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ON THE DISTRIBUTION OF THE SUM OF INDEPENDENT DOUBLY TRUNCATED GAMMA VARIABLES

Prepared under Contract No. NAS 8-11175 by
D. Earl Lavender

UNIVERSITY OF GEORGIA

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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ON THE DISTRIBUTION OF THE SUM OF INDEPENDENT DOUBLY TRUNCATED GAMMA VARIABLES[1]

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ABSTRACT

The density and distribution functions of the sum of N independent doubly truncated Gamma variables is derived for the case where the parameter α is one and N is any positive integer and for the cases where $N = 2$ or $N = 3$ and α is any positive integer.

Tables of critical values for these distributions are given as functions of the truncation points, and a comparison is made between these critical values and the estimated critical values given by Pearson's β_1 and β_2 tables.

[1] The research reported in this paper was submitted as a Ph.D. dissertation directed by Dr. A. C. Cohen at the University of Georgia, Athens, Georgia. This research was performed under NASA Contract NAS8-11175 with the Aerospace Environment Office, Aero-Astrodynamics Laboratory, Marshall Space Flight Center, Huntsville, Alabama.

FOREWORD

This report presents results of an investigation performed by the Department of Statistics, University of Georgia, Athens, Georgia, as a part of NASA Contract NAS8-11175 with the Aerospace Environment Office, Aero-Astrodynamics Laboratory, NASA-George C. Marshall Space Flight Center, Huntsville, Alabama. This research was performed by Mr. D. Earl Lavender under the supervision of Dr. A. C. Cohen, Jr., the contract principal investigator, and was submitted in August 1966, as a PH.D. dissertation, in Mathematics. The NASA contract monitors are Mr. O. E. Smith and Mr. L. W. Falls.

TABLE OF CONTENTS

ON THE DISTRIBUTION OF THE SUM OF INDEPENDENT DOUBLY TRUNCATED GAMMA VARIABLES

SUMMARY

When performing tests of hypotheses, the experimenter usually assumes that the possible range of the measurements applied to his observations is unlimited. Actually this is seldom the case due to physical limitations of measuring devices, or physical restrictions on the elements of a population.

To make allowance for these restrictions, we may assume sampling from a population with a truncated distribution.

In many cases, the test statistic used in a test of hypothesis is some function of the sum of the measurements obtained in a random sample.

The probability density function for the sum of N independent variables, each having a Gamma density function with parameter α is known to be a Gamma density function with parameter $n\alpha$, Cramer [11]. This, however, is not the case if each of the variables has a truncated Gamma density function.

In this paper, the density and distribution functions of the sum of n independent variables, each having a truncated Gamma density function, is derived for the case where the parameter α is one and n is any positive integer and for the cases where $n = 2$ or $n = 3$ and α is any positive integer.

I. INTRODUCTION

The Gamma distribution serves as a model for describing many of the random variables which concern aerospace scientists. In particular, wind velocities and measurements of various physical characteristics of space vehicle components conform to this distribution.

Quite often, restrictions which apply to the observation of sample data from these distributions, in effect produce a truncation which must be taken into account in estimating parameters and in testing hypotheses based on such samples.

Estimation in the truncated Gamma distribution and in special cases of the Gamma distribution has been dealt with by various authors including Cohen [4, 5, 6, 7, 8, 9, 10], Des Raj [16], Iyer [13], Epstein [12], and Sarhan and Greenberg [17].

Aggarwal and Guttman [1, 2], and Williams [18] have examined the loss of power when using tests based on the assumption that the variable being sampled has a complete normal distribution when, in fact, the distribution was a truncated normal distribution.

In this paper we are concerned with hypothesis testing and in particular with the distribution of sums of sample observations from a doubly truncated Gamma distribution.

The distribution of the sum of N independent Gamma variables with parameter α is known to be a Gamma distribution with parameter $N\alpha$, Cramer [11]. However, if the variables have a truncated, rather than a complete, Gamma distribution this is not the case.

In this paper the distribution of the sum of independent doubly truncated Gamma variables is derived for the case where the parameter is one and the sample is of any size N, and for the cases where the sample is of size $N = 2$ or $N = 3$ and the parameter α is any positive integer.

Tables of critical values for these distributions are given as functions of the truncation points, a and b, which were selected so that approximatly 1%, 2%, 3%, and 4% was truncated from the left and right tails respectively.

Tables of the mean μ, the standard derivation σ, and

Pearson's β_1 and β_2 values, Kendall [14], are also given for selected values of N and α.

When the estimated critical values given by β_1 and β_2 tables, Beyer [3], were compared with the actual critical values for the distributions which are derived in this paper, the agreement was found to be quite close. After rounding to one decimal place, none differed by more than one-tenth.

Hence for larger values of N it is felt that the estimated critical values given by β_1 and β_2 can be used confidently.

II. DENSITY AND DISTRIBUTION FUNCTIONS

The distribution function and density function of the Gamma distribution are defined by

2.1 $$G(x;\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt \quad \text{where} \quad \alpha > 0, \quad \text{and}$$

2.2 $$g(x;\alpha) = G'(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad 0 < x < \infty,$$

respectively, where $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$.

More generally the Gamma distribution is sometimes defined by

$G(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t/\beta} dt$. However, if we make the substitution $z = x/\beta$, then $G(z;\alpha)$ is as defined by 2.1.

Tables for the Gamma distribution function as defined by 2.1 have been provided by Pearson [15].

When the distribution is truncated on the left at a and on the right at b, the density function and distribution functions are defined by

2.3 $$f_T(x;\alpha) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & a \leq x \leq b \quad \text{and} \\ 0 & \text{otherwise} \end{cases}$$

2.4 $\quad F_T(x;\alpha) = \frac{1}{I\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, dt$, where

$$I = \frac{1}{\Gamma(\alpha)} \int_a^b x^{\alpha-1}\, e^{-x}\, dx = G(b;\alpha) - G(a;\alpha).$$

We will now consider some special cases of the Gamma distribution.

The density function for the Pearson Type III distribution is defined by

2.5 $\quad g(x) = \frac{c}{\sigma}\left[1 + \frac{3}{2}\left(\frac{x-u}{\sigma}\right)\right]^{4/\alpha_3^2 - 1} e^{-2/\alpha_3\left(\frac{x-u}{\sigma}\right)}, \quad u - \frac{2\sigma}{\alpha_3} < x < \infty,$

where $c = \left(\frac{4}{\alpha_3^2}\right)^{4/\alpha_3^2 - 1} e^{-4/\alpha_3^2}\left[\Gamma\left(\frac{4}{\alpha_3^2}\right)\right]^{-1}.$

If we make the substitution $\alpha = \frac{4}{\alpha_3^2}$ and

$z = \alpha\left(\frac{x-u}{\sigma\sqrt{\alpha}} + 1\right)$, then $g(z) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}, \quad 0 < z < \infty.$

The density function for the Chi-Square distribution is defined by

2.6 $\quad g(x^2) = \frac{1}{2^{r/2}\Gamma\left(\frac{r}{2}\right)} (x^2)^{r/2 - 1} e^{-x^2/2}, \quad r > 0,\ 0 < x^2 < \infty.$

If we make the substitutions $z = \frac{x^2}{2}$ and $\alpha = \frac{r}{2}$,
then $g(z) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}, \quad 0 < z < \infty.$

The density function for the standard normal distribution is defined by

2.7 $\quad f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty.$

If we make the substitutions $\alpha = \frac{1}{2}$ and $x = \frac{y^2}{2}$,
then 2.2 becomes $g(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty.$

For the case $\alpha = 1$ we have $g(x; 1) = e^{-x}$, $0 < x < \infty$. This distribution is known as the exponential distribution and is usually defined by

$$2.8 \quad f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty, \quad \theta > 0.$$

III. THE EXPONENTIAL CASE

Suppose we wish to purchase electron tubes for an electronic system. We want tubes which will last an average of at least μ_0 hours. A manufacturer claims that his tubes will meet this specification. To test his claim we install n tubes furnished by the manufacturer and observe their life spans. We wish to test the null hypothesis $H_0: n\mu = n\mu_0$ against the alternative hypothesis $H_a: n\mu < n\mu_0$ with a probability of Type I error of size .05. To perform this test we assume that the lifetime of the electron tubes obeys an exponential distribution given by $f(x) = \frac{1}{\mu_0} e^{-x/\mu_0}$, $0 < x < \infty$. We make the substitution $z = \frac{x}{\mu_0}$ which gives $f(z) = e^{-z}$ and choose as a test statistic $t = \sum_{i=1}^{n} z_i$. To determine the critical region for the test we find z_0 such that $G(z_0; n) = .05$ since the sum of n independent z variables will have a Gamma distribution with parameter n. We would then reject H_0 if $t < z_0$.

For this test, however, we are assuming that the lifetime of the tubes lies between 0 and infinity hours. It would seem more realistic to assume the lifetime of the tubes lies in some finite interval from a_1 to b_1 hours. Then the distribution would be given by $f_T(x) = \frac{1}{I\mu_0} e^{-x/\mu_0}$, $a_1 \leq x \leq b_1$ and after the substitution by $f_T(z) = \begin{cases} \frac{e^{-z}}{I}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$ where $a = \frac{a_1}{\mu_0}$, $b = \frac{b_1}{\mu_0}$, and $I = \int_a^b f_T(z)\, dz$. Now in order to determine the critical region for the test we must find z_0 such that $F_T(z_0; n) = .05$ where $F_T(z; n)$ is the distribution function for the sum of n independent variables each having the truncated exponential distribution given by $f_T(z)$ above.

It is this distribution function which we derive in this chapter.

The characteristic function of a random variable x with probability density function f is defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx.$$

We note that

$$3.1 \quad \varphi(t + i\ ) = \int_{-\infty}^{\infty} e^{i(t+i\alpha)x} f(x)\, dx = \int_{-\infty}^{\infty} e^{itx}[e^{-\alpha x} f(x)]\, dx.$$

Hence, if $\varphi(t)$ is the characteristic function of $f(x)$, then $\varphi(t + i\alpha)$ is the characteristic function of $e^{-\alpha x} f(x)$.

For the rectangular distribution, defined by

$$3.2 \quad f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b, \\ 0 & \text{otherwise} \end{cases}$$

the characteristic function is

$$3.3 \quad \varphi(t) = \frac{e^{itb} - e^{ita}}{(b-a)it}\ .$$

It is a well known fact that the characteristic function of the distribution for the sum of n independent variables, each having the same characteristic function $\varphi(t)$, is given by $[\varphi(t)]^n$. Kendall [14].

Hence the characteristic function of the distribution of the sum of n independent variables, each having the density function 3.2 is given by

$$3.4 \quad \varphi(t) = \frac{(e^{itb} - e^{ita})^n}{(b-a)^n (it)^n} .$$

If x is a truncated exponential variables, then the density function for x is defined by

$$3.5 \quad f_T(x) = \begin{cases} \dfrac{1}{e^{-a} - e^{-b}} e^{-x} & a < x < b. \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function of x is defined by

$$3.6 \quad \varphi(t) = \frac{1}{e^{-a} - e^{-b}} \int_a^b e^{itx} e^{-x} dx =$$

$$\frac{1}{e^{-a} - e^{-b}} \int_a^b e^{x(it - 1)} dx = \frac{e^{(it - 1)b} - e^{(it - 1)a}}{(e^{-a} - e^{-b})(it - 1)} .$$

Hence the characteristic function for the sum of n independent truncated exponential variables is given by

$$3.7 \quad \varphi(t) = \frac{(e^{(it - 1)b} - e^{(it - 1)a})^n}{(e^{-a} - e^{-b})^n (it - 1)^n} .$$

If S is the sum of n independent variables, each with density function 3.2, then

$$3.8 \quad g(S) = (b - a)^{-n} \sum_{k=0}^{m} (-1)^k \binom{n}{k} \frac{[S - na - k(b - a)]^{n-1}}{(n - 1)!}$$

for $na + m(b - a) < S < na + (m + 1)(b - a)$ and $m = 0, 1, \cdots, n - 1$. Cramer [11].

Since $it - 1 = i(t + i)$ we have by 3.1 that S, the sum of n independent truncated exponential variables, each having density function 3.3, is defined by

$$3.9 \quad f_T(S) = \frac{(b - a)^n}{(e^{-a} - e^{-b})^n} g(S) e^{-s} =$$

$$\frac{1}{(e^{-a} - e^{-b})^n} \sum_{k=0}^{m} (-1)^k \binom{n}{k} \frac{[s - na - k(b - a)]^{n-1}}{(n - 1)!} e^{-s}$$

for $na + m(b - a) < S < na + (m + 1)(b - a)$, and $m = 0, 1, \cdots, n-1$.

If F_T is the distribution function for S, then to find $F_T(x)$ we must first find m_o such that $na + m_o(b - a) < x < na + (m_o + 1)(b - a)$. This inequality reduces to $m_o < \frac{x - na}{b - a} < m_o + 1$.

Hence we choose m_o as the largest integer less than or equal to $\frac{x - na}{b - a}$. Now we must integrate $f_T(s)$ for $m = o$ from na to $na + (b - a)$, for $m = 1$ from $na + (b - a)$ to $na + 2(b - a)$, and finally for $m = m_o$ from $na + m_o(b - a)$ to x. Since $f_T(s)$ is a sum of $m_o + 1$ terms, the first term, that for $k = o$, will occur in all segments, and hence must be integrated from na to x. The second term occurs from $k = 1$ to $k = m_o$ and hence must be integrated from $na + (b - a)$ to x. Finally the last term occurs for $k = m_o$ only, and hence must be integrated from $na + m_o(b - a)$ to x.

Hence the distribution function F_T is defined by

$$3.10 \qquad F_T(x) = \frac{1}{(e^{-a}-e^{-b})^n} \sum_{k=o}^{m_o} (-1)^k \binom{n}{k} \int_{na+k(b-a)}^{x} \frac{[s-na-k(b-a)]^{n-1}}{(n-1)!} e^{-s} ds.$$

If we let $y = S - na - k(b - a)$ this becomes

$$3.11 \quad F_T(x) = \frac{e^{-na}}{(e^{-a}-e^{-b})^n} \sum_{k=o}^{m_o} (-1)^k \binom{n}{k} e^{-k(b-a)} \frac{1}{(n)} \int_o^{x-na-k(b-a)} y^{n-1} e^{-y} dy.$$

Now let $a_o = \frac{e^{na} \cdot e^{-na}}{(e^{-a})^n (e^{-a}-e^{-b})^n} = \frac{1}{(1-e^{a-b})^n}$

and for $k = 1, 2, \cdots, m$, let $a_k = (-1)^k \binom{n}{k} [e^{a-b}]^k a_o$.

Next let $z_k = x - na - k(b - a)$ for $k = 0, \cdots, m_o$.

Then

3.12 $\quad F_T(x) = \sum_{k=o}^{m_o} a_k \, G(z_k;\, n)$ where $G(x;n)$ is defined by 2.1.

Below we compare the graphs for the distribution of the sum of 5 independent truncated exponential variables with truncation at $a = 1$ and $b = 3$ with the Gamma distribution with parameter $\alpha = 5$ which would be the distribution for the sum of 5 independent untruncated exponential variables.

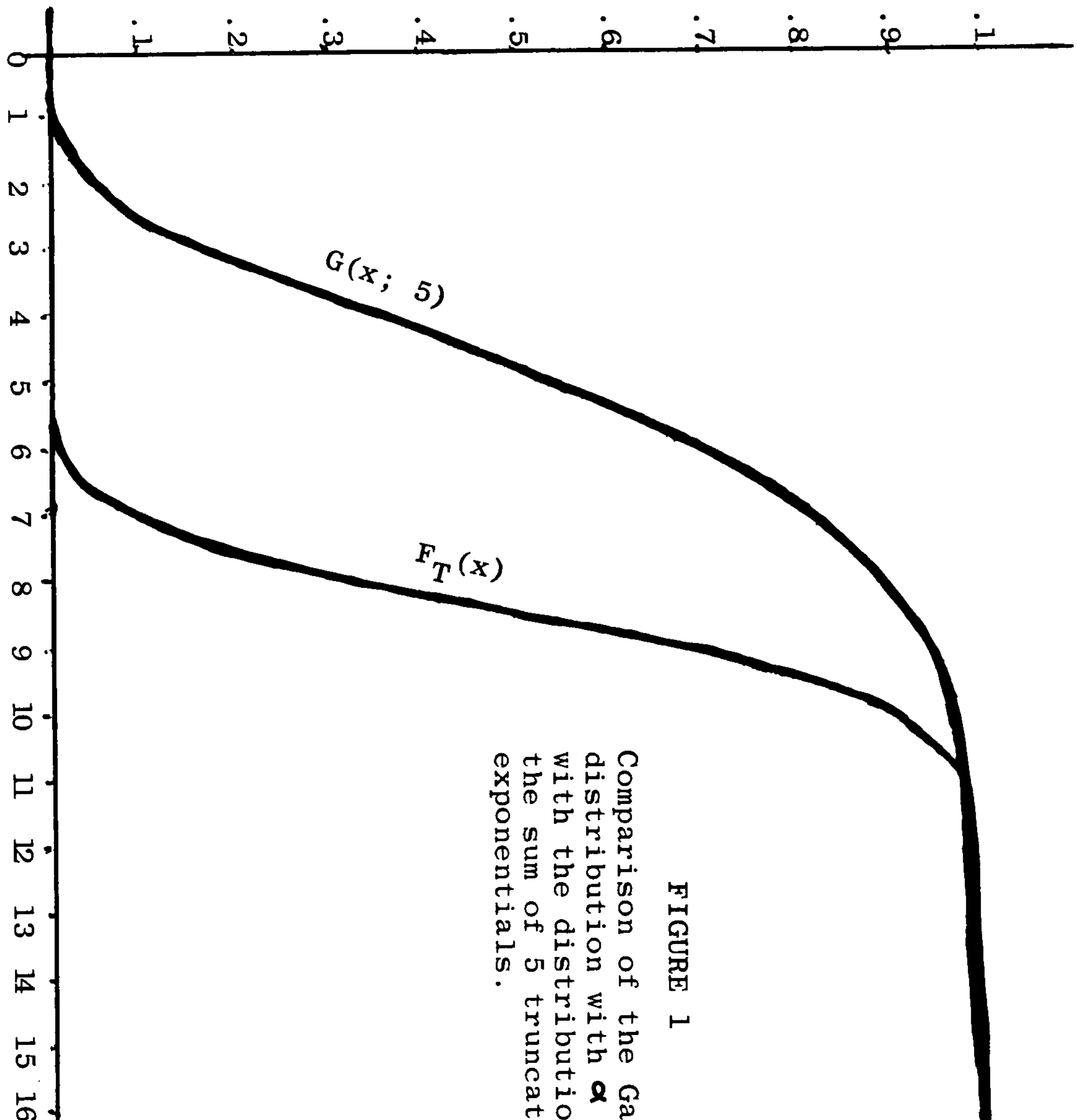


FIGURE 1

Comparison of the Gamma distribution with $\alpha = 5$ with the distribution of the sum of 5 truncated exponentials.

Now, returning to the example at the beginning of the chapter, suppose we have a = .11, b = .91 and n = 5. From table I we find $F(1.47; 5) = .05$ and hence the critical region is given by $t < 1\cdot47$.

If we had conducted the test assuming a complete distribution, when in fact it was truncated as above, the critical region would have been $t < 1\cdot97$ and the size of the critical region would have been $.25 = F(1\cdot97; 5)$.

IV. THE DISTRIBUTION FOR N = 2

A manufacturer of electronic equipment receives a shipment of parts for an electronic component. Since the weight of the component must not exceed a specific maximum weight, he wishes to test whether the mean weight of the parts exceed a specific weight k. From past experience it is known that the weights of these parts obeys a gamma distribution given by $g(x; \alpha_o) = \dfrac{1}{\Gamma(\alpha_o)\beta^{\alpha_o}} x^{\alpha_o - 1} e^{-x/\beta}$,

where α_o is known.

The mean of a complete Gamma distribution is $\beta\alpha$ and the maximum likelihood estimate for $\beta\alpha_o$ is $\sum_{i=1}^{n} x^i/n$.

For a sample of size n we wish to test the null hypothesis H_o: $n\beta\alpha_o = nk$ against the alternative hypothesis H_a: $n\beta\alpha_o > nk$. As a test statistic we would use $t = \sum_{i=1}^{n} x_i$. However, to use available tables we first make the substitution $z_i = \frac{\alpha_o}{k} x_i$ and use as a test statistic $t = \sum_{i=1}^{n} z_i$ since the distribution of $\sum_{i=1}^{n} z_i$ will be a Gamma distribution with parameter $n\alpha_o$. To determine the critical region we therefore would need z_o such that $G(z_o; n\alpha_o) = .95$ if we wished to have a critical region of size .05. We would then reject H_o if $t > z_o$.

This test, however, is conducted assuming that the weights of the parts obeys a complete Gamma distribution with parameters α_o and $\beta = \frac{k}{\alpha_o}$, when it would seem more realistic to assume that the weights of the parts fell in some finite interval a_1 to b_1, and hence obeyed a truncated Gamma distribution which, after the substitution $z = \frac{\alpha_o}{k} x$, would be given by

$$f_T(z;\alpha_o) = \frac{1}{I\Gamma(\alpha_o)} z^{\alpha_o - 1} e^{-z}, \quad a \leq z \leq b \quad \text{where}$$

$$a = \frac{\alpha_o a_1}{k}, \quad b = \frac{\alpha_o b_1}{k}, \quad \text{and} \quad I = \frac{1}{\Gamma(\alpha_o)} \int_a^b t^{\alpha_o - 1} e^{-t}\, dt.$$

Now in order to determine a critical region of size .05, we would need to find z_o such that $F_T(z_o;\, n) = .95$ where $F_T(z;\, n)$ is the distribution function for the sum of n independent truncated variables each with the density function given by $f_T(z;\alpha_o)$ above.

For $n = 2$, this distribution is derived in this chapter and in the following chapter the distribution is derived for $n = 3$.

Let x_1 and x_2 be independent variables, each having the truncated Gamma density function defined by

$$4.1 \qquad f_T(x) = Cx^{\alpha - 1} e^{-x}, \quad a \leq x \leq b \quad \text{where}$$

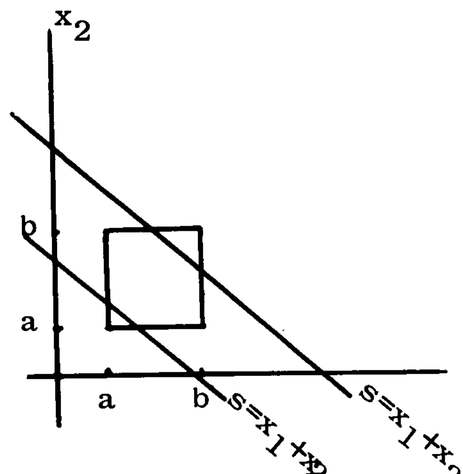
$$C = \frac{1}{\int_a^b t^{\alpha - 1} e^{-t}\, dt.}$$

The joint probability density function of x_1 and x_2 is given by $f_T(x_1, x_2) = f_T(x_1) \cdot f_T(x_2) = c^2 x_1^{\alpha - 1} x_2^{\alpha - 1} e^{-(x_1 + x_2)}$.

Let $s = x_1 + x_2$. Then $dx_1 = ds$ and hence $f_T(s, x_2) = c^2 (s - x_2)^{\alpha - 1} e^{-s} x_2^{\alpha - 1}$

In order to find $f_T(s)$ we must integrate $f_T(s, x_2)$ with respect to x_2 over the proper limits.

The following diagram is helpful in determining the limits of integration.



From the diagram we see that for $2a \le s \le a+b$, x_2 goes from a to $s-a$, and for $a+b \le s \le 2b$, x_2 goes from $s-b$ to b.

Hence $f_T(s)$ is given by

$$f_T(s) = \begin{cases} c^2 e^{-s} \displaystyle\int_a^{s-a} x_2^{\alpha-1}(s-x_2)^{\alpha-1}\,dx_2 & \text{for } 2a \le s \le a+b \\ c^2 e^{-s} \displaystyle\int_{s-b}^{b} x_2^{\alpha-1}(s-x_2)^{\alpha-1}\,dx_2 & \text{for } a+b \le s \le 2b. \end{cases}$$

Let $x_2 = sz$. Then $dx_2 = sdz$ and we have

$$f_T(s) = \begin{cases} c^2 e^{-s} \displaystyle\int_{a/s}^{1-a/s} (sz)^{\alpha-1}(s-sz)^{\alpha-1}\,s\,dz & \text{for } 2a \le s \le a+b \\ c^2 e^{-s} \displaystyle\int_{1-b/s}^{b/s} (sz)^{\alpha-1}(s-sz)^{\alpha-1}\,s\,dz & \text{for } a+b \le s \le 2b. \end{cases}$$

Factoring s out of the integrals above gives the following definition for $f_T(s)$:

$$4.2 \quad f_T(s) = \begin{cases} c^2 e^{-s} s^{2\alpha-1} \displaystyle\int_{a/s}^{1-a/s} z^{\alpha-1}(1-z)^{\alpha-1}dz & \text{for } 2a \le s \le a+b \\ c^2 e^{-s} s^{2\alpha-1} \displaystyle\int_{1-b/s}^{b/s} z^{\alpha-1}(1-z)^{\alpha-1}\,dz & \text{for } a+b \le s \le 2b. \end{cases}$$

If we introduce a negative sign in the second integral defining $f_T(s)$, then the limits of both integrals are of the form k to 1-k, where $k = \frac{a}{s}$ or $k = \frac{b}{s}$. Hence each integral can be represented by

$$I = K \int_k^{1-k} z^{\alpha-1}(1-z)^{\alpha-1}\,dz \quad \text{where} \quad K = c^2 e^{-s} s^{2\alpha-1}.$$

Let $v = \frac{z^2}{\alpha}$ and $u = (1-z)^{\alpha-1}$. Then $dv = z^{\alpha-1}\,dz$ and $du = -(\alpha-1)(1-z)^{\alpha-2}\,dz$, and hence

$$I = K\left[\frac{z^{\alpha}(1-z)^{\alpha-1}}{\alpha}\Bigg]_k^{1-k} + \frac{\alpha-1}{\alpha}\int_k^{1-k} z^{\alpha}(1-z)^{\alpha-2}\,dz\right].$$

Repeated integration by parts gives

$$I = K\left[\left(\frac{z^{\alpha}(1-z)^{\alpha-1}}{\alpha} + \frac{(\alpha-1)}{\alpha(\alpha+1)}z^{\alpha+1}(1-z)^{\alpha-2} + \frac{(\alpha-1)(\alpha-2)}{\alpha(\alpha+1)(\alpha+2)}z^{\alpha+2}(1-z)^{\alpha-3} + \ldots\ldots + \frac{(\alpha-1)(\alpha-2)\ldots 2}{\alpha(\alpha+1)(\alpha+2)\ldots(2\alpha-2)}z^{2\alpha-2}(1-z)\right)\Bigg]_k^{1-k} + \frac{(\alpha-1)(\alpha-2)\cdots 2}{\alpha(\alpha+1)(\alpha+2)\cdots(2\alpha-2)}\int_k^{1-k} z^{2\alpha-2}\,dz\ .\right]$$

Now $\int_k^{1-k} z^{2\alpha-2}\,dz = \frac{z^{2\alpha-1}}{2\alpha-1}\Big]_k^{1-k}$ and, if we assume α is an integer,

$$\frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!}\binom{2\alpha-1}{\alpha-j} = \frac{(\alpha-1)(\alpha-2)\ \ldots\ (\alpha-j)}{(2\alpha-j-1)!} \quad \text{for}$$

$j = 1, 2, \ldots \alpha-1$. Hence

$$I = K\frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!}\Big[(1-k)^{2\alpha-1} + \binom{2\alpha-1}{1}(1-k)^{2\alpha-2}k + \ldots + \binom{2\alpha-1}{\alpha-2}(1-k)^{\alpha+1}k^{\alpha-2} + \binom{2\alpha-1}{\alpha-1}(1-k)^{\alpha}k^{2\alpha-2} - k^{2\alpha-1} -$$

$$\binom{2\alpha-1}{1} k^{2\alpha-2}(1-k) - \dots - \binom{2\alpha-1}{\alpha-2} k^{\alpha+1}(1-k)^{\alpha-2} -$$

$$\left.\binom{2\alpha-1}{\alpha-1}k^{\alpha}(1-k)^{\alpha-1}\right] = K \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!}\left[\sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j}k^j\right.$$

$$\left.(1-k)^{2\alpha-j-1} - \sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j} k^{2\alpha-j-1}(1-k)^j\right],$$ and $f_T(s)$

can be written as $f_T(s) = K \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!}\left[\sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j}k^j\right.$

$\left.(1-k)^{2\alpha-j-1} - \sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j}k^{2\alpha-j-1}(1-k)^j\right]$ for $k = \frac{a}{s}$ and

$2a \le s \le a+b$ or $f_T(s) = K \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!}\left[\sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j}k^{2\alpha-j-1}\right.$

$\left.(1-k)^j - \sum_{j=0}^{\alpha-1}\binom{2\alpha-1}{j}k^j(1-k)^{2\alpha-j-1}\right]$ for $k = \frac{b}{s}$ and

$a+b \le s \le 2b$.

Substituting $\frac{a}{s}$ and $\frac{b}{s}$ for k and taking e^{-s} and $s^{2\alpha-1}$ inside the summation gives

$$f_T(s) = \begin{cases} \dfrac{c^2(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!}\left[\displaystyle\sum_{j=0}^{\alpha-1} e^{-s}\binom{2\alpha-1}{j}[(s-a)^{2\alpha-j-1}a^j - \right. \\ \left.(s-a)^j a^{2\alpha-j-1}]\right] \text{ for } 2a \le s \le a+b \text{ or} \\ \dfrac{c^2(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!}\left[\displaystyle\sum_{j=0}^{\alpha-1} e^{-s}\binom{2\alpha-1}{j}[(s-b)^j b^{2\alpha-j-1} - \right. \\ \left.(s-b)^{2\alpha-j-1} b^j]\right] \text{ for } a+b \le s \le 2b. \end{cases}$$

The distribution function F_T is defined by

$F_T(x) = \int_{2a}^{x} f_T(s)\,ds$ for $2a \le x \le a+b$ or

$$F_T(x) = \int_{2a}^{a+b} f_T(s)\,ds + \int_{a+b}^{x} f_T(s)\,ds \quad \text{for} \quad a+b \le x \le 2b.$$

Therefore, for $2a \le x \le a+b$, $F_T(x) =$

$$c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1} \binom{2\alpha-1}{j} [a^j \int_{2a}^{x} (s-a)^{2\alpha-j-1} e^{-s}\,ds -$$

$a^{2\alpha-j-1} \int_{2a}^{x} (s-a)^j e^{-s}\,ds]$. Let $z = s-a$. Then $dz = ds$, $e^{-s} = e^{-a}e^{-z}$, and hence $F_T(x) = c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1}$

$$\binom{2\alpha-1}{j} [a^j e^{-a} \int_{a}^{x-a} z^{2\alpha-j-1} e^{-z}\,dz - a^{2\alpha-j-1} e^{-a} \int_{a}^{x-a} z^j e^{-z} dz] =$$

$$c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1} \binom{2\alpha-1}{j} e^{-a} \Bigg[\Big[a^j \Gamma(2\alpha-j)[G(x-a;\ 2\alpha-j) -$$

$$G(a;\ 2\alpha-j)]\Big] - \Big[a^{2\alpha-j-1}\Gamma(j+1)[G(x-a;\ j+1) - G(a;\ j+1)]\Big]\Bigg]$$

where $G(x;\alpha)$ is defined by 2.1. Similarly for $a+b \le x \le 2b$, $F_T(x) = F_T(a+b) + c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1}$

$$\binom{2\alpha-1}{j} [b^{2\alpha-j-1} \int_{a+b}^{x} (s-b)^j e^{-s}\,ds - b^j \int_{a+b}^{x} (s-b)^{2\alpha-j-1} e^{-s} ds].$$

Let $z = s - b$. Then $dz = ds$, $e^{-s} = e^{-b}e^{-z}$, and

$$F_T(x) = F_T(a+b) + c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1} \binom{2\alpha-1}{j} e^{-b}$$

$$[b^{2\alpha-j-1} \int_{a}^{x-b} z^j e^{-z}\,dz - b^j \int_{a}^{x-b} z^{2\alpha-j-1} e^{-z}\,dz] =$$

$$F_T(a+b) + c^2 \frac{(\alpha-1)!(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1} \binom{2\alpha-1}{j} e^{-b} \Big[b^{2\alpha-j-1}\Gamma(j+1)$$

$[G(x-b;\ j+1) - G(a;\ j+1)] - b^j\Gamma(2\alpha-j)[G(x-b;\ 2\alpha-j) -$

$G(a;\ 2\alpha-j)]\Big]$ where $G(x;\alpha)$ is defined by 2.1.

By repeated integration by parts it is possible to show that

$$\frac{1}{(\alpha-1)!}\int_a^v x^{\alpha-1}e^{-x}dx = e^{-a}-e^{-v}+\frac{(ae^{-a}-ve^{-v}}{1!}+\frac{a^2e^{-a}-v^2e^{-v}}{2!}+$$

$$\ldots+\frac{(a^{\alpha-1}e^{-a}-v^{\alpha-1}e^{-v})}{(\alpha-1)!}.$$

Hence $\frac{1}{(\alpha-1)!}\int_a^v x^{\alpha-1}e^{-x}dx = \frac{1}{(\alpha-2)!}\int_a^v x^{\alpha-2}e^{-x}dx +$

$$\frac{(a^{\alpha-1}e^{-a}-v^{\alpha-1}e^{-v})}{(\alpha-1)!}.$$

Let $AN(\lambda) = \frac{1}{(\lambda-1)!}\int_a^b x^{\lambda-1}e^{-x}dx,$

$BN(\lambda) = \frac{1}{(\lambda-1)!}\int_a^{x-a} x^{\lambda-1}e^{-x}dx,$ and

$CN(\lambda) = \frac{1}{(\lambda-1)!}\int_a^{x-b} x^{\lambda-1}e^{-x}dx.$ Let

$$G = [C\cdot(\alpha-1)!]^2 = \left[\frac{(\alpha-1)!}{\int_a^b x^{\alpha-1}e^{-x}dx}\right]^2 = \left[\frac{1}{AN(\alpha)}\right]^2.$$

Substituting these expressions and making all possible cancellations gives the following definition for $F_T(x)$, the distribution function for the sum of two independent truncated Gamma variables with truncation at a and b: for $2a \le x \le a+b$,

$$F_T(x) = e^{-a}G\sum_{J=1}^{\alpha}\left[\frac{a^{J-1}BN(2\alpha-J+1)}{(J-1)!} - \frac{a^{2\alpha-J}\,BN(J)}{(2\alpha-J)!}\right]$$

or for $a + b \leq x \leq 2b$,

$$F_T(x) = e^{-b}G \sum_{J=1}^{\alpha} \frac{b^{2\alpha - J}CN(J)}{(2\alpha - J)!} - \frac{b^{J-1}CN(2\alpha - J+1)}{(J-1)!} \quad .$$

Below we compare the graph of the sum of two independent truncated Gamma variables with parameter $\alpha = 2$ and truncation at $a = 1.5$ and $b = 3.5$ with the graph of the Gamma distribution with parameter $\alpha = 4$ which would be the distribution for the sum of two independent Gamma variables with parameter $\alpha = 2$.

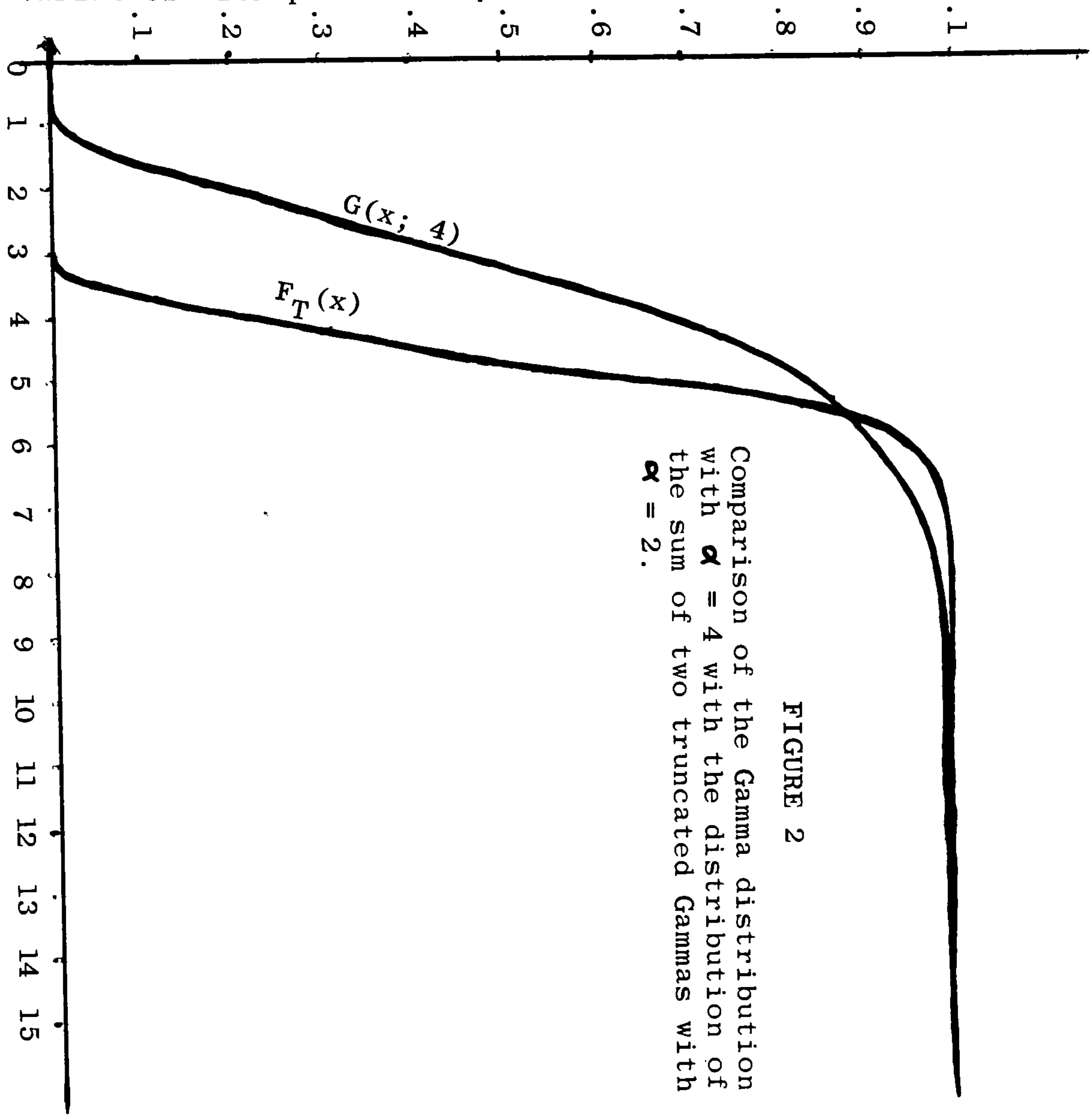


FIGURE 2

Comparison of the Gamma distribution with $\alpha = 4$ with the distribution of the sum of two truncated Gammas with $\alpha = 2$.

Returning to the example at the beginning of the chapter, suppose we have $\alpha_o = 4$, $a = 2.3$, $b = 5.6$, and $n = 2$. Then from table II we find that $F_T(9.76;\ 2) = .95$ and hence the critical region is given by $t > 9.76$.

If we had conducted the test assuming a complete distribution, when in fact it was truncated as above, the critical region would have been $t > 13.15$, and since this is greater than $2b$, the size of the critical region would have been zero.

V. THE DISTRIBUTION FOR N = 3

Let x_1, x_2, and x_3 be independent variables, each having the truncated Gamma distribution with density function given by 4.1. Let $y = x_1 + x_2$ and $S = y + x_3$.

Then y has the probability density function defined by 4.2, and the joint probability density function of y and x_3 is given by

$$f_T(y,\ x_3) = f_T(x_3) \cdot f_T(y) = [c x_3^{\alpha-1} e^{-x_3}] \cdot [c^2 e^{-y} y^{2\alpha-1} \int_{a/y}^{1-a/y} z^{\alpha-1}(1-z)^{\alpha-1} dz$$

for $2a \leq y \leq a+b$, or $f_T(x_3) \cdot f_T(y) = [c x_3^{\alpha-1} e^{-x}] . [c^2 e^{-y} y^{2\alpha-1}$

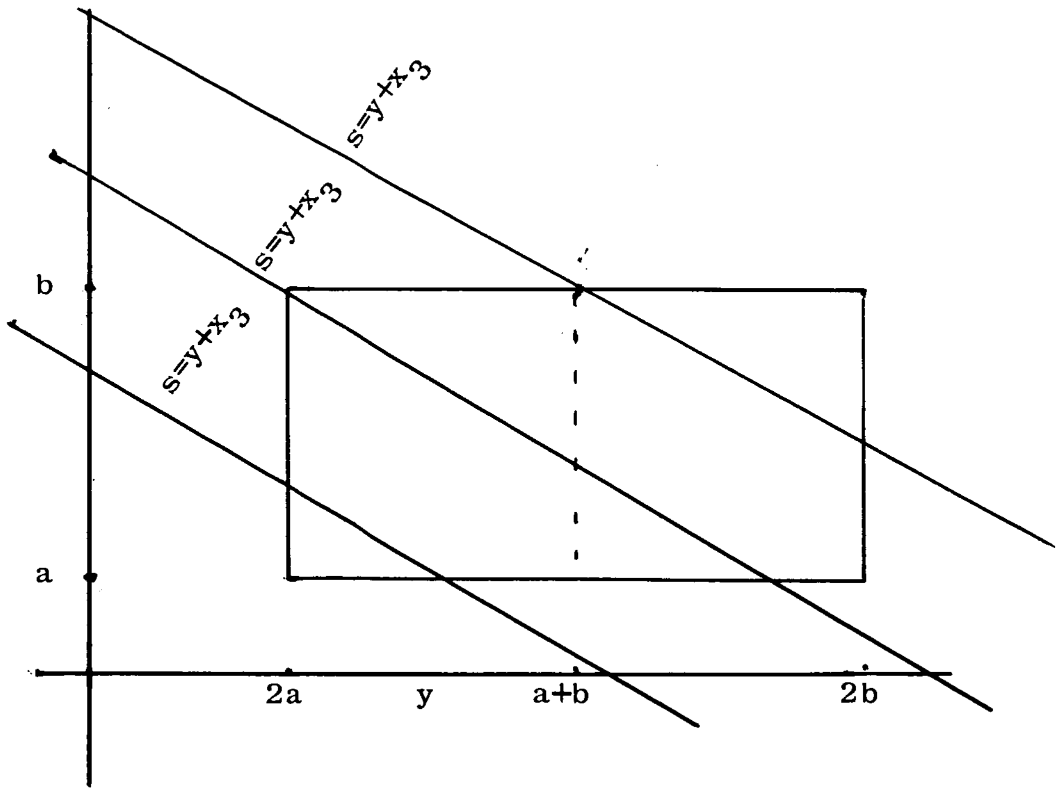
$\int_{1-b/y}^{b/y} z^{\alpha-1}(1 - z)^{\alpha-1} dz]$ for $a+b \leq y \leq 2b$. Hence

$$f_T(s,x_3) = c^3 x_3^{\alpha-1} e^{-x_3}\ e^{-(s-x_3)}(s-x_3)^{2\alpha-1} \int_{a/s-x_3}^{1-a/s-x_3} z^{\alpha-1}(1-z)^{\alpha-1} dz$$

for $a+b \leq S-x_3 \leq 2b$.

To find $f_T(s)$ we must integrate $f_T(s,\ x_3)$ with respect to x_3 over the proper limits.

The following diagram is helpful in determining the limits of x_3.



From the diagram we see that for $3a \leq s \leq 2a+b$, x_3 goes from a to $s - 2a$. For $2a + b \leq s \leq a + 2b$, x_3 goes from a to $s - a - b$ and from $s - a - b$ to b. Finally, from $a + 2b \leq s \leq 3b$, x_3 goes from $s - 2b$ to b. Hence

$$f_T(s) = \begin{cases} \int_a^{s-2a} f_T(s,x_3)\,dx_3 & \text{for } 3a \le s \le 2a + b \\ \int_a^{s-a-b} f_T(s,x_3)\,dx_3 + \int_{s-a-b}^{b} f_T(s,x_3)\,dx_3 & \text{for } 2a + b \le s \le a + b \\ \int_{s-2b}^{b} f_T(s,x_3)\,dx_3 & \text{for } a + 2b \le s \le 3b. \end{cases}$$

Substituting the formulas for $f_T(s,x_3)$ this becomes

$$5.1 \quad f_T(s) = \begin{cases} c^3 e^{-s} \int_a^{s-2a} x_3^{\alpha-1}(s-x_3)^{2\alpha-1} \int_{a/s-x_3}^{1-a/x-x_3} z^{\alpha-1}(1-z)^{\alpha-1} \\ \quad dz\,dx_3 \quad \text{for } 3a \le s \le 2a+b \\ c^3 e^{-s}\Big[\int_a^{s-a-b} x_3^{\alpha-1}(s-x_3)^{2\alpha-1} \int_{1-b/s-x_3}^{b/s-x_3} z^{\alpha-1}(1-z)^{\alpha-1}dz\,dx_3 + \\ \quad \int_{s-a-b}^{b} x_3^{\alpha-1}(s-x_3)^{2\alpha-1} \int_{a/s-x_3}^{1-a/s-x_3} z^{\alpha-1}(1-z)^{\alpha-1}dz\,dx_3\Big] \\ \quad \text{for } 2a + b \le s \le a + 2b \\ c^3 e^{-s} \int_{s-2b}^{b} x_3^{\alpha-1}(s-x_3)^{2\alpha-1} \int_{1-b/s-x_3}^{b/s-x_3} z^{\alpha-1}(1-z)^{\alpha-1}dz\,dx_3 \\ \quad \text{for } a + 2b \le s \le 3b \end{cases}$$

Let $w = (s - x_3)z$. Then $z = w/s-x_3$, $dz = \dfrac{dw}{s-x_3}$,

$$1 - z = \frac{s-w-x_3}{s-x_3}, \quad \text{and } z^{\alpha-1}(1-t)^{\alpha-1}dz = \frac{w^{\alpha-1}(s-w-x_3)^{\alpha-1}}{(s - x_3)^{2\alpha-1}}\,dw.$$

Substituting these results into 5.1 gives the following definition for $f_T(s)$:

$$
5.2\quad f_T(s)=\begin{cases}
c^3e^{-s}\displaystyle\int_a^{s-2a} x_3^{\alpha-1}\int_a^{s-x_3-a} w^{\alpha-1}(s-w-x_3)^{\alpha-1}dw\,dx_3 \\
\quad\text{for } 3a \le s \le 2a+b \\
c^3e^{-s}\left[\displaystyle\int_a^{s-a-b} x_3^{\alpha-1}\int_{s-x_3-b}^{b} w^{\alpha-1}(s-w-x_3)^{\alpha-1}dw\,dx_3 + \right. \\
\quad\left.\displaystyle\int_{s-a-b}^{b} x_3^{\alpha-1}\int_a^{s-x_3-a} w^{\alpha-1}(s-w-x_3)^{\alpha-1}dw\,dx_3\right] \\
\quad\text{for } 2a+b \le s \le a+2b \\
c^3e^{-s}\displaystyle\int_{s-2b}^{b} x_3^{\alpha-1}\int_{s-x_3-b}^{b} w^{\alpha-1}(s-w-x_3)^{\alpha-1}dw\,dx_3 \\
\quad\text{for } a+2b \le s \le 3b
\end{cases}
$$

Let $x_3 = su$ and $w = sv$. Then $dx_3dw = s^2du\,dv$, and $x_3^{\alpha-1}w^{\alpha-1}(s-w-x_3)^{\alpha-1}dwdx_3 = s^{3\alpha-1}u^{\alpha-1}v^{\alpha-1}(1-u-v)^{\alpha-1}$.

Substituting these results into 5.2 gives

$$
5.3\quad f_T(s)=\begin{cases}
c^3e^{-s}s^{3\alpha-1}\displaystyle\int_{a/s}^{1-2a/s} u^{\alpha-1}\int_{a/s}^{1-u-a/s} v^{\alpha-1}dv\,du \quad\text{for } 3a\le s\le 2a+b \\
c^3e^{-s}s^{3\alpha-1}\left[\displaystyle\int_{a/s}^{1-a/s-b/s} u^{\alpha-1}\int_{1-u-b/s}^{b/s} v^{\alpha-1}(1-u-v)^{\alpha-1}dv\,du + \right. \\
\quad\left.\displaystyle\int_{1-a/s-b/s}^{b/s} u^{\alpha-1}\int_{a/s}^{1-u-a/s} v^{\alpha-1}(1-u-v)^{\alpha-1}dv\,du\right] \\
\quad\text{for } 2a+b\le s\le a+2b \\
c^3e^{-s}s^{3\alpha-1}\displaystyle\int_{1-2b/s}^{b/s} u^{\alpha-1}\int_{1-u-b/s}^{b/s} v^{\alpha-1}(1-u-v)^{\alpha-1}dv\,du \quad\text{for} \\
\quad a+2b\le s\le 3b
\end{cases}
$$

If we interchange the limits for both integrals in the second and fourth lines of 5.3, then each of the inner integrals will have limits of the form k to 1-u-k where k = a/s or k = b/s.

Hence each of the inner integrals may be represented by

$$I = \int_k^{1-u-k} v^{\alpha-1}(1-u-v)^{\alpha-1}dv.$$

Let $z = v/1-u$. Then $v = (1-u)z$, $dv = (1-u)dz$, and

$$I = (1-u)^{2\alpha-1} \int_{k/1-u}^{1-k/1-u} z^{\alpha-1}(1-z)^{\alpha-1}dz.$$

Let $v' = z^{\alpha}/\alpha$ and $u' = (1-z)^{\alpha-1}$. Then $dv' = z^{\alpha-1}dz$, $du' = -(\alpha-1)(1-z)^{\alpha-2}dz$, and

$$I = (1-u)^{2\alpha-1}\left[\left(\frac{z^{\alpha}(1-z)^{\alpha-1}}{\alpha}\right)\Bigg]_{k/1-u}^{1-k/1-u} + \frac{\alpha-1}{\alpha}\int_{k/1-u}^{1-k/1-u} z^{\alpha}(1-z)^{\alpha-2}dz\right].$$

After repeated integration by parts we find

$$I = (1-u)^{2\alpha-1}\Bigg[\Bigg(\frac{z^{\alpha}(1-z)^{\alpha-1}}{\alpha} + \frac{\alpha-1}{\alpha(\alpha+1)} z^{\alpha+1}(1-z)^{\alpha-2}+\cdots+z^{\alpha+2}$$

$$(1-z)^{\alpha-3}+\cdots+\frac{(\alpha-1)(\alpha-2)\cdots 2}{\alpha(\alpha+1)(\alpha+2)\cdots(2\alpha-2)} z^{2\alpha-2}(1-z)\Bigg)\Bigg]_{k/1-u}^{1-k/1-u} +$$

$$\frac{(\alpha-1)(\alpha-2)\cdots 1}{(\alpha)(\alpha+1)(\alpha+2)\cdots(2\alpha-2)}\int_{k/1-u}^{1-k/1-u} z^{2\alpha-2}dz\Bigg].$$ If we assume α is an integer, then $\frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!}\binom{2\alpha-1}{j} = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{(2\alpha-j-1)!}$

for $j = 1,\cdots,\alpha-1$, and I can be expressed as

$$I = \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!}(1-u)^{2\alpha-1}\Bigg[z^{2\alpha-1}+\binom{2\alpha-1}{1} z^{2\alpha-2}(1-z) +$$

$$\cdots+\binom{2\alpha-1}{\alpha-1} z^{\alpha}(1-z)^{\alpha-1}\Bigg]_{k/1-u}^{1-k/1-u}\Bigg] = \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!}\Bigg[(1-u-k)^{2\alpha-1}+$$

$$\binom{2\alpha-1}{1}(1-u-k)^{2\alpha-2}+\cdots+\binom{2\alpha-1}{\alpha-1}(1-u-k)^{\alpha}k^{\alpha-1} -$$

$$k^{2\alpha-1} - \binom{2\alpha-1}{1}k^{2\alpha-2}(1-u-k)-\cdots-\binom{2\alpha-1}{\alpha-1}k^{\alpha}(1-u-k)^{\alpha-1}\Bigg]$$

Hence 5.4 $\int_r^t u^{\alpha-1} \int_k^{1-u-k} v^{\alpha-1}(1-u-v)^{\alpha-1} du\, dv = \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!}$

$$\left[\sum_{j=o}^{\alpha-1} \binom{2\alpha-1}{j} k^j \int_r^t u^{\alpha-1}(1-k-u)^{2\alpha-j-1} du - \sum_{j=o}^{\alpha-1} \binom{2\alpha-1}{j} k^{2\alpha-j-1} \int_r^t u^{\alpha-1}(1-k-u)^j du\right]$$

By repeating the same steps which were required to reduce $\int_k^{1-u-k} v^{\alpha-1}(1-u-v)^{\alpha-1} dv$ for $\int_r^t u^{\alpha-1}(1-k-u)^j du$ and $\int_r^t u^{\alpha-1}(1-k-u)^{2\alpha-j-1} du$, we find 5.5 $\int_r^t u^{\alpha-1}(1-k-u)^j du =$

$$\frac{1}{\alpha\binom{\alpha+j}{\alpha}}\left[\sum_{i=o}^{j} \binom{\alpha+j}{i} t^{\alpha+j-i}(1-k-t)^i - \sum_{i=o}^{j} \binom{\alpha+j}{i} r^{\alpha+j-i}(1-k-r)^i\right] \text{ and}$$

$$5.6 \int_r^t u^{\alpha-1}(1-k-u)^{2\alpha-j-1} du = \frac{1}{\alpha\binom{3\alpha-j-1}{\alpha}}\left[\sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} t^{3\alpha-j-i-1}\right.$$

$$\left.(1-k-t)^i - \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} r^{3\alpha-j-i-1}(1-k-r)^i\right].$$

Combining 5.5 and 5.6 with 5.4 we get

$$f_T(s) = c^3 e^{-s} s^{3\alpha-1} \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!}\left[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} k^j \sum_{i=o}^{2\alpha-j-1}\right.$$

$$\binom{3\alpha-j-1}{i} \left[t^{3\alpha-j-i-1}(1-k-t)^i - r^{3\alpha-j-i-1}(1-k-r)^i\right] -$$

$$\left.\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} k^{2\alpha-j-1} \sum_{i=o}^{j} \binom{\alpha+j}{i} \left[t^{\alpha+j-i}(1-k-t)^i - r^{\alpha+j-i}(1-k-s)^i\right]\right]$$

for $r = a/s$, $t = 1 - 2a/s$, $k = a/s$, and $3a \leq s \leq 2a+b$, or

$$f_T(s) = c^3 e^{-s} s^{3\alpha-1} \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} k^j \sum_{i=o}^{2\alpha-j-1}$$

$$\binom{3\alpha-j-1}{i}[t^{3\alpha-j-i-1}(1-k-t)^i - r^{3\alpha-j-i-1}(1-k-r)^i] - \sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}}$$

$$k^{2\alpha-j-1} \sum_{i=o}^{j} \binom{\alpha+j}{i}[t^{\alpha+j-i}(1-k-t)^i - r^{\alpha+j-i}(1-k-s)^i] + \sum_{j=o}^{\alpha-1}$$

$$\frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} k^j \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}[t^{3\alpha-j-i-1}(1-k-t)^i - s^{3\alpha-j-i-1}(1-k-r)^i] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} k^{2\alpha-j-1} \sum_{i=o}^{j} \binom{\alpha+j}{i}[t^{\alpha+j-i}(1-k-t)^i - r^{\alpha+j-i}(1-k-r)^i] \Bigg]$$

for $r = 1 - a/s - b/s$, $t = a/s$ and $k = b/s$ in the first two sums and $r = 1 - a/s - b/s$, $t = b/s$, and $k = a/s$ in the second two sums, and $2a + b \leq s \leq a+2b$, or

$$f_T(s) = c^3 c^{-s} s^{3\alpha-1} \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{i}}{\alpha\binom{3\alpha-j-1}{\alpha}} k^j \sum_{i=o}^{2\alpha-j-1}$$

$$\binom{3\alpha-i-1}{i}[t^{3\alpha-j-i-1}(1-k-t)^i - r^{3\alpha-j-i-1}(1-k-r)^i] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} k^{2\alpha-j-1} \sum_{i=o}^{j} \binom{\alpha+j}{i}[t^{\alpha+j-i}(1-k-t)^i - s^{\alpha+j-i}(1-k-s)^i] \Bigg]$$

for $r = b/s$, $t = 1 - 2b/s$, $k = b/s$, and $a + 2b \leq s \leq 3b$.

Substituting the values for k, r, and t, and taking $s^{3\alpha-1}e^{-s}$ inside the summation gives the following definition for the probability denisty function of the sum of three independent truncated Gamma variables with truncation points a and b:

$$f_T(s) = c^3 \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}$$

$$[a^{j+i}(s-2a)^{3\alpha-j-i-1}e^{-s} - a^{3\alpha-j-1}(s-2a)^{i}e^{-s}] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} [a^{2\alpha+i-j-1}(s-2a)^{\alpha+j-i}e^{-s} - a^{3\alpha-i-1}$$

$$(s-2a)^{i}e^{-s}]\Bigg] \quad \text{for } 3a \leq s \leq a+2b, \text{ or}$$

$$f_T(s) = c^3 \frac{(\alpha-1)!\;(\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}$$

$$[(b^{j}a^{3\alpha-j-i-1} + a^{j}b^{3\alpha-j-i-1})((s-b-a)^{i}e^{-s}) - (a^{j}b^{i} + a^{i}b^{j})$$

$$((s-a-b)^{3\alpha-j-i-1}e^{-s})] - \sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} [(b^{2\alpha-j-1}a^{\alpha+j-i} +$$

$$a^{2\alpha-j-1}b^{\alpha+j-1})((s-a-b)^{i}e^{-s}) - (a^{i}b^{2\alpha-j-1} + b^{i}a^{2\alpha-j-1})$$

$$((s-a-b)^{\alpha+j-i}\;e^{-s})]\Bigg] \quad \text{for } 2a+b \leq s \leq a+2b, \text{ or}$$

$$f_T(s) = c^3 \frac{(\alpha-1)!\;(\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}$$

$$[b^{j+i}(s-2b)^{3\alpha-j-i-1}e^{-s} - b^{3\alpha-i-1}(s-2b)^{i}e^{-s}] - \sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}}$$

$$\sum_{i=o}^{j} \binom{\alpha+j}{i} [b^{2\alpha+i-j-1}(s-2b)^{\alpha+j-i}e^{-s} - b^{3\alpha-i-1}(s-2b)^{i}e^{-s}]\Bigg] \quad \text{for}$$

$$a+2b \leq s \leq 3b.$$

The distribution function $F_T(x) = \int_{3a}^{x} f_T(s)ds$ is defined by

$$F_T(x) = c^3 \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}$$

$$[a^{j+i} \int_{3a}^{x} (s-2a)^{3\alpha-j-i-1} e^{-s} ds - a^{3\alpha-i-1} \int_{3a}^{x} (s-2a)^{i} e^{-s} ds] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} [a^{2\alpha+i-j-1} \int_{3a}^{x} (s-2a)^{\alpha+j-i} e^{-s} ds -$$

$$a^{3\alpha-j-1} \int_{3a}^{x} (s-2a)^{i}\ e^{-s} ds] \Bigg] \quad \text{for } 3a \leq x \leq 2a+b, \text{ or}$$

$$F_T(x) = F_T(2a+b) + c^3 \frac{(\alpha-1)!\ (\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1}$$

$$\binom{3\alpha-j-1}{i} [(b^j a^{3\alpha-j-i-1} + a^j b^{3\alpha-j-i-1}) \int_{2a+b}^{x} (s-b-a)^{i} e^{-s} ds -$$

$$(a^j b^i + a^i b^j) \int_{2a+b}^{x} (s-a-b)^{3\alpha-j-i-1} e^{-s} ds] - \sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{i} \binom{\alpha+j}{i}$$

$$[(b^{2\alpha-j-1} a^{\alpha+j-i} + a^{2\alpha-j-1} b^{\alpha+j-i}) \int_{2a+b}^{x} (s-a-b)^{i} e^{-s} ds -$$

$$(a^i b^{2\alpha-j-1} + b^i a^{2\alpha-j-1}) \int_{2a+b}^{x} (s-a-b)^{\alpha+j-i} e^{-s} ds] \Bigg] \quad \text{for } 2a+b \leq x \leq a+2b,$$

or

$$F_T(x) = F_T(a+2b) c^3 \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!} \sum_{j=0}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=0}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} [b^{j+i}$$

$$\int_{a+2b}^{x} (s-2b)^{3\alpha-j-i-1}\ e^{-s} ds - b^{3\alpha-i-1} \int_{a+2b}^{x} (s-2b)^{i} e^{-s} ds] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} \Big[b^{2\alpha+i-j-1} \int_{a+2b}^{x} (s-2b)^{\alpha+j-i} e^{-s} ds -$$

$$b^{3\alpha-j-i} \int_{a+2b}^{x} (s-2b)^{i} e^{-s} ds \Big] \Bigg] \quad \text{for } a+2b \le x \le 3b.$$

Let $z = s - 2a$. Then $dz = ds$, $e^{-s} = e^{-2a}\, e^{-z}$, and for $3a \le x \le a+b$

$$F_T(x) = c^3 \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!} \Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i}$$

$$\Big[e^{-2a} a^{j+i} \int_{a}^{x-2a} z^{3\alpha-j-i-1} e^{-z} dz - e^{-2a} a^{3\alpha-i-1} \int_{a}^{x-2a} z^{i} e^{-z} dz \Big] -$$

$$\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} \Big[e^{-2a} a^{2\alpha+i-j-1} \int_{a}^{x-2a} z^{\alpha+j-i} e^{-z} dz -$$

$$e^{-2a} a^{3\alpha-i-1} \int_{a}^{x-2a} z^{i} e^{-z} dz \Big] \Bigg] = c^3 e^{-2a} \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!}$$

$$\Bigg[\sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \sum_{i=o}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} \Big[a^{j+i}\, \Gamma(3\alpha-j-i) (G(x-2a;$$

$$3\alpha-j-i) - G(a;\, 3\alpha-j-i)) - a^{3\alpha-i-1}\, \Gamma(i+1) (G(x-2a;\, i+1) -$$

$$G(a;\, i+1)) \Big] - \sum_{j=o}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=o}^{j} \binom{\alpha+j}{i} \Big[a^{2\alpha+i-j-1}\, \Gamma(\alpha+j-i+1)$$

$$(G(x-2a; \alpha+j-i+1) - G(a; \alpha+j-i+1)) - a^{3\alpha-i-1}\, \Gamma(i+1)$$

$$(G(x-2a;\, i+1) - G(a;\, i+1)) \Big] \Bigg]$$ where $G(x;\alpha)$ is defined by 2.1.

Similarly, if we let $z = s-b-a$, then for $2a+b \le x \le a+2b$,

$$F_T(x) = F_T(2a+b) + e^{-b-a}\, c^3 \, \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!} \left[\sum_{j=0}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \right.$$

$$\sum_{i=0}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} \left[(b^j a^{3\alpha-j-i-1} + a^j b^{3\alpha-j-i-1})\, \Gamma(i+1) \right.$$

$$(G(x-a-b;\ i+1) - G(a;\ i+1)) - (a^j b^i + a^i b^j)\, \Gamma(3\alpha-j-i)$$

$$\left.(G(x-a-b;\ 3\alpha-j-i) - G(a;\ 3\alpha-j-i))\right] - \sum_{j=0}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}} \sum_{i=0}^{j} \binom{\alpha+j}{i}$$

$$[(b^{2\alpha-j-1} a^{\alpha+j-i} + a^{2\alpha-j-1}\, b^{\alpha+j-i})\, \Gamma(i+1)(G(x-a-b;\ i+1) -$$

$$G(a;\ i+1)) = (a^i b^{2\alpha-j-1} + b^i a^{2\alpha-j-1})\, \Gamma(\alpha+j-i+1)$$

$$\left.(G(x-a-b;\alpha+j-i+1) - G(a;\alpha+j-j+1))]\right]$$ and if we let $z = s - 2b$, then for $a+2b \leq x \leq 3b$,

$$F_T(x) = F_T(a+2b) + e^{-b} c^3 \, \frac{(\alpha-1)!\,(\alpha-1)!}{(2\alpha-1)!} \left[\sum_{j=0}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{3\alpha-j-1}{\alpha}} \right.$$

$$\sum_{i=0}^{2\alpha-j-1} \binom{3\alpha-j-1}{i} \left[b^{j+i}\, \Gamma(3\alpha-j-i)(G(x-2b;\ 3\alpha-j-i) - G(a;\ 3\alpha-j-i)) - \right.$$

$$\left. b^{3\alpha-i-1}\, \Gamma(i+1)(G(x-2b;\ i+1) - G(a;\ i+1))\right] - \sum_{j=0}^{\alpha-1} \frac{\binom{2\alpha-1}{j}}{\alpha\binom{\alpha+j}{\alpha}}$$

$$\sum_{i=0}^{j} \binom{\alpha+j}{i} \left[b^{2\alpha-i-j-1}\, \Gamma(\alpha+j-i+1)(G(x-2b;\alpha+j-i+1) - \right.$$

$$\left.\left. G(a;\alpha+j-i+1)) - b^{3\alpha-i-1}\, \Gamma(i+1)(G(x-2b;\ i+1) - G(a;\ i+1))\right]\right].$$

As in chapter IV, let $AN(\lambda) = \frac{1}{(\lambda-1)!} \int_a^b x^{\lambda-1} e^{-x} dx$,

$BN(\lambda) = \frac{1}{(\lambda-1)!} \int_a^{x-2a} x^{\lambda-1} e^{-x} dx$, $CN(\lambda) = \frac{1}{(\lambda-1)!} \int_a^{x-b-a} x^{\lambda-1} e^{-x} dx$, and $DN(\lambda) = \frac{1}{(\lambda-1)!} \int_a^{x-2b} x^{\lambda-1} e^{-x} dx.$

Substituting these expressions and making all possible cancellations gives the following definition for $F_T(x)$, the distribution function of the sum of three independent truncated Gamma variables with truncation at a and b:

$$F_T(x) = Ge^{-2a}\left[\sum_{J=1}^{\alpha}\sum_{I=1}^{2\alpha-J+1}\left[\frac{a^{J+I-2}}{(J-1)!}\frac{BN(3\alpha-J-I+2)}{(I-1)!} - \frac{a^{3\alpha-I}}{(J-1)!}\frac{BN(I)}{(3\alpha-J-I+1)!}\right] - \sum_{J=1}^{\alpha}\sum_{I=1}^{J}\left[\frac{a^{2\alpha+I-J-1}}{(2\alpha-J)!}\frac{BN(\alpha+J-I+1)}{(I-1)!} - \frac{a^{3\alpha-I}}{(2\alpha-J)!}\frac{BN(I)}{(\alpha+J-I)!}\right]\right]$$

for $3a \leq x \leq 2a+b$, or

$$F_T(x) = F_T(2a+b) + G\,e^{-a-b}\left[\sum_{J=1}^{\alpha}\sum_{I=1}^{2\alpha-J+1}\left[\frac{(b^{J-1}a^{3\alpha-J-I+1} + a^{J-1}b^{3\alpha-J-I+1})CN(I)}{(J-1)!\ (3\alpha-J-I+1)!} - \frac{(a^{J-1}b^{I-1} + a^{I-1}b^{J-1})\ CN(3\alpha-J-I+2)}{(J-1)!\ (I-1)!}\right] - \sum_{J=1}^{\alpha}\sum_{I=1}^{J}\left[\frac{(b^{2\alpha-J}a^{\alpha+J-I} + a^{2\alpha-J}b^{\alpha+J-I})CN(I)}{(2\alpha-J)!\ (\alpha+J-I)!} - \frac{(a^{I-1}b^{2\alpha-J} + a^{2\alpha-J}b^{I-1})CN(\alpha+J-I+1)}{(2\alpha-J)!\ (I-1)!}\right]\right]$$

for $2a+b \leq x \leq a+2b$, or

$$F_T(x) = F_T(a+2b) + e^{-2b}G\left[\sum_{J=1}^{\alpha}\sum_{I=1}^{2\alpha-J+1}\left[\frac{b^{J+I-2}DN(3\alpha-J-I+2)}{(J-1)!\ (I-1)!} - \right.\right.$$

$$\left.\frac{b^{3\alpha-1}DN(I)}{(J-1)!\,(3\alpha-J-I+1)!}\right] - \sum_{J=1}^{\alpha}\ \sum_{I=1}^{J}\left[\frac{b^{2\alpha+I-J-1}DN(\alpha+J-I+1)}{(2\alpha-J)!\ (I-1)!} - \frac{b^{3\alpha-I}DN(I)}{(2\alpha-J)!\ (\alpha+J-I)!}\right]\Bigg]$$

for $a+2b \leq x \leq 3b$, where $G = \left[\frac{1}{AN(\alpha)}\right]^3$.

Below we compare the graphs of the sum of three independent truncated Gamma variables with parameter $\alpha = 2$ and truncation at $a = 1.5$ and $b = 3.5$ with the graph of the Gamma distribution with parameter $\alpha = 6$ which would be the distribution for the sum of three independent Gamma variables with parameter $\alpha = 2$.

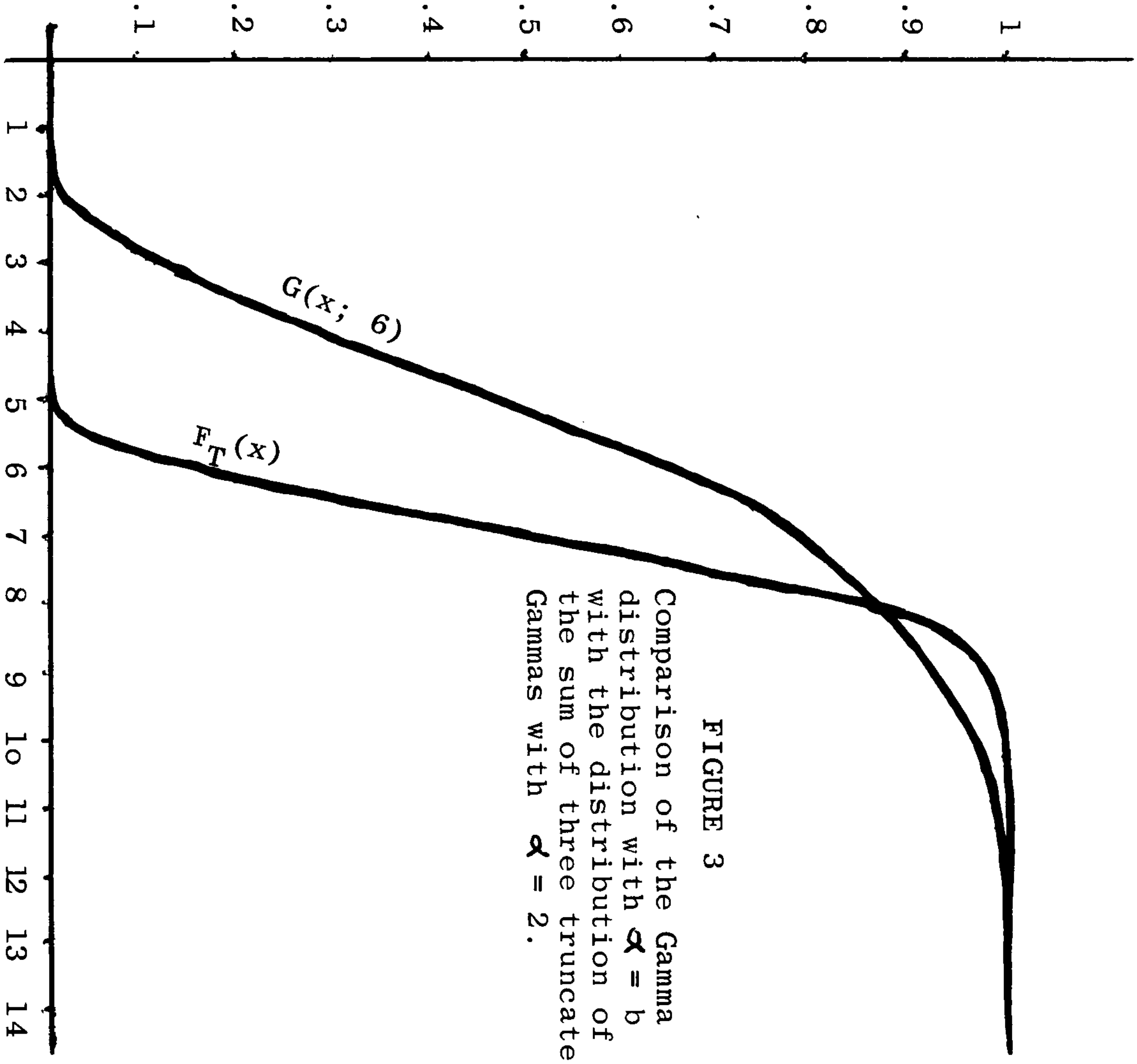


FIGURE 3

Comparison of the Gamma distribution with $\alpha = b$ with the distribution of the sum of three truncated Gammas with $\alpha = 2$.

Returning to the example at the beginning of chapter 4, suppose we have $\alpha_0 = 4$, $a = 2.3$, $b = 5.6$, and $n = 3$. Then from table III we find that $F_T(13.98; 3) = .95$ and hence the critical region for the test would be given by $t > 13.98$.

If we had conducted the test assuming a complete distribution, when in fact it was truncated as above, the critical region would have been $t > 18.20$ and since this is greater than 3b, the size of the critical region would have been zero.

VI. COMPARISON OF CRITICAL VALUES

In the following table the parameters which determine the distributions, a, b, α, and n, are given with the .05 and .95 critical values from the tables obtained from the actual distributions and with the critical values taken from the β_1 and β_2 tables, which are denoted by $X^*_{.05}$ and $X^*_{.95}$.

a	b	α	n	$X_{.05}$	$X^*_{.05}$	$X_{.95}$	$X^*_{.95}$
.54	2.06	2	2	1.55	1.55	3.56	3.54
.54	2.06	2	3	2.59	2.60	5.04	5.03
1.10	2.06	2	2	2.47	2.48	3.77	3.76
1.10	2.06	2	3	3.88	3.88	5.46	5.45
1.07	3.12	3	2	2.85	2.86	5.54	5.52
1.07	3.12	3	3	4.65	4.66	7.93	7.92
2.30	5.60	4	2	5.49	5.45	9.76	9.74
2.30	5.60	4	3	8.78	8.81	13.98	13.97
2.46	5.29	5	2	5.96	5.99	9.67	9.65
2.46	5.29	5	3	9.47	9.49	13.98	13.98
6.36	9.25	8	2	13.58	13.57	17.45	17.41
4.67	9.25	8	3	17.41	17.45	24.60	24.57
1.91	3.64	3	3	6.75	6.77	9.56	9.56
5.25	7.88	6	2	11.18	11.20	14.68	14.65
3.18	9.31	6	3	13.12	13.14	22.17	22.16
5.42	10.58	7	2	12.64	12.08	18.49	18.47
7.20	10.29	9	2	15.33	15.32	19.47	19.42
.11	1.60	5	1	1.94	1.95	4.95	4.95
.37	.91	5	1	2.52	2.52	3.66	3.65
.51	1.20	10	1	7.15	7.14	9.20	9.19
.11	1.60	10	1	4.70	4.71	8.95	8.96
.37	1.60	3	1	1.65	1.67	3.61	3.61
.37	1.60	2	1	.98	1.00	2.58	2.57
.51	2.40	2	1	1.32	1.31	3.62	3.63

In the following table the parameters which determine the distributions, a, b, α, and n, are given with the .01 and .99 critical values from the tables obtained from the actual distributions and with the critical values taken from the β_1 and β_2 tables, which are denoted by $X^*_{.01}$ and $X^*_{.99}$

a	b	α	n	$X_{.01}$	$X^*_{.01}$	$X_{.99}$	$X^*_{.99}$
.11	.91	1	2	.30	.28	1.66	1.64
.11	1.60	1	3	.66	.64	3.77	3.74
.22	2.40	1	5	2.20	2.19	7.88	7.86
.37	1.20	1	10	5.62	5.63	9.04	9.02
1.40	2.45	2	2	2.92	2.92	4.71	4.71
1.10	3.96	2	3	3.97	3.97	9.76	9.74
1.56	4.33	3	2	3.44	2.40	8.04	8.o3
1.91	3.12	3	3	6.17	6.15	8.83	8.80
2.80	4.80	4	2	5.85	5.85	9.25	9.27
2.30	6.80	4	3	8.18	8.20	17.26	17.20
3.62	5.86	5	2	7.53	7.56	11.33	11.34
2.46	6.71	5	3	9.10	9.13	17.96	17.97
4.53	9.31	6	2	9.55	9.58	17.30	17.29
5.20	7.03	6	3	16.38	16.35	20.28	20.25
5.42	9.05	7	2	11.29	11.28	17.38	17.36
4.76	10.58	7	3	16.11	16.06	27.95	27.97
5.60	11.88	8	2	11.94	11.90	22.04	22.00
4.67	10.21	8	3	16.43	16.40	27.95	27.93
5.40	10.29	9	2	11.72	11.67	19.82	19.84
6.42	11.40	9	3	21.12	21.08	31.68	31.66
8.13	11.38	10	2	16.70	16.65	22.22	22.18
8.96	12.65	10	3	28.08	28.09	36.06	36.00
10.46	18.20	15	2	22.31	22.27	34.98	34.99
17.44	23.70	20	2	35.70	35.69	46.21	46.18

TABLE I

Critical Values for the Truncated Distributions as Functions of the Truncation Points a and b for $\alpha = 1$ and N = 2, 3, 5, and 10

N = 2

a	b	.01	.02	.95	.99
.11	.91	.3000	.4051	1.4751	1.6560
.11	1.20	.3169	.4462	1.8774	2.1439
.11	1.60	.3338	.4875	2.3803	2.7786
.11	2.40	.3528	.5853	3.2043	3.8851
.22	.91	.5122	.6066	1.5346	1.6857
.22	1.20	.5311	.6519	1.9485	2.1813
.22	1.60	.5498	.6977	2.4682	2.8289
.22	2.40	.5710	.7507	3.3247	3.9671
.37	.91	.8022	.8781	1.6091	1.7220
.37	1.20	.8220	.9299	2.0381	2.2274
.37	1.60	.8436	.9824	2.5804	2.8915
.37	2.40	.8682	1.0436	3.4822	4.0715
.51	.91	1.0674	1.1281	1.6720	1.7520
.51	1.20	1.0922	1.1866	2.1146	2.2657
.51	1.60	1.1169	1.2462	2.6774	2.9439
.51	2.40	1.1451	1.3159	3.6218	4.1612

N = 3

a	b	.01	.02	.95	.99
.11	.91	.5582	.7378	2.0413	2.2996
.11	1.20	.6085	.8331	2.5817	2.9455
.11	1.60	.6590	.9313	3.2567	3.7672
.11	2.40	.7172	1.0474	4.3654	5.1609
.22	.91	.8654	1.0253	2.1550	2.3744
.22	1.20	.9209	1.1296	2.7118	3.0353
.22	1.60	.9770	1.2376	3.4091	3.8794
.22	2.40	1.0416	1.3661	4.5620	5.3186
.37	1.60	1.4083	1.6514	3.6071	4.0224
.37	2.40	1.4830	1.7986	4.8204	5.5240
.51	1.20	1.7354	1.8953	3.0250	3.2444
.51	2.40	1.8937	2.2000	5.0514	5.7053

N = 5

a	b	.01	.02	.95	.99
.11	.91	1.1843	1.4665	3.1434	3.4865
.11	1.20	1.3335	1.6967	3.9505	4.4267
.11	1.60	1.4876	1.9422	4.9464	5.6082
.11	2.40	1.6705	2.2453	6.5473	7.5704

Table I continued.

.22	.91	1.6679	1.9161	3.3684	3.6620
.22	1.20	1.8310	2.1650	4.2028	4.6283
.22	1.60	2.0004	2.4322	5.2360	5.8467
.22	2.40	2.2027	2.7651	6.9065	7.8814
.37	.91	2.3176	2.5168	3.6588	3.8856
.37	1.20	2.5014	2.7925	4.5302	4.8870
.37	1.60	2.6935	3.0914	5.6144	6.1555
.37	2.40	2.9251	3.4688	7.3818	8.2900
.51	.91	2.9132	3.0639	3.9127	4.0786
.51	1.20	3.1179	3.3661	4.8184	5.1120
.51	1.60	3.3335	3.6967	5.9505	6.4267
.51	2.40	3.5957	4.1196	7.8097	8.6549
			N = 10		
.11	.91	2.9690	3.4067	5.7745	6.2709
.11	1.20	3.4470	4.0211	7.2038	7.8853
.11	1.60	3.9642	4.7039	8.9475	9.8821
.11	2.40	4.6186	5.6089	11.6954	13.1088
.22	.91	3.8622	4.2447	6.2954	6.7217
.22	1.20	4.3767	4.9006	7.7782	8.3895
.22	1.60	4.9369	5.6337	9.5938	10.4593
.22	2.40	5.6552	6.6145	12.4719	13.8226
.37	.91	5.0558	5.3605	6.9728	7.3042
.37	1.20	5.6229	6.0753	8.5289	9.0444
.37	1.60	6.2455	6.8806	10.4438	11.2143
.37	2.40	7.0514	7.9719	13.5041	14.7674
.51	.91	6.1430	6.3731	7.5708	7.8140
.51	1.20	6.7622	7.1447	9.1954	9.6217
.51	1.60	7.4470	8.0211	11.2038	11.8853
.51	2.40	8.3449	9.2232	14.4386	15.6187

TABLE II

Critical Values for the Truncated Distributions as Functions of the Truncation Points a and b for N = 2 and α = 2,3,4,5,6,7,8,9,10,15, and 20.

$\alpha = 2$

a	b	.01	.05	.95	.99
.54	2.06	1.2969	1.5478	3.5613	3.8584
.54	2.45	1.3346	1.6286	4.1264	4.5268
.54	3.04	1.3769	1.7197	4.8955	5.4803
.54	3.96	1.4165	1.8058	5.8863	6.7870

Table II continued.

.84	2.06	1.8550	2.0459	3.6715	3.9116
.84	2.45	1.8905	2.1259	4.2484	4.5895
.84	3.04	1.9307	2.2172	5.0436	5.5620
.84	3.96	1.9687	2.3045	6.0752	6.9076
1.10	2.06	2.3199	2.4703	3.7688	3.9580
1.10	2.45	2.3560	2.5530	4.3588	4.6446
1.10	3.04	2.3970	2.6480	5.1775	5.6345
1.10	3.96	2.4360	2.7396	6.2493	7.0167
1.40	2.06	2.8834	2.9885	3.8863	4.0132
1.40	2.45	2.9220	3.0774	4.4933	4.7104
1.40	3.04	2.9661	3.1806	5.3431	5.7220
1.40	3.96	3.0081	3.2808	6.4694	7.1514
			$\alpha = 3$		
1.07	3.12	2.4775	2.8473	5.5443	5.9169
1.07	3.64	2.5394	2.9735	6.3093	6.8158
1.07	4.33	2.6008	3.0988	7.2264	7.9401
1.07	5.37	2.6580	3.2164	8.3721	9.4368
1.56	3.12	3.3357	3.5958	5.6905	5.9865
1.56	3.64	3.3897	3.7147	6.4720	6.8972
1.56	4.33	3.4440	3.8347	7.4167	8.0432
1.56	5.37	3.4954	3.9490	8.6090	9.5842
1.91	3.12	3.9790	4.1760	5.8134	6.0444
1.91	3.64	4.0319	4.2954	6.6102	6.9653
1.91	4.33	4.0854	4.4171	7.5805	8.1305
1.91	5.37	4.1362	4.5339	8.8169	9.7115
2.28	3.12	4.6693	4.8062	5.9490	6.1076
2.28	3.64	4.7238	4.9306	6.7642	7.0401
2.28	4.33	4.7792	5.0586	7.7657	8.2273
2.28	5.37	4.8320	5.1825	9.0569	9.8553
			$\alpha = 4$		
1.80	4.20	4.0034	4.4460	7.6074	8.0336
1.80	4.80	4.0797	4.6006	8.5010	9.0772
1.80	5.60	4.1562	4.7560	9.5778	10.3893
1.80	6.80	4.2272	4.9007	10.9101	12.1230
2.30	4.20	4.8742	5.2003	7.7477	8.0998
2.30	4.80	4.9412	5.3455	8.6555	9.1538
2.30	5.60	5.0092	5.4934	9.7573	10.4857
2.30	6.80	5.0730	5.6328	11.1327	12.2607

Table II continued.

2.80	4.80	5.8518	6.1630	8.8415	9.2449
2.80	5.60	5.9172	6.3109	9.9763	10.6017
2.80	6.80	5.9791	6.4519	11.4095	12.4294
3.33	5.60	6.9083	7.2240	10.2310	10.7337
3.33	6.80	6.9726	7.3743	11.7400	12.6263

$\alpha = 5$

2.46	5.29	5.4198	5.9563	9.6688	10.1594
2.46	5.86	5.4985	6.1130	10.5255	11.1552
2.46	6.71	5.5898	6.2948	11.6926	12.5649
2.46	8.05	5.6803	6.4754	13.2180	14.5323
3.13	5.29	6.5766	6.9518	9.8476	10.2435
3.13	5.86	6.6433	7.0957	10.7183	11.2497
3.13	6.71	6.7218	7.2652	11.9134	12.6815
3.13	8.05	6.8005	7.4361	13.4918	14.6988
3.62	5.86	7.5316	7.8891	10.8910	11.3335
3.62	6.71	7.6066	8.0568	12.1138	12.7859
3.62	8.05	7.6823	8.2276	13.7450	14.8507
4.21	6.71	8.7032	9.0607	12.3802	12.9224
4.21	8.05	8.7804	9.2395	14.0898	15.0536

$\alpha = 6$

3.18	6.50	6.9584	7.5926	11.9302	12.5046
3.18	7.03	7.0329	7.7393	12.7226	13.4275
3.18	7.88	7.1283	7.9273	13.8934	14.8391
3.18	9.31	7.2315	8.1309	15.5381	16.9519
3.91	6.50	8.2237	8.6742	12.1221	12.5953
3.91	7.03	8.2863	8.8082	12.9270	13.5276
3.91	7.88	8.3675	8.9823	14.1239	14.9601
3.91	9.31	8.4565	9.1735	15.8220	17.1229
4.53	7.03	9.3878	9.7887	13.1375	13.6298
4.53	7.88	9.4644	9.9596	14.3643	15.0847
4.53	9.31	9.5488	10.1493	16.1236	17.3024
5.24	7.88	10.8016	11.1821	14.6824	15.2464
5.24	9.31	10.8876	11.3812	16.5332	17.5411

$\alpha = 7$

3.97	7.30	8.5503	9.2010	13.5704	14.1280
3.97	8.18	8.6804	9.4575	14.9058	15.6725
3.97	9.05	8.7801	9.6535	16.1076	17.1193
3.97	10.58	8.8940	9.8777	17.8773	19.3871

Table II continued.

4.75	8.28	10.0273	10.6179	15.2677	15.9504
4.75	9.05	10.1013	10.7758	16.3478	17.2448
4.75	10.58	10.1996	10.9860	18.1721	19.5641
5.42	8.18	11.2050	11.6500	15.3447	15.8863
5.42	9.05	11.2846	11.8271	16.6014	17.3758
5.42	10.58	11.3774	12.0347	18.4893	19.7521
6.21	9.05	12.7489	13.1629	16.9438	17.5496
6.21	10.58	12.8427	13.3794	18.9288	20.0075

$\alpha = 8$

4.67	8.35	10.0376	10.7695	15.5811	16.1875
4.67	9.25	10.1773	11.0411	16.9519	17.7697
4.67	10.21	10.2931	11.2656	18.2834	19.3693
4.67	11.88	10.4224	11.5164	20.2099	21.8382
5.60	9.25	11.7379	12.3622	17.1940	17.8877
5.60	10.21	11.8343	12.5671	18.5544	19.5106
5.60	11.88	11.9435	12.7995	20.5434	22.0385
6.36	9.25	13.1052	13.5746	17.4474	18.0101
6.36	10.21	13.1949	13.7738	18.8416	19.6585
6.36	11.88	13.2972	14.0025	20.9039	22.2525
6.95	9.25	14.1951	14.5619	17.6652	18.1142
6.95	10.21	14.2841	14.7631	19.0916	19.7854
6.95	11.88	14.3861	14.9973	21.2248	22.4400

$\alpha = 9$

5.40	9.45	11.5861	12.4000	17.6782	18.3400
5.40	10.29	11.7210	12.6598	18.9606	19.8183
5.40	11.40	11.8601	12.9266	20.5044	21.6700
5.40	12.90	11.9841	13.1647	22.2619	23.9113
6.42	9.45	13.3119	13.8610	17.9165	18.4506
6.42	10.29	13.4211	14.0921	19.2154	19.9418
6.42	11.40	13.5355	14.3338	20.7918	21.8199
6.42	12.90	13.6390	14.5527	22.6044	24.1129
7.20	10.29	14.8190	15.3271	19.4676	20.0632
7.20	11.40	14.9248	15.5610	21.0799	21.9686
7.20	12.90	15.0212	15.7751	22.9536	24.3160
7.98	11.40	16.3627	16.8660	21.4081	22.1354
7.98	12.90	16.4584	17.0851	23.3593	24.5481

Table II continued.

6.33	10.56	13.4591	14.2996	19.8302	20.5285
6.33	11.38	13.5877	14.5504	21.0832	21.9724
6.33	12.65	13.7431	14.8516	22.8472	24.0891
6.33	14.23	13.8681	15.0942	24.6787	26.4323
7.27	10.56	15.0558	15.6538	20.0525	20.6318
7.27	11.38	15.1633	15.8806	21.3197	22.0869
7.27	12.65	15.2946	16.1571	23.1166	24.2304
7.27	14.23	15.4017	16.3829	24.9995	26.6224
8.13	11.38	16.7026	17.2389	21.5947	22.2190
8.13	12.65	16.8234	17.5056	23.4342	24.3951
8.13	14.23	16.9226	17.7259	25.3839	26.8476
8.96	12.65	18.3525	18.8929	23.7854	24.5746
8.96	14.23	18.4509	19.1181	25.8179	27.0977

$\alpha = 15$

10.46	15.39	21.8823	22.8924	29.3502	30.1308
10.46	16.77	22.1184	23.3483	31.4974	32.5835
10.46	18.20	22.3065	23.7094	33.5086	34.9827
10.46	20.14	22.4748	24.0319	35.7964	37.8898
11.70	16.77	24.1858	25.0854	31.7952	32.7267
11.70	18.20	24.3424	25.4132	33.8422	35.1555
11.70	20.14	24.4843	25.7101	36.1931	38.1222
12.78	16.77	25.1130	26.7803	32.1292	32.8864
12.78	18.20	26.2557	27.0937	34.2214	35.3499
12.78	20.14	26.3860	27.3813	36.6518	38.3881
13.68	15.39	27.5981	27.8919	30.2219	30.5284
13.68	16.77	27.7672	28.2707	32.4440	33.0355
13.68	18.20	27.9065	28.5843	34.5839	35.5333
13.68	20.14	28.0343	28.8744	37.0990	38.6437

$\alpha = 20$

14.54	21.15	30.4253	31.7897	40.3551	41.4083
14.54	22.05	30.5828	32.0874	41.7392	42.9971
14.54	23.70	30.8162	32.5268	44.0738	45.7736
14.54	25.94	31.0249	32.9189	46.7257	49.1355
16.20	21.15	33.1795	34.0972	40.7141	41.5750
16.20	22.05	33.3043	34.3578	42.1136	43.1764
16.20	23.70	33.4920	34.7486	44.4914	45.9886
16.20	25.94	33.6623	35.1029	47.2212	49.4245

Table II Continued.

17.44	22.05	35.5286	36.3084	42.4895	43.3553
17.44	23.70	35.6985	36.6802	44.9163	46.2054
17.44	25.94	35.8540	37.0214	47.7344	49.7209
18.90	23.70	38.3799	39.1026	45.4983	46.4979
18.90	25.94	38.5319	39.4491	48.4544	50.1298

TABLE III

Critical Values for the Truncated Distributions as Functions of the Truncation Points a and b for N = 3 and α = 2,3,4,5,6,7,8,9, and 10

a	b	.01	.05	.95	.99
			$\alpha = 2$		
.54	2.06	2.2023	2.5943	5.0424	5.4841
.54	2.45	2.3030	2.7668	5.8049	6.3807
.54	3.04	2.4166	2.9647	6.8418	7.6293
.54	3.96	2.5238	3.1555	8.1735	9.2869
.85	2.06	2.9784	3.2861	5.2662	5.6219
.85	2.45	3.0774	3.4609	6.0439	6.5365
.85	3.04	3.1904	3.6639	7.1141	7.8181
.85	3.96	3.2981	3.8621	8.5013	9.5333
1.10	2.06	3.6343	3.8792	5.4575	5.7435
1.10	2.45	3.7364	4.0611	6.2582	6.6752
1.10	3.04	3.8536	4.2744	7.3608	7.9882
1.10	3.96	3.9662	4.4846	8.8018	9.7593
1.40	2.45	4.5427	4.7999	6.5218	6.8438
1.40	3.04	4.6700	5.0319	7.6685	8.1979
1.40	3.96	4.7928	5.2634	9.1821	10.0441
			$\alpha = 3$		
1.07	3.12	4.0939	4.6511	7.9299	8.4943
1.07	3.64	4.2524	4.9119	8.9700	9.7074
1.07	4.33	4.4099	5.1752	10.2067	11.1874
1.07	5.37	4.5577	5.4269	11.7450	13.0944
1.56	3.12	5.2702	5.6816	8.2273	8.6769
1.56	3.64	5.4178	5.9362	9.2899	9.9120
1.56	4.33	5.5668	6.1973	10.5640	11.4294
1.56	5.37	5.7086	6.4506	12.1664	13.4025
1.91	3.12	6.1708	6.4876	8.4773	8.8302
1.91	3.64	6.3184	6.7467	9.5618	10.0854
1.91	4.33	6.4688	7.0154	10.8713	11.6373
1.91	5.37	6.6130	7.2787	12.5331	13.6719

Table III Continued.

2.28	3.12	7.1445	7.3659	8.7544	8.9989
2.28	3.64	7.2981	7.6372	9.8666	10.2782
2.28	4.33	7.4559	7.9218	11.2202	11.8714
2.28	5.37	7.6085	8.2042	12.9553	13.9815
			$\alpha = 4$		
1.80	4.20	6.4569	7.1201	10.9634	11.6143
1.80	4.80	6.6513	7.4373	12.1813	13.0281
1.80	5.60	6.8467	7.7608	13.6355	14.7614
1.80	6.80	7.0287	8.0675	15.4244	16.9746
2.30	4.20	7.6454	8.1555	11.2511	11.7894
2.30	4.80	7.8260	8.4628	12.4879	13.2221
2.30	5.60	8.0101	8.7805	13.9757	14.9895
2.30	6.80	8.1835	9.0855	15.8241	17.2641
2.80	4.20	8.9195	9.2903	11.5932	11.9976
2.80	4.80	9.0971	9.5999	12.8566	13.4553
2.80	5.60	9.2801	9.9242	14.3898	15.2670
2.80	6.80	9.4543	10.2396	16.3165	17.6227
3.33	4.20	10.3099	10.5415	11.9812	12.2324
3.33	4.80	10.4946	10.8662	13.2805	13.7213
3.33	5.60	10.6869	11.2114	14.8734	15.5888
3.33	6.80	10.8719	11.5521	16.9017	18.0488
			$\alpha = 5$		
2.46	5.29	8.6768	9.4704	13.9843	14.7370
2.46	5.86	8.8743	9.7879	15.1544	16.0899
2.46	6.71	9.1036	10.1611	16.7341	17.9617
2.46	8.05	9.3314	10.5384	18.7831	20.4842
3.13	5.29	10.2493	10.8338	14.3526	14.9599
3.13	5.86	10.4284	11.1368	15.5409	16.3317
3.13	6.71	10.6395	11.4982	17.1580	18.2417
3.13	8.05	10.8523	11.8690	19.2797	20.8394
3.62	5.29	11.4894	11.9348	14.6788	15.1577
3.62	5.86	11.6642	12.2372	15.8864	16.5480
3.62	6.71	11.8720	12.6017	17.5415	18.4952
3.62	8.05	12.0834	12.9800	19.7348	21.1668
4.21	5.29	13.0290	13.3170	15.1028	15.4140
4.21	5.86	13.2076	13.6297	16.3405	16.8308
4.21	6.71	13.4221	14.0117	18.0526	18.8312
4.21	8.05	13.6426	14.4142	20.3515	21.6107

Table III Continued.

$\alpha = 6$

3.18	6.50	11.0839	12.0159	17.2906	18.1692
3.18	7.03	11.2691	12.3116	18.3728	19.4216
3.18	7.88	11.5065	12.6950	19.9588	21.2980
3.18	9.31	11.7639	13.1172	22.1689	24.0123
3.92	6.50	12.7968	13.4952	17.6853	18.4087
3.92	7.03	12.9639	13.7762	18.7835	19.6780
3.92	7.88	13.1809	14.1450	20.4041	21.5901
3.92	9.31	13.4193	14.5568	22.6871	24.3801
4.52	6.50	14.3314	14.8551	18.0872	18.6533
4.52	7.03	14.4930	15.1343	19.2052	19.9416
4.52	7.88	14.7047	15.5046	20.8665	21,8939
4.52	9.31	14.9397	15.9232	23.2326	24.7696
5.25	6.50	16.2093	16.5413	18.6072	18.9686
5.25	7.03	16.3747	16.8309	19.7563	20.2844
5.25	7.88	16.5934	17.2199	21.4797	22.2947
5.25	9.31	16.8388	17.6668	23.9695	25.2962

$\alpha = 7$

3.97	7.30	13.4885	14.4421	19.7555	20.6209
3.97	8.18	13.8123	14.9569	21.5837	22.7259
3.97	9.05	14.0599	15.3554	23.2125	24.6508
3.97	10.58	14.3434	15.8187	25.5912	27.5680
4.76	7.30	15.3254	16,0265	20.1652	20.8658
4.76	8.18	15.6456	16.5626	22.2159	23.2286
4.76	9.05	15.8426	16.8962	23.6782	24.9550
4.76	10.58	16.1048	17.3470	26.1316	27.9499
5.42	7.30	16.9843	17.4932	20.5859	21.1183
5.42	8.18	17.2640	17.9741	22.4653	23.2759
5.42	9.05	17.4836	18.3568	24.1678	25.2754
5.42	10.58	17.7407	18.8131	26.7074	28.3593
6.21	7.30	19.0394	19.3335	21.1383	21.4493
6.21	8.18	19.3232	18.8286	23.0647	23.6487
6.21	9.05	19.5487	20.2289	24.8298	25.7068
6.21	10.58	19.8157	20.7136	27.5006	28.9242

$\alpha = 8$

4.67	8.35	15.8015	16.8642	22.7166	23.6601
4.67	9.25	16.1450	17.4053	24.5953	25.8194
4.67	10.21	16.4293	17.8583	26.4016	27.9502
4.67	11.88	16.7471	18.3729	28.9914	31.1262

Table III Continued.

5.60	8.35	17.9420	18.7045	23.1836	23.9385
5.60	9.25	18.2459	19.2110	25.0848	26.1228
5.60	10.21	18.5016	19.6421	26.9289	28.2933
5.60	11.88	18.7918	20.1388	29.6040	31.5583
6.36	8.35	19.8491	20.3890	23.6636	24.2261
6.36	9.25	20.1393	20.8873	25.5947	26.4399
6.36	10.21	20.3863	21.3172	27.4847	28.6560
6.36	11.88	20.6697	21.8191	30.2593	32.0239
6.95	8.35	21.3785	21.7565	24.0714	24.4703
6.95	9.25	21.6692	22.2616	26.0333	26.7120
6.95	10.21	21.9187	22.7021	27.9687	28.9710
6.95	11.88	22.2072	23.2222	30.8391	32.4368

$\alpha = 9$

5.40	9.45	18.2071	19.3815	25.8022	26.8322
5.40	10.29	18.5361	19.8960	27.5611	28.8511
5.40	11.40	18.8746	20.4316	29.6569	31.3202
5.40	12.90	19.1768	20.9170	32.0207	34.2104
6.42	9.45	20.5298	21.3721	26.3002	27.1288
6.42	10.29	20.8183	21.8509	28.0788	29.1704
6.42	11.40	21.1202	22.3574	30.2169	31.6842
6.42	12.90	21.3938	22.8230	32.6555	34.6526
7.20	9.45	22.4754	23.0869	26.7848	27.4191
7.20	10.29	22.7496	23.5560	28.5883	29.4858
7.20	11.40	23.0398	24.0585	30.7751	32.0483
7.20	12.90	23.3054	24.5262	33.2962	35.1021
7.98	9.45	24.4943	24.8910	27.3219	27.7410
7.98	10.29	24.7682	25.3638	29.1601	29.8390
7.98	11.40	25.0609	25.8833	31.4104	32.4618
7.98	12.90	25.3315	26.3706	34.0367	35.6226

$\alpha = 10$

6.33	10.56	21.0443	22.2653	28.9920	30.0775
6.33	11.38	21.3615	22.7643	30.7106	32.0450
6.33	12.65	21.7428	23.3710	33.1051	34.8719
6.33	14.23	22.0502	23.8676	35.5677	37.8885
7.27	10.56	23.1955	24.1109	29.4562	30.3543
7.27	11.38	23.4788	24.5800	31.1916	32.3462
7.27	12.65	23.8244	25.1582	33.6289	35.2133
7.27	14.23	24.1067	25.6378	36.1615	38.3033

Table III Continued.

8.13	10.56	25.3357	25.9958	29.9878	30.6730
8.13	11.38	25.6039	26.4542	31.7481	32.6900
8.13	12.65	25.9348	27.0268	34.2428	35.6149
8.13	14.23	26.2081	27.5078	36.8658	38.7987
8.96	10.56	27.4828	27.9141	30.5593	31.0157
8.96	11.38	27.7503	28.3787	32.3533	33.0635
8.96	12.65	28.0837	28.9669	34.9211	36.0578
8.96	14.23	28.3619	29.4679	37.6561	39.3561

TABLE IV

Tables of μ, σ, β_1, and β_2 for N = 5 and α = 2,3,4,5,6,7,8,9, and 10

N = 5

a	b	α	μ	σ	β_1	β_2
.54	2.06	2	6.3449	.9532	.0018	2.7733
.54	2.45	2	7.0982	1.1842	.0062	2.7822
.54	3.04	2	8.0677	1.5149	.0191	2.8045
.54	3.96	2	9.1966	1.9681	.0547	2.8622
.85	2.06	2	7.1044	.7690	.0029	2.7691
.85	2.45	2	7.8786	1.0064	.0077	2.7781
.85	3.04	2	8.8848	1.3478	.0212	2.8003
.85	3.96	2	10.0709	1.8196	.0581	2.8583
1.10	2.06	2	7.7657	.6139	.0027	2.7665
1.10	2.45	2	8.5647	.8557	.0074	2.7746
1.10	3.04	2	9.6115	1.2053	.0206	2.7957
1.10	3.96	2	10.8589	1.6930	.0572	2.8522
1.40	2.06	2	8.5747	.4242	.0018	2.7635
1.40	2.45	2	9.4110	.6702	.0058	2.7702
1.40	3.04	2	10.5176	1.0289	.0178	2.7888
1.40	3.96	2	11.8540	1.5360	.0528	2.8420
1.07	3.12	3	10.4816	1.2766	.0000	2.7744
1.07	3.64	3	11.5369	1.5814	.0017	2.7810
1.07	4.33	3	12.7244	1.9641	.0092	2.7972
1.07	5.37	3	14.0721	2.4741	.0340	2.8417
1.56	3.12	3	11.5766	.9904	.0009	2.7677
1.56	3.64	3	12.6484	1.3061	.0042	2.7753
1.56	4.33	3	13.8700	1.7042	.0138	2.7932
1.56	5.37	3	15.2764	2.2389	.0418	2.8404

Table IV Continued.

1.91	3.12	3	12.4578	.7738	.0013	2.7650
1.91	3.64	3	13.5547	1.0960	.0048	2.7723
1.91	4.33	3	14.8168	1.5046	.0148	2.7898
1.91	5.37	3	16.2871	2.0581	.0438	2.8370
2.28	3.12	3	13.4254	.5398	.0011	2.7627
2.28	3.64	3	14.5599	.8676	.0043	2.7689
2.28	4.33	3	15.8788	1.2864	.0139	2.7851
2.28	5.37	3	17.4356	1.8606	.0427	2.8305
*1.80	4.20	4	15.0746	1.4964	.0001	2.7741
1.80	4.80	4	16.3244	1.8483	.0007	2.7797
1.80	5.60	4	17.7386	2.2924	.0065	2.7942
1.80	6.80	4	19.3256	2.8795	.0283	2.8358
2.30	4.20	4	16.1618	1.2043	.0003	2.7678
2.30	4.80	4	17.4213	1.5674	.0026	2.7745
2.30	5.60	4	18.8624	2.0269	.0105	2.7907
2.30	6.80	4	20.5004	2.6382	.0357	2.8355
2.80	4.20	4	17.3909	.8955	.0009	2.7646
2.80	4.80	4	18.6779	1.2677	.0036	2.7710
2.80	5.60	4	20.1680	1.7419	.0124	2.7873
2.80	6.80	4	21.8865	2.3794	.0396	2.8329
3.33	4.20	4	18.7620	.5596	.0007	2.7622
3.33	4.80	4	20.0971	.9393	.0033	2.7674
3.33	5.60	4	21.6632	1.4277	.0119	2.7821
3.33	6.80	4	23.5000	2.0943	.0392	2.8261
*2.46	5.29	5	19.5620	1.7578	.0004	2.7760
2.46	5.86	5	20.7766	2.0908	.0001	2.7798
2.46	6.71	5	22.3394	2.5638	.0033	2.7911
2.46	8.05	5	24.1949	3.2249	.0209	2.8276
3.13	5.29	5	20.9802	1.3690	.0002	2.7677
3.13	5.86	5	22.1998	1.7145	.0015	2.7728
3.13	6.71	5	23.7880	2.2070	.0074	2.7863
3.13	8.05	5	25.7021	2.8996	.0296	2.8272
3.62	5.29	5	22.1658	1.0671	.0006	2.7648
3.62	5.86	5	23.4033	1.4202	.0025	2.7698
3.62	6.71	5	25.0302	1.9259	.0093	2.7835
3.62	8.05	5	27.0160	2.6433	.0334	2.8255
4.21	5.29	5	23.6749	.6942	.0007	2.7637
4.21	5.86	5	24.9504	1.0545	.0026	2.7668
4.21	6.71	5	26.6474	1.5747	.0095	2.7793
4.21	8.05	5	28.7536	2.3232	.0343	2.8204

*$\mu_3 < 0$

Table IV Continued.

*3.18	6.50	6	24.4438	2.0544	.0005	2.7776
3.18	7.03	6	25.5682	2.3616	.0000	2.7808
3.18	7.88	6	27.1466	2.8330	.0021	2.7902
3.18	9.31	6	29.1686	3.5394	.0170	2.8234
3.92	6.50	6	25.9744	1.6304	.0001	2.7688
3.92	7.03	6	27.1014	1.9495	.0012	2.7732
3.92	7.88	6	38.7000	2.4403	.0059	2.7849
3.92	9.31	6	30.7775	3.1800	.0256	2.8228
4.53	6.50	6	27.4365	1.2570	.0007	2.7652
4.53	7.30	6	28.5807	1.5843	.0023	2.7697
4.53	7.88	6	30.2192	2.0901	.0081	2.7819
4.53	9.31	6	32.3777	2.8594	.0302	2.8213
5.25	6.50	6	29.2801	.8031	.0008	2.7607
5.25	7.30	6	30.4632	1.1379	.0025	2.7660
5.25	7.88	6	32.1776	1.6597	.0084	2.7774
5.25	9.31	6	34.4770	2.4653	.0314	2.8159
*3.97	7.30	7	28.5331	2.0693	.0013	2.7767
*3.97	8.18	7	30.4518	2.5820	.0000	2.7809
3.97	9.05	7	32.0787	3.0641	.0016	2.7895
3.97	10.58	7	34.2674	3.8205	.0152	2.8212
*4.76	7.30	7	30.1601	1.6100	.0000	2.7674
4.76	8.18	7	32.2783	2.2013	.0013	2.7743
4.76	9.05	7	33.7209	2.6438	.0051	2.7841
4.76	10.58	7	35.9646	3.4353	.0235	2.8206
5.42	7.30	7	31.7241	1.2026	.0002	2.7639
5.42	8.18	7	33.6619	1.7479	.0021	2.7699
5.42	9.05	7	35.3445	2.2648	.0073	2.7811
5.42	10.58	7	37.6706	3.0877	.0283	2.8193
6.21	7.30	7	33.7205	.7015	.0003	2.7587
6.21	8.18	7	35.7159	1.2589	.0024	2.7664
6.21	9.05	7	37.4737	1.7924	.0079	2.7767
6.21	10.58	7	39.9474	2.6544	.0300	2.8145
*4.67	8.35	8	33.0319	2.2797	.0020	2.7785
*4.67	9.25	8	35.0159	2.8018	.0001	2.7820
4.67	10.21	8	36.8317	3.3317	.0010	2.7900
4.67	11.88	8	39.2245	4.1509	.0134	2.8211
*5.60	8.35	8	34.9096	1.7425	.0000	2.7673
5.60	9.25	8	36.8851	2.2867	.0007	2.7728
5.60	10.21	8	38.7176	2.8404	.0044	2.7835
5.60	11.88	8	41.1698	3.7011	.0226	2.8203

* $\mu_3 < 0$

Table IV Continued.

6.36	8.35	8	36.7024	1.2733	.0002	2.7623
6.36	9.25	8	38.6987	1.8318	.0018	2.7688
6.36	10.21	8	40.5734	2.4036	.0068	2.7802
6.36	11.88	8	43.1193	3.3010	.0280	2.8192
6.95	8.35	8	38.1824	.8999	.0003	2.7569
6.95	9.25	8	40.2143	1.4673	.0021	2.7658
6.95	10.21	8	42.1414	2.0519	.0075	2.7773
6.95	11.88	8	44.7921	2.9791	.0298	2.8164
*5.40	9.45	9	37.7117	2.5016	.0024	2.7806
*5.40	10.29	9	39.5771	2.9871	.0004	2.7831
5.40	11.40	9	41.6944	3.5979	.0007	2.7911
5.40	12.90	9	43.9004	4.3386	.0094	2.8151
*6.42	9.45	9	39.7321	1.9173	.0000	2.7693
6.42	10.29	9	41.5854	2.4244	.0004	2.7732
6.42	11.40	9	43.7175	3.0637	.0041	2.7841
6.42	12.90	9	45.9725	3.8427	.0180	2.8132
7.20	9.45	9	41.5515	1.4379	.0001	2.7683
7.20	10.29	9	43.4196	1.9582	.0014	2.7702
7.20	11.40	9	45.5937	2.6176	.0065	2.7813
7.20	12.90	9	47.9247	3.4281	.0230	2.8121
7.98	9.45	9	43.5029	.9449	.0003	2.7863
7.98	10.29	9	45.4099	1.4755	.0018	2.7707
7.98	11.40	9	47.6568	2.1533	.0074	2.7787
7.98	12.90	9	50.1024	2.9964	.0252	2.8090
*6.33	10.56	10	42.7690	2.6204	.0019	2.7784
*6.33	11.38	10	44.5889	3.0962	.0003	2.7813
6.33	12.65	10	47.0035	3.7963	.0010	2.7906
6.33	14.23	10	49.2895	4.5721	.0106	2.8161
*7.27	10.56	10	44.6459	2.0798	.0000	2.7679
7.27	11.38	10	46.4558	2.5738	.0003	2.7724
7.27	12.65	10	48.8858	3.3022	.0042	2.7848
7.27	14.23	10	51.2182	4.1133	.0183	2.8148
8.13	10.56	10	46.6439	1.5523	.0002	2.7615
8.13	11.38	10	48.4677	2.0598	.0013	2.7677
8.13	12.65	10	50.9463	2.8125	.0069	2.7815
8.13	14.23	10	53.3607	3.6583	.0239	2.8138
8.96	10.56	10	48.7189	1.0282	.0004	2.7448
8.96	11.38	10	50.5804	1.5459	.0018	2.7617
8.96	12.65	10	53.1420	2.3201	.0080	2.7776
8.96	14.23	10	55.6771	3.2010	.0263	2.8106

* $\mu_3 < 0$

TABLE V

Tables of μ, σ, β_1 and β_2 for N = 10 and α= 2,3,4, and 5

N = 10

a	b	α	μ	σ	β_1	β_2
.54	2.06	2	12.6898	1.3480	.0009	2.8867
.54	2.45	2	14.1965	1.6748	.0031	2.8911
.54	3.04	2	16.1353	2.1423	.0095	2.9022
.54	3.96	2	18.3931	2.7834	.0274	2.9311
.85	2.06	2	14.2088	1.0875	.0014	2.8846
.85	2.45	2	15.7572	1.4233	.0039	2.8890
.85	3.04	2	17.7695	1.9060	.0106	2.9002
.85	3.96	2	20.1417	2.5733	.0290	2.9292
1.10	2.06	2	15.5314	.8682	.0014	2.8832
1.10	2.45	2	17.1294	1.2102	.0037	2.8873
1.10	3.04	2	19.2230	1.7045	.0103	2.8979
1.10	3.96	2	21.7177	2.3943	.0286	2.9261
1.40	2.06	2	17.1493	.5999	.0009	2.8818
1.40	2.45	2	18.8220	.9478	.0029	2.8851
1.40	3.04	2	21.0352	1.4551	.0089	2.8944
1.40	3.96	2	23.7080	2.1723	.0264	2.9210
1.07	3.12	3	20.9632	1.8054	.0000	2.8872
1.07	3.64	3	23.0737	2.2364	.0008	2.8905
1.07	4.33	3	25.4488	2.7776	.0046	2.8986
1.07	5.37	3	28.1441	3.4990	.0170	2.9208
1.56	3.12	3	23.1533	1.4006	.0005	2.8838
1.56	3.64	3	25.2969	1.8472	.0021	2.8877
1.56	4.33	3	27.7400	2.4102	.0069	2.8966
1.56	5.37	3	30.5528	3.1662	.0209	2.9202
1.91	3.12	3	24.9156	1.0943	.0007	2.8825
1.91	3.64	3	27.1094	1.5500	.0024	2.8862
1.91	4.33	3	29.6337	2.1278	.0074	2.8949
1.91	5.37	3	32.5741	2.9107	.0219	2.9185
2.28	3.12	3	26.8508	.7634	.0005	2.8813
2.28	3.64	3	29.1198	1.2270	.0022	2.8845
2.28	4.33	3	31.7577	1.8193	.0070	2.8925
2.28	5.37	3	34.8712	2.6313	.0213	2.9153

Table V Continued.

*1.80	4.20	4	30.1492	2.1162	.0000	2.8871
1.80	4.80	4	32.6489	2.6139	.0004	2.8898
1.80	5.60	4	35.4771	3.2419	.0033	2.8971
1.80	6.80	4	38.6513	4.0723	.0142	2.9179
2.30	4.20	4	32.3235	1.7032	.0002	2.8839
2.30	4.80	4	34.8427	2.2166	.0013	2.8872
2.30	5.60	4	37.7247	2.8665	.0052	2.8954
2.30	6.80	4	41.0008	3.7310	.0179	2.9178
2.80	4.20	4	34.7817	1.2665	.0004	2.8823
2.80	4.80	4	37.3558	1.7928	.0018	2.8855
2.80	5.60	4	40.3360	2.4634	.0062	2.8936
2.80	6.80	4	43.7730	3.3650	.0198	2.9164
3.33	4.20	4	37.5239	.7915	.0004	2.8811
3.33	4.80	4	40.1942	1.3284	.0017	2.8837
3.33	5.60	4	43.3263	2.0191	.0059	2.8911
3.33	6.80	4	47.0000	2.9617	.0196	2.9131
*2.46	5.29	5	39.1240	2.4859	.0002	2.8880
2.46	5.86	5	41.5532	2.9568	.0000	2.8899
2.46	6.71	5	44.6789	3.6258	.0017	2.8955
2.46	8.05	5	48.3897	4.5607	.0104	2.9138
3.13	5.29	5	41.9605	1.9360	.0001	2.8839
3.13	5.86	5	44.3996	2.4247	.0008	2.8864
3.13	6.71	5	47.5759	3.1212	.0037	2.8931
3.13	8.05	5	51.4043	4.1007	.0148	2.9136
3.62	5.29	5	44.3317	1.5091	.0003	2.8824
3.62	5.86	5	46.8066	2.0084	.0012	2.8849
3.62	6.71	5	50.0604	2.7236	.0046	2.8918
3.62	8.05	5	54.0319	3.7383	.0167	2.9128
4.21	5.29	5	47.3498	.9818	.0003	2.8818
4.21	5.86	5	49.9008	1.4913	.0013	2.8834
4.21	6.71	5	53.2949	2.2270	.0047	2.8897
4.21	8.05	5	57.5071	3.2855	.0172	2.9102

* $\mu_3 < 0$

TABLE VI

Tables of μ, σ, β_1 and β_2 for N = 15
and α = 1,2,3,4

a	b	α	μ	σ	β_1	β_2
.11	.91	1	6.8584	.8804	.0051	2.9271
.11	1.20	1	8.3685	1.1836	.0094	2.9331
.11	1.60	1	10.1474	1.5787	.0174	2.9443
.22	.91	1	7.8845	.7624	.0038	2.9252
.22	1.20	1	9.4683	1.0700	.0076	2.9306
.22	1.60	1	11.3418	1.4731	.0150	2.9409
.22	2.40	1	14.1324	2.1810	.0363	2.9713
.37	.91	1	9.2373	.5994	.0023	2.9232
.37	1.20	1	10.9236	.9123	.0055	2.9276
.37	1.60	1	12.9299	1.3252	.0119	2.9366
.37	2.40	1	15.9462	2.0594	.0317	2.9646
.51	.91	1	10.4505	.4454	.0013	2.9217
.51	1.20	1	12.2345	.7624	.0038	2.9253
.51	1.60	1	14.3685	1.1836	.0094	2.9331
.51	2.40	1	17.6049	1.9411	.0276	2.9588

N = 15

a	b	α	μ	σ	β_1	β_2
.54	2.06	2	19.0348	1.6510	.0006	2.9244
.54	2.45	2	21.2947	2.0512	.0021	2.9274
.54	3.04	2	24.2030	2.6238	.0064	2.9348
.54	3.96	2	27.5897	3.4089	.0182	2.9541
.85	2.06	2	21.3133	1.3320	.0010	2.9230
.85	2.45	2	23.6359	1.7432	.0026	2.9260
.85	3.04	2	26.6543	2.3344	.0071	2.9334
.85	3.96	2	30.2126	3.1517	.0194	2.9528
1.10	2.06	2	23.2971	1.0634	.0009	2.9222
1.10	2.45	2	25.6940	1.4821	.0025	2.9249
1.10	3.04	2	28.8346	2.0876	.0069	2.9319
1.10	3.96	2	32.5766	2.9324	.0191	2.9507
1.40	2.06	2	25.7240	.7374	.0006	2.9212
1.40	2.45	2	28.2330	1.1609	.0019	2.9234
1.40	3.04	2	31.5527	1.7822	.0059	2.9296
1.40	3.96	2	35.5620	2.6605	.0176	2.9473
1.07	3.12	3	31.4448	2.2111	.0000	2.9248
1.07	3.64	3	34.6106	2.7390	.0006	2.9270
1.07	4.33	3	38.1733	3.4018	.0031	2.9324
1.07	5.37	3	42.2162	4.2853	.0118	2.9472

Table VI Continued.

1.56	3.12	3	34.7299	1.7154	.0003	2.9226
1.56	3.64	3	37.9453	2.2623	.0014	2.9251
1.56	4.33	3	41.6099	2.9518	.0046	2.9311
1.56	5.37	3	45.8292	3.8778	.0139	2.9468
1.91	3.12	3	37.3734	1.3403	.0004	2.9217
1.91	3.64	3	40.6641	1.8983	.0016	2.9241
1.91	4.33	3	44.4505	2.6060	.0049	2.9299
1.91	5.37	3	48.8612	3.5648	.0146	2.9457
2.28	3.12	3	40.2762	.9350	.0004	2.9209
2.28	3.64	3	43.6796	1.5027	.0014	2.9230
2.28	4.33	3	47.6365	2.2282	.0046	2.9284
2.28	5.37	3	52.3068	3.2227	.0142	2.9435
*1.80	4.20	4	45.2239	2.5918	.0000	2.9247
1.80	4.80	4	48.9733	3.2014	.0002	2.9266
1.80	5.60	4	53.2157	3.9705	.0022	2.9314
1.80	6.80	4	57.9769	4.9875	.0094	2.9453
2.30	4.20	4	48.4853	2.0860	.0001	2.9226
2.30	4.80	4	52.2640	2.7148	.0009	2.9248
2.30	5.60	4	56.5871	3.5107	.0035	2.9302
2.30	6.80	4	61.5011	4.5695	.0119	2.9452
2.80	4.20	4	52.1726	1.5511	.0003	2.9215
2.80	4.80	4	56.0337	2.1957	.0012	2.9237
2.80	5.60	4	60.5040	3.0170	.0041	2.9291
2.80	6.80	4	65.6595	4.1213	.0132	2.9443
3.33	4.20	4	56.2859	.9693	.0002	2.9207
3.33	4.80	4	60.2913	1.6269	.0011	2.9225
3.33	5.60	4	65.9895	2.4729	.0040	2.9274
3.33	4.20	4	70.5000	3.6274	.0131	2.9420

* $\mu_3 < 0$

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