sigma_{y_i} \quad \text{where } \sigma_{y_i} \text{ is the standard deviation of } Y_i.$$

γ_o — Flight path angle at insertion measured from the normal to the radius vector at insertion, pointing in flight direction, against the direction of motion (i.e., clockwise if motion is counterclockwise), such that the direction at $\gamma_o = 90°$ is radially away from the center of attraction.

θ — True anomaly of the spacecraft at insertion into parking orbit — the angle measured from the perigee radius to the radius vector at insertion in the direction of motion.

λ — A variable in the secular (characteristic) equation of **P**.

$\lambda_1, \lambda_2, \lambda_3$ — Eigenvalues of the covariance matrix **P** or elements of the diagonal matrix $\mathbf{C}^T\mathbf{P}\mathbf{C}$, the covariance matrix of **Y**. We have $\lambda_1 = \sigma_{y_1}^2$, $\lambda_2 = \sigma_{y_2}^2$, and $\lambda_3 = \sigma_{y_3}^2$.

$\mathbf{\Lambda}$ — Covariance matrix of the normal random vector **W** having correlated normal random variables ΔR_o, ΔV_o, and $\Delta \Gamma_o$ as components.

μ — Gravitational constant of the earth (398603.2 km^3/sec^2).

ρ — Coefficient of correlation ($|\rho| \leq 1$).

$\rho_{\Delta r_o \Delta v_o}$ — Coefficient of correlation between ΔR_o and ΔV_o.

$\rho_{\Delta v_o \Delta \gamma_o}$ — Coefficient of correlation between ΔV_o and $\Delta \Gamma_o$.

$\rho_{\Delta r_o \Delta \gamma_o}$ — Coefficient of correlation between ΔR_o and $\Delta \Gamma_o$.

$\rho_{x_1 x_2}$	Coefficient of correlation between X_1 and X_2. (X_1 and X_2 are standardized normal random variables).
$\rho_{x_2 x_3}$	Coefficient of correlation between X_2 and X_3. (X_2 and X_3 are standardized normal random variables).
$\rho_{x_1 x_3}$	Coefficient of correlation between X_1 and X_3. (X_1 and X_3 are standardized normal random variables).
$\sigma_{y_1}, \sigma_{y_2}, \sigma_{y_3}$	Standard deviations of the uncorrelated normal random variables Y_1, Y_2, and Y_3.
$\sigma_{y_1}^2, \sigma_{y_2}^2, \sigma_{y_3}^2$	Variances of the uncorrelated normal random variables Y_1, Y_2, Y_3. Diagonal elements of $C^T PC$.
$\sigma_{r_o}, \sigma_{v_o}, \sigma_{\gamma_o}$	Standard deviations of the correlated normal random variables $(r_o + \Delta R_o)$, $(v_o + \Delta V_o)$, and $(\gamma_o + \Delta\Gamma_o)$.
$\sigma_{\Delta r_o}, \sigma_{\Delta v_o}, \sigma_{\Delta\gamma_o}$	Standard deviations of the correlated normal random variables ΔR_o, ΔV_o, and $\Delta\Gamma_o$.
$\sigma_{\Delta r_o}^2, \sigma_{\Delta v_o}^2, \sigma_{\Delta\gamma_o}^2$	Variances of the correlated normal random variables ΔR_o, ΔV_o, and $\Delta\Gamma_o$. Diagonal elements of $\mathbf{\Lambda}$, the covariance matrix of $\mathbf{W}$.
$\sigma_{\Delta r_o \Delta v_o}$	Covariance between the normal random variables ΔR_o and ΔV_o. Off diagonal element of $\mathbf{\Lambda}$, the covariance matrix of $\mathbf{W}$. We have $\sigma_{\Delta r_o \Delta v_o} = \rho_{\Delta r_o \Delta v_o}\sigma_{\Delta r_o}\sigma_{\Delta v_o}$.
$\sigma_{\Delta v_o \Delta\gamma_o}$	Covariance between the normal random variables ΔV_o and $\Delta\Gamma_o$. Off diagonal element of $\mathbf{\Lambda}$, the covariance matrix of $\mathbf{W}$. We have $\sigma_{\Delta v_o \Delta\gamma_o} = \rho_{\Delta v_o \Delta\gamma_o}\sigma_{\Delta v_o}\sigma_{\Delta\gamma_o}$.
$\sigma_{\Delta r_o \Delta\gamma_o}$	Covariance between the normal random variables ΔR_o and $\Delta\Gamma_o$. Off diagonal element of $\mathbf{\Lambda}$, the covariance matrix of $\mathbf{W}$. We have $\sigma_{\Delta r_o \Delta\gamma_o} = \rho_{\Delta r_o \Delta\gamma_o}\sigma_{\Delta r_o}\sigma_{\Delta\gamma_o}$.
$\sigma_{x_1 x_2}$	Covariance between the standardized normal random variables X_1 and X_2. Off diagonal element of $\mathbf{P}$, the covariance matrix of $\mathbf{X}$ (also called the correlation matrix). We have $\sigma_{x_1 x_2} = \rho_{x_1 x_2}\sigma_{x_1}\sigma_{x_2} = \rho_{x_1 x_2}$ since X_1 and X_2 are standardized variables.

$\sigma_{x_2x_3}$ Covariance between the standardized normal random variables X_2 and X_3. Off diagonal element of **P**, the covariance matrix of **X**. We have $\sigma_{x_2x_3} = \rho_{x_2x_3}\sigma_{x_2}\sigma_{x_3} = \rho_{x_2x_3}$ since X_2 and X_3 are standardized variables.

$\sigma_{x_1x_3}$ Covariance between the standardized normal random variables X_1 and X_3. Off diagonal element of **P**, the covariance matrix of **X**. We have $\sigma_{x_1x_3} = \rho_{x_1x_3}\sigma_{x_1}\sigma_{x_3} = \rho_{x_1x_3}$ since both X_1 and X_3 are standardized variables.

APPENDIX A

EXPRESSION FOR THE PERIGEE RADIUS IN TERMS OF THE INSERTION PARAMETERS

The basic Keplerian two-body equations of motion which relate the insertion parameters r_o, v_o, and γ_o to the orbital elements a, e, and θ (see Figure 1) are:

$$v_o^2 = \mu\left(\frac{2}{r_o} - \frac{1}{a}\right) \tag{A.1}$$

$$(1 - e^2) = \frac{v_o^2\, r_o^2 \cos^2\gamma_o}{\mu a} \tag{A.2}$$

$$r_o = \frac{a\,(1 - e^2)}{1 + e\cos\theta} \tag{A.3}$$

where r_o, v_o, γ_o are respectively the magnitude of the radius vector at insertion, the magnitude of the velocity vector at insertion, and the flight path angle at insertion. The semi-major axis is a, the eccentricity of the parking orbit is e, and the true anomaly of the spacecraft at insertion is θ. The gravitational constant is μ.

The perigee radius r_p is the value of r_o when $\theta = 0°$. From (A.3) we have:

$$r_p = a\,(1 - e)\,. \tag{A.4}$$

Rewriting (A.1)

$$a = \frac{r_o}{2 - \left(\frac{r_o v_o^2}{\mu}\right)} \tag{A.5}$$

Substituting (A.5) into (A.2) and solving for e we have

$$e = \sqrt{\sin^2\gamma_o + \left(\frac{r_o v_o^2}{\mu} - 1\right)^2 \cos^2\gamma_o} \tag{A.6}$$

where the positive sign is taken by definition. Using (A.5) and (A.6) we can write (A.4) as:

$$r_p = \frac{r_o}{2 - \left(\frac{r_o v_o^2}{\mu}\right)} \left\{ 1 - \sqrt{\sin^2 \gamma_o + \left(\frac{r_o v_o^2}{\mu} - 1\right)^2 \cos^2 \gamma_o} \right\}. \tag{A.7}$$

APPENDIX B

THE CUMULATIVE DISTRIBUTION FUNCTION OF PERIGEE ERROR BY NUMERICAL INTEGRATION

The perigee radius r_p can be expressed as a function of the insertion parameters r_o, v_o, and γ_o (Appendix A)

$$r_p = r_p(r_o, v_o, \gamma_o) = \frac{r_o}{2 - \left(\frac{r_o v_o^2}{\mu}\right)} \left\{ 1 - \sqrt{\sin^2 \gamma_o + \left(\frac{r_o v_o^2}{\mu} - 1\right)^2 \cos^2 \gamma_o} \right\}. \tag{B.1}$$

In calculating these insertion parameters from measurements, the errors Δr_o, Δv_o, and $\Delta \gamma_o$ are introduced. Therefore, the calculated perigee radius $r_{P_{cal}}$ will deviate from the actual perigee radius r_p by the amount Δr_p.

$$\Delta r_p = r_{P_{cal}} - r_p = r_p(r_o + \Delta r_o, v_o + \Delta v_o, \gamma_o + \Delta \gamma_o) - r_p(r_o, v_o, \gamma_o). \tag{B.2}$$

Let us denote by ΔR_o, ΔV_o, and $\Delta \Gamma_o$ random variables representing the errors in the calculation of the insertion parameters.[1] Further, let us assume that the random vector

$$\mathbf{W} = \begin{pmatrix} \Delta R_o \\ \Delta V_o \\ \Delta \Gamma_o \end{pmatrix} \tag{B.3}$$

has a trivariate normal distribution with zero mean vector and covariance matrix $\mathbf{\Lambda}$

[1] It should be noted that Δr_o is a value which the random variable ΔR_o can assume. The probability that ΔR_o is less than or equal to Δr_o is a function of Δr_o. Mathematically, we write

$$F_1(\Delta r_o) = P(\Delta R_o \leq \Delta r_o)$$

where $F_1(\Delta r_o)$ is the cumulative distribution function of ΔR_o.

$$\mathbf{\Lambda} = \begin{pmatrix} \sigma^2_{\Delta r_o} & \sigma_{\Delta r_o \Delta v_o} & \sigma_{\Delta r_o \Delta \gamma_o} \\ \sigma_{\Delta r_o \Delta v_o} & \sigma^2_{\Delta v_o} & \sigma_{\Delta v_o \Delta \gamma_o} \\ \sigma_{\Delta r_o \Delta \gamma_o} & \sigma_{\Delta v_o \Delta \gamma_o} & \sigma^2_{\Delta \gamma_o} \end{pmatrix} \tag{B.4}$$

where $\sigma^2_{\Delta r_o}$, $\sigma^2_{\Delta v_o}$, and $\sigma^2_{\Delta \gamma_o}$ are the variances of ΔR_o, ΔV_o, and $\Delta \Gamma_o$ respectively, and $\sigma_{\Delta r_o \Delta v_o}$, $\sigma_{\Delta v_o \Delta \gamma_o}$, and $\sigma_{\Delta r_o \Delta \gamma_o}$ are the covariances between these random variables.[2]

Then $\mathbf{W}$ has the density function[3]

$$(2\pi)^{-\frac{3}{2}} \, |\mathbf{\Lambda}|^{-\frac{1}{2}} \, e^{-\frac{1}{2} \mathbf{W}^T \mathbf{\Lambda}^{-1} \mathbf{W}} \tag{B.5}$$

Since $\mathbf{W}$ has a trivariate normal distribution, ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are normally distributed random variables, and

$$\Delta R_p = r_p (r_o + \Delta R_o, v_o + \Delta V_o, \gamma_o + \Delta \Gamma_o) - r_p (r_o, v_o, \gamma_o) \tag{B.6}$$

is a random variable with a cumulative distribution function $F(\Delta r_p)$ and a probability density function $f(\Delta r_p)$. Although $F(\Delta r_p)$ can be expressed in integral form, the integral cannot be expressed in terms of known or tabulated functions. It must be evaluated by numerical techniques — either a Monte Carlo approach or numerical integration. The latter approach will be described in the following paragraphs. In an effort to simplify notation we will use matrix algebra wherever possible.

Since ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are jointly distributed with a trivariate normal distribution, they can be expressed as linear functions of 3 uncorrelated normally distributed random variables, Y_1, Y_2, and Y_3. And, since uncorrelated normally

(2) By definition $\sigma_{\Delta r_o \Delta v_o} = \rho_{\Delta r_o \Delta v_o} \sigma_{\Delta r_o} \sigma_{\Delta v_o}$ where $\sigma_{\Delta r_o \Delta v_o}$ is the covariance between ΔR_o and ΔV_o, $\rho_{\Delta r_o \Delta v_o}$ is the coefficient of correlation between ΔR_o and ΔV_o, and $\sigma_{\Delta r_o}$ and $\sigma_{\Delta v_o}$ are the standard deviations of ΔR_o and ΔV_o respectively.

(3) $\mathbf{W}^T$ is the transpose of $\mathbf{W}$, $\mathbf{\Lambda}^{-1}$ is the inverse of $\mathbf{\Lambda}$ and $|\mathbf{\Lambda}|$ is the determinant of $\mathbf{\Lambda}$.

distributed random variables are necessarily independent (in the probability sense), their joint probability density function is given by the product of their marginal distributions. This is the essence of the technique. First, a linear transformation is made from the correlated space of ΔR_o, ΔV_o, and $\Delta \Gamma_o$ to an uncorrelated space of Y_1, Y_2, and Y_3. Computations are made in the uncorrelated space to obtain the joint probability density function of Y_1, Y_2, and Y_3. Then, using the inverse linear transformation, the joint density of ΔR_o, ΔV_o, and $\Delta \Gamma_o$ is obtained. Finally, the probability density function of perigee error ΔR_p is calculated using (B.6) and values for r_o, v_o, and γ_o. The cumulative distribution function is obtained by summing the probability density function of ΔR_p.

Let us first express ΔR_o, ΔV_o, and $\Delta \Gamma_o$ in terms of standardized normal random variables X_1, X_2, and X_3.[4] The reason for doing this is to avoid numerical problems in the machine program which diagonalizes the covariance matrix.

$$\begin{aligned} \Delta R_o &= \sigma_{\Delta r_o} X_1 \\ \Delta V_o &= \sigma_{\Delta v_o} X_2 \\ \Delta \Gamma_o &= \sigma_{\Delta \gamma_o} X_3 \end{aligned} \tag{B.7}$$

From (B.7) it is seen that the coefficients of correlation between the variables X_1, X_2, and X_3 are the same as the coefficients of correlation between the variables ΔR_o, ΔV_o, and $\Delta \Gamma_o$. For example, the coefficient of correlation between X_1 and X_2, $\rho_{x_1 x_2}$ is the same as the coefficient of correlation between ΔR_o and ΔV_o, since[5]

$$\rho_{x_1 x_2} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1} \sigma_{x_2}} = \frac{\sigma_{\Delta r_o} \sigma_{\Delta v_o} \mathcal{E}(X_1 X_2)}{\sigma_{\Delta r_o} \sigma_{\Delta v_o}} = \frac{\mathcal{E}(\sigma_{\Delta r_o} X_1 \sigma_{\Delta v_o} X_2)}{\sigma_{\Delta r_o} \sigma_{\Delta v_o}} \tag{B.8}$$

[4] The standardized normal random variable X has zero mean and unit variance, and probability density function $g(x)$ given by:

$$g(x) = (2\pi)^{-1/2} e^{-x^2/2} \quad (-\infty < x < \infty).$$

[5] $\mathcal{E}(X_1)$ is the expectation of a random variable X_1. $\mathcal{E}$ is the expectation operator.

$$= \frac{\mathcal{E}(\Delta R_o \Delta V_o)}{\sigma_{\Delta r_o} \sigma_{\Delta v_o}} = \frac{\sigma_{\Delta r_o \Delta v_o}}{\sigma_{\Delta r_o} \sigma_{\Delta v_o}} = \rho_{\Delta r_o \Delta v_o} \qquad \text{(B.8 cont'd.)}$$

Denoting by $\mathbf{X}$ the random vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{B.9}$$

and by $\mathbf{A}$ the matrix

$$\mathbf{A} = \begin{pmatrix} \sigma_{\Delta r_o} & 0 & 0 \\ 0 & \sigma_{\Delta v_o} & 0 \\ 0 & 0 & \sigma_{\Delta \gamma_o} \end{pmatrix} \tag{B.10}$$

the transformation indicated in (B.7) can be written as

$$\mathbf{W} = \mathbf{AX} \tag{B.11}$$

Thus $\mathbf{X}$ has a trivariate normal distribution with zero mean vector and covariance matrix $\mathbf{P}$ (also called the correlation matrix) given by[6]

$$\mathbf{P} = \mathcal{E}(\mathbf{XX}^T) = \mathcal{E}\left[\mathbf{A}^{-1}\mathbf{W}(\mathbf{A}^{-1}\mathbf{W})^T\right] = \mathcal{E}\left[\mathbf{A}^{-1}\mathbf{WW}^T(\mathbf{A}^{-1})^T\right]$$

$$= \mathbf{A}^{-1}\mathcal{E}(\mathbf{WW}^T)(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}\Lambda(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}\Lambda\mathbf{A}^{-1} \tag{B.12}$$

[6]The covariance matrix of a random vector $\mathbf{X}$ with zero mean is defined as $\mathcal{E}(\mathbf{XX}^T)$ where $\mathcal{E}$ is the expectation operator.

$$= \begin{pmatrix} 1 & \rho_{\Delta r_o \Delta v_o} & \rho_{\Delta r_o \Delta \gamma_o} \\ \rho_{\Delta r_o \Delta v_o} & 1 & \rho_{\Delta v_o \Delta \gamma_o} \\ \rho_{\Delta r_o \Delta \gamma_o} & \rho_{\Delta v_o \Delta \gamma_o} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{x_1 x_2} & \rho_{x_1 x_3} \\ \rho_{x_1 x_2} & 1 & \rho_{x_2 x_3} \\ \rho_{x_1 x_3} & \rho_{x_2 x_3} & 1 \end{pmatrix} \qquad \text{(B.12 cont'd.)}$$

We now wish to express $\mathbf{X}$ as a linear function of a normal random vector $\mathbf{Y}$ having components which are uncorrelated. Then, since $\mathbf{W}$ is a linear function of $\mathbf{X}$, the components of $\mathbf{W}$ will also be linear functions of these same uncorrelated random variables. Denoting by Y_1, Y_2, and Y_3 uncorrelated normally distributed random variables we can write the system of equations as

$$\begin{aligned} X_1 &= c_{11} Y_1 + c_{12} Y_2 + c_{13} Y_3 \\ X_2 &= c_{21} Y_1 + c_{22} Y_2 + c_{23} Y_3 \\ X_3 &= c_{31} Y_1 + c_{32} Y_2 + c_{33} Y_3 \end{aligned} \qquad \text{(B.13)}$$

or in matrix form

$$\mathbf{X} = \mathbf{C}\mathbf{Y} \qquad \text{(B.14)}$$

where

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \qquad \text{(B.15)}$$

and $\mathbf{C}$ is the nonsingular matrix of coefficients

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \qquad \text{(B.16)}$$

Since $\mathbf{C}$ is a nonsingular square matrix, the inverse $\mathbf{B} = \mathbf{C}^{-1}$ exists, and we may write the $\mathbf{Y}$ vector as[7]

$$\mathbf{Y} = \mathbf{C}^{-1}\mathbf{X} = \mathbf{BX} \tag{B.17}$$

or

$$\begin{aligned} Y_1 &= b_{11} X_1 + b_{12} X_2 + b_{13} X_3 \\ Y_2 &= b_{21} X_1 + b_{22} X_2 + b_{23} X_3 \\ Y_3 &= b_{31} X_1 + b_{32} X_2 + b_{33} X_3 \end{aligned} \tag{B.18}$$

Since Y_1, Y_2, and Y_3 are uncorrelated, the following relations hold among the elements of $\mathbf{B}$ and the variances and covariances of X_1, X_2, X_3 and Y_1, Y_2, and Y_3.

$$\begin{aligned} \sigma_{y_1}^2 &= b_{11}^2 + b_{12}^2 + b_{13}^2 + 2\,(b_{11} b_{12} \rho_{x_1 x_2} + b_{12} b_{13} \rho_{x_2 x_3} + b_{11} b_{13} \rho_{x_1 x_3}) \\ \sigma_{y_2}^2 &= b_{21}^2 + b_{22}^2 + b_{23}^2 + 2\,(b_{21} b_{22} \rho_{x_1 x_2} + b_{22} b_{23} \rho_{x_2 x_3} + b_{21} b_{23} \rho_{x_1 x_3}) \\ \sigma_{y_3}^2 &= b_{31}^2 + b_{32}^2 + b_{33}^2 + 2\,(b_{31} b_{32} \rho_{x_1 x_2} + b_{32} b_{33} \rho_{x_2 x_3} + b_{31} b_{33} \rho_{x_1 x_3}) \\ \sigma_{y_1 y_2} &= b_{11} b_{21} + b_{12} b_{22} + b_{13} b_{23} + (b_{11} b_{22} + b_{21} b_{12}) \rho_{x_1 x_2} \\ &\quad + (b_{12} b_{23} + b_{22} b_{13}) \rho_{x_2 x_3} + (b_{11} b_{23} + b_{21} b_{13}) \rho_{x_1 x_3} = 0 \\ \sigma_{y_2 y_3} &= b_{21} b_{31} + b_{22} b_{32} + b_{23} b_{33} + (b_{21} b_{32} + b_{31} b_{22}) \rho_{x_1 x_2} \\ &\quad + (b_{22} b_{33} + b_{32} b_{23}) \rho_{x_2 x_3} + (b_{21} b_{33} + b_{31} b_{23}) \rho_{x_1 x_3} = 0 \end{aligned} \tag{B.19}$$

[7] b_{ij} is the element of $\mathbf{B}$ in the i^{th} row and j^{th} column.

$$\sigma_{y_1 y_3} = b_{11} b_{31} + b_{12} b_{32} + b_{13} b_{33} + (b_{11} b_{32} + b_{31} b_{12}) \rho_{x_1 x_2}$$

(B.19 cont'd.)

$$+ (b_{12} b_{33} + b_{32} b_{13}) \rho_{x_2 x_3} + (b_{11} b_{33} + b_{31} b_{13}) \rho_{x_1 x_3} = 0$$

In matrix notation (B.19) is equivalent to the requirement that

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \tag{B.20}$$

have a covariance matrix $\mathbf{Q}$ given by

$$\mathbf{Q} = \mathcal{E}(\mathbf{YY}^T) = \mathcal{E} \begin{pmatrix} Y_1^2 & Y_1 Y_2 & Y_1 Y_3 \\ Y_2 Y_1 & Y_2^2 & Y_2 Y_3 \\ Y_3 Y_1 & Y_3 Y_2 & Y_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{E}(Y_1^2) & \mathcal{E}(Y_1 Y_2) & \mathcal{E}(Y_1 Y_3) \\ \mathcal{E}(Y_1 Y_2) & \mathcal{E}(Y_2^2) & \mathcal{E}(Y_2 Y_3) \\ \mathcal{E}(Y_1 Y_3) & \mathcal{E}(Y_2 Y_3) & \mathcal{E}(Y_3^2) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{y_1}^2 & \sigma_{y_1 y_2} & \sigma_{y_1 y_3} \\ \sigma_{y_1 y_2} & \sigma_{y_2}^2 & \sigma_{y_2 y_3} \\ \sigma_{y_1 y_3} & \sigma_{y_2 y_3} & \sigma_{y_3}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{y_1}^2 & 0 & 0 \\ 0 & \sigma_{y_2}^2 & 0 \\ 0 & 0 & \sigma_{y_3}^2 \end{pmatrix} \tag{B.21}$$

However $\mathbf{Q}$ can be written also as

$$\mathbf{Q} = \mathcal{E}(\mathbf{YY}^T) = \mathbf{B}\mathcal{E}(\mathbf{XX}^T)\mathbf{B}^T = \mathbf{C}^{-1}\mathbf{P}(\mathbf{C}^{-1})^T \tag{B.22}$$

Thus, the problem becomes that of finding a nonsingular matrix which diagonalizes $\mathbf{P}$. From matrix theory, since $\mathbf{P}$ is a real symmetric square matrix of order 3, there exists an orthogonal matrix $\mathbf{E}$ of order 3 such that $\mathbf{EPE}^T = \mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{B.23}$$

is a diagonal matrix of order 3.

Moreover, since $\mathbf{P}$ is also positive definite, the diagonal elements (eigenvalues) λ_1, λ_2, and λ_3 are all positive. Any diagonal matrix which is obtained by another orthogonal transformation of $\mathbf{P}$, has the same diagonal elements λ_1, λ_2, and λ_3, possibly in a different arrangement. Hence, in the above we can let $\mathbf{B}$ be an orthogonal matrix. Then λ_1, λ_2, and λ_3 are the variances of Y_1, Y_2, and Y_3, or $\sigma_{y_1}^2$, $\sigma_{y_2}^2$, $\sigma_{y_3}^2$.

As an example, let $\rho_{\Delta r_o \Delta v_o} = \rho_{\Delta v_o \Delta \gamma_o} = \rho_{\Delta r_o \Delta \gamma_o} = \rho$. $(0 \leq \rho < 1)$ Then $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix} \tag{B.24}$$

To find the eigenvalues of $\mathbf{P}$ we must find the roots of the secular equation (characteristic equation)

$$|\mathbf{P} - \lambda \mathbf{I}| = 0 \tag{B.25}$$

Here $\mathbf{I}$ is the identity matrix of order 3. Note that the eigenvalues of a real symmetric matrix are always real.

Equation (B.25) can be written in expanded form as

$$(\lambda - 1)^3 - 2\rho^3 - 3\rho^2(\lambda - 1) = 0 \tag{B.26}$$

Solving (B.26) for the roots we obtain

$$\begin{aligned} \lambda_1 &= \sigma_{y_1}^2 = 1 + 2\rho \\ \lambda_2 &= \sigma_{y_2}^2 = 1 - \rho \\ \lambda_3 &= \sigma_{y_3}^2 = 1 - \rho \end{aligned} \tag{B.27}$$

Since two of the roots are equal, the orthogonal matrix for transforming $\mathbf{P}$ into diagonal form is not unique. However, it can be constructed. Let us now construct the orthogonal matrix $\mathbf{C}$. Since $\mathbf{C}$ is orthogonal, $\mathbf{C}^{-1} = \mathbf{C}^T$. Therefore, $\mathbf{C} = \mathbf{B}^T$ or

$$\mathbf{C} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \tag{B.28}$$

The eigenvector, $\vec{x}_1 = (c_{11}c_{21}c_{31})^T$ corresponding to $\lambda_1 = (1 + 2\rho)$ must be such that the components c_{j1} $(j = 1, 2, 3)$ must satisfy the matrix equation

$$\mathbf{P}\vec{x}_1 = \lambda_1\vec{x}_1$$

or

$$\begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} = (1+2\rho) \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix} \tag{B.29}$$

or expanding the above

$$\begin{aligned} -2c_{11} + c_{21} + c_{31} &= 0 \\ c_{11} - 2c_{21} + c_{31} &= 0 \\ c_{11} + c_{21} - 2c_{31} &= 0 \end{aligned} \tag{B.30}$$

From the first two equations of (B.30) we obtain $c_{11} = c_{21}$ and from the second two equations $c_{21} = c_{31}$. Since equation (B.29) is linear, we may make the length of the vector $\vec{x}_1$ unity, thus requiring

$$c_{11}^2 + c_{21}^2 + c_{31}^2 = 1 \tag{B.31}$$

Equation (B.31) is satisfied if

$$c_{11} = c_{21} = c_{31} = -\frac{1}{\sqrt{3}} \tag{B.32}$$

The eigenvector $\vec{x}_2$ corresponding to $\lambda_2 = (1 - \rho)$ must be such that the components of c_{j2} (j = 1, 2, 3) satisfy the matrix equation

$$\mathbf{P} \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix} = (1-\rho) \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix} \tag{B.33}$$

from which we obtain three equivalent scalar equations in the c_{j2}'s

$$c_{12} + c_{22} + c_{32} = 0$$

$$c_{12} + c_{22} + c_{32} = 0 \tag{B.34}$$

$$c_{12} + c_{22} + c_{32} = 0$$

We may choose for $\vec{x}_2 = \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \end{pmatrix}$ and $\vec{x}_3 = \begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \end{pmatrix}$ any two linearly independent vectors subject to the condition that they are orthogonal and that they satisfy the above equations in (B.34). Setting $c_{12} = 0$ in (B.34) we obtain $c_{22} = -c_{32}$. Thus, if $c_{22} = \frac{1}{\sqrt{2}}$ and $c_{32} = -\frac{1}{\sqrt{2}}$, $\vec{x}_2$ will be normalized. Furthermore, we will have the dot product $(\vec{x}_1) \cdot (\vec{x}_2) = 0$.[8] Hence

$$\vec{x}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{B.35}$$

To obtain the components of $\vec{x}_3$ we use the fact that we must have $\vec{x}_1 \cdot \vec{x}_3 = 0$ and $\vec{x}_2 \cdot \vec{x}_3 = 0$ simultaneously, that is

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_3 &= c_{13} + c_{23} + c_{33} = 0 \\ \vec{x}_2 \cdot \vec{x}_3 &= c_{23} - c_{33} = 0 \end{aligned} \tag{B.36}$$

(8) It should be noted that the eigenvectors of a real symmetric matrix are necessarily orthogonal.

from which we obtain

$$c_{13} = -2c_{23}$$

and (B.37)

$$c_{23} = c_{33}$$

Normalizing we obtain

$$\vec{x}_3 = \begin{pmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B.38}$$

Thus, the orthogonal matrix **C** is

$$\mathbf{C} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B.39}$$

and therefore **B** is

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B.40}$$

It can be verified that the elements of **B** satisfy the equations in (B.19) which relate the variances and covariances between the components of **X** and **Y** with $\rho_{x_1 x_2} = \rho_{x_2 x_3} = \rho_{x_1 x_3} = \rho$. The linear equations for this example are therefore, from (B.13)

$$X_1 = \left(-\frac{1}{\sqrt{3}}\right)Y_1 + (o)\ Y_2 + \left(-\frac{2}{\sqrt{6}}\right)Y_3$$

$$X_2 = \left(-\frac{1}{\sqrt{3}}\right)Y_1 + \left(\frac{1}{\sqrt{2}}\right)Y_2 + \left(\frac{1}{\sqrt{6}}\right)Y_3 \qquad \text{(B.41)}$$

$$X_3 = \left(-\frac{1}{\sqrt{3}}\right)Y_1 + \left(-\frac{1}{\sqrt{2}}\right)Y_2 + \left(\frac{1}{\sqrt{6}}\right)Y_3$$

For the other combinations of signed values for the correlation coefficients $\rho_{x_1x_2}$, $\rho_{x_2x_3}$, and $\rho_{x_1x_3}$ which keep the covariance matrix of **X** nonsingular we have the following orthogonal matrices C_1, C_2, and C_3 which diagonalize $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ respectively.

For the covariance matrix

$$\mathbf{P}_1 = \begin{pmatrix} 1 & -\rho & -\rho \\ -\rho & 1 & \rho \\ -\rho & \rho & 1 \end{pmatrix} \qquad \text{(B.42)}$$

the orthogonal matrix

$$C_1 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \qquad \text{(B.43)}$$

diagonalizes $\mathbf{P}_1$.

For the covariance matrix

$$\mathbf{P}_2 = \begin{pmatrix} 1 & -\rho & \rho \\ -\rho & 1 & -\rho \\ \rho & -\rho & 1 \end{pmatrix} \qquad \text{(B.44)}$$

the orthogonal matrix

$$C_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{B.45}$$

diagonalizes P_2.

And, for the covariance matrix

$$P_3 = \begin{pmatrix} 1 & \rho & -\rho \\ \rho & 1 & -\rho \\ -\rho & -\rho & 1 \end{pmatrix} \tag{B.46}$$

the orthogonal matrix

$$C_3 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \tag{B.47}$$

diagonalizes P_3.

Each of the orthogonal matrices indicated above was obtained in the same manner. It should be noted that in this special case where each coefficient of correlation has the same absolute value, each covariance matrix has the same eigenvalues: $\lambda_1 = 1 + 2\rho$, $\lambda_2 = 1 - \rho$, $\lambda_3 = 1 - \rho$, inasmuch as the characteristic equation is the same in each instance.

In the same manner as was done for the random vector **W** we may express **Y** in terms of a standardized normal vector **Z** having uncorrelated standardized normal random variables Z_1, Z_2, and Z_3 as components.

$$Y_1 = \sigma_{y_1} Z_1$$

$$Y_2 = \sigma_{y_2} Z_2 \quad \text{(B.48)}$$

$$Y_3 = \sigma_{y_3} Z_3$$

Finally, we may write ΔR_o, ΔV_o, and $\Delta \Gamma_o$ as linear combinations of Z_1, Z_2, and Z_3.

$$\Delta R_o = \left(\sigma_{\Delta r_o} \sigma_{y_1}\right) c_{11} Z_1 + \left(\sigma_{\Delta r_o} \sigma_{y_2}\right) c_{12} Z_2 + \left(\sigma_{\Delta r_o} \sigma_{y_3}\right) c_{13} Z_3$$

$$\Delta V_o = \left(\sigma_{\Delta v_o} \sigma_{y_1}\right) c_{21} Z_1 + \left(\sigma_{\Delta v_o} \sigma_{y_2}\right) c_{22} Z_2 + \left(\sigma_{\Delta v_o} \sigma_{y_3}\right) c_{23} Z_3 \quad \text{(B.49)}$$

$$\Delta \Gamma_o = \left(\sigma_{\Delta \gamma_o} \sigma_{y_1}\right) c_{31} Z_1 + \left(\sigma_{\Delta \gamma_o} \sigma_{y_2}\right) c_{32} Z_2 + \left(\sigma_{\Delta \gamma_o} \sigma_{y_3}\right) c_{33} Z_3$$

Now, since Y_1, Y_2, and Y_3 are uncorrelated normal random variables, they are independent in the probability sense.[9] Therefore, their joint probability density function $f(y_1, y_2, y_3)$ factors into the product of the marginal distributions of Y_1, Y_2, and Y_3, or $f_1(y_1)$, $f_2(y_2)$, and $f_3(y_3)$.

$$f(y_1, y_2, y_3) = f_1(y_1)\, f_2(y_2)\, f_3(y_3) \quad \text{(B.50)}$$

Denoting by $g(x_1, x_2, x_3)$ the density of X_1, X_2, X_3, and noting that by (B.13) the transformation from the y-space to the x-space is one-to-one, as well as from the x-space to the y-space (B.18), the density of X_1, X_2 and X_3 is

$$g(x_1, x_2, x_3) = f\left[y_1(x_1, x_2, x_3), y_2(x_1, x_2, x_3), y_3(x_1, x_2, x_3)\right] J(x_1 x_2, x_3) \quad \text{(B.51)}$$

[9] We note that uncorrelated random variables are not always independent. However, independent random variables are always uncorrelated.

where $J(x_1, x_2, x_3)$ is the Jacobian

$$J(x_1, x_2, x_3) = \text{mod} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} \tag{B.52}$$

and "mod" means the absolute value of the expression following it. Using (B.18) we can compute $J(x_1, x_2, x_3)$ as follows:

$$J(x_1, x_2, x_3) = \text{mod} \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \text{mod } |\mathbf{B}| \tag{B.53}$$

However, since $\mathbf{B}$ is an orthogonal matrix, $J(x_1, x_2, x_3) = 1$. Therefore, using (B.50) and (B.51) the joint density of X_1, X_2, and X_3 with the aid of (B.13) is given by

$$P\{x_1 < X_1 \leq x_1 + \Delta x_1,\ x_2 < X_2 \leq x_2 + \Delta x_2,\ x_3 < X_3 \leq x_3 + \Delta x_3\}$$

$$= g(x_1, x_2, x_3)\Delta x_1 \Delta x_2 \Delta x_3 = f(y_1, y_2, y_3)\Delta y_1 \Delta y_2 \Delta y_3$$

$$= P\{y_1 < Y_1 \leq y_1 + \Delta y_1, y_2 < Y_2 \leq y_2 + \Delta y_2, y_3 < Y_3 \leq y_3 + \Delta y_3\} \Delta y_1 \Delta y_2 \Delta y_3$$

$$= P\{c_{11}x_1 + c_{21}x_2 + c_{31}x_3 < Y_1 \leq c_{11}(x_1 + \Delta x_1) + c_{21}(x_2 + \Delta x_2) + c_{31}(x_3 + \Delta x_3)\}$$

$$\times P\{c_{12}x_1 + c_{22}x_2 + c_{32}x_3 < Y_2 \leq c_{12}(x_1 + \Delta x_1) + c_{22}(x_2 + \Delta x_2) + c_{32}(x_3 + \Delta x_3)\}$$

$$\times P\{c_{13}x_1 + c_{23}x_2 + c_{33}x_3 < Y_3 \leq c_{13}(x_1 + \Delta x_1) + c_{23}(x_2 + \Delta x_2) + c_{33}(x_3 + \Delta x_3)\}$$

$$\times \Delta y_1 \Delta y_2 \Delta y_3$$

$$= f_1(c_{11}x_1 + c_{21}x_2 + c_{31}x_3)\, f_2(c_{12}x_1 + c_{22}x_2 + c_{32}x_3)\, f_3(c_{13}x_1 + c_{23}x_2 + c_{33}x_3)$$
$$\times\ (c_{11}\Delta x_1 + c_{21}\Delta x_2 + c_{31}\Delta x_3) \times (c_{12}\Delta x_1 + c_{22}\Delta x_2 + c_{32}\Delta x_3)$$
$$\times\ (c_{13}\Delta x_1 + c_{23}\Delta x_2 + c_{33}\Delta x_3) \tag{B.54}$$

For purposes of machine computation, the range of each random variable $Y_i (i = 1, 2, 3)$ was approximated by $(2N + 1)$ mutually exclusive intervals of length $\left(\frac{L}{N}\right)\sigma_{y_i}$ extending from $-L\sigma_{y_i}$ to $L\sigma_{y_i}$, and the probabilities of falling within these intervals approximated in the following manner.

First, $(2N + 1)$ values for the normal probability density function were calculated for each Y_i

$$q_{i_k} = \left(2\pi\sigma_{y_i}^2\right)^{-\frac{1}{2}} e^{-(y'_{i_k})^2 / 2\sigma_{y_i}^2} \tag{B.55}$$

where

$$y'_{i_k} = \left(\frac{L}{N}\right)\left[k - (N + 1)\right]\sigma_{y_i}$$

$$\big(k = 1, 2, 3, \ldots, (2N + 1)\big)$$
$$(i = 1, 2, 3)$$

Then, the sum S_i was formed

$$S_i = \sum_{k=1}^{2N+1} q_{i_k} \tag{B.56}$$

Next, the q_{i_k}'s were normalized, i.e., p_{i_k}'s were formed where

$$p_{i_k} = \frac{q_{i_k}}{S_i} \tag{B.57}$$
$$\big(k = 1, 2, 3, \ldots, (2N + 1)\big)$$
$$(i = 1, 2, 3)$$

Next, defining Y'_1, Y'_2, and Y'_3 as discrete random variables approximating the normal random variables Y_1, Y_2, and Y_3 we may write

$$P\left(Y'_i = y'_{i_k}\right) = p_{i_k} \tag{B.58}$$

$$(i = 1, 2, 3)$$

$$\left(k = 1, 2, 3, \ldots, (2N + 1)\right)$$

And, we may write the probability that Y_i should fall within the interval

$$\left(y'_{i_k} - \left(\frac{1}{2}\right)\Delta y_i,\ y'_{i_k} + \left(\frac{1}{2}\right)\Delta y_i\right) \tag{B.59}$$

where

$$\Delta y_i = \left(\frac{L}{N}\right)\sigma_{y_i}$$

as follows

$$P\left\{y'_{i_k} - \left(\frac{1}{2}\right)\Delta y_i < Y_i \leq y'_{i_k} + \left(\frac{1}{2}\right)\Delta y_i\right\} \approx P\left(Y'_i = y'_{i_k}\right) = p_{i_k} \tag{B.60}$$

$$(i = 1, 2, 3)$$

$$\left(k = 1, 2, 3, \ldots, (2N + 1)\right)$$

Thus, each normally distributed random variable Y_i (i = 1, 2, 3) has been approximated by a discrete random variable Y'_i (i = 1, 2, 3) in the range $-L\sigma_{y_i}$ to $+L\sigma_{y_i}$ and having probability mass points p_{i_k} (i = 1, 2, 3; k = 1, 2, 3, . . . (2N + 1)) at the center of each subdivision $\left(y'_{i_k} - \left(\frac{1}{2}\right)\Delta y_i,\ y'_{i_k} + \left(\frac{1}{2}\right)\Delta y_i\right)$.
It is easily verified that the probabilities for each Y'_i sum to 1, since

$$\sum_{k=1}^{2N+1} P\left\{Y'_i = y'_{i_k}\right\} = \sum_{k=1}^{2N+1} p_{i_k} = \left(\frac{1}{S_i}\right)\sum_{k=1}^{2N+1} q_{i_k} = 1 \tag{B.61}$$

Since each Y_i' can assume $(2N + 1)$ possible values, y'_{i_k}, each with probability p_{i_k} $(k = 1, 2, \ldots 2N + 1)$ there will be a total of $(2N + 1)^3$ possible values for the approximation to the joint probability density function of Y_1, Y_2, and Y_3. Since the Y_i 's are independent, the probability of occurrence p_{jkl} of each combination of values for y'_{1_j}, y'_{2_k}, and y'_{3_l} $(j, k, l = 1, 2, 3, \ldots, (2N + 1))$ is given by the product of p_{1_j}, p_{2_k}, and p_{3_l}, i.e.,

$$p_{jkl} = p_{1_j} \; p_{2_k} \; p_{3_l} \tag{B.62}$$

$$\big(j, k, l = 1, 2, 3, \ldots, (2N + 1)\big)$$

In order to distinguish between the continuous random variables and the discrete approximations to these variables, let X_1', X_2', X_3', $\Delta R_o'$, $\Delta V_o'$, $\Delta \Gamma_o'$, and $\Delta R_p'$ be the discrete random variables which approximate X_1, X_2, X_3, ΔR_o, ΔV_o, $\Delta \Gamma_o$, and ΔR_p, respectively.

For each combination of values for Y_1', Y_2', and Y_3', the probability of occurrence was obtained using (B.62), the inverse transformation to the correlated set X_1', X_2', and X_3' made using (B.13), and values calculated for $\Delta R_o'$, $\Delta V_o'$, and $\Delta \Gamma_o'$ using (B.7). Since there is a one-to-one correspondence from the uncorrelated y-space to the correlated space of $\Delta R_o'$, $\Delta V_o'$, and $\Delta \Gamma_o'$, the joint density function approximation of ΔR_o, ΔV_o, and $\Delta \Gamma_o$ will have $(2N + 1)^3$ possible values, and for each combination of values for $\Delta R_o'$, $\Delta V_o'$, and $\Delta \Gamma_o'$ we may obtain a value for $\Delta R_p'$, using (B.6). Therefore, the discrete approximation to the probability density function of ΔR_p will also have $(2N + 1)^3$ possible values. At this point we should note that ΔR_p as given in (B.6) depends on <u>actual</u> values r_o, v_o, and γ_o.

We will now discuss how the actual values for r_o, v_o, and γ_o were calculated.

First, actual values for the semi-major axis a, the true anomaly of the spacecraft at insertion θ, and the eccentricity of the parking orbit e, were assumed.

The insertion radius r_o was obtained using (A.3) of Appendix A.

$$r_o = \frac{a(1 - e^2)}{1 + e \cos \theta} \tag{B.63}$$

The insertion speed v_o was obtained from (A.1) of Appendix A.

$$v_o = \sqrt{\mu\left(\frac{2}{r_o} - \frac{1}{a}\right)} \tag{B.64}$$

In order to obtain the flight path angle at insertion γ_o, let us rewrite (A.2) of Appendix A.

$$1 - e^2 = \left(\frac{r_o v_o^2}{\mu}\right)\left(\frac{r_o}{a}\right)\cos^2\gamma_o \tag{B.65}$$

From (A.1)

$$\frac{r_o v_o^2}{\mu} = 2 - \left(\frac{r_o}{a}\right) \tag{B.66}$$

Substituting (B.66) into (B.65) gives

$$1 - e^2 = \left(2 - \frac{r_o}{a}\right)\left(\frac{r_o}{a}\right)\cos^2\gamma_o \tag{B.67}$$

Solving for $\left(\frac{r_o}{a}\right)$ in (B.63) and substituting the result in (B.67), we have

$$\gamma_o = \tan^{-1}\left(\frac{e \sin\theta}{1 + e\cos\theta}\right) \tag{B.68}$$

The actual (or assumed) perigee radius r_p was obtained using (B.1).

Thus, with assumed actual values for r_o, v_o, γ_o, and r_p, and by using all the $(2N + 1)^3$ possible combinations of values for Y_1', Y_2', and Y_3' with their corresponding $(2N + 1)^3$ probabilities $p_{jkl} = p_{1_j} p_{2_k} p_{3_l}$ $(j, k, l = 1, 2, 3, \ldots, (2N + 1))$, an approximation to the probability density function of ΔR_p through the transformations indicated in (B.13), (B.7), and (B.6) was obtained.

It is easily seen that the probability density function approximation for ΔR_p sums to 1.

$$\int_{-\infty}^{\infty} f(\Delta r_p)\, d(\Delta r_p) \approx \sum_{i,j,k=1}^{2N+1} p_{ijk}$$

$$= \sum_{i,j,k=1}^{2N+1} p_{1_i}\, p_{2_j}\, p_{3_k} = \left(\sum_{i=1}^{2N+1} p_{1_i}\right)\left(\sum_{j=1}^{2N+1} p_{2_j}\right)\left(\sum_{k=1}^{2N+1} p_{3_k}\right) = 1 \tag{B.69}$$

where $f(\Delta r_p)$ is the probability density function of ΔR_p.

The cumulative distribution function of ΔR_p or $F(\Delta r_p)$ is given by

$$F(\Delta r_p) = \int_{-\infty}^{\Delta r_p} f(x)\, dx \tag{B.70}$$

For purposes of machine computation, the range of ΔR_p was subdivided into 2000 mutually exclusive intervals or "bins" one tenth of a kilometer wide and extending from -80 kilometers to +120 kilometers. For all values of $\Delta R_p'$ falling within one of these intervals, the associated probabilities were summed. This can be done since all of the $(2N + 1)^3$ possible combinations of values for $\Delta R_o'$, $\Delta V_o'$, and $\Delta \Gamma_o'$ are mutually exclusive. In this fashion we obtain an approximation to the probability density function of ΔR_p. The cumulative distribution function approximation for ΔR_p was then obtained by summing the probability density function approximation for ΔR_p.

A Fortran IV computer program was written and used to calculate the probability distribution of perigee error on the UNIVAC 1108 computer. For $L = 4$ and $N = 50$, i.e., approximating the normal curve from -4 standard deviations to +4 standard deviations in $(2N + 1) = 101$ equally spaced intervals (0.08 standard deviations in width), the calculations took approximately 4 minutes to obtain the probability density function and cumulative distribution function approximation to ΔR_p for a given set of insertion conditions.

Expression For the Error in Perigee For a Circular Orbit

The expression for the error in perigee for a circular orbit is somewhat less complicated than the expression in (B.2). We will now treat the circular case.

Let us first rewrite equations (A.1), (A.2), and (A.4) of Appendix A.

$$1 - e^2 = \frac{v_o^2 \, r_o^2 \cos^2 \gamma_o}{\mu a} \tag{B.71}$$

$$\frac{1}{a} = \left(\frac{2}{r_o}\right) - \left(\frac{v_o^2}{\mu}\right) \tag{B.72}$$

$$r_p = a(1 - e). \tag{B.73}$$

Substitute (B.72) into (B.71) and also into (B.73)

$$1 - e^2 = \frac{v_o^2 \, r_o^2 \cos^2 \gamma_o}{\mu} \left(\frac{2}{r_o} - \frac{v_o^2}{\mu}\right) \tag{B.74}$$

$$r_p \left(\frac{2}{r_o} - \frac{v_o^2}{\mu}\right) = 1 - e \tag{B.75}$$

or

$$e = 1 + \left(\frac{v_o^2 \, r_p}{\mu}\right) - \frac{2 r_p}{r_o} \tag{B.76}$$

Rewrite (B.34) and multiply by r_p.

$$r_p (1 - e)(1 + e) = \left(\frac{v_o^2 \, r_o^2 \cos^2 \gamma_o}{\mu}\right) r_p \left(\frac{2}{r_o} - \frac{v_o^2}{\mu}\right). \tag{B.77}$$

Cancel (1 - e) in (B.77) using (B.75), and substitute (B.76) into (B.77) for (e + 1). The result is

$$2 r_o r_p + \left(\frac{v_o^2 r_o r_p^2}{\mu}\right) = 2 r_p^2 + \left(\frac{v_o^2 r_o^3 \cos^2 \gamma_o}{\mu}\right) . \qquad \text{(B.78)}$$

Substituting $(r_o + \Delta r_o)$ for r_o, $(v_o + \Delta v_o)$ for v_o, $(\gamma_o + \Delta\gamma_o)$ for γ_o, $(r_p + \Delta r_p)$ for r_p, and collecting terms, we have

$$\left[2 r_o + \left(\frac{2}{\mu}\right) v_o^2 r_o r_p - 4 r_p\right] \Delta r_p + \left[2 + \left(\frac{2}{\mu}\right) v_o^2 r_p\right] \Delta r_o \Delta r_p$$

$$+ \left[\left(\frac{4}{\mu}\right) r_o v_o r_p\right] \Delta v_o \Delta r_p + \left[\left(\frac{1}{\mu}\right) v_o^2 r_o - 2\right] (\Delta r_p)^2$$

$$= \left[- 2 r_p - \left(\frac{1}{\mu}\right) v_o^2 r_p^2 + \left(\frac{3}{\mu}\right) v_o^2 r_o^2 \cos^2 \gamma_o\right] \Delta r_o$$

$$+ \left[\left(\frac{2}{\mu}\right) v_o r_o^3 \cos^2 \gamma_o - \left(\frac{2}{\mu}\right) v_o r_o r_p^2\right] \Delta v_o$$

$$+ \left[- \left(\frac{2}{\mu}\right) v_o^2 r_o^3 \sin \gamma_o \cos \gamma_o\right] \Delta\gamma_o$$

$$+ \left[- \left(\frac{2}{\mu}\right) v_o r_p^2 + \left(\frac{6}{\mu}\right) v_o r_o^2 \cos^2 \gamma_o\right] \Delta r_o \Delta v_o$$

$$+ \left[- \left(\frac{4}{\mu}\right) v_o r_o^3 \sin \gamma_o \cos \gamma_o\right] \Delta v_o \Delta\gamma_o$$

$$+ \left[- \left(\frac{6}{\mu}\right) v_o^2 r_o^2 \sin \gamma_o \cos \gamma_o\right] \Delta r_o \Delta\gamma_o$$

$$+\left[\left(\frac{3}{\mu}\right) v_o^2\, r_o \cos^2 \gamma_o\right] (\Delta r_o)^2 + \left[\left(\frac{1}{\mu}\right) r_o^3 \cos^2 \gamma_o - \left(\frac{1}{\mu}\right) r_o\, r_p^2\right] (\Delta v_o)^2$$

$$+ \left[-\left(\frac{1}{\mu}\right) v_o^2\, r_o^3 \cos\,(2\gamma_o)\right] (\Delta\gamma_o)^2 + \text{higher order terms}. \tag{B.79}$$

For a circular orbit $\gamma_o = 0°, \left(\frac{r_o v_o^2}{\mu}\right) = 1$, and $r_p = r_o$. Making these substitutions in (B.79), and neglecting terms of order higher than the second, we have

$$\left(\frac{4}{r_o^2}\right) \Delta r_o \Delta r_p - \left(\frac{\Delta r_p}{r_o}\right)^2 - 4\left(\frac{\Delta v_o}{v_o}\right)\left(\frac{\Delta r_o}{r_o}\right)$$

$$+ 4\left(\frac{\Delta v_o}{v_o}\right)\left(\frac{\Delta r_p}{r_o}\right) = 3\left(\frac{\Delta r_o}{r_o}\right)^2 - (\Delta\gamma_o)^2 \tag{B.80}$$

and finally,

$$\Delta r_p = 2\left[\left(\frac{r_o}{v_o}\right)\Delta v_o + \Delta r_o\right] \pm \sqrt{\left[2\left(\frac{r_o}{v_o}\right)\Delta v_o + \Delta r_o\right]^2 + (r_o\,\Delta\gamma_o)^2}\,. \tag{B.81}$$

The minus sign has to be used in (B.81) since $\Delta\gamma_o$ has to produce a negative Δr_p if $\Delta r_o = \Delta v_o = 0$. (The plus sign is applicable to the error in apogee.) Thus, the perigee error for a circular orbit becomes:

$$\Delta r_p = 2\left[\left(\frac{r_o}{v_o}\right)\Delta v_o + \Delta r_o\right] - \sqrt{\left[\left(\frac{2 r_o}{v_o}\right)\Delta v_o + \Delta r_o\right]^2 + (r_o\,\Delta\gamma_o)^2}\,. \tag{B.82}$$

Derivation of the Probability Density Function of Perigee Error For a Circular Parking Orbit When the Insertion Errors are Perfectly Correlated and Normally Distributed.

As a check against the calculations performed by the computer, the probability density function of the error in perigee for a circular orbit was obtained, assuming the insertion errors to be perfectly correlated (coefficients of correlation between the insertion errors of +1.0) and normally distributed.

If ΔR_o, ΔV_o and $\Delta \Gamma_o$ are normally distributed random variables mutually correlated with coefficients of correlation of +1.0, there is complete linear dependence between them. Thus, they will vary in the same sense. Let X denote the standardized normal random variable (zero mean and unit variance). We may then write (B.82), substituting ΔR_p for Δr_p, ΔR_o for Δr_o, ΔV_o for Δv_o, and $\Delta \Gamma_o$ for $\Delta \gamma_o$, as

$$\Delta R_p = 2\left[\left(\frac{r_o}{v_o}\right)\Delta V_o + \Delta R_o\right] - \sqrt{\left[\left(\frac{2\,r_o}{v_o}\right)\Delta V_o + \Delta R_o\right]^2 + (r_o\,\Delta\Gamma_o)^2}$$

$$= 2\left[\left(\frac{r_o}{v_o}\right)\sigma_{\Delta v_o} + \sigma_{\Delta r_o}\right] X - \sqrt{\left[\left(\frac{2\,r_o}{v_o}\right)\sigma_{\Delta v_o} + \sigma_{\Delta r_o}\right]^2 + (r_o\,\sigma_{\Delta\gamma_o})^2}\;|X|$$

(B.83)

where $\sigma_{\Delta r_o}$, $\sigma_{\Delta v_o}$, $\sigma_{\Delta \gamma_o}$ are the standard deviations of ΔR_o, ΔV_o and $\Delta \Gamma_o$ respectively.

Letting $l = 2\left[\left(\frac{r_o}{v_o}\right)\sigma_{\Delta v_o} + \sigma_{\Delta r_o}\right]$ and

$$m = \sqrt{\left[2\left(\frac{r_o}{v_o}\right)\sigma_{\Delta v_o} + \sigma_{\Delta r_o}\right]^2 + (r_o\,\sigma_{\Delta\gamma_o})^2},$$

we can write (B.83) as

$$\Delta R_p = 1\,X - m\,|X| \tag{B.84}$$

Letting $F(\Delta r_p)$ denote the distribution function of ΔR_p, $G(x)$ the distribution function of X, and assuming $m > 1$, we have:

$$F(\Delta r_p) = \text{Prob}\left\{\Delta R_p \leq \Delta r_p\right\} = \text{Prob}\left\{1\,X - m\,|X| \leq \Delta r_p\right\} \tag{B.85}$$

For $X < 0$, $|X| = -X$, and we have

$$\text{Prob}\left\{1\,X - m\,|X| \leq \Delta r_p\right\} = \text{Prob}\left\{X \leq \frac{\Delta r_p}{1 + m}\right\} = G\left(\frac{\Delta r_p}{1 + m}\right) \tag{B.86}$$

For $X \geq 0$, $|X| = X$, and we have

$$\text{Prob}\left\{1\,X - m\,|X| \leq \Delta r_p\right\} = \text{Prob}\left\{(1 - m)X \leq \Delta r_p\right\}$$

$$= \text{Prob}\left\{(m - 1)X > -\Delta r_p\right\} = 1 - \text{Prob}\left\{X \leq \frac{-\Delta r_p}{(m - 1)}\right\}$$

$$= 1 - G\left(\frac{-\Delta r_p}{(m - 1)}\right) \tag{B.87}$$

Since the events $X < 0$ and $X \geq 0$ are mutually exclusive, the total probability, or $F(\Delta r_p)$ is given by the sum of (B.86) and (B.87)

$$F(\Delta r_p) = \text{Prob}\left\{1\,X - m\,|X| \leq \Delta r_p\right\}$$

$$= \text{Prob}\left\{X \leq \frac{\Delta r_p}{(1 + m)}\right\} + \text{Prob}\left\{X > \frac{-\Delta r_p}{(m - 1)}\right\} \tag{B.88}$$

$$= 1 + G\left(\frac{\Delta r_p}{(1 + m)}\right) - G\left(\frac{-\Delta r_p}{(m - 1)}\right) \qquad \text{(B.88) (cont'd.)}$$

Letting f (Δr_p) be the probability density function of ΔR_p, and g(x) the standardized normal density function, then we have the following relationship between f (Δr_p) and g(x).

$$f(\Delta r_p) = F'(\Delta r_p) = \left(\frac{1}{1 + m}\right) g \left(\frac{\Delta r_p}{1 + m}\right) + \left(\frac{1}{m - 1}\right) g \left(\frac{-\Delta r_p}{m - 1}\right) \qquad \text{(B.89)}$$

These functions f (Δr_p) and F (Δr_p) are shown in Figures 17 and 18.

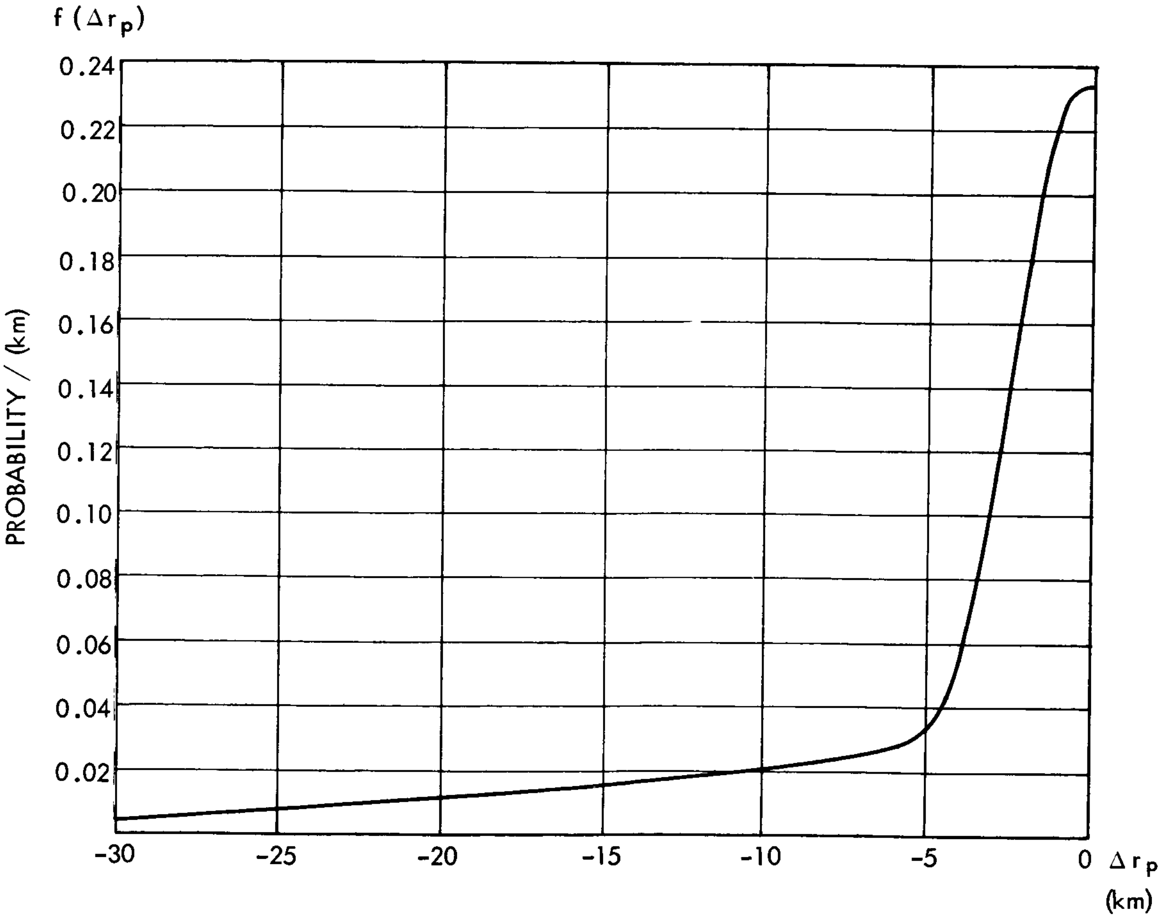

Figure 17. The Probability Density Function f of Perigee Error ΔR_p for a Circular Orbit when the Insertion Errors Are Perfectly Correlated ($\rho_{\Delta r_o \Delta v_o} = \rho_{\Delta v_o \Delta \gamma_o} = \rho_{\Delta r_o \Delta \gamma_o} = +1.0$). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.

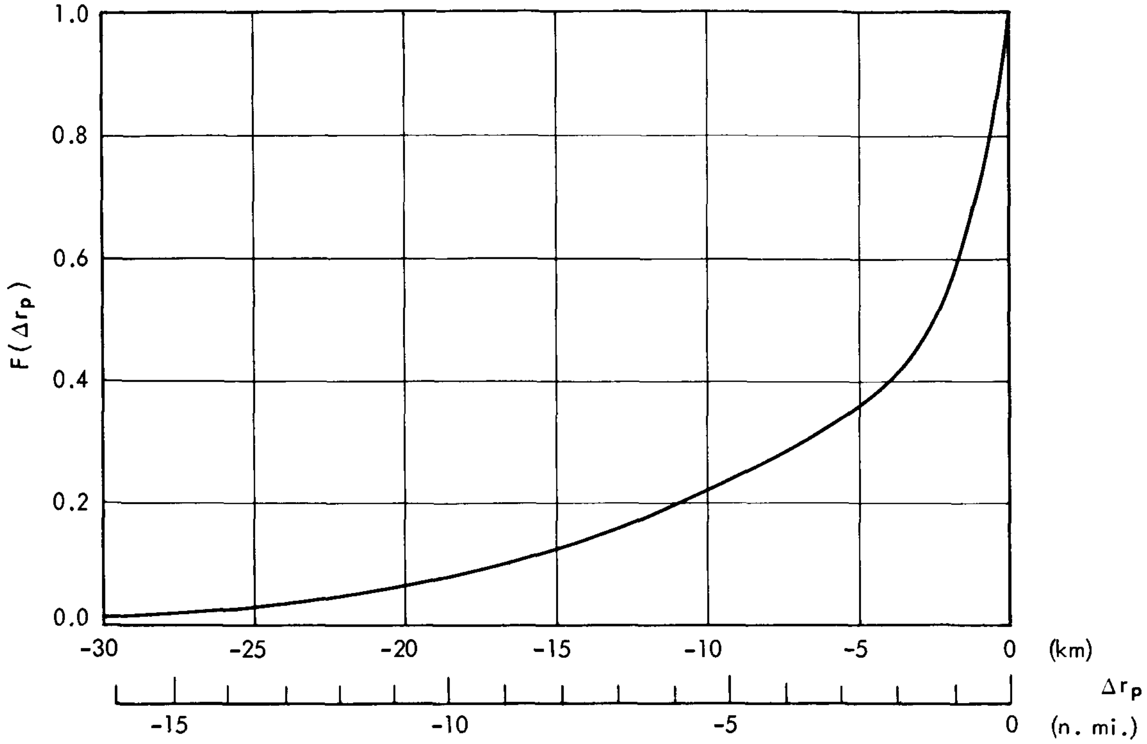

Figure 18. The Cumulative Distribution Function F of Perigee Error ΔR_p for a Circular Orbit when the Insertion Errors Are Perfectly Correlated ($\rho_{\Delta r_o \Delta v_o} = \rho_{\Delta v_o \Delta \gamma_o} = \rho_{\Delta r_o \Delta \gamma_o} = 1.0$). Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.0.

Expression for the Perigee Error for Eccentricities In the Range $0.005 \le e \le \le 0.05$

For eccentricities in the range of approximately 0.005 to 0.05, the following expression taken from Reference 5 (equation (9)) is a good approximation for the error in perigee. It is repeated here.

$$\Delta R_p \approx (2 - \cos\theta)\Delta R_o + 2(1 - \cos\theta)\, r_o \left(\frac{1}{V_o}\right)\Delta V_o - (r_o \sin\theta)\Delta\Gamma_o \tag{B.90}$$

It is evident that, if ΔR_o, ΔV_o, and $\Delta\Gamma_o$ are all normally distributed, then, for a fixed θ (true anomaly of the spacecraft at insertion) and e (eccentricity),

ΔR_p will also be normally distributed, having a zero mean, and a variance given by

$$\sigma^2_{\Delta r_p} = (2 - \cos\theta)^2 \sigma^2_{\Delta r_o} + 4(1 - \cos\theta)^2 \left(\frac{r_o}{v_o}\right)^2 \sigma^2_{\Delta\gamma_o}$$

$$+ r_o^2 \sin^2\theta \, \sigma^2_{\Delta\gamma_o} + 4(2 - \cos\theta)(1 - \cos\theta)\left(\frac{r_o}{v_o}\right) \rho_{\Delta r_o \Delta v_o} \sigma_{\Delta r_o} \sigma_{\Delta v_o}$$

$$- 4(1 - \cos\theta)\left(\frac{r_o}{v_o}\right)(r_o \sin\theta) \, \rho_{\Delta v_o \Delta\gamma_o} \sigma_{\Delta v_o} \sigma_{\Delta\gamma_o} \tag{B.91}$$

$$- 2(2 - \cos\theta)(r_o \sin\theta) \, \rho_{\Delta r_o \Delta\gamma_o} \sigma_{\Delta r_o} \sigma_{\Delta\gamma_o}$$

In order to compare the distribution of ΔR_p as calculated by numerical integration with the normal distribution approximation, we note Figures 19 and 20. In each instance, the semi-major axis is $a = 6563$ km, the eccentricity $e = 0.01$, and the standard deviations of ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are

$$\sigma_{\Delta r_o} = 0.8 \text{ n.mi. } (1.4816 \text{ km})$$

$$\sigma_{\Delta v_o} = \left(\frac{16}{3}\right) \text{ft/second } (1.62 \text{ m/sec}) \tag{B.92}$$

$$\sigma_{\Delta\gamma_o} = \left(\frac{0.16}{3}\right)^\circ \quad (9.3 \text{ mrad})$$

In Figure 19, the coefficients of correlation between the insertion errors are all +0.9. In Figure 20, the insertion errors are uncorrelated. Both figures show the 99.5% probability points as a function of true anomaly θ. The agreement between the two curves is good.

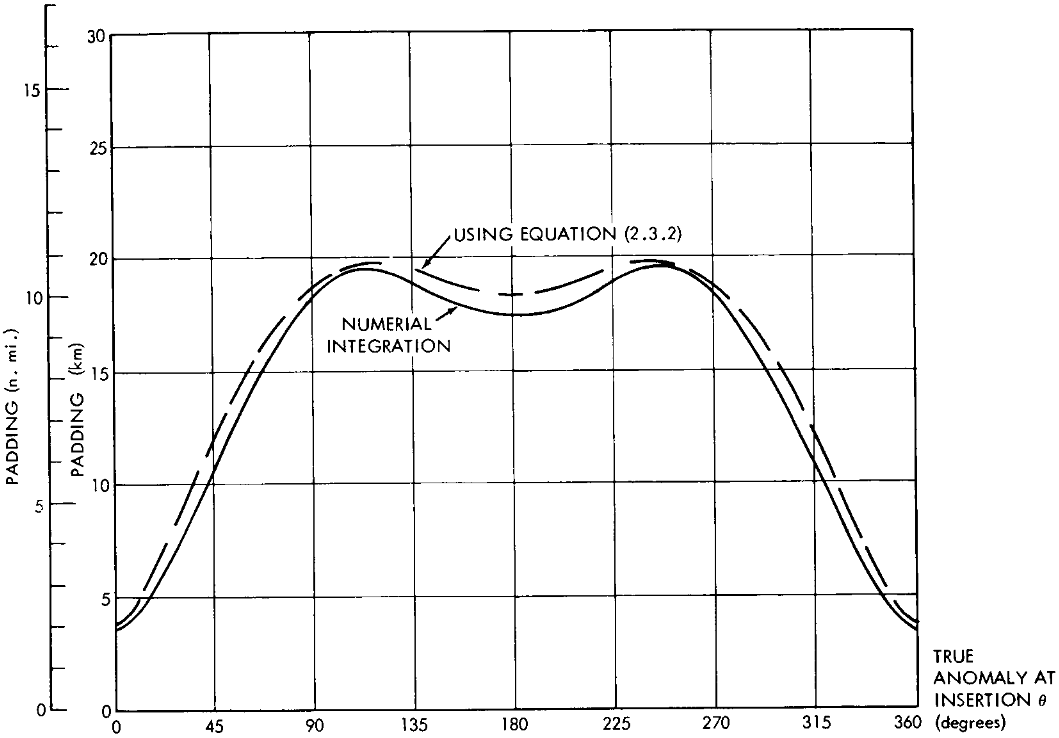

Figure 19. Comparison Between the Required 99.5% Padding as Calculated by Numerical Integration and the Required Padding Using a Normal Distribution Approximation. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01. Insertion Errors Are Positively Correlated (Coefficients of Correlation = +0.9).

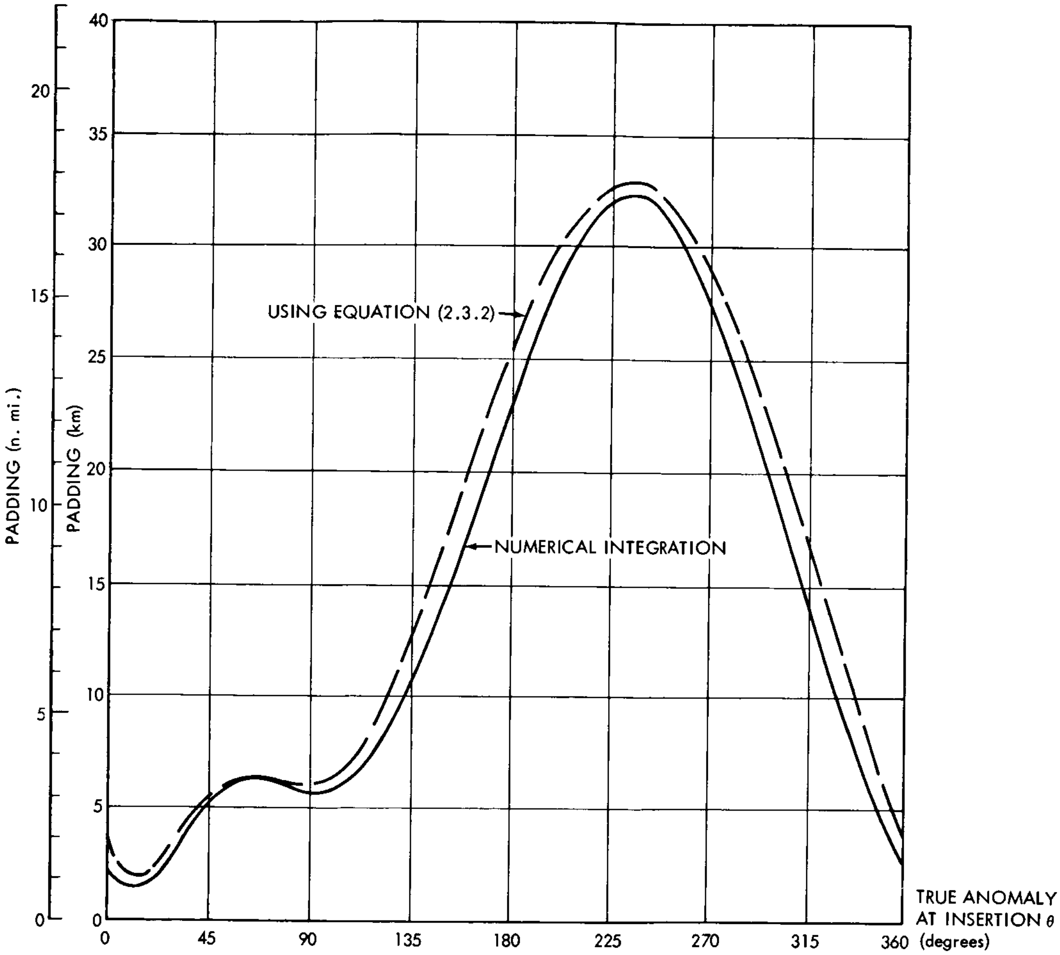

Figure 20. Comparison Between the Required 99.5% Padding as Calculated by Numerical Integration and the Required Padding Using a Normal Distribution Approximation. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.01. Insertion Errors Are Uncorrelated.

APPENDIX C

THE CUMULATIVE DISTRIBUTION FUNCTION OF THE ERROR IN ECCENTRICITY.

The eccentricity e can be expressed as a function of r_o, v_o, and γ_o. From Appendix A (equation A.6)

$$e = e(r_o, v_o, \gamma_o) = \sqrt{\sin^2\gamma_o + \left(\frac{r_o v_o^2}{\mu} - 1\right)^2 \cos^2\gamma_o} \qquad \text{(C.1)}$$

In calculating the insertion parameters r_o, v_o, γ_o, the errors Δr_o, Δv_o, and $\Delta\gamma_o$ are introduced. Therefore, the calculated eccentricity e_{cal} will deviate from the actual eccentricity e, by the amount Δe

$$\Delta e = e_{cal} - e = e(r_o + \Delta r_o, v_o + \Delta v_o, \gamma_o + \Delta\gamma_o) - e(r_o, v_o, \gamma_o) \qquad \text{(C.2)}$$

Letting ΔE be a random variable representing the error in eccentricity, and ΔR_o, ΔV_o, $\Delta\Gamma_o$ random variables representing the errors in insertion radius, speed, and flight path angle respectively, we may write with the aid of (C.2)

$$\Delta E = e(r_o + \Delta R_o, v_o + \Delta V_o, \gamma_o + \Delta\Gamma_o) - e(r_o, v_o, \gamma_o) \qquad \text{(C.3)}$$

The probability density function approximation for ΔE was obtained in an analogous manner as was done for ΔR_p (Appendix B). In an effort to shorten machine calculation time, however, each normal probability density function (in the uncorrelated y-space) was approximated by a discrete probability distribution having 81 mass points (instead of 101), extending from -4 standard deviations to +4 standard deviations. The width of each bin for the error in eccentricity was taken as 10^{-5}.

The Error In Eccentricity for a Circular Orbit

Equation (C.1) may be written

$$1 - e^2 = \left(\frac{2r_o v_o^2}{\mu}\right)\cos^2\gamma_o - \left(\frac{r_o^2 v_o^4}{\mu^2}\right)\cos^2\gamma_o \qquad \text{(C.4)}$$

Substitute $(e + \Delta e)$ for e, $(r_o + \Delta r_o)$ for r_o, $(v_o + \Delta v_o)$ for v_o, $(\gamma_o + \Delta\gamma_o)$ for γ_o and collect terms.

$$1 - e^2 - 2e\Delta e - (\Delta e)^2 = \left(\frac{2}{\mu}\right) r_o v_o^2 \cos^2\gamma_o$$

$$+\left\{\left(\frac{1}{\mu^2}\right) r_o^2 v_o^4 \cos(2\gamma_o) - \left(\frac{2}{\mu}\right) r_o v_o^2 \cos(2\gamma_o)\right\}(\Delta\gamma_o)^2$$

$$+\left\{\left(\frac{2}{\mu^2}\right) r_o^2 v_o^4 \sin\gamma_o \cos\gamma_o - \left(\frac{4}{\mu}\right) r_o v_o^2 \sin\gamma_o \cos\gamma_o\right\}\Delta\gamma_o$$

$$+\left\{\left(\frac{4}{\mu}\right) r_o v_o \cos^2\gamma_o - \left(\frac{4}{\mu^2}\right) r_o^2 v_o^3 \cos^2\gamma_o\right\}\Delta v_o$$

$$+\left\{\left(\frac{8}{\mu^2}\right) r_o^2 v_o^3 \sin\gamma_o \cos\gamma_o - \left(\frac{8}{\mu}\right) r_o v_o \sin\gamma_o \cos\gamma_o\right\}\Delta v_o \Delta\gamma_o$$

$$+\left\{\left(\frac{2}{\mu}\right) r_o \cos^2\gamma_o - \left(\frac{6}{\mu^2}\right) r_o^2 v_o^2 \cos^2\gamma_o\right\}\Delta v_o^2$$

$$+\left\{\left(\frac{2}{\mu}\right) v_o^2 \cos^2\gamma_o - \left(\frac{2}{\mu^2}\right) r_o v_o^4 \cos^2\gamma_o\right\}\Delta r_o$$

$$+\left\{\left(\frac{4}{\mu^2}\right) r_o v_o^4 \sin\gamma_o \cos\gamma_o - \left(\frac{4}{\mu}\right) v_o^2 \sin\gamma_o \cos\gamma_o\right\}\Delta r_o \Delta\gamma_o$$

$$+\left\{\left(\frac{4}{\mu}\right) v_o \cos^2\gamma_o - \left(\frac{8}{\mu^2}\right) r_o v_o^3 \cos^2\gamma_o\right\}\Delta r_o \Delta v_o$$

$$-\left(\frac{1}{\mu^2}\right) r_o^2 v_o^4 \cos^2\gamma_o \qquad \text{(C.5)}$$

$$-\left\{\left(\frac{1}{\mu^2}\right) v_o^4 \cos^2\gamma_o\right\}(\Delta r_o)^2 + \text{higher order terms} \qquad \text{(C.5 cont'd.)}$$

For a circular orbit $e = 0, \left(\frac{r_o v_o^2}{\mu}\right) = 1$, and $\gamma = 0°$. Making these substitutions in (C.5), and neglecting terms higher than the second order, gives,

$$1 - (\Delta e)^2 = 2 - (\Delta\gamma_o)^2 - \left(\frac{4}{v_o^2}\right)(\Delta v_o)^2 - \left(\frac{4}{r_o v_o}\right)\Delta r_o \Delta v_o - 1 - \left(\frac{1}{r_o^2}\right)(\Delta r_o)^2 \qquad \text{(C.6)}$$

Solving for Δe in (C.6), we have,

$$\Delta e = \pm\sqrt{\left\{\left(\frac{2}{v_o}\right)\Delta v_o + \left(\frac{1}{r_o}\right)\Delta r_o\right\}^2 + (\Delta\gamma_o)^2} \qquad \text{(C.7)}$$

Since $e = 0$ for a circular orbit and the calculated eccentricity must be positive (by the definition of eccentricity), Δe in (C.7) must also be positive. Therefore, the error in eccentricity is,

$$\Delta e = +\sqrt{\left[\left(\frac{2}{v_o}\right)\Delta v_o + \left(\frac{1}{r_o}\right)\Delta r_o\right]^2 + (\Delta\gamma_o)^2} \qquad \text{(C.8)}$$

APPENDIX D

THE CUMULATIVE DISTRIBUTION FUNCTION OF THE ERROR IN TRUE ANOMALY.

Using (A.3) we may calculate θ

$$\theta = \cos^{-1}\left\{\frac{a(1-e^2) - r_o}{r_0 e}\right\} \tag{D.1}$$

where a and e are functions of r_o, v_o, and γ_o by (A.5) and (A.6)

$$a = \frac{r_o}{2 - \left(\frac{r_o v_o^2}{\mu}\right)} \tag{A.5}$$

$$e = \sqrt{\sin^2\gamma_o + \left(\frac{r_o v_o^2}{\mu} - 1\right)^2 \cos^2\gamma_o} \tag{A.6}$$

The correct quadrant for θ may be resolved from inspection of (B.68)

$$\tan\gamma_o = \frac{e \sin\theta}{1 + e\cos\theta} \tag{B.68}$$

We see that for $\gamma_o \geq 0°$, $0 \leq \theta \leq 180°$ and for $\gamma_o \leq 0°$, $-180° \leq \theta \leq 0°$. Thus, we may write

$$\theta = \theta(r_o, v_o, \gamma_o) \tag{D.2}$$

Letting ΔR_o, ΔV_o, $\Delta\Gamma_o$ be random variables representing the errors in insertion radius, speed, and flight path angle, with the aid of (D.2) we may write

$$\Delta\Theta = \theta(r_o + \Delta R_o, v_o + \Delta V_o, \gamma_o + \Delta\Gamma_o) - \theta(r_o, v_o, \gamma_o) \tag{D.3}$$

where $\Delta\Theta$ is a random variable representing the error in true anomaly for a non-circular orbit.

The cumulative distribution function of $\Delta\Theta$

$$K(\Delta\theta) = P(\Delta\Theta \leq \Delta\theta) \tag{D.4}$$

may be seen in Figures 21 and 22. These were obtained in analogous fashion as for ΔR_p and ΔE.

Figure 21 is for uncorrelated insertion errors, an eccentricity of e = 0.001 and true anomalies at insertion of θ = 0°, 90°, 180°, and 270°.

Figures 22a and 22b are for the same conditions of true anomaly at insertion and correlation, except the eccentricity is 0.005.

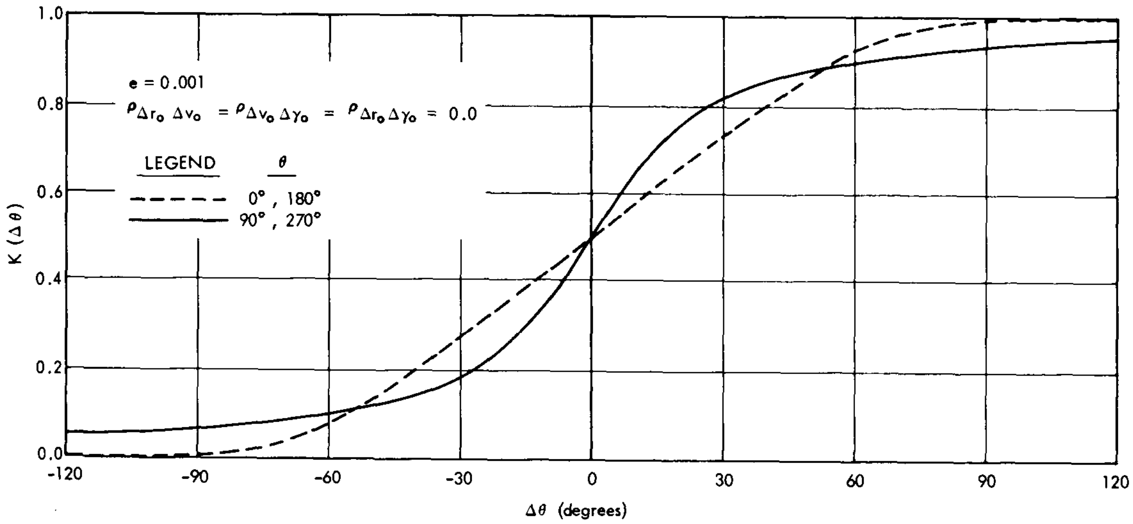

Figure 21. Cumulative Distribution Functions of the Error in True Anomaly for Various Actual Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.001. Uncorrelated Insertion Errors.

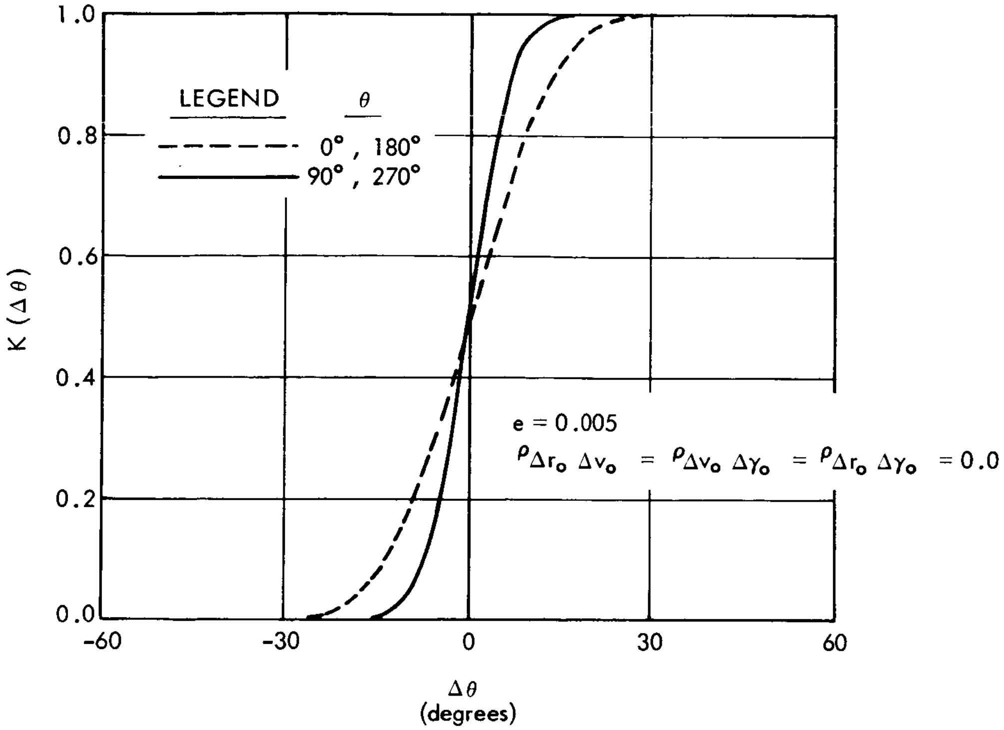

Figure 22a. Cumulative Distribution Functions of the Error in True Anomaly for Various Actual Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Uncorrelated Insertion Errors.

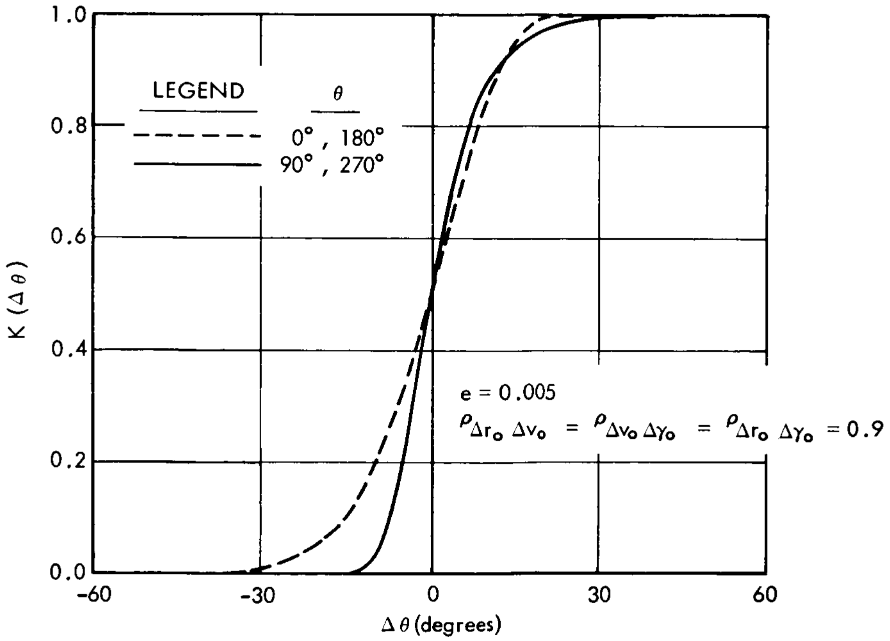

Figure 22b. Cumulative Distribution Functions of the Error in True Anomaly for Various Actual Values of True Anomaly at Insertion. Insertion Parameters: a = 3544 n.mi. (6563 km), e = 0.005. Correlated Insertion Errors ($\rho_{\Delta r_o \Delta v_o}$ = +0.9, $\rho_{\Delta v_o \Delta \gamma_o}$ = $\rho_{\Delta r_o \Delta \gamma_o}$ = ±0.9)

The "3-sigma" limits used for ΔR_o, ΔV_o, and $\Delta \Gamma_o$ are

$$
\begin{aligned}
3\sigma_{\Delta r_o} &= 2.4 \text{ n. mi. } (4.44 \text{ km}) \\
3\sigma_{\Delta v_o} &= 16 \text{ ft/sec } (4.87 \text{ m/sec}) \qquad \text{(D.5)} \\
3\sigma_{\Delta \gamma_o} &= 0.16° \ (2.79 \text{ mrad})
\end{aligned}
$$

From inspection of the curves it can be seen that for parking orbits with low eccentricities ($e \leq 0.001$) true anomaly at insertion θ cannot be determined accurately. For example, due to tracking errors, for $e = 0.001$, $\theta = 0°$ or $\theta = 180°$, and uncorrelated insertion errors (dashed curve of Figure 21) there is a 90% probability that the error in true anomaly will be between -66° and +66°.

From Figures 22a and 22b it can be seen that true anomaly can be determined more accurately if the eccentricity is greater.