NASA TECHNICAL NOTE

NASA TN D-4244

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NASA TN D-4244

PRODUCTS OF SOME GENERALIZED FUNCTIONS

by Robert C. Costen

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Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • NOVEMBER 1967

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SUMMARY

The purpose of this report is to define a new class of generalized functions which includes products of all members of the class. Application is made to the families of generalized functions derived from $\tanh nx$ (hyperbolic tangent family) and from $ne^{-n^2x^2}$ (Gaussian family), where n is a sequence index and x the independent variable. Elementary uses of the product analysis in physics are also presented.

The hyperbolic tangent family of generalized functions has the following features: The Dirac delta functions $\delta_n(x)$ and its derivatives are polynomials in the Heaviside unit step function $H_n(x)$ multiplied by powers of the sequence index n. Conversely, powers of $H_n(x)$ may be expressed as linear sums of $H_n(x)$, $\delta_n(x)$, $\delta_n'(x)$, . . . with coefficients containing powers of n. Products of these generalized functions are similarly expressed either as polynomials in $H_n(x)$ or as linear sums of $H_n(x)$, $\delta_n(x)$, $\delta_n'(x)$, ... The Gaussian family of generalized functions has analogous features, but products are not the same in the two families.

INTRODUCTION

The mathematics of generalized functions, although highly developed, does not usually treat null generalized functions nor products of generalized functions. (An exception is the work of Schmieden and Laugwitz (ref. 1) which enlarges the concept of real numbers to include infinitesimally small and infinitely large numbers.) The purpose of this report is to develop a formalism which treats products of certain generalized functions and to show that such products (and also null generalized functions) are potentially useful in mathematical physics. The latter objective shall be deferred to the examples at the end.

It is possible to include products in the theory of generalized functions by extending the definition of regular sequences presented by Lighthill (ref. 2) while simultaneously imposing certain restrictions. This is done in a subsequent section, and the extension is a natural one for some useful generalized function sequences.

A detailed product analysis is carried out for the family of generalized functions derived from the hyperbolic tangent representation for the Heaviside step function

$H_n(x) = \frac{1 + \tanh nx}{2}$, where n is the sequence index. An alternative hyperbolic tangent formulation in terms of the signum function $\mathrm{sgn}_n x = \tanh nx$ is presented in appendix A. In appendix B a brief product analysis of generalized functions derived from the Gaussian representation for the Dirac delta function $\delta_n(x) = \frac{ne^{-n^2x^2}}{\sqrt{\pi}}$ is presented, and features of the Gaussian and hyperbolic tangent families are compared. Other useful sequences for the delta function, which are not treated in this report, include $\delta_n = \frac{\sin nx}{\pi x}$ (ref. 3) and $\delta_n(x) = \frac{n}{\pi\left(n^2x^2 + 1\right)}$ (ref. 4).

SYMBOLS AND NOTATIONS

A,B,C,a,b	constants
$a_{\alpha\beta}, b_{\mu\nu}$	coefficients appearing in equations (8) and (10) and given by tables I and II
C(n)	integration constant
c	speed of light in vacuum, meters per second
D	electric excitation, coulombs per meter
E	electric field strength, volts per meter
$\mathcal{E}$	electrostatic energy, joules per meter^2
F(x)	a good function, one which is everywhere differentiable any number of times and such that it and all its derivatives are $O\left(\lvert x\rvert^{-N}\right)$ as $\lvert x\rvert \to \infty$ for all N
$H_n(x)$	hyperbolic tangent sequence for the Heaviside unit step function with sequence index n, $\frac{1}{2}(1 + \tanh nx)$
$\mathcal{H}_\alpha(x)$	Hermite polynomials of degree α and argument x
M,N,k,l,p,r,s, $\alpha,\beta,\gamma,\zeta,\eta,\mu,\nu,\xi$	integers

n sequence index for generalized functions

$O(g(n))$ order $g(n)$ as $n \to \infty$; $y = O(g(n))$ implies that there exists an integer M for which $|y| < M|g(n)|$ as $n \to \infty$

$P_m(u)$ Legendre polynomial of argument u and index m

$r_{\alpha\beta}, s_{\mu\nu}$ coefficients appearing in equations (28) and (31) and given by tables III and IV

t time, seconds

W electromagnetic energy density, joules per meter^3

$\Delta_n^{(\alpha)}(+), \Delta_n^{(\alpha)}(-)$ definite generalized function pairs of related symmetry such that $\Delta_n^{(\alpha)}(+) - \Delta_n^{(\alpha)}(-) = \delta_n^{(\alpha)}$ for indefinite generalized functions $\delta_n^{(\alpha)}$ (eqs. (5) and fig. 3)

$\delta_n(x)$ sequence for the Dirac delta function; hyperbolic tangent sequence (eqs. (3a) and (26)) is employed throughout this presentation, except in appendix B where the Gaussian sequence (eq. (36)) is used

$\delta_{\alpha\beta}$ Kronecker delta, $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ when $\alpha \neq \beta$

$\epsilon(x)$ dielectric function (eq. (22)), farads per meter

ϵ_o dielectric constant of vacuum, farads per meter

ρ electric charge density, coulombs per meter^3

τ parameter of $O(n^{-1})$ as $n \to \infty$, meters

$\varphi_o(x), \varphi_1(x)$ linear electric scaler potential functions appearing in sketch 1, volts

$\mathrm{sgn}_n x$ hyperbolic tangent sequence for the signum function with sequence index n, $\tanh nx$

$$\left.\begin{array}{l} f'(x) = df/dx \\ f''(x) = d^2f/dx^2 \\ f^{(\alpha)}(x) = d^\alpha f/dx^\alpha \end{array}\right\} \text{primes and parenthetical superscripts denote derivatives}$$

$|\ |$ absolute value

Vector notation is not used, since electromagnetic fields have only one component in the examples presented.

EXTENDED CLASS OF GENERALIZED FUNCTIONS

In this section a useful class of generalized functions is defined which includes products of all members of the class. The presentation is similar to that given by Lighthill (ref. 2) but is not nearly as complete. Generalizations and departures from the previous theory are pointed out. Many of the properties of this class of generalized functions – including their Fourier transforms – are not considered. However, the formalism given is adequate for the product analysis contained in the rest of this paper.

Definition 1: A good function is one which is everywhere differentiable any number of times and such that it and all its derivatives are $O\left(|x|^{-N}\right)$ as $|x| \to \infty$ for all N (ref. 2, p. 15).

Definition 2: A fairly good function is one which is everywhere differentiable any number of times and such that it and all its derivatives are $O\left(|x|^{N}\right)$ as $|x| \to \infty$ for some N (ref. 2, p. 15).

Theorem 1: The derivative of a good function is a good function. The sum of two good functions is a good function. The product of two good functions is a good function. The product of a fairly good function and a good function is a good function. (Proof omitted.)

Definition 3: A sequence $f_n(x)$ of good functions is called a k regular sequence if for any good function $F(x)$ there exist integers l such that $\lim_{n\to\infty} n^{-l} \int_{-\infty}^{\infty} f_n(x)\, F(x)\, dx$ exists, and if k is the least value of l for which this limit exists. If there is no least value of l, $f_n(x)$ is termed a null sequence.

Regular sequences as defined by Lighthill (ref. 2) correspond to $k \leqq 0$ regular sequences in definition 3. Products of generalized functions (defined subsequently) are frequently positive k regular sequences. Negative k regular sequences correspond to null generalized functions in previous treatments. (See ref. 5, pp. 24-29.) It is necessary to retain negative k regular sequences in the present treatment because the product of two negative k regular sequences may be a $k = 0$ regular sequence or a positive k regular sequence. For such examples see section entitled "Elementary Applications."

The notion of equivalent sequences, as presented by Lighthill (ref. 2, p. 17), is necessarily omitted in the present treatment. This is because two sequences $f_n(x)$ and $g_n(x)$, which satisfy the equivalence relation

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(f_n(x) - g_n(x)\right) F(x)\, dx = 0$$

for any good function $F(x)$, will in general differ by a negative k regular sequence. Since negative k regular sequences are elevated to equal status with $k = 0$ and positive k regular sequences in the present treatment, the concept of equivalent sequences cannot be retained. For further discussion of this point see appendix B.

Theorem 2: If $f_n(x)$ is a k regular sequence of good functions, its derivative $f_n'(x)$ is a k regular sequence of good functions.

Proof: By theorem 1 $f_n'(x)$ is a sequence of good functions. Also for any good function $F(x)$

$$\lim_{n\to\infty} n^{-k} \int_{-\infty}^{\infty} f_n'(x)\ F(x)\ dx = \lim_{n\to\infty} -n^{-k} \int_{-\infty}^{\infty} f_n(x)\ F'(x)\ dx$$

and the limit on the right-hand side exists by definition 3 and theorem 1. Hence, by definition 3, $f_n'(x)$ is a k regular sequence of good functions.

Definition 3 substantially enlarges the class of sequences considered regular. But the product of any two regular sequences thus defined is not necessarily a regular sequence. For example, the sequence $\frac{e^n}{\sqrt{\pi}} e^{-x^2 e^{2n}}$ is one of the representations of the delta function and is a $k = 0$ regular sequence. But its square $\frac{e^n}{\sqrt{2\pi}}\left(\sqrt{\frac{2}{\pi}}\, e^n e^{-2x^2 e^{2n}}\right)$ is not regular, for the term in parentheses (which is another representation for the delta function) is multiplied by e^n. (In this case the index substitution $m = e^n$ renders the square a $k = 1$ regular sequence in the new index m.) A sufficient condition that the product of two regular sequences of good functions be a regular sequence of good functions is that the two sequences are of integral order, in the sense of the following definition:

Definition 4: A sequence of good functions is said to be of order r if r is the least integral value of p for which there exists an integer M such that $n^{-p}|f_n(x)| < M$ for any $n \geqq 1$ and all x.

Theorem 3: A sequence $f_n(x)$ of good functions of order r is a k regular sequence with $k \leqq r$.

Proof: By definition 4 there exists an integer M such that $n^{-r}|f_n(x)| < M$ for any $n \geqq 1$ and all x. Hence, for any good function $F(x)$

$$\lim_{n\to\infty} n^{-r} \int_{-\infty}^{\infty} f_n(x)\, F(x)\, dx < M \int_{-\infty}^{\infty} F(x)\, dx$$

Since the right-hand side of the inequality is finite by definition 1, the limit exists. Therefore, by definition 3, $f_n(x)$ is a k regular sequence with k at most equal to r.

On the other hand, a regular sequence of good functions (definition 3) is not necessarily of integral order (definition 4). For example, the delta function sequence $\frac{e^n}{\sqrt{\pi}} e^{-x^2 e^{2n}}$ is a $k = 0$ regular sequence, but it is not of integral order. (In this case the substitution $m = e^n$ renders the sequence of order 1 in the new index m.) Thus, sequences of good functions of integral order are seen to be a subclass of all regular sequences of good functions.

Theorem 4: If sequences of good functions $f_n(x)$ and $g_n(x)$ are of orders r and s, respectively, with $s \geqq r$, their sum $f_n(x) + g_n(x)$ is a sequence of good functions of order at most s. (Proof omitted.)

Theorem 5: If sequences of good functions $f_n(x)$ and $g_n(x)$ are of orders r and s, respectively, their product $f_n(x)\, g_n(x)$ is a sequence of good functions of order at most $r + s$.

Proof: By definition 4 there exist integers M and N such that

$$n^{-r}|f_n(x)| < M$$
$$n^{-s}|g_n(x)| < N$$

for any $n \geqq 1$ and all x. Hence

$$n^{-r-s}|f_n(x)\, g_n(x)| < MN$$

for any $n \geqq 1$ and all x. Therefore, by theorem 1 and definition 4 $f_n(x)\, g_n(x)$ is a sequence of good functions of order at most $r + s$.

Some sequences of integral order have derivatives which are not sequences of integral order (although they are regular sequences by theorem 2). For example, the sequence $e^{-x^2 e^{2n}}$ of order 0 has for its first derivative $-2e^{2n} x e^{-x^2 e^{2n}}$, which is not a sequence of integral order. (Again the index substitution $m = e^n$ makes the derivative sequence of order 1 in the new index m.) To eliminate such sequences from consideration, "proper" sequences are defined as a subclass of all sequences of integral order, as follows:

Definition 5: A sequence $f_n(x)$ of good functions is termed a proper sequence of good functions if the sequence itself and all of its derivatives are sequences of good functions of integral order.

Definition 6: A generalized function is defined as a proper sequence $f_n(x)$ of good functions.

Definition 7: (1) The derivative of a generalized function $f_n(x)$ is defined by the sequence $f'_n(x)$. (2) The sum of two generalized functions $f_n(x)$ and $g_n(x)$ is defined by the sequence $f_n(x) + g_n(x)$. (3) The product of two generalized functions $f_n(x)$ and $g_n(x)$ is defined by the sequence $f_n(x)\, g_n(x)$. (4) The product of a fairly good function $\varphi(x)$ and a generalized function $f_n(x)$ is defined by the sequence $\varphi(x)\, f_n(x)$. (5) The linear substitution $x = ay + b$ in the generalized function $f_n(x)$ is defined by the sequence $f_n(ay + b)$.

Proof of consistency for definition 7: In each case it must be shown that the sequence defined is a proper sequence of good functions. (1) By definition 5 for $f_n(x)$ the sequence $f'_n(x)$ and all its derivatives are sequences of good functions of integral order. Hence $f'_n(x)$ is a proper sequence of good functions. (2) The individual sequences and their derivatives, represented by $f_n^{(\alpha)}(x)$ and $g_n^{(\alpha)}(x)$ with $\alpha = 0, 1, 2, \ldots$, are sequences of good functions of integral order by definition 5. Also the derivatives of $f_n(x) + g_n(x)$ are given by

$$\frac{d^\alpha}{dx^\alpha}\Big(f_n(x) + g_n(x)\Big) = f_n^{(\alpha)}(x) + g_n^{(\alpha)}(x)$$

Hence by theorem 4 and definition 5 the sequence $f_n(x) + g_n(x)$ is a proper sequence of good functions. (3) By theorem 1, $\frac{d^\alpha}{dx^\alpha}\Big(f_n(x)\, g_n(x)\Big)$ with $\alpha = 0, 1, 2, \ldots$, are sequences of good functions. Also $\frac{d^\alpha}{dx^\alpha}\Big(f_n(x)\, g_n(x)\Big) = f_n^{(\alpha)}(x)\, g_n(x) + \alpha f_n^{(\alpha-1)}(x)\, g_n^{(1)}(x) + \ldots$. Hence, by theorems 5 and 4, $\frac{d^\alpha}{dx^\alpha}\Big(f_n(x)\, g_n(x)\Big)$ with $\alpha = 0, 1, 2, \ldots$ are sequences of good functions of integral order. By definition 5, therefore, $f_n(x)\, g_n(x)$ is a proper sequence of good functions. (4) and (5) Proofs omitted.

FAMILY OF GENERALIZED FUNCTIONS DERIVED FROM tanh nx

Consider the hyperbolic tangent representation of the Heaviside unit step function

$$H_n(x) = \frac{1}{2}(1 + \tanh nx) \tag{1}$$

where n is the sequence index. Plots of this representation for $H_n(x)$ (and also of the corresponding signum function representation $\mathrm{sgn}_n x = \tanh nx$) against x are shown in figure 1 for various values of n. This particular choice for $H_n(x)$ results in the recurrence relation

$$\frac{d}{dx} H_n(x) = 2n\left[H_n(x) - H_n^2(x)\right] \tag{2}$$

Therefore the delta function $\delta_n(x) = \frac{d}{dx} H_n(x)$ belonging to this family of generalized functions may be represented by

$$\delta_n(x) = 2n\left[H_n(x) - H_n^2(x)\right] \tag{3a}$$

Further differentiation and reapplication of equation (2) yields the following equations:

$$\delta_n' = 4n^2\left(H_n - 3H_n^2 + 2H_n^3\right) \tag{3b}$$

$$\delta_n'' = 8n^3\left(H_n - 7H_n^2 + 12H_n^3 - 6H_n^4\right) \tag{3c}$$

$$\delta_n^{(3)} = 16n^4\left(H_n - 15H_n^2 + 50H_n^3 - 60H_n^4 + 24H_n^5\right) \tag{3d}$$

Hence sequences for the delta function and its derivatives in this family are polynomials in $H_n(x)$ multiplied by powers of the sequence index n. Graphs of a number of these generalized functions against $H_n(x)$ (eq. (1)) and $\mathrm{sgn}_n x = \tanh nx$ are shown in figure 2. A feature of these figures is that they apply for all values of n.

Set (3) may be inverted to obtain

$$\left.\begin{aligned}
H_n^2 &= H_n - \frac{1}{2}\frac{\delta_n}{n} \\
H_n^3 &= H_n - \frac{3}{4}\frac{\delta_n}{n} + \frac{1}{8}\frac{\delta_n'}{n^2} \\
H_n^4 &= H_n - \frac{11}{12}\frac{\delta_n}{n} + \frac{1}{4}\frac{\delta_n'}{n^2} - \frac{1}{48}\frac{\delta_n''}{n^3} \\
H_n^5 &= H_n - \frac{25}{24}\frac{\delta_n}{n} + \frac{35}{96}\frac{\delta_n'}{n^2} - \frac{5}{96}\frac{\delta_n''}{n^3} + \frac{1}{384}\frac{\delta_n^{(3)}}{n^4}
\end{aligned}\right\} \tag{4}$$

Thus, powers of $H_n(x)$ are expressed as linear sums of $H_n(x)$ and its derivatives, with coefficients containing powers of the sequence index n.

The generalized functions of set (3) are characterized by order (definition 4) and by degree of the polynomial in H_n. Thus $\delta_n''(x)$ (eq. (3c)) is of order 3 and degree 4; and $\delta_n^{(p)}(x)$ is of order $p + 1$ and degree $p + 2$.

All of the generalized functions of set (3) are $k = 0$ regular sequences (definition 3). But within the bounds of definition 6, one may combine arbitrary powers of n with arbitrary polynomials in $H_n(x)$ to obtain generalized functions of any desired order and shape. (Of course, such sequences will be good functions of x (definition 1) only if the polynomials in $H_n(x)$ vanish at $H_n(x) = 0$ and at $H_n(x) = 1$.) The generalized functions thus formed may alternatively be expressed in terms of the functions H_n, δ_n, δ_n', . . ., multiplied by appropriate powers of n, by substituting set (4). In fact, any generalized function $G_n\big(H_n(x)\big)$ which is composed of continuous functions of $H_n(x)$ may be expressed by a series of polynomials in H_n through Legendre expansion and also as a series in H_n, δ_n, δ_n', . . . through substitution of set (4). A related theorem for a different class of generalized functions is cited by Gel'fand and Shilov (ref. 6). $\Big($To carry out the Legendre expansion of $G_n\big(H_n(x)\big)$, first substitute $H_n(x) = \frac{1 + \mathrm{sgn}_n x}{2}$, where $\mathrm{sgn}_n x = \tanh nx$. Since the range of $\mathrm{sgn}_n x$ is $-1 \leqq \mathrm{sgn}_n x \leqq 1$, a series of Legendre polynomials in the signum function $P_m(\mathrm{sgn}_n x)$ is suitable for the expansion. (See Sneddon, ref. 7, pp. 46-60.)$\Big)$ An alternative form of sets (3) and (4) in terms of $\mathrm{sgn}_n x$ (in place of $H_n(x)$) is presented in sets (26) and (27) of appendix A.

INDEFINITE GENERALIZED FUNCTIONS δ_n', δ_n'', . . . EXPRESSED AS DIFFERENCES OF DEFINITE GENERALIZED FUNCTIONS

Each of the indefinite generalized functions of set (3) may be represented by the difference of two definite generalized functions of related symmetry, as follows:

$$\delta_n' = \left(n\delta_n + \frac{\delta_n'}{2}\right) - \left(n\delta_n - \frac{\delta_n'}{2}\right) \equiv \Delta_n'(+) - \Delta_n'(-) \tag{5a}$$

$$\delta_n'' = \left(2n^2\delta_n + \delta_n''\right) - \left(2n^2\delta_n\right) \equiv \Delta_n''(+) - \Delta_n''(-) \tag{5b}$$

$$\delta_n^{(3)} = \left(12n^3\delta_n + 2n\delta_n'' + \frac{\delta_n^{(3)}}{2}\right) - \left(12n^3\delta_n + 2n\delta_n'' - \frac{\delta_n^{(3)}}{2}\right) \tag{5c}$$

$$\delta_n^{(4)} = \left(40n^4\delta_n + 20n^2\delta_n'' + \delta_n^{(4)}\right) - \left(40n^4\delta_n + 20n^2\delta_n''\right) \tag{5d}$$

Plots of these pairs of definite generalized functions against $H_n(x)$ and $\mathrm{sgn}_n x$ are shown in figure 3. The notation $\Delta_n^{(\gamma)}(\pm)$ of definite generalized functions is intended only to correlate them with the indefinite generalized function $\delta_n^{(\gamma)}$ from which they are derived. Obviously $\frac{d}{dx}\,\Delta_n'(\pm) \neq \Delta_n''(\pm)$.

Correlated definite and indefinite generalized functions in set (5) are of like order (definition 6), but the generalized functions on the left-hand side of set (5) are all

$k = 0$ regular sequences (definition 3) while the terms in parentheses on the right are all positive k regular sequences.

PRODUCTS OF GENERALIZED FUNCTIONS DERIVED FROM tanh nx

The product of two generalized functions is given in definition 7. For the hyperbolic tangent family of generalized functions (set (3)) it is clear that all products will be polynomials in $H_n(x)$ multiplied by powers of the sequence index n. For example

$$H_n \delta_n = 2n\left(H_n^2 - H_n^3\right) \tag{6}$$

But substitutions from set (4) give alternatively

$$H_n \delta_n = \frac{\delta_n}{2} - \frac{\delta_n'}{4n} \tag{7a}$$

Therefore any product of the generalized functions of set (3) may also be represented by a linear sum of these generalized functions with appropriate powers of n in the coefficients. A number of such products are

$$\delta_n^2 = \frac{n\delta_n}{3} - \frac{\delta_n''}{12n} \tag{7b}$$

$$H_n \delta_n' = \frac{\delta_n'}{2} - \frac{n\delta_n}{3} - \frac{\delta_n''}{6n} \tag{7c}$$

$$\delta_n \delta_n' = \frac{n\delta_n'}{6} - \frac{\delta_n^{(3)}}{24n} \tag{7d}$$

$$\left(\delta_n'\right)^2 = \frac{4n^3\delta_n}{15} - \frac{\delta_n^{(4)}}{60n} \tag{7e}$$

$$H_n \delta_n'' = -\frac{n\delta_n'}{2} + \frac{\delta_n''}{2} - \frac{\delta_n^{(3)}}{8n} \tag{7f}$$

$$\delta_n \delta_n'' = -\frac{4n^3\delta_n}{15} + \frac{n\delta_n''}{6} - \frac{\delta_n^{(4)}}{40n} \tag{7g}$$

$$\delta_n' \delta_n'' = \frac{2n^3\delta_n'}{15} - \frac{\delta_n^{(5)}}{120n} \tag{7h}$$

$$\left(\delta_n''\right)^2 = \frac{16n^5\delta_n}{21} - \frac{2n^3\delta_n''}{15} - \frac{\delta_n^{(6)}}{280n} \tag{7i}$$

Here the product of two generalized functions, having individual orders r and s (definition 6), is seen to have order $r + s$, in agreement with theorem 5.

It should be emphasized that equations (3) to (7) hold only for the particular family of generalized functions derived from the hyperbolic tangent representation (1). Analogous, but different, relations apply to the family of generalized functions derived from the Gaussian representation (36). These are presented for comparison in appendix B.

EXTENSION OF RELATIONS

Equations (3) may be written

$$\frac{H_n^{(\alpha)}}{(2n)^\alpha} = \sum_{\beta=1}^{\alpha+1} a_{\alpha\beta} H_n^\beta \qquad \begin{pmatrix}\alpha = 0, 1, 2, \ldots \\ \beta = 1, 2, 3, \ldots\end{pmatrix} \tag{8}$$

where the coefficients $a_{\alpha\beta}$ are given in table I.

TABLE I.- COEFFICIENTS $a_{\alpha\beta}$

$\alpha \downarrow$ \ $\beta \rightarrow$	1	2	3	4	5	6	7
0	1	.	.	.	.	.	.
1	1	-1	.	.	.	.	.
2	1	-3	2	.	.	.	.
3	1	-7	12	-6	.	.	.
4	1	-15	50	-60	24	.	.
5	1	-31	180	-390	360	-120	.
6	1	-63	602	-2100	3360	-2520	720

The formula

$$a_{\alpha\beta} = \beta a_{\alpha-1,\beta} - (\beta - 1)a_{\alpha-1,\beta-1} \tag{9}$$

continues table I indefinitely. Similarly, equations (4) are given by

$$H_n^\mu = \sum_{\nu=0}^{\mu-1} b_{\mu\nu} \frac{H_n^{(\nu)}}{(2n)^\nu} \qquad \begin{pmatrix}\mu = 1, 2, 3, \ldots \\ \nu = 0, 1, 2, \ldots\end{pmatrix} \tag{10}$$

where the coefficients $b_{\mu\nu}$ are given in table II.

TABLE II.- COEFFICIENTS $b_{\mu\nu}$

$\mu \downarrow$ / $\nu \rightarrow$	0	1	2	3	4	5	6
1	1	.	.	.	.	.	.
2	1	-1	.	.	.	.	.
3	1	-3/2	1/2	.	.	.	.
4	1	-11/6	1	-1/6	.	.	.
5	1	-25/12	35/24	-5/12	1/24	.	.
6	1	-137/60	15/8	-17/24	1/8	-1/120	.
7	1	-49/20	203/90	-49/48	35/144	-7/240	1/720

Table II may also be extended indefinitely by the formula

$$b_{\mu\nu} = b_{\mu-1,\nu} - \frac{b_{\mu-1,\nu-1}}{\mu - 1} \tag{11}$$

The coefficients $a_{\mu\nu}$ and $b_{\mu\nu}$ satisfy the relations

$$\sum_{\gamma=1}^{\alpha+1} a_{\alpha\gamma} b_{\gamma\beta} = \sum_{\eta=0}^{\alpha-1} b_{\alpha\eta} a_{\eta\beta} = \delta_{\alpha\beta} \tag{12}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

Products of generalized functions expressed as polynomials in $H_n(x)$ (eq. (6)) are given by

$$\frac{H_n^{(\alpha)} H_n^{(\beta)}}{(2n)^{\alpha+\beta}} = \sum_{\xi=1}^{\alpha+1} \sum_{\eta=1}^{\beta+1} a_{\alpha\xi} a_{\beta\eta} H_n^{\xi+\eta} \tag{13}$$

Products expressed as linear sums of generalized functions (eqs. (7)) are given by

$$\frac{H_n^{(\alpha)} H_n^{(\beta)}}{(2n)^{\alpha+b}} = \sum_{\xi=1}^{\alpha+1} \sum_{\eta=1}^{\beta+1} \sum_{\zeta=0}^{\xi+\eta-1} a_{\alpha\xi} a_{\beta\eta} b_{\xi+\eta,\zeta} \frac{H_n^{(\zeta)}}{(2n)^{\zeta}} \tag{14}$$

DISCUSSION

The product equations (7) are useful conceptually in that products are given in terms of more familiar functions – the step function and its derivatives. These relations are also useful mathematically because (a) indefinite integrals of these products are easily obtained for all values of n and (b) definite integrals of these products, multiplied by any analytic good function, are obtained to highest and next highest orders of n in the limit as $n \to \infty$.

Example 1: Determine the indefinite integral $\int \delta_n(x)\delta_n''(x)\,dx$.

Integrating equation (7g) gives

$$\int \delta_n(x)\delta_n''(x)\,dx = -\frac{4n^3 H_n(x)}{15} + \frac{n\delta_n'(x)}{6} - \frac{\delta_n^{(3)}(x)}{40n} + C(n) \tag{15}$$

a result which is valid for all values of n, where $C(n)$ is an integration constant.

Example 2: Find the asymptotic behavior for large n of the definite integral $\int_{-\infty}^{\infty} H_n(x)\,\delta_n'(x)\,F(x)\,dx$, where $F(x)$ is any good function which is also analytic in the neighborhood of $x = 0$.

By equation (7c)

$$\int_{-\infty}^{\infty} H_n(x)\,\delta_n'(x)\,F(x)\,dx = \int_{-\infty}^{\infty}\left(\frac{\delta_n'(x)}{2} - \frac{n\delta_n(x)}{3} - \frac{\delta_n''(x)}{6n}\right)F(x)\,dx \tag{16}$$

The generalized functions $\delta_n(x)$, $\delta_n'(x)$, and $\delta_n''(x)$ (eqs. (3)) decay as $e^{-2n|x|}$ for $|x| > Mn^{-1}$, where M is some integer. Therefore it is easily shown, by using the Maclaurin expansion of $F(x)$, that as $n \to \infty$

$$\left.\begin{aligned}
&\int_{-\infty}^{\infty} \delta_n'(x)\,F(x)\,dx \to -F'(0) + 0\left(\frac{F^{(3)}(0)}{n^2}\right)\\
&\int_{-\infty}^{\infty} n\delta_n(x)\,F(x)\,dx \to nF(0) + 0\left(\frac{F''(0)}{n}\right)\\
&\int_{-\infty}^{\infty} \frac{\delta_n''(x)}{n}\,F(x)\,dx \to \frac{F''(0)}{n} + 0\left(\frac{F^{(4)}(0)}{n^3}\right)
\end{aligned}\right\} \tag{17}$$

Hence

$$\int_{-\infty}^{\infty} H_n(x)\ \delta_n'(x)\ F(x)\ dx \to -\frac{nF(0)}{3} - \frac{F'(0)}{2} + 0\left(\frac{F''(0)}{n}\right) \tag{18}$$

as $n \to \infty$. If $F(0) = 0$ the limit of the integral exists as $n \to \infty$, and if $F'(0) = 0$ also the limit of the integral vanishes. Equation (18) indicates that the generalized function $H_n(x)\ \delta_n'(x)$ is a $k = 1$ regular sequence (definition 3).

ELEMENTARY APPLICATIONS

The foregoing analysis is convenient for treating physical systems whose fields are approximated by the sequence functions of appropriate generalized functions. The finite amplitude and narrowness of all physical systems correspond to finite (although possibly large) values of the sequence index n. Energy density and other product quantities are readily found. Certain physical fields correspond naturally to negative k regular sequences (previously regarded as null generalized functions (ref. 5, pp. 24-29)). The equal status of negative k regular sequences with $k = 0$ regular sequences in product calculations is illustrated in the following two elementary examples.

Electromagnetic Wave Pulses

The product analysis allows a simple comparison of the energies of plane electromagnetic wave pulses in vacuum having different shapes but the same order of amplitude and narrowness. Let the electric fields of three pulses be given, for example, by

$$\left.\begin{aligned} E_1(x,t) &= A\delta_n(x - ct) \\ E_2(x,t) &= \frac{B}{n}\,\delta_n'(x - ct) \\ E_3(x,t) &= \frac{C}{n^2}\,\delta_n''(x - ct) \end{aligned}\right\} \tag{19}$$

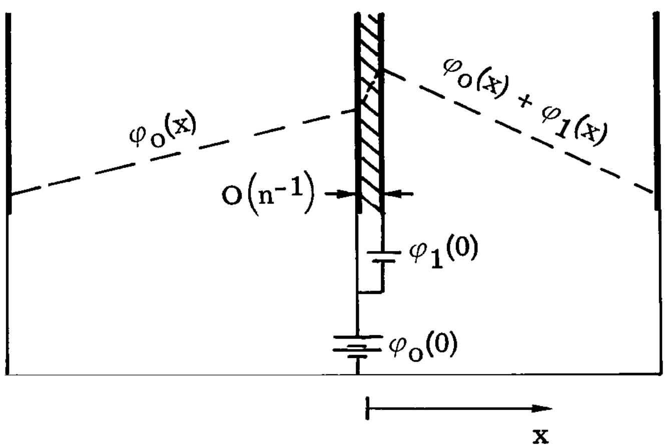
where A, B, and C are constants and c is the speed of light in vacuum. The generalized functions δ_n, δ_n'/n, and δ_n''/n^2 are $k = 0$, $k = -1$, and $k = -2$ regular sequences, respectively (definition 3); but they are all of order 1 (definition 4).

The energy density of a plane electromagnetic wave in vacuum (including the magnetic part) is given by $W = \epsilon_0 E^2$. For the three pulses of set (19), then, the energy densities are, by equations (7)

$$\left.\begin{aligned}
W_1(x,t) &= \epsilon_o A^2 \delta_n^2(x - ct) = \epsilon_o A^2\left[\frac{n}{3}\,\delta_n(x - ct) - \frac{1}{12n}\,\delta_n''(x - ct)\right] \\
W_2(x,t) &= \frac{\epsilon_o B^2}{n^2}\left(\delta_n'(x - ct)\right)^2 = \epsilon_o B^2\left[\frac{4n}{15}\,\delta_n(x - ct) - \frac{1}{60n^3}\,\delta_n^{(4)}(x - ct)\right] \\
W_3(x,t) &= \frac{\epsilon_o C^2}{n^4}\left(\delta_n''(x - ct)\right)^2 = \epsilon_o C^2\left[\frac{16n}{21}\,\delta_n(x - ct) - \frac{2}{15n}\,\delta_n''(x - ct) - \frac{1}{280n^5}\,\delta_n^{(6)}(x - ct)\right]
\end{aligned}\right\} \tag{20}$$

Integration of expressions (20) shows that the pulses have equal energy when $A^2 = \frac{4}{5} B^2 = \frac{16}{7} C^2$. These results hold for all values of n; consequently, they apply to pulses of small amplitude and large width (small n) as well as to tall and narrow pulses (large n).

Thin Dielectric Filled Capacitor With External Fields Applied



Sketch 1

Let the electrostatic system of sketch 1 (within the outer plates) be idealized by the following representations for the electric scaler potential

$$\varphi(x) = \varphi_o(x) + H_n(x)\,\varphi_1(x) \tag{21}$$

and for the dielectric medium

$$\epsilon(x) = \epsilon_o\left[1 + A\,\frac{\delta_n(x)}{n}\right] \tag{22}$$

where φ_0 and φ_1 are linear functions of x, and where $\epsilon(x)$ has a peak value of $\epsilon_0\left(1+\frac{A}{2}\right)$ inside the thin capacitor and equals ϵ_0 outside. The generalized function $\delta_n(x)/n$ in the definition of the dielectric is a $k = -1$ regular sequence (definition 3) of order 0 (definition 4).

The electric field $E(x) = \frac{d\varphi}{dx}$, electric excitation $D(x) = \epsilon(x)\, E(x)$, and charge density $\rho(x) = \frac{dD}{dx}$ are found to be

$$\left.\begin{aligned}
E(x) &= \left(\varphi_0' + H_n\varphi_1'\right) + \delta_n\varphi_1 \\
D(x) &= \epsilon_0\left[\varphi_0' + H_n\varphi_1' + \frac{\delta_n}{n} A\left(\varphi_0' + \frac{\varphi_1'}{2}\right) - \frac{\delta_n'}{4n^2} A\varphi_1'\right] + \epsilon_0\left[\delta_n\varphi_1\left(1+\frac{A}{3}\right) - \frac{\delta_n''}{12n^2} A\varphi_1\right] \\
\rho(x) &= \epsilon_0\left[\delta_n\varphi_1'\left(2+\frac{A}{3}\right) + \frac{\delta_n'}{n} A\left(\varphi_0' + \frac{\varphi_1'}{2}\right) - \frac{\delta_n''}{3n^2} A\varphi_1'\right] + \epsilon_0\left[\delta_n'\varphi_1\left(1+\frac{A}{3}\right) - \frac{\delta_n^{(3)}}{12n^2} A\varphi_1\right]
\end{aligned}\right\} \tag{23}$$

where terms of like order have been grouped together.

Features of this approach using generalized functions are (a) expressions (23) apply to the entire region between the outer plates, (b) these expressions exhibit the detailed structure of the fields in the neighborhood of the origin, and (c) Maxwell's equations in differential form suffice without the jump (discontinuity) conditions. Comparable detail could be obtained with ordinary functions only by using jump conditions at the plates of the thin capacitor to piece together solutions for the three separate regions included between the outer plates.

The expressions for energy density $\frac{E(x)\, D(x)}{2}$ and for electric force density $\rho(x)\, E(x) - \frac{E^2(x)\, \epsilon'(x)}{2}$ (ref. 8) are also readily obtained. The energy within the thin capacitor (per unit area) $\mathcal{E} = \frac{1}{2}\int_{-\tau}^{\tau} E(x)\, D(x)\, dx$, where $\tau = O\left(n^{-1}\right)$ in the limit as $n \to \infty$, is given to highest and next highest orders of n by

$$\mathcal{E} = \frac{n}{6}\,\epsilon_0\varphi_1^2(0)\left(1+\frac{2A}{5}\right) + \epsilon_0\varphi_1(0)\left(1+\frac{A}{3}\right)\left(\varphi_0' + \frac{\varphi_1'}{2}\right) + O\left(n^{-1}\right) \tag{24}$$

To highest order in n, therefore, the capacitance of the thin capacitor (per unit area) is $\frac{n\epsilon_0}{3}\left(1 + \frac{2A}{5}\right) + O(1)$. The thin dielectric layer, represented by the negative k regular sequence (22), is seen to affect the highest order term in the capacitance formula.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 16, 1967,
129-02-01-01-23.

APPENDIX A

ALTERNATIVE FORMULATION OF HYPERBOLIC TANGENT FAMILY OF GENERALIZED FUNCTIONS IN TERMS OF $\mathrm{sgn}_n x$

The substitution

$$H_n(x) = \frac{1}{2}\left(1 + \mathrm{sgn}_n x\right) \tag{25}$$

where $\mathrm{sgn}_n x = \tanh nx$ in accord with equation (1), reveals symmetries in the generalized functions derived from $\tanh nx$. Equations (3) become

$$\left.\begin{aligned}
\delta_n(x) &= \frac{n}{2}\left(1 - \mathrm{sgn}_n^2 x\right) \\
\delta_n'(x) &= n^2\left(-\mathrm{sgn}_n x + \mathrm{sgn}_n^3 x\right) \\
\delta_n''(x) &= n^3\left(-1 + 4\ \mathrm{sgn}_n^2 x - 3\ \mathrm{sgn}_n^4 x\right) \\
\delta_n^{(3)}(x) &= 4n^4\left(2\ \mathrm{sgn}_n x - 5\ \mathrm{sgn}_n^3 x + 3\ \mathrm{sgn}_n^5 x\right)
\end{aligned}\right\} \tag{26}$$

The inverse of set (26) is

$$\left.\begin{aligned}
\mathrm{sgn}_n^2 x &= 1 - \frac{2\delta_n}{n} \\
\mathrm{sgn}_n^3 x &= \mathrm{sgn}_n x + \frac{\delta_n'}{n^2} \\
\mathrm{sgn}_n^4 x &= 1 - \frac{8\delta_n}{3n} - \frac{\delta_n''}{3n^3} \\
\mathrm{sgn}_n^5 x &= \mathrm{sgn}_n x + \frac{5\delta_n'}{3n^2} + \frac{\delta_n^{(3)}}{12n^4}
\end{aligned}\right\} \tag{27}$$

Equations (26) may be written

$$\frac{\mathrm{sgn}_n^{(\alpha)} x}{n^\alpha} = \sum_{\beta=0}^{\alpha+1} r_{\alpha\beta}\ \mathrm{sgn}_n^\beta x \qquad \begin{pmatrix} \alpha = 00,\ 0,\ 1,\ 2,\ \ldots \\ \beta = 0,\ 1,\ 2,\ \ldots \end{pmatrix} \tag{28}$$

where the special definitions

$$\frac{\mathrm{sgn}_n^{(00)}x}{n^{00}} = \mathrm{sgn}_n^0 x = 1 \tag{29}$$

are employed and where the coefficients $r_{\alpha\beta}$ are presented in table III.

TABLE III.- COEFFICIENTS $r_{\alpha\beta}$

$\alpha \downarrow$ \ $\beta \rightarrow$	0	1	2	3	4	5	6	7
00	1	.	.	.	.	.	.	.
0	.	1	.	.	.	.	.	.
1	1	.	-1	.	.	.	.	.
2	.	-2	.	2	.	.	.	.
3	-2	.	8	.	-6	.	.	.
4	.	16	.	-40	.	24	.	.
5	16	.	-136	.	240	.	-120	.
6	.	-272	.	1232	.	-1680	.	720

Table III may be extended indefinitely by means of the formula

$$r_{\alpha\beta} = (\beta + 1)r_{\alpha-1,\beta+1} - (\beta - 1)r_{\alpha-1,\beta-1} \qquad (\alpha > 0) \tag{30}$$

Equations (27) may be written

$$\mathrm{sgn}_n^{\mu}x = \sum_{\nu=00}^{\mu-1} s_{\mu\nu} \frac{\mathrm{sgn}_n^{(\nu)}x}{n^{\nu}} \tag{31}$$

where the coefficients $s_{\mu\nu}$ are given in table IV.

TABLE IV.- COEFFICIENTS $s_{\mu\nu}$

$\mu \downarrow$ \ $\nu \rightarrow$	00	0	1	2	3	4	5	6
0	1	.	.	.	.	.	.	.
1	.	1	.	.	.	.	.	.
2	1	.	-1	.	.	.	.	.
3	.	1	.	1/2	.	.	.	.
4	1	.	-4/3	.	-1/6	.	.	.
5	.	1	.	5/6	.	1/24	.	.
6	1	.	-23/15	.	-1/3	.	-1/120	.
7	.	1	.	49/45	.	7/72	.	1/720

The formula

$$s_{\mu\nu} = s_{\mu-2,\nu} - \frac{s_{\mu-1,\nu-1}}{(\mu - 1)} \qquad (\nu > 0) \qquad (32)$$

is applicable to table IV for $\nu > 0$. The coefficients $r_{\mu\nu}$ and $s_{\mu\nu}$ satisfy the relations

$$\sum_{\gamma=0}^{\alpha+1} r_{\alpha\gamma} s_{\gamma\beta} = \sum_{\eta=00}^{\alpha-1} s_{\alpha\eta} r_{\eta\beta} = \delta_{\alpha\beta} \qquad (33)$$

Product formulas in terms of $\mathrm{sgn}_n x$ are

$$\frac{\mathrm{sgn}_n^{(\alpha)} x \ \mathrm{sgn}_n^{(\beta)} x}{n^{\alpha+\beta}} = \sum_{\xi=0}^{\alpha+1} \sum_{\eta=0}^{\beta+1} r_{\alpha\xi} r_{\beta\eta} \ \mathrm{sgn}_n^{\xi+\eta} x \qquad (34a)$$

and

$$\frac{\mathrm{sgn}_n^{(\alpha)} x \ \mathrm{sgn}_n^{(\beta)} x}{n^{\alpha+\beta}} = \sum_{\xi=0}^{\alpha+1} \sum_{\eta=0}^{\beta+1} \sum_{\zeta=00}^{\xi+\eta-1} r_{\alpha\xi} r_{\beta\eta} s_{\xi+\eta,\zeta} \frac{\mathrm{sgn}_n^{(\zeta)} x}{n^{\zeta}} \qquad (34b)$$

which correspond to equations (13) and (14), respectively. A number of products are

$$
\left.\begin{aligned}
\operatorname{sgn}_n x \ \delta_n(x) &= -\frac{\delta_n'(x)}{2n} \\
\operatorname{sgn}_n x \ \delta_n'(x) &= -\frac{2n\delta_n(x)}{3} - \frac{\delta_n''(x)}{3n} \\
\operatorname{sgn}_n x \ \delta_n''(x) &= -n\delta_n'(x) - \frac{\delta_n^{(3)}(x)}{4n}
\end{aligned}\right\} \tag{35}
$$

Products such as δ_n^2, $\delta_n\delta_n'$, $\left(\delta_n'\right)^2$, which do not contain $\operatorname{sgn}_n x$ as a factor, are the same as in set (7).

APPENDIX B

FAMILY OF GENERALIZED FUNCTIONS DERIVED FROM $ne^{-n^2x^2}$

The Gaussian representation for the delta function is

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \tag{36}$$

The derivatives of this generalized function are given by

$$\delta_n^{(\alpha)}(x) = (-1)^\alpha n^\alpha H_\alpha(nx)\delta_n(x) \tag{37}$$

where $H_\alpha(nx)$ are Hermite polynomials of argument nx. (See ref. 7, pp. 132-134.) When written out for comparison with sets (3) and (26), equation (37) becomes

$$\left.\begin{aligned}
\delta_n' &= (-2nx)n\delta_n \\
\delta_n'' &= \left(-2 + 4n^2x^2\right)n^2\delta_n \\
\delta_n^{(3)} &= \left(12nx - 8n^3x^3\right)n^3\delta_n \\
\delta_n^{(4)} &= \left(12 - 48n^2x^2 + 16n^4x^4\right)n^4\delta_n
\end{aligned}\right\} \tag{38}$$

The inverse of set (38) is, for comparison with sets (4) and (27)

$$\left.\begin{aligned}
x\delta_n &= -\frac{\delta_n'}{2n^2} \\
x^2\delta_n &= \frac{\delta_n}{2n^2} + \frac{\delta_n''}{4n^4} \\
x^3\delta_n &= -\frac{3}{4}\frac{\delta_n'}{n^4} - \frac{\delta_n^{(3)}}{8n^6} \\
x^4\delta_n &= \frac{3}{4}\frac{\delta_n}{n^4} + \frac{3}{4}\frac{\delta_n''}{n^6} + \frac{\delta_n^{(4)}}{16n^8}
\end{aligned}\right\} \tag{39}$$

Products of the generalized functions of set (38) are polynomials in nx multiplied by $\delta_n^2(x)$ and powers of n. But

$$\delta_n^2(x) = \frac{n}{\sqrt{2\pi}} \delta_{\sqrt{2}n}(x) \tag{40a}$$

by equation (36), and set (39) may be substituted to express the products alternatively as linear sums of the delta function of modified index $\delta_{\sqrt{2}n}(x)$, and its derivatives, with coefficients containing powers of n, as illustrated:

$$\delta_n\delta_n' = \frac{n}{2\sqrt{2\pi}}\,\delta_{\sqrt{2}n} \tag{40b}$$

$$\delta_n\delta_n'' = -\frac{n^3}{\sqrt{2\pi}}\,\delta_{\sqrt{2}n} + \frac{n}{4\sqrt{2\pi}}\,\delta''_{\sqrt{2}n} \tag{40c}$$

$$\left(\delta_n'\right)^2 = \frac{n^3}{\sqrt{2\pi}}\,\delta_{\sqrt{2}n} + \frac{n}{4\sqrt{2\pi}}\,\delta''_{\sqrt{2}n} \tag{40d}$$

$$\delta_n'\delta_n'' = \frac{n^3}{2\sqrt{2\pi}}\,\delta'_{\sqrt{2}n} + \frac{n}{8\sqrt{2\pi}}\,\delta^{(3)}_{\sqrt{2}n} \tag{40e}$$

$$\left(\delta_n''\right)^2 = \frac{3n^5}{\sqrt{2\pi}}\,\delta_{\sqrt{2}n} + \frac{n^3}{2\sqrt{2\pi}}\,\delta''_{\sqrt{2}n} + \frac{n}{16\sqrt{2\pi}}\,\delta^{(4)}_{\sqrt{2}n} \tag{40f}$$

A comparison of the Gaussian and hyperbolic tangent families of generalized functions shows the following:

(1) Definite integrals of similar product quantities in the Gaussian family (set (40)) and in the hyperbolic tangent family (set (7)) agree only in powers of n. For example,

$$\int_{-\infty}^{\infty} \delta_n^2(x)\,dx = \begin{cases} n/\sqrt{2\pi} & \text{(Gaussian equation (40a))} \\ n/3 & \text{(hyperbolic tangent equation (7b))} \end{cases}$$

This stems from the fact that

$$\left(\delta_n(x)\right)_G - \left(\delta_n(x)\right)_{ht} = n\left(\frac{e^{-n^2x^2}}{\sqrt{\pi}} - \frac{1 - \tanh^2 nx}{2}\right) \tag{41}$$

is a $k = -2$ regular sequence of order 1, and it exemplifies the remarks following definition 3. The subscripts G and ht in equation (41) denote Gaussian and hyperbolic tangent, respectively.

(2) Products involving the unit step function $H_n(x)$ are tractable in the hyperbolic tangent family but not in the Gaussian family. (Compare sets (7) and (40).)

(3) Product terms have precisely the same form as linear terms in the hyperbolic tangent family (eqs. (7)), and this permits direct comparison of linear terms with nonlinear terms. But in the Gaussian family multiplication changes the sequence index as in equations (40).

APPENDIX B

(4) Definite and indefinite integrals of the form $\int_{-\infty}^{\infty} F(x)\, g_n(x)\, dx$ or $\int F(x)\, g_n(x)\, dx$, where $F(x)$ is an analytic good function and $g_n(x)$ is a generalized function of the Gaussian family, may be determined to any desired degree of accuracy for finite n. (This is accomplished by substituting for $g_n(x)$ from set (38), expanding $F(x)$ in a Maclaurin series, multiplying terms, and substituting eqs. (39).) The hyperbolic tangent family of generalized functions allows definite integrals of this type to be determined only to the highest and next highest orders of n in the limit as $n \to \infty$, as illustrated by equation (18).

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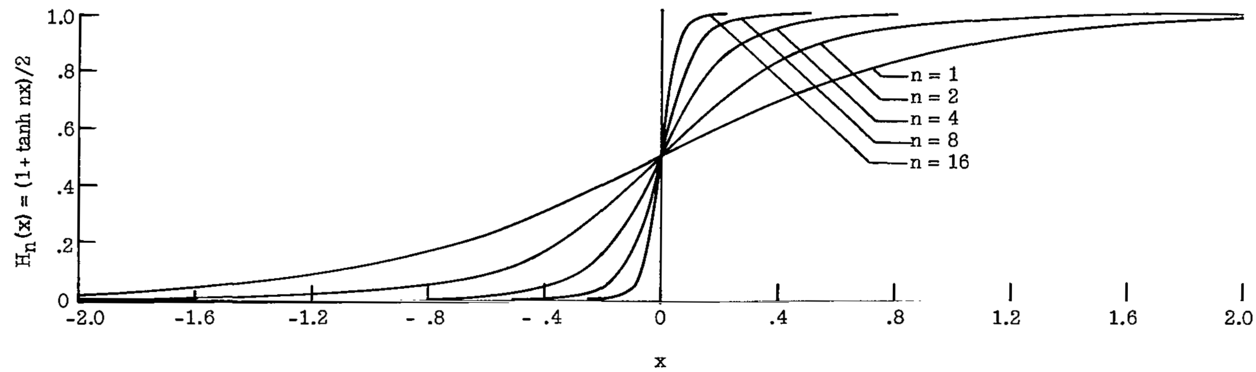
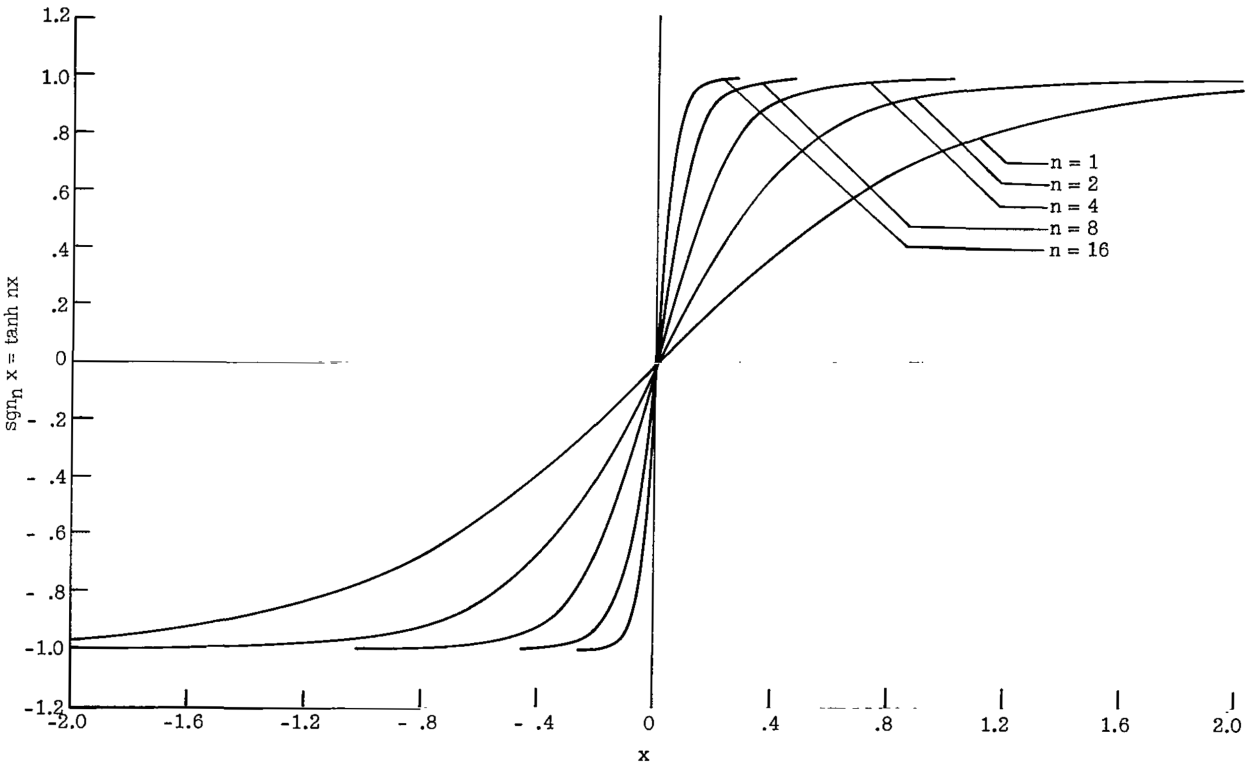
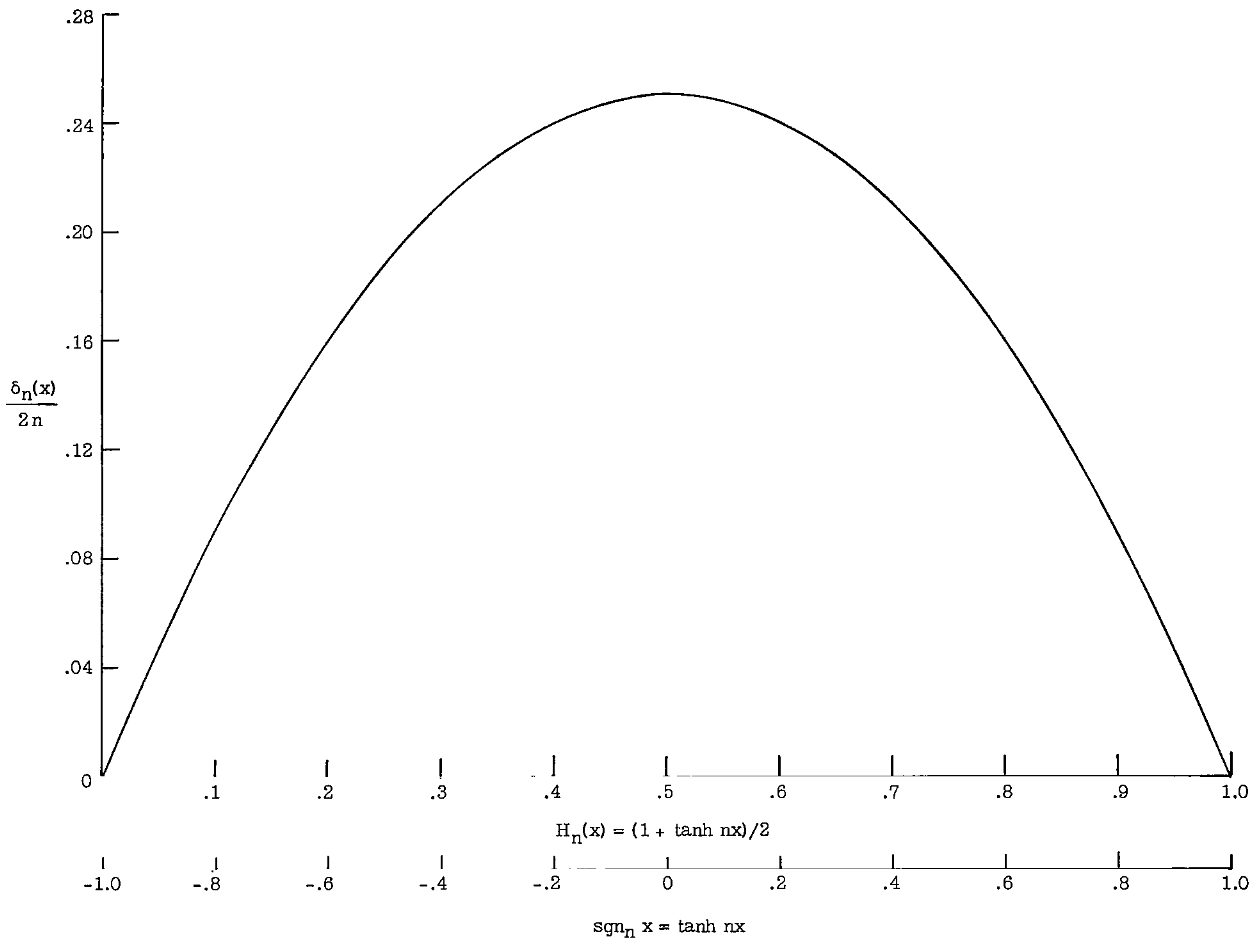


(a) Hyperbolic tangent representation for Heaviside function $H_n(x)$ plotted against x for various n.



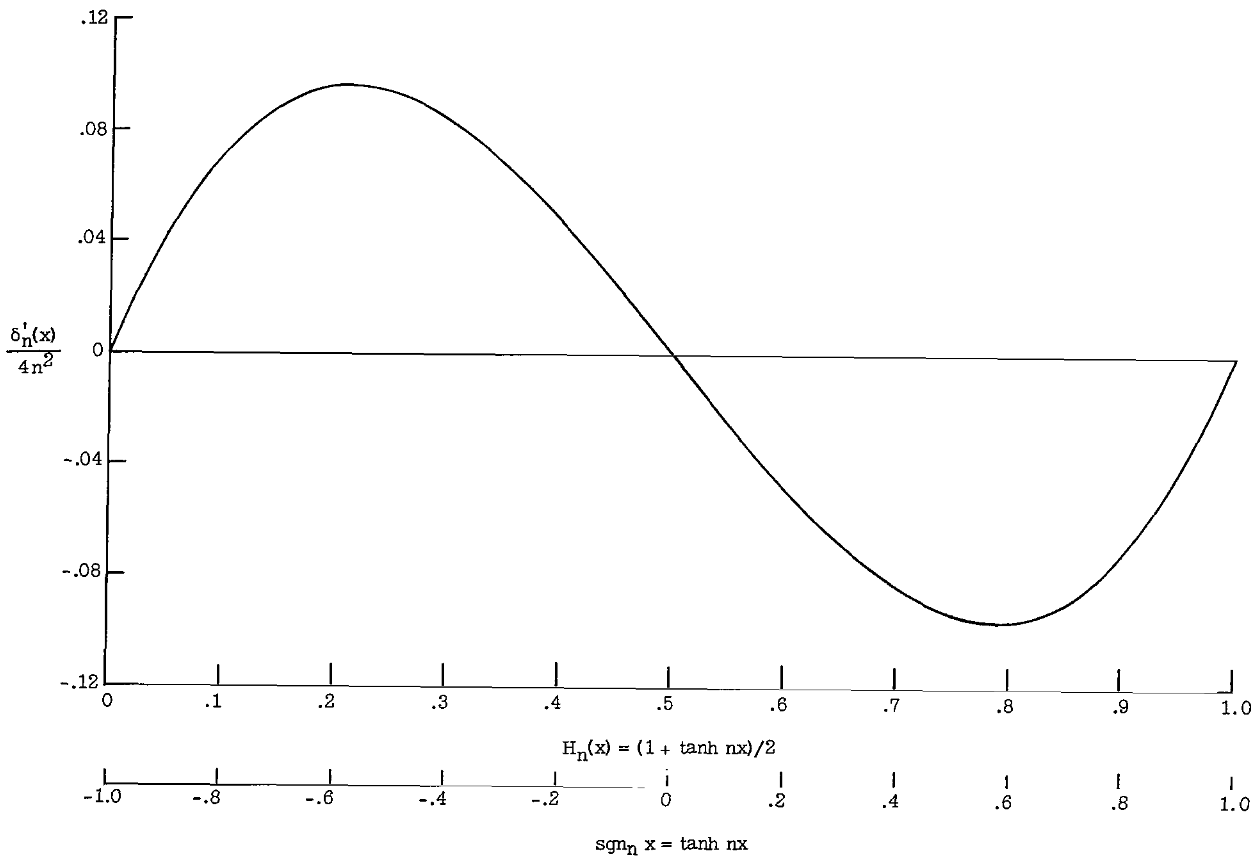
(b) Hyperbolic tangent representation for signum function $\mathrm{sgn}_n x$ plotted against x for various n.

Figure 1.- Hyperbolic tangent step function representations.



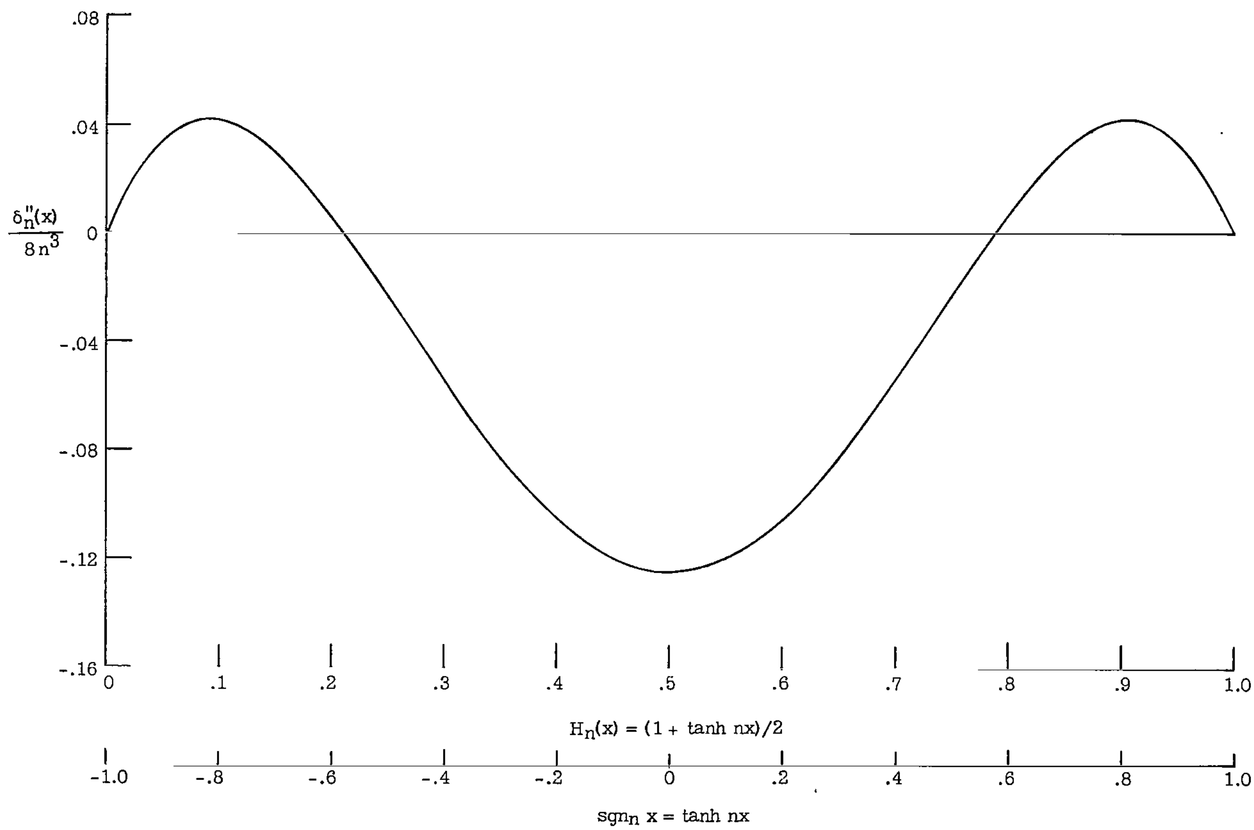
(a) $\gamma = 0$.

Figure 2.- Graphs of hyperbolic tangent generalized functions $\frac{\delta_n^{(\gamma)}(x)}{(2n)^{\gamma+1}}$ plotted against $H_n(x)$ and $\mathrm{sgn}_n x$ for $\gamma = 0$ to 4.



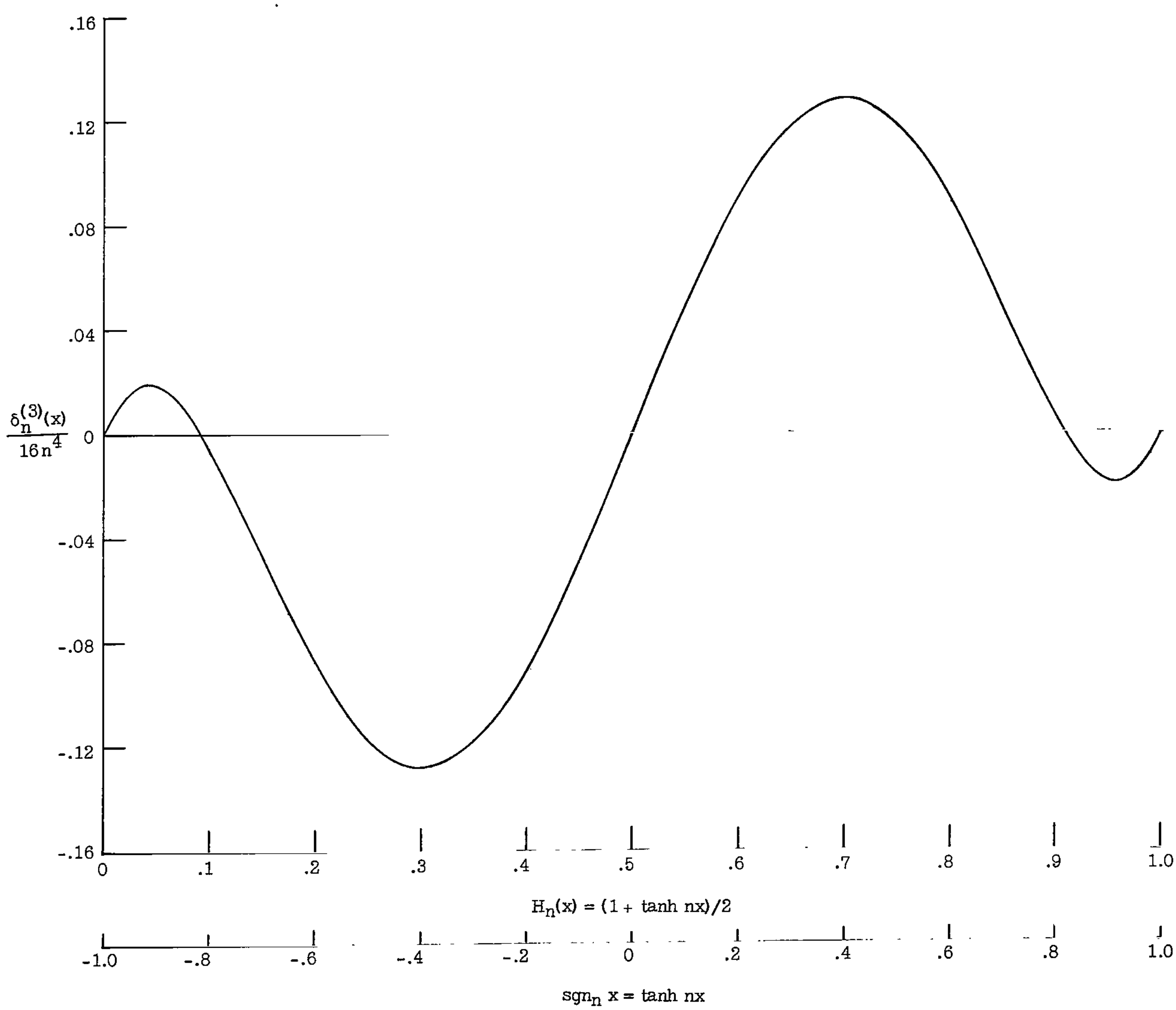
(b) $\gamma = 1.$

Figure 2.- Continued.



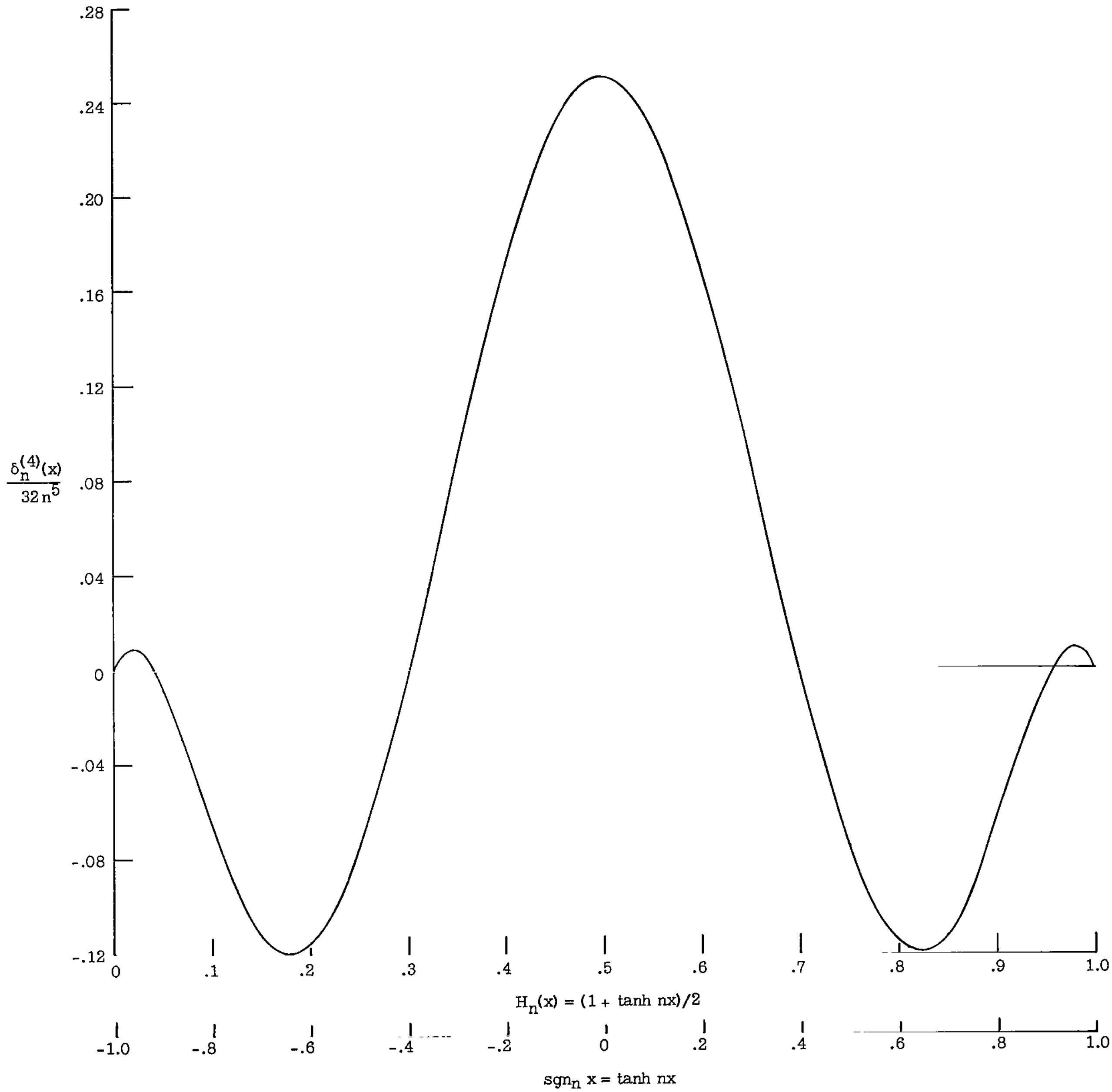
(c) $\gamma = 2$.

Figure 2.- Continued.



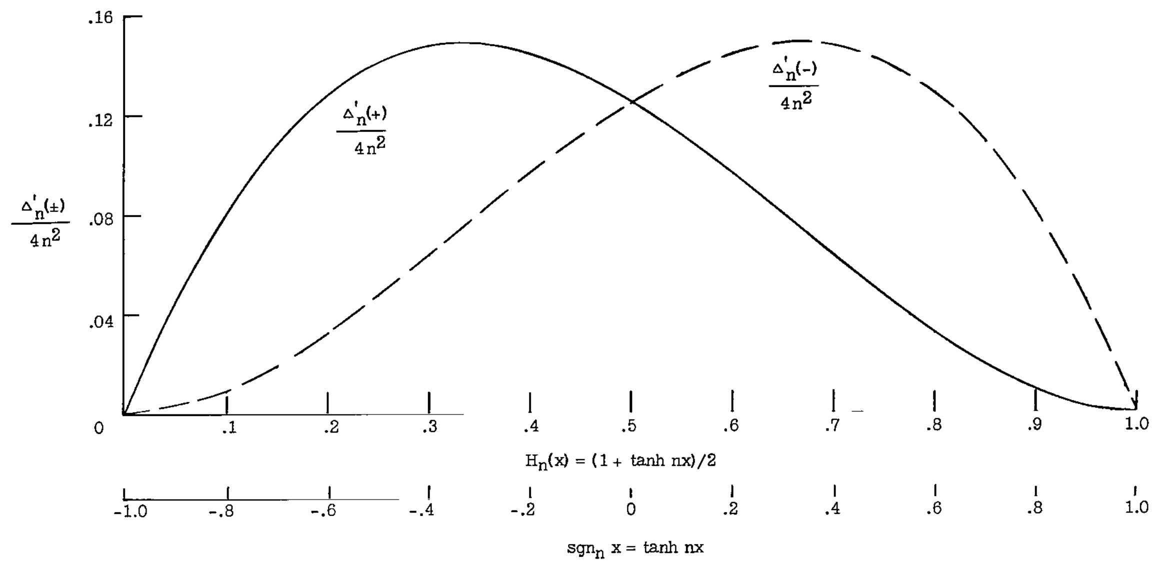
(d) $\gamma = 3$.

Figure 2.- Continued.

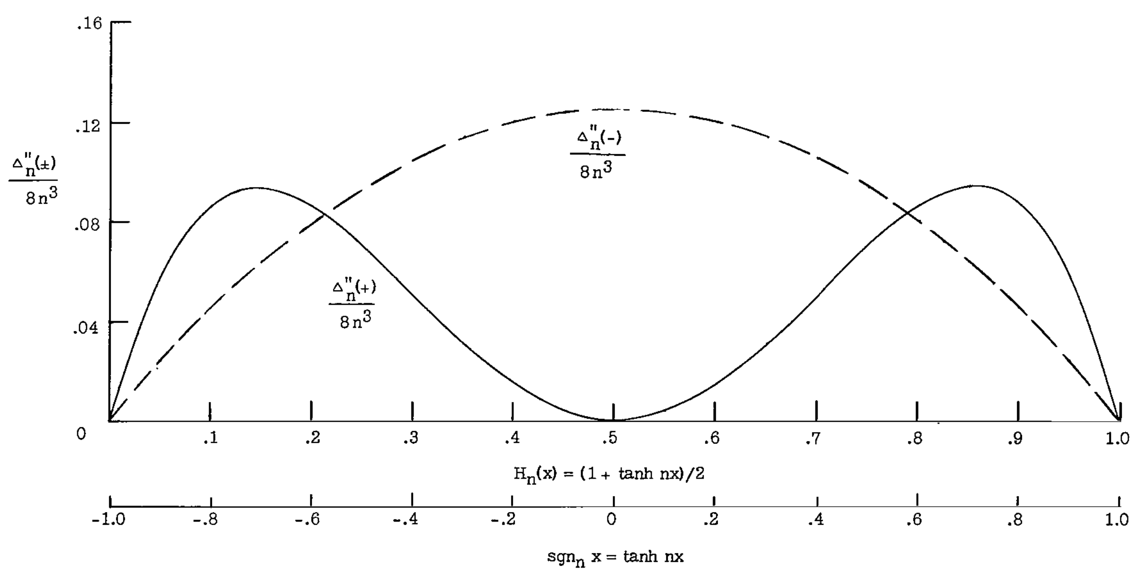


(e) $\gamma = 4$.

Figure 2.- Concluded.

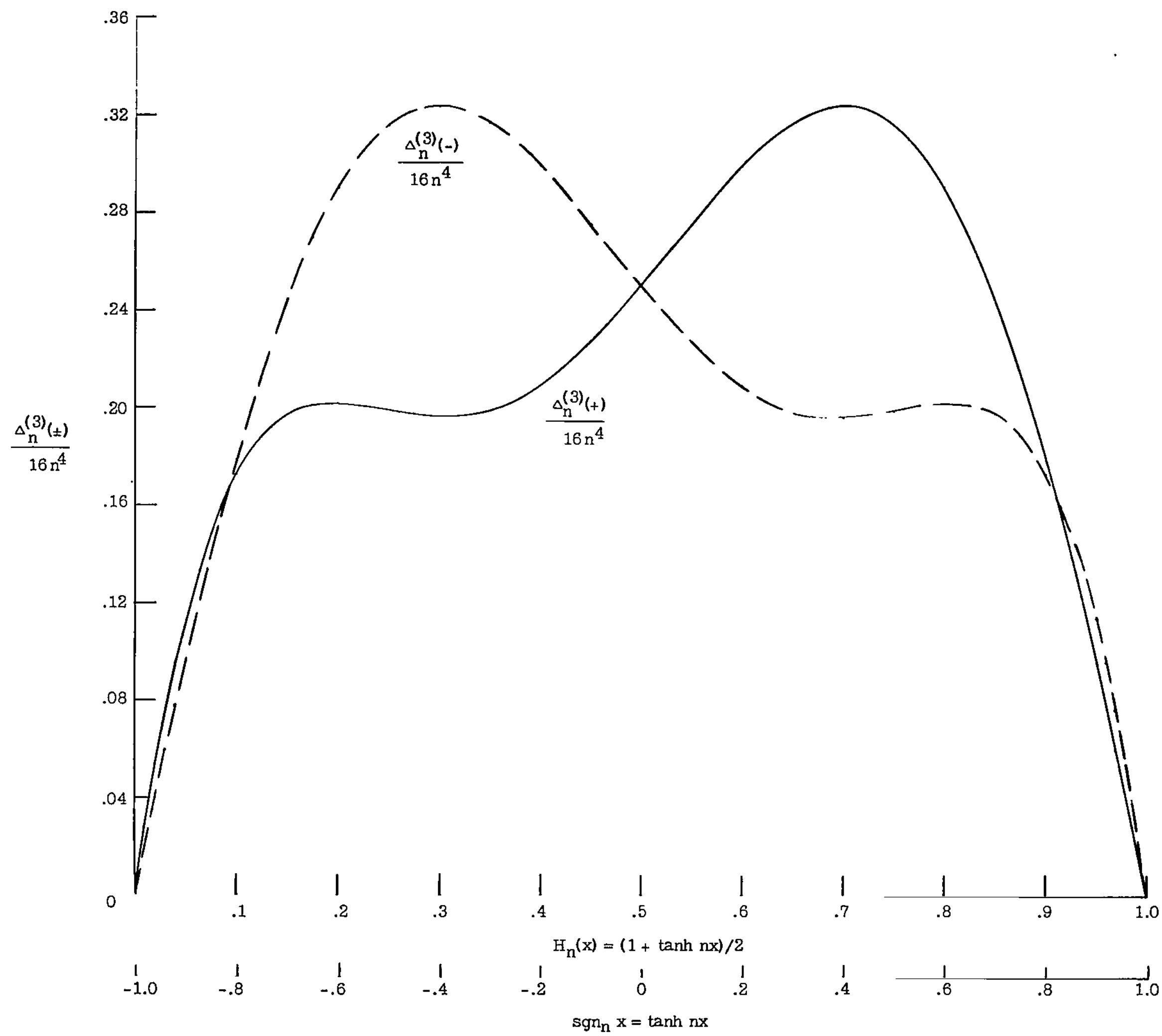


(a) $\gamma = 1$.



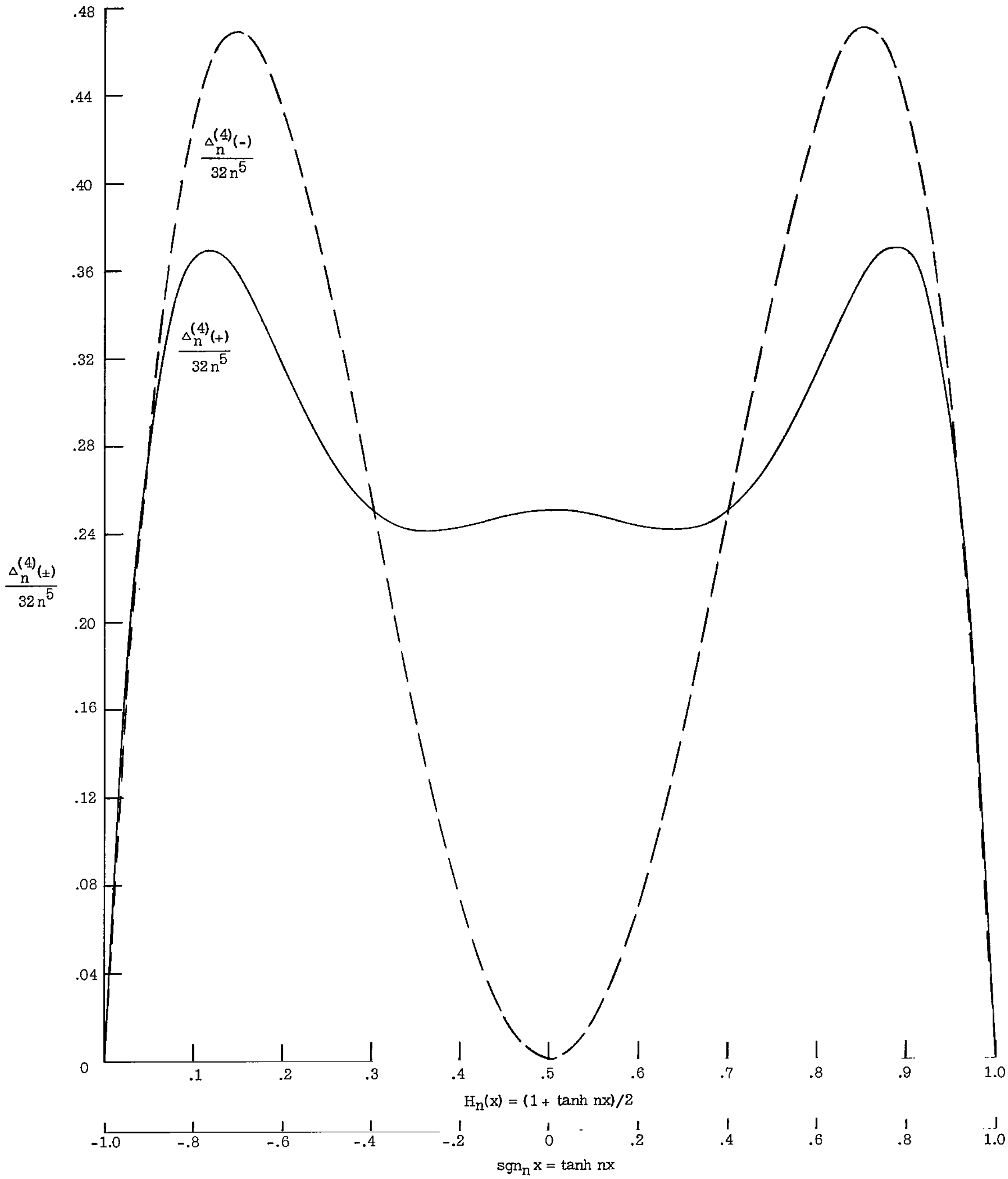
(b) $\gamma = 2$.

Figure 3.- Graphs of definite generalized function pairs $\dfrac{\Delta_n^{(\gamma)}(+)}{(2n)^{\gamma+1}}$ and $\dfrac{\Delta_n^{(\gamma)}(-)}{(2n)^{\gamma+1}}$ plotted against $H_n(x)$ and $\mathrm{sgn}_n x$ for $\gamma = 1$ to 4; where $\delta_n^{(\gamma)} = \Delta_n^{(\gamma)}(+) = \Delta_n^{(\gamma)}(-)$.



(c) $\gamma = 3$.

Figure 3.- Continued.



(d) $\gamma = 4$.

Figure 3.- Concluded.

NASA-Langley, 1967 —— 19 L-5351

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—National Aeronautics and Space Act of 1958

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