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STATISTICAL TESTS FOR SIGNAL DETECTION

by James C. Morakis

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

Some of the problems arising from maximum likelihood decision systems in time varying channels are solved by first introducing simple statistics which are functions of the outputs of the correlators of an n-ary detection system. Then methods are developed to use the statistics to a) attach a confidence level to each decision in a maximum likelihood decision system, b) control the error rate at the expense of data rejection, and c) define the critical region for an optimum generalized decision system. These improvements and optimizations are accomplished by taking advantage of the information already available in the sample representing the outputs of n-ary detection systems. The six statistics that are investigated are simple functions of the largest correlator output, the mean, the standard deviation, the sample mean, the sample variance, the next to the largest correlator output, and the smallest correlator output.

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INTRODUCTION

Present-day detection systems employ maximum likelihood decision for correlation detectors or ideal observer detectors for decisions on coded messages (References 1 through 4). Under certain conditions, maximum likelihood decision maximizes the probability of correct detection P_c, and, if the distributions of the signal and noise are known, P_c can be determined (Appendix B) for either coherent or noncoherent systems. For a gaussian channel, the outputs of the coherent correlation detector are normally distributed with means equal to zero or A and variance equal to σ^2, where $A/\sigma = \sqrt{2m\,E/N_0}$ and m is the length of the word.

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} ;$$

and

$$f_s(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-A)^2/2\sigma^2} ;$$

where $-\infty < x < \infty$.

Similarly, the distributions of the outputs of noncoherent detectors are

$$f_x(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$$

and

$$f_s(r) = \frac{r}{\sigma^2} e^{-(r^2+A^2)/2\sigma^2} I_0\left(\frac{rA}{\sigma^2}\right),$$

*This report was prepared previously as a thesis submitted to the Faculty of the Graduate School of the University of Maryland in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

where $0 \leq r < \infty$. Under the maximum likelihood decision scheme, the largest sample point is chosen as the useful signal, irrespective of its relative value with respect to the value of the other sample points; thus under the maximum likelihood decision scheme, $x_s = \max(x_1, x_2, \cdots x_n) = x_{(n)}$. This function is certainly simple and powerful, but, unfortunately, it does not take full advantage of the information contained in the n outputs.

For example, some of the observations that could supply additional information are the values of $x_{(n)}$, $x_{(n-1)}$, and $x_{(1)}$; the mean μ, the variance σ^2; the sample mean $\bar{x}$; the sample variance s^2; and a multitude of other functions of the foregoing sample. Obviously there is a limitation in regard to the number of statistics involving any of the $(x_1, \cdots, x_n)$; consequently, to choose relatively simple statistics that could extract most of the information contained in the $(x_1, \cdots, x_n)$, some of the tests used for outliers will be introduced (Reference 5). The intended application of outliers was to reject observation points that do not come from the same distribution as the rest of the sample points Since a measure was needed for the rejection of these points, each rejection was accompanied by a confidence level depending on the parent distribution of the good data. The outlier approach gives good results if the distribution of the contaminator (the signal in our case) is not known or if there is no indication that a signal is transmitted. However, if something about the signal is known, this knowledge can be used to modify the test for outliers and thus make it more powerful.

As was shown earlier, the maximum likelihood decision scheme can be supplemented by measuring some function of the outputs of the correlators and, on the basis of the value of this function, placing a confidence level on each maximum likelihood decision. One of the drawbacks of the maximum likelihood decision scheme is the inability to control the error rate. Coding systems that can correct up to t errors per block have word-error probability

$$P_w = \sum_{i=t+1}^{m} \binom{m}{i} P_e^{\,i} \left(1 - P_e\right)^{m-i} ,$$

where P_e is the bit-error probability, and m is the word length. If the error rate increases beyond a certain limit, the coding scheme breaks down, resulting in abnormal error rates (References 6 and 7). To insure an upper bound on the maximum probability of error, the relationships between thresholds and the corresponding maximum tolerable probabilities of error will be given. These thresholds will be functions of the observed outputs of the correlator and will separate the critical region from the acceptance region in a generalized decision scheme.

If there is no restriction on the maximum probability of error in the foregoing generalized decision scheme, the value of the threshold that will optimize the overall loss could be chosen provided that the relative loss functions that correspond to the different types of erroneous and correct decisions are known. It was previously mentioned that relatively simple statistics containing most of the information in the sample will be searched to use them either for determining confidence levels on maximum likelihood decisions or to use them as thresholds for more general decision schemes. Some simple statistics will be analyzed for comparison.

In summary, this report has four main objectives.

1. Attach a confidence level to each maximum likelihood decision, based on additional information that could be extracted from the data. This result could be used to determine the usefulness of the received data, the effectiveness of coding, and possibly the need for retransmission.

2. Provide a technique that defines the acceptance and critical regions, based on the maximum tolerable error rate or minimum probability of correct decision or any other requirement which is a function of the probabilities of error and correct decision.

3. Assign loss functions to the errors and correct decisions and optimize the decision scheme by finding the value of the statistic that minimizes the average risk function.

4. Examine some statistics (hereinafter referred to as threshold ratios) and determine their relative performance under conditions of varying signal strength, number of signals, and different distributions. These statistics do not require any new information besides the information already available in n-ary detection systems.

TESTING SIMPLE HYPOTHESIS—DECISION CRITERIA

In general two distinct areas of statistics are applied to communications—estimation and hypothesis testing (References 8 and 9). Estimation is employed by communications engineers for the extraction of signals from noise or for the estimation of one or more of the signal parameters. Hypothesis testing is employed for signal detection. This technique is employed when some information is known about the signal. The philosophy of the difference between estimation and hypothesis testing reveals a very important point—that one should search for tests that utilize as much of the information contained in the sample as possible.

Hypothesis testing is a decision process. The parameter space is divided into two subspaces. The parameter space under the null hypothesis H_0 is ω, and $\Omega - \omega$ is the parameter space under the alternative hypothesis H_1. The sample space X is also partitioned into the critical region X_c and the acceptance region $X - X_c$. The critical region is used to make the decision d on whether to accept or reject H_0. A typical example of choosing the decision space is to make decision d_1 (reject H_0) if $x \in X_c$ and d_0 (accept H_0) if $x \notin X_c$. Although this choice of d_0 and d_1 is the logical choice, one is not restricted to this decision combination (Reference 8). Thus, one can consider a number of different decision strategies and choose the strategy that proves to be the least costly. To accomplish this, a cost function must be assigned to each decision-parameter pair. In simple hypothesis testing, there are two decisions (two sample spaces) and two parameter spaces, thus yielding the following four cost functions.

$C_1(d_1; \theta)$ the cost of accepting H_1 when $\theta \in \Omega - \omega$

$C_2(d_1; \theta)$ the cost of accepting H_1 when $\theta \in \omega$

$C_3(d_0; \theta)$ the cost of accepting H_0 when $\theta \in \omega$

$C_4(d_0; \theta)$ the cost of accepting H_0 when $\theta \in \Omega - \omega$

If α is defined as the probability of rejecting H_0 when $\theta \in \omega$ and β the probability of rejecting H_0 when $\theta \in \Omega - \omega$, or $P(x \in X_c / \theta \in \Omega - \omega)$, then the expected cost (also referred to as the expected loss or as the risk function) is given by

$$r(d; \theta_0) = C_2 \alpha + C_3 (1 - \alpha); \; r(d; \theta_1) = -C_1 \beta + C_4 (1 - \beta) .$$

The term α is sometimes referred to as the size, the type I error, or the false alarm probability; $1 - \beta$ is referred to as the type II error, and β is the power of the test.

For example, consider the case where the signal is either gaussian noise or useful signal. The parameter space Ω consists of the two points $\mu = 0$, $\sigma^2 = 1$ and $\mu = A$, $\sigma^2 = 1$; ω is the point $\mu = 0$, $\sigma^2 = 1$; and $\Omega - \omega$ is the point $\mu = A$, $\sigma^2 = 1$. Letting the critical region be all points such that $x > A/2$ is equivalent to deciding that the signal is present if $x > A/2$, and that the signal is absent if $x \leq A/2$. Four different cases are thus generated:

$x > A/2$ and $\mu = A$,

$x > A/2$ and $\mu = 0$,

$x \leq A/2$ and $\mu = 0$, and

$x \leq A/2$ and $\mu = A$.

According to the decision scheme, it is decided that the signal is present in cases 1 and 2; this is correct in case 1 and incorrect in case 2. The probability of case 1 is designated as β, and the probability of case 2 as α; β is the probability of correctly accepting x as the signal, and α is the probability of incorrectly accepting x as the signal. It can be shown that, if the probability of case 1 is β and the probability of case 2 is α, then the probability of case 3 is $1 - \alpha$, and the probability of case 4 is $1 - \beta$. This simple hypothesis example can be described by using Table 1; the average probability of correct detection is

$$P_c = h\beta + \bar{h}(1 - \alpha) . \tag{1a}$$

Table 1

The Meaning of α and β in the Binary Case.

Decision	Parameter	
	$\mu = 0$	$\mu = A$
$x \leq A/2$ accept H_0, the hypothesis that no signal is present	$1 - \alpha$ probability of correctly deciding that no signal is present	$1 - \beta$ Probability (type II error)
$x > A/2$ accept H_1, the hypothesis that signal is present	Probability (type I error) $= \alpha$	Probability of correctly deciding that signal is present, power P_c

Similarly the average probability of error is

$$P_e = \bar{h}\alpha + h(1-\beta) , \tag{1b}$$

where $h = P(\mu = A)$, $\bar{h} = P(\mu = 0)$, and $h + \bar{h} = 1$.

The best test is the one that minimizes the average risk for all $\theta \in \Omega$. Unfortunately since this is not in general possible, one is usually satisfied in finding the best test for certain θ's (the ones of interest). Another problem with the expression given earlier for the expected cost is the assignment of values to C_i. In most of the cases, the problem can be simplified by setting $C_3 = C_1 = 0$ and obtaining

$$r(d; \theta_0) = C_2 \alpha; \; r(d; \theta_1) = C_4 (1-\beta) . \tag{2}$$

It appears that the risk can be minimized if α and $1-\beta$, the two types of errors, are minimized. The common procedure is to maximize β, the power, while α is kept below a certain value which is determined by C_2, C_4 and the expected value of β. Thus the best test is the one with the highest β for a given upperbound on α.

Tests are usually designed for maximum power (given the same α_{max}), and the search for a test is usually the search for a critical region which has some correspondence to the decision space. A criterion for finding the critical region for testing a simple hypothesis (a single null hypothesis and single alternative) is the Neyman-Pearson Criterion; it is a likelihood ratio inequality:

$$\frac{f(x_1, x_2 \cdots\cdots, x_n; \theta_1)}{f(x_1, x_2 \cdots\cdots, x_n; \theta_0)} > C \quad \text{for} \quad \theta \in \Omega - \omega .$$

If the expression for the expected loss can be minimized with respect to both α and β with the cost functions $C_4 = C_2$, $C_1 = C_3 = 0$, then this minimization is the Ideal Observer Criterion.

When the *a priori* probabilities of H_1 and H_0 are known (h and $\bar{h} = 1-h$ respectively), this knowledge can be used to advantage by properly weighing α and $1-\beta$ to minimize the average risk function, which is defined as

$$\mathcal{B}(d) = E[r(d; \theta)] = \bar{h}r(d; \theta_0) + hr(d; \theta_1) , \tag{3}$$

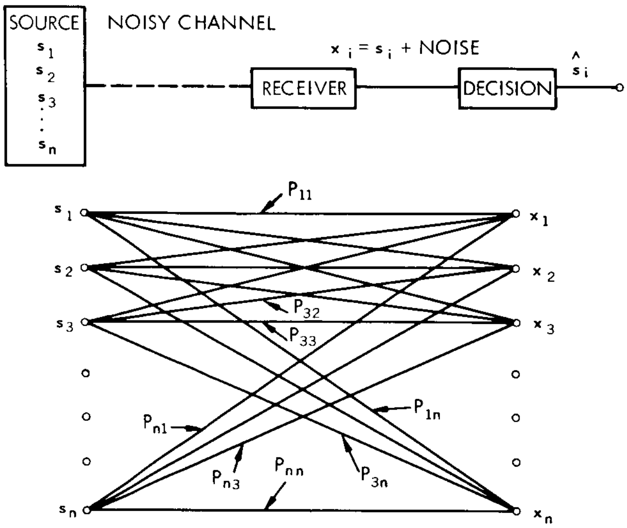
Equation 3 is possible because h is known and some kind of loss function is assigned to C_2, C_4 depending on the specific problem. Minimization of this expression is called the Bayes Strategy; it yields a solution for the decision criterion

$$\frac{P(x/\theta_1)}{P(x/\theta_0)} \left(\frac{h}{\bar{h}}\right) > \frac{C_2}{C_4} , \quad \text{where} \quad \theta \in \Omega - \omega .$$

This inequality defines the critical region; it should be noticed that this inequality is a weighted likelihood ratio.

N-ARY DETECTION: MAXIMUM LIKELIHOOD DECISION

An n-ary detection system is illustrated in Figure 1, where $s_1 \ldots\ldots s_n$ are the source symbols, and $x_1 \ldots\ldots x_n$ are the received symbols. The received symbols are random variables with a given probability density function depending on the detection scheme, the signal transmitted, and the channel noise distribution. A transition matrix (Figure 2a) is associated with the input symbols; each element of the transition matrix P_{ij} represents the probability that s_i will be received as x_j. Finally a decision process is defined whereby one or more x_j is associated with a given s_i.



$P_{ij} = \begin{cases} p \text{ FOR } i = j \\ q \text{ FOR } i \neq j \end{cases} \quad i = 1, 2, 3 \ldots\ldots\ldots, n.$

$p + (n-1)\, q = 1$

$P(s_1) = P(s_2) = \ldots\ldots\ldots\ldots\ldots = P(s_n)$

Figure 1—An n-ary detection system with a representation of the noisy channel illustrating the possible reception of a signal s_i as x_i and the associated conditional probabilities.

The transition matrix is a function of the channel noise and detection system, and the decision is a function of the average cost of error. To formalize this discussion, a cost matrix can be constructed as illustrated in Figure 2b, where C_{ij} is the cost of deciding that s_j was transmitted, when in reality s_i was transmitted. Thus the risk function, which is the average cost of making a decision d_j when s_j is observed, is

$$r(d_j) = \sum_i C_{ij}\, P(\hat{s}_i / x_j) \,. \tag{4}$$

This equation was obtained under the decision criterion that $s = \hat{s}_j$ was transmitted if x_j was received.

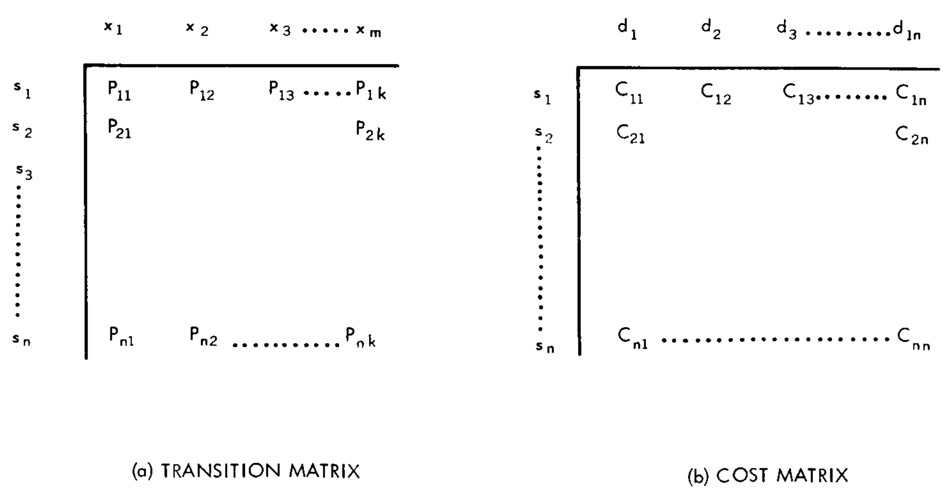


Figure 2—The transition and cost matrices of the channel of Figure 1.

The best decision criterion for this system is the one that minimizes the average risk for each j; but it becomes complex. Fortunately, for our application

$$C_{ij} = C\,\delta_{ij}\,, \text{ where } \delta_{ij} = \begin{cases} 0 & \text{for } i = j \\ 1 & \text{for } i \neq j \end{cases}, \tag{5}$$

because the cost of an erroneous decision is the same, irrespective of the signal decided on. When C_{ij} is given by Equation 5, this decision criterion is identified as the ideal observer decision scheme. In this case, the average risk becomes

$$E(r) = \sum_j P(x_j)\, r(d_j) = \sum_j P(x_j) \sum_{i,\, i\neq j} P(\hat{s}_i/x_j)$$

Now

$$\sum_i P(\hat{s}_i/x_j) = 1 \;;$$

thus

$$E(r) = \sum_j P(x_j)\left[1 - P(\hat{s}_j/x_j)\right] , \tag{6}$$

and $E(r) = 1 - P$ (correct detection). The left-hand side of Equation 6 is identified as the probability of error; thus, the ideal observer decision scheme minimizes the probability of error or maximizes the probability of correct detection; in other words, the ideal observer decision scheme, for each output x_j, chooses an input s that maximizes the inverse probability $p(s/x_j)$ because

$$p(x_j) = \sum_i P(s_i)\, P(x_j/s_i)$$

is a function of the *a priori* and transition probabilities only, and thus is independent of the decision scheme.

Using Bayes' rule for any given x gives

$$P(s_i/x) = \frac{P(s_i)\, P(x/s_i)}{P(x)} \,.$$

If one does not know much about the source *(a priori)* distribution, then one can assume $P(s_i) = 1/n$. For the foregoing case, maximizing the inverse probability is equivalent to maximizing $P(x/s_i)$.

Thus, for uniform *a priori* distributions, the ideal observer decision criterion selects the input s_i which maximizes the likelihood function $p(x/s_i)$; this is called the maximum likelihood decision scheme and is implemented by choosing the largest of the outputs as the signal transmitted. (The detector design depends on the probability density functions of the signals and of the channel noise and results in correlation detectors for gaussian noise and signals of known phase (Reference 10).)

TESTS FOR OUTLIERS

A class of tests applicable to the problems of n-ary detection (one out of n signals is the transmitted signal) is that of outliers (Reference 5). An outlier is an atypical point among a collection of points with the same characteristics. Although the outlier tests have been developed for the purpose of rejecting contaminators (i.e., sample points that come from another distribution), they can be used here to make a maximum likelihood decision (greatest of) in favor of the outlier whose distribution is $N(\mu_1, \sigma^2)$ in contrast with the remaining $n - 1$ sample points which come from the distribution $N(\mu_0, \sigma^2)$. The development of the different tests for outliers is based on order statistics. Depending on the application and the specific interest, the values of the remaining $n - 1$ samples (correlator outputs) could be utilized for more information if needed. In general, if none of the samples are destroyed in the process of detection, order statistics are the most powerful. However, for simplicity, only three or four out of n statistics could be used to extract most of the information. In this case, the power decreases, but it may still be high enough.

Tests for outliers may vary depending on the way that one regards the atypical point. Generally, the outlier is considered to come from a different distribution, and most of the time it is regarded as contamination. Consequently, the tests for outliers have been designed to decide whether a point(s) belongs to the same distribution as the rest of the data (with a given significance level) or whether it comes from a different sample space.

While tests for outliers are normally designed to search for a possible contaminator, in detection theory one knows that the outlier (signal) is present and must decide which point is the contaminator. The foregoing knowledge should be appropriately used either to find the proper outlier test or to extend the outlier theory for this specific application. Also, exactly $n - 1$ samples are drawn from the same distribution, and only one sample is drawn from another distribution. This information could be used similarly in parametric or nonparametric hypothesis testing for distribution or some other equivalent hypotheses.

SOME NEW STATISTICAL TESTS

Maximum likelihood decision is not the optimum scheme for all applications. By the use of the risk function, the introduction of certain statistical tests allow P_e and P_c to be controlled for optimum operation, based on the cost functions. The risk function also illustrates a method of comparing the performance of tests without the knowledge of the cost functions. On the basis of these results, it can be shown that for the applications where the maximum likelihood decision scheme is optimum, it can be further improved by using the aforementioned ratios for the determination of P_e. This will provide each maximum likelihood decision with a confidence level.

If the signal is not known or if the signal-to-noise ratio is very small, it is believed that outlier theory, with some modifications, can be successfully applied for establishing confidence levels. Obviously, in the extreme case where one does not know whether or not the signal is being received, outlier theory is applicable with no need for approximations.

A sample of points is usually represented by $(x_1, x_2, \cdots x_n)$, the subscripts signifying ordering in time or space. An ordered sample is represented by $(x_{(1)}, x_{(2)}, \cdots x_{(n)})$. In this case the subscripts signify the order of the magnitude in reference to the rest of the sample. For example, $x_{(i)}$ is the point with the ith highest value.

Maximum Likelihood Decision Scheme from the Hypothesis Testing Point of View

A measure for comparison of tests in simple hypothesis testing is the power β for a given α_{max}. For n-ary detection where the useful signal has the distribution $f_s(x_s)$ and the remaining $n-1$ noise signals have the distribution $f_x(x)$, we want to test the hypothesis that the largest signal $x_{(n)}$ is not the useful signal x_s (the useful signal is defined as the output of the correlator that corresponds to the transmitted signal; the noise signals are the outputs of any of the remaining $n-1$ correlator outputs). To simplify the hypothesis testing, let us confine the number of competing signals to two, i.e., the useful signal x_s and the largest of the noise signals x_L; thus,

$$H_0: \theta_n \in \omega \Rightarrow \left\{ \begin{array}{l} x_{(n)} \text{ has the distribution } f_L(x_L), \\ \text{i.e., } x_{(n)} \text{ is the largest of the noise} \\ \text{signals;} \end{array} \right\}$$

and

$$H_1: \theta_n \in \Omega - \omega \rightarrow \left\{ \begin{array}{l} x_{(n)} \text{ has the distribution } f_s(x_s), \text{ i.e.,} \\ x_{(n)} \text{ is the useful signal.} \end{array} \right\}$$

If $x_{(n)}$ falls in the critical region X_c, then hypothesis H_0 is rejected, and H_1 is accepted; X_c is part of the X space, the collection of all possible values that $x_{(n)}$ can take. Thus if $x_{(n)}$ can take all values from $-\infty$ to ∞, $X = \{-\infty, \infty\}$, let $X_c = \{B_T, \infty\}$, where B_T is some value on the x-axis. The foregoing decision scheme states that if $x_{(n)} > B_T$, then we reject H_0 and decide that $x_{(n)}$ is the useful signal and that if $x_n \le B_T$, we accept H_0. Unfortunately, although we decided that the largest signal $x_{(n)}$ is a noise signal, we cannot find which is the useful signal because this is a composite hypothesis. If this were a simple hypothesis with a simple alternative (H_0 and H_1 each involve one signal), then in accepting H_0 one could point out the signal. Thus in the case of n-ary detection, if $x_{(n)} > B_T$, we decide that $x_{(n)} = x_s$; if $x_{(n)} \le B_T$, then all we can say is that $x_{(n)}$ is not the useful signal; we shall call this action an indecision.

In view of the foregoing difficulty we shall redefine the probability of correct decision to be

$$P_c = h\beta \tag{7}$$

(instead of $P_c = h\beta + h(1-\alpha)$ as given by Equation 1a) and the probability of incorrect decision (error) as

$$P_e = \bar{h}\alpha \tag{8}$$

(instead of $P_e = \bar{h}\alpha + h(1-\beta)$ as given by Equation 1b) where

$$\alpha = P\left(x_{(n)} \in X_c / \theta_n \in \omega\right) , \tag{9}$$

$$\beta = P\left(x_{(n)} \in X_c / \theta_n \in \Omega - \omega\right) , \tag{10}$$

$$h = P\left(\theta_n \in \Omega - \omega\right) , \tag{11}$$

and

$$\bar{h} = P\left(\theta_n \in \omega\right) . \tag{12}$$

The remaining two terms of the binary P_c and P_e are $\bar{h}(1-\alpha)$ and $h(1-\beta)$. These two terms are combined to give the probability of no decision,

$$\begin{aligned} P_{ND} &= \bar{h}(1-\alpha) + h(1-\beta) = \bar{h} + h - \bar{h}\alpha - h\beta \\ &= 1 - P_e - P_c . \end{aligned} \tag{13}$$

Substituting Equations 9, 10, 11 and 12 into Equations 7 and 8 gives

$$P_c = P\left(x_{(n)} \in X_c , \theta_n \in \Omega - \omega\right) \tag{14}$$

and

$$P_e = P\left(x_{(n)} \in X_c , \theta_n \in \omega\right) . \tag{15}$$

Since $\theta_n \in \Omega - \omega$ implies $x_S > x_L$ and since $\theta_n \in \omega$ implies $x_L > x_S$, Equations 14 and 15 may be rewritten as

$$P_e = P\left(x_{(n)} \in X_c , x_L > x_s\right) \tag{16}$$

and

$$P_c = P\left(x_{(n)} \in X_c , x_s > x_L\right) . \tag{17}$$

The maximum likelihood decision scheme can be derived from the foregoing scheme by setting $X_c = X$. This sets the critical region equal to the whole outcome-space, which implies that $B_T = -\infty$ and $x_{(n)}$ always falls in the critical region. Consequently H_0 is always rejected, and the largest signal is the useful signal irrespective of the value of $x_{(n)}$. Thus,

$$P_e = P\left(x_{(n)} \in X_c, \theta_n \in \omega\right) = P\left(\theta_n \in \omega\right)$$

because $X_c = X$ implies that $P\left(x_{(n)} \in X_c\right) = 1$ and similarly

$$P_c = P\left(x_{(n)} \in X_c, \theta_n \in \Omega - \omega\right) = P\left(\theta_n \in \Omega - \omega\right) .$$

Since ω and $\Omega - \omega$ are disjoint sets, $P(\theta_n \in \Omega - \omega) = 1 - P(\theta_n \in \omega)$; this implies that for maximum likelihood decision $P_c = 1 - P_e$. P_c was found to be (Appendix B)

$$P_c = \int_{-\infty}^{\infty} f_s\left(x_s\right)\left[\int_{-\infty}^{x_s} f_x(x)\, dx\right]^{n-1} dx_s .$$

The foregoing analysis indicates that P_c is a function of P_e and that the probability of error is only dependent on the parameters of the signal and noise. Furthermore, the probability of correct detection becomes maximum, but no control can be exerted on the probability of error. This drawback of the maximum likelihood decision could be eliminated by returning to the more general problem of hypothesis-testing with a more restricted critical region.

A Decision Strategy Based on Thresholds

Let B stand for some ratio N/V where N is a function of the difference between $x_{(n)}$ and some function of the sample such as the mean or the next to the largest sample point, and V is some function of the spread of the sample such as range, standard deviation, or variance. The new critical region could then be formed by adding the condition that $B > B_T$ in order for H_0 to be rejected. Thus the critical region is now restricted to the interval $\left(B_T, \infty\right)$.

The hypotheses for n-ary detection become:

H_0: $\theta_n \in \omega$ where ω is the noise parameter space;

H_1: $\theta_n \in \Omega - \omega$ where $\Omega - \omega$ is the signal parameter space.

The critical region is the interval $\left(B_T, \infty\right)$; thus $x_{(n)} \in X_c$ can be replaced by $B_T < B_L < \infty$ in Equation 16 and by $B_T < B_s < \infty$ in Equation 17. The resultant equations are

$$P_e = P\left(B_L > B_T, x_L > x_s\right) \tag{18}$$

and

$$P_c = P\left(B_s > B_T, x_L < x_s\right) . \tag{19}$$

The new criterion could be used to lower P_e to any predetermined value; but at the same time it lowers P_c and increases the probability of *no decision* (accepting H_0) which is given by $P_{ND} = 1 - P_e - P_c$. This is a more general scheme than the maximum likelihood decision scheme (MLDS) because one can reduce it to an MLDS by assigning an appropriate value to B_T (such as $B_T = -\infty$). Thus by having control of P_e, we can adjust it, depending on the loss functions to get the best strategy. This implies that one can control the error rate at the expense of a lower correct detection rate and a higher indecision rate. By introducing cost values for error, for correct decision and for indecision, one can solve for the appropriate B_T that minimizes the average risk of Equation 3, which becomes

$$\begin{aligned} \mathbb{B} &= -C_1 h\beta + C_2 \bar{h}\alpha + C_3 \bar{h}(1-\alpha) + C_4 h(1-\beta) \\ &= -C_1 P_c + C_2 P_e + C_3 \left(1 - P_e\right) + C_4 \left(1 - P_c\right) - C_3 h - C_4 \bar{h} . \end{aligned} \tag{20}$$

Finally, monitoring B will give an indication of the strength of the transmitted signal, thus placing a confidence level on the decision which can be used as an indication of decreasing signal strength.

The threshold ratios of interest are (see Appendix C):

$$B_1 = \frac{x_{(n)} - \mu}{\sigma} ,$$

$$B_2 = \frac{x_{(n)} - \bar{x}}{\sigma} ,$$

$$B_3 = \frac{x_{(n)} - \bar{x}}{s} ,$$

$$B_4 = \frac{x_{(n)} - x_{(n-1)}}{\sigma} ,$$

$$B_5 = \frac{x_{(n)} - x_{(n-1)}}{s} ,$$

and

$$B_6 = \frac{x_{(n)} - x_{(n-1)}}{x_{(n)} - x_{(1)}} .$$

Some of these ratios can only be used under certain conditions. For example, B_1 needs the knowledge of μ and σ, and B_2 and B_4 need the knowledge of σ.

It should be noticed that the six ratios could be classified into two groups, one with the numerator equal to the difference between the largest order statistic and some measure of the mean, and the other having as the numerator the difference between the largest and the next to the largest order statistics. As will be proven later, the latter group is more powerful than the former because of the fact that for our application the means of the statistics depend heavily on the mean of the noise, which is assumed to be constant. Consequently, the information contained in either the mean or the sample mean is almost zero. On the contrary, the information contained in the next to the largest order statistic is very high when only one of the signals (the useful one) has a different distribution.

Of the first three ratios, B_1 is the ratio with the most information if μ and σ are known. If μ and σ are not known, then it is weak, relative to B_2 and B_3. B_2 is the next best because it assumes that σ is known. This ratio is more realistic in a practical situation because σ can be approximated by s_ν, where s_ν has been computed by independent (previous) samples, with ν very large (ideally infinite). B_3 is the studentized form and is the most powerful of the three if μ and σ are not known because it makes no assumptions about μ and σ. Thus if μ and σ are known, B_1, B_2, and B_3 is the order in which the ratios should be chosen; B_3, B_2, B_1 if μ and σ are not known, and B_2 is the best if only σ is known. In practical applications, μ and σ are not known, and B_3 is the most powerful.

Whatever holds for the relative performance of B_2 and B_3 holds for B_4 and B_5 since their denominators undergo the same change (σ to s). However, there is a basic difference between B_6 and the former in that B_6 is a function of only three samples. One would thus expect the information conveyed by B_6 to be much lower than that of B_4 or B_5. This is not true because the denominator of B_6 is the range which is very closely related to the variance, and also because finding $x_{(1)}$ requires knowing the values of the remaining statistics. Thus B_6 is expected to be almost as powerful as B_4 or B_5.

Comparative Performance of Maximum Likelihood Decision Scheme and a Threshold-Type Decision Scheme

The probability of error of a maximum likelihood detector is given by Equation B1 and depends solely on the signal-to-noise ratio; on the other hand, the probability of error in the threshold detector is decreased to a lower value by the extra condition that $B > B_T$; to illustrate this by means of an example, if a total of M decisions are made in a MLDS, then the number of correct decisions is $P_c M$, and the number of incorrect decisions is $P_e M$. To lower the number of erroneous decisions, let us specify that the decision that $x_s = x_{(n)}$ will be made only if $B > B_T$. This scheme will reduce the number of incorrect decisions to some value $P_{e_1}M$. This does not mean that the remaining $\left(P_e - P_{e_1}\right)M$ decisions will be correct. It merely means that the remaining $\left(P_e - P_{e_1}\right)M$ cases will be indecisions; i.e., we do not have strong enough evidence to reject H_0.

For the same reason, this strategy will lower the number of correct decisions $\left(P_c M\right)$ to a new number $\left(P_{c_1} M\right)$ and will generate $\left(P_c - P_{c_1}\right)M$ indecisions. Summarizing, the threshold B_T produced a gain by changing $\left(P_e - P_{e_1}\right)M$ of the errors to indecisions, and a loss by changing $\left(P_c - P_{c_1}\right)M$ of the correct decisions to indecisions. The total number of indecisions is thus

$$\left(P_e + P_c - P_{e_1} - P_{c_1}\right)M = \left(1 - P_{e_1} - P_{c_1}\right)M .$$

Letting $C_3 = C_4 = C_{ND}$ in Equation 20 gives

$$\mathcal{B} = -C_1 P_c + C_{ND}\left(1 - P_c - P_e\right) + C_2 P_e , \tag{21}$$

and the change in the average risk in the foregoing example is

$$\Delta\mathcal{B} = C_2\left(P_e - P_{e_1}\right) + C_{ND}\left(1 - P_{e_1} - P_{c_1}\right) - C_1\left(P_c - P_{c_1}\right) .$$

By assigning values to the loss functions C_1, C_{ND} (the cost of indecision), and C_2, one can optimize the gain by choosing the proper threshold B_T, which in turn determines $P_{e_1}(B_T)$ and the corresponding $P_{c_1}(B_T)$. In the binary case, this is known as the binary erasure channel where error detection and correction could be used to improve the system performance. To illustrate this, consider the average risk with $C_{ND} = 0$. Thus $\mathcal{B} = -C_1 P_c + C_2 P_e$. Differentiating $\mathcal{B}$ with respect to P_e and setting it equal to zero, reveals that the minimum risk occurs at the point where

$$\frac{\partial P_c}{\partial P_e} = \frac{C_2}{C_1} .$$

Now refer to Figure 3 which is a plot of P_c vs. P_e for a typical threshold decision scheme. The points along the plot signify the values of B_T to which the point (P_e, P_c) corresponds. As expected, the plot ends at the $P_c = 1 - P_e$ line, which is the locus of the maximum likelihood decision points (for additional cases, refer to Appendix E).

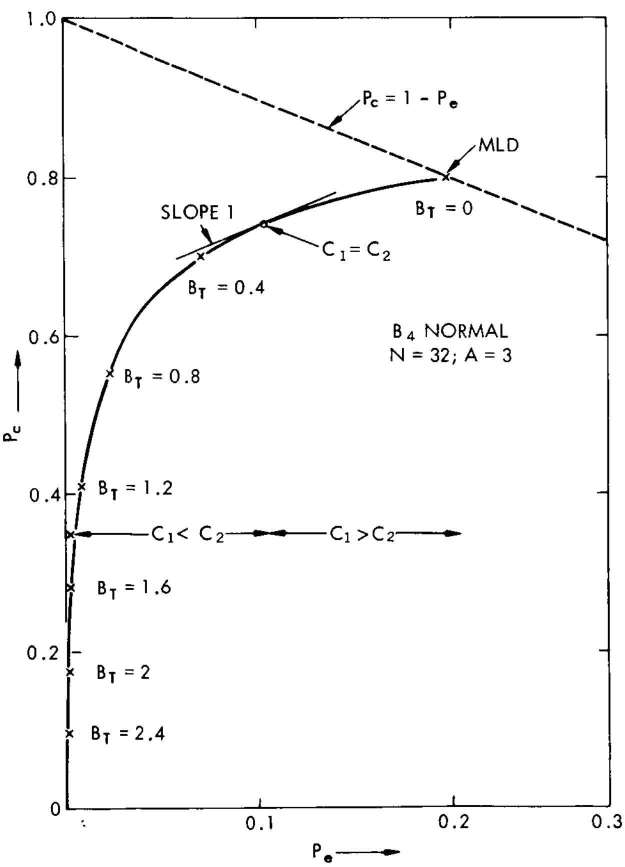


Figure 3—P_c vs. P_e for a typical threshold decision scheme.

Let us now optimize $\mathcal{B}$ for three pairs of cost functions:

1. $C_1 \gg C_2$; this is the case where one is mainly interested in maximizing P_c with no concern about P_e. This is equivalent to minimizing $1 - \beta$ (the type II error) or saying that the type II error is much more costly than the type I error. Under this condition, the optimum point is the one with a slope close to zero. Since the point with the lowest slope is the maximum likelihood decision point M the system becomes optimum at this point.

2. $C_1 = C_2$; in this case, the decrease in P_c is as costly as the increase in P_e. In the binary case, this condition assigns equal loss to both types of error (α and $1 - \beta$).

3. $C_1 < C_2$; in this case, one places more value on the loss due to P_e than the gain due to P_c, and is consequently willing to sacrifice detectability for lower error rates.

The foregoing example shows that the maximum likelihood decision scheme cannot be compared in general to a threshold decision scheme unless the loss functions are known. A much more important point is the ability to control the error with a minimum cost. Another application of the added restriction on the critical region is to use B for determining P_e. This technique conserves the desirable properties of maximum likelihood decision but adds a feature that maximum likelihood alone does not possess. This feature is the ability to know with a certain confidence whether signals are being received and possibly if the signal and noise distributions have changed. This last property can be used to advantage in the case of an adaptive system.

Relative Performance of the Threshold Ratios

The performances of two ratios may be compared without the knowledge of the cost functions, by comparing the P_c's for the same P_e. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are the risks of test no. 1 and test no. 2 respectively, then

$$\mathcal{B}_1 = C_e P_e + C_{ND}\left(1 - P_e - P_{c_1}\right) + C_c P_{c_1} ,$$

$$\mathcal{B}_2 = C_e P_e + C_{ND}\left(1 - P_e - P_{c_2}\right) + C_c P_{c_2} ,$$

and

$$\mathcal{B}_1 - \mathcal{B}_2 = \left(C_{ND} - C_c\right)\left(P_{c_2} - P_{c_1}\right) ;$$

$C_{ND} - C_c$ is positive because the cost of no decision is larger than the cost of correct detection. Thus

$$\mathcal{B}_1 - \mathcal{B}_2 = C\left(P_{c_2} - P_{c_1}\right)$$

if P_c of test no. 2 is larger than P_c of test no. 1, then $\mathcal{B}_2$ is smaller than $\mathcal{B}_1$, or test no. 2 is better than test no. 1 for the value of P_e for which P_c was calculated. This equation shows that P_c is a measure of the goodness of the test when P_c of each test is calculated for the same P_e. The procedure of finding P_c for all P_e's in the case of B_1 is as follows.

1. Find $g(B_L)$, the pdf of B_1 under H_0 and $x_L > x_s$.
2. Find $g(B_s)$, the pdf of B_1 under H_1 and $x_L < x_s$.
3. Find P_e and P_c from

$$P_e = \int_{B_T}^{\infty} g(B_L)\, dB_L ,$$

$$P_c = \int_{B_T}^{\infty} g(B_s)\, dB_s ;$$

P_e is the cross-hatched area in Figure 4, while P_c is the shaded area.

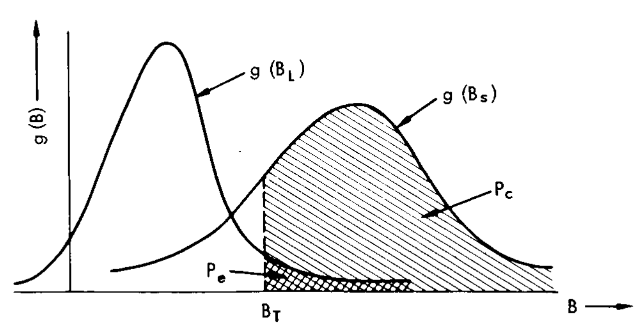


Figure 4—P_c and P_e as functions of threshold B_T.

Applications to N-ary Detection

It was mentioned previously that the maximum likelihood decision scheme could not be compared to the testing by ratios because in maximum likelihood decision, the critical region is the whole X space. This choice of critical region results in the single decision space; i.e., that $x_{(n)}$ is the useful signal. It was also shown that the MLDS is superior for some classes of applications, the main disadvantage being the complete disregard for the probability of error P_e. This disadvantage could be eliminated if one is willing to exploit information which is contained in the sample but which is not used by the MLDS. Given a sample of n outputs, the only information extracted by the MLDS is the maximum of $x_1, \cdots, x_n$. If in addition to this information, one utilizes some other information about the sample such as the value of the maximum $(x_1 \cdots x_n)$, the sample mean $\bar{x}$, the value of the next highest statistic, etc., one could be in a position to attach a confidence level to each maximum likelihood decision. Any of the statistics mentioned in the paragraph on decision strategy based on thresholds can be used to make a maximum likelihood decision and to determine P_e so that a confidence level $(1 - P_e)$ can be attached to each decision. Although the power of the different ratios differs, the confidence level does not depend on the power of the test, or on the probability of correct detection. Thus by adding the ratio test to the MLDS, in addition to the maximization of the probability of correct detection offered by the MLDS, one is given the opportunity to monitor the probability of error, a feature not provided by the MLDS.

The confidence level is a function of P_e which in turn is a function of the noise distribution, the signal distribution, and the value of the statistic B. In outlier theory, the distribution of the contaminator (signal) is not known. Consequently P_e depends only on the noise distribution and the statistic B. This type of testing could be applied if we do not know whether the signal is being transmitted and if we do not know $f_s(x_s)$. However, if we know anything at all about $f_s(x_s)$ or the fact that the signal is being transmitted, then this extra information could be used to modify the outlier testing to make it more powerful. Consequently the difference in decision between outliers and our application is that we shall always accept $x_{(n)}$ as the contaminator or signal, but we will calculate P_e each time, i.e., we will calculate the probability that $x_{(n)}$ could be a noise signal; then $1 - P_e$ is the confidence that we can place on our decision.

DISTRIBUTIONS

The distributions of the ratios are closely related to P_c and P_e. Thus the knowledge of P_c and P_e completely specifies the distribution of the ratios. The three equations of interest are

$$P_e\left[B_L, f_s(x_s), f_x(x)\right] = P(B_L > B_T, x_L > x_s) \, , \tag{22}$$

$$P_c\left[B_s, f_s(x_s), f_x(x)\right] = P(B_s > B_T, x_s > x_L) \, , \tag{23}$$

and

$$a_0\left[B_L, f_x(x)\right] = P(B_L > B_T) \, . \tag{24}$$

The first two are exact parameters and depend on the distribution of the useful signal. The third equation is only a function of the noise and does not depend on the signal. This is the type of equation that is used in outlier theory and is very useful for small signals or when one does not know whether or not the signal is present; this equation can also be used to make inferences regarding the signal. These equations are the basic equations that will be used for determining the average probability of error P_e and the average probability of correct detection P_c for both the exact and other cases. The threshold ratios given earlier will be compared by finding the P_c's for a given P_e.

It was previously mentioned that the confidence level of a maximum likelihood decision could be found by solving for the distribution of the confidence ratios. Each confidence ratio has two probability distribution functions, one under the correct decision and one under the incorrect decision. The former is called $G(B_s)$ and the latter $G(B_L)$; $G(B_L) = 1 - P_e$, and $G(B_s) = 1 - P_c$. The ratios of interest are, in general, functions of $x_{(n)}$, the largest order statistic; $x_{(n-1)}$, the next to the largest; $x_{(1)}$, the lowest; $\bar{x}$, the sample mean; s, the sample variance; and, of course, μ and σ if they are known. This implies that to solve for $G(B_L)$ and $G(B_s)$ one must know the distribution and parameters of the useful signal and the noise signals. For these distributions called $f_s(x_s)$ and $f_x(s)$, respectively, the null hypothesis is: the largest order statistic comes from the noise distribution $f_x(x)$. The null hypothesis is slightly more complicated by the fact that the useful signal x_s is smaller than the largest of the $n - 1$ noise signals x_L. Thus in calculating P_e, one could find the probability that x_L will be larger than c_1 (a constant), ignoring the fact that there is a useful signal present which is less than x_L, one could take into consideration the fact that x_s is smaller than x_L. The former approach is the outlier approach, while the latter results in the exact value of P_e and in general depends on the distribution and parameters of x_s.

Hypotheses Employed in the Two Approaches

The exact approach employs the hypotheses

$$H_0: \theta_n \in \omega: \text{ the parameter space of the } n-1 \text{ noise signals } x \text{ and}$$

$$H_1: \theta_n \in \Omega - \omega: \text{ the parameter space of the useful signal } x_s.$$

Thus H_0 is the hypothesis that $x_{(n)}$ came from the distribution of the $n - 1$ noise signals and that the useful signal is in the sample but is not the largest signal $x_{(n)}$. For this hypothesis, P_e and P_c become $P_e = P(B > B_T, x_L > x_s)$ and $P_c = P(B > B_T, x_L < x_s)$.

In the outlier approach, the hypothesis is $x_{(n)}$ came from the noise distribution of n noise signals with no reference to any useful signal. This is a good approach if one does not know if x_s is being transmitted or if the signal-to-noise ratio is very small; the hypotheses are

$$H_0;\ \theta_n \in \omega \quad \text{the parameter space of the } n \text{ noisy signals and}$$

$$H_1;\ \theta_n \in \Omega - \omega \quad \text{another parameter space, not the one of the } n \text{ signals.}$$

In this case, α_0 is the probability that the nth noise signal is such that $B > B_T$ and consequently the probability of incorrectly deciding that the nth largest noise signal did not come from the distribution of the remaining $n - 1$ signals; thus, $P_e = P\left[B > B_T \text{ for a given } f_x\left(x_{(n)}\right)\right]$. There is a fine point here, i.e., we started with a null hypothesis that all n signals are noisy, and, if $B > B_T$, we decide that only $n - 1$ signals are noisy and the nth largest is from another distribution. Obviously no use was made of the fact that x_s is one of those signals and that we cannot distinguish it because it is not the largest. As a result of this omission, the power of this outlier test is expected to be lower for the same α (i.e., α is expected to be higher for the same β).

The outlier approach gives a simple decision scheme which is independent of the distribution of the useful signal. Unfortunately, this method results in a loss of power because of the complete omission of x_s. A test having the desirable properties of the outlier tests, such as ignoring the distribution of x_s, but which does not ignore the presence of x_s is the described outlier test with n replaced by $n - 1$. The hypotheses of this modified outlier test are

H_0; $\theta_n \in \omega$ the parameter space of the $n - 1$ noise signals and

H_1: $\theta_n \in \Omega - \omega$ another parameter space not the one of the $n - 1$ noise signals and P_e is given by $\alpha = P\left(B > B_T | x_L = x_{(n)}\right)$.

The size and power can be expressed in terms of the distributions of B under correct and incorrect detection as

$$\alpha = P\left[B > B_T | G(B_L)\right] , \tag{25}$$

$$\beta = P\left[B > B_T | G(B_s)\right] , \tag{26}$$

$$\alpha = P\left(B > B_T | n \text{ noise signals}\right) , \tag{27}$$

and

$$\alpha = P\left(B > B_T | n - 1 \text{ noise signals}\right) . \tag{28}$$

Clearly Equation 28 acknowledges the presence of the useful signal x_s but does not utilize its distribution in solving for α; thus it is more powerful if it is known that the signal is present, and Equation 27 is preferred if no such knowledge exists.

Generalized Distributions of B

Although B could be any function of $(x_1, \cdots, x_n)$, the ratios of interest will be functions of μ, σ, s, $\bar{x}$, $x_{(1)}$, $x_{(n-1)}$, $x_{(n)}$ ($x_{(i)}$ is the ith order statistic; $x_{(1)}$ is the smallest; $x_{(n)}$ is the largest; and $x_{(n-1)}$ is the next to the largest).

In general, $B = Q(x_1, \cdots, x_n)$; if we apply a transformation such that

$$x_i = T_i\left(y_1, y_2, \cdots, y_{n-1}, B\right) \qquad i = 1, \cdots, n$$

with a Jacobian J, then the pdf of B, $g(B)$ becomes

$$g(B) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{R_{n-1}\,\text{dim space}} Q\left[x_{(1)} = T_1\left(y_1, \cdots y_{n-1}, B\right) x_{(2)} = T_2, \cdots x_{(n)} = T_n\right] |J| \, dy_1 \cdots dy_{n-1} \; .$$

To find $g(B_s)$, let $x_{(n)} = x_s$, $f\left(x_{(n)}\right) = f_s\left(x_s\right)$. Then

$$g(B_s) = \int_{-\infty}^{x_s = T_i} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} Q_s\left(x_{(1)} = T_1, \cdots, x_s = T_i, \cdots x_{(n-1)} = y_{n-1}\right) |J| \, dy_1 \cdots dy_{n-1} \; .$$

It should be noticed in the foregoing equation that, in order to incorporate the condition that $x_s > x_L = x_{(n-1)}$, it was necessary during the transformation to preserve $x_{(n-1)}$. Then the probability of correct detection $P_c\left(B_T\right)$ becomes

$$P_c\left(B_T\right) = P_c = \int_{B_T}^{\infty} g\left(B_s\right) dB_s \; .$$

The procedure of finding $f\left(B_L\right)$ is slightly more complicated because B_L is the ratio with $x_{(n)} = x_L$, where x_L is the largest noise signal; but the condition $x_L > x_s$ implies the union of the disjoint events $x_L > x_s > x_{L-1}$, $x_{L-1} > x_s > x_{L-2}$, $\cdots$, $x_{(1)} > x_s$. Thus letting $g_i\left(B_L\right)$ be the case $x_{(i-1)} < x_s < x_{(i)}$,

$$g_i\left(B_L\right) = \int_{-\infty}^{\infty} \cdots \int_{T_{i-1}}^{x_i = T_i} \cdots \int_{-\infty}^{\infty} Q_i\left(x_{(1)} = T_1, \cdots x_s = Y_s, x_i = T_i \cdots x_{(n-1)} = T_{n-1}\right) |J| \, dy_1 \cdots dy_{n-1} \; ,$$

and the probability of error becomes

$$P_e = \sum_{i=1}^{n-1} \int_{B_T}^{\infty} g_i\left(B_L\right) dB_L \; .$$

When B is a function of all the order statistics as in the case where $\bar{x}$ and s are involved,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and

$$s = \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2 \; .$$

The expressions for P_e and P_c are as complex as they look; this is the case of B_2, B_3, and B_5. When B is a function of one, two, or three order statistics such as in the case of B_1, B_4, and B_6, then the expressions for P_e and P_c become relatively simpler with only three to five integrations. For the outlier case, $f(B_L)$ is obtained by replacing $x_{(n)}$ by x_L and ignoring x_s, by assuming that all other signals came from $f_x(x)$. For the modified outlier case, $G(B_L)$ is obtained by letting n equal to k, the number of noise signals, and replacing $x_{(n)}$ by x_L.

The Distribution of Indecision

When the threshold decision is used, the sum of the probability of error and probability of correct decision is not in general equal to unity. Since $P_e + P_c$ represents the probability of decision, the probability of indecision is given by

$$P_{ND} = 1 - (P_e + P_c) .$$

Although P_{ND} can be solved from the foregoing equation, once P_e and P_c are known, it will be shown in this section that $P_{decision}$ where $P_{decision} = P_e + P_c$ can be found in a more direct way without the knowledge of P_e or P_c. This objective will be reached by the application of a theorem on the probability of the realization of at least one among n equiprobable events A_i.

From set theory, it can be shown (Reference 11) that the probability of the union of n events is

$$\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i}^{n} A_i - \sum_{i}\sum_{j}(A_i \cap A_j) + \sum_{i}\sum_{j}\sum_{k}(A_i \cap A_j \cap A_k) - + \cdots ,$$

if $P(A_i)$ is the same for all i, and $P(A_i \cap A_j)$ is the same for all i, j, and so on; then

$$\left[P \bigcup_{i=1}^{n} A_i\right] = n\,P(A_i) - \binom{n}{2}P(A_i \cap A_j) + \binom{n}{3}P(A_i \cap A_j \cap A_k) \cdots (-1)^n P(A_1 \cap A_2 \cdots A_n)$$

or

$$P \sum_{i}^{n} A_i = \sum_{i}^{n} (-1)^{i-1} \binom{n}{i} P_i , \tag{29}$$

where P_i designates the i-fold joint probability density function of the i out of n events. Equation 29 applies in the case where the density function of all statistics are the same.

If the pdf's are not all the same, the form is slightly different; for example, in the case where $n - 1$ statistics have the same pdf, and one statistic has a different pdf, Equation 29 becomes

$$P\left[\bigcup_{i=1}^{n} A_i \Bigg/ \begin{matrix}\text{a signal}\\ \text{is}\\ \text{present}\end{matrix}\right] = \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} \left[(n-1) P_i + P_{i-1,s}\right] , \tag{30}$$

where $P_{i-1,s}$ is the i-fold joint of the different pdf statistic with $i - 1$ out of $n - 1$ remaining statistics. For independent events, $P_i = \left[P(A_i)\right]^i$. Letting A_S be the event that $x_s > B_T$, and A_i the event that any of the $n - 1$ remaining signals is greater than B_T, Equation 30 will give the probability that any of the $n - 1$ noise signals and the useful signal will be greater than B_T. Thus for

$$A_S = \text{the event } \{x_s > B_T\},\ P(A_S) = \int_{B_T}^{\infty} f_s(x_s)\, dx_s \ ;$$

$$A_i = \text{the event } \{x > B_T\},\ P(A_i) = \int_{B_T}^{\infty} f_x(x)\, dx \ \text{ for } i = 2, 3, \cdots n .$$

Equation 30 gives the probability of decision $P_e + P_c$ under the exact hypothesis that $f_s(x_s)$ is known. If $f_s(x_s)$ is not known, the probability of decision is given by Equation 29 with $n = k$; k takes the values determined by the discussion in Equation 24.

The usefulness of Equations 29 and 30 can be extended to finding P_e or P_c if one of them is known. In certain cases, an approximate evaluation is possible with only one to three of the first terms of the series, but in others (especially for large n) the binomial coefficients become excessively large, thus introducing relatively large errors even when double-precision computation is used.

Summary

Three different procedures for setting a confidence level were outlined in this section. The first one was the exact method, which requires knowledge of the distribution of both the noise and the useful signals. In this case, P_e and P_c are given by Equations 22 and 23. The outlier case described the outlier technique, where the presence of the useful signal is completely disregarded. Since P_c depends on $f_s(x_s)$, one cannot find an expression for P_c unless one is willing to specify $f_s(x_s)$, an action that defeats the objectives of outlier theory (P_c can still be found experimentally for testing purposes or analytically for reason of comparison). In this case, α_0 is given by

$$\alpha_0 = P\left\{B > B_T \,|\, f\left[x_L(k)\right]\right\} ,$$

where $f(x)$ signifies the pdf of the n noise signals. The modified outlier case is a compromise which makes use of the fact that x_s is present by simply using the correct number of noise signals, i.e., $n = k$. This method still conforms with the outlier philosophy by disregarding the pdf of x_s.

In this case, α is given by

$$\alpha = P\left\{B > B_T \mid f\left[x_L (k-1)\right]\right\} .$$

THEORETICAL RESULTS

Obviously the aforementioned ratios belong to two distinct groups depending on the numerator function. The first group is the one where the numerator is the extreme deviate, $x_{(n)} - q(\bar{x})$, and the second has a numerator of the form $x_{(n)} - x_{(n-1)}$; obviously as $n \to \infty$, $\bar{x} \to \mu$, $s \to \sigma$, and the second and third statistics converge to the first statistic. There exists a similar relationship between B_4 and B_5. B_6 belongs to the same class as B_5 since the range is a measure of the standard deviation. The cdf of these ratios can only be obtained by numerical methods. B_2, B_3, and B_5 become even more complex with n-fold integrations because of the dependence of $\bar{x}$ and/or s on $x_{(n)}$.

Since we are interested in values of n ranging from 16 to 128, this could mean as many as 128-fold integrals for numerical integration. This prohibitive complexity and the fact that B_2 and B_3 converge in probability to B_1 as $n \to \infty$ and B_5 converges in probability to B_4 as $n \to \infty$, are considered to be sufficient justification to limit our investigation to B_1, B_4, and B_6. Thus by application of the equations given in the section on distributions we have derived (see Appendix C) the distributions for B_1, B_4, and B_6 in terms of the unspecified probability density functions.

When the channel noise is normally distributed, it can be shown (Reference 10) that the correlator outputs are normally distributed in the case of coherent detection; for the case of non-coherent detection, the pdf's are Rician. In this section, normal and Rician pdf's are substituted for the unspecified pdf's in order to derive the probability of error P_e and the probability of correct detection P_c as a function of the value of the threshold ratio B_T.

The Extreme Deviate B_1

For the coherent case, substituting $f_x(x) = Ce^{-x^2/2}$ and $f_s(x) = Ce^{-(x-A)^2/2}$ for $-\infty < x < \infty$ in Equations C7 and C8 gives

$$P_c = \int_{B_T}^{\infty} Ce^{-(x-A)^2/2} \left(\int_{-\infty}^{x} Ce^{-t^2/2} dt \right)^{n-1} dx \tag{31}$$

and

$$P_e = \int_{B_T}^{\infty} C(n-1) e^{-x^2/2} \left(\int_{-\infty}^{x} Ce^{-t^2/2} dt \right)^{n-2} dx \int_{-\infty}^{x} Ce^{-(x_s-A)^2/2} dx_s , \tag{32}$$

where $C = 1/\sqrt{2\pi}$. The outlier confidence level is from Equation D4:

$$1 - \alpha_0 = \left(\int_{-\infty}^{B_T} Ce^{-x^2/2}\, dx \right)^k , \tag{33}$$

where k is the number of noise signals.

For the noncoherent case, substituting $f_x(r) = re^{-r^2/2}$ and $f_s(r) = re^{-(r^2+A^2)/2} I_0(rA)$ for $0 \leq r < \infty$ in Equations C7 and C8 gives

$$P_c = \int_{B_T}^{\infty} re^{-(r^2+A^2)/2} I_0(rA) \left(1 - e^{-r^2/2}\right)^{n-1} dr \tag{34}$$

and

$$P_e = \int_{B_T}^{\infty} (n-1) \left(1 - e^{-r^2/2}\right)^{n-2} re^{-r^2/2}\, dr \int_0^r xe^{-(x^2+A^2)/2} I_0(xA)\, dx . \tag{35}$$

In this case the outlier confidence level becomes

$$1 - \alpha_0 = \left(\int_0^{B_T} re^{-r^2/2}\, dr \right)^k = \left(1 - e^{-B_T^2/2}\right)^k . \tag{36}$$

In general, the confidence level is in the form $1 - \alpha$, and once α is known for some B_T one can determine the confidence level by using Equations 32 or 35 if $f_s(x_s)$ is known, and Equations 33 or 36 if $f_s(x_s)$ is not known; thus $\alpha = P(B > B_T/x_s < x_L)$. Letting $k = n - 1$: $\alpha_0 = P(B > B_T)$. From these relationships, it is obvious that $\alpha_0 \geq \alpha$ or $1 - \alpha_0 \leq 1 - \alpha$.

Thus, if $f_s(x_s)$ is known, P_e can be found for the exact case; having found P_e one can find the confidence level $1 - P_e$. If $f_s(x_s)$ is not known, the outlier approach must be used, resulting in an overestimated α_0 and a conservative confidence level.

B_u

For the coherent case, substituting $f_s(x_s)$ and $f_x(x)$ in Equations C11 and C14 gives

$$P_c = \int_{B_T}^{\infty} (n-1)\, dB \int_{-\infty}^{\infty} C^2 e^{-(x-A)^2/2} e^{-(x-B)^2/2} \left(\int_{-\infty}^{x-B} Ce^{-t^2/2}\, dt \right)^{n-2} dx$$

and

$$P_e = \int_{B_T}^{\infty} (n-1)\, dB \left[\int_{-\infty}^{\infty} (n-2)\, C^2 e^{-x^2/2} e^{-(x-B)^2/2} \left(\int_{-\infty}^{x-B} Ce^{-t^2/2}\, dt \right)^{n-3} dx \int_{-\infty}^{x-B} Ce^{-(y-A)^2/2}\, dy \right.$$

$$\left. + \int_{-\infty}^{\infty} C^2 e^{-x^2/2} e^{-(x-A-B)^2/2} \left(\int_{-\infty}^{x-B} Ce^{-t^2/2}\, dt \right)^{n-2} dx \right] .$$

Similarly, substituting $f_x(x)$ in Equation D6 gives the confidence level in the outlier case:

$$1 - \alpha_0 = \int_{-\infty}^{B_T} k(k-1)\, dB \int_{-\infty}^{\infty} C^2 e^{-x^2/2} e^{-(x-B)^2/2} \left(\int_{-\infty}^{x-B} Ce^{-t^2/2}\, dt \right)^{k-2} dx .$$

For the noncoherent case, substituting $f_s(x)$ and $f_x(x)$ in Equations C11 and C14 gives

$$P_c = \int_{B_T}^{\infty} (n-1)\, dB \int_0^{\infty} r^2 e^{-(r^2+A^2)/2} I_0(rA)\, e^{-(r-B)^2/2} \left[1 - e^{-(r-B)^2/2} \right]^{n-2} dr$$

and

$$P_e = \int_{B_T}^{\infty} (n-1)\, dB \left\{ \int_0^{\infty} r^2 e^{-r^2/2} e^{-\left[(r-B)^2+A^2\right]/2} I_0(rA - BA) \left[1 - e^{-(r-B)^2/2} \right]^{n-2} dr \right.$$

$$\left. + \int_0^{\infty} (n-2)\, r^2 e^{-r^2/2} e^{(r-B)^2/2} \left[1 - e^{-(r-B)^2/2} \right]^{n-3} dr \int_0^{r-B} te^{-(t^2+A^2)/2} I_0(tA)\, dt \right\} .$$

By using Equation D6, the confidence level in the noncoherent case is

$$1 - \alpha_0 = \int_0^{B_T} k(k-1)\, dB \int_0^{\infty} r^2 e^{-r^2/2} e^{-(r-B)^2/2} \left[1 - e^{-(r-B)^2/2} \right]^{k-2} dr .$$

B_6

For the coherent case, substitution of $f_s(x)$ and $f_x(x)$ in Equations C16, C20, and D8 yields

$$P_c = \int_{B_T}^{1} (n-1)(n-2)\, dB \int_{-\infty}^{\infty} Ce^{-(x-A)^2/2}\, dx \int_0^{\infty} \left(\int_{x-w}^{x-Bw} Ce^{-t^2/2}\, dt \right)^{n-3} C^2 e^{-(x-w)^2/2} e^{-(x-Bw)^2/2}\, w\, dw ,$$

$$P_e = \int_{B_T}^{1} (n-1)(n-2)\, dB \Bigg[\int_{-\infty}^{\infty} Ce^{-x^2/2}\, dx \int_0^{\infty} C^2 e^{-(x-wB-A)^2/2} e^{-(x-w)^2/2}\, w \left(\int_{x-w}^{x-Bw} Ce^{-t^2/2}\, dt\right)^{n-3} dw$$

$$+ (n-3)\int_{-\infty}^{\infty} dx \int_0^{\infty} \left(\int_{x-w}^{x-Bw} Ce^{-t^2/2}\, dt\right)^{n-4} C^3 e^{-(x-w)^2/2} e^{-(x-wB)^2/2} e^{-x^2/2}\, w\, dw \int_{-\infty}^{x-wB} Ce^{-(y-A)^2/2}\, dy\Bigg] ,$$

and

$$1 - \alpha_0 = \int_0^{B_T} \frac{k!}{(k-3)!}\, dB \int_{-\infty}^{\infty} dx \int_0^{\infty} w \left(\int_{x-w}^{x-Bw} Ce^{-t^2/2}\, dt\right)^{k-3} C^3 e^{-\left[x^2+(x-w)^2+(x-Bw)^2\right]/2}\, dw ,$$

For the noncoherent case, substitution of $f_s(x)$ and $f_x(x)$ in Equations C16, C20, and D8 results in

$$P_c = \int_{B_T}^{1} (n-1)(n-2)\, dB \int_0^{\infty}\int_0^{\infty} r^3 e^{-\left[r^2+A^2+(r-w)^2+(r-wB)^2\right]/2} I_0(rA) \left(\int_{r-w}^{r-wB} te^{-t^2/2}\, dt\right)^{n-3} w\, dw\, dr ,$$

$$P_e = \int_{B_T}^{1} (n-1)(n-2)\, dB \Bigg[\int_0^{\infty}\int_0^{\infty} r^3 e^{-\left[r^2+(r-wB)^2+A^2+(r-w)^2\right]/2} I_0(rA - wBA)\, w \left(\int_{r-w}^{r-wB} te^{-t^2/2}\, dt\right)^{n-3} dw\, dr$$

$$+ (n-3)\int_0^{\infty}\int_0^{\infty} r^3 e^{-\left[r^2+(r-w)^2+(r-wB)^2\right]/2} \left(\int_{r-w}^{r-wB} te^{-t^2/2}\, dt\right)^{n-4} dw\, dr \int_0^{r-wB} ye^{-(y^2+A^2)/2} I_0(yA)\, dy\Bigg] ,$$

and

$$1 - \alpha_0 = \int_0^{B_T} \frac{k!}{(k-3)!}\, dB \int_0^{\infty}\int_0^{\infty} r^3 e^{-\left[r^2+(r-w)^2+(r-wB)^2\right]/2} \left(\int_{r-w}^{r-wB} te^{-t^2/2}\, dt\right)^{k-3} dw\, dr ,$$

Discussion

The majority of the equations derived so far are impossible to evaluate or reduce to closed form, especially if $n \geq 16$. For some limited values of n, A, and B_T, the cdf's can be expressed in terms of polynomial approximations (References 12 and 13), especially for combinations of n and A that result in the tails of the distributions. To obtain a complete picture of the dependence of P_e and P_c on n, A, and B_T, the expressions for P_e and P_c were numerically integrated for all combinations of A = 1, 2, 3, 4, 5, 6, 8, 10 and n = 16, 32, 64, 128. The ratios of interest were B_1, B_4, and B_6 for both the coherent and noncoherent detection; some of the results are tabulated in Tables E1 through E6. These tables were in turn used to plot Figures E1 through E6. Each figure contains plots of the probability of correct decision P_c vs. the probability of erroneous decision P_e for some interesting combinations of n and A.

It should be noticed that the detectability is degraded as the number of signals is increased; this result was expected and becomes obvious if Figure 5 is observed, where the pdf of the highest of n noise signals is plotted. (From the information point of view, the detectability must deteriorate as n is increased because, by increasing n, we increase the information rate, thus operating either closer to channel capacity or exceeding channel capacity.) It should be noticed that for $n = 127$, the mean of x_L (the highest noise signal) is roughly 2.5, and the standard deviation is roughly 0.5; this indicates that unless the useful signal has a mean greater than 2.5 the probability of error will be larger than the probability of correct detection on the average. This can be verified by looking up P_e and P_c for $A = 2$ and $A = 3$ with $n = 128$. It should also be noticed that as B_T increases, both P_e and P_c decrease and that the maximum P_e and P_c occurs for B_T zero or $-\infty$ (depending on the function defined by B), and they lie on the straight line defined by $P_c = 1 - P_e$. This is the line that represents P_e and P_c for the maximum likelihood decision scheme which is obviously a special case of the more general decision scheme proposed in this paper.

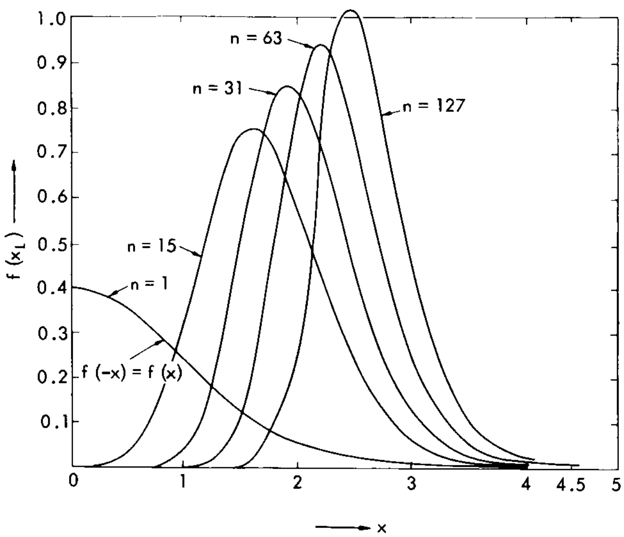


Figure 5—The pdf of the largest of n normally distributed, independent, random variables with zero mean and unity variance.

Some graphs representing B_1, B_4, and B_6 were superimposed in Figure E7 in order to illustrate the relative performance of these ratios. It can be seen from this figure that B_4 performs as well or better than B_1 and that B_6 performs as well as B_1. The difference in performance is not large enough to justify the preference of one ratio over the other based on performance alone. The choice of the ratio should be based on the amount and type of knowledge about the signals so that all assumptions are satisfied. For example, B_6 does not require any knowledge about μ and σ of the signals; B_4 requires knowledge of σ, and B_1 requires knowledge of both μ and σ.

Applications

The plots of Appendix E can be used in many ways depending on the type of problem. If one is only interested in maximum likelihood decision scheme, then by observing the value of B, one can attach a confidence level on the decision. For example, consider coherent detection using the maximum likelihood decision scheme with $A = 2$, $n = 32$, $B = B_4$. Suppose that we made a maximum likelihood decision and that $x_{(n)} = 5$, $x_{(n-1)} = 4.6$, and $\sigma = 1$; then $B_4 = (5\text{-}4.6)/1 = 0.4$. Thus by making use of the information contained in the value of $x_{(n)}$ and $x_{(n-1)}$ we can say that $x_{(n)}$ is the useful signal with a confidence about 82.5 percent ($P_e = 0.2$).

If errors in excess of one in ten cannot be tolerated (this could be the case where an error correction code will break down, and it is preferable to retransmit if the error rate is larger than 0.1), assuming $A = 3$, $n = 32$, and the ratio $B = B_6$ noncoherent, refer to Figure E6 and look up B_T for $P_e = 0.1$; extrapolating, $B_T = 1.6$; thus we can set a threshold and either accept $x_{(n)}$ as the useful signal if $B_6 > 1.6$, or ask for retransmission if $B_6 \leq 1.6$ (of course we need the values of $x_{(n)}$, $x_{(n-1)}$, and $x_{(1)}$ to calculate B_6). If the values of the cost functions C_1, C_2, C_3, and C_4 are known, the decision system can be optimized by minimizing the average risk:

$$\begin{aligned} \mathcal{B} &= -C_1 P_c + C_2 P_e + C_3 (1-P_e) + C_4 (1-P_c) - C_3 h - C_4 \bar{h} \\ &= -P_c (C_1 + C_4) + P_e (C_2 - C_3) + C_3 \bar{h} + C_4 h \,. \end{aligned}$$

Since $\mathcal{B}$, P_e, P_c are functions of B_T, differentiate this equation with respect to B_T and set it equal to zero. The result is

$$\frac{\partial P_c}{\partial P_e} = \frac{C_2 - C_3}{C_1 + C_4} \,.$$

This result indicates that the optimum threshold B_T for any $B(B_1, B_2, \cdots B_6)$ is the point on the appropriate curve which has a slope equal to $(C_2 - C_3)/(C_1 + C_4)$. Since the curves are monotonic, the solution is unique. Since the maximum likelihood decision points lie on the $P_c = 1 - P_e$ line, obviously they do not in general represent the point of optimum operation. For example, let $C_3 = C_4 = 0$ and $C_1 = C_2$; then $(C_2 - C_3)/(C_1 + C_4) = 1$, and the optimum B_T is the one corresponding to unity slope, which for B_1 coherent occurs at the points indicated in Figure 3. It can be seen that for the cost functions mentioned, the maximum likelihood decision scheme is optimum for $A \geq 4$; however, for values of $A \leq 3$, the maximum likelihood decision scheme is not optimum (the optimum points are for $A = 3$, $n = 32$; $B_T \approx 2.0$; for $A = 3$, $n = 64$; $B_T \approx 2.4$; for $A = 3$, $n = 128$, $B_T \approx 2.8$; for $A = 2$, $n = 16$: $B_T \approx 2.0$; for $A = 2$, $n = 32$: $B_T \approx 2.6$ and so on). The last result shows that only the relative magnitudes of C_1, C_2, C_3, and C_4 are sufficient for the determination of the point of optimum operation.

EXPERIMENTAL RESULTS

As was mentioned earlier, the solutions of P_e and P_c for the ratios B_2, B_3, and B_5 become impractical; it was shown, however, that for large n, the distributions of these ratios converge to those of the ratios discussed in the section on theoretical results (B_1, B_4, and B_6). Since the n's of interest ($n = 16, 32, 64, 128$) are fairly large, the performance of B_2 and B_3 is expected to be comparable to that of B_1 and the performance of B_5 comparable to that of B_4. To substantiate this argument a simulation was conducted where normally distributed numbers were generated with mean zero and variance one; half of the histogram of 100,000 numbers from this distribution is shown on Table 2. It can be seen that it is very close to the theoretical histogram.

Table 2

The Histogram of the Random Numbers Which Were Generated for the Simulation.

Interval	Exper	Theor	Interval	Exper	Theor	Interval	Exper	Theor	Interval	Exper	Theor
0-0.04	1607	1595	1-1.04	948	949	2-2.04	207	207	3-3.04	15	17
0.04-0.08	1588	1593		909	910		187	191		15	15
0.08-0.12	1590	1587		869	871		175	177		12	13
0.12-0.11	1586	1580		825	833		159	161		12	12
0.11-0.2	1568	1571	1.16-1.2	797	796	2.16-2.2	148	149	3.2-3.24	9	10
0.20-0.24	1562	1557		760	758		136	135		9	9
0.24-0.28	1549	1543		716	721		122	125		9	8
0.28-0.32	1525	1525		687	686		112	113		6	7
0.32-0.36	1509	1507		650	650		103	103		6	6
0.36-0.4	1489	1484	1.36-1.4	625	616	2.36-2.4	95	94	3.4-3.44	6	5
	1452	1461		578	583		87	85		3	5
	1444	1436		549	549		76	78		6	4
	1408	1408		519	518		69	70		3	4
	1376	1379		489	488		63	63		3	3
0.56-6	1349	1349	1.56-1.6	457	458	2.56-2.6	57	57	3.6-3.64	3	3
	1316	1316		429	430		52	52		0	3
	1276	1284		406	402		45	46		4	2
	1252	1248		375	376		43	42		3	2
	1215	1214		352	352		36	37		0	2
0.76-0.8	1178	1177		330	327	2.76-2.8	36	34	3.8-3.84	0	2
	1141	1140		307	304		27	30		3	2
	1100	1103		282	284		27	27		0	1
	1064	1064		265	262		25	24		0	1
	1031	1026		244	243		21	21		0	1
0.96-1	991	987	1.96-2	221	225	2.96-3.0	18	19	3.96-4.00	3	1

A sample of n of these numbers was drawn, and the first one was given a bias A, $(x_s = x_1 + A)$, resulting in $n - 1$ numbers from $N(0, 1)$ and one from $N(A, 1)$. This experiment was repeated 1000 times, and each time the ratios R_1, R_2, R_3, R_4, R_5, R_6, $\bar{x}$, s, etc. were calculated, and histograms were made, both for the case where $x_s = x_{(n)}$ and when $x_s < x_{(n)}$ (in the latter case the largest signal is the largest noise signal x_L). The foregoing histograms were converted into cdf's. Thus the value of the resulting cdf was identified with $1 - P_c$ when $x_{(n)} = x_s$ and $1 - P_e$ when $x_{(n)} = x_L$. Parts of the experimental results are tabulated in Tables E7 to E18, and the corresponding graphs were plotted in Figures E8 to E19 to show the deviation of B_2, B_3, and B_5 from B_1 and B_4.

This experiment was also repeated by generating a Rician distribution. It should be noticed from the appropriate tables or plots that the agreement between the theoretical and experimental results for B_1, B_4, and B_6 is excellent. A further look into the plots containing B_1, B_2, B_3 on the same graph and B_4, B_5, B_6 on the same graph substantiates the argument that the performances of B_1, B_2, and B_3, are comparable and so are those of B_4, B_5, and B_6. The foregoing results were obtained for all combinations of $n = 16, 32, 64, 128$, and $A = 1, 2, 3, 4, 5, 6, 8, 10$; however, since for $A \geq 5$ the probability of error approaches zero and the probability of correct detection approaches unity, graphical representation for the cases $A \geq 5$ became difficult; consequently the cases for which $A \geq 5$ were omitted.

CONCLUSION

The aims of this paper have been to introduce some functions of the observations at the correlator outputs which could be used to place confidence levels on our decisions, control the error rate or determine the optimum decision scheme with the aid of loss functions and to analyze the performance of the aforementioned functions (ratios) in order to obtain numerical results for the first objective, and to compare their relative performance. A generalized decision scheme was introduced. The decision space consisted of two points—d_1: $x_{(n)}$ is the useful signal, and d_2: $x_{(n)}$ is not the useful signal. The decisions depended on the value of a function of the correlator outputs B. If $B > B_T$, the decision was d_1; if $B \leq B_5$, the decision was d_2. The probability of error is P_e for d_1 and $1 - P_c$ for d_2. The probability of correct decision is P_c for d_1 and $1 - P_e$ for d_2; the probability of d_1 is $P_e + P_c$, and the probability of d_2 is $1 - P_e - P_c$. Since the decision that $x_{(n)}$ is not the useful signal is an action that leads to no decision, d_2 is an indecision for our specific application, and $1 - P_e - P_c$ is the probability of indecision. Similarly, $P_e + P_c$ is the probability of decision that $x_{(n)}$ is the useful signal; P_e is the probability of erroneously selecting $x_{(n)}$ as the useful signal; and P_c is the probability of correctly selecting $x_{(n)}$ as the useful signal. If the loss function C_1, C_2, C_3, C_4 is known, the value of B_T that results in the optimum decision scheme can be determined graphically from the P_c vs. P_e plots by choosing the point on the appropriate curve with slope $(C_2 - C_3)/(C_1 + C_4)$. On the other hand, if one's only concern is to keep the probability of erroneous decision P_e below a certain value P_{e_T}, then B_T is the B that corresponds to this value of P_e, P_{e_T}. If B_T is set equal to $-\infty$, then the probability of d_1 becomes unity, and the probability of d_2 becomes zero. Thus

$$P_e + P_c = 1 \quad \text{and} \quad 1 - P_e - P_c = 0 .$$

The probability of error in d_1, (P_e) is equal to the probability of error in d_2, $(1 - P_c)$, if there was a d_2 (but $P(d_2) = 0$ with $B_T = -\infty$). This value of B_T results in a degenerate case of the generalized decision scheme that we started with, and it is identified as the maximum likelihood decision scheme because under the maximum likelihood decision scheme we always make decision d_1 irrespective of the value of B (the critical region $B_T < B < \infty$ contains all values of B when $B_T = -\infty$). As can be observed from the graphs, this technique results in the maximum P_c but also maximum P_e, with the probability of indecision $1 - P_e - P_c$ equal to zero. Since, under the maximum likelihood decision scheme, we have no control on B_T and consequently on the probability of error, the only improvement that we can add to this technique is to calculate B for each decision and, by searching the graphs or tables for the corresponding value of P_e, to determine the confidence level of each decision by calculating $1 - P_e$. This number could be used for determining the degree of the validity of decisions and or assumptions about the signal and noise parameters. Finally the results indicate that the performance of B_4 is as good or better than that of B_1, which was expected because it was assumed that the parameters μ and σ were known. If μ is not known, then B_4 is superior. If nothing is known about μ or σ^2 of the noise, then B_6 should be used. This ratio is a function of only three statistics $x_{(1)}$, $x_{(n-1)}$, and $x_{(n)}$; it is relatively simple to calculate, and performed almost as well as B_4, the ratio with the best performance.

Finally the experimental results confirmed the theoretical results on B_1, B_4, and B_6 and substantiated the argument that for $n \geq 16$, B_2 and B_3 perform as well as B_1, and B_5 performs as well as B_4. The difference in power of the six ratios was so small that the preference of one ratio over another should not be based on performance alone but rather on the satisfaction of the assumptions associated with each ratio and the simplicity of calculating the ratio.

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Appendix A

LIST OF SYMBOLS

α: The significance level (also known as the type I error).

P_e = Probability of error.

β: The power (also known as the probability of rejecting H_0/H_1).

P_c = The probability of correct detection.

ϵ Means "an element of."

$\not\epsilon$ Means "not an element of."

Ω Is used for the parameter space.

ω Is used for the parameter space under H_0.

H_0: The null hypotheses; unless otherwise specified, H_0 is the hypothesis that the largest signal is noise.

H_1: The alternative (hypothesis); unless otherwise specified, H_1 is the hypothesis that the largest signal is the useful signal.

$N(a, b^2)$ Means normal distribution with mean a and variance b^2.

$x_{(i)}$: The ith order statistic.

x_L: The largest of the $n - 1$ noise signals.

B_T: The value of the ratio that separates the acceptance from the critical region.

pdf: Abbreviation for probability density function.

cdf: Abbreviation for cumulative probability distribution function.

$f_x(x)$, $f_s(x)$: pdf's of the noise and useful signal respectively.

$F_x(x)$, $F_s(x)$: cdf's of the noise and useful signals respectively.

E: The average signal energy per bit.

N_0: The noise power density per unit bandwidth.

$\bar{x}$: The sample mean.

s: The sample standard deviation.

n, N: The number of words.

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Appendix B

PROBABILITY OF ERROR FOR THE MAXIMUM LIKELIHOOD DECISION SCHEME

Let x_1, x_2, x_{n-1} from $f(x)$ and x_s from $g(x)$. Let x_i be the ith largest sample point from $f(x)$. The probability density function of x_L, the largest sample point from $f(x)$, is (Reference 14)

$$h(x_L)\,dx = (n-1)\left[F(x)\right]^{n-2} f(x)\,dx \,, \tag{B1}$$

where

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \,;$$

since

$$f(x)\,dx = dF(x) \,,$$

Equation B1 can be written as

$$h(x_L)\,dx = h(x_{n-1})\,dx = (n-1)\left[F(x)\right]^{n-2} dF(x) = d\left[F(x)\right]^{n-1} \,. \tag{B2}$$

The probability of correct detection in maximum likelihood decision is equal to the probability that the largest of the noise signals x_L is less than the useful signal x_s. Thus

$$\begin{aligned} P_c &= \int_{-\infty}^{\infty} g(x_s) \int_{-\infty}^{x_s} d\left[F(x)\right]^{n-1} dx_s \\ &= \int_{-\infty}^{\infty} g(x_s)\left[F(x_s)\right]^{n-1} dx_s = \int_{-\infty}^{\infty} g(x_s)\left[\int_{-\infty}^{x_s} f(x)\,dx\right]^{n-1} dx_s \,. \end{aligned} \tag{B3}$$

The probability of error $P_e = 1 - P_c$. It should be noticed that Equation B3 could be derived from Equation C7 by letting $B_T = -\infty$.

Coherent Detection

If

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty ,$$

and

$$g(x_s) = \frac{1}{\sqrt{2\pi}} e^{-(x_s-A)^2/2} \qquad -\infty < x_s < \infty ,$$

then

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x_s-A)^2/2} \, dx_s \left[\int_{-\infty}^{x_s} f(x) \, dx \right]^{n-1} .$$

Results of this integral have been tabulated (Reference 3) for several combinations of n and A.

Noncoherent Detection

If

$$f(r) = re^{-r^2/2} \qquad 0 \le r < \infty$$

and

$$g(r) = re^{-(r^2+A^2)/2} \, I_0(rA) \qquad 0 \le r < \infty ,$$

then

$$P_c = \int_0^{\infty} re^{-(r^2+A^2)/2} \, I_0(rA) \left(\int_0^{r} te^{-t^2/2} \, dt \right)^{n-1} dr$$

$$= \int_0^{\infty} re^{-(r^2+A^2)/2} \, I_0(rA) \left(1 - e^{-r^2/2}\right)^{n-1} dr$$

$$= \int_0^{\infty} re^{-(r^2+A^2)/2} \, I_0(rA) \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-kr^2/2} \, dr$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} re^{-\left[r^2(1+k)+A^2\right]/2} I_0(rA) \, dr .$$

By using new variables r', A' such that $r'A' = rA$ and $r^2(1+k) = r'^2$,

$$r\sqrt{1+k}\,A' = rA \ ,$$

$$A' = \frac{A}{\sqrt{1+k}} \ ,$$

and

$$P_e = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{e^{-A^2 k/2(1+k)}}{1+k} \int_0^{\infty} r' e^{-\left(r'^2 + A'^2\right)/2} I_0(r'A')\,dr' \ ;$$

$$P_c = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{e^{-kA^2/[2(1+k)]}}{1+k} \ .$$

Appendix C

DISTRIBUTIONS OF B_1, B_4, AND B_6

The expressions for the probability of error P_e and the probability of correct decision P_c are derived for each of the ratios mentioned in the section on new statistical tests by using the results of the section on distributions. The first statistic is the ratio of the difference between the largest sample point and the population mean to the population standard deviation. Since population mean and variance is used, it has been assumed in this case that μ and σ are known; thus

$$B_1 = \frac{x_{(n)} - \mu}{\sigma}, \tag{C1}$$

$$B_{1s} = \frac{x_s - \mu}{\sigma}, \tag{C2}$$

and

$$B_{1L} = \frac{x_L - \mu}{\sigma}. \tag{C3}$$

Without loss of generality, one can let $\mu = 0$ and $\sigma = 1$; the statistics involved here are x_L and x_s. Since x_L and x_s are independent,

$$\begin{aligned} f(x_L, x_s) &= f_L(x_L) f_s(x_s) \\ &= (n-1)\left[\int_{-\infty}^{x_L} f_x(t)\,dt\right]^{n-2} f_x(x_L) f_s(x_s), \end{aligned} \tag{C4}$$

where $f_x(t)$, $F_x(t)$ are the noise pdf and cdf respectively, and $f_s(t)$, $F_s(t)$ are the signal pdf and cdf respectively. Applying the transformation

$$x_s = B_{1s};$$

$$|J| = 1,$$

gives

$$f(B_{1s}) = \int_{-\infty}^{B_{1s}} (n-1) f_s(B_{1s}) f_x(x_L)\left[\int_{-\infty}^{x_L} f_x(t)\,dt\right]^{n-2} dx_L. \tag{C5}$$

Identifying

$$(n-1)\, f_x\left(x_L\right) \int_{-\infty}^{x_L} f_x\,(t)\,dt^{\;n-2}\; dx_L \tag{C6}$$

as

$$d\left\{\left[F_x\left(x_L\right)\right]^{n-1}\right\}$$

gives

$$f\left(B_{1s}\right) = f_s\left(B_{1s}\right) \int_{-\infty}^{B_{1s}} d\left\{\left[F_x\left(x_L\right)\right]^{n-1}\right\} = f_s\left(B_{1s}\right)\left[F_x\left(B_{1s}\right)\right]^{n-1} \; ; \qquad -\infty < B_{1s} < \infty$$

and

$$P_c = \int_{B_T}^{\infty} f\left(B_{1s}\right) dB_{1s} = \int_{B_T}^{\infty} f_s\left(B_{1s}\right)\left[F_x\left(B_{1s}\right)\right]^{n-1} dB_{1s} \; ; \tag{C7}$$

P_e is the probability that $B_L > B_T$ and $x_L > x_s$. Applying the transformation

$$x_L = B_L \; ;$$

$$|J| = 1$$

and using Equation C4 gives

$$f\left(B_{1L}\right) = \int_{-\infty}^{x_L = B_L} (n-1)\, f_x\left(B_L\right)\left[F_x\left(B_L\right)\right]^{n-2} f_s\left(x_s\right) dx_s \; ; \qquad -\infty < B_{1L} < \infty$$

and

$$P_e = \int_{B_T}^{\infty} f\left(B_{1L}\right) dB_{1L} = (n-1)\int_{B_T}^{\infty}\int_{-\infty}^{B_L} f_x\left(B_L\right)\left[F_x\left(B_L\right)\right]^{n-2} f_s\left(x_s\right) dx_s\, dB_L$$

$$= \int_{B_T}^{\infty} (n-1)\, f_x\left(B_L\right)\left[F_x\left(B_L\right)\right]^{n-2} dB_L \int_{-\infty}^{B_L} f_s\left(x_s\right) dx_s \, . \tag{C8}$$

The first ratio of the second group is the one having σ in the denominator, thus assuming that σ is known. Thus,

$$B_4 = \frac{x_{(n)} - x_{(n-1)}}{\sigma} . \tag{C9}$$

For $\sigma = 1$,

$$B_{4s} = x_s - x_L$$

and

$$B_{4L} \begin{cases} = Y = x_L - x_{L-1} , & \text{if} \quad x_s < x_{L-1} ; \\ = Z = x_L - x_s , & \text{if} \quad x_{L-1} < x_s < x_L . \end{cases} \tag{C10}$$

Applying the transformation,

$$x_s = x ;$$

$$x_L = x - B_{4s}$$

$$|J| = 1 .$$

Using the results of the section on distributions and Equation C4 gives

$$f(B_{4s}) = \int_{-\infty}^{\infty} f(x_s = x , x_L = x - B_{4s})\, dx$$

$$= \int_{-\infty}^{\infty} (n-1) f_s(x) f_x(x - B_{4s}) \left[\int_{-\infty}^{x-B_{4s}} f_x(t)\, dt \right]^{n-2} dx \qquad -\infty < B_{4s} < \infty$$

and

$$P_c = \int_{B_T}^{\infty} (n-1)\, dB \int_{-\infty}^{\infty} f_s(x) f_x(x - B_{4s}) \left[\int_{-\infty}^{x-B_{4s}} f_x(t)\, dt \right]^{n-2} dx . \tag{C11}$$

To find P_e, we have to use the joint density of x_s, x_L, and x_{L-1} which is (References 11, 14)

$$f(x_L, x_{L-1}, x_s) = f(x_L, x_{L-1}) f_s(x_s) ,$$

$$= (n-1)(n-2) \left[F_x(x_{L-1}) \right]^{n-3} f_x(x_L) f_x(x_{L-1}) f_s(x_s) ,$$

where

$$-\infty < x_{L-1} < \infty, \qquad -\infty < x_L < \infty, \quad \text{and} \quad -\infty < x_s < \infty .$$

If $x_s < x_{L-1}$, apply the transformation

$$x_L = x$$

$$x_{L-1} = x - Y ;$$

$$|J| = 1 ;$$

thus

$$P_{e_1} = \int_{B_T}^{\infty} f\left(Y, x, x_s\right) dY$$

$$= \int_{B_T}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{x_{L-1}=x-Y} f\left(x_L = x, x_{L-1} = x - Y, x_s = x_s\right)|J|\, dx_s\, dx\, dY$$

$$= \int_{B_T}^{\infty} dY \int_{-\infty}^{\infty} (n-1)(n-2)\left[F_x(x-Y)\right]^{n-3} f_x(x) f_x(x-Y)\, dx \int_{-\infty}^{x-Y} f_s\left(x_s\right) dx_s . \qquad \text{(C12)}$$

If $x_{L-1} < x_s < x_L$, apply the transformation

$$x_L = x$$

$$x_s = x - Z ;$$

$$|J| = 1 ;$$

thus

$$P_{e_2} = \int_{B_T}^{\infty} f\left(Z, x, x_{L-1}\right) dz = \int_{B_T}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{x_s=x-Z} f\left(x_L = x, x_{L-1} = t, x_s = x - Z\right)|J|\, dt\, dx\, dZ$$

$$= \int_{B_T}^{\infty} dZ \int_{-\infty}^{\infty} (n-1)(n-2) f_x(x) f_s(x-Z)\, dx \int_{-\infty}^{x-Z} f_x(t)\left[F_x(t)\right]^{n-3} dt$$

$$= \int_{B_T}^{\infty} dZ \int_{-\infty}^{\infty} (n-1) f_x(x) f_s(x-Z)\left[F_x(x-Z)\right]^{n-2} . \qquad \text{(C13)}$$

Of course,

$$P_e = P_{e_1} + P_{e_2} . \tag{C14}$$

The last ratio to be theoretically investigated is B_6. Although this ratio appears to depend on only three statistics, it is a function of all n order statistics because they must all be known in order to find $x_{(1)}$, $x_{(n-1)}$, and $x_{(n)}$. Since

$$B_6 = \frac{x_{(n)} - x_{(n-1)}}{x_{(n)} - x_{(1)}} ,$$

when $x_s = x_{(n)}$,

$$B_{6s} = \frac{x_s - x_L}{x_s - x_{(1)}} . \tag{C15}$$

In this case, we need $f\left(x_L, x_{(1)}\right)$ which is

$$f\left(x_L, x_{(1)}\right) = (n-1)(n-2)\left[F_x\left(x_L\right) - F_x\left(x_1\right)\right]^{n-3} f\left(x_{(1)}\right) f\left(x_L\right) ; \qquad -\infty < x_{(1)} < \infty, \qquad -\infty < x_L < \infty .$$

Applying the transformation

$$\left.\begin{aligned} x_s &= x \\ x_s - x_{(1)} &= w \\ x_s - x_L &= wB_{6s} \end{aligned}\right\} \rightarrow \begin{aligned} x_s &= x \\ x_1 &= x - w \\ x_L &= x - wB_{6s} \end{aligned}$$

$$|J| = |w|$$

gives

$$P_c = \int_{B_T}^{1} f\left(B_{6s}\right) dB_{6s}$$

$$= \int_{B_T}^{1}\int_{-\infty}^{\infty}\int_{0}^{\infty} (n-1)(n-2) f\left(x_1 = x - w,\ x_L = x - wB_{6s},\ x_s = x\right) |w|\, dw\, dx\, dB_{6s} .$$

The upper limit of the first integral is 1 because $0 \le B_6 \le 1$, and the lower limit of the last integral is zero because the range w is always non-negative; making the proper substitutions:

$$P_c = \int_{B_T}^{1} (n-1)(n-2)\, dB_{6s} \int_{-\infty}^{\infty} f_s(x)\, dx \int_{0}^{\infty} \left[\int_{x_{(1)}=x-w}^{x_L=x-wB} f_x(t)\, dt\right]^{n-3} f_x(x-w)\, f_x(x-wB)\, w\, dw . \tag{C16}$$

For the P_e's we shall need $f(x_{(1)}, x_{L-1}, x_L)$ which is

$$f(x_{(1)}, x_{L-1}, x_L) = (n-1)(n-2)(n-3)\left[F_x(x_{L-1}) - F_x(x_{(1)})\right]^{n-4} f_x(x_{(1)})\, f_x(x_{L-1})\, f_x(x_L)$$

for $x_{(1)}, x_L, x_{L-1}$ in $(-\infty, \infty)$. Thus

$$B_{6L} = \begin{cases} Y = \dfrac{x_L - x_{L-1}}{x_L - x_{(1)}}, & \text{if } x_s \le x_{L-1} \text{ and} \\[2ex] Z = \dfrac{x_L - x_s}{x_L - x_{(1)}}, & \text{if } x_{L-1} \le x_s \le x_L . \end{cases} \tag{C17}$$

If $x_s \le x_{L-1}$ apply the transformation

$$\left.\begin{aligned} x_L &= x \\ x_L - x_{(1)} &= w \\ x_L - x_{L-1} &= wY \end{aligned}\right\} \rightarrow \begin{aligned} x_L &= x \\ x_{(1)} &= x - w \\ x_{L-1} &= x - wY \end{aligned}$$

$$|J| = |w| = w .$$

The joint density of $x_{(1)}, x_{L-1}, x_L$ is

$$f(x_{(1)}, x_{L-1}, x_L) = (n-1)(n-2)(n-3)\left[F_x(x_{L-1}) - F_x(x_{(1)})\right]^{n-4} f_x(x_{(1)})\, f_x(x_{L-1})\, f_x(x_L)$$

for $x_{(1)}, x_{L-1}, x_L$ in $(-\infty, \infty)$;

$$f_1(B_{6L}) = f(Y) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{x_{L-1}=x-wY} f(x_{(1)} = x-w,\ x_{L-1} = x-wY,\ x_L = x,\ x_s = x_s)\,|J|\,dx_s\,dw\,dx$$

for $0 \le B_{6L} \le 1$, and P_{e_1} becomes:

$$P_{e_1} = \int_{B_T}^{1}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{(n-1)!}{(n-4)!}\left[\int_{x-w}^{x-wY} f_x(t)\,dt\right]^{n-4} f_x(x-w)\, f_x(x-wY)\, f_x(x)\, w\,dw\,dx\,dY \int_{-\infty}^{x-wY} f_s(x_s)\,dx_s\ . \tag{C18}$$

If $x_{L-1} < x_s < x_L$, apply the transformation

$$x_L = x$$

$$x_{(1)} = x - w$$

$$x_s = x - wZ$$

$$|J| = w\ .$$

Then $f_2(B_{6L})$ becomes

$$f_2(B_{6L}) = f(Z) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{x_s = x - wZ} f\left(x_{(1)} = x - w,\ x_{L-1} = x_{L-1},\ x_L = x,\ x_s = x - wZ\right)|J|\ dx_{L-1}\ dw\ dx\ ,$$

$0 \le B_{6L} \le 1$ and P_{e_2} becomes

$$P_{e_2} = \frac{(n-1)!}{(n-4)!}\int_{B_T}^{1}\int_{-\infty}^{\infty}\int_{0}^{\infty} f_x(x)\, f_x(x-w)\, f_s(x-wZ)\, w\ dw\ dx\ dZ \int_{x-w}^{x-wZ} f_x\left(x_{L-1}\right)\left[\int_{x-w}^{x_{L-1}} f_x(t)\, dt\right]^{n-4} dx_{L-1}$$

$$= (n-1)(n-2)\int_{B_T}^{1} dZ \int_{-\infty}^{\infty} f_x(x)\, dx \int_{0}^{\infty} f_s(x-wZ)\, f_x(x-w)\, w\left[\int_{x-w}^{x-wZ} f_x(t)\, dt\right]^{n-3} dw\ . \qquad \text{(C19)}$$

The probability of error is

$$P_e = P_{e_1} + P_{e_2}\ . \qquad \text{(C20)}$$

Appendix D

OUTLIERS

In this type of approach, we shall assume k noise signals and find the probability of erroneously deciding that the largest noise statistic comes from another distribution. This probability has been previously defined as α_0; thus

$$\alpha_0 = P\left(B > B_T \mid \text{all } k \text{ signals came from } \theta_0 \in \omega\right) . \tag{D1}$$

Having found α_0, $1 - \alpha_0$ is the confidence with which we accept $x_{(k)}$ as the signal. Thus

$$\alpha_0 = \int_{B_T}^{\infty} f\left(B_L\right) dB_L . \tag{D2}$$

For $B = B_1$,

$$\alpha_0 = \int_{B_T}^{\infty} k\, f_x\left(x_L\right) \left[\int_{-\infty}^{x_L} f_x(t)\, dt\right]^{k-1} dx_L ; \tag{D3}$$

$$1 - \alpha_0 = \int_{-\infty}^{B_T} k\, f_x(x) \left[\int_{-\infty}^{x} f_x(t)\, dt\right]^{n-1} dx = \left[F_x\left(B_T\right)\right]^n . \tag{D4}$$

The only pdf is the noise pdf; therefore we shall drop the subscript in this appendix. To simplify notation, we shall also drop the parentheses about the subscripts.

For $B = B_4$, apply the transformation

$$x_k = x$$

$$x_{k-1} = x - B$$

$$|J| = 1 .$$

Thus

$$f(B) = \int_{-\infty}^{\infty} f\left(x_k = x,\ x_{k-1} = x - B\right) dx \qquad -\infty < B < \infty$$

$$= \int_{-\infty}^{\infty} k(k-1)\left[F\left(x_{k-1}\right)\right]^{k-2} f\left(x_k\right) f\left(x_{k-1}\right) dx_k$$

$$= \int_{-\infty}^{\infty} k(k-1)\left[\int_{-\infty}^{x-B} f(t)\,dt\right]^{k-2} f(x)\, f(x-B)\, dx \ ,$$

$$\alpha_0 = \int_{B_T}^{\infty} f(B)\, dB \ , \tag{D5}$$

and

$$1 - \alpha_0 = \int_{-\infty}^{B_T} f(B)\, dB \ . \tag{D6}$$

For $B = B_6$, apply the transformation

$$x_k = x$$

$$x_1 = x - w$$

$$x_{k-1} = x - Bw$$

$$|J| = |w| \ .$$

Thus

$$f(B) = \int_{-\infty}^{\infty} k(k-1)(k-2)\left[F\left(x_{k-1}\right) - F\left(x_1\right)\right]^{k-3} f\left(x_1\right) f\left(x_{k-1}\right) f\left(x_k\right) \ , \qquad 0 \le B \le 1 \ ;$$

then

$$\alpha_0 = \int_{B_T}^{1} k(k-1)(k-2)\, dB \int_{-\infty}^{\infty} dx \int_0^{\infty} w\left[\int_{x-w}^{x-Bw} f(t)\,dt\right]^{k-3} f(x)\, f(x-w)\, f(x-Bw)\, dw \tag{D7}$$

and

$$\left(1-\alpha_0\right) = \int_0^{B_T} k(k-1)(k-2)\,dB \int_{-\infty}^{\infty} dx \int_0^{\infty} w \left[\int_{x-w}^{x-Bw} f(t)\,dt\right]^{k-3} f(x)\,f(x-w)\,f(x-Bw)\,dw \ . \qquad \text{(D8)}$$

Appendix E

TABULATED AND GRAPHICAL RESULTS

This appendix contains both the theoretical and the experimental results in the form of tables and graphs.

Table E1

Theoretical Results, B_1 Coherent.

B_T	n = 16: A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
-0.2	0.739	0.260	0.405	0.594	0.133	0.866	0.025	0.974
0.2	0.739	0.260	0.405	0.594	0.133	0.866	0.025	0.974
0.6	0.736	0.260	0.404	0.594	0.133	0.866	0.025	0.974
1.0	0.704	0.254	0.395	0.591	0.132	0.865	0.024	0.974
1.4	0.575	0.227	0.346	0.570	0.123	0.859	0.024	0.974
1.8	0.360	0.169	0.242	0.504	0.098	0.830	0.021	0.969
2.2	0.169	0.102	0.128	0.390	0.062	0.759	0.016	0.952
2.6	0.062	0.051	0.052	0.267	0.030	0.640	0.009	0.911
3.0	0.018	0.021	0.016	0.152	0.011	0.490	0.004	0.834
3.4	0.004	0.007	0.004	0.077	0.003	0.336	0.001	0.718
3.8	0.000	0.002	0.000	0.034	0.000	0.206	0.000	0.571
4.2		0.000		0.013		0.111		0.412
4.6				0.004		0.052		0.267
5.0				0.001		0.021		0.153
5.4				0.000		0.007		0.077
5.8						0.002		0.034
6.2						0.000		0.013
6.6								0.004
7.0								0.001
7.4								0.000

B_T	n = 32: A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
-0.2	0.828	0.171	0.517	0.482	0.197	0.802	0.042	0.957
0.2	0.828	0.171	0.517	0.482	0.197	0.802	0.042	0.957
0.6	0.828	0.171	0.517	0.482	0.197	0.802	0.042	0.957
1.0	0.825	0.171	0.516	0.482	0.197	0.802	0.042	0.957
1.4	0.779	0.166	0.498	0.478	0.194	0.801	0.041	0.957
1.8	0.589	0.141	0.405	0.449	0.171	0.788	0.039	0.955
2.2	0.316	0.094	0.241	0.369	0.118	0.737	0.031	0.943
2.6	0.123	0.049	0.104	0.256	0.060	0.632	0.019	0.907
3.0	0.037	0.021	0.034	0.151	0.023	0.487	0.009	0.833
3.4	0.009	0.007	0.009	0.077	0.007	0.336	0.003	0.718
3.8	0.002	0.002	0.002	0.034	0.001	0.205	0.001	0.571
4.2	0.000	0.000	0.000	0.013	0.000	0.111	0.000	0.412
4.6				0.004		0.052		0.267
5.0				0.001		0.021		0.153
5.4				0.000		0.007		0.077
5.8						0.002		0.034
6.2						0.000		0.013
6.6								0.004
7.0								0.001
7.4								0.000

B_T	n = 64: A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
-0.20	0.889	0.110	0.618	0.381	0.271	0.728	0.065	0.934
0.20	0.889	0.110	0.618	0.381	0.271	0.728	0.065	0.934
0.60	0.889	0.110	0.618	0.381	0.271	0.728	0.065	0.934
1.00	0.889	0.110	0.618	0.381	0.271	0.728	0.065	0.934
1.40	0.885	0.110	0.617	0.381	0.270	0.728	0.065	0.934
1.80	0.807	0.105	0.577	0.374	0.261	0.725	0.064	0.933
2.20	0.533	0.081	0.412	0.333	0.206	0.699	0.056	0.927
2.60	0.235	0.047	0.198	0.246	0.116	0.616	0.038	0.898
3.00	0.075	0.020	0.068	0.149	0.047	0.483	0.019	0.829
3.40	0.019	0.007	0.018	0.077	0.014	0.335	0.073	0.717
3.80	0.004	0.002	0.004	0.034	0.003	0.205	0.021	0.571
4.20	0.000	0.000	0.000	0.013	0.000	0.111	0.000	0.412
4.60				0.004		0.052		0.267
5.00				0.001		0.021		0.153
5.40				0.007		0.007		0.077
5.80						0.002		0.034
6.20						0.000		0.013
6.60								0.004
7.00								0.001
7.40								0.000

B_T	n = 128: A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
-0.20	0.929	0.069	0.707	0.293	0.352	0.648	0.097	0.903
0.20	0.929	0.069	0.707	0.293	0.352	0.648	0.097	0.903
0.60	0.929	0.069	0.707	0.293	0.352	0.648	0.097	0.903
1.00	0.929	0.069	0.707	0.293	0.352	0.648	0.097	0.903
1.40	0.929	0.069	0.707	0.293	0.352	0.648	0.097	0.903
1.80	0.920	0.069	0.701	0.293	0.349	0.648	0.097	0.903
2.20	0.772	0.063	0.609	0.281	0.318	0.640	0.092	0.900
2.60	0.416	0.042	0.353	0.228	0.209	0.589	0.069	0.882
3.00	0.146	0.020	0.133	0.146	0.091	0.475	0.037	0.823
3.40	0.038	0.007	0.037	0.076	0.029	0.334	0.014	0.715
3.80	0.008	0.002	0.008	0.034	0.007	0.205	0.004	0.570
4.20	0.001	0.000	0.001	0.013	0.014	0.111	0.001	0.412
4.60	0.000		0.000	0.004	0.000	0.052	0.000	0.267
5.00				0.001		0.021		0.153
5.40				0.000		0.007		0.077
5.80						0.002		0.034
6.20						0.000		0.013
6.60								0.004
7.00								0.001
7.40								0.000

Table E2

Theoretical Results, B_4 Coherent.

B_T	n = 16								n = 32							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.594	0.134	0.866	0.025	0.974	0.830	0.171	0.518	0.482	0.198	0.802	0.042	0.957
0.4	0.334	0.159	0.172	0.455	0.052	0.777	0.088	0.947	0.334	0.094	0.200	0.343	0.717	0.689	0.013	0.915
0.8	0.134	0.087	0.065	0.321	0.184	0.660	0.027	0.899	0.118	0.046	0.068	0.221	0.023	0.554	0.041	0.847
1.2	0.047	0.043	0.022	0.206	0.057	0.525	0.000	0.825	0.037	0.020	0.020	0.129	0.657	0.411	0.010	0.749
1.6	0.015	0.019	0.066	0.119	0.016	0.386		0.722	0.010	0.007	0.054	0.067	0.016	0.279	0.000	0.623
2.0	0.004	0.007	0.001	0.625	0.000	0.260		0.594	0.002	0.002	0.001	0.031	0.000	0.171		0.482
2.4	0.001	0.002	0.000	0.029		0.159		0.455	0.000	0.000	0.000	0.012		0.094		0.343
2.8	0.000	0.000		0.012		0.087		0.321				0.004		0.046		0.221
3.2				0.004		0.043		0.206				0.001		0.020		0.129
3.6				0.001		0.019		0.119				0.000		0.007		0.067
4.0				0.000		0.007		0.062						0.002		0.031
4.4						0.002		0.029						0.000		0.012
4.8						0.000		0.012								0.004
5.2								0.004								0.001
5.6								0.001								0.000
6.0								0.000								

B_T	n = 64								n = 128							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.891	0.110	0.620	0.381	0.272	0.728	0.066	0.934	0.932	0.069	0.708	0.298	0.352	0.648	0.097	0.902
0.4	0.323	0.055	0.218	0.251	0.090	0.597	0.020	0.874	0.306	0.032	0.227	0.180	0.108	0.505	0.028	0.824
0.8	0.102	0.024	0.067	0.149	0.026	0.452	0.056	0.785	0.088	0.012	0.064	0.098	0.029	0.360	0.007	0.714
1.2	0.028	0.009	0.018	0.079	0.006	0.313	0.001	0.665	0.022	0.004	0.015	0.048	0.007	0.233	0.001	0.578
1.6	0.006	0.003	0.004	0.037	0.001	0.196	0.000	0.525	0.004	0.001	0.003	0.020	0.001	0.135	0.000	0.431
2.0	0.001	0.000	0.000	0.015	0.000	0.110		0.381	0.000	0.000	0.000	0.007	0.000	0.069		0.293
2.4	0.000			0.005		0.055		0.251				0.002		0.032		0.180
2.8				0.001		0.024		0.014				0.000		0.012		0.098
3.2				0.000		0.009		0.079						0.004		0.048
3.6						0.003		0.037						0.001		0.020
4.0						0.000		0.015						0.000		0.007
4.4								0.005								0.002
4.8								0.001								0.000
5.2								0.000								

Table E3

Theoretical Results, B_6 Coherent.

	n = 16						n = 32						n = 64			
	A = 2		A = 3		A = 4		A = 2		A = 3		A = 4		A = 2		A = 4	
B_T	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0	0.399	0.601	0.130	0.890	0.026	0.974	0.520	0.480	0.190	0.810	0.041	0.959	0.617	0.383	0.064	0.936
0.1	0.184	0.468	0.053	0.789	0.010	0.947	0.179	0.326	0.062	0.683	0.012	0.880	0.187	0.235	0.014	0.845
0.2	0.067	0.317	0.017	0.644	0.004	0.874	0.044	0.173	0.014	0.468	0.003	0.756	0.025	0.070	0.003	0.661
0.3	0.018	0.177	0.004	0.446	0.002	0.707	0.007	0.057	0.002	0.230	0.001	0.531	0.002	0.013	0.001	0.382
0.4	0.003	0.078	0.001	0.239	0.001	0.466	0.000	0.010	0.000	0.069	0.001	0.247	0.000	0.001	0.000	0.097
0.5	0.000	0.013	0.000	0.064	0.001	0.198		0.001		0.008	0.000	0.052		0.000		0.007
0.6		0.001		0.009		0.039				0.000		0.003				0.000
0.7		0.000		0.000		0.002						0.000				
0.8						0.000										
0.9																
1.0																

Table E4

Theoretical Results, B_1 Noncoherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.858	0.141	0.603	0.396	0.275	0.723	0.722	0.927
0.4	0.858	0.141	0.603	0.396	0.275	0.723	0.722	0.927
0.8	0.858	0.141	0.603	0.396	0.275	0.723	0.722	0.927
1.2	0.858	0.141	0.603	0.396	0.275	0.723	0.722	0.927
1.6	0.854	0.141	0.602	0.395	0.275	0.723	0.722	0.927
2.0	0.783	0.134	0.566	0.388	0.266	0.720	0.711	0.926
2.4	0.536	0.103	0.416	0.346	0.215	0.694	0.631	0.919
2.8	0.250	0.059	0.211	0.255	0.127	0.611	0.440	0.890
3.2	0.084	0.026	0.077	0.154	0.053	0.476	0.229	0.818
3.6	0.022	0.009	0.021	0.078	0.017	0.325	0.901	0.700
4.0	0.005	0.002	0.004	0.034	0.004	0.196	0.272	0.549
4.4	0.000	0.000	0.000	0.012	0.000	0.103	0.000	0.389
4.8				0.004		0.047		0.246
5.2				0.001		0.019		0.137
5.6				0.000		0.006		0.067
6.0						0.001		0.028
6.4						0.000		0.010
6.8								0.003
7.2								0.000

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
0.4	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
0.8	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
1.2	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
1.6	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
2.0	0.906	0.085	0.698	0.297	0.363	0.635	0.109	0.890
2.4	0.779	0.077	0.619	0.285	0.336	0.628	0.104	0.888
2.8	0.446	0.052	0.380	0.233	0.230	0.579	0.081	0.870
3.2	0.167	0.025	0.152	0.149	0.106	0.466	0.045	0.809
3.6	0.046	0.009	0.044	0.077	0.035	0.323	0.018	0.697
4.0	0.010	0.002	0.010	0.034	0.008	0.195	0.005	0.548
4.4	0.001	0.000	0.001	0.012	0.001	0.103	0.001	0.389
4.8	0.000		0.000	0.004	0.000	0.047	0.000	0.246
5.2				0.001		0.019		0.137
5.6				0.000		0.006		0.067
6.0						0.001		0.028
6.4						0.000		0.010
6.8								0.003
7.2								0.000

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
0.4	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
0.8	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
1.2	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
1.6	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
2.0	0.948	0.051	0.781	0.218	0.453	0.546	0.154	0.844
2.4	0.926	0.050	0.767	0.217	0.448	0.545	0.153	0.844
2.8	0.694	0.042	0.597	0.198	0.372	0.527	0.136	0.837
3.2	0.309	0.023	0.282	0.140	0.199	0.448	0.086	0.794
3.6	0.091	0.009	0.087	0.076	0.069	0.320	0.036	0.693
4.0	0.020	0.002	0.020	0.033	0.017	0.195	0.011	0.547
4.4	0.003	0.000	0.003	0.012	0.003	0.103	0.002	0.389
4.8	0.000		0.000	0.004	0.000	0.047	0.000	0.246
5.2				0.001		0.019		0.137
5.6				0.000		0.006		0.067
6.0						0.001		0.028
6.4						0.000		0.010
6.8								0.003
7.2								0.000

n = 128

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
0.4	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
0.8	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
1.2	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
1.6	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
2.0	0.969	0.030	0.841	0.158	0.538	0.461	0.207	0.791
2.4	0.968	0.030	0.841	0.158	0.538	0.461	0.207	0.791
2.8	0.897	0.029	0.788	0.154	0.514	0.458	0.202	0.790
3.2	0.525	0.020	0.481	0.125	0.342	0.417	0.150	0.767
3.6	0.176	0.008	0.168	0.073	0.134	0.312	0.070	0.684
4.0	0.041	0.002	0.040	0.033	0.035	0.194	0.022	0.545
4.4	0.007	0.000	0.007	0.012	0.007	0.103	0.005	0.388
4.8	0.001		0.001	0.004	0.001	0.047	0.001	0.246
5.2	0.000		0.000	0.001	0.000	0.019	0.000	0.137
5.6				0.000		0.006		0.067
6.0						0.001		0.028
6.4						0.000		0.010
6.8								0.003
7.2								0.000

Table E5

Theoretical Results, B_4 Noncoherent.

B_T	n = 16								n = 32							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.858	0.141	0.604	0.396	0.276	0.723	0.072	0.927	0.914	0.085	0.702	0.297	0.364	0.635	0.109	0.890
0.4	0.334	0.070	0.227	0.259	0.097	0.588	0.023	0.863	0.316	0.038	0.237	0.179	0.118	0.487	0.033	0.804
0.8	0.112	0.031	0.074	0.152	0.030	0.441	0.006	0.767	0.094	0.015	0.069	0.096	0.033	0.341	0.008	0.687
1.2	0.033	0.011	0.021	0.080	0.008	0.301	0.001	0.642	0.024	0.005	0.017	0.045	0.008	0.215	0.002	0.546
1.6	0.008	0.004	0.005	0.037	0.001	0.185	0.000	0.499	0.005	0.001	0.003	0.019	0.001	0.122	0.000	0.398
2.0	0.001	0.001	0.001	0.015	0.000	0.102		0.355	0.001	0.000	0.000	0.007	0.000	0.061		0.264
2.4	0.000	0.000	0.000	0.005		0.050		0.229	0.000			0.002		0.027		0.157
2.8				0.001		0.021		0.133				0.000		0.010		0.083
3.2				0.000		0.008		0.069						0.003		0.039
3.6						0.002		0.031						0.001		0.016
4.0						0.000		0.012						0.000		0.005
4.4								0.004								0.001
4.8								0.001								0.000
								0.000								

B_T	n = 64								n = 128							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.949	0.051	0.781	0.218	0.453	0.546	0.154	0.844	0.970	0.030	0.842	0.158	0.539	0.461	0.208	0.791
0.4	0.295	0.021	0.239	0.122	0.134	0.395	0.043	0.738	0.274	0.011	0.235	0.081	0.146	0.314	0.054	0.667
0.8	0.079	0.007	0.063	0.060	0.034	0.259	0.010	0.604	0.066	0.003	0.056	0.037	0.034	0.192	0.012	0.521
1.2	0.018	0.002	0.014	0.026	0.007	0.152	0.002	0.455	0.014	0.001	0.011	0.014	0.007	0.105	0.002	0.371
1.6	0.003	0.000	0.002	0.009	0.001	0.079	0.000	0.311	0.002	0.000	0.002	0.005	0.001	0.050	0.000	0.239
2.0	0.000		0.000	0.003	0.000	0.036		0.192	0.000		0.000	0.001	0.000	0.021		0.137
2.4				0.000		0.014		0.105				0.000		0.007		0.070
2.8						0.005		0.051						0.002		0.031
3.2						0.001		0.022						0.000		0.012
3.6						0.000		0.008								0.004
4.0								0.002								0.001
4.4								0.000								0.000
4.8																
5.2																

Table E6

Theoretical Results, B_6 Noncoherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0	0.857	0.143	0.610	0.390	0.279	0.721	0.073	0.927
0.1	0.541	0.099	0.365	0.313	0.154	0.638	0.036	0.882
0.2	0.292	0.061	0.188	0.226	0.073	0.529	0.015	0.814
0.3	0.130	0.032	0.079	0.141	0.028	0.395	0.005	0.703
0.4	0.043	0.013	0.025	0.077	0.008	0.247	0.001	0.536
0.5	0.009	0.003	0.005	0.024	0.001	0.112		0.319
0.6	0.001	0.000	0.000	0.005	0.000	0.029		0.116
0.7	0.009	0.000		0.000		0.003		0.015
0.8						0.000		0.000
0.9								
1.0								

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0	0.902	0.106	0.703	0.297	0.364	0.636	0.109	0.891
0.1	0.481	0.052	0.357	0.213	0.172	0.528	0.047	0.823
0.2	0.204	0.026	0.145	0.133	0.065	0.399	0.016	0.723
0.3	0.064	0.010	0.043	0.067	0.018	0.255	0.004	0.570
0.4	0.013	0.003	0.008	0.024	0.003	0.122	0.001	0.365
0.5	0.001	0.000	0.001	0.005	0.000	0.035	0.000	0.155
0.6	0.000	0.000	0.000	0.000		0.004		0.001
0.7						0.000		0.000
0.8								
0.9								
1.0								

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0	0.949	0.051	0.782	0.218	0.453	0.547	0.160	0.840
0.1	0.416	0.027	0.333	0.141	0.183	0.423	0.057	0.754
0.2	0.138	0.011	0.107	0.075	0.055	0.287	0.016	0.621
0.3	0.030	0.003	0.023	0.030	0.011	0.154	0.003	0.437
0.4	0.003	0.001	0.002	0.007	0.001	0.055	0.000	0.227
0.5	0.000	0.000	0.000	0.001	0.000	0.009		0.064
0.6				0.000		0.000		0.005
0.7								0.000
0.8								
0.9								
1.0								

Table E7

Experimental Results, B_1 Coherent.

B_T	n = 16								n = 32							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.4	0.739	0.260	0.405	0.595	0.120	0.880	0.022	0.978	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.8	0.727	0.259	0.402	0.594	0.120	0.880	0.022	0.978	0.811	0.189	0.519	0.491	0.202	0.798	0.050	0.950
1.2	0.665	0.245	0.380	0.591	0.117	0.878	0.022	0.978	0.798	0.187	0.512	0.480	0.201	0.798	0.050	0.950
1.6	0.475	0.204	0.310	0.558	0.099	0.866	0.021	0.976	0.692	0.177	0.560	0.470	0.194	0.793	0.049	0.949
2.0	0.261	0.151	0.192	0.466	0.075	0.826	0.015	0.965	0.474	0.139	0.317	0.420	0.152	0.768	0.042	0.943
2.4	0.111	0.075	0.088	0.339	0.046	0.731	0.010	0.939	0.191	0.078	0.163	0.319	0.085	0.683	0.030	0.924
2.8	0.031	0.033	0.029	0.214	0.022	0.590	0.002	0.883	0.063	0.029	0.056	0.220	0.037	0.545	0.017	0.869
3.2	0.003	0.017	0.009	0.112	0.011	0.429	0.000	0.785	0.017	0.010	0.021	0.120	0.014	0.395	0.008	0.762
3.6	0.001	0.009	0.002	0.046	0.002	0.283		0.652	0.001	0.004	0.004	0.057	0.003	0.258	0.002	0.640
4.0	0.000	0.003	0.000	0.016	0.000	0.163		0.501	0.000	0.002	0.000	0.024	0.000	0.137	0.001	0.510
4.4		0.002		0.007		0.088		0.324		0.000		0.008		0.072	0.000	0.355
4.8		0.000		0.001		0.036		0.208				0.006		0.036		0.214
				0.000		0.016		0.114				0.001		0.015		0.118
						0.006		0.051				0.001		0.004		0.060
						0.003		0.026				0.000		0.002		0.030
						0.001		0.010						0.001		0.009
						0.000		0.002						0.000		0.005
								0.000								0.000

B_T	n = 64								n = 128							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.4	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.8	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
1.2	0.881	0.118	0.620	0.380	0.270	0.730	0.076	0.924	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
1.6	0.862	0.118	0.612	0.377	0.269	0.730	0.076	0.924	0.935	0.064	0.709	0.291	0.383	0.617	0.089	0.911
2.0	0.700	0.103	0.529	0.358	0.245	0.721	0.071	0.921	0.894	0.064	0.679	0.290	0.380	0.617	0.087	0.911
2.4	0.382	0.069	0.313	0.284	0.160	0.671	0.052	0.907	0.623	0.050	0.492	0.266	0.312	0.600	0.081	0.907
2.8	0.128	0.038	0.120	0.190	0.070	0.574	0.038	0.868	0.294	0.026	0.226	0.189	0.176	0.509	0.045	0.877
3.2	0.044	0.019	0.034	0.101	0.024	0.415	0.017	0.771	0.097	0.009	0.077	0.109	0.059	0.387	0.022	0.790
3.6	0.007	0.007	0.008	0.048	0.008	0.273	0.004	0.643	0.023	0.003	0.019	0.049	0.015	0.247	0.008	0.674
4.0	0.000	0.000	0.003	0.016	0.002	0.155	0.000	0.501	0.006	0.002	0.004	0.024	0.002	0.144	0.001	0.515
			0.001	0.003	0.000	0.085		0.353	0.001	0.000	0.001	0.005	0.000	0.073	0.000	0.379
			0.000	0.000		0.035		0.198	0.000		0.001	0.002		0.022		0.241
						0.014		0.099			0.000	0.000		0.010		0.141
						0.003		0.036						0.008		0.079
						0.003		0.017						0.002		0.032
						0.000		0.005						0.002		0.015
								0.000						0.000		0.009
																0.001
																0.000

Table E8

Experimental Results, B_2 Coherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.4	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.8	0.740	0.260	0.404	0.595	0.120	0.880	0.022	0.978
1.2	0.691	0.253	0.389	0.593	0.118	0.880	0.022	0.978
1.6	0.507	0.213	0.312	0.568	0.105	0.864	0.020	0.976
2.0	0.300	0.155	0.194	0.479	0.076	0.825	0.016	0.966
2.4	0.119	0.085	0.092	0.349	0.040	0.720	0.010	0.937
2.8	0.038	0.040	0.030	0.215	0.018	0.591	0.004	0.877
3.2	0.007	0.015	0.009	0.118	0.006	0.427	0.000	0.779
3.6	0.002	0.006	0.002	0.054	0.200	0.288		0.642
4.0	0.000	0.003	0.000	0.021		0.177		0.494
		0.002		0.008		0.096		0.349
		0.001		0.002		0.044		0.218
		0.000		0.000		0.016		0.123
						0.006		0.059
						0.003		0.028
						0.001		0.008
						0.001		0.003
						0.000		0.001
								0.001
								0.000

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.4	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.8	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
1.2	0.811	0.189	0.517	0.481	0.201	0.798	0.050	0.950
1.6	0.732	0.179	0.468	0.470	0.190	0.795	0.050	0.950
2.0	0.476	0.139	0.329	0.417	0.147	0.769	0.042	0.943
2.4	0.205	0.071	0.162	0.326	0.083	0.677	0.027	0.920
2.8	0.068	0.032	0.058	0.220	0.033	0.559	0.017	0.869
3.2	0.018	0.011	0.021	0.118	0.016	0.403	0.007	0.766
3.6	0.001	0.005	0.006	0.056	0.004	0.259	0.002	0.644
4.0	0.000	0.000	0.000	0.022	0.001	0.143	0.001	0.507
				0.010	0.000	0.072	0.000	0.347
				0.006		0.036		0.221
				0.002		0.015		0.127
				0.001		0.006		0.070
				0.000		0.002		0.033
						0.001		0.008
						0.000		0.004
								0.001
								0.000

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.4	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.8	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
1.2	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
1.6	0.874	0.118	0.615	0.377	0.270	0.729	0.076	0.924
2.0	0.710	0.100	0.530	0.365	0.246	0.725	0.070	0.922
2.4	0.378	0.074	0.315	0.285	0.155	0.672	0.053	0.910
2.8	0.140	0.040	0.121	0.182	0.062	0.565	0.035	0.867
3.2	0.043	0.021	0.039	0.106	0.019	0.482	0.014	0.773
3.6	0.009	0.007	0.009	0.044	0.006	0.269	0.003	0.649
4.0	0.002	0.000	0.001	0.014	0.002	0.161	0.000	0.506
	0.000		0.001	0.002	0.001	0.086		0.346
			0.000	0.000	0.000	0.039		0.205
						0.014		0.100
						0.004		0.037
						0.002		0.016
						0.000		0.005
								0.000

n = 128

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.4	0.935	0.065	0.709	0.291	0.388	0.617	0.089	0.911
0.8	0.935	0.065	0.709	0.291	0.388	0.617	0.089	0.911
1.2	0.935	0.065	0.709	0.291	0.388	0.617	0.089	0.911
1.6	0.935	0.064	0.709	0.291	0.388	0.617	0.089	0.911
2.0	0.895	0.064	0.684	0.289	0.382	0.617	0.087	0.911
2.4	0.642	0.052	0.508	0.265	0.311	0.599	0.080	0.906
2.8	0.296	0.025	0.212	0.190	0.168	0.514	0.046	0.881
3.2	0.093	0.011	0.079	0.111	0.056	0.394	0.020	0.794
3.6	0.027	0.003	0.018	0.054	0.017	0.248	0.005	0.664
4.0	0.007	0.002	0.004	0.024	0.002	0.144	0.002	0.514
	0.002	0.000	0.001	0.008	0.000	0.068	0.000	0.370
	0.000		0.000	0.002		0.022		0.236
				0.000		0.012		0.138
						0.008		0.077
						0.003		0.033
						0.002		0.015
						0.000		0.009
								0.002
								0.000

Table E9

Experimental Results, B_3 Coherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.4	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.8	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
1.2	0.739	0.260	0.403	0.595	0.120	0.880	0.022	0.978
1.6	0.655	0.252	0.367	0.584	0.111	0.872	0.021	0.976
2.0	0.458	0.203	0.242	0.524	0.086	0.839	0.017	0.971
2.4	0.248	0.128	0.136	0.414	0.046	0.763	0.008	0.949
2.8	0.117	0.080	0.071	0.291	0.019	0.634	0.002	0.888
3.2	0.049	0.052	0.034	0.199	0.005	0.494	0.001	0.805
3.6	0.020	0.030	0.015	0.134	0.001	0.384	0.000	0.695
4.0	0.011	0.020	0.006	0.089	0.001	0.286		0.574
4.4	0.007	0.009	0.002	0.051	0.001	0.195		0.463
4.8	0.002	0.004	0.001	0.031	0.001	0.126		0.351
5.2	0.000	0.001	0.000	0.014	0.001	0.084		0.252
		0.000		0.007	0.001	0.060		0.184
				0.001	0.001	0.035		0.137
				0.001	0.001	0.017		0.074
				0.001	0.000	0.012		0.062
				0.000		0.007		0.045
						0.004		0.033
						0.003		0.018
						0.002		0.011
						0.001		0.011
						0.000		0.008

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.4	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.8	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
1.2	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
1.6	0.784	0.188	0.508	0.478	0.198	0.798	0.050	0.950
2.0	0.599	0.160	0.383	0.444	0.166	0.784	0.041	0.944
2.4	0.316	0.101	0.199	0.359	0.093	0.703	0.026	0.923
2.8	0.123	0.040	0.089	0.260	0.043	0.582	0.016	0.862
3.2	0.037	0.020	0.031	0.142	0.014	0.440	0.006	0.775
3.6	0.010	0.007	0.008	0.076	0.003	0.304	0.003	0.660
4.0	0.003	0.002	0.003	0.040	0.001	0.189	0.001	0.537
4.4	0.000	0.001	0.001	0.028	0.000	0.116	0.000	0.413
4.8		0.000	0.000	0.016		0.062		0.278
5.2				0.008		0.032		0.192
				0.002		0.018		0.126
				0.000		0.007		0.081
						0.002		0.041
						0.001		0.024
						0.000		0.014
								0.009
								0.004
								0.001
								0.000

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.4	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.8	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
1.2	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
1.6	0.881	0.118	0.620	0.380	0.270	0.730	0.076	0.924
2.0	0.797	0.110	0.573	0.375	0.255	0.725	0.070	0.923
2.4	0.472	0.087	0.356	0.310	0.167	0.691	0.058	0.908
2.8	0.211	0.045	0.146	0.221	0.071	0.578	0.035	0.874
3.2	0.069	0.022	0.050	0.127	0.019	0.443	0.016	0.786
3.6	0.022	0.009	0.012	0.061	0.006	0.313	0.004	0.677
4.0	0.010	0.003	0.003	0.024	0.003	0.182	0.001	0.513
	0.003	0.000	0.001	0.008	0.001	0.121	0.000	0.361
	0.001		0.001	0.003	0.000	0.064		0.241
	0.000		0.001	0.000		0.032		0.135
			0.000			0.014		0.069
						0.006		0.035
						0.002		0.012
						0.000		0.008
								0.005
								0.001
								0.001
								0.000

n = 128

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.4	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.8	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
1.2	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
1.6	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
2.0	0.921	0.063	0.701	0.291	0.383	0.617	0.088	0.911
2.4	0.718	0.058	0.546	0.266	0.329	0.604	0.080	0.905
2.8	0.342	0.027	0.264	0.203	0.167	0.516	0.044	0.873
3.2	0.118	0.015	0.088	0.118	0.063	0.392	0.020	0.796
3.6	0.034	0.007	0.019	0.058	0.011	0.272	0.005	0.675
4.0	0.010	0.003	0.004	0.027	0.001	0.161	0.001	0.527
	0.004	0.001	0.002	0.011	0.001	0.075	0.000	0.387
	0.002	0.000	0.001	0.003	0.000	0.035		0.254
	0.000		0.000	0.001		0.019		0.153
				0.001		0.011		0.089
				0.000		0.007		0.049
						0.003		0.021
						0.000		0.007
								0.001
								0.000

Table E10

Experimental Results, B_4 Coherent.

B_T	n = 16								n = 32							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.4	0.357	0.168	0.172	0.466	0.047	0.789	0.003	0.954	0.314	0.108	0.192	0.340	0.069	0.680	0.015	0.897
0.8	0.133	0.091	0.078	0.336	0.018	0.686	0.002	0.910	0.109	0.041	0.063	0.228	0.022	0.539	0.004	0.828
1.2	0.049	0.050	0.027	0.211	0.006	0.536	0.001	0.839	0.028	0.014	0.017	0.132	0.004	0.399	0.002	0.734
1.6	0.011	0.018	0.008	0.124	0.001	0.393	0.000	0.747	0.005	0.005	0.005	0.063	0.001	0.248	0.000	0.621
2.0	0.001	0.007	0.001	0.062	0.001	0.271		0.610	0.002	0.003	0.000	0.032	0.000	0.145		0.489
2.4	0.000	0.003	0.000	0.025	0.000	0.163		0.447	0.000	0.000		0.017		0.072		0.355
2.8		0.001		0.008		0.091		0.326				0.008		0.032		0.234
3.2		0.000		0.003		0.045		0.209				0.003		0.018		0.142
3.6						0.022		0.122				0.001		0.010		0.074
4.0						0.004		0.064				0.001		0.001		0.036
4.4						0.001		0.026				0.000		0.000		0.015
4.8						0.000		0.010								0.006
5.2								0.006								0.001
								0.002								0.000
								0.001								
								0.000								

B_T	n = 64								n = 128							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.4	0.310	0.061	0.224	0.249	0.082	0.605	0.033	0.868	0.321	0.036	0.226	0.171	0.116	0.491	0.025	0.837
0.8	0.105	0.026	0.061	0.144	0.018	0.454	0.011	0.786	0.099	0.013	0.060	0.095	0.028	0.342	0.006	0.718
1.2	0.026	0.011	0.012	0.075	0.006	0.332	0.005	0.665	0.027	0.004	0.014	0.047	0.002	0.216	0.003	0.582
1.6	0.007	0.001	0.004	0.027	0.003	0.210	0.001	0.521	0.007	0.002	0.004	0.017	0.001	0.124	0.000	0.460
2.0	0.003	0.000	0.002	0.011	0.001	0.116	0.000	0.378	0.004	0.000	0.001	0.008	0.001	0.059		0.310
2.4	0.000		0.000	0.001	0.000	0.055		0.244	0.000		0.001	0.002	0.000	0.026		0.201
2.8				0.001		0.023		0.143			0.000	0.000		0.014		0.118
3.2				0.000		0.008		0.060						0.003		0.058
3.6						0.004		0.025						0.001		0.029
4.0						0.001		0.016						0.001		0.013
						0.000		0.002						0.001		0.001
								0.000						0.000		0.001
																0.000

Table E11

Experimental Results, B_5 Coherent.

n = 16								
B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.4	0.395	0.184	0.189	0.469	0.046	0.792	0.008	0.955
0.8	0.197	0.106	0.097	0.356	0.021	0.695	0.002	0.907
1.2	0.084	0.064	0.049	0.256	0.008	0.564	0.001	0.836
1.6	0.036	0.040	0.021	0.164	0.003	0.439	0.000	0.739
2.0	0.013	0.024	0.005	0.112	0.001	0.341		0.644
2.4	0.006	0.015	0.003	0.070	0.001	0.239		0.516
2.8	0.003	0.007	0.001	0.043	0.001	0.158		0.396
3.2	0.002	0.004	0.001	0.028	0.001	0.107		0.294
3.6	0.000	0.001	0.000	0.010	0.001	0.069		0.216
4.0		0.001		0.005	0.001	0.047		0.158
		0.000		0.002	0.000	0.026		0.115
				0.001		0.017		0.086
				0.000		0.010		0.057
						0.006		0.039
						0.003		0.027
						0.002		0.016
						0.002		0.011
						0.000		0.009
								0.007
								0.005
								0.002
								0.000

n = 32								
B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.4	0.346	0.112	0.206	0.345	0.073	0.686	0.015	0.894
0.8	0.137	0.048	0.074	0.233	0.028	0.543	0,005	0.828
1.2	0.043	0.016	0.027	0.147	0.005	0.415	0.001	0.726
1.6	0.012	0.007	0.010	0.081	0.003	0.281	0.001	0.619
2.0	0.004	0.004	0.002	0.046	0.000	0.176	0.000	0.507
2.4	0.000	0.002	0.000	0.026		0.100		0.377
2.8		0.001		0.017		0.054		0.277
3.2				0.010		0.027		0.192
3.6				0.003		0.019		0.118
4.0				0.001		0.008		0.078
				0.000		0.002		0.044
						0.001		0.025
						0.000		0.015
								0.008
								0.003
								0.000

n = 64								
B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.4	0.329	0.063	0.229	0.257	0.082	0.608	0.033	0.866
0.8	0.123	0.028	0.070	0.153	0.019	0.456	0.012	0.790
1.2	0.042	0.011	0.016	0.072	0.006	0.344	0.006	0.666
1.6	0.012	0.001	0.004	0.037	0.003	0.217	0.002	0.525
2.0	0.007	0.000	0.001	0.015	0.002	0.131	0.000	0.385
2.4	0.002		0.001	0.004	0.000	0.073		0.269
2.8	0.000		0.000	0.001		0.039		0.153
3.2				0.000		0.015		0.086
3.6						0.005		0.044
4.0						0.002		0.026
						0.002		0.012
						0.002		0.004
						0.000		0.002
								0.001
								0.000

n = 128								
B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.4	0.337	0.036	0.233	0.171	0.121	0.492	0.023	0.836
0.8	0.107	0.013	0.065	0.095	0.031	0.346	0.006	0.713
1.2	0.032	0.007	0.015	0.050	0.004	0.228	0.004	0.589
1.6	0.010	0.004	0.003	0.020	0.001	0.133	0.000	0.462
2.0	0.004	0.000	0.001	0.008	0.001	0.064		0.323
2.4	0.002		0.001	0.002	0.000	0.030		0.210
2.8	0.000		0.001	0.001		0.019		0.128
3.2			0.000	0.000		0.005		0.064
3.6						0.002		0.035
4.0						0.001		0.018
						0.001		0.003
						0.000		0.001
								0.001
								0.000

Table E12

Experimental Results, B_6 Coherent.

n = 16

B_T	A = 1 P_e	A = 1 P_c	A = 2 P_e	A = 2 P_c	A = 3 P_e	A = 3 P_c	A = 4 P_e	A = 4 P_c
0.	0.740	0.260	0.405	0.595	0.120	0.880	0.022	0.978
0.1	0.394	0.182	0.189	0.473	0.047	0.787	0.008	0.950
0.2	0.161	0.100	0.085	0.332	0.013	0.658	0.001	0.885
0.3	0.041	0.047	0.024	0.174	0.004	0.458	0.000	0.751
0.4	0.006	0.017	0.002	0.073	0.001	0.239		0.511
0.5	0.000	0.001	0.001	0.012	0.000	0.078		0.224
0.6		0.000	0.000	0.000		0.012		0.052
0.7						0.000		0.002
0.8								0.000

n = 32

B_T	A = 1 P_e	A = 1 P_c	A = 2 P_e	A = 2 P_c	A = 3 P_e	A = 3 P_c	A = 4 P_e	A = 4 P_c
0.	0.811	0.189	0.519	0.481	0.202	0.798	0.050	0.950
0.1	0.314	0.101	0.180	0.322	0.063	0.665	0.013	0.882
0.2	0.074	0.027	0.045	0.193	0.013	0.471	0.001	0.768
0.3	0.010	0.005	0.006	0.066	0.002	0.227	0.001	0.563
0.4	0.000	0.001	0.000	0.015	0.000	0.057	0.000	0.292
0.5		0.000		0.002		0.007		0.071
0.6				0.000		0.000		0.005
0.7								0.000
0.8								

n = 64

B_T	A = 1 P_e	A = 1 P_c	A = 2 P_e	A = 2 P_c	A = 3 P_e	A = 3 P_c	A = 4 P_e	A = 4 P_c
0.	0.882	0.118	0.620	0.380	0.270	0.730	0.076	0.924
0.1	0.259	0.050	0.177	0.225	0.055	0.557	0.021	0.840
0.2	0.058	0.012	0.028	0.079	0.007	0.357	0.006	0.671
0.3	0.008	0.001	0.002	0.016	0.001	0.140	0.001	0.382
0.4	0.001	0.000	0.000	0.000	0.000	0.024	0.000	0.114
0.5	0.000					0.002		0.010
0.6						0.000		0.000
0.7								

n = 128

B_T	A = 1 P_e	A = 1 P_c	A = 2 P_e	A = 2 P_c	A = 3 P_e	A = 3 P_c	A = 4 P_e	A = 4 P_c
0.	0.935	0.065	0.709	0.291	0.383	0.617	0.089	0.911
0.1	0.221	0.024	0.114	0.141	0.072	0.427	0.017	0.770
0.2	0.028	0.005	0.017	0.041	0.003	0.207	0.002	0.545
0.3	0.003	0.001	0.002	0.008	0.001	0.051	0.000	0.261
0.4	0.000	0.000	0.000	0.000	0.000	0.003		0.052
0.5						0.000		0.001
0.6								0.000
0.7								

Table E13

Experimental Results, B_1 Noncoherent.

	n = 16								n = 32							
B_T	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.4	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.8	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
1.2	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
1.6	0.845	0.147	0.617	0.380	0.253	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
2.0	0.780	0.137	0.583	0.376	0.243	0.742	0.070	0.928	0.896	0.092	0.698	0.299	0.362	0.637	0.118	0.882
2.4	0.522	0.110	0.439	0.335	0.205	0.714	0.065	0.920	0.768	0.083	0.622	0.283	0.343	0.629	0.110	0.880
2.8	0.224	0.064	0.220	0.243	0.127	0.623	0.043	0.894	0.413	0.050	0.380	0.236	0.231	0.572	0.091	0.862
3.2	0.075	0.027	0.086	0.136	0.062	0.488	0.020	0.819	0.147	0.023	0.156	0.151	0.107	0.463	0.051	0.796
3.6	0.018	0.012	0.024	0.074	0.023	0.332	0.008	0.697	0.033	0.005	0.046	0.087	0.029	0.314	0.023	0.687
4.0	0.002	0.005	0.008	0.025	0.007	0.202	0.000	0.542	0.009	0.002	0.013	0.034	0.006	0.170	0.004	0.545
4.4	0.000	0.002	0.000	0.009	0.002	0.110	0.000	0.373	0.002	0.001	0.004	0.016	0.002	0.095	0.003	0.400
4.8		0.000		0.002		0.052		0.238	0.000	0.000	0.001	0.006	0.000	0.044	0.000	0.247
5.2				0.000		0.026		0.137			0.000	0.003		0.019		0.139
5.6						0.010		0.064				0.001		0.007		0.074
6.0						0.003		0.029				0.000		0.004		0.039
6.4						0.001		0.012						0.001		0.010
6.8						0.000		0.003						0.000		0.005
7.2								0.000								0.001

	n = 64								n = 128							
B_T	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.4	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.8	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.2	0.045	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.6	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
2.0	0.945	0.055	0.794	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
2.4	0.927	0.054	0.777	0.203	0.440	0.552	0.163	0.835	0.976	0.023	0.844	0.156	0.565	0.435	0.184	0.816
2.8	0.690	0.048	0.584	0.184	0.356	0.526	0.147	0.828	0.908	0.022	0.777	0.153	0.540	0.433	0.177	0.815
3.2	0.302	0.027	0.258	0.120	0.193	0.446	0.087	0.791	0.571	0.017	0.452	0.127	0.361	0.392	0.125	0.796
3.6	0.097	0.014	0.067	0.069	0.072	0.317	0.040	0.678	0.196	0.007	0.157	0.074	0.137	0.282	0.061	0.697
4.0	0.018	0.006	0.015	0.022	0.027	0.194	0.008	0.543	0.045	0.002	0.040	0.037	0.028	0.176	0.025	0.570
4.4	0.004	0.000	0.004	0.004	0.006	0.109	0.000	0.398	0.012	0.001	0.007	0.010	0.013	0.088	0.005	0.432
4.8	0.001		0.000	0.001	0.000	0.048		0.234	0.004	0.000	0.000	0.004	0.000	0.029	0.001	0.275
5.2				0.000		0.018		0.121	0.002			0.000		0.014	0.000	0.159
5.6						0.007		0.049	0.000					0.008		0.094
6.0						0.005		0.019						0.003		0.045
6.4						0.001		0.005						0.002		0.017
6.8						0.000		0.001						0.000		0.009
								0.000								0.002
																0.000

Table E14

Experimental Results, B_2 Noncoherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929
0.4	0.852	0.147	0.620	0.380	0.254	0.746	0.071	0.929
0.8	0.811	0.140	0.600	0.379	0.246	0.746	0.070	0.928
1.2	0.534	0.110	0.447	0.336	0.195	0.712	0.062	0.921
1.6	0.227	0.058	0.200	0.240	0.114	0.622	0.042	0.885
2.0	0.075	0.027	0.069	0.134	0.048	0.472	0.016	0.804
2.4	0.018	0.012	0.019	0.068	0.016	0.311	0.003	0.688
2.8	0.003	0.004	0.004	0.028	0.006	0.193	0.000	0.512
3.2	0.001	0.001	0.000	0.009	0.000	0.106		0.359
3.6	0.000	0.000		0.003		0.052		0.227
				0.001		0.022		0.132
				0.000		0.007		0.059
						0.003		0.028
						0.001		0.009
						0.000		0.003
								0.000

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.4	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.8	0.906	0.092	0.700	0.299	0.363	0.637	0.118	0.882
1.2	0.772	0.086	0.628	0.288	0.338	0.628	0.111	0.879
1.6	0.394	0.051	0.356	0.229	0.208	0.560	0.087	0.857
2.0	0.131	0.017	0.140	0.143	0.091	0.444	0.050	0.773
2.4	0.030	0.006	0.041	0.080	0.022	0.293	0.018	0.665
2.8	0.008	0.001	0.010	0.031	0.006	0.164	0.004	0.524
3.2	0.002	0.001	0.003	0.014	0.000	0.083	0.003	0.367
3.6	0.000	0.000	0.001	0.005		0.041	0.000	0.237
			0.000	0.001		0.016		0.123
				0.001		0.009		0.067
				0.000		0.001		0.032
						0.001		0.010
						0.000		0.004
								0.001
								0.000

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
0.4	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
0.8	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
1.2	0.928	0.054	0.782	0.203	0.441	0.551	0.164	0.835
1.6	0.671	0.047	0.557	0.181	0.340	0.523	0.138	0.827
2.0	0.273	0.026	0.226	0.120	0.169	0.438	0.080	0.778
2.4	0.084	0.011	0.055	0.059	0.059	0.308	0.032	0.652
2.8	0.019	0.003	0.012	0.019	0.019	0.183	0.006	0.534
3.2	0.004	0.000	0.001	0.004	0.003	0.106	0.000	0.373
3.6	0.000		0.000	0.001	0.000	0.044		0.211
4.0				0.000		0.015		0.119
						0.006		0.045
						0.005		0.015
						0.001		0.004
						0.000		0.001
								0.000

n = 128

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.4	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.8	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.2	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.6	0.895	0.022	0.763	0.152	0.536	0.433	0.173	0.814
2.0	0.519	0.018	0.397	0.121	0.326	0.379	0.114	0.786
2.4	0.172	0.006	0.138	0.069	0.116	0.275	0.053	0.688
2.8	0.034	0.002	0.030	0.034	0.021	0.171	0.022	0.541
3.2	0.011	0.001	0.003	0.009	0.009	0.081	0.005	0.411
3.6	0.004	0.000	0.001	0.003	0.000	0.025	0.001	0.252
4.0	0.001		0.000	0.000		0.013	0.000	0.142
	0.000					0.007		0.089
						0.003		0.041
						0.002		0.016
						0.000		0.007
								0.002
								0.000

Table E15

Experimental Results, B_3 Noncoherent.

B_T	n = 16								n = 32							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.4	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.8	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
1.2	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
1.6	0.828	0.145	0.597	0.376	0.244	0.746	0.070	0.928	0.906	0.092	0.701	0.299	0.363	0.637	0.118	0.882
2.0	0.661	0.129	0.499	0.346	0.208	0.726	0.066	0.925	0.834	0.089	0.651	0.291	0.347	0.629	0.114	0.880
2.4	0.444	0.095	0.333	0.294	0.151	0.665	0.049	0.899	0.598	0.070	0.498	0.264	0.272	0.595	0.097	0.864
2.8	0.260	0.071	0.204	0.227	0.087	0.586	0.029	0.851	0.349	0.047	0.307	0.209	0.163	0.524	0.069	0.820
3.2	0.143	0.043	0.111	0.171	0.044	0.487	0.013	0.788	0.184	0.021	0.166	0.148	0.087	0.442	0.042	0.756
3.6	0.081	0.027	0.057	0.123	0.026	0.382	0.005	0.721	0.091	0.015	0.086	0.102	0.033	0.343	0.017	0.685
4.0	0.047	0.018	0.033	0.088	0.013	0.307	0.001	0.636	0.047	0.007	0.040	0.074	0.014	0.241	0.008	0.599
	0.023	0.005	0.016	0.058	0.007	0.235	0.001	0.553	0.023	0.005	0.016	0.052	0.009	0.175	0.021	0.512
	0.017	0.004	0.010	0.038	0.005	0.190	0.000	0.481	0.012	0.002	0.007	0.029	0.002	0.132	0.000	0.421
	0.010	0.004	0.004	0.023	0.003	0.157		0.389	0.003	0.002	0.003	0.023	0.001	0.083		0.334
	0.006	0.003	0.000	0.017	0.001	0.106		0.315	0.003	0.001	0.000	0.011	0.001	0.051		0.274
	0.004	0.002		0.014	0.001	0.086		0.259	0.000	0.000		0.006	0.001	0.035		0.205
	0.002	0.001		0.009	0.001	0.058		0.203				0.005	0.000	0.023		0.147
	0.001	0.001		0.005	0.000	0.049		0.148				0.004		0.016		0.105
	0.001	0.000		0.004		0.037		0.116				0.004		0.009		0.076
	0.001			0.001		0.028		0.082				0.003		0.006		0.052
	0.001			0.001		0.021		0.064				0.002		0.004		0.025
	0.001			0.001		0.012		0.049				0.001		0.002		0.016
	0.000			0.001		0.007		0.036				0.000		0.002		0.010
				0.000		0.004		0.029						0.001		0.004
						0.003		0.020						0.000		0.002
						0.000		0.000								0.001

B_T	n = 64								n = 128							
	A = 1		A = 2		A = 3		A = 4		A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.4	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.8	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.2	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
1.6	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
2.0	0.932	0.055	0.786	0.205	0.444	0.550	0.164	0.834	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
2.4	0.832	0.052	0.683	0.196	0.384	0.536	0.151	0.828	0.945	0.023	0.828	0.155	0.555	0.435	0.181	0.816
2.8	0.549	0.043	0.434	0.172	0.272	0.489	0.110	0.814	0.785	0.021	0.664	0.142	0.465	0.418	0.160	0.806
3.2	0.312	0.028	0.225	0.117	0.155	0.433	0.066	0.762	0.493	0.018	0.379	0.113	0.289	0.362	0.100	0.771
3.6	0.124	0.018	0.092	0.068	0.068	0.342	0.036	0.687	0.259	0.008	0.190	0.081	0.142	0.295	0.052	0.705
4.0	0.077	0.008	0.042	0.041	0.029	0.256	0.015	0.596	0.105	0.005	0.074	0.050	0.050	0.224	0.026	0.617
	0.030	0.005	0.015	0.024	0.015	0.181	0.007	0.503	0.042	0.003	0.034	0.033	0.018	0.156	0.012	0.519
	0.009	0.002	0.005	0.012	0.006	0.123	0.001	0.403	0.018	0.001	0.011	0.021	0.008	0.100	0.005	0.441
	0.005	0.001	0.001	0.006	0.003	0.080	0.001	0.311	0.007	0.001	0.003	0.005	0.005	0.053	0.002	0.341
	0.003	0.001	0.000	0.003	0.001	0.052	0.000	0.225	0.003	0.000	0.002	0.003	0.001	0.033	0.000	0.259
	0.000	0.000		0.001	0.000	0.034		0.162	0.000		0.001	0.000	0.001	0.018		0.166
				0.000		0.018		0.117			0.000		0.000	0.013		0.127
						0.010		0.063						0.010		0.092
						0.008		0.038						0.003		0.069
						0.007		0.023						0.003		0.041
						0.005		0.013						0.002		0.026
						0.002		0.006						0.000		0.011
						0.002		0.004								0.007
						0.001		0.001								0.002
						0.001		0.001								0.002
						0.000		0.000								0.000

Table E16

Experimental Results, B_4 Noncoherent.

n = 16

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.749	0.071	0.929
0.4	0.312	0.081	0.239	0.255	0.099	0.598	0.027	0.866
0.8	0.105	0.035	0.081	0.141	0.029	0.460	0.006	0.758
1.2	0.039	0.014	0.023	0.082	0.010	0.296	0.002	0.647
1.6	0.008	0.004	0.004	0.032	0.003	0.192	0.000	0.496
2.0	0.001	0.001	0.000	0.014	0.000	0.112		0.343
2.4	0.000	0.000		0.005		0.055		0.216
2.8				0.002		0.026		0.128
3.2				0.001		0.012		0.073
3.6				0.000		0.002		0.033
4.0						0.000		0.011
4.4								0.006
4.8								0.000

n = 32

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.4	0.295	0.032	0.262	0.175	0.126	0.478	0.036	0.779
0.8	0.086	0.012	0.069	0.090	0.038	0.336	0.014	0.662
1.2	0.019	0.005	0.015	0.047	0.010	0.201	0.002	0.542
1.6	0.005	0.001	0.003	0.027	0.001	0.105	0.000	0.402
2.0	0.001	0.000	0.000	0.007	0.000	0.050		0.261
2.4	0.000			0.004		0.024		0.161
2.8				0.002		0.007		0.077
3.2				0.001		0.005		0.041
3.6				0.000		0.002		0.017
4.0						0.000		0.006
4.4								0.000

n = 64

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
0.4	0.291	0.028	0.220	0.101	0.124	0.382	0.044	0.733
0.8	0.095	0.009	0.050	0.049	0.033	0.269	0.012	0.588
1.2	0.019	0.002	0.011	0.016	0.011	0.151	0.002	0.453
1.6	0.005	0.000	0.002	0.005	0.002	0.086	0.000	0.309
2.0	0.003		0.000	0.001	0.000	0.039		0.177
2.4	0.000			0.000		0.015		0.091
2.8						0.007		0.039
3.2						0.004		0.013
3.6						0.003		0.005
4.0						0.000		0.001
4.4								0.000

n = 128

B_T	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.977	0.023	0.844	0.156	0.565	0.435	0.185	0.816
0.4	0.310	0.011	0.221	0.080	0.152	0.282	0.040	0.683
0.8	0.076	0.004	0.048	0.036	0.029	0.183	0.012	0.542
1.2	0.018	0.000	0.018	0.020	0.005	0.079	0.005	0.401
1.6	0.005		0.004	0.004	0.000	0.039	0.001	0.276
2.0	0.001		0.000	0.001		0.014	0.000	0.152
2.4	0.001			0.000		0.008		0.095
2.8	0.000					0.004		0.050
3.2						0.002		0.027
3.6						0.001		0.010
4.0						0.000		0.002
4.4								0.000

Table E17

Experimental Results, B_5 Noncoherent.

B_T	n = 16							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929
0.4	0.505	0.112	0.388	0.301	0.148	0.663	0.042	0.885
0.8	0.287	0.074	0.206	0.235	0.081	0.561	0.020	0.836
1.2	0.173	0.048	0.117	0.174	0.038	0.490	0.011	0.761
1.6	0.105	0.033	0.066	0.126	0.024	0.383	0.002	0.707
2.0	0.068	0.020	0.039	0.087	0.015	0.299	0.001	0.635
2.4	0.035	0.008	0.023	0.066	0.007	0.236	0.000	0.533
2.8	0.019	0.004	0.011	0.041	0.005	0.196		0.450
3.2	0.012	0.004	0.005	0.028	0.004	0.153		0.391
3.6	0.009	0.004	0.003	0.017	0.002	0.122		0.308
4.0	0.005	0.003	0.000	0.016	0.001	0.082		0.259
	0.002	0.001		0.013	0.001	0.066		0.204
	0.002	0.000		0.006	0.000	0.050		0.153
	0.001			0.003		0.041		0.122
	0.001			0.002		0.030		0.097
	0.001			0.001		0.024		0.072
	0.001			0.001		0.017		0.051
	0.000			0.001		0.010		0.041
				0.000		0.007		0.029
						0.005		0.024
						0.003		0.016
						0.001		0.010
						0.001		0.005
						0.000		0.005
								0.003
								0.000

B_T	n = 32							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.4	0.469	0.050	0.391	0.224	0.190	0.524	0.065	0.818
0.8	0.245	0.028	0.208	0.153	0.102	0.446	0.029	0.745
1.2	0.129	0.016	0.101	0.108	0.043	0.347	0.015	0.666
1.6	0.064	0.011	0.044	0.076	0.024	0.259	0.006	0.588
2.0	0.030	0.006	0.024	0.052	0.009	0.190	0.003	0.498
2.4	0.012	0.003	0.008	0.035	0.004	0.137	0.001	0.421
2.8	0.007	0.001	0.002	0.021	0.002	0.093	0.001	0.342
3.2	0.003	0.001	0.000	0.015	0.001	0.059	0.000	0.274
3.6	0.002	0.000		0.010	0.001	0.039		0.209
4.0	0.001			0.006	0.001	0.026		0.161
	0.000			0.005	0.000	0.016		0.112
				0.003		0.009		0.088
				0.003		0.007		0.058
				0.002		0.005		0.038
				0.001		0.003		0.019
				0.000		0.003		0.008
						0.001		0.006
						0.000		0.003
								0.002
								0.001
								0.001
								0.001
								0.001
								0.000

B_T	n = 64							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_c	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
0.4	0.467	0.039	0.357	0.145	0.213	0.434	0.081	0.768
0.8	0.231	0.024	0.161	0.083	0.083	0.351	0.027	0.695
1.2	0.117	0.012	0.067	0.056	0.037	0.276	0.014	0.596
1.6	0.056	0.005	0.027	0.029	0.018	0.199	0.006	0.512
2.0	0.023	0.002	0.013	0.015	0.010	0.137	0.003	0.419
2.4	0.009	0.000	0.003	0.008	0.005	0.099	0.001	0.329
2.8	0.005		0.001	0.002	0.002	0.058	0.000	0.246
3.2	0.003		0.000	0.001	0.000	0.043		0.179
3.6	0.003			0.001		0.025		0.124
4.0	0.000			0.000		0.015		0.086
						0.009		0.046
						0.008		0.024
						0.007		0.016
						0.004		0.010
						0.002		0.006
						0.002		0.002
						0.001		0.002
						0.001		0.001
						0.001		0.001
						0.000		0.000

B_T	n = 128							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_c	P_c	P_e	P_c
0.	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.4	0.487	0.017	0.373	0.104	0.263	0.327	0.079	0.732
0.8	0.223	0.007	0.167	0.071	0.096	0.256	0.028	0.637
1.2	0.093	0.004	0.053	0.037	0.035	0.189	0.013	0.545
1.6	0.037	0.002	0.031	0.025	0.013	0.115	0.006	0.463
2.0	0.014	0.001	0.014	0.015	0.004	0.074	0.001	0.364
2.4	0.008	0.000	0.006	0.007	0.001	0.047	0.001	0.300
2.8	0.003		0.003	0.001	0.001	0.026	0.000	0.211
3.2	0.001		0.001	0.001	0.000	0.016		0.147
3.6	0.001		0.000	0.000		0.010		0.108
4.0	0.000					0.006		0.072
						0.005		0.054
						0.002		0.040
						0.001		0.025
						0.001		0.012
						0.000		0.005
								0.003
								0.000

Table E18

Experimental Results, B_6 Noncoherent.

B_T	n = 16							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.853	0.147	0.620	0.380	0.254	0.746	0.071	0.929
0.1	0.516	0.112	0.399	0.303	0.147	0.662	0.040	0.885
0.2	0.267	0.066	0.195	0.218	0.076	0.544	0.019	0.823
0.3	0.117	0.039	0.082	0.135	0.027	0.408	0.005	0.715
0.4	0.043	0.009	0.026	0.071	0.010	0.249	0.000	0.549
0.5	0.012	0.004	0.004	0.026	0.003	0.132		0.329
0.6	0.000	0.001	0.000	0.004	0.000	0.039		0.130
0.7				0.000		0.003		0.011
0.8						0.000		0.000

B_T	n = 32							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.908	0.092	0.701	0.299	0.363	0.637	0.118	0.882
0.1	0.454	0.047	0.375	0.217	0.180	0.512	0.060	0.807
0.2	0.196	0.025	0.162	0.136	0.073	0.392	0.021	0.713
0.3	0.065	0.007	0.043	0.071	0.024	0.247	0.007	0.556
0.4	0.010	0.003	0.003	0.027	0.004	0.114	0.001	0.374
0.5	0.002	0.000	0.000	0.008	0.001	0.025	0.000	0.174
0.6	0.000			0.000	0.000	0.005		0.030
0.7						0.000		0.001
0.8								0.000

B_T	n = 64							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.945	0.055	0.795	0.205	0.448	0.552	0.165	0.835
0.1	0.910	0.035	0.316	0.124	0.185	0.412	0.069	0.751
0.2	0.154	0.013	0.098	0.065	0.049	0.292	0.020	0.628
0.3	0.039	0.003	0.019	0.022	0.013	0.154	0.004	0.446
0.4	0.005	0.000	0.001	0.002	0.002	0.065	0.001	0.226
0.5	0.001		0.000	0.001	0.000	0.013	0.000	0.060
0.6	0.000			0.000		0.003		0.005
0.7						0.000		0.000

B_T	n = 128							
	A = 1		A = 2		A = 3		A = 4	
	P_e	P_c	P_e	P_c	P_e	P_c	P_e	P_c
0.	0.977	0.023	0.844	0.156	0.565	0.435	0.184	0.816
0.1	0.397	0.015	0.301	0.094	0.207	0.298	0.060	0.698
0.2	0.103	0.005	0.066	0.042	0.044	0.189	0.012	0.542
0.3	0.020	0.000	0.019	0.015	0.007	0.072	0.003	0.365
0.4	0.002		0.002	0.001	0.000	0.016	0.000	0.156
0.5	0.000		0.000	0.000		0.004		0.042
0.6						0.000		0.000
0.7								

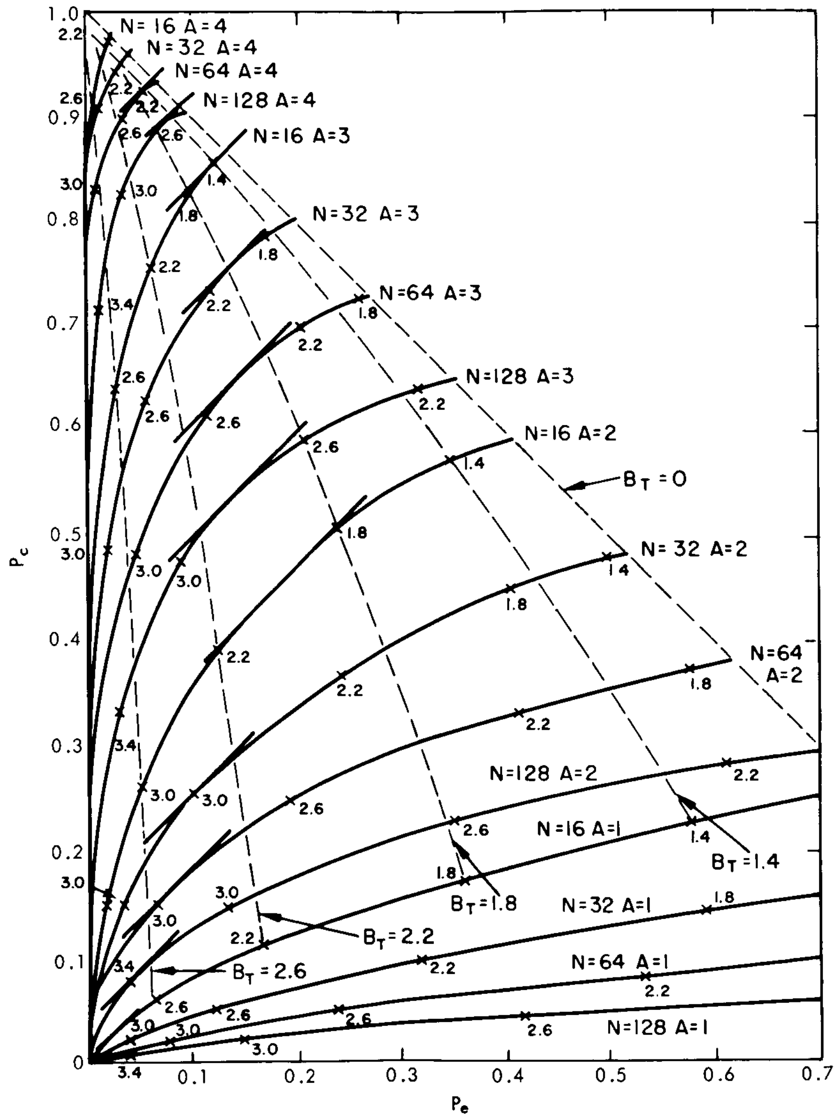

Figure E1—P_c vs. P_e for B_1 coherent (theoretical).

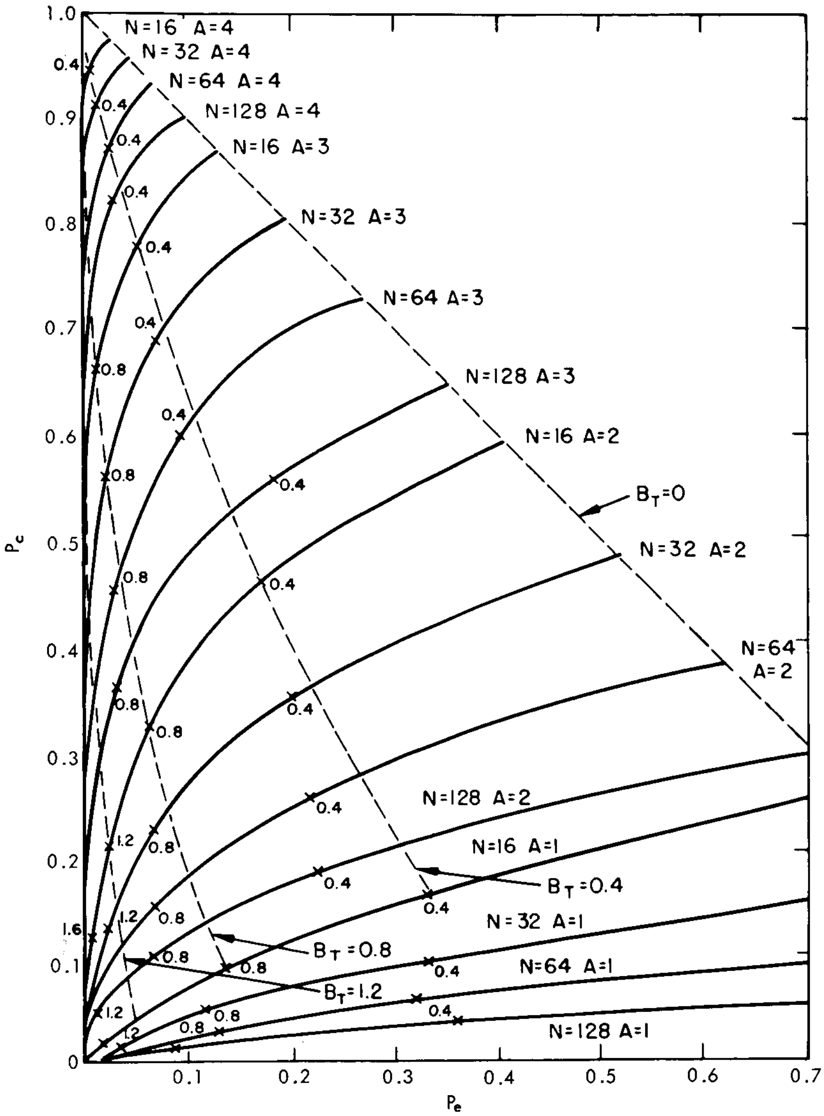

Figure E2—P_c vs. P_e for B_4 coherent (theoretical).

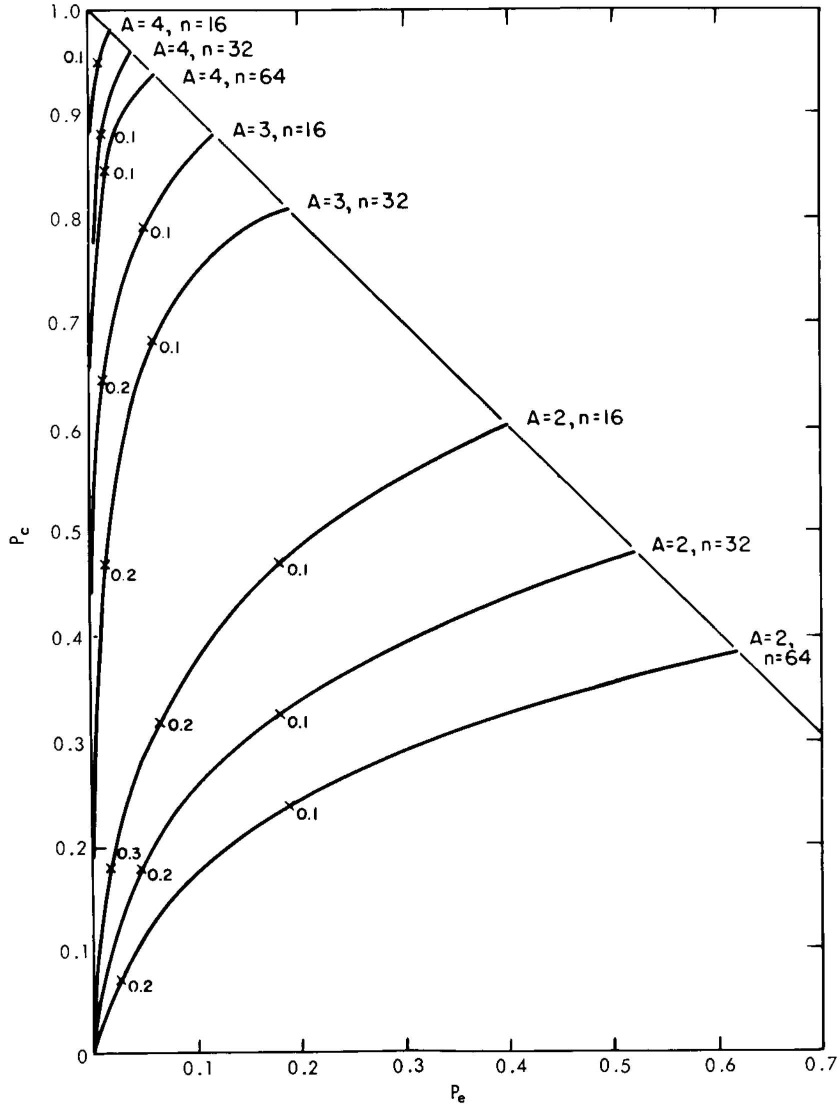

Figure E3— P_c vs. P_e for B_6 coherent (theoretical).

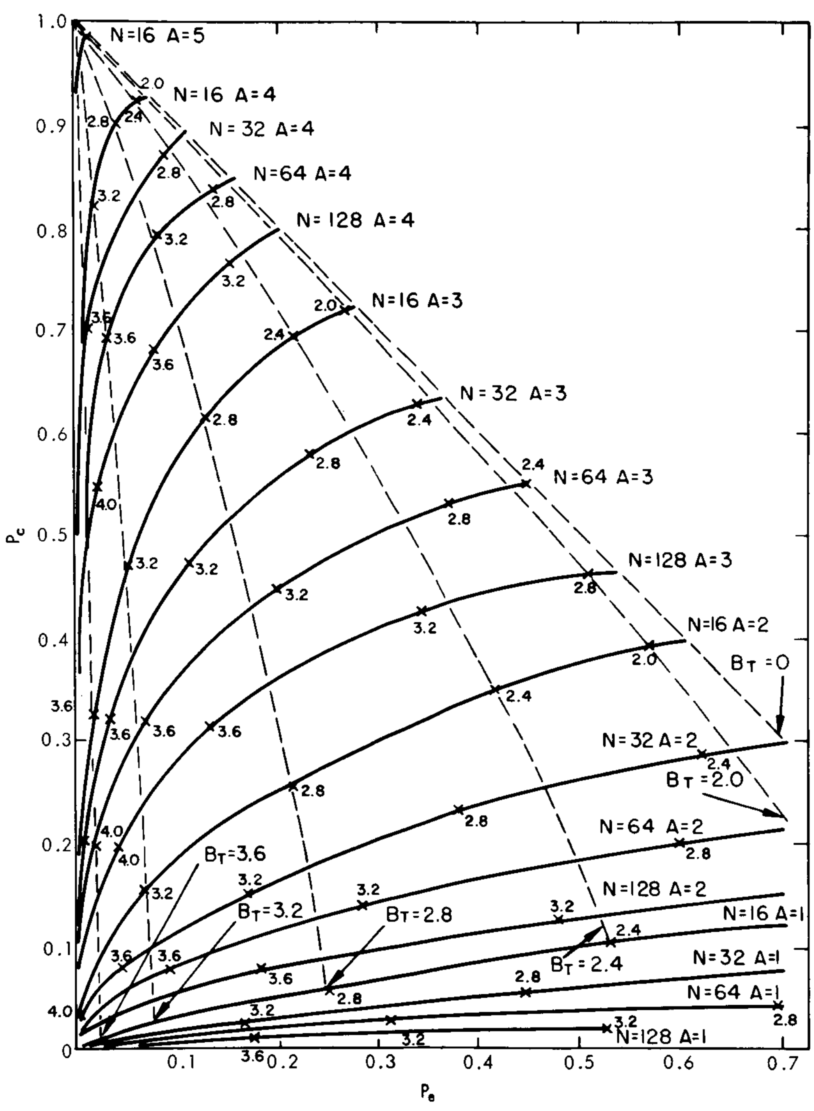

Figure E4— P_c vs. P_e for B_1 noncoherent (theoretical).

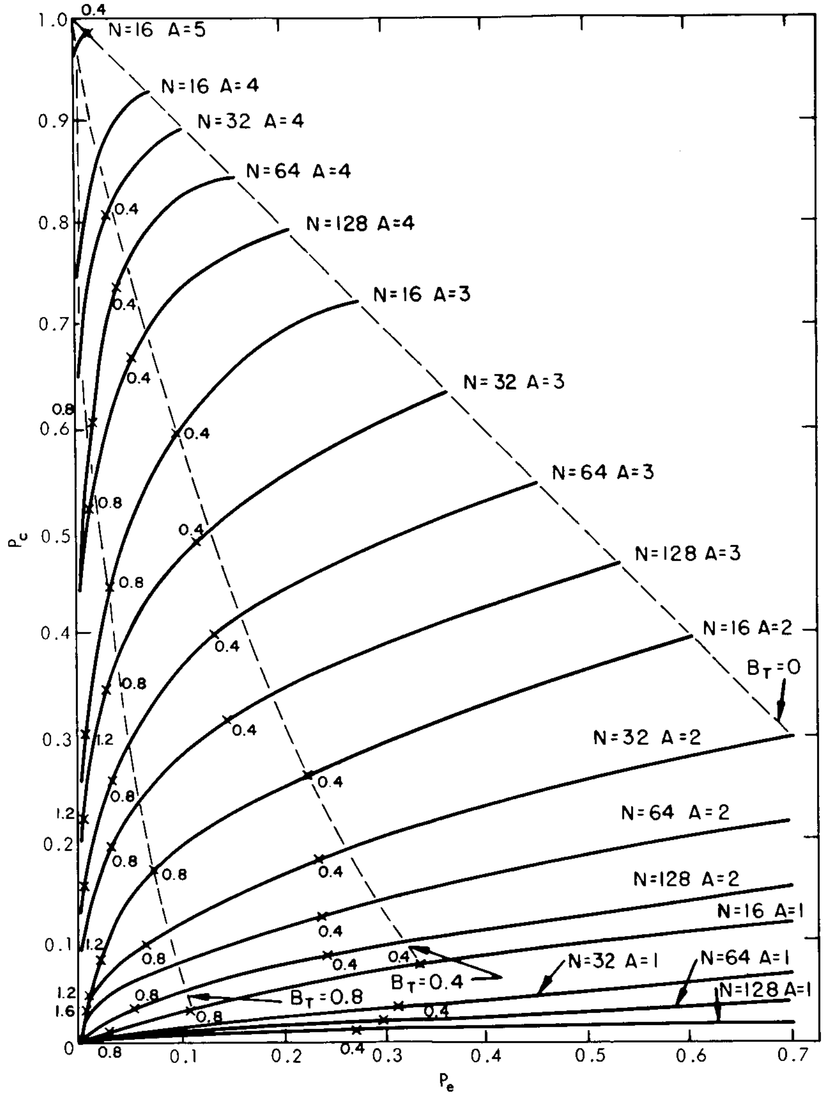

Figure E5— P_c vs. P_e for B_4 noncoherent (theoretical).

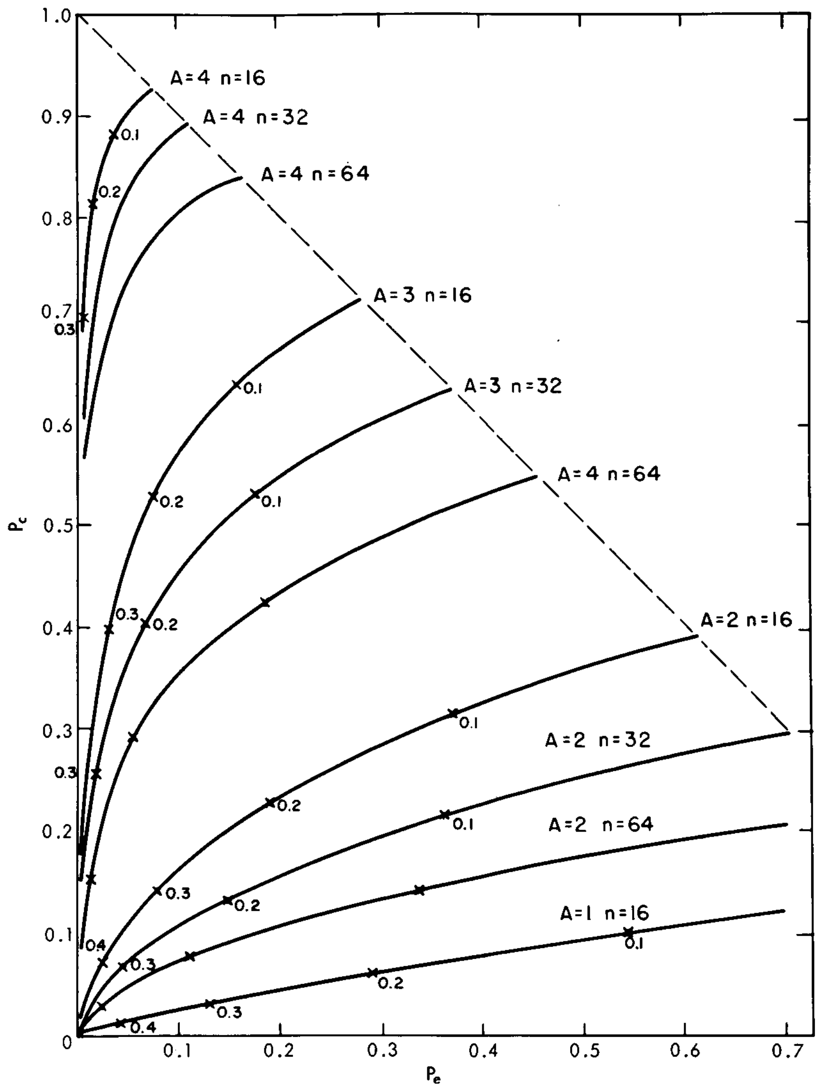

Figure E6— P_c vs. P_e for B_6 noncoherent (theoretical).

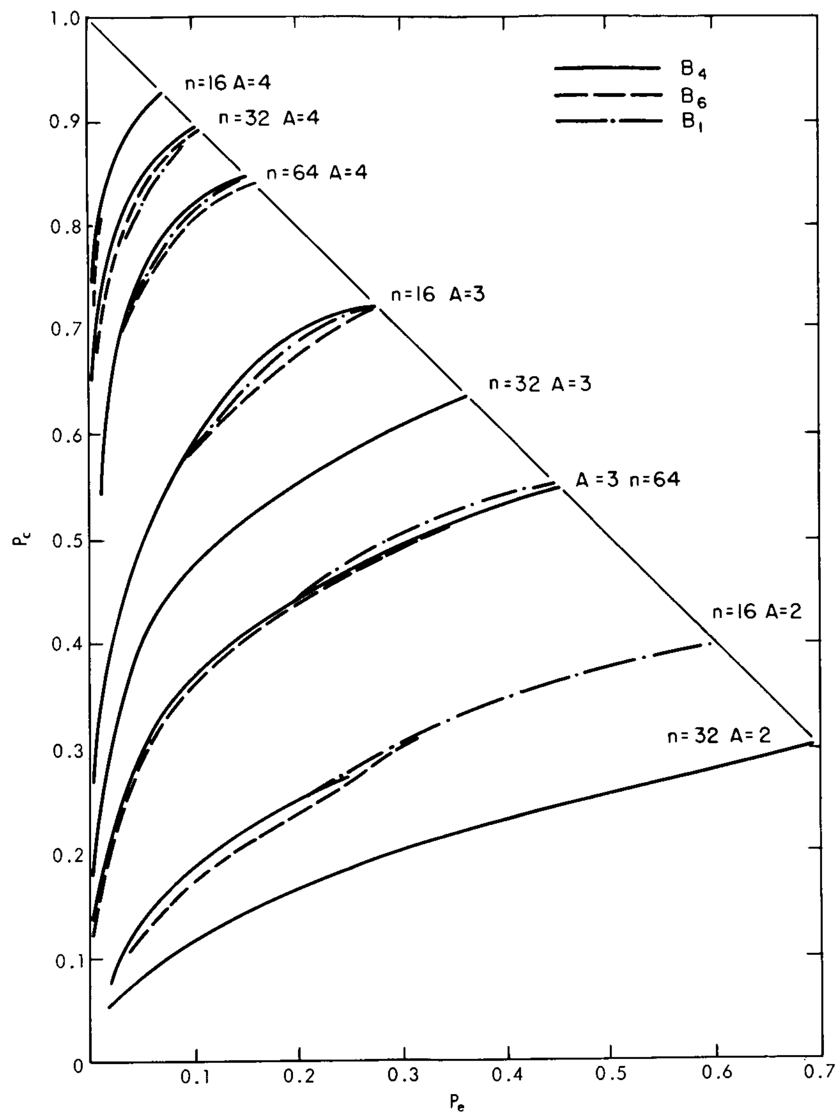

Figure E7—Comparison of B_1, B_4, and B_6 (theoretical).

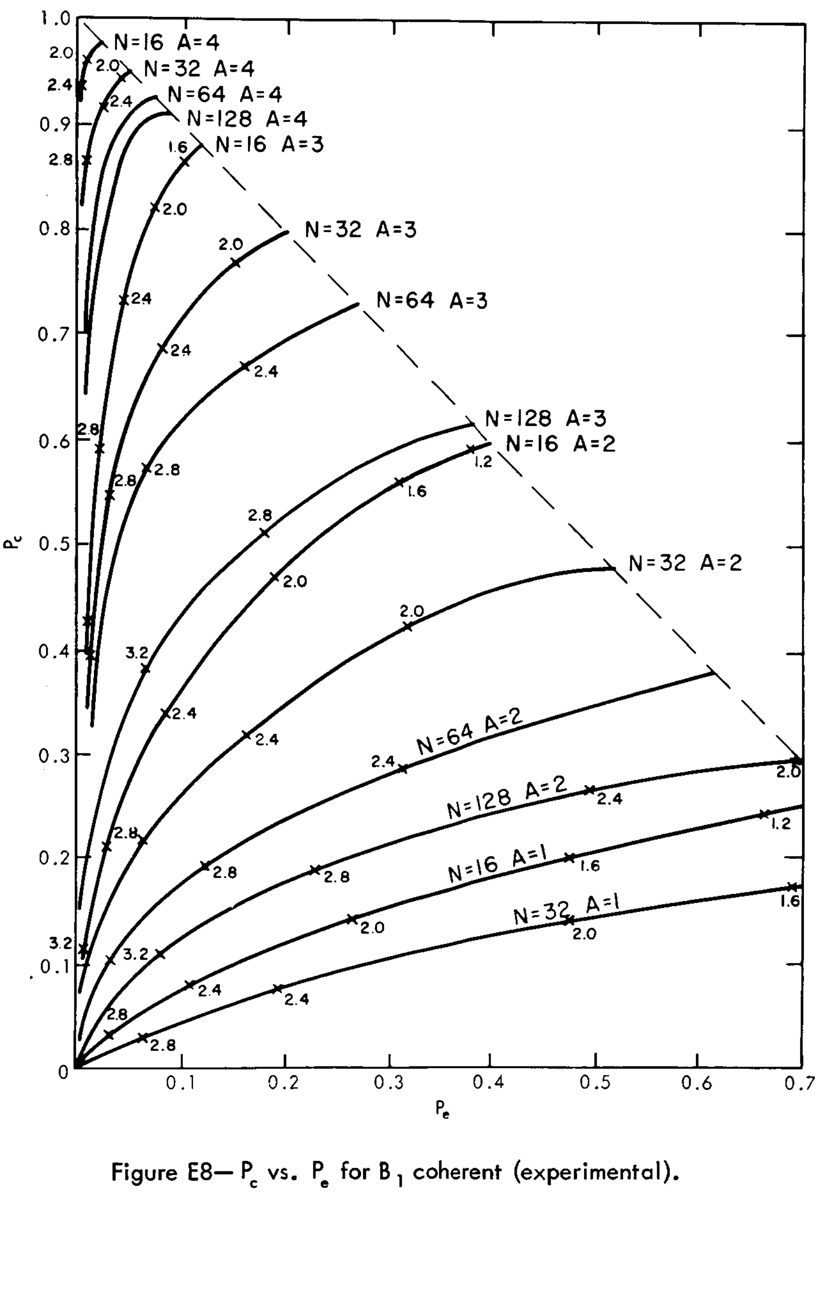

Figure E8— P_c vs. P_e for B_1 coherent (experimental).

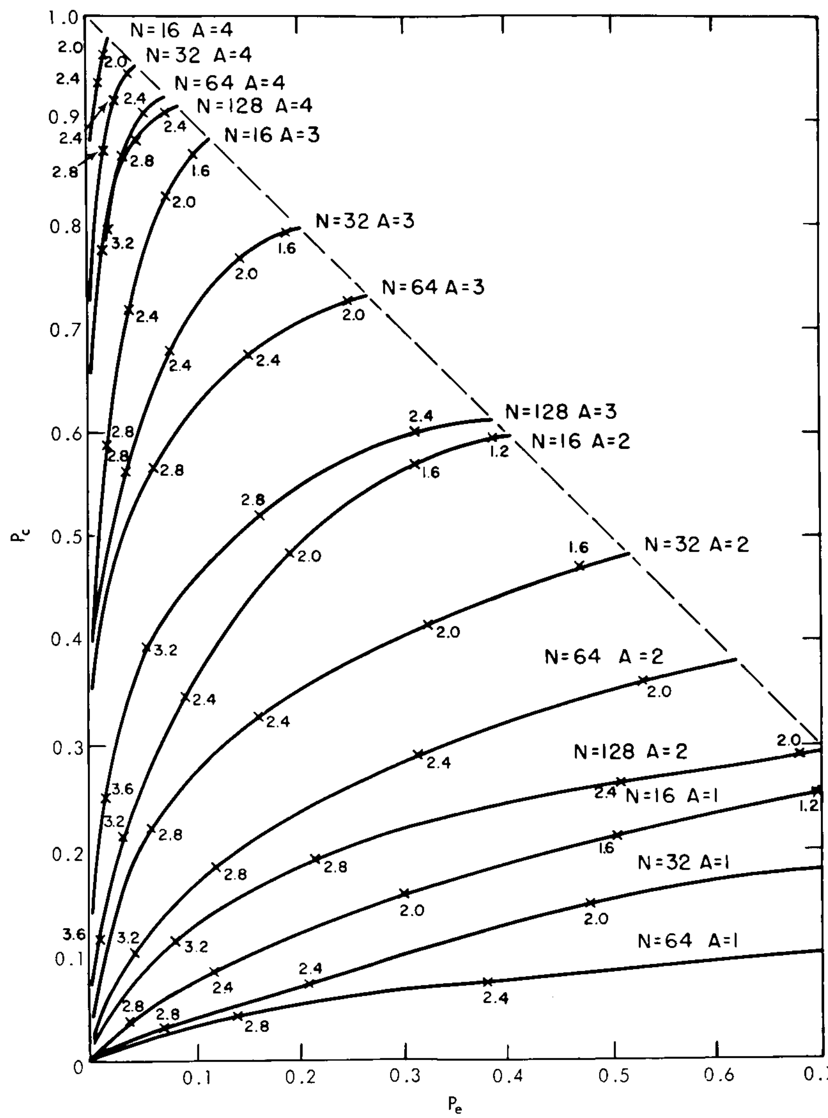

Figure E9— P_c vs. P_e for B_2 coherent (experimental).

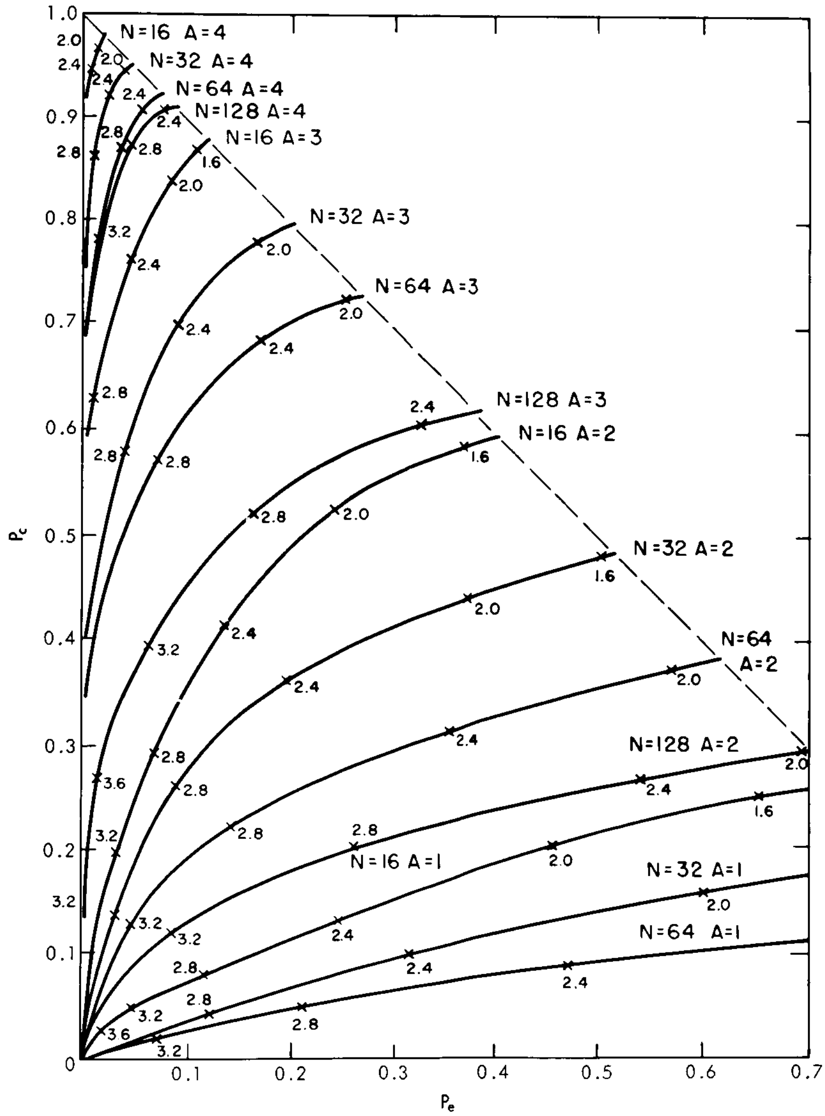

Figure E10— P_c vs. P_e for B_3 coherent (experimental).

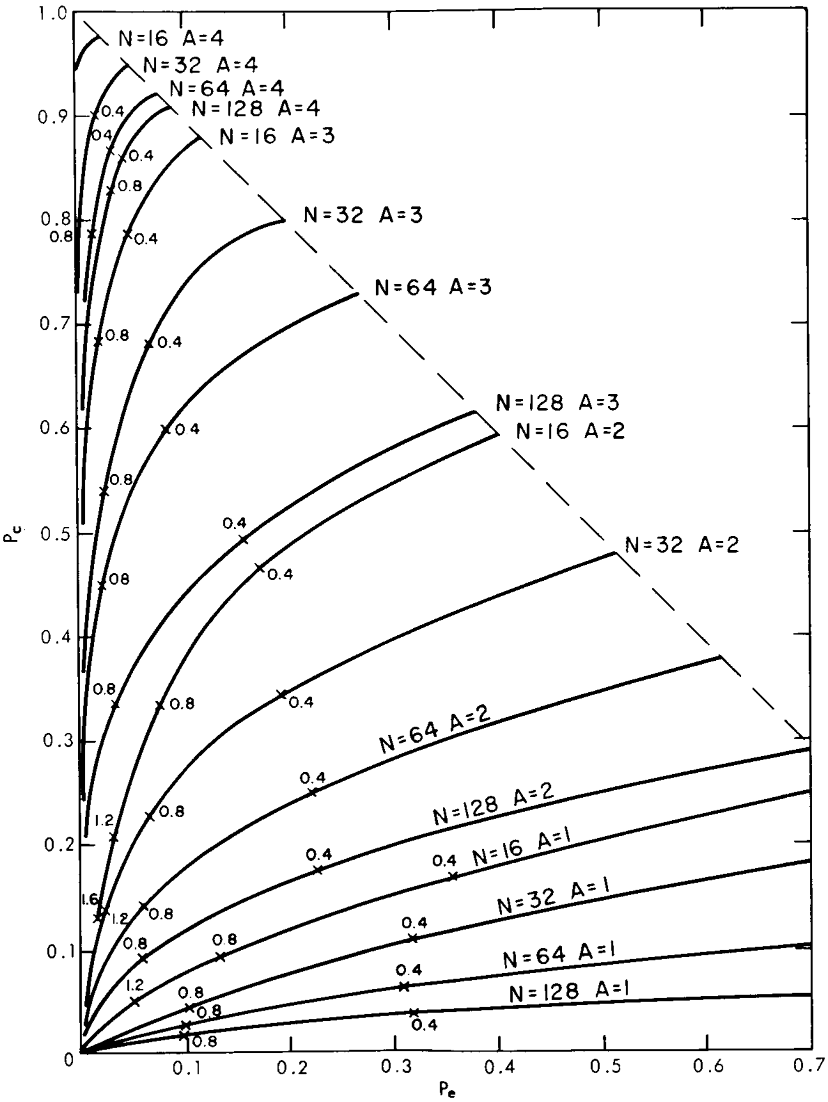

Figure E11— P_c vs. P_e for B_4 coherent (experimental).

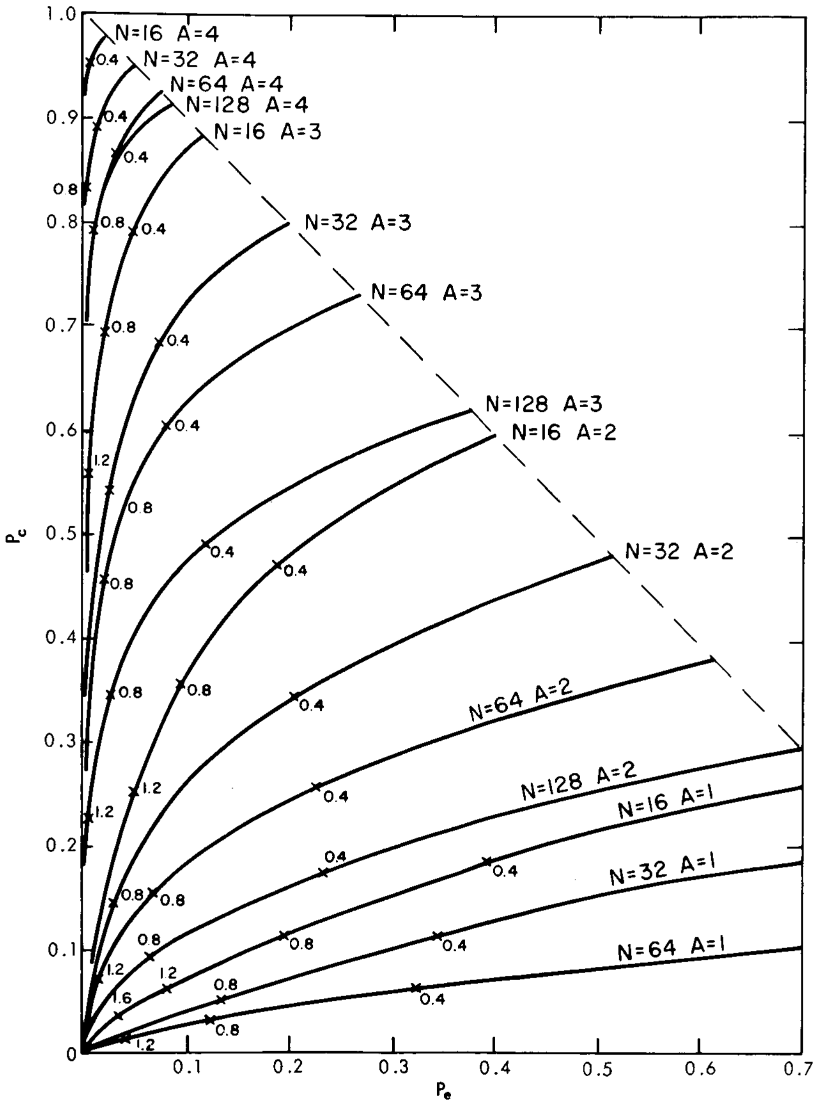

Figure E12— P_c vs. P_e for B_5 coherent (experimental).

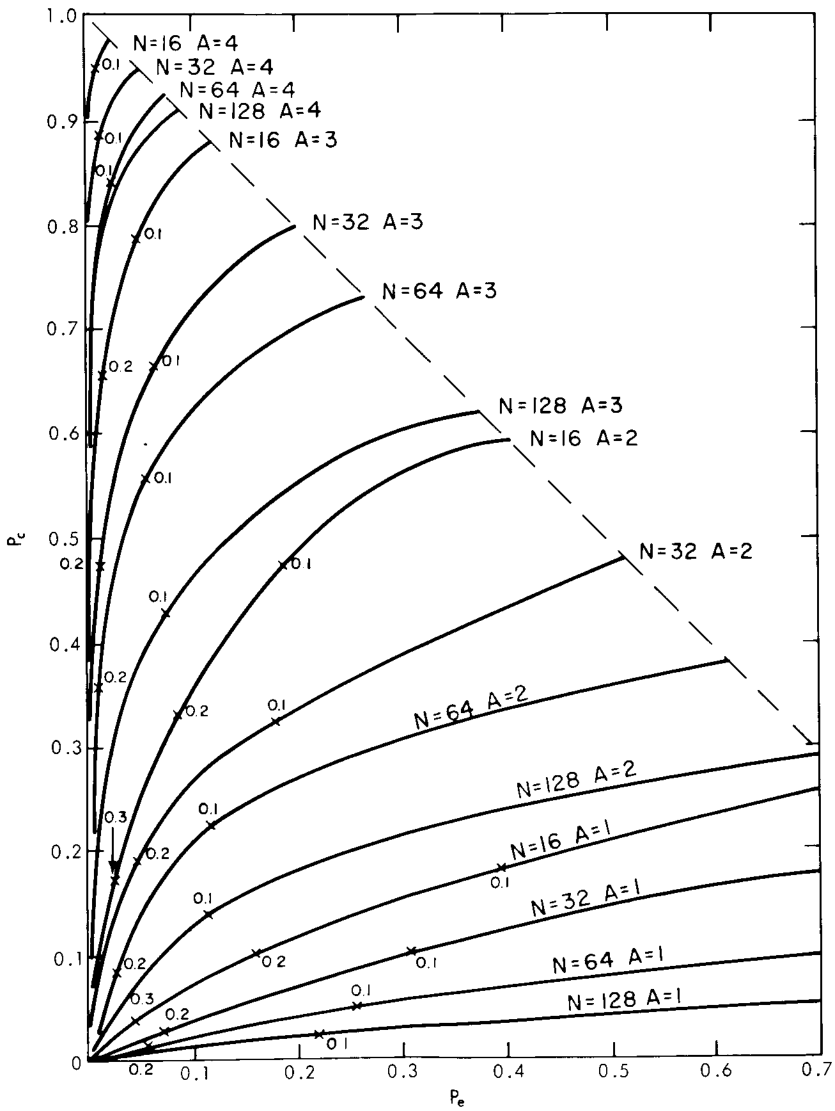

Figure E13— P_c vs. P_e for B_6 coherent (experimental).

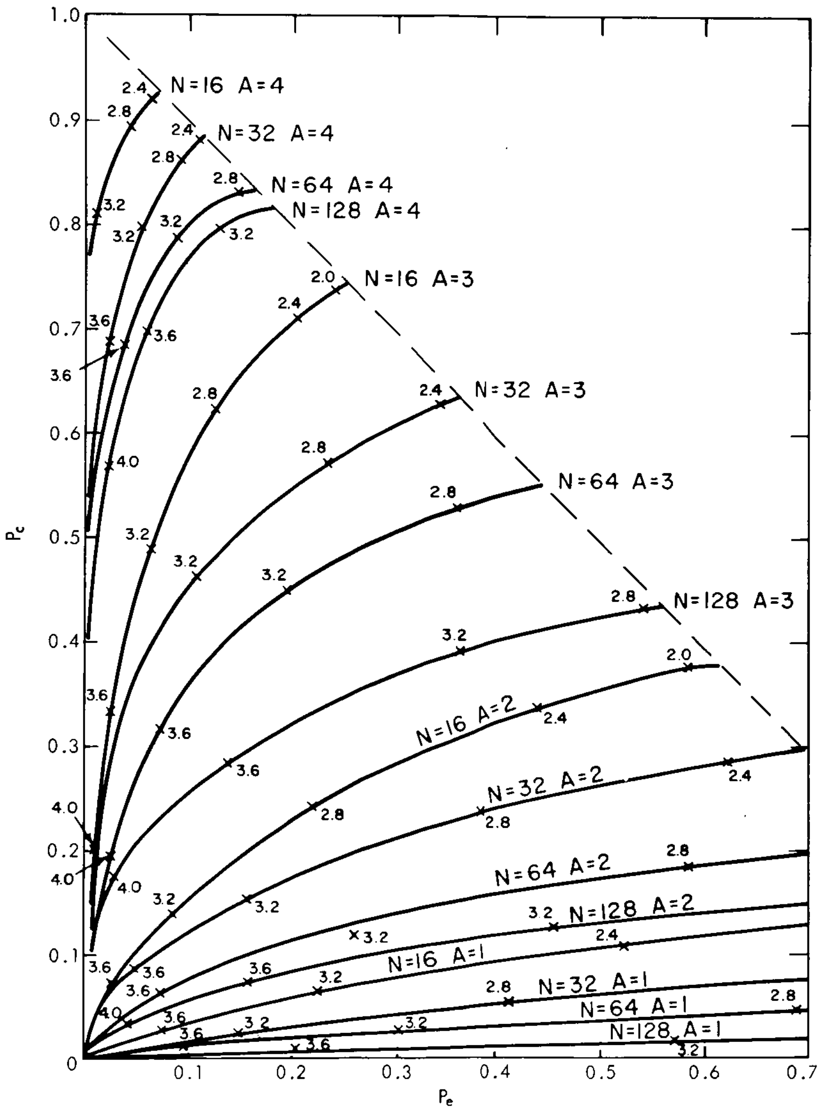

Figure E14— P_c vs. P_e for B_1 noncoherent (experimental).

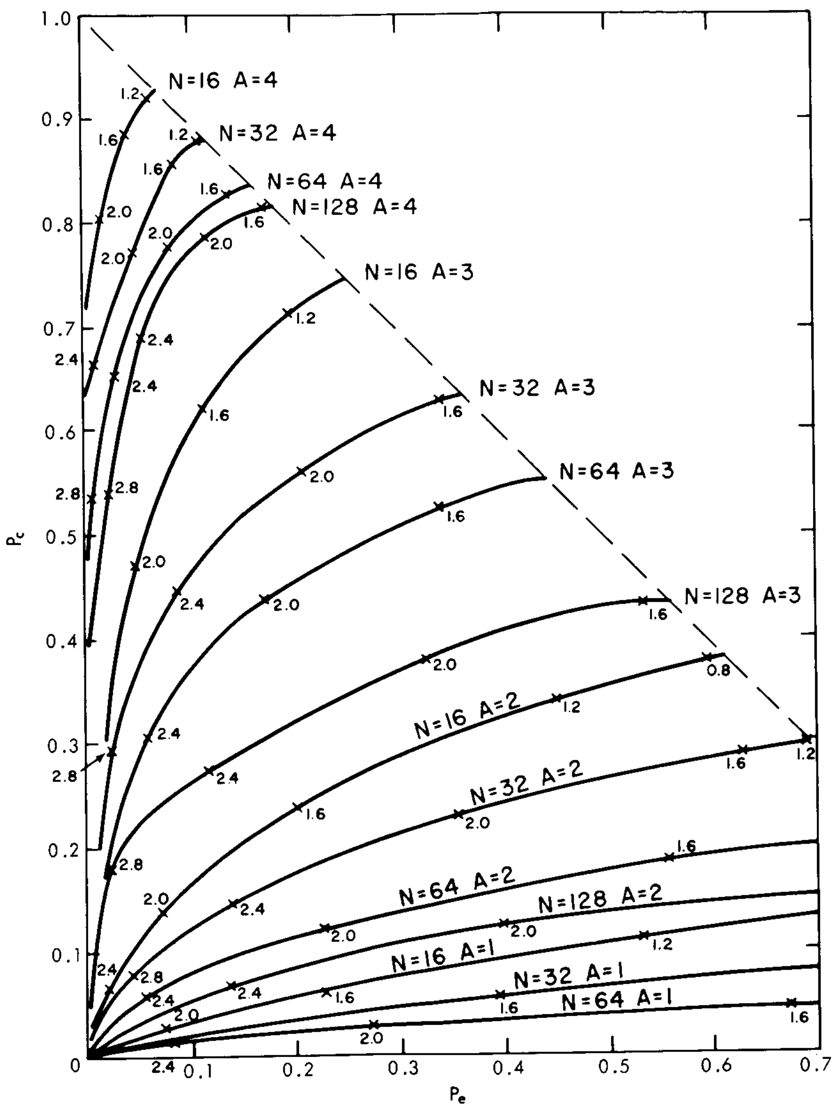

Figure E15— P_c vs. P_e for B_2 noncoherent (experimental).

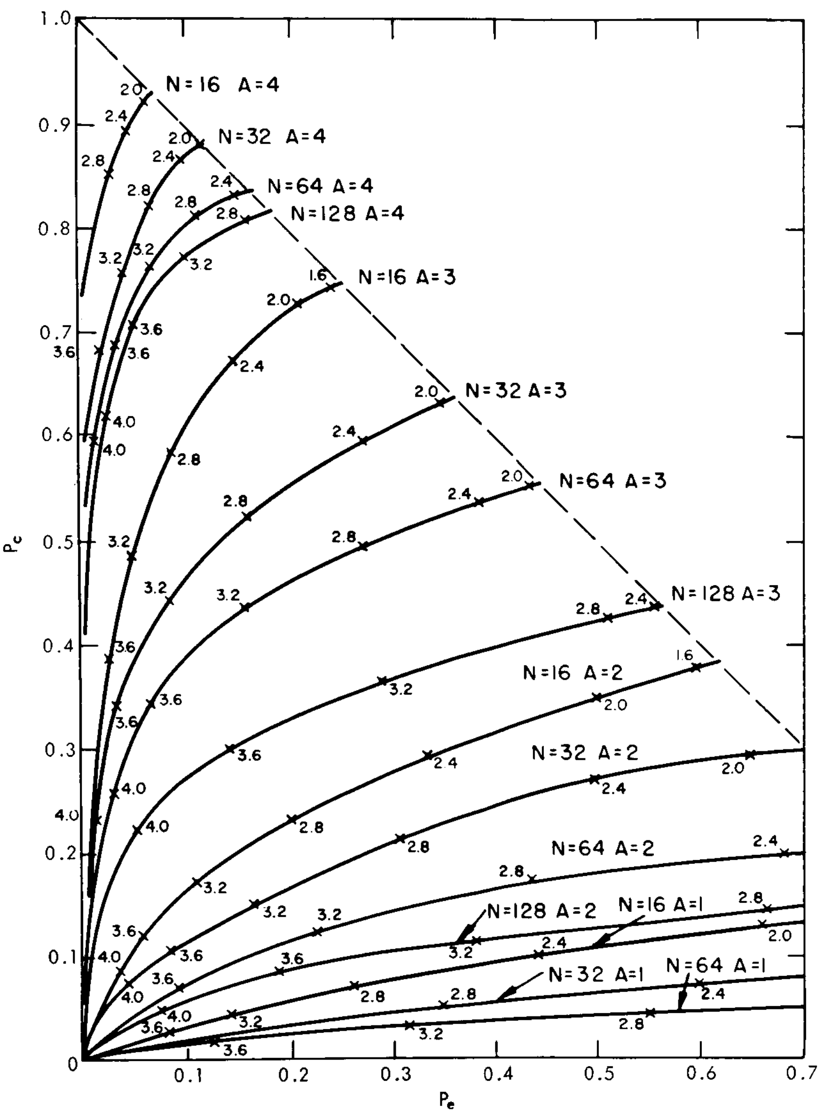

Figure E16—P_c vs. P_e for B_3 noncoherent (experimental).

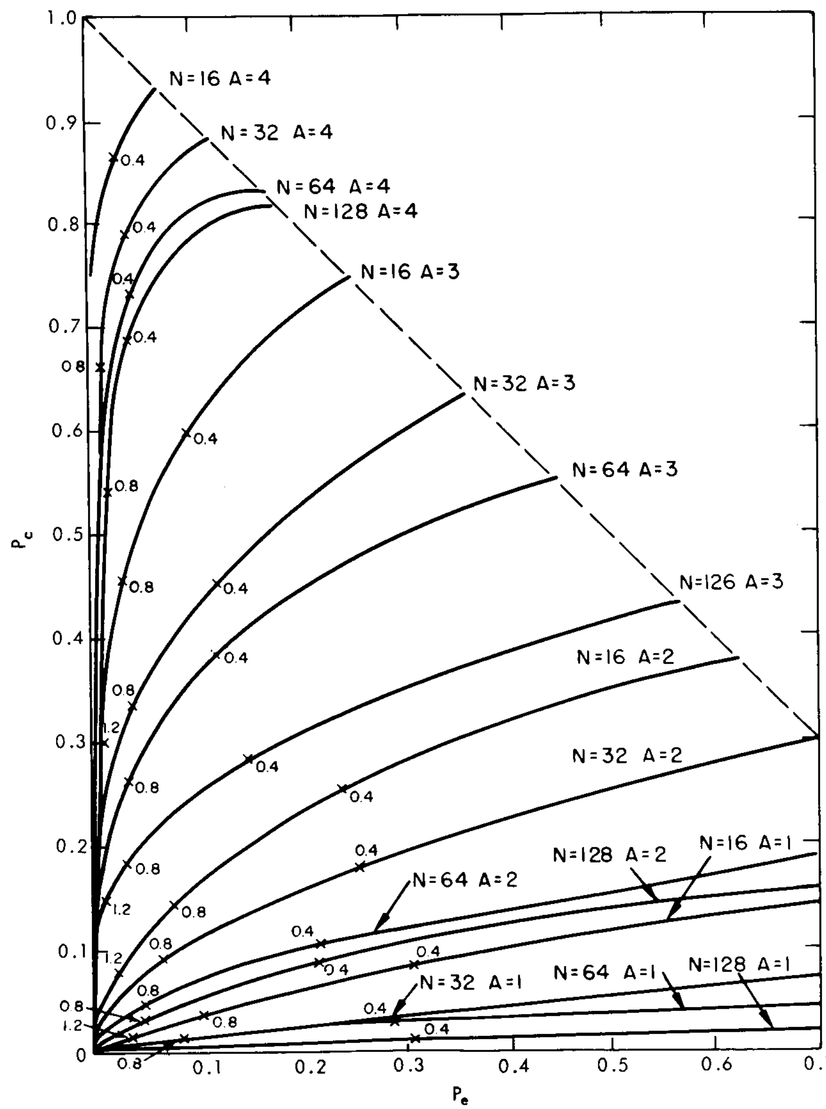

Figure E17—P_c vs. P_e for B_4 noncoherent (experimental).

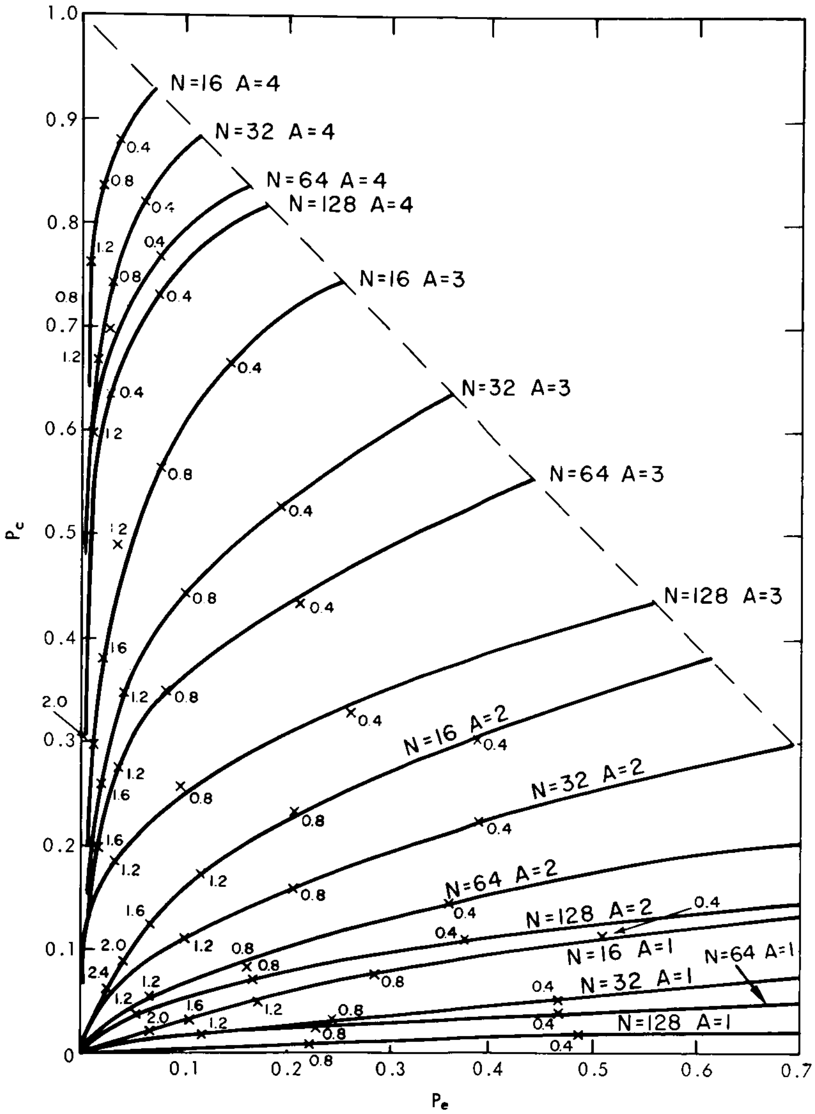

Figure E18— P_c vs. P_e for B_5 noncoherent (experimental).

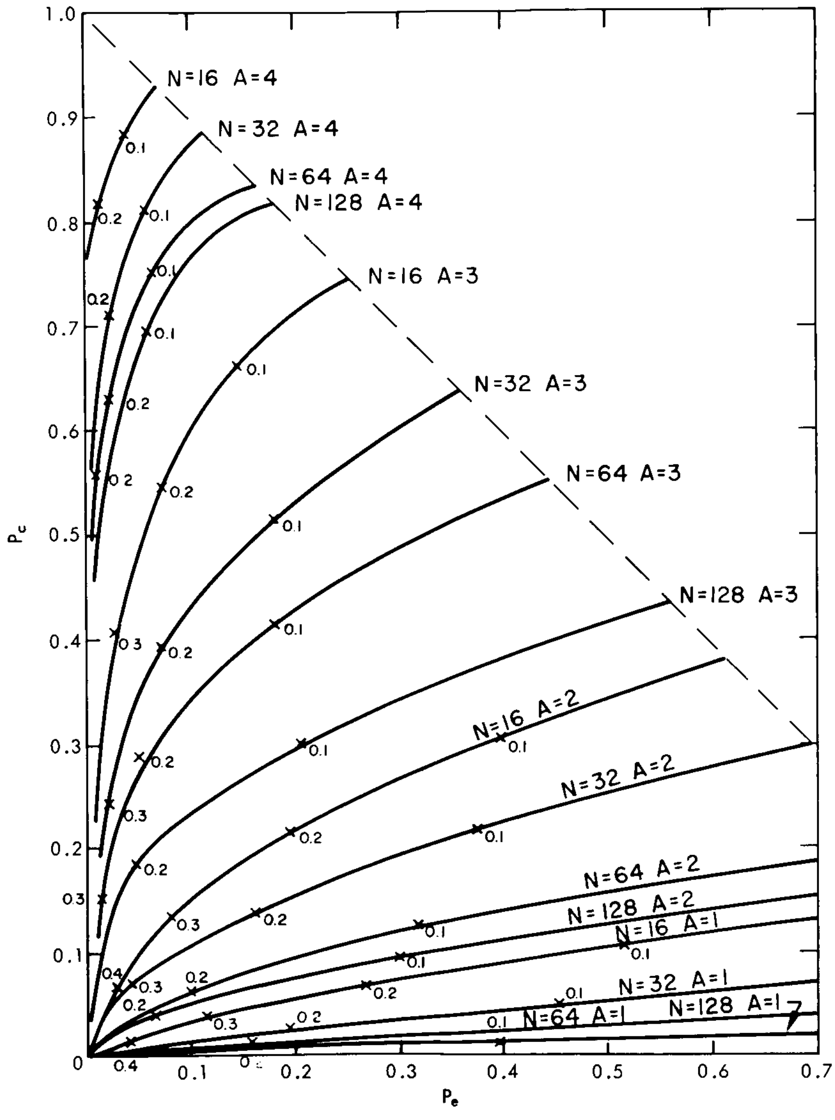

Figure E19— P_c vs. P_e for B_6 noncoherent (experimental).

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