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ABSTRACT

A simple and quantitatively accurate representation of the current distribution in a dipole antenna is derived. Numerical data are given and are found to be in good agreement with the experiment when $h \geqq 0.15\lambda$.

1. Introduction

The problem of the thin dipole antenna, whether center-driven or asymmetrically driven, for antenna lengths that are short, long, or infinitely long, has already been treated by many authors [1]-[13]. For example, King-Middleton's iteration procedure [4] gives accurate input admittances; the so-called three-term theory [12] provides simple expressions of current distributions on antennas less than a wavelength long; the Wiener-Hopf procedure [10] predicts successfully the behavior of longer dipole antennas; and if more accurate theoretical data are desired, solving the problem numerically on a high-speed computer is quite feasible now [11]-[13].

The purpose of this study is to give, for practical reasons, a theory of the dipole antenna which can give a simple yet accurate alegebraic expression for the current distribution everywhere on the antenna even when antennas are not short or are asymmetrically driven. Such an expression is found provided the generator is not less than 0.15 wavelengths from either end and the antenna is not too thick. Since the formula is simple and general, it is particularly useful in discussing the properties of antennas with multiple loadings or excitations or whenever a superposition of current distributions is needed, such as in the case of calculating the transient response of a dipole to a short pulse applied at its driving-point. These possibilities are being studied. Note that in the work of King and Sandler [14], trigonometric functions are used to represent currents on long antennas. However, their theory is specifically for resonant dipoles that are not excessively long. There are no such limitations in the present theory.

To pave the way for the simple theory, the current on an infinitely long cylindrical antenna is obtained. An approximate formula which involves a logarithmic term is found and is shown to be accurate all along the antenna. Since the current reflected at the end of a semi-infinite antenna behaves just like that on an infinitely long antenna [8], the current on a finite dipole may be expressed as the superposition of the outgoing and the reflected waves, both of which can be represented in terms of a universal function — the distribution function of the current on an infinitely long antenna. The numerical results for this simple theory are compared with the experiment. It is shown that the agreement is remarkably good. The good agreement justifies a physical description of the mechanism of a dipole antenna which is shown to be analogous to a transmission line. Thus a simple and quantitatively accurate picture of a dipole antenna can be drawn even when it is asymmetrically driven.

2. Current on an Infinitely Long Antenna

The vector potential $A(z)$ on an infinitely long tubular antenna driven by a delta-function source $e^{-i\omega t}\delta(z)$ located at $z = 0$ is

$$A(z) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dz' I_\infty(z')K(z - z') \qquad (1)$$

where

$$K(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{ikR}/R \ , \quad R = [z^2 + (2a \sin\theta/2)^2]^{\frac{1}{2}}$$

The geometry of the antenna is shown in Fig. 1a. In (1), $I_\infty(z)$ is the current distribution, a is the radius of the antenna, k is the free-space wave-number, and μ_o is the free-space permeability.

On the other hand, assuming that the dipole is perfectly conducting, the axial component of the electric field vanishes at the surface of the antenna:

$$E_z = \frac{i\omega}{k^2}\left(\frac{d^2}{dz^2} + k^2\right) A(z) = -\delta(z) \tag{2}$$

Direct application of Fourier transform theory to (1) and (2) gives the current on an infinitely long antenna.

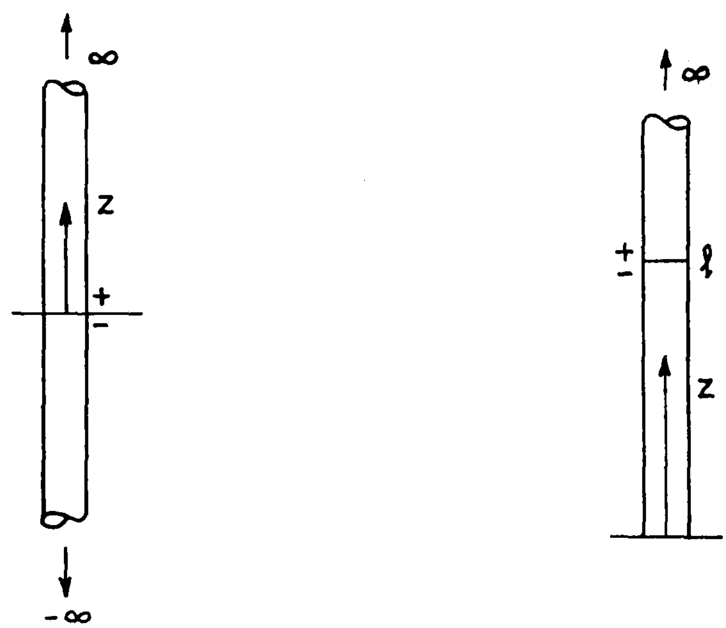
$$I_\infty(z) = \frac{2ik}{\zeta_o}\int_{C_o} \frac{e^{i\zeta z}}{(k^2 - \zeta^2)\overline{K}(\zeta)}\, d\zeta \tag{3}$$

where $\overline{K}(\zeta) = \pi i J_o H_o^{(1)}$ and J_o and $H_o^{(1)}$ are Bessel functions with arguments $(k^2 - \zeta^2)^{\frac{1}{2}}$. The branch cuts are shown in Fig. 2a together with the contour C_o. ζ_o is the intrinsic impedance of the free-space. If the contour C_o and the branch cuts are deformed as shown in Fig. 2b, the current distribution can be expressed in a form which is particularly convenient for carrying out a numerical integration.

$$I_\infty(z) = \frac{4}{\pi i\zeta_o}\int_0^\infty \frac{e^{-kz\eta}}{(1+\eta^2)(J_o^2 + Y_o^2)}\, d\eta + \frac{4}{\pi\zeta_o}\int_0^1 \frac{e^{ikz\eta}}{(1-\eta^2)(J_o^2 + Y_o^2)}\, d\eta$$

$$- \frac{4ika}{\zeta_o}\sum_{n=1}^{\infty} \frac{e^{-z\sqrt{t_n^2 - a^2k^2}/a}}{t_n J_1(t_n) Y_o(t_n)\sqrt{t_n^2 - a^2k^2}} \tag{4}$$

In (4), J_o, J_1 and Y_o are Bessel functions with arguments $ka\sqrt{1+\eta^2}$ in the first integral and $ka\sqrt{1-\eta^2}$ in the second integral, and t_n is the n-th root of $J_o(x)$.



(a) Infinite antenna

(b) Semi-infinite antenna

FIG. 1 INFINITE AND SEMI-INFINITE ANTENNAS

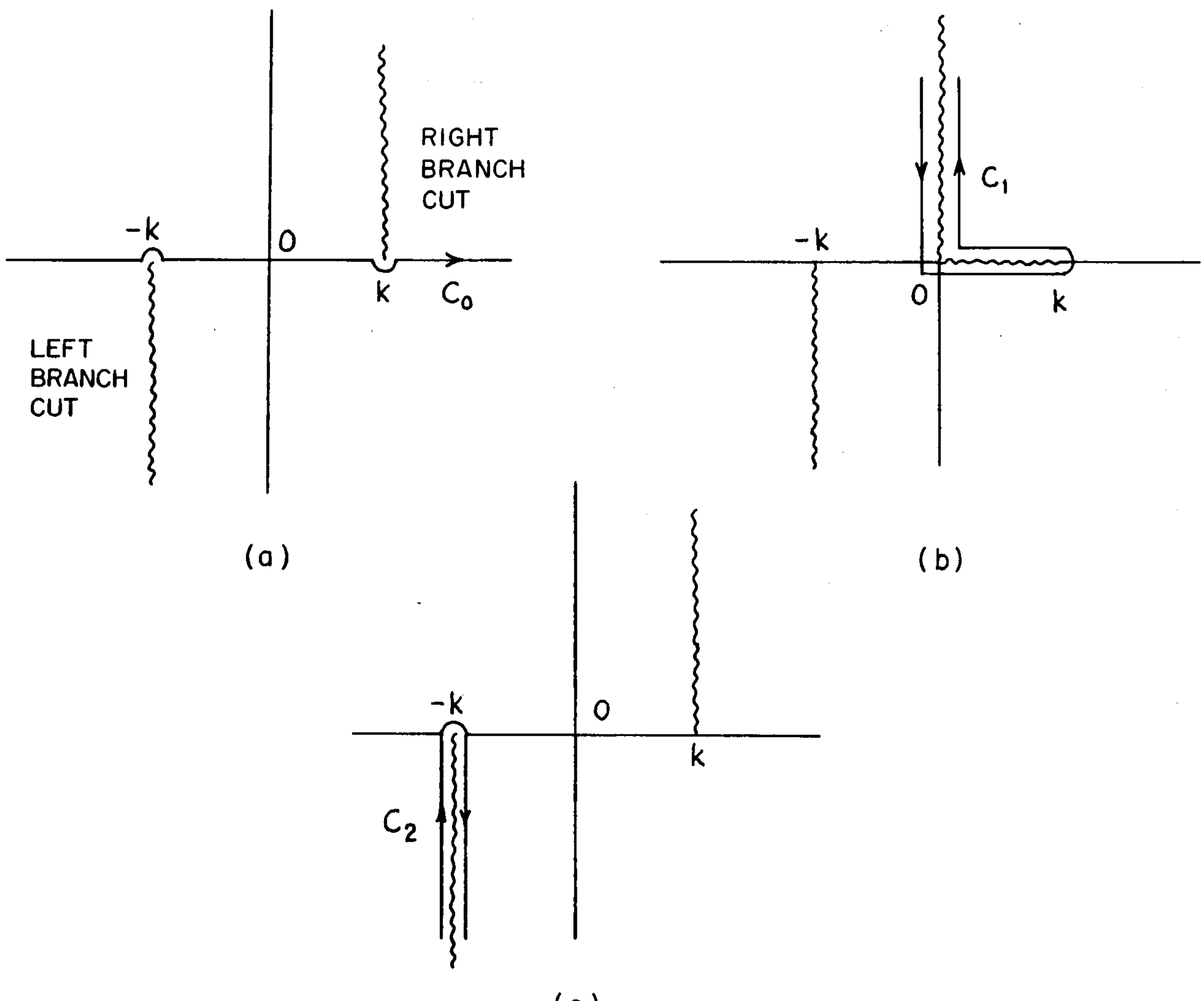


FIG. 2 THE BRANCH CUTS AND CONTOURS ON ζ-PLANE

The results of a numerical integration of (4) for different values of ka are shown in Fig. 3. Note that the e^{ikz} factor has been suppressed to reveal the traveling-wave nature of the current on the antenna. The phase of the current, when the suppressed e^{ikz}-factor is taken into account, is seen to be very close to a pure traveling-wave even when kz is small. In other words, it is seen that the effect of the generator is localized. This observation is very useful to explain the final conclusion of this study that the current on a finite dipole can be represented by outgoing and reflected waves even when the length of the dipole is not very long.

Since $I_\infty(z)$ plays a very important role in the present theory, it is necessary to find a simple representation for it. For this purpose, the contour C_o is deformed into C_2 as shown in Fig. 2c. Suppose that kz is not too small so that the contribution to the integral comes from near $\zeta = k$, $I_\infty(z)$ can be approximated by the following integral.

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_o} \int_0^\infty \frac{e^{-2kz\eta}}{\eta} \left[\frac{1}{2C_w - \log\eta + i\frac{3}{2}\pi} - \frac{1}{2C_w - \log\eta - i\frac{\pi}{2}} \right] d\eta \qquad (5)$$

where $C_w = \log(\frac{1}{ka}) - \gamma$, and $\gamma = 0.57721566$. Due to the exponential factor, the limit of the integration can be "cut off" to some number D' which is approximately $e^{-\gamma}$ (see Appendix).

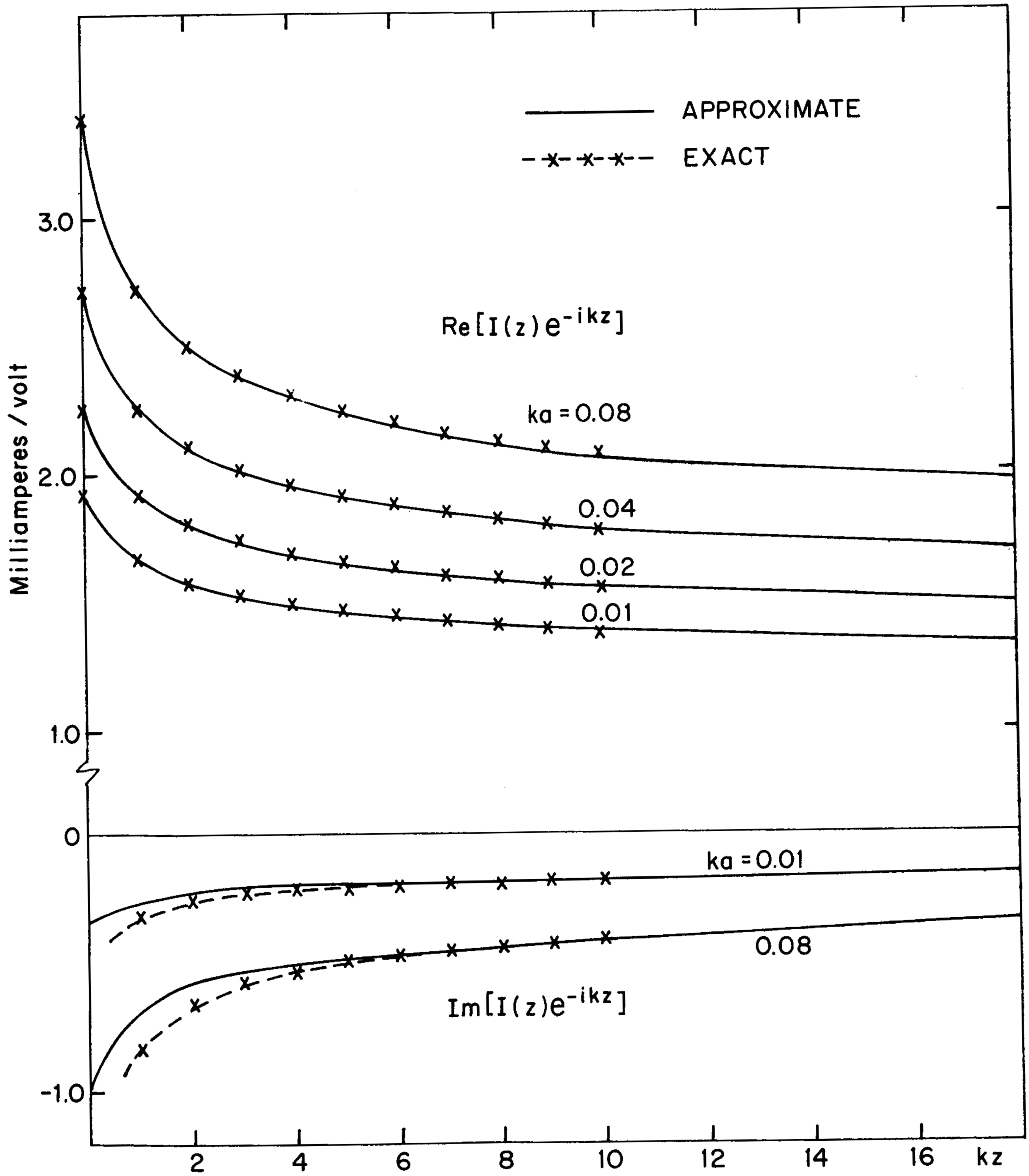


FIG. 3 CURRENT DISTRIBUTION ON INFINITELY LONG ANTENNA .

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_o} \int_0^{\frac{D'}{2kz}} \frac{d\eta}{\eta} \left[\frac{1}{2C_w - \log\eta + i\frac{3}{2}\pi} \right.$$

$$\left. - \frac{1}{2C_w - \log\eta - i\frac{\pi}{2}} \right] \qquad (6)$$

The integral of (6) can be carried out to get

$$I_\infty(z) \cong \frac{ie^{ikz}}{\zeta_o} \log\left[1 - \frac{2\pi i}{2C_w + \log 2kz + \gamma + i\frac{3}{2}\pi} \right] \qquad (7)$$

(for large kz; see (9) for general case)

The input conductance of the infinitely long antenna is known for small ka [10] to be approximately

$$G_\infty = \frac{1}{\zeta_o} \operatorname{Im}\left\{ \log\left(1 + \frac{i\pi}{C_w}\right) + \frac{\pi^2}{12}\left[\frac{1}{(C_w - \log 2)^2} \right.\right.$$

$$\left.\left. - \frac{1}{(C_w - \log 2 + \pi i)^2} \right]\right\} \qquad (8)$$

In order to obtain a formula for $I_\infty(z)$ which is good even for small kz, the term $\log 2kz$ in (7) is changed to $\log\left(kz + \sqrt{(kz)^2 + D''}\right)$. The constant D'' is determined by requiring that when $kz = 0$, the real part of $I_\infty(0)$ as given by (7) and G_∞ as given by (8) are not to differ by more than $\frac{1}{(C_w)^3}$ in the limit that ka approaches zero.

If this is done, D'' is found to be equal to $e^{-2\gamma}$. Thus for all kz, $I_\infty(z)$ can be expressed as

$$I_\infty(z) = \frac{ie^{ikz}}{\zeta_o} \log\left[1 - \frac{2\pi i}{2C_w + \log(kz + \sqrt{(kz)^2 + e^{-2\gamma}}) + \gamma + i\frac{3}{2}\pi}\right] \tag{9}$$

The above equation can be thought of as an interpolating formula between (7) and (8). To illustrate the accuracy, numerical data from (9) are plotted in Fig. 3 together with the exact values. It can be seen that (9) remains in excellent agreement with the exact value until kz is less than unity where a discrepancy in the imaginary part begins to appear; the real part is good to the driving-point.

3. Reflected Current at the End of a Semi-infinite Antenna

Fig. 1b shows the geometry of a semi-infinite antenna and the coordinate system. A generator with voltage V is located at $z = \ell$. To simplify the problem, eventually ℓ and V will be allowed to go to infinity so that the amplitude of the incident current is unity, that is,

$$I^{inc}(z) = e^{-ikz} \; . \tag{10}$$

This problem was studied by Hallen [8].

If the conductor of the semi-infinite antenna existed for $z < 0$ (see Fig. 1b), the current would be $I_\infty(z - \ell)$ which satisfies the following equation

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{+\infty} I_\infty(z' - \ell) K(z - z')dz' = \frac{4\pi ik}{\zeta_o} V\delta(z - \ell)$$

$$\text{for all } z \tag{11}$$

Since the conductor is not there, let the scattered current be $i(z)$, then

$$i(z) = -I_\infty(z - \ell) \qquad \text{for } z < 0 \tag{12}$$

and

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{+\infty} [I_\infty(z' - \ell) + i(z')]K(z - z')dz'$$

$$= \frac{4\pi i k V}{\zeta_o} \delta(z - \ell) \qquad \text{for } z > 0 \tag{13}$$

Comparing (13) with (11), the following equation is obtained

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{+\infty} i(z')K(z - z')dz' = 0 \qquad \text{for } z > 0 \tag{14}$$

In other words,

$$\int_{-\infty}^{+\infty} i(z')K(z - z')dz' = Ce^{ikz} \qquad \text{for } z > 0 \tag{15}$$

where C is a constant. Define

$$\int_{-\infty}^{+\infty} i(z')K(z - z')dz' = \begin{cases} Ce^{ikz} & z > 0 \\ G(z) & z < 0 \end{cases} \tag{16}$$

$$\overline{I}_-(\zeta) = \int_0^\infty i(z)e^{-i\zeta z}\, dz \tag{17}$$

$$\overline{I}_+(\zeta) = \int_{-\infty}^0 i(z)e^{-i\zeta z}\, dz \tag{18}$$

$$\overline{K}(\zeta) = \pi i J_o H_o^{(1)}[a(k^2 - \zeta^2)^{\frac{1}{2}}] \tag{19}$$

$$\overline{G}_+(\zeta) = \int_{-\infty}^0 G(z)e^{-i\zeta z}\, dz \tag{20}$$

Apply Fourier transformations on both sides of (16), to obtain

$$[\overline{I}_-(\zeta) + \overline{I}_+(\zeta)]\overline{K}(\zeta) = \frac{-C}{i(k-\zeta)} + \overline{G}_+(\zeta) \tag{21}$$

The kernel $\overline{K}(\zeta)$ is split into two fractions,

$$\overline{K}(\zeta) = \overline{L}_-(\zeta) \Big/ \overline{L}_+(\zeta) \tag{22}$$

where

$$\log \overline{L}_\pm(\zeta) = \frac{-1}{2\pi i} \int_{-\infty \mp \epsilon}^{+\infty \mp \epsilon} dz' \, \frac{\log \overline{K}(\zeta')}{\zeta' - \zeta} \tag{23}$$

so that $\overline{L}_+(\zeta)$ and $\overline{L}_-(\zeta)$ are analytic in the upper and lower half of the ζ-plane, respectively. Note that

$$\overline{L}_\pm(\zeta) \rightarrow |\zeta|^{\pm\frac{1}{2}} \qquad \text{as } |\zeta| \rightarrow \infty \tag{23a}$$

and $$\overline{L}_+(\zeta) = \frac{1}{\overline{L}_-(-\zeta)} \; .$$

Substitute (22) into (21) and re-arrange terms. The following equation is obtained

$$[\overline{I}_+(\zeta) + \overline{I}_-(\zeta)]\overline{L}_-(\zeta) = \left[\frac{iC}{k-\zeta} + \overline{G}_+(\zeta)\right]\overline{L}_+(\zeta).$$

At this point the simplification is introduced that $V \rightarrow \infty$ and $\ell \rightarrow \infty$ so that (10) holds. Thus

$$\overline{I}_-(\zeta)\overline{L}_-(\zeta)(k^2 - \zeta^2) + \frac{1}{i}(k-\zeta)\overline{L}_-(\zeta)$$

$$= iC(k+\zeta)\overline{L}_+(\zeta) + \overline{G}_+(\zeta)(k^2 - \zeta^2)\overline{L}_+(\zeta) \tag{24}$$

The left-hand side of (24) is analytic in the lower ζ-plane and the right-hand side is analytic in the upper ζ-plane. They both equal an entire function which is found to be a constant due to the behavior at infinity. Thus

$$\bar{I}_-(\zeta)\bar{L}_-(\zeta)(k^2-\zeta^2)+\frac{1}{i}(k-\zeta)\bar{L}_-(\zeta) = B.$$

Let $\zeta = -k$, B is found to be

$$B = -2ik\bar{L}_-(-k).$$

Therefore the Fourier transform of the reflected current is found:

$$\bar{I}_-(\zeta) = \frac{2k}{i}\bar{L}_-(-k)\frac{1}{(k^2-\zeta^2)\bar{L}_-(\zeta)} - \frac{1}{i}\frac{1}{k+\zeta} \tag{25}$$

For $z > 0$, after applying Fourier transform theory to (25),

$$i(z) = \frac{k}{\pi i}\bar{L}_-(-k)\int_{C_o}\frac{e^{i\zeta z}\,d\zeta}{(k^2-\zeta^2)\bar{L}_-(\zeta)} \qquad \text{for } z > 0 \tag{26}$$

If C_o is deformed as shown in Fig. 2c, and for kz not too small, the contribution to the integral of (26) comes mainly from near $\zeta = k$ so that

$$i(z) \cong \frac{k}{\pi i}[\bar{L}_+(+k)]^{-2}\int_{C_2}\frac{e^{i\zeta z}}{(k^2-\zeta^2)\bar{K}(\zeta)}\,d\zeta \tag{27}$$

Note that here relations (22) and (23a) have been used. Comparing (27) with (3) gives

$$i(z) = -R\,I_\infty(z) \qquad \text{for large kz} \tag{28}$$

where $R = \left|\frac{\zeta_o}{2\pi}\right| [L_+(k)]^{-2}$. The approximate value of $[\bar{L}_+(k)]^2$ can be shown to be

$$[\bar{L}_+(k)]^2 = \frac{1}{2}\frac{1}{C_w + i\frac{\pi}{2}} \tag{29}$$

What (28) means is that, when away from the end, the reflected current distribution behaves just like that of the current on an infinitely long antenna driven by a fictitious generator of voltage -R located at the point of reflection.

4. Current on a Center-driven Dipole Antenna

The foregoing analysis suggests that the current on a finite center-driven antenna may be expressed as the superposition of the outgoing wave, which is $I_\infty(z)$, and the reflected waves, which are proportional to $I_\infty(h+z)$ and $I_\infty(h-z)$ respectively. That is,

$$I(z) = I_\infty(z) + C_R[I_\infty(h+z) + I_\infty(h-z)] \qquad (30)$$

The simplest way to determine the constant C_R is to evaluate (30) at $z = 0$, because the left-hand side of (30) is the admittance of a center-driven antenna and is well-tabulated. The constant which is found this way is

$$C_{R1} = \frac{Y - Y_\infty}{2I_\infty(h)}\ , \qquad (31)$$

where Y is the admittance obtained by any other accurate theory. The second method to determine C_R is independent of other theories as follows: Assume the reflection coefficient $R(\ell)$ at the end of an antenna to be a slowly varying function of the parameter ℓ (the distance between the generator and the reflecting end) so that the limiting values of R as ℓ goes to infinity (as obtained in the previous section) can be used universally for all ℓ. Then from (30), the incident current at $z = -h$ is known to be $I_\infty(h) + C_R I_\infty(2h)$; the amplitude of the reflected wave is C_R. Thus

$$-R = \frac{C_R}{I_\infty(h) + C_R I_\infty(2h)} \qquad (32)$$

since $-R$ is the amplitude of the reflected wave due to unit incident wave. Solving (32) for C_R and denoting the value as C_{R2}, it is found that

$$C_{R2} = \frac{-RI_\infty(h)}{1 + RI_\infty(2h)} \tag{33}$$

Note that (31) can also be written in the same form as (33) with the "reflection coefficient" defined by

$$R(h) = \frac{-(Y - Y_\infty)}{(Y - Y_\infty)I_\infty(2h) + 2[I_\infty(h)]^2} \tag{31a}$$

Needless to say, R(h) is now a function of h. Fig. 4 shows numerical values of R(h) when the King-Middleton admittance [15] and the admittance calculated from the long antenna theory [10] are substituted into (31a). It is observed that if the admittance from the long antenna theory is used, R(h) is virtually a constant, while if the admittance from [15] is used, R(h) begins to deviate from the constant value only when h is decreased further than 0.15λ. Thus it makes little difference in the numerical value whether C_R is determined by (31) or (33). Hereafter, (33) is used.

The following equations are presented as a summary of the foregoing discussion:

$$I(z) = I_\infty(z) - \frac{I_\infty(h)}{\frac{1}{R} + I_\infty(2h)} [I_\infty(h+z) + I_\infty(h-z)] \tag{34a}$$

$$Y = Y_\infty - \frac{2[I_\infty(h)]^2}{\frac{1}{R} + I_\infty(2h)} \tag{34b}$$

where $\frac{1}{R} = \frac{\pi}{\zeta_o} \frac{1}{C_w + \pi\frac{i}{2}}$ and $I_\infty(z)$ is given by (9). (34a) and (34b) are similar to those obtained by Chen and Keller [16] except for a factor 2 in (34b). However, here the formula for $I_\infty(z)$ is good for all z and therefore (34a) and (34b) are valid everywhere on the antenna.

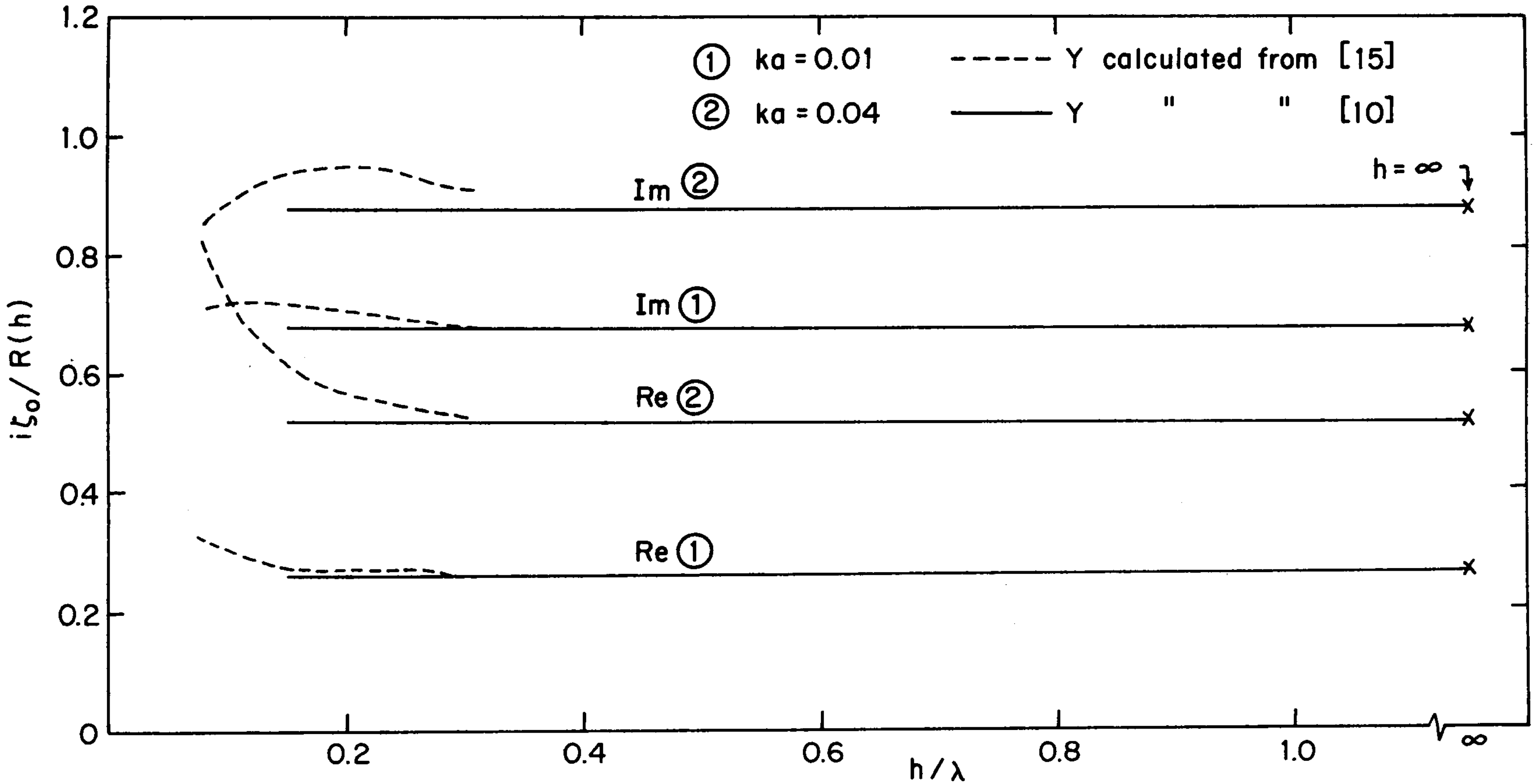


FIG. 4 REFLECTION COEFFICIENT R(h)

The input admittance and the current distribution calculated from (34a, b) are plotted in Figs. 5 and 6 together with experimental data which are obtained from works of Mack [17] and Chen [18]. The experimental data for current distribution obtained by Mack are normalized at the driving-point while those obtained by Chen are normalized at quarter-wavelength from the end of the antenna. Note that in Chen's set-up the antenna was supported by a polyfoam slab. The agreement between present theory and experiment is seen to be remarkably good. Some of the discrepancy may be due to the experimental difficulty of absolute measurements and in cases of long antennas (Fig. 6(f), (g)), as pointed out by Chen, the presence of the polyfoam is responsible for a small increase of the effective length of the antenna.

5. Current on an Asymmetrically Driven Antenna

Let the coordinate be chosen so that the ends of the antenna are at $z = -h_1$ and $z = h_2$, and the driving-point is at $z = 0$. The current is represented by out-going and reflected waves just as in Section 4, except that in the present case the amplitudes of the reflected currents are not equal.

$$I(z) = I_\infty(z) + C_d I_\infty(h_1 + z) + C_u I_\infty(h_2 - z) \tag{35}$$

It is assumed that the reflection coefficient is very insensitive to the distance between the generator and the reflecting end. This assumption has been justified in the sense discussed in the previous section. The amplitude of the incident wave at $z = -h_1$ is $I_\infty(h_1) + C_u I_\infty(h_1 + h_2)$ while that of the reflected wave is C_d, thus

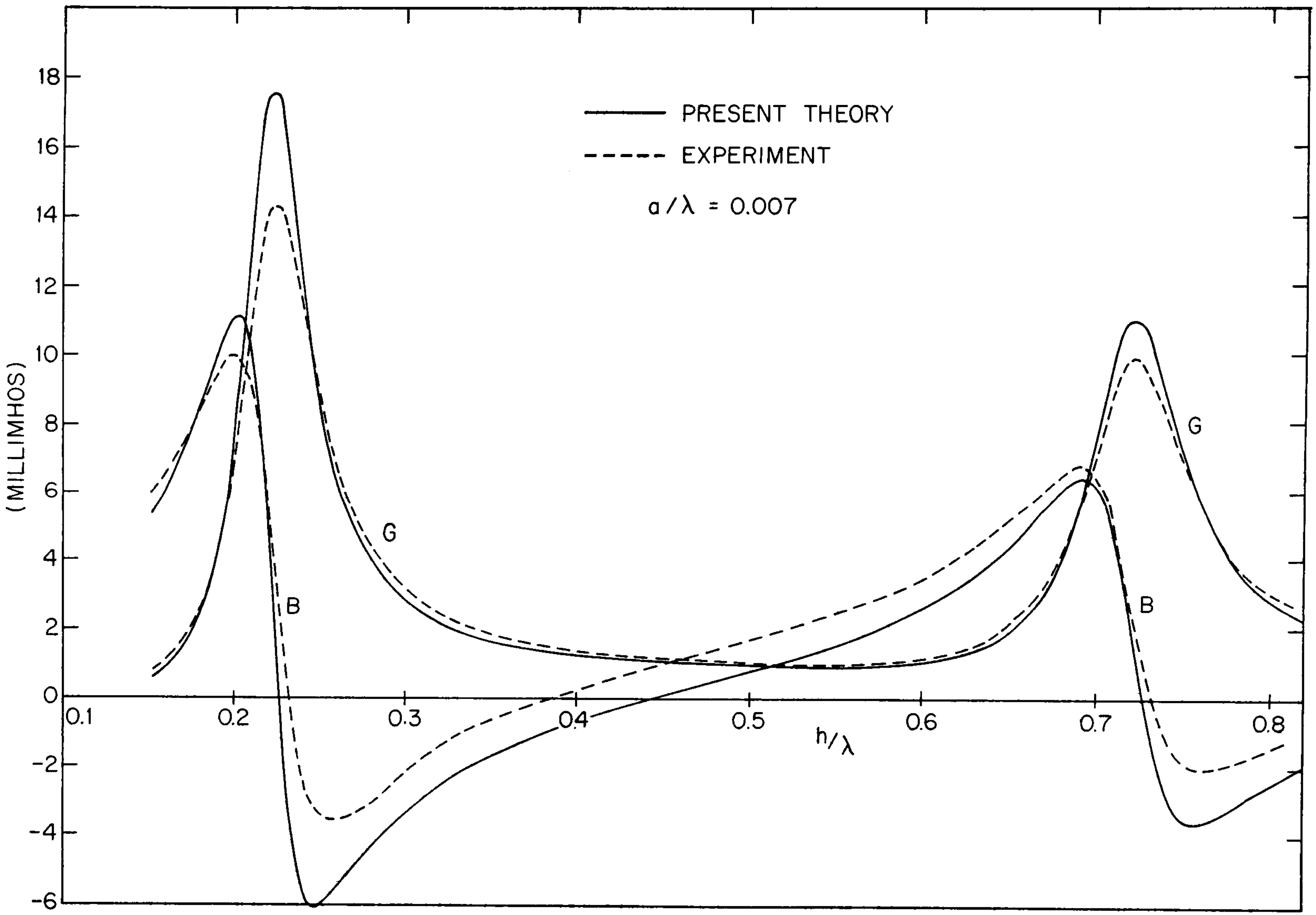


FIG. 5 EXPERIMENTAL AND THEORETICAL ADMITTANCES

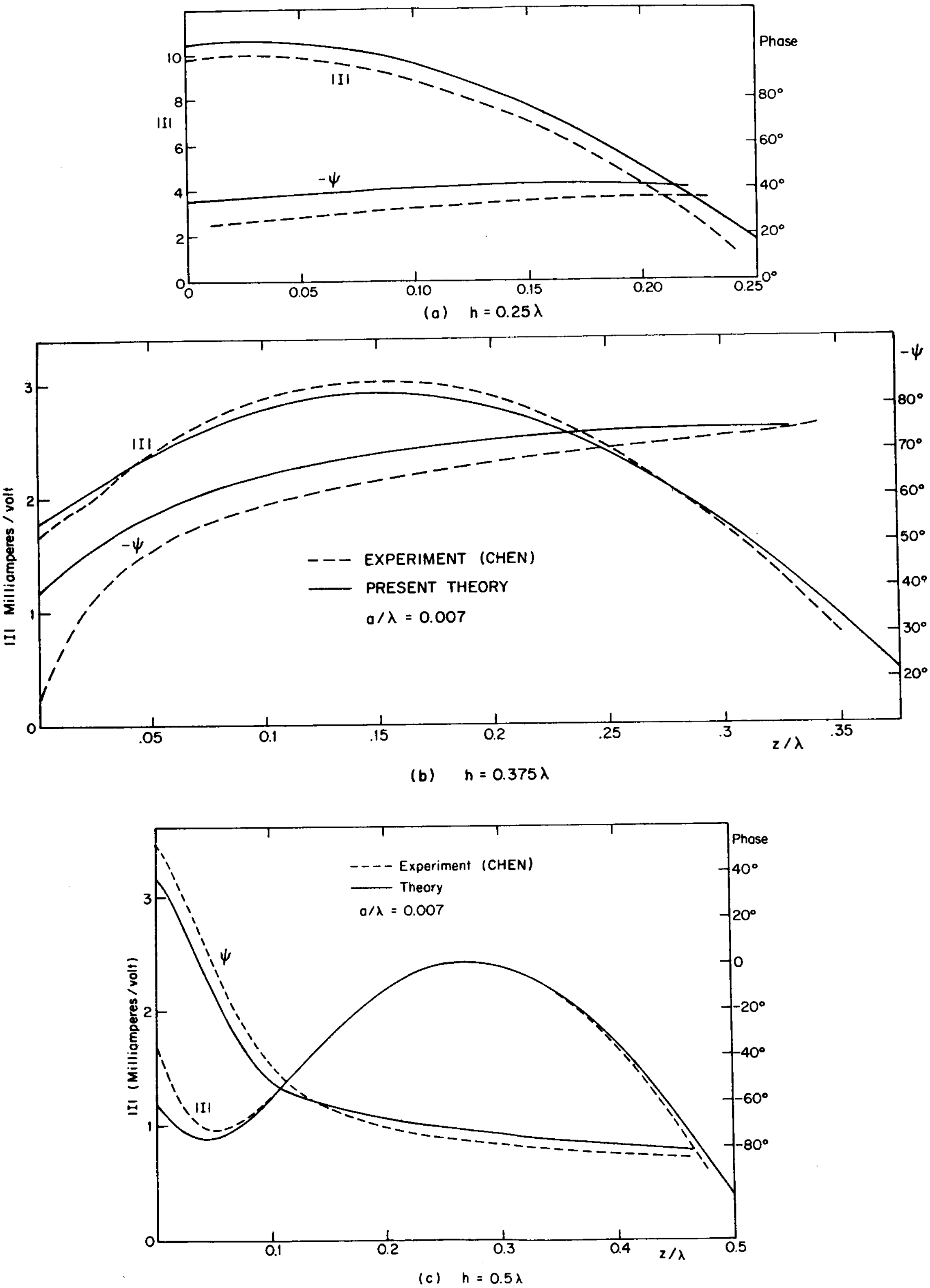
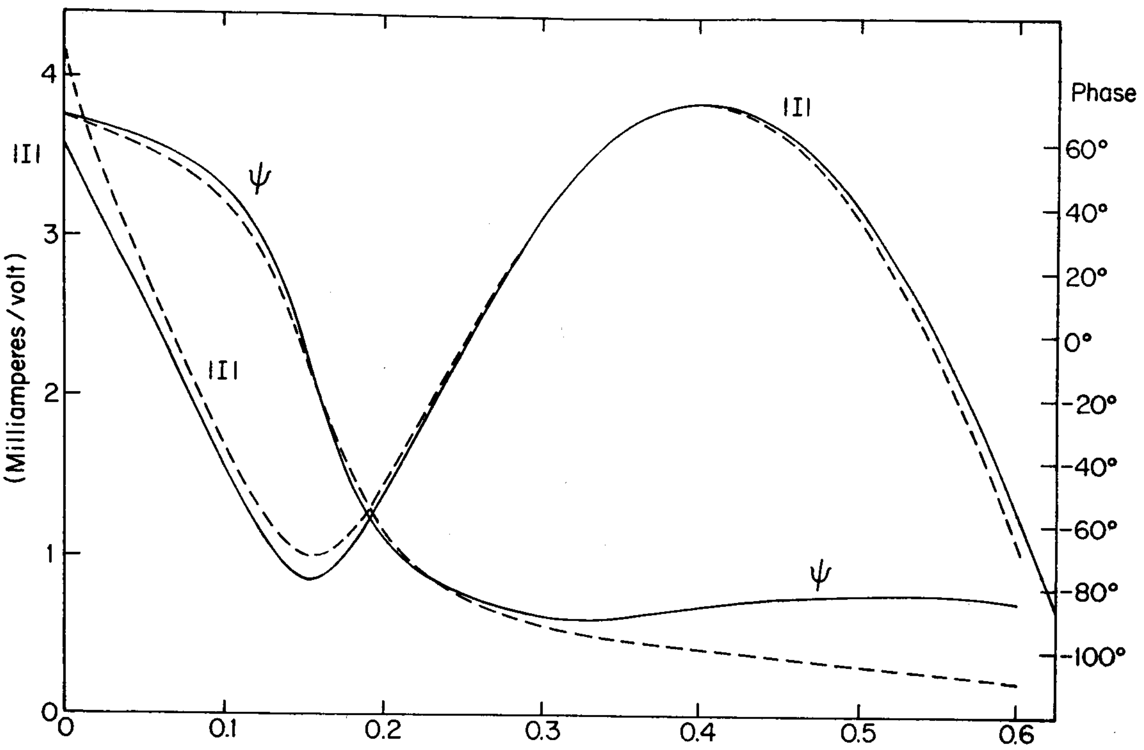
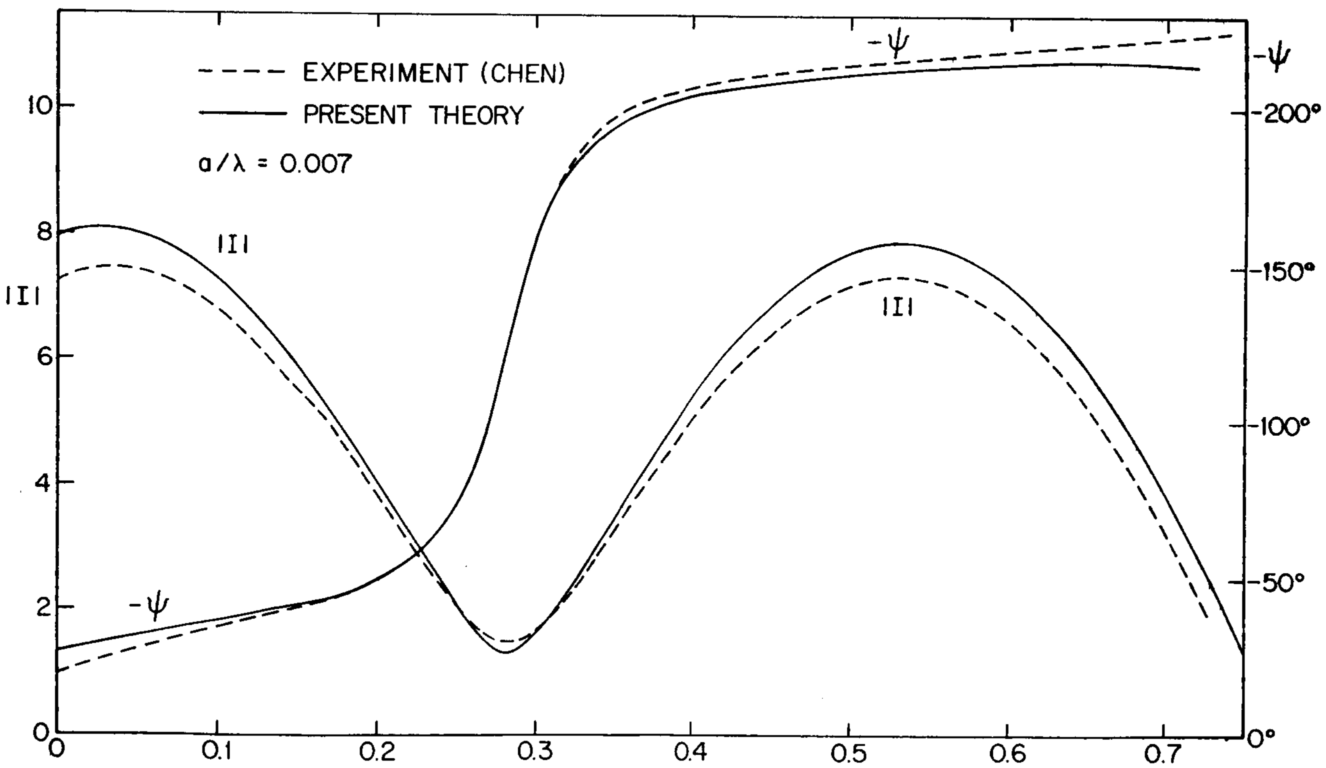


FIG. 6 CURRENT DISTRIBUTIONS



(d) h = 0.623 λ



(e) h = 0.75 λ

FIG. 6 CURRENT DISTRIBUTIONS

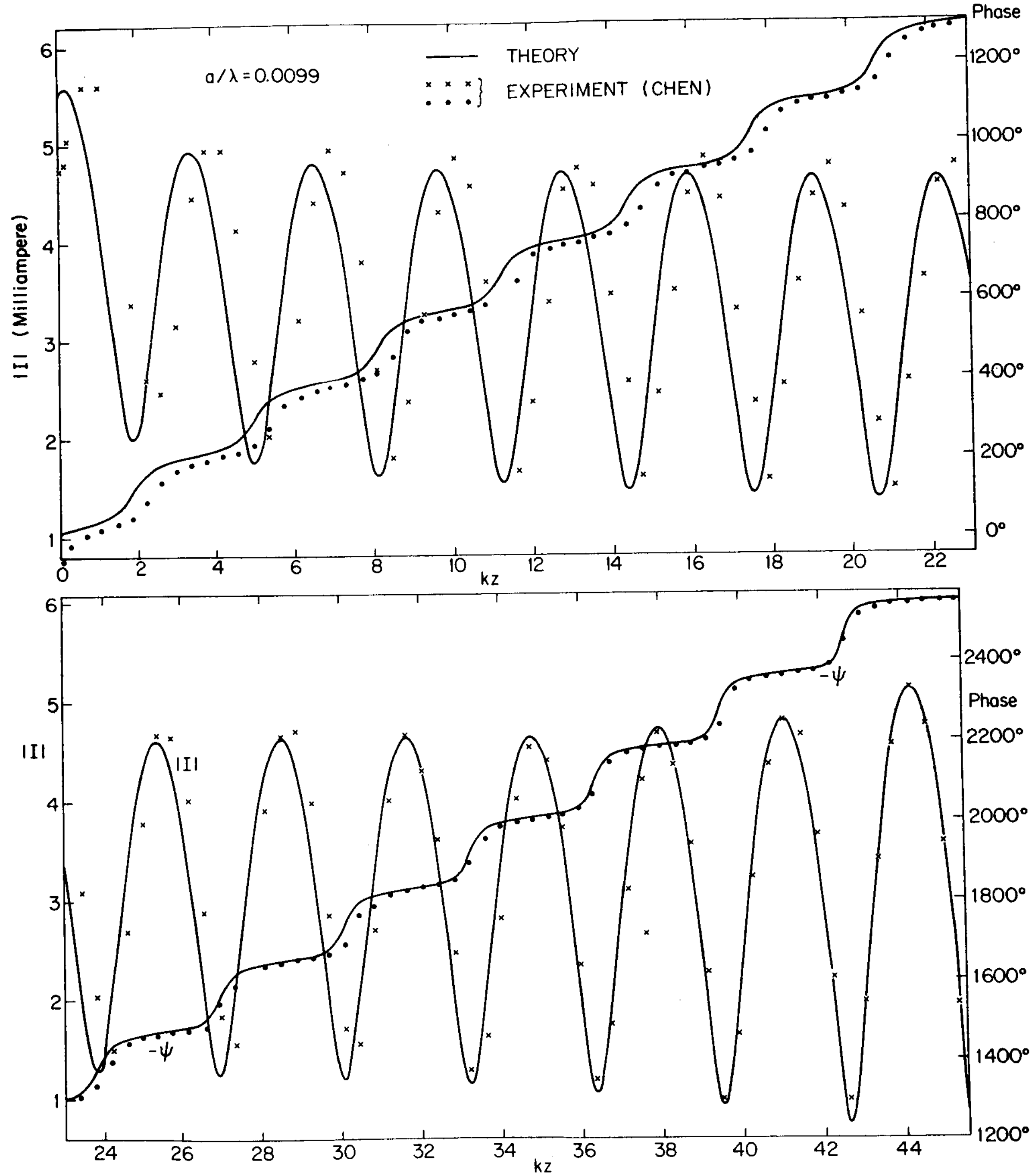


FIG. 6(f) CURRENT DISTRIBUTIONS h = 7.25λ

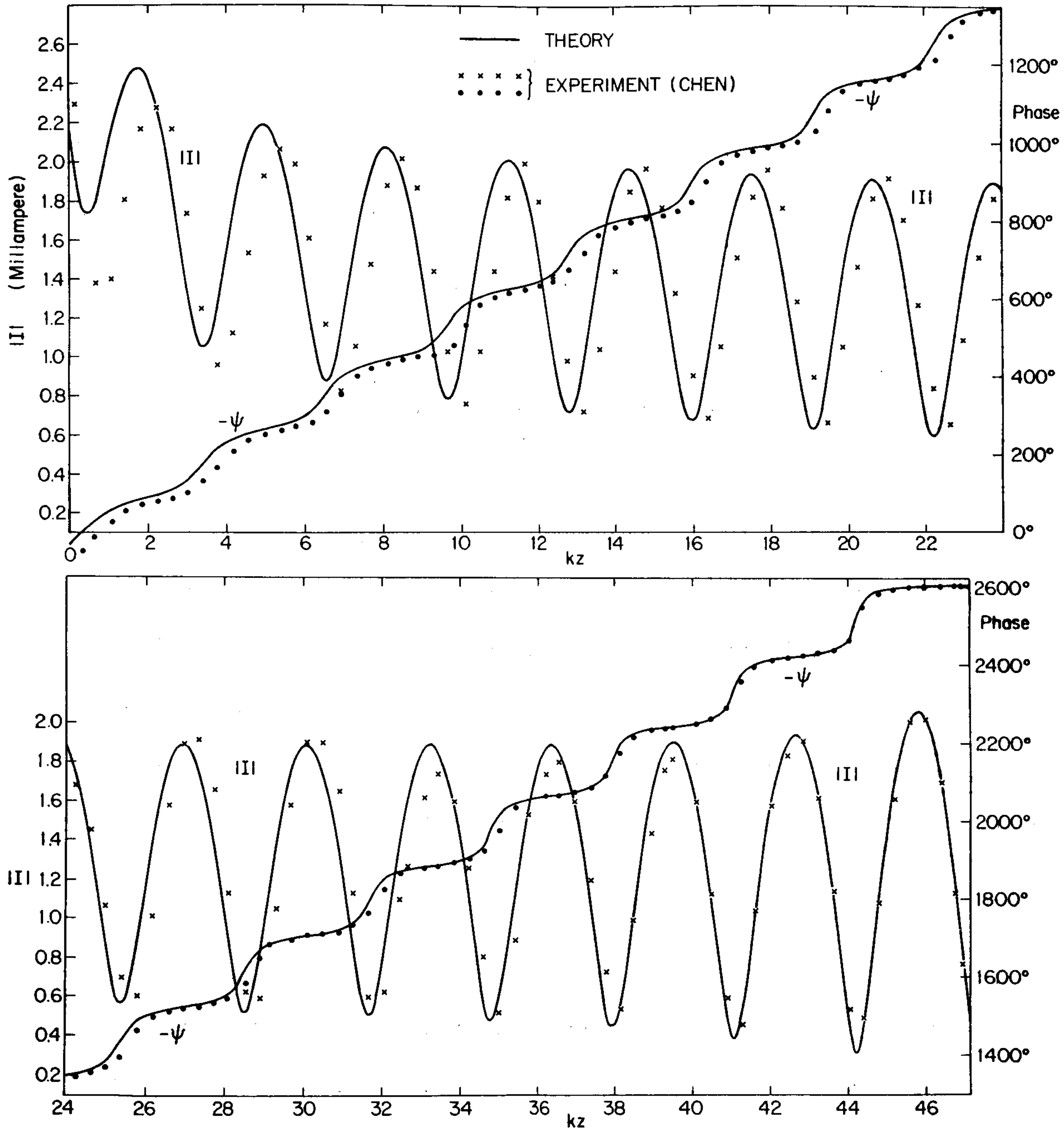


FIG. 6 (g) CURRENT DISTRIBUTIONS h = 7.5λ

$$-R = \frac{C_d}{I_\infty(h_1) + C_u I_\infty(h_1 + h_2)} \tag{36}$$

Similarly at $z = h_2$,

$$-R = \frac{C_u}{I_\infty(h_2) + C_d I_\infty(h_1 + h_2)} \tag{37}$$

Solving (36) and (37):

$$C_d = -R\left\{\frac{I_\infty(h_1) - RI_\infty(h_2)I_\infty(h_1 + h_2)}{1 - R^2[I_\infty(h_1 + h_2)]^2}\right\} \tag{38}$$

$$C_u = -R\left\{\frac{I_\infty(h_2) - RI_\infty(h_1)I_\infty(h_1 + h_2)}{1 - R^2[I_\infty(h_1 + h_2)]^2}\right\} \tag{39}$$

where R is given by (34) and $I_\infty(z)$ is given by (9).

In Fig. 5 it is observed that the admittance of a symmetrically-driven antenna calculated by present theory agrees with experiment when h is greater than 0.15λ. If this length is accepted as a limit to the present theory, then in the asymmetrically-driven case, (35) should be good when the generator is no less than 0.15λ away from either end of the antenna.

6. Conclusions

The foregoing discussion seems to support the following picture of a dipole antenna. An outgoing traveling-wave of current is generated along the dipole antenna when it is driven by a time-harmonic source. It travels along the two arms of the dipole with a speed almost equal to the speed of light and decays slowly in a manner described by (9) as a result of radiation. It is reflected at the ends of the dipole with the reflection coefficient given by (34). After it is reflected, the current wave travels in the opposite direction with the same speed and decays in the same manner as before. The current distribution on the antenna is just the result of the superposition of the outgoing current wave and all the reflected waves. This description of the current along a dipole antenna is analogous to that for a lossless transmittion-line, for which the traveling-wave of current does not decay. This qualitative picture is, of course, not new [5]; the main point of the present paper is that quantitatively accurate results are obtained when a good approximation of the current in an infinite antenna is used. Numerical results are quite accurate when $h \geqq 0.15\lambda$.

Appendix

In this appendix an approximation to the following integral is demonstrated. This approximating method is used in deriving (27) from (26)

$$I = \int_0^\infty \frac{e^{-\eta z}}{\eta} f(\eta)\, d\eta \tag{A1}$$

where $f(\eta)$ is slowly varying and is zero at $\eta = 0$ so that the integral is well-defined. I is to be approximated by the following integral

$$I \cong \int_0^{D'/z} \frac{d\eta}{\eta} f(\eta) \tag{A2}$$

The task is to find D'.

This problem is approximately equivalent to determining the constant D in the following equation

$$\int_{\frac{D}{z}}^{\infty} \frac{d\eta}{\eta} e^{-\eta z} = \int_{0}^{\frac{D}{z}} \frac{d\eta}{\eta}(1 - e^{-\eta z}) \tag{A3}$$

(A3) is equivalent to

$$\int_{D}^{\infty} \frac{d\eta}{\eta} e^{-\eta} = \int_{0}^{D} \frac{d\eta}{\eta} (1 - e^{-\eta}) \tag{A4}$$

Since

$$-\gamma = \int_{0}^{\infty} (e^{-t} - \frac{1}{1+t}) \frac{dt}{t} \tag{A5}$$

(see p. 17, Higher Transcendental Functions, Vol. I, by Erdelyi, McGraw-Hill Book Co.), substituting (A4) in (A5) yields

$$-\gamma = \log D$$

or

$$D = e^{-\gamma}.$$

This is an exact solution. If $f(\eta)$ is slowly varying, it should be a good approximation to let $D' = D = e^{-\gamma}$.

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