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INVESTIGATIONS IN HANSEN'S PLANETARY THEORY

by Peter Musen

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

Hansen's planetary theory represents the position of the disturbed planet as a deviation in time and space from the position of a fictitious planet, whose motion is Keplerian relative to pseudo-time. This work contains an investigation about the nature of the basic function W of Hansen's theory, which serves to determine the deviation mentioned. The standard way to obtain the differential equation for W is based on the application of the method of variation of the astronomical constants. However, Hansen's coordinates of a planet are more intimately connected with the perturbations in the position vector than with the perturbations in the elements. For this reason, following von Zeipel, dW/dt is obtained here as the projection of the perturbations in the acceleration vector on a variable vector $\vec{N}$, which is selected in such a way that dW/dt is totally integrable.

The form of W thus obtained contains an arbitrary constant and an arbitrary function of the osculating areal velocity. If dW/dt is obtained in terms of a disturbing force, then it can clearly be seen that the presence of the arbitrary elements in W suggests the separate introduction of a vector $\vec{S}$ and of a scalar h_0/h, bypassing the formation of W. In fact, W represents a fusion of three independent series into one, although no actual gain is expected from such a fusion in terms of programming or computing time.

The "barred" Hansen function $\overline{W}$, however, is the most essential part of the theory. It is intimately connected with the determination of the perturbations in Hansen's coordinates and with the formation of the integrating operator. $\overline{W}$ is retained in this exposition. Furthermore, the geometric characteristics of Hansen's theory favor the expansion of the perturbations into trigonometric series with the disturbed mean anomalies of planets as arguments. Such a form of expansion eliminates the need to develop $\overline{W}$ and the disturbing force in powers of the perturbations of the mean anomalies. It contracts the series and speeds up the convergence. We economize, especially, in the expansions of the odd negative powers of the mutual planetary distances.

The motion of the disturbed planet is related to an inertial frame of reference situated in the orbital plane of the fictitious planet. The standard use of the rotating ideal frame of reference instead of an inertial frame does not simplify the actual numerical procedure. Hansen himself was compelled to introduce the perturbations in the "third coordinate." In the final instance this is equivalent to the use of a fixed frame of reference of the type mentioned.

The integration operator is expanded into a series, whose form favors the application of the method of iteration. At each step of the iterative process the perturbations are obtained by solving a linear partial differential equation with constant coefficients.

CONTENTS

NOTATION

P - the disturbed planet,

m - the mass of P,

M - the mass of the Sun,

f - the gravitational constant,

$\mu^2 = f(M+m)$,

P_1 - the auxiliary planet moving in a fixed plane in accordance with Kepler's laws,

M, a_0, e_0, $\cdots$ etc. the standard elliptic elements of P_1,

P' - the disturbing planet,

$\vec{s}$ - the heliocentric position vector of P,

$\vec{r}$ - the projection of $\vec{s}$ on the orbital plane of P,

$r = |\vec{r}|$,

$\vec{\bar{r}}$ - the heliocentric position vector of P_1,

$\bar{r} = |\vec{\bar{r}}|$,

$\bar{f}$ - the true anomaly of P_1,

E - the eccentric anomaly of P_1,

u - the distance of P from the orbital plane of P_1,

$\vec{P}$ - the unit vector directed toward the perihelion of P_1,

$\vec{R}$ - the unit vector normal to the orbital plane of P_1,

$\vec{Q} = \vec{R} \times \vec{P}$,

$1+\nu$ - the ratio $r/\bar{r}$,

t - time,

z - the pseudo-time (the disturbed time). It appears instead of t in Kepler's equation for P_1,

$\vec{\bar{v}} = \frac{d\vec{\bar{r}}}{dz}$ - the velocity of $\vec{\bar{r}}$ relative to z,

$\delta z = z - t$ - the perturbations of time,

$n_0\,\delta z$ - the perturbations in the mean anomaly,

The notations for P' corresponding to the above are designated by the "primed" symbols,

J - the mutual inclination of the orbital planes of two auxiliary planets,

∇' - the gradient operator relative to $\vec{\bar{r}}'$,

$\vec{\ell} = \vec{\bar{r}}' - \vec{\bar{r}}$ - the mutual distance between two auxiliary planets,

$\delta\vec{\bar{r}} = \nu\vec{\bar{r}} + u\vec{R}$ - Hansen perturbations of $\vec{\bar{r}}$,

$\vec{\bar{r}} + \delta\vec{\bar{r}} = \vec{r}$,

$\delta\vec{\bar{r}}' = \nu'\vec{\bar{r}}' + u'\vec{R}'$ - Hansen perturbations of $\vec{\bar{r}}'$,

$\vec{r}' = \vec{\bar{r}}' + \delta\vec{\bar{r}}'$,

$D' = \nabla' \exp(\delta\vec{\bar{r}} \cdot \nabla')$,

$D'' = \nabla' \exp(\delta\vec{\bar{r}} \cdot \nabla')$,

I - the operator of projection on the orbital plane of P_1,

$1/h$ - the areal velocity of the projection of P on the orbital plane of P_1,

$1/h_0$ - the areal velocity of P_1,

$X = h_0/h$,

W and $\bar{W}$ - Hansen's functions.

INVESTIGATIONS IN HANSEN'S PLANETARY THEORY

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INTRODUCTION

This work presents an investigation concerning the foundations of Hansen's planetary theory (1857), and a system of formulas which can be used in actual computations of the first and higher order general perturbations in Hansen's coordinates.

As the first approximation to the motion of the planet P we take, following the suggestion by Oppolzer (1883) and Andoyer (1926), the motion of a fictitious auxiliary planet P_1 moving in a constant ellipse in a fixed plane (xy) in accordance with Kepler's laws. The position of P is determined by its deviation from the position of P_1 in time and space.

More precisely: at the moment t the projection of the position vector of P on the (xy)-plane will have the same direction as the position vector of P_1 at the moment z. Designating by $\vec{s}$ the heliocentric position vector of P, by $\vec{r}$ the heliocentric position vector of the projection of P on (xy), by u the distance of P from (xy), by $\vec{\bar{r}}$ the heliocentric position vector of P_1 and finally by $\vec{R}$ the unit vector normal to (xy), we can write the basic relations of the theory in the form:

$$\vec{s}(t) = \vec{r}(t) + u(t)\vec{R} ,$$

$$\vec{r}(t) = (1+\nu)\vec{\bar{r}}(z) , \tag{1}$$

The fictitious planet should be so chosen that Hansen's perturbations ν, u, and δz in the semi-major axis, in the "third coordinate", and in time, respectively, will be small—of the order of the perturbations. The introduction of the pseudo-time z and of the "perturbations of time" δz is the main characteristic of Hansen's theory which was considered peculiar during his lifetime and which, surprisingly, is sometimes considered difficult to accept even now. Another important characteristic of Hansen's theory is that the perturbations ν and δz in the orbit plane of P_1, are determined by one single function $\bar{W}$. The mean anomalies of the fictitious planets corresponding to the disturbed and to the disturbing planets are the basic arguments in expansions of the perturbations into periodic series. The role of the mean anomaly of P_1 is twofold: It enters into the expansion of the perturbations of Hansen's coordinates and it is also the argument in the expansion of the elliptic coordinates of P_1. In order to separate the perturbative effects from the elliptic motion of P_1 Hansen uses a special notation for each mean anomaly. Then he forms a function W and its derivative dW/dt in which these anomalies are kept separated. The derivative dW/dt is formed in terms of the components of the disturbing force. The W-function is obtained by integrating dW/dt. After the integration, the $\bar{W}$-function is obtained by removing the distinction between the two types of mean anomalies.

The operator for converting W into $\overline{W}$ is Hansen's "bar-operator," and its use represents the third main characteristic of Hansen's theory. Normally the differential equation governing the variation of W is deduced on the basis of the variation of elements. Thus, the theory of Hansen can be considered as an ingenious transfer of the perturbations in the elements to the perturbations in the coordinates. However, an approach without resort to the variation of elements and based solely on the equations of motion is also possible. Such an approach was suggested by von Zeipel (1902). He used the Lagrangian form of the equations of motion in polar coordinates with the eccentric anomaly as the independent variable.

Looking closely at dW/dt, one can recognize easily that it is the projection of the perturbations in the acceleration of P on a variable vector selected so that dW/dt is totally integrable and W is a linear function of $d\nu/dt$. W contains an arbitrary vector $\vec{c}$ which can be taken to be a function of the "elliptic" mean anomaly. Hansen's solution corresponds to the special case when $\vec{c}$ is normal to the position vector of the auxiliary planet. We follow here in von Zeipel's footsteps with some modifications. The direct use of Equation (1) leads immediately to the expression for the perturbations of the acceleration vector of P and to the general forms of dW/dt and W which facilitate the kinematical conclusions.

Looking at these general forms of W and dW/dt we recognize that $\overline{W}$, rather than W, represents the main feature of the method. The $\overline{W}$-function has a direct kinematical meaning, whereas W represents merely an artificial device to combine these series into one. At the present time the expansion of the perturbations into series is done on electronic machines. Neither the length of the series, nor the computing time can be reduced by employing W instead of the three series. For this reason we propose to compute the three parts of W separately. After the computation is completed, we form the $\overline{W}$-function.

We deviate from Hansen in the method of expanding the disturbing force and in the method of integration. In Hansen's work the expansion of the disturbing function and of the disturbing force is done in powers of ν, ν', $n_0\,\delta z$, $n_0'\,\delta z'$, u and u'. The expansion in powers of ν, ν' and u, u' is associated with the expansion of the disturbing force in odd negative powers of the mutual distance of the planets, while the expansion in powers of $n_0\,\delta z$ and $n_0'\,\delta z'$ is associated with the expansion of the odd negative powers of ℓ as functions of time. The perturbations $n_0\,\delta z$ and $n_0'\,\delta z'$ normally are the largest ones. Their determination is associated with the possible appearance in the process of integration of the squares of the small divisors caused by the commensurability of the mean motions

Thus, the convergence in powers of $n_0\,\delta z$ and $n_0'\,\delta z'$ is a relatively slow process as compared to the convergence in ν, ν', u, u'. To speed up convergence we discard the development in powers of $n_0\,\delta z$ and $n_0'\,\delta z'$. Then the angular arguments in the periodic series representing the perturbations or their derivatives will become the linear functions of the perturbed mean anomalies M and M'. The integration of such series is then reduced either to solving a partial differential equation if we use the process of iteration, or to solving a chain of linear partial differential equations, all of the same type, if we resort to the standard asymptotic expansion of perturbations in powers of masses.

We will not follow in Hansen's and von Zeipel's footsteps and do not employ the eccentric anomaly as the independent variable. However, only a slight modification is required in the method we present if it is desired to switch to either the eccentric or the true anomaly as the basic variable.

The computational scheme is arranged so that the process of iteration is applicable and thus the programming is homogeneous. If preferred, the computational scheme for the standard asymptotic expansion in powers of masses can be easily deduced.

THE DIFFERENTIAL EQUATIONS OF MOTION AND THE EXPANSION OF THE DISTURBING FORCE

The motion of the planet P_1 is Keplerian and its position vector $\vec{\bar{r}}(z)$ satisfies the differential equation

$$\frac{d^2 \vec{\bar{r}}}{dz^2} = -\frac{\mu^2 \vec{\bar{r}}}{\bar{r}^3} . \tag{2}$$

The equation of motion of P can be written in the form (Musen, 1965)

$$\frac{d^2}{dt^2}(\vec{r} + u\vec{R}) = -\frac{\mu^2}{(r^2 + u^2)^{3/2}}(\vec{r} + u\vec{R}) + fm'\left(-D''\frac{1}{\ell} + D'\frac{1}{\bar{r}'}\right), \tag{3}$$

where

$$\ell = |\vec{\ell}| , \tag{4}$$

$$\vec{\ell} = \vec{\bar{r}}' - \vec{\bar{r}} , \tag{5}$$

$$D'' = \nabla' \exp(\delta\vec{\ell} \cdot \nabla') ,$$

$$D' = \nabla' \exp(\delta\vec{\bar{r}} \cdot \nabla') , \tag{6}$$

$$\delta\vec{\ell} = \nu' \vec{\bar{r}}' - \nu \vec{\bar{r}} + u' \vec{R}' - u\vec{R} = \delta \vec{\bar{r}}' - \delta \vec{\bar{r}} , \tag{7}$$

$$\delta\vec{\bar{r}} = \nu \vec{\bar{r}} + u\vec{R} ,$$

$$\delta\vec{\bar{r}}' = \nu' \vec{\bar{r}}' + u' \vec{R}' , \tag{8}$$

and ∇' is the gradient operator relative to $\vec{r}'$. From the author's previous (1965) work, we take the expansions:

$$D'' \frac{1}{\ell} = -\frac{\vec{\ell}}{\ell^3} + \left(\frac{3}{\ell^5}\vec{\ell}\vec{\ell}\cdot\delta\vec{\ell} - \frac{1}{\ell^3}\delta\vec{\ell}\right) + \left[-\frac{15}{\ell^7}\vec{\ell}(\vec{\ell}\cdot\delta\vec{\ell})^2 + \frac{6}{\ell^5}\vec{\ell}\cdot\delta\vec{\ell}\,\delta\vec{\ell} + \frac{3}{\ell^5}\vec{\ell}\,\delta\vec{\ell}\cdot\delta\vec{\ell}\right] + \left\{+\frac{105}{\ell^9}\vec{\ell}(\vec{\ell}\cdot\delta\vec{\ell})^3 - \frac{45}{\ell^7}\left[(\vec{\ell}\cdot\delta\vec{\ell})^2\,\delta\vec{\ell} + \vec{\ell}(\vec{\ell}\cdot\delta\vec{\ell})\,(\delta\vec{\ell}\cdot\delta\vec{\ell})\right] + \frac{9}{\ell^5}\delta\vec{\ell}\cdot\delta\vec{\ell}\,\delta\vec{\ell}\right\} + \cdots \;; \tag{9}$$

and

$$D' \frac{1}{\bar{r}'} = -\frac{\vec{r}'}{\bar{r}'^3} + \left(\frac{3}{\bar{r}'^5}\vec{r}'\,\vec{r}'\cdot\delta\vec{r}' - \frac{1}{\bar{r}'^3}\delta\vec{r}'\right) + \left[-\frac{15}{\bar{r}'^7}\vec{r}'(\vec{r}'\cdot\delta\vec{r}')^2 + \frac{6}{\bar{r}'^5}\vec{r}'\cdot\delta\vec{r}'\,\delta\vec{r}' + \frac{3}{\bar{r}'^5}\vec{r}\,\delta\vec{r}'\cdot\delta\vec{r}'\right] + \left\{+\frac{105}{\bar{r}'^9}\vec{r}'(\vec{r}'\cdot\delta\vec{r}')^3 - \frac{45}{\bar{r}'^7}\left[(\vec{r}'\cdot\delta\vec{r}')^2\,\vec{r}' + \vec{r}'\,(\vec{r}'\cdot\delta\vec{r}')\,(\delta\vec{r}'\cdot\delta\vec{r}')\right] + \frac{9}{\bar{r}'^5}(\delta\vec{r}'\cdot\delta\vec{r}')\,\delta\vec{r}'\right\} + \cdots \,. \tag{10}$$

In the process of expanding (9) and (10) and the components of the disturbing force in terms of the disturbed mean anomalies, we shall use the following auxiliary quantities:

$$A = a_0^2\left(\frac{\bar{r}}{a_0}\right)^2 \,, \tag{11}$$

$$A' = a_0'^2\left(\frac{\bar{r}'}{a_0'}\right)^2 \,, \tag{12}$$

$$B = B' = \vec{r}\cdot\vec{r}' = a_0\,a_0'\,(\alpha cc' + \beta cs' + \alpha' sc' + \beta' ss') \,, \tag{13}$$

$$F = \vec{r}'\cdot\vec{R} = a_0'\,(\gamma c' + \gamma' s') \,, \tag{14}$$

$$F' = \vec{r}\cdot\vec{R}' = a_0\,(\kappa c + \kappa' s) \,, \tag{15}$$

$$\Lambda' = \vec{R}\cdot\vec{r}\times\vec{r}' = a_0\,a_0'\,(\alpha' cc' - \alpha sc' + \beta' cs' - \beta ss') \,, \tag{13'}$$

$$\Theta' = \vec{r} \cdot \vec{R}' \times \vec{R} = a_0 (\kappa' c - \kappa s) , \tag{14'}$$

$$\Gamma = \vec{v} \cdot \vec{r} = \frac{n_0 a_0^2 e_0}{\sqrt{1 - e_0^2}} s , \tag{11''}$$

$$\Gamma' = \vec{v} \cdot \vec{r}' = \frac{n_0}{\sqrt{1 - e_0^2}} \frac{a_0^2}{\bar{r}^2} \left[e_0 s B + \sqrt{1 - e_0^2} \, A' \right] , \tag{13''}$$

$$\Phi' = \vec{v} \cdot \vec{R}' = \frac{n_0}{\sqrt{1 - e_0^2}} \frac{a_0^2}{r^2} \left[e_0 s F' + \Theta' \left(1 - e_0^2\right)\right] , \tag{15''}$$

where

$$c = \frac{\bar{r}}{a_0} \cos \bar{f} , \qquad s = \frac{\bar{r}}{a_0} \sin \bar{f} ,$$

$$c' = \frac{\bar{r}'}{a_0'} \cos \bar{f}' , \qquad s' = \frac{\bar{r}'}{a_0'} \sin \bar{f}' ,$$

$$\alpha = \vec{P} \cdot \vec{P}' , \qquad \alpha' = \vec{Q} \cdot \vec{P}' ,$$

$$\beta = \vec{P} \cdot \vec{Q}' , \qquad \beta' = \vec{Q} \cdot \vec{Q}' ,$$

$$\gamma = \vec{R} \cdot \vec{P}' , \qquad \gamma' = \vec{Q}' \cdot \vec{R} ,$$

$$\kappa = \vec{P} \cdot \vec{R}' , \qquad \kappa' = \vec{Q} \cdot \vec{R}' .$$

$$G = \vec{r} \cdot \vec{\ell} = B - A , \tag{16}$$

$$G' = \vec{r}' \cdot \vec{\ell} = A' - B' . \tag{17}$$

The expansion of (11) - (17) in terms of M and M' is performed in the standard manner, either using the classical analytical formulas or by employing harmonic analysis.

The explicitly written portions of (9) and (10) permit the development of the general perturbations up to the fourth order relative to the disturbing masses. We have from (7):

$$\vec{\ell} \cdot \delta\vec{\ell} = G' \nu' - G\nu - F' u' - Fu , \tag{18}$$

$$\delta\vec{\ell} \cdot \delta\vec{\ell} = A'\nu'^2 + A\nu^2 - 2B\nu'\nu - 2F\nu'u - 2F'\nu u' + \left(u^2 + u'^2 - 2uu'\cos J\right) , \tag{19}$$

$$\vec{\bar{r}}' \cdot \delta\vec{\bar{r}}' = A'\nu' , \tag{20}$$

$$\delta\vec{\bar{r}}' \cdot \delta\vec{\bar{r}}' = A'\nu'^2 + u'^2 . \tag{21}$$

Substituting these expressions into (9) and (10) we deduce the expansions:

$$-D''\frac{1}{\ell} + D'\frac{1}{\bar{r}'} = K\vec{r} + K'\vec{\bar{r}}' + H\vec{R} + H'\vec{R}' ; \tag{22}$$

$$K = \sum K^{(n)}_{j,j',k,k'} \nu^j \nu'^{j'} u^k u'^{k'} ; \tag{23}$$

$$K' = \sum K'^{(n)}_{j,j',k,k'} \nu^j \nu'^{j'} u^k u'^{k'} ; \tag{24}$$

$$H = \sum H^{(n)}_{j,j',k,k'} \nu^j \nu'^{j'} u^k u'^{k'} , \tag{25}$$

$$H' = \sum H'^{(n)}_{j,j',k,k'} \nu^j \nu'^{j'} u^k u'^{k'} , \tag{26}$$

$$j, j', k, k' \geqq 0 , \qquad j + j' + k + k' = n$$

$$n = 0, 1, 2, \cdots .$$

In the following table we give the values of the coefficients K and H which are different from zero for $n = 0, 1, 2$.

$$K^{(0)}_{0000} = -\frac{1}{\ell^3} ,$$

$$K'^{(0)}_{0000} = \frac{1}{\ell^3} - \frac{1}{\bar{r}'^3} ,$$

$$K^{(1)}_{1000} = -\frac{1}{\ell^3} - \frac{3G}{\ell^5} , \qquad K'^{(1)}_{1000} = +\frac{3G}{\ell^5} ,$$

$$K^{(1)}_{0100} = +\frac{3G'}{\ell^5}, \qquad K'^{(1)}_{0100} = \frac{1}{\ell^3} - \frac{3G'}{\ell^5} + \frac{2}{\bar{r}'^3},$$

$$K^{(1)}_{0010} = -\frac{3F}{\ell^5}, \qquad K'^{(1)}_{0010} = +\frac{3F}{\ell^5},$$

$$K^{(1)}_{0001} = -\frac{3F'}{\ell^5}, \qquad K'^{(1)}_{0001} = +\frac{3F'}{\ell^5},$$

$$H^{(1)}_{0010} = -\frac{1}{\ell^3}, \qquad H'^{(1)}_{0001} = \frac{1}{\ell^3} - \frac{1}{\bar{r}'^3},$$

$$K^{(2)}_{2000} = -\frac{15}{\ell^7} G^2 - \frac{6G}{\ell^5} + \frac{3A}{\ell^5},$$

$$K^{(2)}_{0200} = -\frac{15}{\ell^7} G'^2 + \frac{3A'}{\ell^5},$$

$$K^{(2)}_{0020} = -\frac{15}{\ell^7} F^2 + \frac{3}{\ell^5},$$

$$K^{(2)}_{0002} = -\frac{15}{\ell^7} F'^2 + \frac{3}{\ell^5},$$

$$K^{(2)}_{1100} = +\frac{30}{\ell^7} GG' + \frac{6G'}{\ell^5} - \frac{6B}{\ell^5},$$

$$K^{(2)}_{0101} = +\frac{30}{\ell^7} G'F',$$

$$K^{(2)}_{0110} = +\frac{30}{\ell^7} FG' - \frac{6F}{\ell^5},$$

$$K^{(2)}_{1001} = -\frac{30}{\ell^7} GF' - \frac{12F'}{\ell^5},$$

$$K^{(2)}_{1010} = -\frac{30}{\ell^7} GF - \frac{6F}{\ell^5},$$

$$K^{(2)}_{0011} = -\frac{30}{\ell^7} FF' - \frac{6\cos J}{\ell^5},$$

$$K'^{(2)}_{0200} = +\frac{15}{\ell^7} G'^2 - \frac{6}{\ell^5} G' - \frac{3A'}{\ell^5} - \frac{6}{r'^3} ,$$

$$K'^{(2)}_{2000} = +\frac{15}{\ell^7} G^2 - \frac{3A}{\ell^5} ,$$

$$K'^{(2)}_{0002} = +\frac{15}{\ell^7} F'^2 - \frac{3}{\ell^5} + \frac{3}{r'^5} ,$$

$$K'^{(2)}_{0110} = -\frac{30}{\ell^7} G' F + \frac{12F}{\ell^5} ,$$

$$K'^{(2)}_{1001} = +\frac{30}{\ell^7} GF' + \frac{6}{\ell^5} F' ,$$

$$K'^{(2)}_{1010} = +\frac{30}{\ell^7} GF ,$$

$$K'^{(2)}_{0011} = +\frac{30}{\ell^7} FF' + \frac{6}{\ell^5} \cos J ,$$

$$H^{(2)}_{0110} = +\frac{6G'}{\ell^5} ,$$

$$H^{(2)}_{1010} = -H'^{(2)}_{1001} = -\frac{6G}{\ell^5} ,$$

$$H^{(2)}_{0011} = -H'^{(2)}_{0002} = -\frac{6F'}{\ell^5} ,$$

$$H^{(2)}_{0020} = -H'^{(2)}_{0011} = -\frac{6F}{\ell^5} ,$$

$$H'^{(2)}_{0101} = -\frac{6G'}{\ell^5} + \frac{6}{r'^3} .$$

This table permits one to develop the perturbations up to the third order. This accuracy is sufficient for the majority of planets in our solar system. The inclusion of the perturbative effects of the fourth and higher orders, if necessary, can be easily accomplished by expanding the additional terms in (9) and (10).

The choice of the orbital planes of the auxiliary planets can be made in such a way that the orders of the magnitudes of u and u′ numerically will be higher than the orders of magnitudes of

ν, ν', $n_0\,\delta z$, $n_0'\,\delta z'$. Thus, the terms in $\vec{F}$ which contain the factors $m'\nu u$, $m'\nu u'$, $m'\nu' u$, $m'\nu' u'$, $m' u^2$, $m' uu'$, $m' u'^2$ can usually be omitted.

From (3) we have

$$\frac{d^2 \vec{r}}{dt^2} = -\frac{\mu^2}{r^3}\vec{r} + \vec{F}\,, \tag{27}$$

$$\frac{d^2 u}{dt^2} = -\frac{\mu^2 u}{\left(r^2 + u^2\right)^{3/2}} + Z\,, \tag{28}$$

where we set

$$\vec{F} = \mu^2 \left[\frac{1}{r^3} - \frac{1}{\left(r^2 + u^2\right)^{3/2}}\right]\vec{r} + fm'\, I \cdot \left(-D'' \frac{1}{\ell} + D' \frac{1}{r'}\right)\,, \tag{29}$$

$$Z = \vec{R} \cdot \left(-D'' \frac{1}{\ell} + D' \frac{1}{r'}\right)\,, \tag{30}$$

I is the operator of projection on the (xy)-plane. It can be represented in several ways, either as

$$\begin{aligned} I &= h_0 \vec{R} \times \left(\vec{r}\vec{v} - \vec{v}\vec{r}\right) \\ &= h_0 \left(\vec{r}\vec{v} - \vec{v}\vec{r}\right) \times \vec{R}\,, \end{aligned} \tag{31}$$

or as

$$I = \frac{1}{r^2}\left(\vec{r}\,\vec{r} - \vec{R} \times \vec{r}\,\vec{r} \times \vec{R}\right)\,, \tag{32}$$

or as

$$I = \vec{P}\vec{P} + \vec{Q}\vec{Q}\,, \tag{33}$$

according to the kind of decomposition of $\vec{F}$ we prefer. The representation (31) of I is preferable if we want to separate the terms with small divisors. The first terms in (29) and (30) can be written as

$$\mu^2 \left[\frac{1}{r^3} - \frac{1}{\left(r^2 + u^2\right)^{3/2}}\right]\vec{r} = \frac{3}{2}\mu^2 \frac{u^2}{r^5}\left(1 + \nu\right)^{-4}\vec{r} + \cdots \tag{34}$$

and

$$\mu^2 - \frac{\mu^2 u}{\left(r^2 + u^2\right)^{3/2}} = -\frac{\mu^2 u}{r^3} + \frac{3}{2}\mu^2 \frac{u^2}{r^5} + \cdots .$$

The last quantity normally is negligible.

Taking (11) - (15") into account, we deduce the following expressions for Z and for the components of $\vec{F}$:

$$Z = fm' (K' F + H + H' \cos J) , \tag{35}$$

$$\vec{r} \cdot \vec{F} = fm' (KA + K' B + H' F') + \frac{3}{2}\mu^2 \frac{u^2}{r^3} (1+\nu)^{-4} + \cdots , \tag{36}$$

$$\vec{R} \times \vec{r} \cdot \vec{F} = fm' (K' \Lambda' + H' \Theta') , \tag{37}$$

$$\vec{v} \cdot \vec{F} = fm' (K\Gamma + K' \Gamma' + H' \Phi') + \frac{3}{2}\mu^2 \frac{u^2}{r^3} (1+\nu)^{-4} \Gamma + \cdots , \tag{38}$$

$$\vec{P} \cdot \vec{F} = fm' \left[Ka_0 c + K' a_0' (\alpha c' + \beta s') + H' \kappa\right] + \frac{3}{2}\mu^2 a_0 \frac{u^2}{r^5} (1+\nu)^{-4} c + \cdots , \tag{36'}$$

$$\vec{Q} \cdot \vec{F} = fm' \left[Ka_0 s + K' a_0' (\alpha' c' + \beta' s') + H' \kappa'\right] + \frac{3}{2}\mu^2 a_0 \frac{u^2}{r^5} (1+\nu)^{-4} s + \cdots . \tag{37'}$$

The system of formulas (35) - (37'), because of its symmetry, provides us with a scheme which is convenient for the use of the process of iteration as well as for programming.

PERTURBATIONS IN THE ORBITAL PLANE OF THE AUXILIARY PLANET

Differentiating (1) twice with respect to time and taking (2) and (27) into account, we have

$$\vec{v} = \dot{\nu} \vec{\bar{r}} + (1+\nu) \vec{\bar{v}} \dot{z} , \tag{39}$$

$$\left[\ddot{\nu} - \frac{\mu^2 (1+\nu) \dot{z}^2}{\bar{r}^3}\right] \vec{\bar{r}} + \frac{\vec{\bar{v}}}{1+\nu} \frac{d}{dt} \left[(1+\nu)^2 \dot{z}\right] = -\frac{\mu^2 \vec{r}}{r^3} + \vec{F} . \tag{40}$$

Introducing Hansen's notations for the sectorial velocities,

$$\vec{R} \cdot \vec{r} \times \vec{v} = \frac{1}{h} , \tag{41}$$

$$\vec{R} \cdot \vec{\bar{r}} \times \vec{\bar{v}} = \frac{1}{h_0} , \tag{42}$$

we deduce from (39) the classical formula

$$\dot{z} = \frac{X}{(1+\nu)^2} , \tag{43}$$

where we set

$$\frac{h_0}{h} = X . \tag{44}$$

Taking (43) into account we can rewrite (40) as

$$\left[\ddot{\nu} - \frac{X^2 - (1+\nu)}{(1+\nu)^3} \frac{\mu^2}{\bar{r}^3}\right] \vec{\bar{r}} + \frac{\vec{\bar{v}}}{1+\nu} \dot{X} = \vec{F} . \tag{45}$$

We now determine a vector $\vec{N}$ so that the scalar product of $\vec{N}$ with the left side of (45) represents a totally integrable function.

For this purpose a convenient form of the decomposition of $\vec{N}$ is

$$\vec{N} = p\vec{\bar{v}} \times \vec{R} + q\vec{R} \times \vec{\bar{r}} , \tag{46}$$

where p and q are functions of X, ν, z. From (42) we have

$$p = h_0 \vec{\bar{r}} \cdot \vec{N} , \tag{47}$$

$$q = h_0 \vec{\bar{v}} \cdot \vec{N} . \tag{48}$$

We determine a function

$$W = W(X, \nu, \dot{\nu}, z) \tag{49}$$

such that, in accordance with (45) - (48), we have

$$\frac{dW}{dt} = \frac{\partial W}{\partial X}\dot{X} + \frac{\partial W}{\partial \dot{\nu}}\ddot{\nu} + \left(\frac{\partial W}{\partial \nu}\dot{\nu} + \frac{\partial W}{\partial z}\dot{z}\right)$$

$$= \frac{p}{h_0}\left[\ddot{\nu} - \frac{X^2 - (1+\nu)}{(1+\nu)^3} \cdot \frac{\mu^2}{r^3}\right] + \frac{q}{h_0}\frac{1}{1+\nu}\dot{X} . \tag{50}$$

In this respect we follow in von Zeipel's footsteps. However, since we are using time, and not the eccentric anomaly, as the independent variable, the straight forward application of (2) and (27) makes the exposition simpler. In addition, the kinematical meaning of the results becomes more direct.

From the last equation we deduce:

$$\frac{\partial W}{\partial X} = \frac{1}{h_0}\frac{q}{1+\nu} , \tag{51}$$

$$\frac{\partial W}{\partial \dot{\nu}} = \frac{p}{h_0} , \tag{52}$$

$$\frac{\partial W}{\partial \nu}\dot{\nu} + \frac{\partial W}{\partial z}\frac{X}{(1+\nu)^2} = \frac{(1+\nu) - X^2}{(1+\nu)^3} \cdot \frac{\mu^2}{r^3}\frac{p}{h_0} . \tag{53}$$

From (52) we conclude that W must have the form

$$W = \frac{p}{h_0}\dot{\nu} + \frac{b(X, \nu, z)}{h_0} . \tag{54}$$

Substituting (54) into (51) and (53) we obtain

$$\frac{\partial p}{\partial \nu} = 0 , \qquad \frac{\partial p}{\partial X} = 0 ,$$

$$\frac{\partial b}{\partial z} = \left(\frac{1}{X} - \frac{X}{1+\nu}\right)\frac{\mu^2 p}{r^3} , \tag{55}$$

$$\frac{\partial b}{\partial \nu} = -\frac{X}{(1+\nu)^2}\frac{\partial p}{\partial z} , \tag{56}$$

$$\frac{\partial b}{\partial X} = \frac{q}{1+\nu} . \tag{57}$$

Thus p is a function of z only and from (55) and (56), we deduce

$$\frac{d^2 p}{dz^2} = -\frac{\mu^2 p}{\bar{r}^3} . \tag{58}$$

The general solution of (58) is

$$p = h_0 \vec{c} \cdot \vec{\bar{r}} , \tag{59}$$

where $\vec{c}$ is an arbitrary vector lying in the orbital plane of P_1. From (59), we also deduce

$$\frac{dp}{dz} = h_0 \vec{c} \cdot \vec{\bar{v}} . \tag{60}$$

From (55), (56) and (60), we have

$$b = h_0 \left(\frac{X}{1+\nu} - \frac{1}{X}\right) \vec{c} \cdot \vec{\bar{v}} + h_0 f(X) , \tag{61}$$

where $f(X)$ is an arbitrary function.

From (57) we have

$$q = h_0 \left(1 + \frac{1+\nu}{X^2}\right) \vec{c} \cdot \vec{\bar{v}} + h_0 (1+\nu) f'(X) . \tag{62}$$

Substituting (61) and (62) into (46) and (54) we obtain

$$\vec{N} = h_0 \vec{c} \cdot \vec{\bar{r}}\vec{\bar{v}} \times \vec{R} + h_0 \left[\left(1 + \frac{1+\nu}{X^2}\right) \vec{c} \cdot \vec{\bar{v}} + (1+\nu) f'\right] \vec{R} \times \vec{\bar{r}} , \tag{63}$$

and the general form of the W-function is

$$W = \dot{\nu} \vec{\bar{r}} \cdot \vec{c} + \left(\frac{X}{1+\nu} - \frac{1}{X}\right) \vec{\bar{v}} \cdot \vec{c} + f . \tag{64}$$

Taking (31) into account, we obtain

$$\vec{N} = \vec{c} \cdot \left(I + \frac{h_0 \vec{\bar{v}}}{X^2} \vec{R} \times \vec{\bar{r}}\right) + h_0 f' \vec{R} \times \vec{\bar{r}} . \tag{65}$$

From (45) and (50), we deduce

$$\frac{dW}{dt} = \vec{N} \cdot \vec{F} .$$

Substituting (64) and (65) into the last equation, and because $\vec{c}$ and f are arbitrary, we obtain

$$\frac{dX}{dt} = h_0 \vec{R} \cdot \vec{r} \times \vec{F} , \tag{66}$$

$$\frac{d}{dt} \left[\dot{\nu} \vec{\bar{r}} + \left(\frac{X}{1+\nu} - \frac{1}{X} \right) \vec{\bar{v}} \right] = \vec{F} + \frac{h_0}{X^2} \vec{\bar{v}} \vec{R} \cdot \vec{r} \times \vec{F} . \tag{67}$$

The equation (66) is classical. The equation (67) is new. It is convenient to bypass the use of W and introduce the vector $\vec{S}$ defined by the differential equation

$$\frac{d\vec{S}}{dt} = h_0 \vec{R} \times \left(\vec{F} + \frac{h_0}{X^2} \vec{\bar{v}} \vec{R} \cdot \vec{r} \times \vec{F} \right) . \tag{68}$$

Of course, this formula can be connected with the variation of astronomical constants.

We have from (67)

$$\dot{\nu} \vec{\bar{r}} + \left(\frac{X}{1+\nu} - \frac{1}{X} \right) \vec{\bar{v}} = \vec{S} \times \vec{R} h_0^{-1} . \tag{69}$$

Taking (42) into consideration we deduce from the last equation, after some easy vectorial transformations,

$$\nu' = + \vec{\bar{v}} \cdot \vec{S} , \tag{70}$$

$$\frac{X}{1+\nu} - \frac{1}{X} = - \vec{\bar{r}} \cdot \vec{S} . \tag{71}$$

We can obtain ν from (70) by integration. Then a constant of integration will appear which depends upon the constant in X. Perhaps it is preferable to determine ν from (71) which does not involve any additional integration. We can re-write (71) in the form

$$\nu = (1+\nu) \left[\left(1 - \frac{1}{X^2} \right) + \frac{1}{X} \vec{\bar{r}} \cdot \vec{S} \right] \tag{72}$$

which is convenient for the application of the process of iteration.

At this step we introduce Hansen's $\overline{W}$-function in order to determine $n_0 \delta z$. We have

$$\overline{W} = -1 - X + \frac{2X}{1+\nu} . \tag{73}$$

The quantity X differs from unity by the order of perturbations and we can set

$$X = 1 + \delta X . \tag{74}$$

Using (71) we obtain

$$\overline{W} = -\left(1 + \frac{2}{X}\right) \delta X - 2\vec{\overline{r}} \cdot \vec{S} , \tag{75}$$

and the perturbations in the mean anomaly and z can be obtained by using Hill's (1881) formulas:

$$\frac{dn_0 \delta z}{dt} = n_0 \frac{\overline{W} + \nu^2}{1 - \nu^2} , \tag{76}$$

$$\frac{dz}{dt} = \frac{1 + \overline{W}}{1 - \nu^2} . \tag{77}$$

PERTURBATIONS IN THE THIRD COORDINATE

Taking into account the identity

$$\frac{d^2 u}{dt^2} \equiv \left[\frac{d^2 u}{dz^2} - \frac{du}{dt}\frac{dt}{dz}\frac{d}{dt}\left(\frac{dt}{dz}\right)\right]\left(\frac{dz}{dt}\right)^2 ,$$

we can rewrite (28) in the form:

$$\frac{d^2 u}{dz^2} = -\frac{\mu^2 u}{\overline{r}^3} + C \frac{dt}{dz} , \tag{78}$$

where we set

$$C = \mu^2 \left[\frac{1}{\overline{r}^3}\frac{dz}{dt} - \frac{1}{\left(r^2 + u^2\right)^{3/2}}\frac{dt}{dz}\right] u + Z \frac{dt}{dz} + \frac{du}{dt}\frac{d}{dt}\left(\frac{dt}{dz}\right) . \tag{79}$$

Expanding the first term in powers of u and using (71), (43) and (77), we obtain

$$C = \frac{1 - \nu^2}{1 + \overline{W}} Z - \mu^2 \frac{1}{1 + \nu} \frac{u}{\overline{r}^3} \vec{\overline{r}} \cdot \vec{S} + \frac{du}{dt} \frac{d}{dt} \left(\frac{dt}{dz}\right) + \frac{3}{2} \mu^2 \frac{u^3}{\overline{r}^5} \quad . \tag{80}$$

From (43) we have

$$\frac{d}{dt} \left(\frac{dt}{dz}\right) = \frac{2(1+\nu)}{X} \frac{d\nu}{dt} - \frac{(1+\nu)^2}{X^2} \frac{dX}{dt} \quad . \tag{81}$$

From (73) we obtain with the needed accuracy

$$\frac{1 + \nu}{X} = 1 - \frac{1}{2} (\overline{W} + \delta X) \ ;$$

and substituting this value into (81) we have

$$\frac{d}{dt} \left(\frac{dt}{dz}\right) = (2 - \overline{W} - \delta X) \frac{d\nu}{dt} - (1 - \overline{W} - \delta X) \frac{dX}{dt} \quad .$$

Thus (80) becomes

$$C = \frac{1 - \nu^2}{1 + \overline{W}} Z - \mu^2 (1 - \nu) \frac{u}{\overline{r}^3} \vec{\overline{r}} \cdot \vec{S} + \left[(2 - \overline{W} - \delta X) \frac{d\nu}{dt} - (1 - \overline{W} - \delta X) \frac{dX}{dt}\right] \frac{du}{dt} + \frac{3}{2} \mu^2 \frac{u^3}{\overline{r}^5} \quad . \tag{82}$$

For the majority of planets the perturbations of the second order in u will suffice, providing that the orbital plane of P′ is chosen in a proper way. Then we can put

$$C = (1 - \overline{W}) Z + \left(2 \frac{d\nu}{dt} - \frac{dX}{dt}\right) \frac{du}{dt} - \frac{\mu^2 u}{\overline{r}^3} \vec{\overline{r}} \cdot \vec{S} \ . \tag{83}$$

The integration of (78) gives

$$u = \int h_0 C \overline{r} \overline{\rho} \sin (\overline{\phi} - \overline{f}) \, dt \ . \tag{84}$$

At this point it was convenient to use the Hansen and Hill device: the auxiliary mean anomaly (M) in the expressions of

$$\frac{\overline{\rho}}{a_0} \cos \overline{\phi} \quad \text{and} \quad \frac{\overline{\rho}}{a_0} \sin \overline{\phi}$$

is considered as a temporary constant. After the integration is completed (M) must be replaced by M. In computing the terms of the second order in (83) the elliptic values of the coordinates can be used.

THE PROCESS OF INTEGRATION

The integration of the differential equations of the problem is accomplished by the method of successive approximation using the expansion of the integrating operator into series. The operator

$$D = \frac{d}{dt}$$

can be decomposed into a sum of two operators

$$D = T_0 + T , \tag{85}$$

where

$$T_0 = n_0 \frac{\partial}{\partial M} + n_0' \frac{\partial}{\partial M'} \tag{85'}$$

and

$$\begin{aligned} T &= \frac{d(n_0\,\delta z)}{dt} \frac{\partial}{\partial M} + \frac{d(n_0'\,\delta z')}{dt} \frac{\partial}{\partial M'} \\ &= n_0 \frac{\overline{W} + \nu^2}{1 - \nu^2} \frac{\partial}{\partial M} + n_0' \frac{\overline{W}' + \nu'^2}{1 - \nu'^2} \frac{\partial}{\partial M'} . \end{aligned} \tag{85''}$$

If only the perturbative effects of the first and second orders are needed then we set $\nu^2 = \nu'^2 = 0$ in (85') and the operator T takes a simpler form. Putting (85) in the form

$$D = T_0 \left(I_0 + T_0^{-1} T\right) ,$$

where I_0 is the identity operator, we obtain the expansion of the integrating operator.

$$\begin{aligned} Q = D^{-1} &= \left(I_0 + T_0^{-1} T\right)^{-1} T_0^{-1} \\ &= T_0^{-1} - T_0^{-1} T T_0^{-1} + T_0^{-1} T T_0^{-1} T T_0^{-1} - \cdots . \end{aligned} \tag{86}$$

The explicitly written terms in this expansion suffice for computing of the perturbations up to the third order. The central part of the integration procedure is the application of the integrating operator T_0^{-1}. The basic formulas to be used are either

$$T_0^{-1}\left(z^s z'^{s'}\right) = \frac{1}{s+1} z^{s+1} z'^{s'} - \frac{s'}{(s+1)(s+2)} z^{s+2} z'^{s'-1} + \frac{s'(s'-1)}{(s+1)(s+2)(s+3)} z^{s+3} z'^{s'-2} - \cdots$$

$$= \frac{1}{s'+1} z^s z'^{s'+1} - \frac{s}{(s'+1)(s'+2)} z^{s-1} z'^{s'+2} + \frac{s(s-1)}{(s'+1)(s'+2)(s'+3)} z^{s-2} z'^{s'+3} - \cdots$$

and

$$T_0^{-1}\left\{z^s z'^{s'} \exp(jM + j'M')\right\} = \frac{\exp(jM + j'M')}{jn_0 + j'n_0'} \sum_{k=0}^{s+s'} \frac{(-1)^k}{\left(jn_0 + j'n_0'\right)^k} \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right)^k \left(z^s z'^{s'}\right)$$

or their modifications. In practical application of these formulas, we usually do not go beyond $m = 2$.

After ν', u, $n_0 \, \delta z$ are obtained, the time is determined from the equation

$$n_0 t = n_0 z + c_0 - n_0 \delta z = M - n_0 \delta z .$$

If the t is given then the corresponding value of z is determined from the last equation by means of successive approximations. The disturbed position vector $\vec{s}$ is determined using the standard equation

$$\vec{s} = (1+\nu)\left[a_0 \vec{P}\left(\cos E - e_0\right) + a_0 \sqrt{1-e_0^2}\, \vec{Q} \sin E\right] + u\vec{R} .$$

CONCLUSION

The geometrical characteristics of Hansen's planetary theory favor the expansion of perturbations in terms of the disturbed mean anomalies of the auxiliary planets. This approach eliminates the need to expand in powers of $n_0 \, \delta z$, $n_0' \, \delta z'$, · · · and speeds up the convergence of series representing Hansen's coordinates. We economize especially on the expansions of odd negative powers of the mutual distances. Thus, we are not compelled to form the derivatives of the first and higher orders of the disturbing function relative to the pseudo-time. Hansen's theory is connected more intimately with perturbations in the rectangular coordinates than with perturbations of the elements. For this reason we suggest here the form of the theory which does not appeal to the method of variation of astronomical constants. We prefer instead to make direct use of the differential equations of motion in rectangular coordinates relative to an inertial frame of reference.

We did this because the classical use of the moving ideal system of coordinates evidently does not simplify the actual numerical procedure.

Hansen himself was compelled to introduce the perturbations of the "third coordinate." To obtain them he used a method which combines the direct kinematical considerations with the method of variation of constants. Furthermore, we do not make use of Hansen's W-function because of the way the operations with periodic series are performed on electronic machines. However, we consider employing $\overline{W}$ as the most essential part of Hansen's theory. The $\overline{W}$-function is intimately connected with the determination of the perturbations in the mean anomaly as well as with the formation of the integrating operators. For these reasons we have retained the $\overline{W}$-function in the present exposition.

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