THE APPLICATION OF THE METHOD OF QUASILINEARIZATION TO THE COMPUTATION OF OPTIMAL CONTROL

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by

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FOREWORD

The research described in this report, "The Application of the Method of Quasilinearization to the Computation of Optimal Control," No. 67-49, by Garrett Paine, was carried out under the direction of C. T. Leondes in the Department of Engineering, University of California, Los Angeles.

One of the principal goals of this work was the development of the method of quasilinearization so that it could be used as an effective computational tool in the generation of optimal control. The generation of optimal control is of special importance to advanced ballistic systems, conventional aircraft systems problems, advanced space systems problems, and numerous other important areas.

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This report is based on the Doctor of Philosophy dissertation submitted by the author.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF SYMBOLS

y - state vector. May include both physical and adjoint variables.

t - time.

y_o - initial value of y. May not specify all components of y.

y_f - final value of y. May not specify all components of y.

$\hat{y}$ - solution of the inhomogeneous QL equation. A vector.

Y - solution of the homogeneous QL equation. A matrix.

T - final time. May be unspecified.

α - a constant vector in QL. Or a constant in Chapter 3.

c - a vector of unknown constants.

q - a vector of functions of the constants, the state variables and time.

Z - a matrix solution of the inhomogeneous QL with contrast equation.

β - a constant vector in QL with constraints.

τ - a dummy independent variable.

u - the control vector.

A - an operator.

ρ - a norm. Except Chapter 7.

B - an operator.

ϵ - a constant used in expanding the range of convergence of QL or "is a member of" in Chapter 3.

J - a matrix of the partials of F with respect to y.

S - a metric space.

G - the kernel for the integral solution of the second order differential equation $\ddot{y} = f(y, g)$.

H - the Hamiltonian.

p - the adjoint vector.

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (Cont.)

CHAPTER 1

INTRODUCTION

1.1 Control Systems Theory and Optimization Techniques

Currently a large amount of effort is being devoted to the optimization of control systems. The effort is usually directed either towards the proof of fundamental properties of control systems, or, as in this dissertation, towards the improvement of computational techniques.

Without improved computational techniques much of the theory must go unused as its application to a real problem is unwieldy.

There are many ways in which an optimal control problem can be formulated and solved: the method of steepest descents, the method of second variations, dynamic programming, quasilinearization, and others. Most have been applied to solve engineering problems of sufficient difficulty to be considered significant.

The dissertation deals with the application of the method of quasilinearization to the optimal control problem, and to the improvement of its applicability in problems where either the control cannot be solved for explicitly, or where there are bounds on the control. The method of quasilinearization is also called the generalized Newton-Raphson method.

1.2 The Current Uses of Quasilinearization

Several authors have applied the method of quasilinearization (QL) to a variety of problems. One of the earliest applications belongs to Hestenes,[1] Bellman,[2] and Kalaba,[3] who have shown convergence proofs using the maximization operation, have employed

QL to solve a variety of simple, nonlinear differential equations. McGill[4,5] and Kenneth[6] have shown convergence proofs using the contraction mapping operation and have employed QL in the solution of several simple nonlinear differential equations. McGill[22] has worked a simple bounded state space problem using the penalty function approach.

Kopp and Moyer,[11] and McGill[12] have compared QL with several other techniques and have used QL to solve some trajectory analysis problems.

Sylvester and Meyer[7,8] combined the method of quasilinearization with a first order integration procedure and have used it to solve some problems in mechanics and trajectory analysis. Long[9,10] has shown how to apply the method of QL where the final time is free and has applied his results to several estimation problems.

The references above that are concerned with the application of QL to the control problem either assume that the maximum principle can be used to find the control explicitly so that the control can be eliminated from the differential equations, or that continuous variations about the control vector can be taken. Neither of these approaches can handle the case of bounded control directly; instead, Valentine's[13] method must be used which results in an extra term to be carried through the calculations. Kopp and Moyer[11] have noted difficulties in applying either the method of second variations or the method of QL to solve problems with bounded control.

This work is directed towards the development of techniques that will facilitate the application of the method of QL to control problems where bounds on the control exist, and where the final time is free. Towards this end the convergence proofs of McGill and Kenneth have been extended to cover bounded continuous control directly by means of the addition of another term to QL. The work of Long has been modified to provide more accurate integration while preserving its usefulness in solving problems where the final time is free.

1.3 The Scope of the Dissertation

The dissertation is directed towards the goals of formulating the method of quasilinearization so that optimization problems can be handled directly, and of showing that this is indeed so by means of a non-trivial example.

In Chapter 2 the application of the method of quasilinearization to the solution of a two point boundary value problem is discussed. The technique by which unknown constants can be found is included here. In particular it is shown how the period of integration with respect to a dummy variable can be fixed by the addition of one parameter even if the final time is free.

A quadratic convergence proof is given in Chapter 3. The proof is an extension of that found in the literature as it allows bounds on the control to be handled directly.

A method for extending the region over which the method converges is detailed in Chapter 4. The theoretical advantage of the extended method is shown.

In Chapter 5, the numerical techniques used to solve practical problems are discussed. These techniques are applied to the problems described below, and can form the basis for any numerical application of the method of quasilinearization.

Chapter 6 and Chapter 7 cover the two numerical examples used. In Chapter 6 the classical Brachistochrone is solved. Here the solution is used to show that the solution converged much more rapidly for the free time problem using the technique of Chapter 2 than that found in the literature.[6] And it is shown that Method 3 of Chapter 4 extends the normal region of convergence. The Brachistochrone is also used to examine the effects of a modified integration scheme.

In Chapter 7 the general problem of determining the trajectory of a reentering space vehicle to minimize heating is described. Two cases are presented. The reentry vehicle problem is solved for one case and is also used to show that the technique for handling bounded control, discussed in Chapter 3, works well.

The conclusions and areas for further research are presented in Chapter 8.

Hereafter the method of quasilinearization will be abbreviated to the method of QL.

CHAPTER 2

THE METHOD OF QUASILINEARIZATION

2.1 The Elementary Technique

The method of QL is an iterative method for solving two point boundary value problems governed by a system of nonlinear ordinary first or second order differential equations. Here there are two quantities to be satisfied, the differential equations and the boundary values. In control problems there is a third quantity to be satisfied: the Maximum Principle. Iterative techniques have been built around satisfying one or two of these identically and then iterating to satisfy the third.

For example, the methods of the second variation used by Breakwell, Speyer and Bryson,[23] Kelley[24] and Scharmack[15, 16] satisfy the differential equations, the Maximum Principle and some of the boundary conditions exactly. The solution is then iterated until the remaining boundary conditions are satisfied. In the method of steepest descent the differential equations are satisfied exactly but neither the Maximum Principle nor the boundary conditions are satisfied. As the iterations proceed these are satisfied more and more completely.

In the method of quasilinearization the boundary conditions and the Maximum Principles are satisfied exactly. It must be kept in mind that the Maximum Principle is applied along a trajectory that is governed by the equations of quasilinearization, not by the state equations. The differential equations are then satisfied more and more nearly on each iteration. And if convergence occurs it is quadratic.

The computational procedure of QL can be described simply. A solution for an n dimensional system of first order differential equations is desired. Without any loss of generality it is assumed that half of the boundary conditions are specified at each end and that n is even

$$\begin{aligned} &\dot{y} = f(y,t) \\ &y_i(0) = y_{o_i} \ ; \ y_i(T) = y_{f_i} \ ; \ i = 1, 2, \ldots, \frac{n}{2} \end{aligned} \tag{2.1}$$

Where y is the n vector $\begin{Bmatrix} y_i \\ \vdots \\ y_n \end{Bmatrix}$, where y_{o_i} and y_{f_i}, $i = 1, \ldots, \frac{n}{2}$ are the initial and final boundary conditions, and where T, the final time is specified beforehand. It is further assumed, a priori, that a solution does exist.

To initialize the process of QL an approximate solution to the differential equations is chosen which satisfies the boundary conditions. A left superscript indicates the iteration number.

$$y^o(t) \quad 0 \leq t \leq T \tag{2.2}$$

Now for iteration number $k(k = 0, 1, \ldots, N)$ solve the following sets of equations. N is usually not specified before the process is started but is determined by the actual convergence of the solution.

Inhomogeneous Equation: $$\dot{\hat{y}}^{k+1} = f(\hat{y}^k, t) + \left(\frac{\partial f}{\partial y}\right)^k (\hat{y}^{k+1} - y^k); \ \hat{y}^{k+1}(0) = y^k(0) \tag{2.3}$$

Homogeneous Equation: $$\dot{Y}^{k+1} = \left(\frac{\partial f}{\partial y}\right)^k Y^{k+1}, \ Y^{k+1}(0) = I \tag{2.4}$$

where $\hat{y}$ is an n vector, and Y is an nxn matrix with elements Y_{ij}. Find a constant n dimensional vector α^{k+1}, such that the final boundary conditions are satisfied by $y^{k+1}(T)$ where

$$y^{k+1} = \hat{y}^{k+1} + Y^{k+1}\alpha^{k+1} \tag{2.5}$$

i.e.,

$$y_i^{k+1}(T) = y_{f_i}, \quad i = 1, 2, \ldots, \frac{n}{2} \tag{2.6}$$

Note that if we choose $\alpha_i^{k+1} = 0$, $i = 1, \ldots, \frac{n}{2}$ then by construction the initial boundary conditions will be satisfied by y:

$$y_i(0) = y_{o_i}, \quad i = 1, 2, \ldots, \frac{n}{2} \tag{2.7}$$

The vector α^{k+1} can be found by employing a matrix equation based on elements of the homogeneous solution, Y^{k+1} at the final time T, and the difference between the solution of the inhomogeneous equation, $\hat{y}^{k+1}$ and the desired solution, y_{f_i}, $i = 1, \ldots, \frac{n}{2}$:

$$\begin{pmatrix} y_{f_1} \\ \cdot \\ \cdot \\ \cdot \\ y_{f\frac{n}{2}} \end{pmatrix} = \left.\begin{pmatrix} \hat{y}_1(T) \\ \cdot \\ \cdot \\ \cdot \\ \hat{y}_{\frac{n}{2}}(T) \end{pmatrix}\right|^{k+1} + \left.\begin{pmatrix} Y_{1,\frac{n}{2}+1} & \ldots & Y_{1n} \\ & \ldots & \\ Y_{\frac{n}{2},\frac{n}{2}+1} & \ldots & Y_{\frac{n}{2},n} \end{pmatrix}\right|^{k+1} \left.\begin{pmatrix} \alpha_{\frac{n}{2}+1} \\ \ldots \\ \alpha_n \end{pmatrix}\right|^{k+1} \tag{2.8}$$

This matrix, Y, evaluated at the final time is called the transition matrix and it relates small perturbations of the initial conditions to small perturbations of the final state.

If the elements of Y^{k+1} are independent then a unique value for the vector constant α^{k+1} can be found.

The columns of Y from 1 to $\frac{n}{2}$ are not used and they need not be actually computed in a computer program. These columns are not needed because no perturbations are allowed on the variables 1 to $\frac{n}{2}$, as these variables are specified by the initial conditions.

Since the complete solution of $y(t)$ is required for the next iteration either the following computation at every step of the integration can be performed,

$$y^{k+1}(t) = \hat{y}^{k+1}(t) + Y^{k+1}(t)\,\alpha^{k+1} \tag{2.9}$$

or, preferably, the system equations can be reintegrated using $y^{k+1}(0)$ as the initial condition. By reintegrating the system of differential equations and comparing the desired final conditions with the actual final conditions a check is made on the complete computational process.

The foregoing procedure is repeated until satisfactory convergence is obtained. Since it is difficult to discover the actual accuracy of the solution from the equations of QL, the following method of verification can be used.

Once the method of QL has converged the solution may be verified by integrating the following system.

$$\begin{aligned} \dot{y}^{N+1} &= f(y^N, t) \quad 0 \leq t \leq T \\ y^{N+1}(0) &= y^N(0) \end{aligned} \tag{2.10}$$

Compare $y^{N+1}(T)$ with y_{f_i}, $i = 1, \ldots, \frac{n}{2}$. If they do not differ by an amount greater than that which might be expected from an examination of integration procedure, then the solution y^N may be taken as exact. If not, then the entire computational process must be examined for errors.

2.2 The Technique With Undetermined Constants

A system of unknown constants c, where c is a p dimensional vector, can be found using the regular QL procedure with only slight modifications to it. A solution to the following system is desired:

$$\begin{aligned} &\dot{y} = f(y, c, t) \quad 0 \leq t \leq T \\ &y_i(0) = y_{o_i}, \; y_i(T) = y_{f_i}, \; i = 1, \ldots, \frac{n}{2} \end{aligned} \tag{2.11}$$

Also

$$q_i(y, c, 0) = 0, \quad i = 1, \ldots, p \tag{2.12}$$

The q's represent the p additional constraints needed to determine the c's, which for the sake of notational simplicity are assumed to all be defined at time 0.

Choose an initial trajectory y^o satisfying the boundary conditions, and an initial estimate of the constants, c^o. To find the solution iterate the following system of equations:

$$\begin{aligned} &\dot{\hat{y}}^{k+1} = f\Big|^k + \left(\frac{\partial f}{\partial y}\right)\Big|^k \left(\hat{y}^{k+1} - y^k\right), \; \hat{y}^{k+1}(0) = y^k(0) \\ &\dot{Y}^{k+1} = \left(\frac{\partial f}{\partial y}\right)\Big|^k Y^{k+1}, \; Y^{k+1}(0) = I \\ &\dot{Z}^{k+1} = \left(\frac{\partial f}{\partial y}\right)\Big|^k Z + \left(\frac{\partial f}{\partial c}\right)\Big|^k, \; Z(0) = O \end{aligned} \tag{2.13}$$

Where Y^{k+1} is an nxn matrix, Z is a nxp matrix, $\left(\frac{\partial f}{\partial y}\right)\Big|^k$ is an nxn matrix, and $\left(\frac{\partial f}{\partial c}\right)\Big|^k$ is an nxp matrix.

Constant vectors α^{k+1} (m dimensional) and β^{k+1} (p dimensional) are now determined such that

$$
\begin{aligned}
y^{k+1} &= \hat{y}^{k+1} + Y^{k+1}\alpha^{k+1} + Z^{k+1}\beta^{k+1} \\
c^{k+1} &= c^{k} + \beta^{k+1}
\end{aligned}
\tag{2.14}
$$

satisfies the initial and final conditions, and linear approximations to the q's.

This y^{k+1} satisfies the modified QL equation:

$$
\dot{y}^{k+1} = f\Big|^{k} + \left(\frac{\partial f}{\partial y}\right)\Big|^{k}\left(y^{k+1} - y^{k}\right) + \left(\frac{\partial f}{\partial c}\right)\Big|^{k}\left(c^{k+1} - c^{k}\right)
$$

$$
y_i(0) = y_{o_i}, \quad y_i(T) = y_{f_i}, \quad i = 1, \ldots, \frac{n}{2} \tag{2.15}
$$

and

$$
q\Big|^{k} + \left(\frac{\partial q}{\partial y}\right)\Big|^{k}\left(Y^{k+1}(0)\,\alpha^{k}\right) + \left(\frac{\partial q}{\partial c}\right)\Big|^{k}\beta^{k+1} = 0
$$

The constant vectors α^{k+1} and β^{k+1} may be found by inverting the following system of equations. Again α^{k+1} is constructed such that $\alpha_i^{k+1} = 0$, $i = 1, \ldots, \frac{n}{2}$.

$$
\begin{pmatrix} y_{f1} \\ \vdots \\ y_{f\frac{n}{2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}
=
\begin{pmatrix} \hat{y}_1(T) \\ \vdots \\ \hat{y}_{\frac{n}{2}}(T) \\ q_1 \\ \vdots \\ q_p \end{pmatrix}\Bigg|^{k+1}
+
\begin{pmatrix}
Y_{1\frac{n}{2}+1}, \ldots, Y_{1n} & Z_{11} & & Z_{1p} \\
\vdots & & & \\
Y_{\frac{n}{2}\frac{n}{2}+1}, \ldots, Y_{\frac{n}{2}n} & Z_{\frac{n}{2}} & & Z_{\frac{n}{2}p} \\
\dfrac{\partial q_1}{\partial y_{\frac{n}{2}+1}}, \ldots, \dfrac{\partial q_1}{\partial y_n} & \dfrac{\partial q_1}{\partial c_1}, & \ldots, & \dfrac{\partial q_1}{\partial c_p} \\
\vdots & & & \\
\dfrac{\partial q_p}{\partial y_{\frac{n}{2}+1}}, \ldots, \dfrac{\partial q_p}{\partial y_n} & \dfrac{\partial q_p}{\partial c_1} & & \dfrac{\partial q_1}{\partial c_p}
\end{pmatrix}\Bigg|^{k}
\begin{pmatrix} \alpha_{\frac{n}{2}+1} \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}\Bigg|^{k}
\tag{2.16}
$$

The existence of an inverse depends on the independence of the Y's, the Z's and the partials of q with respect to β. Here the independence is assumed a priori.

When the vectors α^{k+1} and β^{k+1} have been found and y^{k+1} generated in accordance with the previous formula by either of the two methods suggested in the discussion of the regular QL procedure, it will be found that y^{k+1} satisfies the specified boundary conditions and more nearly satisfies $q = 0$.

2.3 The Problem of Undetermined Final Time

In many engineering problems the final time is left unspecified and it may be a quantity that the optimization procedure is required to find.

With some methods there is no problem in determining the final time since it falls out naturally through some stopping condition on the integration of the equations of motion. In QL on the other hand, data from prior trajectories is required and either the integration must take place over a fixed interval or some method of approximating the required data must be found.

The solution of the two point boundary value problem is complicated by the need for the derivative f and $\partial f/\partial y$ to be defined at all points along the trajectory in the case where the final time is not specified beforehand.

McGill and Kenneth[6] solved this difficulty by treating the free final time problem as a series of fixed time problems. Their method is a procedure for iterating on both the solution of the differential equation and on satisfying a boundary condition to determine the correct final time.

Since the method involves fixing the final time one of the other boundary conditions at the final time must be temporarily relaxed. If this were not done then the system would be overdetermined.

To start the process a final time is guessed and the solution is iterated until the norm of the error satisfied some tolerance. Then perturbing this guessed final time the solution is again iterated until the tolerance is satisfied. At this point a recursion formula is employed to find a new final time, t_f^{k+1}:

$$t_f^{k+1} = t_f^k + \frac{t_f^k - t_f^{k-1}}{y_f^k - y_f^{k-1}}\left((y_\ell^k - y_{f_\ell}\right) \tag{2.17}$$

the solution is now iterated with the new final time until the tolerance is satisfied. The recursion formula on final time is then employed. These last two steps are repeated until the change in final time is less than some other tolerance and $y_\ell^k\left(t_f\right) \approx y_{f_\ell}$ at which point convergence is assumed. Here y_ℓ is the variable whose terminal value y_{f_ℓ} has been relaxed.

Recently Long[9, 10] described a technique which allows the free final time problem to be handled directly by the method of QL. The technique is a specialized case of the method stated in Section 2.2 for handling undetermined constants. It is this method which has been used here to solve free final time problems unless otherwise noted.

The method for handling a free final time problem consists of changing the problem independent variable from time t to a dummy variable τ, which is a scalar multiple of t. Thus we have the equations

$$\begin{aligned} \dot{y} &= f(y, t) \\ \frac{dt}{d\tau} &= c_t \end{aligned} \tag{2.18}$$

where c_t relates t to τ.

The variable τ is now taken as the independent variable and the integration is performed with respect to it.

$$\frac{dy}{d\tau} = c_t f(y, t)$$
$$\frac{dt}{d\tau} = c_t \tag{2.19}$$

The range of τ is fixed: $0 \leq \tau \leq 1$. With τ fixed the integration is simplified considerably. The constant c_t is now included in the usual manner in the QL framework:

Equations to be solved:

$$\dot{y} = f(y, t), \; y(0) = y_o, y(\tau) = y_f \tag{2.20}$$

Let y^k be the k^{th} approximation to the solution. Note that y^k is constructed to satisfy the boundary conditions.

Three integrations are performed, $0 < t \leq T$

$$\dot{\hat{y}} = c_t^k f(y^k, \tau) + c_t^k \left(\frac{\partial f}{\partial y}\right)\Big|_n \left(\hat{y}^{k+1} - y^k\right); \; \hat{y}^{k+1}(0) = y^k(0)$$

$$\dot{Y}^{k+1} = c_t^k \left(\frac{\partial f}{\partial y}\right)\Big|^k Y^{k+1} \; ; \; Y^{k+1}(0) = I \tag{2.21}$$

$$Z^{k+1} = f(y^k, \tau) + c_t^k \left(\frac{\partial f}{\partial y}\right)\Big|^k Z^k \; ; \; Z^k(0) = 0$$

A vector α and a constant β are now chosen so that

$$y^{k+1} = \hat{y}^{k+1} + Y^{k+1} \alpha^T + \beta Z^{k+1} \tag{2.22}$$

satisfy

$$y^{k+1}(0) \overset{*}{=} y_o, \; y^{k+1}(T) \overset{*}{=} y_f \tag{2.23}$$

($\overset{*}{=}$ denotes equality only on the components of the right-hand vector which are specified

y^{k+1} is seen to satisfy

$$\dot{y}^{k+1} = c_t^{k+1} f(y^k, \tau) + c_t^{k+1} \left(\frac{\partial f}{\partial y}\right)\Big|^{k} (y^{k+1} - y^k)$$

with
$$c_t^{k+1} = c_t^k + \beta \;;\; y^{k+1}(0) = \hat{y}^{k+1}(0) + \alpha \qquad (2.24)$$

The method can be modified by making the time factor variable as has been suggested by Johnson.[29] In Chapter 6 this is referred to as the modified integration method.

$$\frac{dt}{d\tau} = \frac{c_t}{(1 + a\,\dot{y}^T\dot{y})^{\frac{1}{2}}} \qquad (2.25)$$

This variable factor reduces $\frac{dt}{d\tau}$ when $\dot{y}$ is large which is equivalent to decreasing the integration step size when the functions are varying rapidly and increase it when the functions are varying slowly.

The constant a must be chosen by experience.

For more generality it may be desirable to use

$$\frac{dt}{d\tau} = \frac{c_t}{(1 + \dot{y}^T A\dot{y})^{\frac{1}{2}}} \qquad (2.26)$$

where A is a diagonal matrix. The elements of the diagonal may then be chosen to weight the effect of the various state derivatives.

The additional terms generated by the variable time factor while complicated to write out, are simple for a computer to generate since they consist only of products of terms that are required in any case.

Equations: $\dot{y} = f(y, t)$

$$\frac{dt}{d\tau} = c_t\left(1 + a(\dot{y}^T\dot{y})\right)^{\frac{1}{2}} \qquad (2.27)$$

Integration with respect to τ

$$\frac{dy}{d\tau} = c_t \left(1 + a(\dot{y}^T \dot{y})\right)^{\frac{1}{2}} f(y,t)$$

$$\frac{dt}{d\tau} = c_t \left(1 + a(\dot{y}^T \dot{y})\right)^{\frac{1}{2}} \tag{2.28}$$

Equations to be solved by QL

$$\frac{dy^{k+1}}{d\tau} = c_t^k \left(1 + a(\dot{y}^{kT} \dot{y}^k)\right)^{-\frac{1}{2}} f(y,t) +$$

$$c_t^k \left[\left(1 + a(y^{kT} \cdot y^k)\right)^{-\frac{1}{2}} \frac{\partial f}{\partial y}^k + a\, \dot{y}^k \left(\frac{\partial \dot{y}^k}{\partial y}\right)^T (1 + a\, \dot{y}^{kT} \cdot \dot{y}^k)^{-\frac{3}{2}} f^k \right]$$

$$\cdot (y^{k+1} - y^k) \tag{2.29}$$

The method used in solving the free final time problem is much more generally applicable than might at first be thought. Often a control problem has a discontinuity, and a very fine integration mesh must be employed to maintain integration accuracy. In many cases it may be much more expedient to use different time constants in different parts of the integration to guarantee that the discontinuities fall at an integration step where they may be handled conveniently.

This technique is used by Long[9] with the restriction that the number of breakpoints is known before hand. The technique becomes considerably more powerful if that restriction is avoided by letting the computer determine the number of integration intervals and their associated time constants.

CHAPTER 3

A QUADRATIC CONVERGENCE PROOF

3.1 The Contraction Mapping Principle

A proof of the convergence of the method of quasilinearization can be shown for second order differential equations in general. The proof of convergence is shown here using the contraction mapping principle of functional analysis (see Kolmogorov and Fomin, for example).[14]

Definition: Contraction Mapping

Consider a complete metrix space S, an operator A, and a metric ρ. The mapping A is said to be a contraction mapping if:

1) if $y \in R \Rightarrow Ay \in S$
2) if $y_1, y_2 \in S$

then

$$\rho(Ay_1, Ay_2) = \alpha\, \rho(y_1, y_2) \text{ and } \alpha < 1 \tag{3.1}$$

Theorem (The Contraction Mapping Principle)

"Every contraction mapping defined in a complete metric space S has one and only one fixed point, i.e., the equation $Ay = y$ has one and only one fixed point."*

3.2 The Proof of Quadratic Convergence

The proof given here for the quadratic convergence of the method of QL follows McGill and Kenneth[4] with the addition of one term. The importance of the extra term becomes apparent when practical control problems are to be solved.

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*Reference 14 page 43.

It is this additional term which allows the control to be handled directly and the imposition of constraints on the control without the addition of multipliers or the use of a penalty function.

In the proof a method for finding the solution to a system of second order, ordinary nonlinear differential equations will be shown. The proof is for a fixed end time problem.

$$\begin{array}{ll} \text{System:} & y'' = f(y,t,u), \ 0 < t < T \\ \text{Boundary conditions:} & y(0) = y_0, \ y(T) = y_f \\ \text{Dimensionality:} & y \text{ and } f \text{ are } n \text{ dimensional} \\ & u \text{ is } m \text{ dimensional} \end{array} \tag{3.2}$$

It is assumed here that u is of the form:

$$u = g(y, t)$$

Or that, if $g_1(u, y, t) = 0$ must be solved for u, then that g_1 has continuous second partial derivatives.

Employing vector notation consider the following system whose solution is to be found.

$$\begin{aligned} y'' &= f(y,t,u) \quad 0 \le t \le T \\ u &= g(y,t) \\ y(0) &= y_o, \ y(T) = y_f \end{aligned} \tag{3.3}$$

Now consider the sequence of solutions to the system

$$\begin{aligned} y''^{k+1} &= J(y^k,t)\,(y^{k+1} - y^k) + f(y^k,t) \quad k = 0, 1, \dots, \\ y^k(0) &= y_o, \ y^k(T) = y_f \quad \text{all } k \end{aligned} \tag{3.4}$$

where

$$J(y^k, t) = \left(\frac{\partial f_i}{\partial y_j} + \sum_{\ell=1}^{m} \frac{\partial f_i}{\partial g_\ell} \frac{\partial g_\ell}{\partial y_j} \right) \Bigg|_{y=y^k} = (J_{ij}) \tag{3.5}$$

and

$$\max_{t,i} \left| y_i^o(t) - Y_{T_i}(t) \right| \leq k \quad 0 \leq t \leq T \tag{3.6}$$

with

$$\max_{i,t} \left| y_i - y_{T_i} \right| \leq k \quad 0 \leq t \leq T \tag{3.7}$$

and

$$y_{T_i} = \left(y_{f_i} - y_{o_i} \right) \frac{t}{T} - y_{f_i} \tag{3.8}$$

Now provided that

a) f is continuous

b) $\frac{\partial f}{\partial y}$ exists and is continuous

c) $\frac{\partial f}{\partial g}$ exists and is continuous

d) $\frac{\partial g}{\partial y}$ exists and is continuous

e) f and $\frac{\partial f}{\partial y} + \sum_{\ell=1}^{m} \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial y}$ are Lipschitzian with respect to y

Then

a) A unique solution to (3.3) exists.

b) The series (3.4) converges to it.

c) The error bound is given by

$$\rho(y^{k+1}, y) \leq k_2 \, \rho^2(y^{k+1}, y^k) \text{ where } k_2 \text{ is defined later and}$$

$$0 \leq k_2 \leq 1 \tag{3.9}$$

Note that assumption e) is quite restrictive compared to the assumption Kenneth and McGill[4] make at this point. However, it must be kept in mind that the addition of this term allows certain computational short cuts and that the primary goal of this paper is to provide computational results.

Define the constants P_i, R_{ij}, Q_i, and M_{ij}:

$$\max_{y,t} \left| f_i \quad y, t, g(y,t) \right| \leq P_i \tag{3.10}$$

$$\max_{y,t} \left| \frac{\partial f_i}{\partial y_j} + \sum_{\ell=1}^{m} \frac{\partial f_i}{\partial g_\ell} \cdot \frac{\partial g_\ell}{\partial y_i} \right| \leq R_{ij} \tag{3.11}$$

$$\left| F_i(y,t) - F_i(y_2,t) \right| \leq Q_i \sum_{j=1}^{n} \left| y_{ij} - y_{2j} \right| \tag{3.12}$$

$$\left| \left(\frac{\partial f_i}{\partial y_j} + \sum_{\ell=1}^{m} \frac{\partial f_i}{\partial g_\ell} \cdot \frac{\partial g_\ell}{\partial y_j} \right) \right|_{y=y_1} - \left(\frac{\partial f_i}{\partial y_j} + \sum_{\ell=1}^{m} \frac{\partial f_i}{\partial g_\ell} \cdot \frac{\partial g_\ell}{\partial y_j} \right) \Bigg|_{y=y_2} \Bigg|$$

$$\leq M_{ij} \left(\sum_{k=1}^{n} \left| y_{1k} - y_{2k} \right| \right) \tag{3.13}$$

Define

$$m = \max_{i,j} \left(P_i, R_{ij}, Q_i, M_{ij} \right) \tag{3.14}$$

A metric space S is defined as

$$S \cdot \{ y(t) \mid \quad \text{a) } y_i(t) \text{ are continuous} \}$$

$$\text{b) } y(0) = y_o, \ y(T) = y_f$$

$$\text{c) } \max_{i,t} \left| y_i(t) - y_{T_i}(t) \right| < K$$

with a metric

$$\rho(y_1, y_2) = \sum_{i=1}^{n} \max_{t} \left| y_{1j}(t) - y_{2i}(t) \right| \tag{3.15}$$

Use the symbol A to denote an operator performing a single quasilinearization iteration on some $y \epsilon S$.

To show that the contraction mapping principle applies it is necessary to show that the operator A has the following properties.

a) if $y \epsilon S$ then $Ay \epsilon S$

b) if $y_1 \epsilon S$ and $y_2 \epsilon S$ then

$$\rho(Ay_1, Ay_2) \leq \alpha \rho(y_1, y_2), \quad 0 \leq \alpha \leq 1$$

First show that if $y \epsilon S$ then $Ay \epsilon S$.

By employing the Green's function, $G(t, s)$, the solution of (3.4) can be written

$$Ay = y_T - \int_0^T G(t,s)\{J(y,s)[Ay(s) - y(s)] + f(y,s)\}ds \tag{3.16}$$

$$G(t,s) = \begin{cases} \frac{s-T}{T}\, t & \text{for } t \leq s \\ \frac{s}{T}(t-T) & \text{for } t > s \end{cases} \tag{3.17}$$

Note

$$|G(t,s)| \leq \frac{T}{4}$$

Now

$$Ay - y_T = \int_0^T G(t,s)\{J(y,s)[Ay(s) - y(s)] + f(y,s)\}ds \tag{3.18}$$

or

$$\rho(Ay, y_T) \leq \sum_{i=1}^{n} \max_i \left| \int_0^T G(t,s) \left\{ \sum_{j=1}^{n} J_{ij}(y,s)[Ay_j(s) - y_j(s)] + f_i(y,s) \right\} ds \right.$$

$$\leq \sum_{i=1}^{n} \max_i \left| \left(\frac{T}{4}\right) \{ \max_j R_{ij}\, \rho(Ay, y) + P_i \} T \right|$$

$$\leq n\, m\, \frac{T^2}{4} \left(\rho(Ay, y) + \right)$$

$$\leq n\, m\, \frac{T^2}{4} \left(\rho(Ay, y_T) + \rho(y_T, y) + 1 \right) \tag{3.19}$$

So

$$\rho(Ay, y_T) \leq \frac{n\, m \frac{T^2}{4} (nK+1)}{1 - n\, m \frac{T^2}{4}} \quad \rho(y_T, y) \tag{3.20}$$

Now for T sufficiently small

$$\max_{i,t} \left| Ay_i, y_{T_i} \right| \leq K \tag{3.21}$$

Hence $Ay \epsilon S$.

Second, it must be shown that

$$\rho(Ay_1, Ay_2) \leq \alpha \rho(y_1, y_2), \quad 0 \leq \alpha < 1 \tag{3.22}$$

Consider

$$Ay_1 - Ay_2 = \int_o^T G(t,s) \{ J(y_2, s)(Ay_2 - y_2) - J(y,s)(Ay_1 - y_1)$$

$$+ f(y_1, s) - f(y_2, s) \} \, ds \tag{3.23}$$

$$= \int_o^T G(t,s) \{ J(y_1, s)[Ay_1 - Ay_2] + J(y_2, \hat{s})[y_1 - y_2]$$

$$+ [J(y_1, s) - J(y_2, \hat{s})](y_1 - Ay_2) + f(y_1, s) - f(y_2, s) \} \, ds$$

So

$$\rho(Ay_1, Ay_2) = \sum_{i=1}^{n} \max_t \left| Ay_{1i} - Ay_{2i} \right| \tag{3.24}$$

$$\leq n\, m \frac{T^2}{4} [\rho(Ay_1, Ay_2) + \rho(y_1, y_2) + \rho(y_1, y_2)\rho(y_1, Ay_2) + \rho(y_1, y_2)]$$

$$\leq n\, m \frac{T^2}{4} \left[\rho(Ay_1, Ay_2) + \rho(y_1, y_2)\left(2 + \rho(y_1, y_T) + \rho(y_T, Ay_1)\right)\right]$$

$$\leq n\, m \frac{T^2}{4} [\rho(Ay_1, Ay_2) + \rho(y_1, y_2)(2 + 2\, nK)] \tag{3.25}$$

or

or

$$\rho(Ay_1, Ay_2) \leq \frac{nm\frac{T^2}{2}(nK+1)}{1 - nm\frac{T^2}{4}} \rho(y_1, y_2) \tag{3.26}$$

Thus if T is restricted still further then

$$\rho(Ay_1, Ay_2) \leq \alpha\rho(y_1, y_2), \quad 0 \leq \alpha < 1 \tag{3.27}$$

and the operator A satisfies both conditions to be a contraction mapping.

Therefore the sequence

$$y^{k+1} = Ay^k \tag{3.28}$$

has a unique solution.

To prove quadratic convergence examine:

$$y^{k+1} - y = \int_0^T G(t,s)\left\{J(y^k,s)\left[y^{k+1}(s) - y^k(s)\right] + \left[f(y^k,s) - f(y,s)\right]\right\} ds \tag{3.29}$$

This equation arises from taking the difference between

$$\ddot{y}^{k+1} = f\Big|_{y=y^k} + J\Big|_{y=y^k} (y^{k+1} - y^k) \tag{3.30}$$

and

$$\ddot{y} = f\Big|_{y=y} \tag{3.31}$$

Apply the mean value theorem to the last term on the right:

$$f_j(y^k,s) - f_j(y,s) = \left(\frac{\partial f_j}{\partial y} + \delta\frac{\partial f_j}{\partial g_i}\cdot\frac{\partial g_i}{\partial y}\right)\Big|_{y=\bar{y}^k} (y^k - y) \tag{3.32}$$

where ${}^{k-}y$ may differ for each f_j, and y^k. So

$$\rho(y^{k+1}, y) \le m \frac{T^2}{4} \Big[\rho(y^{k+1}, y^n)\{\rho(y^k, {}^1y) + \ldots + \rho(y^k, {}^{n-}y)\}$$

$$+ n\, \rho(y^{k+1}, y)\Big] \tag{3.33}$$

Now $\rho(y^k, {}^jy) \le \rho(y^k, y)$ so that

$$\rho(y^{k+1}, y) \le n\, m \frac{T^2}{4} \left[\rho(y^{k+1}, y^k)\, \rho(y^k, y) + \rho(y^{k+1}, y)\right]$$

But

$$\rho(y^k, y) \le \rho(y^{k+1}, y) + \rho(y^k, y) \le 2\rho(y^k, y)$$

so

$$\rho(y^{k+1}, y) \le n\, m \frac{T^2}{4} \left[2\rho^2(y^k, y) + \rho(y^{k+1}, y)\right]$$

or

$$\rho(y^{k+1}, y) \le \frac{n\, m \frac{T^2}{2}}{1 - n\, m \frac{T^2}{4}} \rho^2(y^k, y) \tag{3.34}$$

Defining

$$k_2 \equiv \frac{n\, m \frac{T^2}{2}}{1 - n\, m \frac{T^2}{4}} \tag{3.35}$$

Then

$$\rho(y^{k+1}, y) \le k_2\, \rho^2(y^k, y) \tag{3.36}$$

and the quadratic convergence is immediately apparent.

A more useful form can be found using

$$\rho(y^k, y) \le \frac{1}{1-k_2} \rho(y^{k+1}, y^k) \tag{3.37}$$

This comes from

$$\begin{aligned}\rho(y^k, y) &\leq \rho(y^k, y^{k+1}) + \rho(y^{k+1}, y^{k+2}) + \ldots + \\ &\leq \rho(y^k, y^{k+1})[1 + \alpha + \alpha^2 + \ldots] \\ &\leq \rho(y^k, y^{k+1}) \frac{1}{1-\alpha} \text{ as } 0 \leq \alpha \leq 1 \end{aligned} \tag{3.38}$$

and as $\alpha = k_2$ from (3.27).

$$\rho(y^{k+1}, y) \leq \frac{k_2}{(1-k_2)^2} \rho^2(y^{k+1}, y^k) \tag{3.39}$$

CHAPTER 4

THE CHOICE OF AN INITIAL TRAJECTORY FOR THE METHOD OF QUASILINEARIZATION

4.1 The Natural Region of Convergence

The choice of an initial trajectory for the method of QL is often difficult. Usually part of the trajectory is known quite well: the physical variables are constrained and a priori it may be known, for instance, that they follow a roughly elliptical path as in the case of a reentering space vehicle. The remainder of the trajectory is usually completely unknown unless a similar problem has been worked before. There are few guides to use in approximating the adjoint variables. Consequently the range of convergence of the technique is of great practical importance.

Three different methods were examined to see how much they could extend the range of convergence. All the methods were compared with the usual method of QL. The computational results are given in detail in Chapter 6, where the Brochistochrone problem is discussed.

4.2 Method 1

Using a technique employed by Breakwell, Speyer and Bryson, the corrections to the initial conditions (the corrections which insure that the boundary conditions at both ends are satisfied) were multiplied by some fraction before use:

$$y^{k+1}(o) = y^k(o) + \epsilon\,\alpha^k \quad , \quad 0 < \epsilon \leq 1 \ , \quad k=0,1,\ldots,P$$

$$y^{k+1}(o) = y^k(o) + \alpha^k \quad , \quad k = P+1,\ldots,N \qquad (4.1)$$

where α^k is the correction vector. Needless to say, the use of this multiplier destroys the quadratic convergence of the method of QL (if the multiplier is not equal to one). The constant P, above, is either fixed as some small number less than the total number of iterations expected, or is changed by the computer program depending on the change in the trajectory between two successive iterations. Here no attempt was made to optimize the choice of P and only the former of the two choices was tried.

4.3 Method 2

The second method tested is a variation of the first. Here

$$\begin{aligned} & y^{k+1}(o) = y^{k}(o) + \alpha^{k}_{\epsilon} \\ & 0 \le \epsilon \le 1 \quad \text{if} \quad k = 0, \ldots, P \\ & \quad\;\; \epsilon = 1 \quad \text{if} \quad k = P+1, \ldots, N \end{aligned} \tag{4.2}$$

as in the usual QL technique. However, the α^{k}_{ϵ} is found in this case from:

$$\epsilon \left| \begin{pmatrix} y_{f_1} \\ \cdot \\ \cdot \\ \cdot \\ y_{f_{\frac{n}{2}}} \end{pmatrix} - \begin{pmatrix} y_1^{k+1}(T) \\ \\ \\ \\ y_{\frac{n}{2}}^{k+1}(T) \end{pmatrix} \right| = \left| \begin{pmatrix} Y_{1,\frac{n}{2}+1} & \ldots & Y_{1n} \\ \\ \\ Y_{\frac{n}{2},\frac{n}{2}+1} & \ldots & Y_{\frac{n}{2},n} \end{pmatrix} \right|^{k+1} \begin{pmatrix} \alpha^{k+1}_{\epsilon_{n/2+1}} \\ \\ \\ \alpha^{k+1}_{\epsilon_n} \end{pmatrix} \tag{4.3}$$

and $\alpha^{k+1}_{\epsilon_1} = \ldots = \alpha^{k+1}_{\epsilon_{n/2}} = 0$.

4.4 Method 3

The third method of increasing the range of convergence is similar to the one presented above but retains more of the elements of QL. Again a constant ϵ was used to slow down the speed of convergence. Here also if $\epsilon \neq 1$ quadratic convergence is lost;

$$y^{k+1} = \epsilon\, \tilde{y}^{k+1} + (1-\epsilon) y^{k}, \quad 0 < \epsilon \le 1\ ; \quad k = 1, \ldots, P$$
$$y^{k+1} = \tilde{y}^{k+1} \qquad k = P+1, \ldots, N \tag{4.4}$$

where

y^{k} is the k trajectory

$\tilde{y}^{k+1}$ is the k+1st trajectory as computed in the usual manner of QL

y^{k+1} is the k+1st trajectory stored and used for computing the matrix of partial derivatives.

It can be shown theoretically that this method extends the range of convergence. Using the notation of Chapter 3 and following that proof as a guide consider the one dimensional case. The m-dimensional case follows directly.

First the theorem from Chapter 3 is simplified without proof to the case where y is a one dimensional vector. Then the effect of Equation (4.4) is shown in terms of this theorem.

Problem

A solution to the following second order differential equation is desired:

$$\begin{aligned} y'' &= f(y,t,u) && 0 \le t \le T \\ u &= g(y,t) && \\ y(o) &= y_o \quad , && y(t) = y_f \end{aligned} \tag{4.5}$$

For simplicity assume that g is an explicit function of y and t. Then

$$\begin{aligned} y'' &= f(y,t) && 0 \le t \le T \\ y(o) &= y_o \quad , && y(T) = y_f \end{aligned} \tag{4.6}$$

Theorem 4.1

Now consider the sequence of solutions to the system:

$$y''^{k+1} = \frac{\partial f}{\partial y}\Big|^{k} (y^{k+1} - y^{k}) + f^{k} \qquad , \qquad k = 0, 1, \ldots, N$$

$$y^{k}(o) = y_o \quad , \quad y^{k}(T) = y_f \qquad \text{all } k \tag{4.7}$$

where $|^k$ indicates that the quantity immediately to its left is evaluated on the k^{th} trajectory, and

$$\max_t \left| y^{o}(t) - y_T(t) \right| \leq K \qquad 0 \leq t \leq T \tag{4.8}$$

with

$$y_T = (y_f - y_o)\frac{t}{T} - y_f \tag{4.9}$$

Now provided that:

a) f is continuous

b) $\frac{\partial f}{\partial y}$ exists and is continuous

c) f and $\frac{\partial f}{\partial y}$ are Lipschitzian with respect to y

Then

a) A unique solution to (4.6) exists

b) The series (4.7) converges to it

c) The bound necessary for a contraction mapping is given by k_2:

$$0 \leq k_2 < 1$$

$$k_2 = \frac{m\,T^2/4}{1 - m\,T^2/4}\,(2 + 2K) \tag{4.10}$$

where m is defined in Theorem 4.2. Now the theorem from Chapter 3 is modified to cover the new algorithm generated by Equation (4.4).

Theorem 4.2

The system for which a solution is desired is still described by (4.6). The algorithm of (4.7) is modified to give:

$$\tilde{y}''^{k+1} = \frac{\partial f}{\partial y}^{k} (\tilde{y}^{k+1} - y^{k}) + f^{k} \qquad k = 0, 1, \ldots, N$$

$$y^{k}(o) = y_o \quad y^{k}(T) = y_f \qquad \text{all } k \tag{4.11}$$

$$y^{k+1} = \epsilon\, y^{k} + (1-\epsilon)\, \tilde{y}^{k+1} \qquad \text{all } k,\ 0 \leq \epsilon < 1$$

and

$$\max \left| y^{o}(t) - y(t) \right| \leq K \qquad 0 \leq t \leq T \tag{4.12}$$

and

$$y_T = (y_f - y_o)\frac{t}{T} - y_f \tag{4.13}$$

Now provided that

a) f is continuous

b) $\frac{\partial f}{\partial y}$ exists and is continuous

c) f and $\frac{\partial f}{\partial y}$ are Lipschitzian with respect to y

Then:

a) A unique solution to (4.6) exists

b) The series (4.11) converges to it

c) The bound necessary for a contraction mapping is given by k_3:

$$0 \leq k_3 < 1$$

$$k_3 = \frac{m T^2/4}{1 - m T^2/4} \left(1 + \epsilon\,(2K+1) - \frac{mT^2}{4}(1-\epsilon) \right) \tag{4.14}$$

Proof

Define

$$\max_{y,t} |f| < P$$

$$\max_{y,t} \left|\frac{\partial f}{\partial y}\right| < R \tag{4.15}$$

$$|f(y_1, t) - f(y_2, t)| < Q |y_1 - y_2|$$

$$\left| \frac{\partial f}{\partial y}\Big|_{y=y_1} - \frac{\partial f}{\partial y}\Big|_{y=y_2} \right| < M |y_1 - y_2|$$

$$m = \max (P, R, Q, M) \tag{4.16}$$

A metric space S is defined

$$S = \{y(t) | \quad \text{a) } y \text{ is continuous}\}$$
$$\text{b) } y(o) = y_o, \; y(T) = y_f$$
$$\text{c) } \max_t |y - y_T| < K$$

with a metric

$$\rho(y_1, y_2) = \max_t |y_1 - y_2| \tag{4.17}$$

The symbol A is used to denote the usual quasilinearization operation of Theorem 4.1.

The symbol B is used to denote the operator in (4.11).

To show that the contraction mapping principle applies it is necessary to show that the operator B has the following properties

a) if $y \epsilon S$ then $B\, y \epsilon S$

b) if $y_1 \epsilon S$ and $y_2 \epsilon S$ then

$$\rho(By_1, By_2) \leq \alpha \rho(y_1, y_2) \qquad 0 \leq \alpha < 1 \tag{4.18}$$

First show that if $y \epsilon S$ that $By \epsilon S$

$$By = (1-\epsilon)\, y + \epsilon\, Ay \quad , \qquad 0 \leq \epsilon < 1 \tag{4.19}$$

Since $y \epsilon S$ then $Ay \epsilon S$ and so directly $By \epsilon S$ by convexity. Thus the contraction mapping principle may be applied. Second, show that

$$\rho(By_1, By_2) \leq \alpha\, \rho(y_1, y_2) \; , \qquad 0 \leq \alpha < 1 \tag{4.20}$$

Consider

$$By_1 - By_2 = \int_o^T G(t,s) \left\{ f_y\Big|_{y_2} (By_2 - y_2) - f_y\Big|_{y_1} (By_1 - y_1) - \left(f\Big|_{y_1} - f\Big|_{y_2} \right) \right\} ds \tag{4.21}$$

where G is the Green's function

$$G(t,s) = \begin{cases} \dfrac{s-T}{T}\, t & \text{for } t \leq s \\ \dfrac{s}{T}(t-T) & \text{for } t > s \end{cases} \tag{4.22}$$

and

$$|G(t,s)| \leq \frac{T}{4} \tag{4.23}$$

So employing the definition of B:

$$By_1 - By_2 = \int_o^T G(t,s) \left\{ f_y\Big|_{y_2} (\epsilon\, Ay_2 - \epsilon\, y_2) - f_y\Big|_{y_1} (\epsilon\, Ay_1 - \epsilon\, y_1) \left(f\Big|_{y_1} - f\Big|_{y_2} \right) \right\} ds \tag{4.24}$$

or

$$By_1 - By_2 = \epsilon \int_0^T G(t,s) \left\{ f_y\Big|_{y_2} (Ay_2 - y_2) - f_y\Big|_{y_1} (Ay_1 - y_1) \right.$$

$$\left. - \left(f\Big|_{y_1} - f\Big|_{y_2} \right) \right\} ds$$

$$- (1-\epsilon) \int_0^T G(t,s) \left(f\Big|_{y_1} - f\Big|_{y_2} \right) ds \tag{4.25}$$

Recalling the derivation of $\rho(Ay_1, Ay_2)$:

$$\rho(By_1, By_2) \leq \epsilon\, \rho(Ay_1, Ay_2) + (1-\epsilon) \frac{mT^2}{4} \rho(y_1, y_2) \tag{4.26}$$

now

$$\rho(Ay_1, Ay_2) \leq \frac{\frac{mT^2}{4}}{1 - \frac{mT^2}{4}} (2K + 2)\, \rho(y_1, y_2) \tag{4.27}$$

so

$$\rho(By_1, By_2) \leq \frac{\frac{mT^2}{4}}{1 - \frac{mT^2}{4}} \left(1 + \epsilon(2K+1) - (1-\epsilon) \frac{mT^2}{4} \right) \rho(y_1, y_2) \tag{4.28}$$

with

$$k_3 = \frac{\frac{mT^2}{4}}{1 - \frac{mT^2}{4}} \left(1 + \epsilon(2K+1) - (1-\epsilon) \frac{mT^2}{4} \right) \tag{4.29}$$

$$\rho(By_1, By_2) \leq k_3\, \rho(y_1, y_2) \tag{4.30}$$

This completes the proof if $0 < k_3 < 1$.

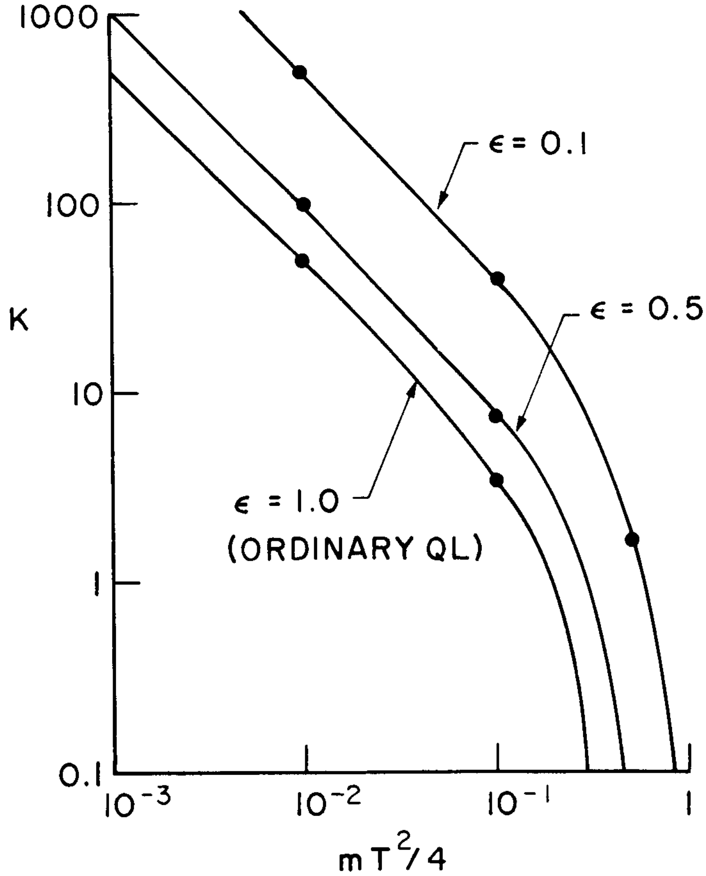
No attempt is made to show quadratic convergence, since it occurs only fortuitously in this case. The loss of quadratic convergence is the price paid for extending the range of convergence.

4.5 The Extended Range of Convergence

To show that Method 3 allows a wider range of convergence than unmodified QL a plot was drawn. Figure 4.1 gives the largest K for a particular $mT^2/4$ which will still result in $0 \leq k_3 < 1$. This shows theoretically that as ϵ is decreased that the range of convergence increases.

The curve $\epsilon = 1$ is the curve for k_2 (the ordinary QL procedure). Thus if $0 \leq \epsilon \leq 1$, Method 3 has a larger range of convergence.

The practical usage of Method 3 is described in detail in Chapter 6 using the Brachistochrone as an example, and in general with the reentry problem treated in Chapter 7.



LARGEST K ALLOWING CONVERGENCE

VS. $\frac{mT^2}{4}$ AND ϵ

FIGURE 4.1

CHAPTER 5

COMPUTATIONAL METHODS

5.1 Procedure Organization

To test the foregoing theories two sample problems, the Brachistochrone, and the reentry vehicle trajectory determination problem, have been solved using a large scale digital computer. In this chapter the techniques by which these problems were solved is discussed.

In order to apply the method of QL to the control problem it is necessary to integrate the equations of motion either analytically or numerically. Few real problems can be solved analytically, and so numerical procedures must be resorted to.

Each iteration of the method requires seven steps:

(1) Choose an approximate trajectory

(2) Compute the derivatives along this trajectory

(3) Integrate both the inhomogeneous equations and the appropriate homogeneous equations along the approximate trajectory

(4) Determine the error in the stopping conditions

(5) Invert the transition matrix

(6) Perturb the initial conditions of the inhomogeneous equations so that the final errors should be nulled out.

(7) Reintegrate the inhomogeneous equations of motion to check that the final errors are zero using the perturbed initial conditions.

The new trajectory obtained from (7) is used as the next approximate trajectory for (1). Repeat (1) to (7) until convergence is obtained. The trajectory of (7) may be modified before use in (1) if the method of widening the range of convergence discussed in Chapter 4 is employed.

5.2 Integration

Because the method of QL requires that the previous trajectory and various derivatives along this trajectory be available at each iteration, care must be taken to select a reasonable integration scheme. There are a variety of methods discussed in the literature (see, for example, 2, 5, and 9). For each of the methods discussed there is a trade off between speed of computation and required computer memory capacity. The methods which require a large computer memory generally are faster than those which require less and vice versa.

The program written stores the points along the trajectory and the necessary derivatives. Thus it is a relatively "fast" method and makes use of the large core memory of the IBM 7094.

After the trade off between computational speed and required memory capacity has been fixed, there are still several alternative ways to proceed because different integration schemes demand different amounts of storage and computation time. To achieve a high integration accuracy it is either necessary to use a very small step size with a simple integration formula or use a larger step size with a more complex formula. The trade off's are discussed in many places, for example, 18, 19, 20, and 21.

The integration is performed in three main blocks. The first block crudely initializes the second. The second block is iterated until fourth order accuracy is obtained and is used to initialize the third block. It is in the third block that the bulk of the integration occurs. The integration procedure uses a fixed number of steps.

The first block consists of three steps of Euler integration to provide four data points for each variable. The second block is a fourth order method and uses:

$$
\begin{aligned}
y_0 &= y_o \\
\bar{y}_1 &= y_o + \frac{h}{2}\left(9\,y'_o + 19y'_1 - 5y'_2 + y'_3\right) \\
\bar{y}_2 &= y_o + \frac{h}{3}\left(y'_o + 4y'_1 + y'_2\right) \\
\bar{y}_3 &= y_o + \frac{h}{8}\left(3y'_o + 9y'_1 + 9y'_2 + 3y'_3\right)
\end{aligned}
\tag{5.1}
$$

where h is the step size, the subscripts refer to steps in the integration and the primes denote the derivative.

y_o is the starting value of y

y_1, y_2, y_3 are provided initially by the Euler integration

$\bar{y}_1, \bar{y}_2, \bar{y}_3$ are the next choices for y_1, y_2, and y_3

The procedure is iterated until the method of 5.1 has converged. As a check on the convergence it is wise to save $|\bar{y}_i - y_i|$ for i = 1, 2, 3. An examination of these quantities will show whether or not the method has converged.

The values of y_1, y_2, and y_3 thus obtained are used to start the modified Hamming method:

$$
\begin{aligned}
p_{n+1} &= y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right) \\
m_{n+1} &= p_{n+1} - \frac{112}{121}(p_n - c_n) \\
m'_{n+1} &= f(m_{n+1}) \\
c_{n+1} &= \frac{1}{8}\left[9y_n - y_{n-2} + 3h\left(m'_{n+1} + 2y'_n - y'_{n-1}\right)\right] \\
y_{n+1} &= c_{n+1} + \frac{9}{121}(p_{n+1} - c_{n+1})
\end{aligned}
\tag{5.2}
$$

The vectors p, m, and c in the above represent the predicted, modified, and corrected approximations to the succeeding step.

This method is used for the remainder of the integration. It is also a fourth order method.

The stability of the modified Hamming method has been examined by Chase[21] and found to be quite good. The use of the initializing formulas, 5.1, is discussed by Ralston and Wilf[19] in connection with this method in particular.

5.3 Matrix Inversion and Determination of the Eigenvalues

To invert the transition matrix a program using Gaussian Elimination (19) was written. The program was written with double precision arithmetic in order to eliminate the round-off error which results from the procedure used. Otherwise this round-off error will destroy the accuracy of the inverse matrix. The elimination of round-off error is particularly important here since Gaussian Elimination is a direct procedure and there is no convenient indication of the accuracy of the inverse.

The importance of using double precision arithmetic when processing a transition matrix cannot be understated either.

In the case of the reentry problem, the transition matrix is so nearly singular that, the use of single precision arithmetic results in eigenvalues that are an order of magnitude off.

The eigenvalues were found by first reducing the transition matrix to upper Hessenberg form. The resulting matrix was then reduced by a series of QR transformations.[25,26,27,28] The program written follows SHARE program 3006 closely but uses double precision arithmetic.

5.4 The Determination of Partial Derivatives

If the Hamiltonian can be used to solve for an analytic expression for the control, then the control can be eliminated from the problem being solved, and the application of the method of QL is straightforward.

On the other hand, if the partial derivative of the Hamiltonian with respect to control cannot be inverted explicitly to find the control as a function of state, then the problem of applying the method of QL becomes much more complex. This occurs if the control is bounded and it is not desired to have to include additional multipliers in the problem formulation.

The method of Chapter 3 can then be used to advantage. In the method of Chapter 3 the system

$$\dot{y} = f(y, u, t) \tag{5.3}$$

is solved iteratively using

$$\dot{y}^{k+1} = f^k + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot \frac{\partial u^T}{\partial y} \right)^k (y^{k+1} - y^k) \tag{5.4}$$

The treatment of the boundary conditions is omitted as they are dealt with in Chapter 3.

Associated with either set of equations is:

$$H(y, t, u) \tag{5.5}$$

where it must be kept in mind that y represents both the state and the adjoint variables.

The Maximum Principle states that H should be minimized if the criterion function is minimized and the multiplier associated with the criterion function is positive.

If u is unbounded this results in the necessary condition (when $\partial H/\partial u$ contains u):

$$\frac{\partial H}{\partial u} = 0 \tag{5.6}$$

which must be solved for u.

If u is bounded this same equation will result if Valentine's method is used.

However, there is another approach. If u can be determined from the Maximum Principle directly:

$$u = u\{\min_u H(y,t)\} \tag{5.7}$$

then the term $\partial u/\partial y$ may be found by perturbations:

$$\frac{\partial u}{\partial y} \approx \frac{\delta u}{\delta y} = \left[u\left\{\min_u \left(H(y+\delta y)\right)\right\} - u\left\{\min_u \left(H(y)\right)\right\}\right]/\delta y \tag{5.8}$$

Equation (5.4) can then be applied, and the modified method of QL used to solve the problem.

In cases where the control is bounded it may be more convenient to recognize that on the bound:

$$\frac{\partial u}{\partial y} = 0 \tag{5.9}$$

and that off the bound $\partial u/\partial y$ may be known explicitly if

$$\frac{\partial H}{\partial u} = 0 \tag{5.10}$$

can be solved for the necessary derivatives.

CHAPTER 6

THE BRACHISTOCHRONE

6.1 Properties of the Brachistochrone

The Brachistochrone is a suitable choice for a test problem. It has many attractive features:

(1) It is governed by a set of six ordinary nonlinear differential equations

(2) An analytic solution is available

(3) It is computationally simple.

With the Brachistochrone a variety of tests were made on the foregoing theories:

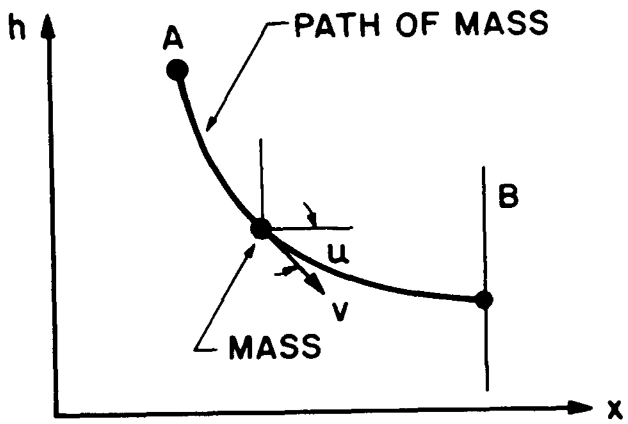
A comparison was made between the rates of convergence of the undetermined parameter method for handling the free final time problem and that used by McGill and Kenneth.

Experiments were conducted to see how well the methods of Chapter 4 succeeded in extending the range over which the method of QL would converge.

The effect using the modification of Chapter 3 instead of eliminating the control from the problem was also tested on the Brachistochrone.

6.2 The Equations of Motion

The problem of the Brachistochrone is the problem of finding the path that a mass, accelerated by gravity alone, should follow to fall from one point to another for minimum time, T. There are many variations on the Brachistochrone problem and the one of finding the path of a mass going from a point A to a vertical line B is worked here. The coordinate system is shown in Figure 6.1.



BRACHISTOCHRONE COORDINATE SYSTEM

FIGURE 6.1

The differential equation of motion are:

$$\dot{x} = v \cos u$$
$$\dot{h} = v \sin u \qquad (6.1)$$
$$\dot{v} = -g \sin u$$

The initial equations are:

$$x(o) = 0$$
$$h(o) = h_o \qquad (6.2)$$
$$v(o) = v_o$$

The final conditions are:

$$x(T) = x_f$$
$$h(T) = \text{unspecified} \qquad (6.3)$$
$$v(T) = \text{unspecified}$$

Changing from (x, h, v) to (y_1, y_2, y_3) and adding the adjoint variables (y_4, y_5, y_6) to get $y^T = (y_1, \ldots, y_6)$.

The Hamiltonian for the time optimal control problems is:

$$H = 1 + y_4(y_3 \cos u) + y_5(y_3 \sin u) + y_6(-g \sin u)$$

The differential equations for the adjoint variables are

$$\dot{y}_4 = 0$$
$$\dot{y}_5 = 0 \qquad (6.4)$$
$$\dot{y}_6 = -y_4 \cos u - y_5 \sin u$$

Since h(T) and v(T) are unspecified there are transversality conditions to be satisfied.

$$y_4(0) = \text{unspecified} \qquad y_4(T) = \text{unspecified}$$
$$y_5(0) = \text{unspecified} \qquad y_5(T) = 0$$
$$y_6(0) = \text{unspecified} \qquad y_6(T) = 0 \tag{6.5}$$

The two specified conditions on the adjoint variables complete the boundary conditions of the system of differential equations formulated. There are now six differential equations and six boundary conditions.

The Hamiltonian may be solved for the optimal control directly

$$\frac{\partial H}{\partial u} = 0 = -y_4 y_3 \sin u + y_5 y_3 \cos u - y_6 g \cos u \tag{6.6}$$

or

$$\tan u = \frac{y_5 y_3 - y_6 g}{-y_4 y_3} \tag{6.7}$$

Using this solution for the control, complete solutions may be found in the following way.

First find T and T' from the two simultaneous transcendental equations:

$$\frac{\pi v_o}{2Tg} = \cos \frac{\pi T'}{2T} \tag{6.8}$$

and

$$0 = \frac{\pi}{g} (x_f - x_o) - T' - \frac{T}{\pi} \sin \frac{\pi T'}{T} \tag{6.9}$$

Equations (6.8) and (6.9) result from (6.1), (6.2), and (6.7) directly. The intermediate steps are skipped.

Second calculate w

$$w = \frac{\pi}{2T} \tag{6.10}$$

Now

$$x = x_o + \frac{g}{2w}\left[t + \frac{1}{w}\cos w(t-2T')\sin wt\right]$$

$$h = h_o + \frac{g}{w^2}\sin w(t-2T')\sin wt \tag{6.11}$$

$$v = \frac{g}{w}\cos w(t-T')$$

$$u = w(t-T')$$

These may be used to compare the calculated solution with the true solution.

In Figures 6.2, 6.3, and 6.4 where it was desired to have a small integration error, 48 steps were used in the integration. In each of the cases, either 15 or 16 iterations were employed to see how the methods stabilized (or did not).

The difference in required IBM 7094 machine time to run these problems varied a few percent from one variation to another. The longest was the modified integration scheme which took 10 percent more time than either of the others. The programs written were in no sense optimized with respect to computer execution time and required about 85 seconds for 16 iterations.

As a check on the 48-step integration accuracy the solution was iterated until the only error sources were round-off and truncation errors. Table 6.1 shows a comparison between the calculated and the true values.

6.3 The Effect of Numerical Partial Derivatives

By the term numerical partial derivatives it is meant that in place of solving the Hamiltonian for the control explicitly to get an analytic expression for it, and the partial derivatives $\partial u/\partial y$, that these quantities are found by perturbations as discussed in 5.4.

TABLE 6.1

BRACHISTOCHRONE SOLUTION

(QL vs True)

	QL	True
x_o	0.0	0.0 ft
h_o	6.0	6.0 ft
v_o	1.0	1.0 ft/sec
$p_1(0)$	-0.06378313	-0.06378321
$p_3(0)$	-0.03101964	-0.03101964
h_f	6.0	6.0 ft
y_f	2.195396	2.19540160 ft
v_f	15.67811	15.678108 ft/sec
t_f	0.7343791	0.7343789 sec
$H(o)$	0.149×10^{-7}	0.0
$H(t_f)$	0.124×10^{-5}	0.0

Note: p_1 is constant, $\dot{p}_2$ is zero, $g_o = 32.172$ ft/sec

Note: 48 steps in QL integration

The truncation error caused by the finite steps taken in the differences results in the convergence rate being slowed down in the middle as shown in Figure 6.2. Neither the initial convergence rate nor the final accuracy were effected.

6.4 Effect of Variable Time Parameter

A comparison was made between the different techniques of Section 2.2 for handling the problem of free final time.

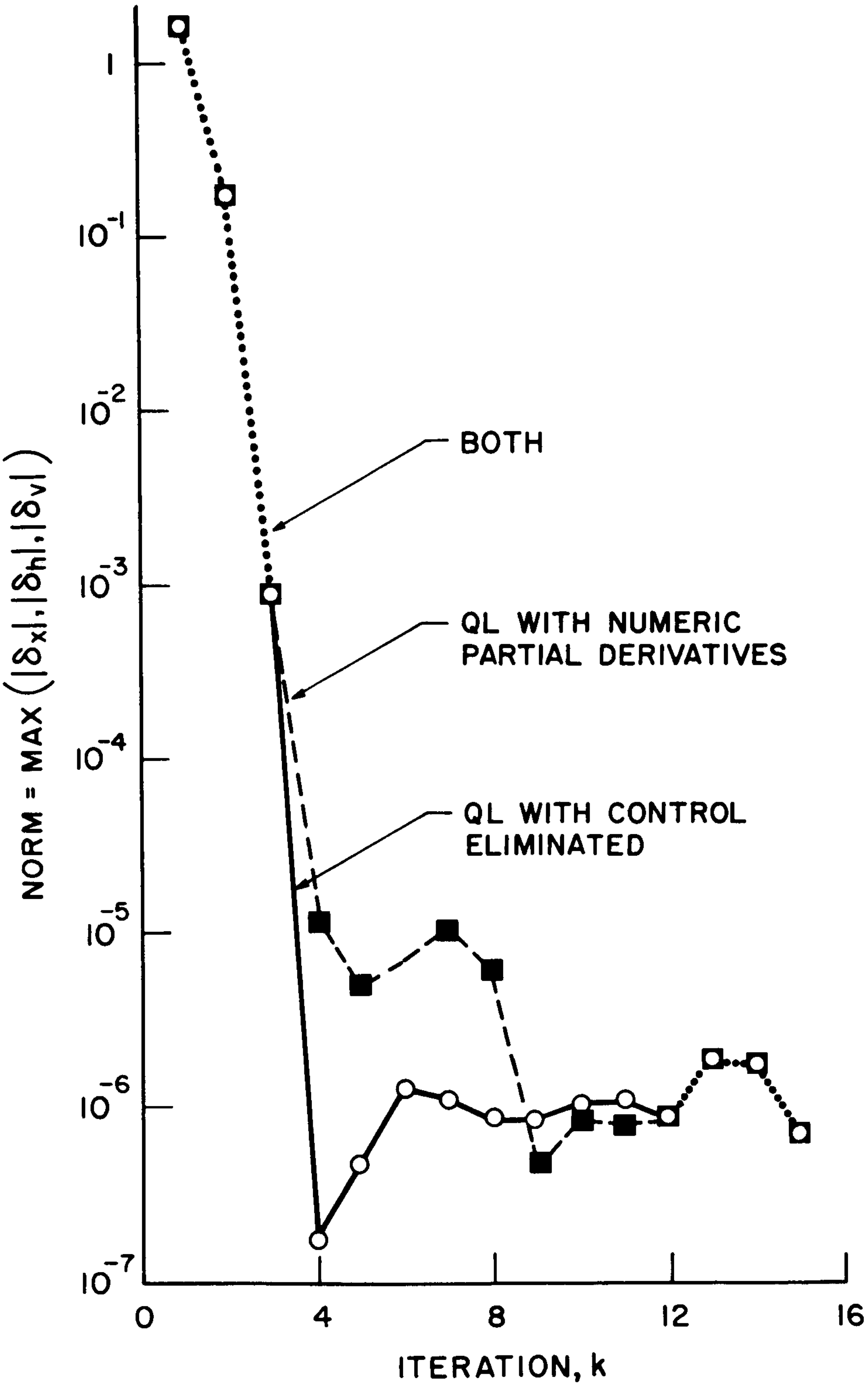
In Figure 6.3 the method of using a free parameter for solving the free final time case, Equations (2.18) to (2.24) are compared with McGill and Kenneth's method, Equation (2.17). It is seen immediately that the method of using a dummy independent variable converges much more rapidly. There is no difference in the computing time required per iteration.

In Figure 6.4 the convergence and ultimate accuracy of using Equations (2.27) to (2.29) in place of Equations (2.18) to (2.24) is illustrated. The more erratic convergence of the modified method can be attributed to two sources, the lack of any sharp changes in the derivatives in the problem solved and the addition of extra computations. In a problem as simple as this, the modification adds a few percent to the computing time per iteration.

6.5 The Extension of the Range of Convergence

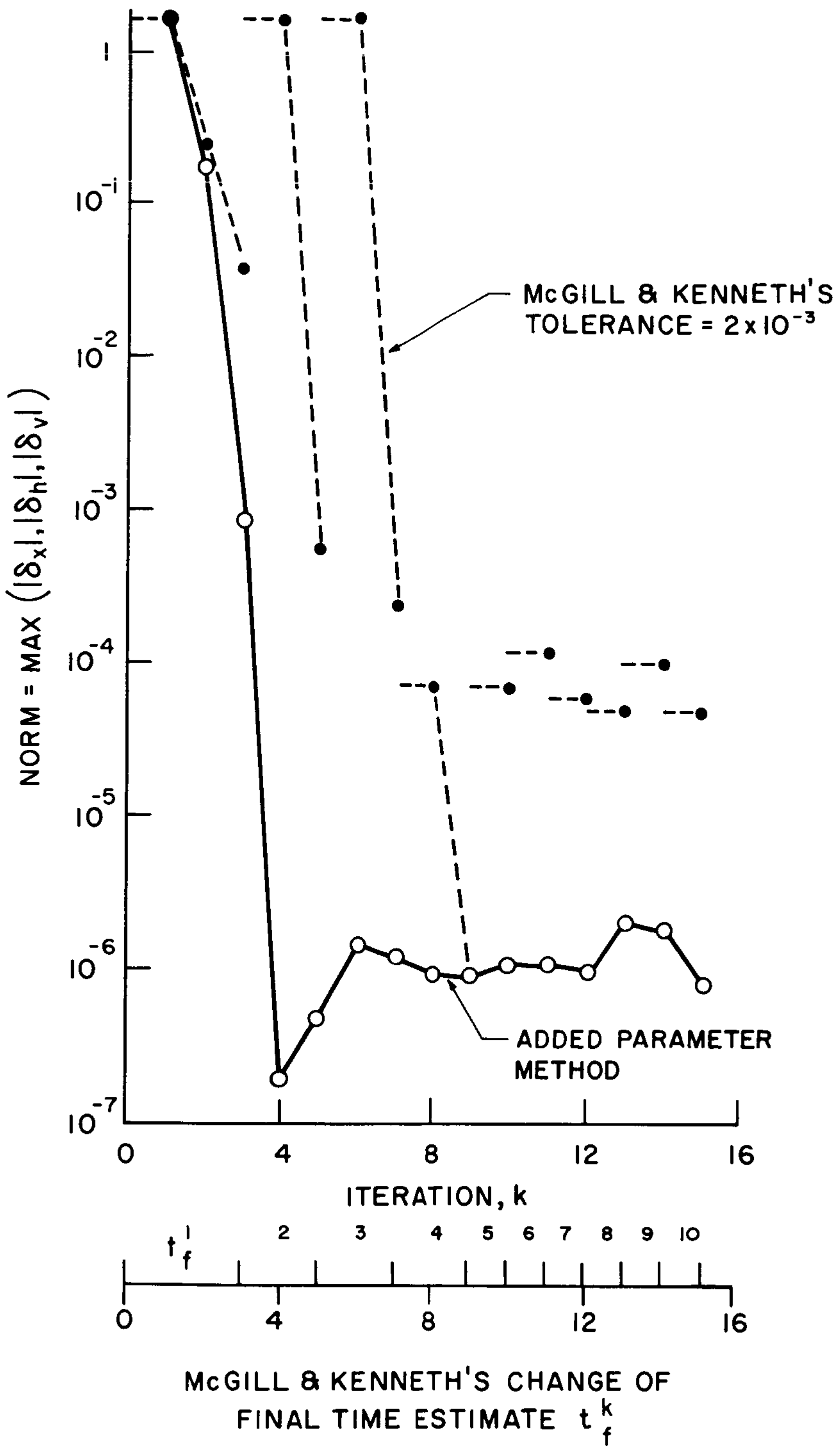
Employing the methods of Chapter 4 a sequence of tests were made to see how far the convergence could be extended.

The number of iterations, P, made with the convergence constant ϵ, not equal to one was chosen a priori to see how the methods of extending the range of convergence affects the speed of convergence. While choosing P a priori is not as computationally efficient as it might be it is much more expletive.



BRACHISTOCHRONE: EFFECT OF NUMERICAL PARTIAL DERIVATIVES ON QL

FIGURE 6.2



BRACHISTOCHRONE: COMPARISON OF ADDED PARAMETER FOR SOLVING THE FREE END TIME CASE WITH McGILL AND KENNETH'S METHOD

FIGURE 6.3

To examine the speed of convergence a quantity called Normax (from maximum norm) was examined after a fixed number of iterations.

$$\text{Normax} = \underset{i,t}{\text{Max}} \left| y_i^{k+1}(t) - y_i^k(t) \right| \tag{6.12}$$

then $y_i^k(t)$ is the time history of the i^{th} variable, including both state and adjoint variable, on the k^{th} iteration.

Method 1 of Chapter 4 was examined using the Brachistochrone as a test problem and it was found that for $\epsilon = 0.5$ and 0.75 that this technique did not have a convergence range even as large as that of the usual QL. No results are displayed.

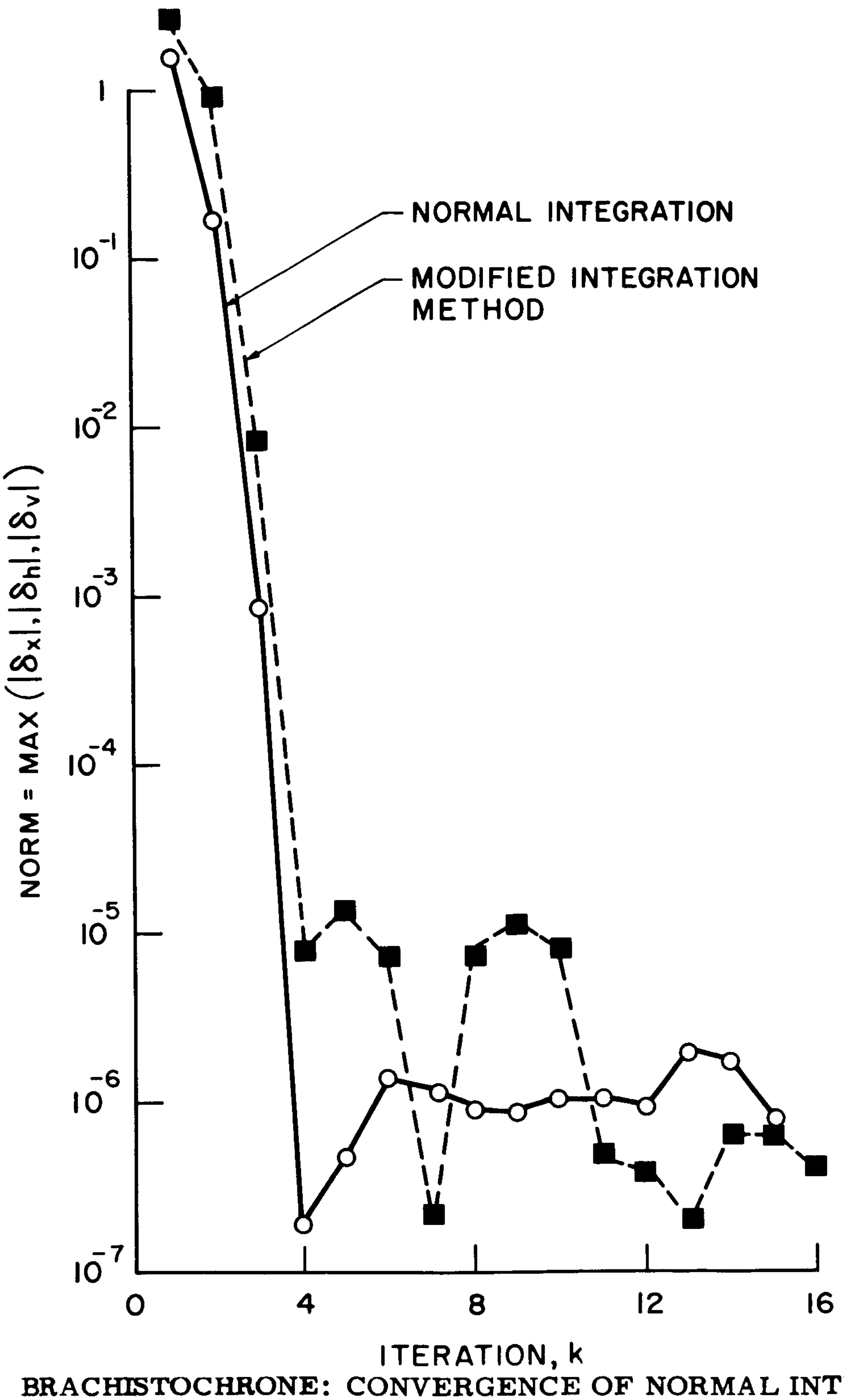
The results of the examination of Method 2 are shown in Figure 6.5. Since P was fixed a priori the method does not converge as rapidly as ordinary QL although it does converge over a somewhat much wider region. The test problem was the Brachistochrone with $\epsilon = 0.5$.

The results from Method 3 are shown in Figures 6.6 and 6.7. This method of extending the range of convergence works far better than that of the other methods examined, and considerably better than the usual QL.

In Figure 6.6 two iterations of Method 3 ($\epsilon = 0.5$) were followed by three of the usual QL procedure for a total of five iterations. The results are compared with five iterations of the usual QL procedure.

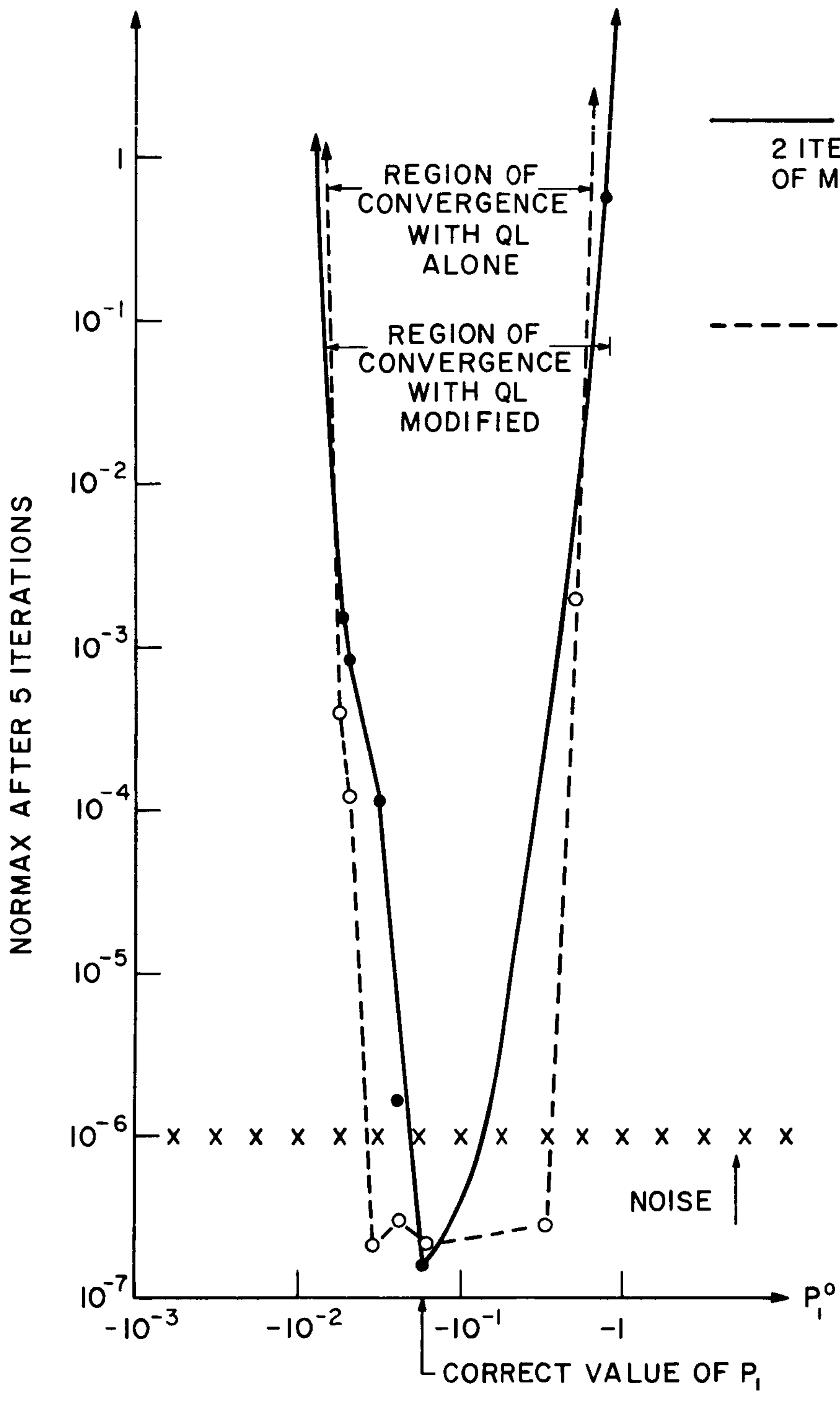
In Figure 6.7 four iterations of Method 3 ($\epsilon = 0.25$) were followed by three of the usual QL procedure for a total of seven iterations. The same standard of comparison is used as in Figure 6.6.

Both Figure 6.6 and Figure 6.7 show that at the expense of computing time a much wider range of convergence can be obtained with Method 3 than is available in ordinary QL and that the more is paid the wider the range of convergence is obtained.



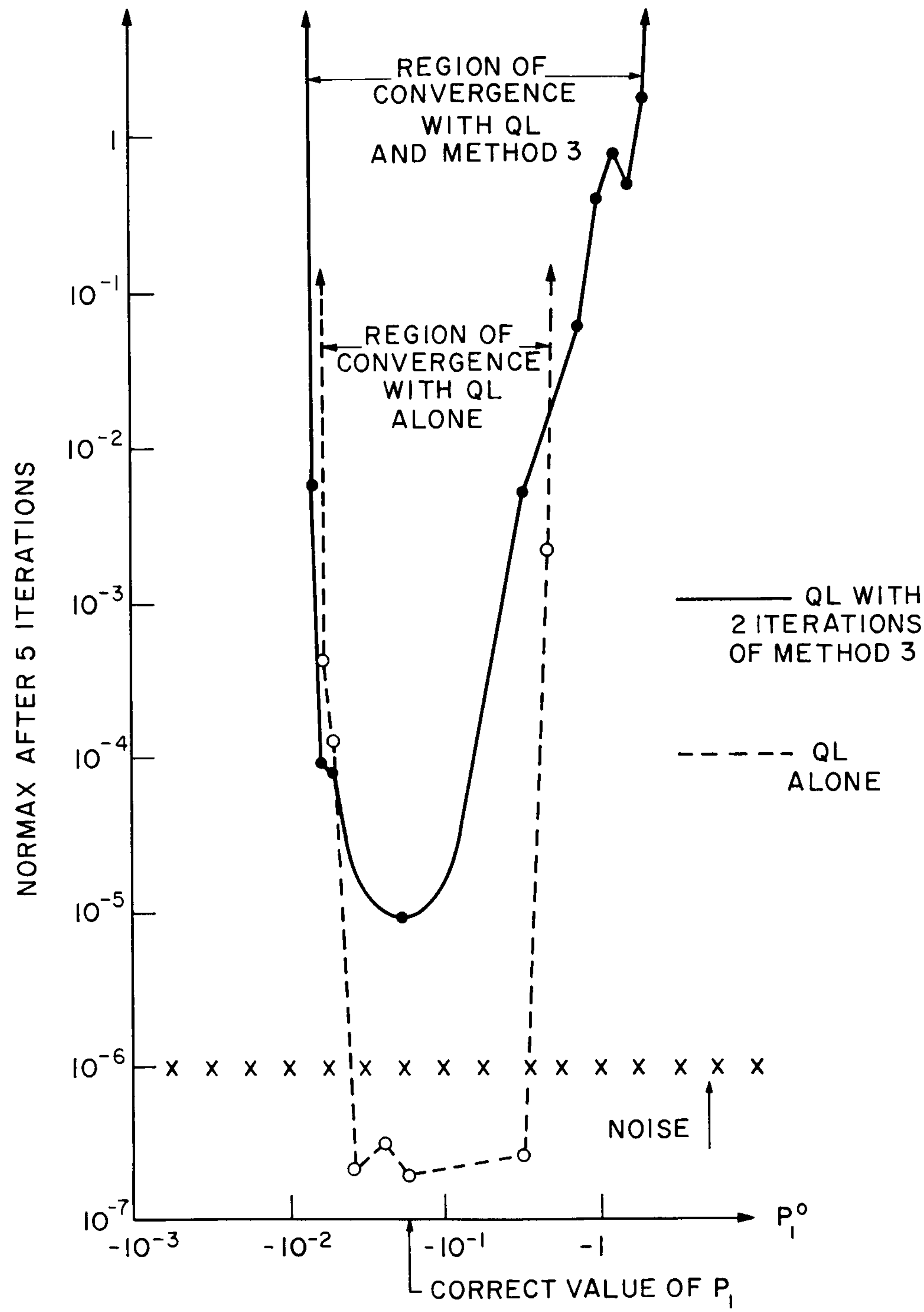
BRACHISTOCHRONE: CONVERGENCE OF NORMAL INTEGRATION COMPARED WITH THE MODIFIED INTEGRATION METHOD

FIGURE 6.4



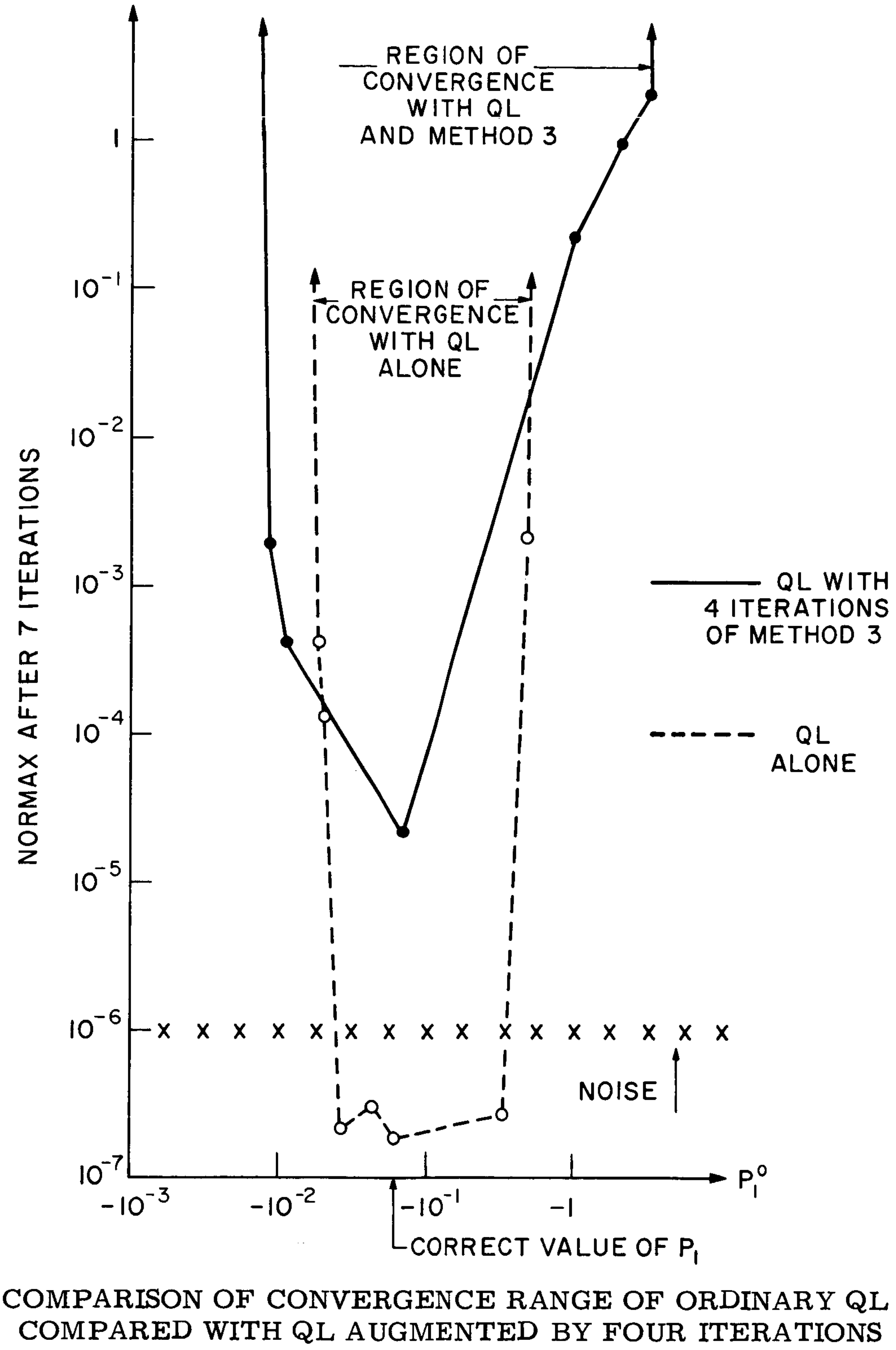
COMPARISON OF CONVERGENCE RANGE OF ORDINARY QL COMPARED WITH QL AUGMENTED BY TWO ITERATIONS OF METHOD 2 (ϵ=0.5) FOR A TOTAL OF FIVE ITERATIONS

FIGURE 6.5



COMPARISON OF CONVERGENCE RANGE OF ORDINARY QL COMPARED WITH QL AUGMENTED BY TWO ITERATIONS OF METHOD 3 (ϵ=1/2) FOR A TOTAL OF FIVE ITERATIONS

FIGURE 6.6



COMPARISON OF CONVERGENCE RANGE OF ORDINARY QL COMPARED WITH QL AUGMENTED BY FOUR ITERATIONS OF METHOD 3 (ϵ=0.25) FOR A TOTAL OF SEVEN ITERATIONS

FIGURE 6.7

CHAPTER 7

THE REENTRY TRAJECTORY PROBLEM

7.1 The Reentry Problem

The solution of the reentry vehicle trajectory problem is an appropriate choice for a more complex problem. It is well known to be computationally difficult because of integration instability and the sensitivity of the adjoint variables.

The equations of motion are those used by Scharmack,[15,16] Breakwell, Speyer, and Bryson,[23] and are similar to those used by Payne.[18] These equations are considerably simplified but are still realistic physically. The reentry vehicle is assumed to have a low lift-drag ratio.

The simplifications include the use of an exponential model of the atmosphere in place of a more complex pressure-altitude relationship and the use of a simple lift drag polar.

7.2 Two Cases

Two cases were formulated to see what differences would arise. The first is similar to both Breakwell, Speyer and Bryson[23] and Payne[18] and the second case is that solved by Scharmack.[15,16] The primary differences are those of initial and final conditions, control polar, and in the case of Payne the use of a more accurate gravity approximation here. The criterion function also varies: Scharmack used both convective and radiative heating, Breakwell used velocity, while Payne considered convective heating and sensed acceleration.

The two cases are defined more explicitly in Table 7.1.

TABLE 7.1

TWO REENTRY CASES

	Case 1	Case 2
Initial velocity	35,000 ft/sec^2	36,000 ft/sec^2
Initial height	400,000 ft	400,000 ft
Initial flight path angle	-8.1^o	-5.7^o
Final velocity	27,000 ft/sec	1650 ft/sec
Final height	250,000 ft	75,530 ft
Final flight path angle	0	Free
Final range	Free	5,170,000 ft
Drag, C_D	$0.274+1.8 \sin^2 u^1$	$0.88+0.52 \cos u$
Lift, C_L	$1.2 \sin u^1 \cos u^1$	$-0.505 \sin u$
Criterion Function	Integral convective heating	Integral convective and radiative heating

7.3 The Equation of Motion

The lift drag polar employed in Case 2 was

$$C_D = C_{DO} + C_{DL} \cos u$$
$$C_L = C_{LO} \sin u \tag{7.1}$$

where u is the control angle (angle of attack).

This is completely equivalent to that used by Case 1:

$$C_D = C^1_{DO} + C^1_{DL} \sin^2 u^1$$
$$C_L = C^1_{LO} \sin u^1 \cos u^1 \tag{7.2}$$

with the following substitutions in the above equations the equivalence can be seen:

$$C^1_{DO} = C_{DO} + C_{DL}$$
$$C^1_{DL} = -2\, C_{DL}$$
$$C^1_{LO} = 2\, C_{LO} \tag{7.3}$$
$$u^1 = u/2$$

The quantity to be optimized (minimized) is the total stagnation point heating per unit area including either the convective term alone or both radiative and convective terms:

$$J = \int_o^T \dot{q}\, dt \tag{7.4}$$

$\dot{q} = \dot{q}_r + \dot{q}_c$ for both radiative and convective heating

$\dot{q}_c = c\, v^3 \sqrt{\frac{\rho}{N}}$ convective term (7.5)

$$\dot{q}_r = 7.5N \left(\frac{\rho}{\rho_o}\right)^{3/2} \left(\frac{v}{10^4}\right)^{12.5} \qquad \text{radiative term} \tag{7.6}$$

The meaning of the symbols is given in Table 3.

The equations of motion are defined with respect to a two dimensional spherical earth. See Figure 7.1. They are:

$$\begin{aligned}
\dot{v} &= -\frac{S}{2m}\rho v^2 C_D(u) - \frac{g_o \sin\gamma}{(1+\xi)^2} \\
\dot{\gamma} &= \frac{S}{2m}\rho v C_L(u) + \frac{v\cos\gamma}{R(1+\xi)} - \frac{g_o\cos\gamma}{v(1+\xi)^2} \\
\dot{\xi} &= \frac{v}{R}\sin\gamma \\
\dot{\zeta} &= \frac{v}{1+\xi}\cos\gamma \\
\rho &= \rho_o e^{-\beta R\xi} \\
a_p &= \frac{S\rho v^2}{2m\, g_o}\sqrt{C_L^2(u) + C_D^2(u)} \\
|u| &\le u_1
\end{aligned} \tag{7.7}$$

7.4 The Choice of Control

The Hamiltonian is defined as

$$H = p_0\,\dot{q} + p_1\,\dot{v} + p_2\,\dot{\gamma} + p_3\dot{\xi} + p_4\,\dot{\zeta} \tag{7.8}$$

and

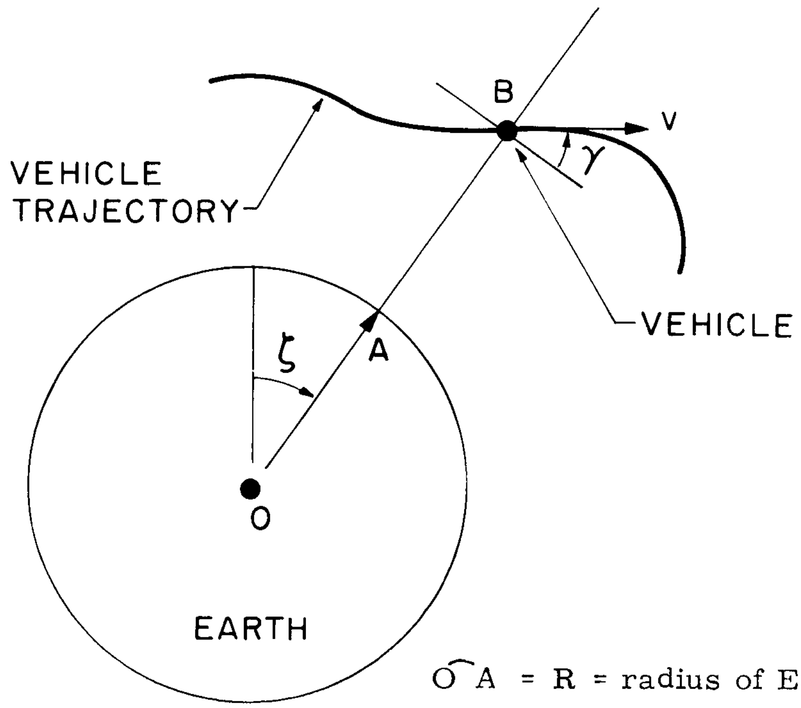
$$p_0 = 1$$

With this choice of sign for p_0 the Hamiltonian must be minimized if the integral of $\dot{q}$ is to be minimized. If the opposite

TABLE 7.2

TABLE OF SYMBOLS IN REENTRY PROBLEMS

v	-	velocity
ξ	-	normalized altitude
γ	-	flight path angle
ζ	-	down range distance
$\frac{S}{M}$	-	ratio of frontal area to vehicle mass
g_o	-	gravitational constant
ρ_o	-	air density at sea level
β	-	exponential constant
R	-	earth radius
$C_D(u)$	-	drag coefficient
$C_L(u)$	-	lift coefficient
C_{DL}, C_{DO}, C_{LO}	-	drag and lift coefficients
$\bar{c}$	-	convective constant
N	-	radius of vehicle nose
a_p	-	sensed acceleration
u_1	-	control constraint



$\widehat{OA}$ = R = radius of Earth

$\widehat{AB}$ = h = altitude of vehicle above Earth

v = velocity vector

γ = angle between local horizon and velocity vector

ζ = down range distance

$\xi = \frac{h}{R}$ = normalized altitude

REENTRY VEHICLE COORDINATE SYSTEM

FIGURE 7.1

sign for p_0 was chosen the Hamiltonian would have to be maximized to minimize the integral of $\dot{q}$.

The choice of p_0 as plus one means that not only must $\frac{\partial H}{\partial u} = 0$ for unbounded control but also:

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \tag{7.9}$$

From this necessary condition it is possible to choose the sign of the control that will minimize the Hamiltonian.

To find the control for the case where the bound is not reached set

$$\frac{\partial H}{\partial u} = 0 \tag{7.10}$$

or

$$0 = p_1 \frac{S\rho v^2}{2m} C_{DL} \sin u + p_2 \frac{S\rho v}{2m} C_{LO} \cos u$$

where H is taken from Appendix A so

$$\tan u = - \frac{C_{DO}}{C_{DL}} \frac{p_2}{p_1 v} \tag{7.11}$$

To find the corresponding value of the second partial of the Hamiltonian with respect to u

$$\frac{\partial^2 H}{\partial u} = p_1 \frac{S\rho v^2}{2m} C_{DL} \cos u - p_2 \frac{S\rho v}{2m} C_{LO} \sin u \tag{7.12}$$

$$= p_1 \frac{S\rho v^2}{2m} C_{DL} \cos u \left(1 - \frac{p_2}{p_1 v} \frac{C_{LO}}{C_{DL}} \tan u\right)$$

$$= p_1 \frac{S\rho v^2}{m} C_{DL} \cos u \tag{7.13}$$

Thus for small u the sign of the second derivative will be the product of the signs of p_1 and C_{DL}.

If C_{DL} is negative then p_1 must be negative for small u and conversely if C_{DL} is positive the p_1 must be positive for small u.

Case 2 uses

$$\begin{aligned} C_D(u) &= 0.88 + 0.52 \cos u \\ C_L(u) &= -0.505 \sin u \end{aligned} \tag{7.14}$$

while Case 1 uses[23]

$$\begin{aligned} C_D(u^1) &= 0.274 + 1.8 \sin^2 u^1 \\ C_L(u^1) &= 1.2 \sin u^1 \cos u^1 \end{aligned} \tag{7.15}$$

or

$$\begin{aligned} C_D(u) &= 1.174 - 0.9 \cos 2u \\ C_L(u) &= 0.6 \sin 2u \end{aligned} \tag{7.16}$$

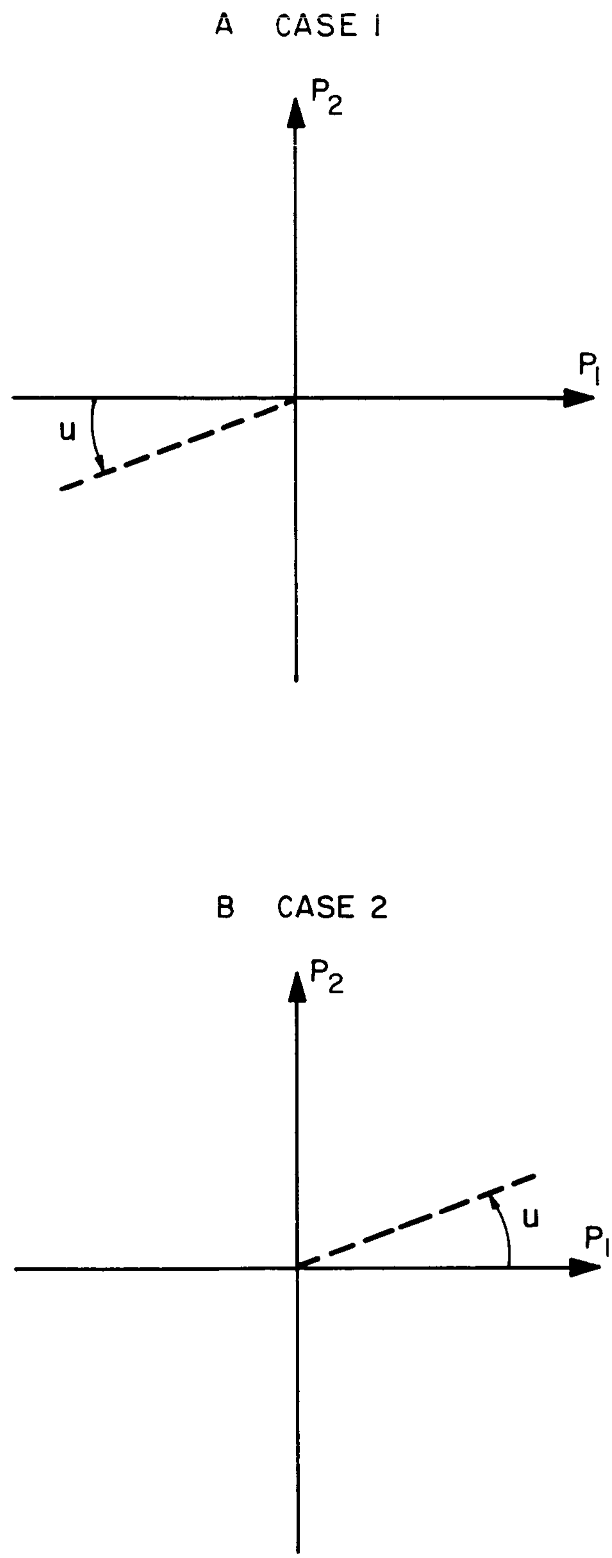
which result in different polars being used to determine the control as C_{DL} has different signs in the two cases.

The polars are illustrated in Figure 7.2.

The equations for the adjoint variables are listed in Appendix A, as are the elements of the matrix of partials necessary for QL. The computer program used to solve the equations is described in Chapter 5.

7.5 Numerical Results

Case 1 was solved with and without a bound on the control. The constants used are listed in Appendix B. Since the computer program was written to handle control in the form of Case 2, the



CONTROL POLAR DIAGRAMS

FIGURE 7.2

values of the lift-drag parameters correspond to Equation (7.16) when they are tabulated.

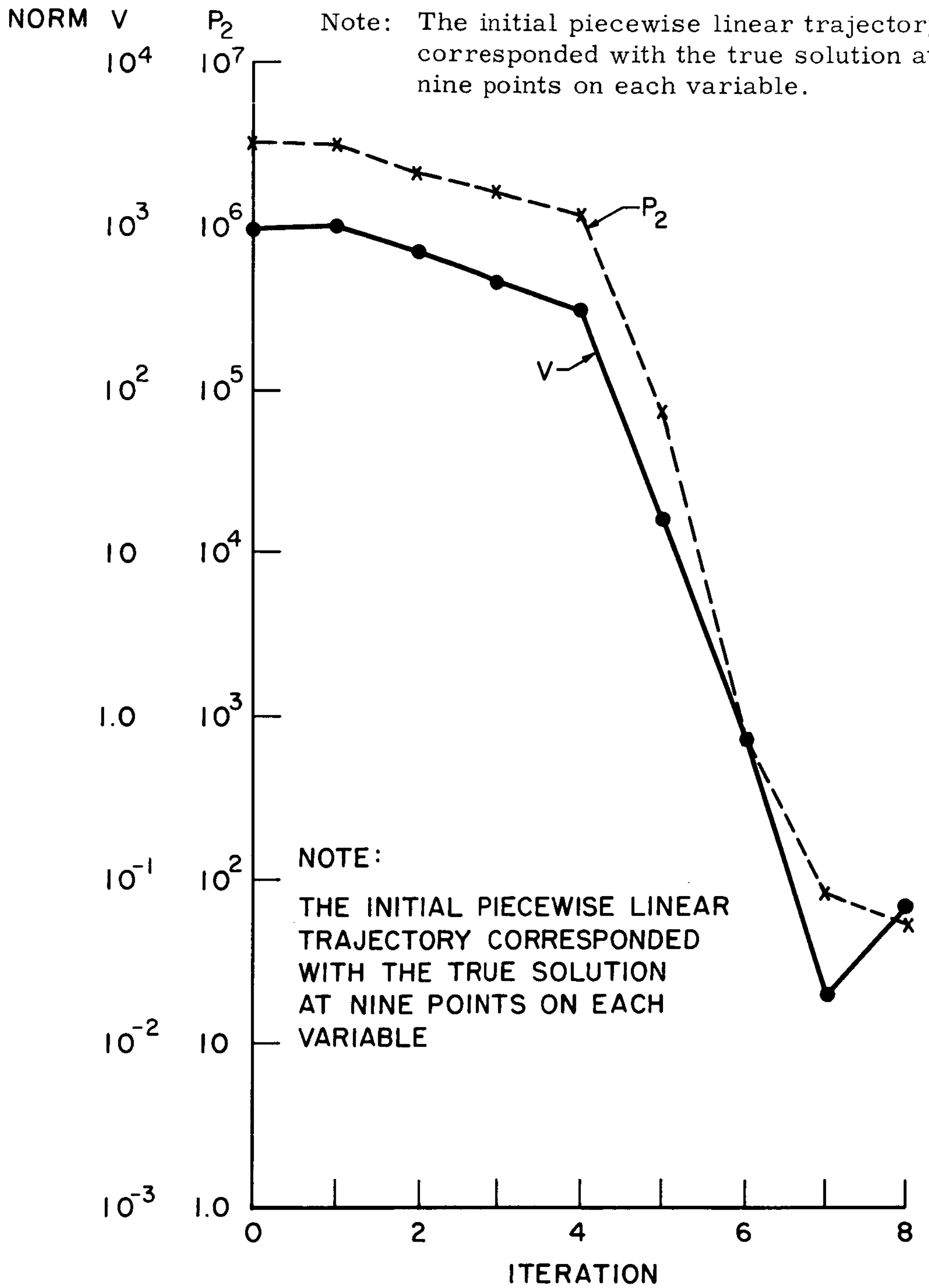
Irregardless of whether or not the control was bounded the range of convergence is quite small. Even if a piecewise linear approximation to the actual solution was employed as the initial guess, the convergence was slow. In fact, it was always necessary to employ the convergence improving constant described in Chapter 4 for the first few iterations, or the method would diverge even with eight sections in the approximation of each variable.

The convergence rates for velocity and the second adjoint variable are shown in Figure 7.3. The figure was generated by using an eight section approximation to the correct solution and a convergence factor of 0.33 for the first three iterations of the method. The remainder of the iterations were run with a convergence factor of 1.0 (ordinary QL). The convergence factor of 0.33 produces the flat portions of Figures 7.3 and 7.11.

To simplify the convergence problem initial guesses were made on the adjoint variables. These guesses were then integrated by a fourth order Runge-Kutta integration scheme.[19,20] When a guess was made that generated a solution near the optimal, then QL was employed. In the application of QL a small convergence factor was used on the first few iterations before switching to a convergence factor, ϵ, of 1.0. If the guess was far from optimal a new improved guess was made.

The final trajectory for Case 1 is shown in Table 7.3, and Figures 7.4 and 7.5. The final trajectory for the control bounded at $\pm$ 22.5 degrees is shown in Table 7.4 and Figures 7.6 and 7.7. The convective heating rates are compared in Figure 7.8, the sensed accelerations in Figure 7.9, and the controls in Figure 7.10. The

Note: The initial piecewise linear trajectory corresponded with the true solution at nine points on each variable.



CASE 1

CONVERGENCE RATE FROM A PIECEWISE LINEAR APPROXIMATION TO THE FINAL SOLUTION UNBOUNDED CONTROL

FIGURE 7.3

TABLE 7.3

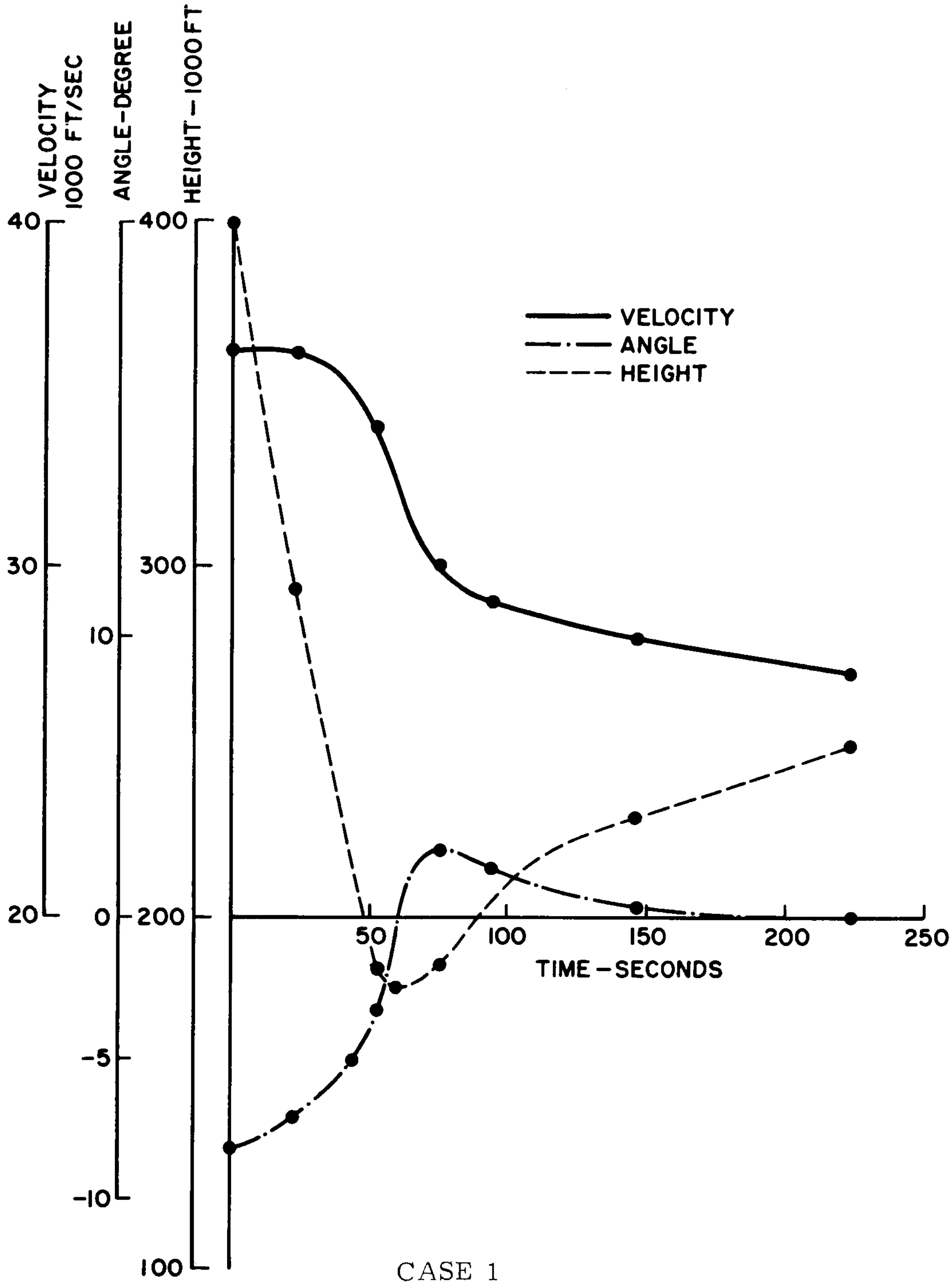
REENTRY VEHICLE - CASE 1
UNBOUNDED CONTROL

INCLUDES CONVECTIVE HEATING TERM ALONE
BACKWARD INTEGRATION WITH 240 STEPS

TIME SEC	VEL FT/SEC	ANGLE DEG	H 1CCO FT	RANGE 1000 FT	P1	P2	P3	P4	CONTROL DEG	HEAT BTU/FT2	ACCEL G
224.4	27000.0	C.	250.C	6695.	-0.137E 02	0.134E 07	-0.128E 08	0.0	-33.8	0.	0.39
216.9	27077.3	0.06	249.9	6495.	-0.128E 02	0.123E 07	-0.117E 08	0.0	-33.5	375.	0.39
209.4	27155.3	0.12	249.6	6295.	-0.120E 02	0.112E 07	-0.107E 08	0.0	-33.3	754.	0.39
202.0	27234.6	0.17	249.1	6094.	-0.112E 02	0.103E 07	-C.978E 07	0.0	-33.0	1140.	0.40
194.5	27315.7	0.23	248.3	5892.	-0.106E 02	0.937E 06	-0.898E 07	0.0	-32.6	1535.	0.41
187.0	27399.4	0.29	247.4	5690.	-0.993E 01	0.855E 06	-0.825E 07	0.0	-32.2	1941.	0.42
179.5	27486.3	0.34	246.3	5487.	-0.936E 01	0.780E 06	-C.760E 07	0.0	-31.8	2359.	0.44
172.0	27577.2	0.4C	245.0	5284.	-0.884E 01	0.710E 06	-C.701E 07	0.0	-31.4	2792.	0.46
164.6	27673.1	C.47	243.4	5079.	-0.836E 01	0.645E 06	-0.648E 07	0.0	-30.9	3244.	0.49
157.1	27775.1	0.54	241.6	4874.	-0.792E 01	0.585E 06	-0.601E 07	0.0	-30.3	3718.	0.53
149.6	27884.4	0.62	239.4	4669.	-0.752E 01	0.529E 06	-0.560E 07	0.C	-29.6	4217.	0.57
142.1	28C02.8	0.71	237.0	4462.	-0.714E 01	0.477E 06	-C.524E 07	0.0	-28.9	4748.	0.62
134.6	28132.3	C.81	234.3	4254.	-0.680E 01	0.427E 06	-0.494E C7	0.0	-28.0	5316.	0.69
127.2	28275.6	0.93	231.1	4046.	-0.649E 01	C.380E 06	-0.471E 07	0.0	-27.0	593C.	0.77
119.7	28436.3	1.07	227.4	3836.	-0.620E 01	0.335E 06	-0.456E 07	0.0	-25.8	6601.	0.87
112.2	28618.9	1.24	223.1	3625.	-0.593E 01	0.290E 06	-0.452E 07	0.0	-24.4	7345.	1.00
104.7	28829.6	1.44	218.1	3412.	-0.568E 01	0.244E 06	-0.462E C7	0.0	-22.4	8184.	1.17
97.2	29075.8	1.70	212.2	3198.	-0.544E 01	0.196E 06	-0.494E 07	C.0	-19.8	9148.	1.38
89.8	29365.1	2.01	205.1	2982.	-0.522E 01	0.143E 06	-C.564E 07	0.0	-15.9	10287.	1.60
82.3	29702.3	2.32	196.8	2763.	-0.501E 01	0.782E 05	-0.696E 07	0.0	-9.7	11672.	1.72
74.8	30103.9	2.40	187.4	2542.	-0.481E 01	-0.629E 04	-0.925E 07	0.0	0.8	13412.	1.90
67.3	30772.3	1.58	179.1	2317.	-0.460E 01	-0.122E 06	-0.122E 08	0.0	14.9	15634.	5.09
59.8	32208.8	-0.68	176.8	2084.	-0.431E 01	-0.265E 06	-C.128E 08	0.0	25.9	18407.	9.65
52.4	34102.7	-3.34	185.7	1838.	-0.403E 01	-0.395E 06	-0.880E 07	0.0	31.2	21457.	8.81
44.9	35391.1	-5.14	205.4	1581.	-0.390E 01	-0.475E 06	-0.456E 07	0.0	33.2	24056.	4.35
37.4	35912.4	-6.07	231.8	1318.	-0.392E 01	-0.518E 06	-0.270E 07	0.0	33.9	2577C.	1.49
29.9	36058.0	-6.61	261.6	1053.	-0.404E 01	-0.548E 06	-C.224E 07	0.0	34.1	26738.	0.42
22.4	36076.9	-7.01	293.6	789.	-0.420E 01	-0.576E 06	-0.221E 07	0.0	34.2	27243.	0.11
15.0	36059.3	-7.38	327.4	525.	-0.440E 01	-0.604E 06	-0.225E 07	0.0	34.3	27494.	0.03
7.5	36031.3	-7.74	362.9	262.	-0.461E 01	-0.633E 06	-0.229E 07	0.0	34.3	27614.	0.01
0.	35999.9	-8.09	400.0	-0.	-0.483E 01	-0.661E 06	-0.231E 07	0.0	34.2	27669.	0.00

CONSTANTS

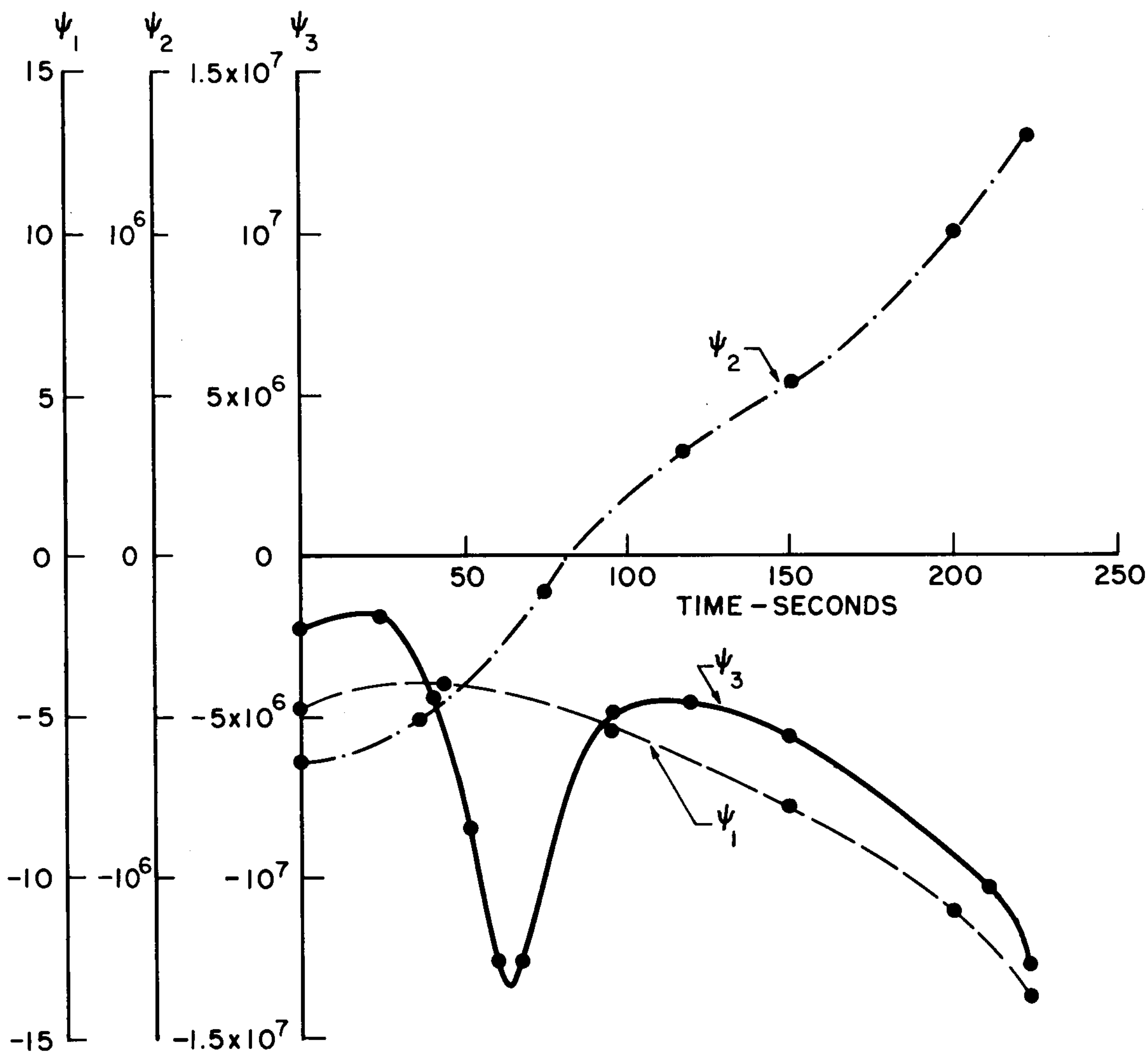
BETA 1/FT	RHO SLUG/FT3	G FT/SEC2	N FT	R FT	C	S FT2	M LB	CDO	CDL	CLO
0.42600E-04	0.27040E-02	32.1720	4.0000	0.20900E 08	0.2E-07	1.0640	2.0000	1.1740	-0.9000	0.6000



CASE 1

REENTRY VEHICLE
STATE VARIABLES VS TIME
UNBOUNDED CONTROL

FIGURE 7.4



CASE 1

REENTRY VEHICLE
ADJOINT VARIABLES
UNBOUNDED CONTROL

FIGURE 7.5

TABLE 7.4

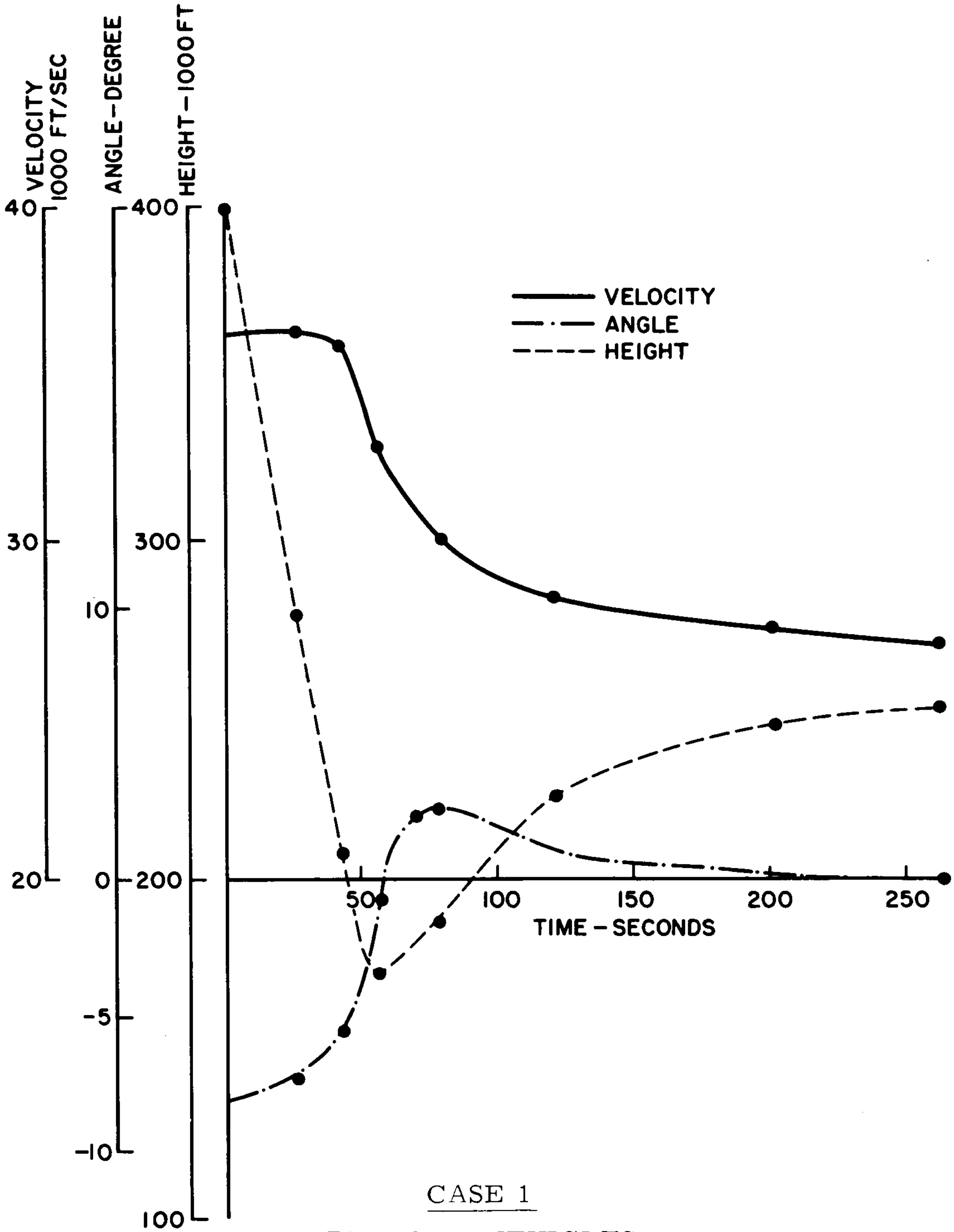
REENTRY VEHICLE - CASE 1
CONTROL BOUNDED AT 22.5 DEGREES

INCLUDES CONVECTIVE HEATING TERM ALONE
FORWARD INTEGRATION WITH 360 STEPS

TIME SEC	VEL FT/SEC	ANGLE DEG	H 1000 FT	RANGE 1000 FT	P1	P2	P3	P4	CONTROL DEG	HEAT BTU/FT2	ACCEL G
0.	36000.0	-8.09	400.0	0.	-0.841E 01	-0.106E 07	-0.374E 07	0.0	22.5	0.	0.00
8.7	36036.7	-7.68	356.8	306.	-0.800E 01	-0.101E 07	-0.371E 07	0.0	22.5	70.	0.00
17.5	36069.1	-7.26	315.8	614.	-0.761E 01	-0.955E 06	-0.367E 07	0.0	22.5	241.	0.03
26.2	36086.4	-6.82	277.1	923.	-0.725E 01	-0.901E 06	-0.368E 07	0.0	22.5	640.	0.15
35.0	36039.0	-6.31	241.0	1232.	-0.696E 01	-0.845E 06	-0.412E 07	0.0	22.5	1522.	0.69
43.7	35734.6	-5.48	208.5	1541.	-0.678E 01	-0.771E 06	-0.653E 07	0.0	22.5	3327.	2.72
52.5	34671.4	-3.70	183.2	1846.	-0.686E 01	-0.631E 06	-0.137E 08	0.0	22.5	6496.	7.52
61.2	32550.7	-0.50	171.8	2138.	-0.731E 01	-0.381E 06	-0.209E 08	0.0	22.5	10577.	10.74
69.9	30826.2	2.12	176.7	2411.	-0.781E 01	-0.119E 06	-0.176E 08	0.0	9.1	14161.	4.24
78.7	30098.4	2.54	188.1	2675.	-0.815E 01	0.640E 05	-0.118E 08	0.0	-4.9	16811.	2.05
87.4	29610.4	2.19	198.9	2933.	-0.849E 01	0.186E 06	-0.849E 07	0.0	-13.1	18772.	1.85
96.2	29222.1	1.78	207.8	3188.	-0.885E 01	0.277E 06	-0.693E 07	0.0	-17.8	20290.	1.54
104.9	28907.9	1.45	215.0	3439.	-0.924E 01	0.354E 06	-0.627E 07	0.0	-20.7	21525.	1.26
113.7	28648.5	1.19	220.7	3688.	-0.965E 01	0.426E 06	-0.609E 07	0.0	-22.5	22569.	1.04
122.4	28438.1	0.98	225.4	3935.	-0.101E 02	0.496E 06	-0.621E 07	0.0	-22.5	23481.	0.84
131.2	28265.2	0.83	229.3	4180.	-0.106E 02	0.568E 06	-0.652E 07	0.0	-22.5	24297.	0.70
139.9	28118.5	0.71	232.7	4424.	-0.111E 02	0.645E 06	-0.697E 07	0.0	-22.5	25040.	0.60
148.6	27991.1	0.61	235.5	4666.	-0.117E 02	0.727E 06	-0.754E 07	0.0	-22.5	25726.	0.53
157.4	27878.3	0.53	237.9	4908.	-0.123E 02	0.815E 06	-0.822E 07	0.0	-22.5	26366.	0.47
166.1	27776.7	0.47	240.0	5148.	-0.130E 02	0.912E 06	-0.900E 07	0.0	-22.5	26969.	0.43
174.9	27684.1	0.41	241.9	5388.	-0.138E 02	0.102E 07	-0.988E 07	0.0	-22.5	27541.	0.39
183.6	27598.7	0.36	243.5	5627.	-0.146E 02	0.113E 07	-0.109E 08	0.0	-22.5	28087.	0.36
192.4	27519.2	0.31	244.9	5865.	-0.156E 02	0.126E 07	-0.120E 08	0.0	-22.5	28611.	0.34
201.1	27444.6	0.27	246.1	6103.	-0.166E 02	0.140E 07	-0.132E 08	0.0	-22.5	29117.	0.32
209.8	27373.9	0.23	247.1	6340.	-0.178E 02	0.156E 07	-0.146E 08	0.0	-22.5	29606.	0.31
218.6	27306.5	0.19	248.0	6576.	-0.191E 02	0.173E 07	-0.161E 08	0.0	-22.5	30083.	0.29
227.3	27241.8	0.15	248.7	6811.	-0.205E 02	0.191E 07	-0.177E 08	0.0	-22.5	30548.	0.28
236.1	27179.2	0.12	249.3	7047.	-0.221E 02	0.212E 07	-0.196E 08	0.0	-22.5	31003.	0.28
244.8	27118.4	0.08	249.7	7281.	-0.238E 02	0.235E 07	-0.217E 08	0.0	-22.5	31451.	0.27
253.6	27058.8	0.04	249.9	7515.	-0.258E 02	0.260E 07	-0.240E 08	0.0	-22.5	31893.	0.27
262.3	27000.1	0.00	250.0	7749.	-0.279E 02	0.288E 07	-0.266E 08	0.0	-22.5	32330.	0.26

CONSTANTS

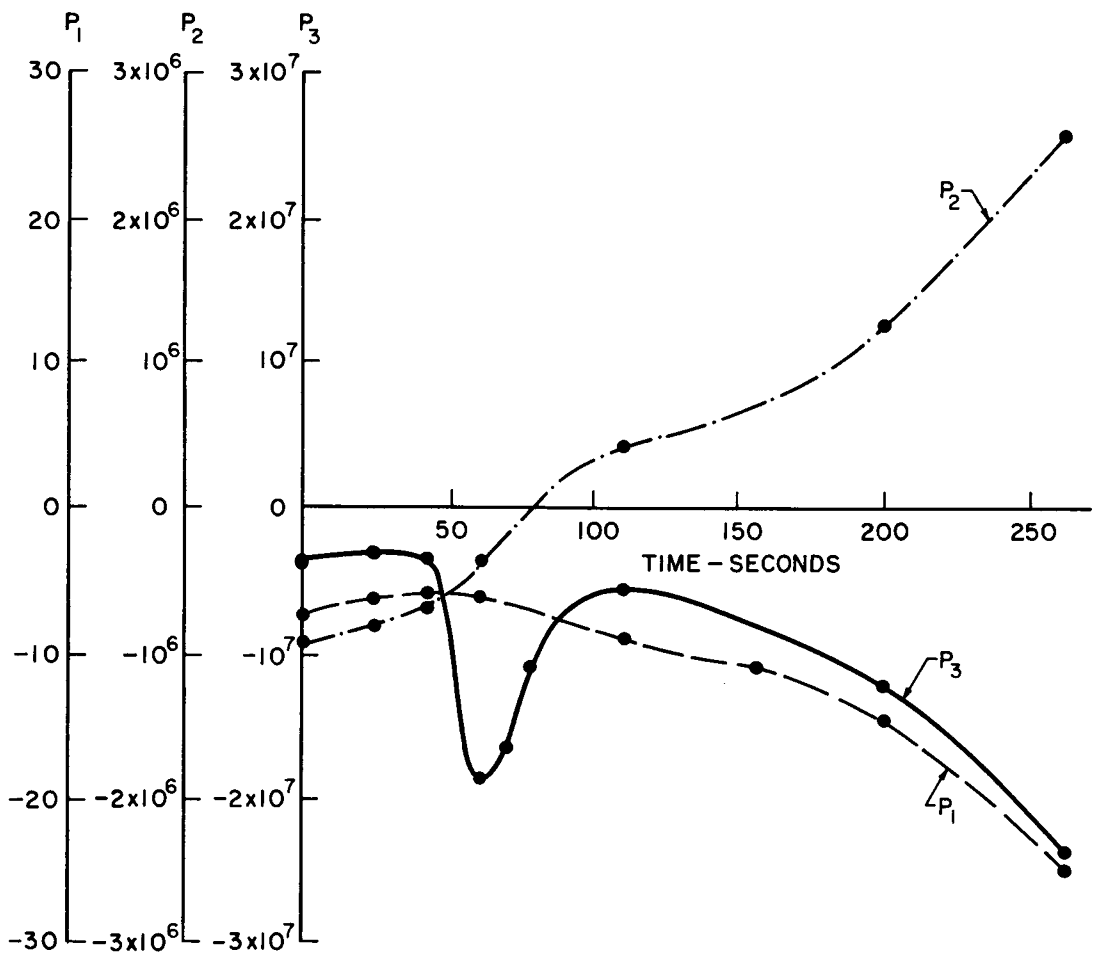
BETA 1/FT	RHO SLUG/FT3	G FT/SEC2	N FT	R FT	C	S FT2	M LB	CDO	CDL	CLO
0.42600E-04	0.27040E-02	32.1720	4.0000	0.20900E 08	0.2E-07	1.0640	2.0000	1.1740	-0.9000	0.6000



CASE 1

REENTRY VEHICLES
STATE VARIABLES VS TIME
BOUNDED CONTROL

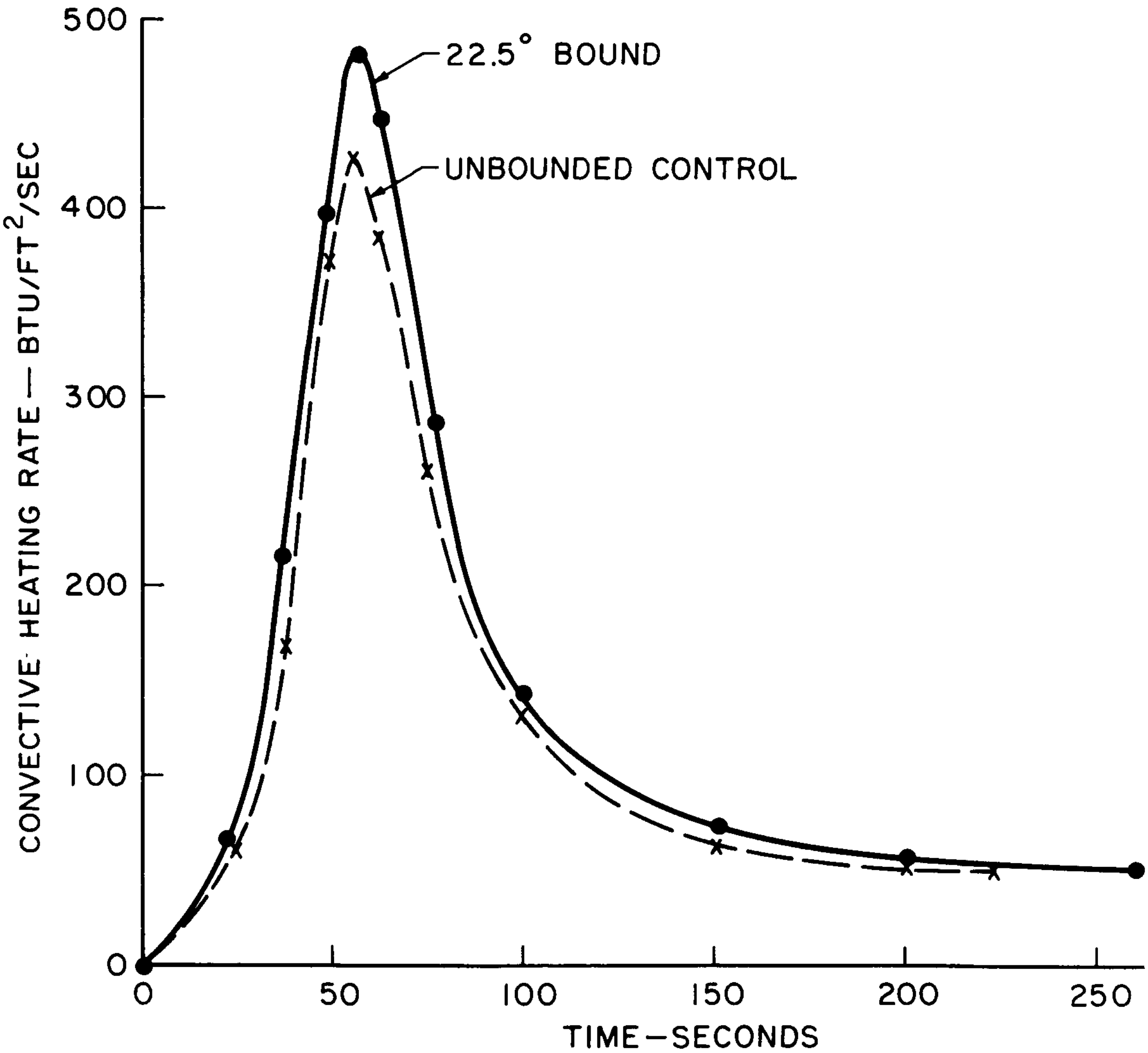
FIGURE 7.6



CASE 1

REENTRY VEHICLES
ADJOINT VARIABLES VS TIME
BOUNDED CONTROL

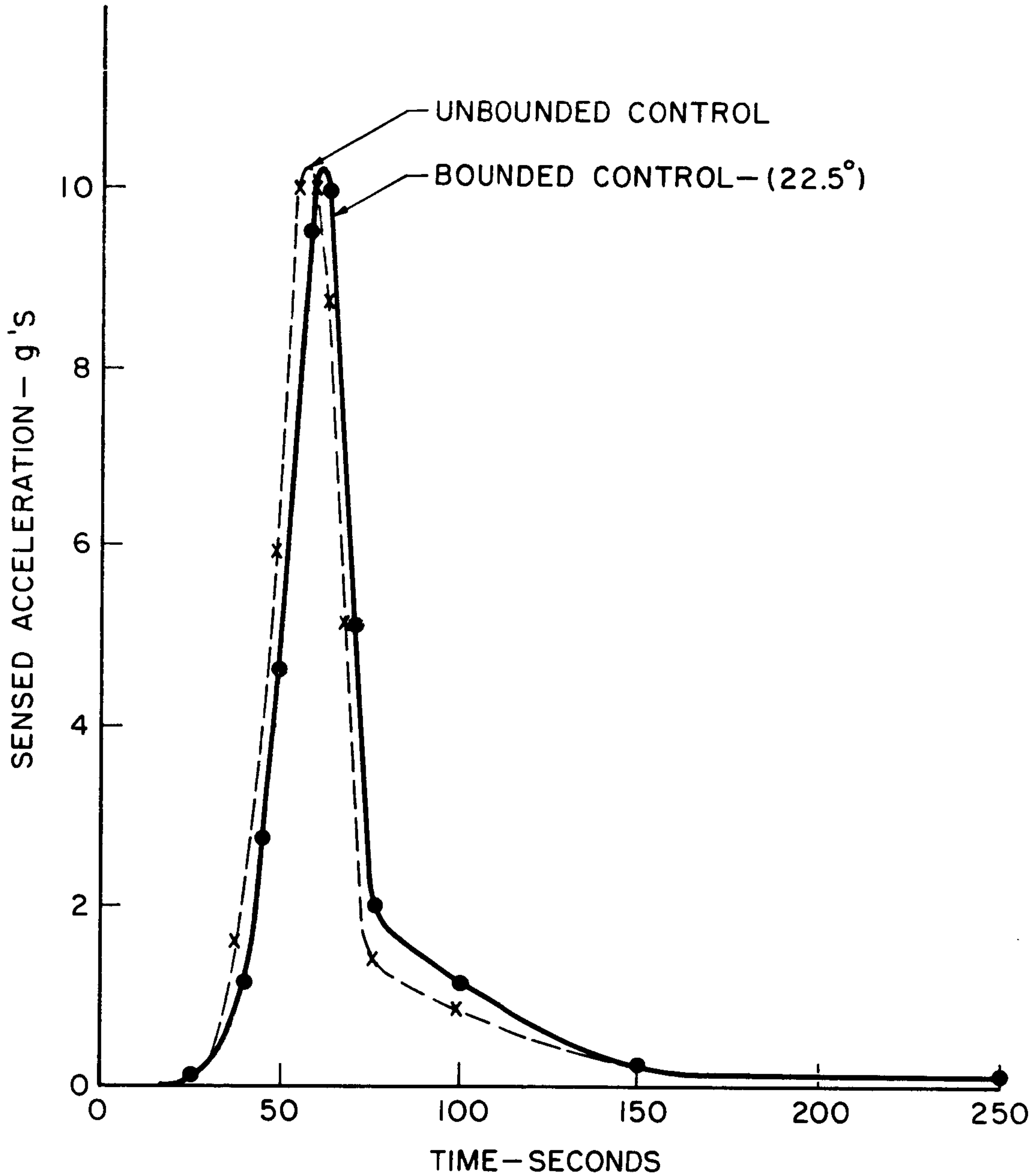
FIGURE 7.7



CASE 1

CONVECTIVE HEATING RATES
UNBOUNDED CONTROL VS 22.5° BOUND

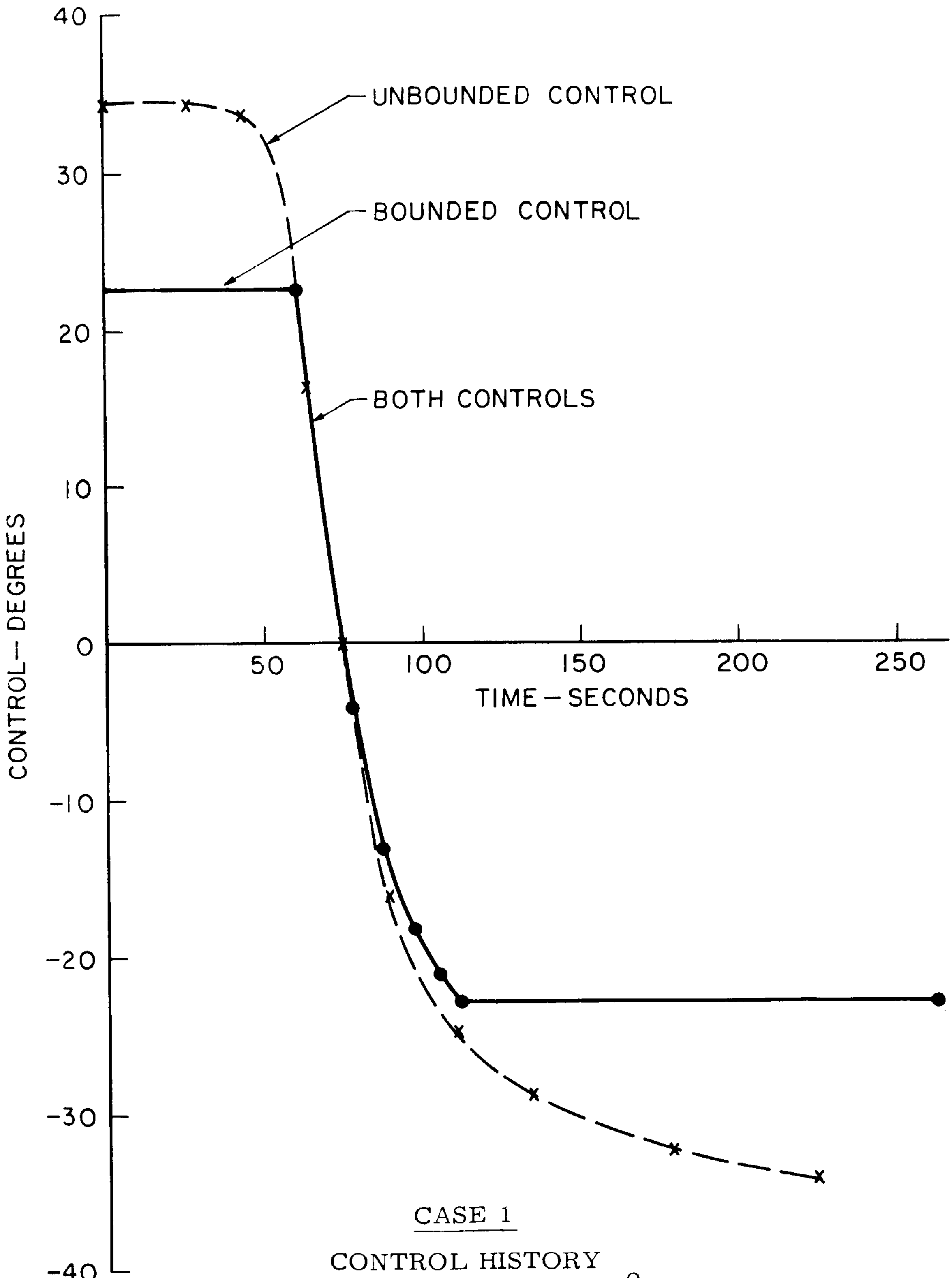
FIGURE 7.8



CASE 1

SENSED ACCELERATION
UNBOUNDED CONTROL VS 22.5° BOUND

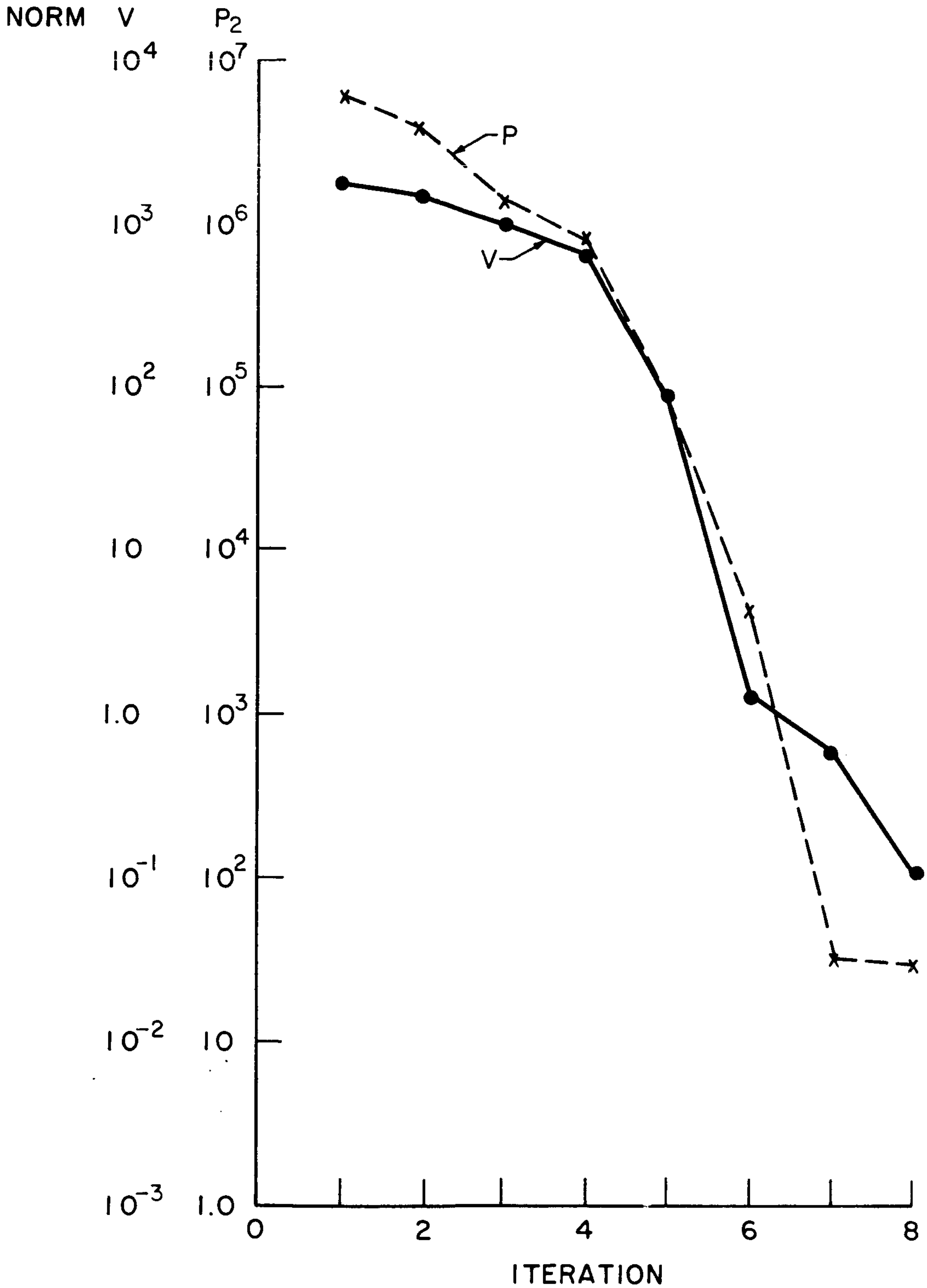
FIGURE 7.9



CASE 1

CONTROL HISTORY
UNBOUNDED CONTROL VS 22.5° BOUND

FIGURE 7.10



CASE 1

CONVERGENCE RATE FOR BOUNDED CONTROL

FIGURE 7.11

convergence rate to the bounded control case from an approximate solution generated by Runge-Kutta integration is shown in Figure 7.11.

The method outlined in Chapter 3 was employed to handle the bounds on the control rather than employing either a penalty function approach, or, Valentine's method. It can be seen by comparing Figures 7.3 and 7.11 that the use of this method does not slow down the rate of convergence.

Part of the reason for the small range of convergence becomes apparent when the transition matrix.is examined. The matrix is shown in Table 7.5. It can be seen that the rows (or columns) are very nearly multiples of each other. The ratio of the largest eigenvalue to the smallest is about 10^{14}. The eigenvalues for the transition matrices that arise is several different cases, are shown in Table 7.6. It is seen that for forward integration the ratio is worse (larger) than for backward integration. And that the bounded control case is worse than unbounded cases which explains the more severe convergence problems in the case of bounded control.

TABLE 7.5

TRANSITION MATRIX

Case 1 – Forward Integration

Unbounded Control

$$\begin{bmatrix} \frac{\partial v}{\partial p_1} & \frac{\partial v}{\partial p_2} & \frac{\partial v}{\partial p_3} & \frac{\partial v}{\partial c_t} \\ \frac{\partial \gamma}{\partial p_1} & \frac{\partial \gamma}{\partial p_2} & \frac{\partial \gamma}{\partial p_3} & \frac{\partial \gamma}{\partial c_t} \\ \frac{\partial \xi}{\partial p_1} & \frac{\partial \xi}{\partial p_2} & \frac{\partial \xi}{\partial p_2} & \frac{\partial \xi}{\partial c_t} \\ \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} & \frac{\partial H}{\partial p_3} & \frac{\partial H}{\partial c_t} \end{bmatrix}$$

OR

$$\begin{bmatrix} -0.194894\text{E } 05 & 0.260198\text{E-}00 & -0.219472\text{E-}01 & -0.256987\text{E } 04 \\ -0.616592\text{E } 00 & 0.819931\text{E-}05 & -0.680890\text{E-}06 & -0.369433\text{E-}01 \\ -0.404747\text{E-}01 & 0.537361\text{E-}06 & -0.443860\text{E-}07 & -0.197759\text{E-}03 \\ -0.404644\text{E } 01 & 0.768588\text{E-}03 & -0.226645\text{E-}03 & 0. \end{bmatrix}$$

TABLE 7.6

TRANSITION MATRIX EIGENVALUES

Conditions ↓ Eigenvalues →	λ_1	λ_2	λ_3	λ_4
Unbounded Control Forward Integration	-.1949E5	-.5336	.6612E-4	-.1533E-9
Unbounded Control Backward Integration	.1906E5	-.5265	-.6721E-4	.1562E-9
22.5° Control Bound Forward Integration	-.1796E5	-.9806	.3605E-4	-.1538E-10
22.5° Control Bound Backward Integration	.1825E5	-.3873	-.9724E-4	.1502E-10

CHAPTER 8

SUMMARY AND CONCLUSIONS

8.1 Summary of the Results

The proof in Chapter 3 demonstrates that the modified method for handling control directly will converge quadratically, under the specified assumptions, to the final solution. This allows the method of quasilinearization to be applied to problem involving bounded continuous control without having to turn to additional multipliers or a penalty function approach,

The method for extending the range of convergence of the method given in Chapter 4 has proven its worth many times over in the application of quasilinearization to practical problems. It is of particular importance since finding the initial guess can be a major part of getting a solution.

The Brachistochrone was used as a test problem to evaluate the effectiveness of Method 3 of Chapter 4 (see Figures 6.6 and 6.7); to show how the use of a numerical approximation to the partial derivatives effected the rate of convergence (see Figure 6.2) and for other tests of the theory. It can be seen from the mentioned figures that the method of Chapter 4 provides an effective way to extend the range of convergence and that numerically approximating the partial derivatives neither slows down the rate of convergence nor does it effect the terminal accuracy. The Brachistrochrone was also used to demonstrate that the method of an undetermined time scale is a far more efficient technique for solving the problem of an undetermined final time than that presented in the literature.

The method was applied to find the trajectory that minimizes the convective heating in a reentering space vehicle to see how the method behaved on a complex engineering problem. It was on this complex problem that the method of Chapter 3 was applied to find the trajectory for the case where the control was bounded as well as unbounded. The fact that there is no difference in the rates of convergence can be seen by comparing Figures 7.3 and 7.11, and proves that the method of Chapter 3 is indeed of practical importance for continuous control problems. Kopp and Moyer[11] stated that other ways of handling control bounds with QL are not straightforward which the method of Chapter 3 is.

The method of quasilinearization has been shown effective in the solution of complex engineering problems where the control cannot be eliminated from the problem itself.

8.2 Comments of the Method of Quasilinearization

The method of quasilinearization is a useful tool in solving the two point boundary value problem that grows out of most optimization techniques.

Despite its advantages of programming ease, computational speed, and rapid convergence,[11] it does suffer from two main defects. The first, it shares with all other quadratic methods and that is the small range of convergence. For most problems of significance the method will not converge, even with the method of Chapter 4, from a purely arbitrary initial guess on the values of the state and adjoint variables. The second difficulty arises out of and compounds the first. If the method does not converge the user of QL is no wiser than he was before. That is to say that because the method obeys only an approximate differential equation, the numbers obtained from a divergent solution give few clues as to what the next initial guess should be like.

Once a convergent solution is obtained it is very easy to vary a parameter and to see how the solution changes. Little difficulty has been found in staying in a rapidly convergent region while making studies of parameters, such as observing the effects of different control bounds in the reentry example.

8.3 Conclusions

Quasilinearization has been demonstrated here to be a powerful technique for solving engineering problems that are complex and sensitive at the same time by its ability to solve the reentry vehicle trajectory problem. Modifications to the method worked well in solving the same problem when bounds were placed on the control.

There are two main areas for future research on this method. The first is the extension of the proofs of Chapters 3 and 4 to first order ordinary nonlinear differential equations with boundary conditions at both ends. Without this step the theory cannot be regarded as complete as, while second order equations can be expressed as first order ones, the converse is not always true.

The second area is the extension of the method to handle jumps whether they be in the state variables, the adjoint variables, or the control. Without this extension the method cannot be applied conveniently to the class of problems where bounds are placed on the states.

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APPENDIX A

COMPLETE REENTRY EQUATIONS

Abbreviations:

$$C_D = C_{DO} + C_{DL} \cos u$$

$$C_L = C_{LO} \sin u$$

$$\rho = \rho_o e^{-\beta R \xi}$$

Equations of motion:

$$\dot{v} = -\frac{S\rho v^2}{2m} C_D - \frac{g \sin\gamma}{(1+\xi)^2}$$

$$\dot{\gamma} = \frac{S\rho v}{2m} C_L + \frac{v \cos\gamma}{R(1+\xi)} - \frac{g \cos\gamma}{v(1+\xi)^2}$$

$$\dot{\xi} = \frac{v \sin\gamma}{R}$$

$$\dot{\zeta} = \frac{v \cos\gamma}{1+\xi}$$

Hamiltonian:

$$H = cv^3 \left(\frac{\rho}{N}\right)^{\frac{1}{2}} + 7.5kN\rho^{3/2} \left(\frac{v}{10000}\right)^{12.5}$$

$$+ p_1 \left(-\frac{S\rho v^2}{2m} C_D - \frac{g \sin\gamma}{(1+\xi)^2}\right)$$

$$+ p_2 \left(\frac{S\rho v}{2m} C_L + \frac{v \cos\gamma}{R(1+\xi)} - \frac{g \cos\gamma}{v(1+\xi)^2}\right) + p_3 \frac{v \sin\gamma}{R}$$

$$+ p_4 \frac{v \cos\gamma}{(1+\xi)}$$

Criterion Function:

$k = 0$ for no radiative heating, $k = 1$ otherwise.

Adjoint variables:

$$\dot{p}_1 = f_5 = -\ 3cv^2 \left(\frac{\rho}{N}\right)^{\frac{1}{2}} - 93.75kN\rho^{3/2} \frac{v^{11.5}}{(10000)^{12.5}}$$

$$+ p_1 \frac{S\rho v}{m} C_D - p_2 \left(\frac{S\rho}{2m} C_L + \frac{\cos\gamma}{R(1+\xi)} + \frac{g\cos\gamma}{v^2(1+\xi)^2}\right)$$

$$- p_3 \frac{\sin\gamma}{R} - p_4 \frac{\cos\gamma}{1+\xi}$$

$$\dot{p}_2 = f_6 = + p_1 \frac{g\cos\gamma}{(1+\xi)^2} + p_2 \left(\frac{v\sin\gamma}{R(1+\xi)} + \frac{g\sin\gamma}{v(1+\xi)^2}\right)$$

$$- p_3 \frac{v\cos\gamma}{R} + p_4 \frac{v\sin\gamma}{1+\xi}$$

$$\dot{p}_3 = f_7 = -\frac{\beta R}{2}\left(cv^3\left(\frac{\rho}{N}\right)^{\frac{1}{2}} + 22.5Nk\rho^{3/2}\left(\frac{v}{10000}\right)^{12.5}\right)$$

$$- p_1\left(\frac{\beta RS\rho v^2}{2m} C_D + \frac{2g\sin\gamma}{(1+\xi)^3}\right)$$

$$+ p_2\left(\frac{\beta RS\rho v}{2m} C_L + \frac{v\cos\gamma}{R(1+\xi)^2} - \frac{2g\cos\gamma}{v(1+\xi)^3}\right)$$

$$+ p_4 \frac{v\cos\gamma}{(1+\xi)^2}$$

$$\dot{p}_4 = f_8 = 0$$

Control:

The control is found from

$$\frac{\partial H}{\partial u} = 0$$

and

$$\frac{\partial^2 H}{\partial u^2} < 0 \qquad \text{so} \qquad \tan u = -\frac{C_{DL}}{C_{LO}} \frac{p_2}{p_1 v}$$

Partial Derivatives

The following simplifications are employed

$$\frac{\partial f_i(y,u)}{\partial y_j} = \frac{\partial f_i}{\partial y_j} + \frac{\partial f_i}{\partial u} \cdot \frac{\partial u}{\partial y_j}$$

Now

$$\frac{\partial f_i}{\partial y_i} = 0 \qquad \begin{array}{l} i = 1,4 \\ j = 5,8 \end{array}$$

and

$$\frac{\partial f_i}{\partial y_j} = - \frac{\partial f_{j-4}}{\partial y_{i-4}} \qquad \begin{array}{l} i = 5,8 \\ j = 5,8 \end{array}$$

With

$$y = (y_1, \ldots, y_8) = (v, \gamma, \xi, \zeta, p_1, p_2, p_3, p_4)$$

$$\frac{\partial f_1}{\partial v} = - \frac{S\rho v}{m} C_D$$

$$\frac{\partial f_1}{\partial \gamma} = - \frac{g \cos\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_1}{\partial \xi} = \frac{\beta R S \rho v^2}{2m} C_D + \frac{2g \sin\gamma}{(1+\xi)^3}$$

$$\frac{\partial f_1}{\partial \zeta} = 0$$

$$\frac{\partial f_2}{\partial v} = \frac{S\rho}{2m} C_L + \frac{\cos\gamma}{R(1+\xi)} + \frac{g\cos\gamma}{v^2(1+\xi)^2}$$

$$\frac{\partial f_2}{\partial \gamma} = - \frac{v \sin\gamma}{R(1+\xi)} + \frac{g \sin\gamma}{v(1+\xi)^2}$$

$$\frac{\partial f_2}{\partial \xi} = -\frac{\beta R S \rho v}{2m} C_D - \frac{v \cos\gamma}{R(1+\xi)^2} + 2\frac{g \cos\gamma}{v(1+\xi)^3}$$

$$\frac{\partial f_2}{\partial \zeta} = 0$$

$$\frac{\partial f_3}{\partial v} = \frac{\sin\gamma}{R}$$

$$\frac{\partial f_3}{\partial \gamma} = \frac{v \cos\gamma}{R}$$

$$\frac{\partial f_3}{\partial \xi} = 0$$

$$\frac{\partial f_3}{\partial \zeta} = 0$$

$$\frac{\partial f_4}{\partial v} = \frac{\cos\gamma}{1+\xi}$$

$$\frac{\partial f_4}{\partial \gamma} = -\frac{v \sin\gamma}{1+\xi}$$

$$\frac{\partial f_4}{\partial \xi} = -\frac{v \cos\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_4}{\partial \zeta} = 0$$

$$\frac{\partial f_5}{\partial v} = -6cv\left(\frac{\rho}{N}\right)^{\frac{1}{2}} - 107.8125\ kN\rho^{3/2}\frac{v^{10.5}}{(10000)^{12.5}}$$

$$+ p_1 \frac{S\rho}{m} C_D + p_2 \frac{2g \cos\gamma}{v^3(1+\xi)^2}$$

$$\frac{\partial f_5}{\partial \gamma} = p_2\left(\frac{\sin\gamma}{R(1+\xi)} + \frac{2g \sin\gamma}{v^2(1+\xi)^2}\right) + p_3 \frac{\cos\gamma}{R}$$

$$\frac{\partial f_5}{\partial \xi} = + 1.5\,\beta R\left(cv^2 \left(\frac{\rho}{N}\right)^{\frac{1}{2}} + 93.75\ kN\rho^{3/2}\ \frac{v^{11.5}}{(10000)^{12.5}}\right)$$

$$- p_1 \frac{\beta RS\rho v}{m}\ C_D + p_2\left(\frac{\beta RS\rho}{2m}\ C_L + \frac{\cos\gamma}{R(1+\xi)}\right.$$

$$\left. + \frac{2g\cos\gamma}{v^2(1+\xi)^3}\right) + p_4\ \frac{\cos\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_5}{\partial \zeta} = 0$$

$$\frac{\partial f_6}{\partial v} = p_2\left(\frac{\sin\gamma}{R(1+\xi)} + \frac{2g\sin\gamma}{v^2(1+\xi)^2}\right) - p_3\ \frac{\cos\gamma}{R}$$

$$\frac{\partial f_6}{\partial \gamma} = - p_1\ \frac{g\sin\gamma}{(1+\xi)^2} + p_2\left(\frac{v\cos\gamma}{R(1+\xi)} - \frac{g\cos\gamma}{v(1+\xi)^2}\right)$$

$$+ p_3\ \frac{v\sin\gamma}{R} + p_4\ \frac{v\cos\gamma}{1+\xi}$$

$$\frac{\partial f_6}{\partial \xi} = - p_1\ \frac{2g\cos\gamma}{(1+\xi)^3} - p_2\left(\frac{v\sin\gamma}{R(1+\xi)^2} - \frac{2g\sin\gamma}{v(1+\xi)^3}\right) - p_4\,\frac{\sin\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_6}{\partial \zeta} = 0$$

$$\frac{\partial f_7}{\partial v} = \frac{3}{2}\,\beta R\left(cv^2 \left(\frac{\rho}{N}\right)^{\frac{1}{2}} + 93.75\ kN\rho^{3/2}\ \frac{v^{11.5}}{(10000)^{12.5}}\right)$$

$$- p_1\ \frac{\beta RS\rho v}{m}\ C_D + p_2\left(\frac{\beta RS\rho}{2m}\ C_L + \frac{\cos}{R(1+\xi)}\right.$$

$$\left. - \frac{2g\cos\gamma}{v^2(1+\xi)^3}\right) + p_4\ \frac{\cos\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_7}{\partial \gamma} = - p_1 \frac{2g \cos\gamma}{(1+\xi)^3} - p_2\left(\frac{v \sin\gamma}{R(1+\xi)^2} - \frac{2g \sin\gamma}{v(1+\xi)^3}\right) - p_4 \frac{v \sin\gamma}{(1+\xi)^2}$$

$$\frac{\partial f_7}{\partial \xi} = - \frac{(\beta R)^2}{4}\left[cv^3\left(\frac{\rho}{N}\right)^{\frac{1}{2}} + 67.5\ kN\rho^{3/2}\left(\frac{v}{10000}\right)^{12.5}\right]$$

$$+ p_1\left(\frac{(\beta R)^2 S\rho v^2}{2m} C_D + \frac{6g \sin\gamma}{(1+\xi)^4}\right)$$

$$- p_2\left(\frac{(\beta R)^2 S\rho v}{2m} C_L + \frac{2v \cos\gamma}{R(1+\xi)^3} - \frac{6g \cos\gamma}{v(1+\xi)^4}\right)$$

$$- p_4 \frac{2v \cos\gamma}{(1+\xi)^3}$$

$$\frac{\partial f_7}{\partial \zeta} = 0$$

$$\frac{\partial f_8}{\partial v} = \frac{\partial f_8}{\partial \gamma} = \frac{\partial f_8}{\partial \xi} = \frac{\partial f_8}{\partial \zeta} = 0$$

$$\frac{\partial f_1}{\partial u} = \frac{S\rho v^2}{2m} C_{DL} \sin u$$

$$\frac{\partial f_2}{\partial u} = \frac{S\rho v}{2m} C_{LO} \cos u$$

$$\frac{\partial f_3}{\partial u} = 0$$

$$\frac{\partial f_4}{\partial u} = 0$$

$$\frac{\partial f_5}{\partial u} = -p_1 \frac{S\rho v}{m} C_{DL} \sin u - p_2 \frac{S\rho}{2m} C_{LO} \cos u$$

$$\frac{\partial f_6}{\partial u} = 0$$

$$\frac{\partial f_7}{\partial u} = p_1 \frac{\beta R S \rho v^2}{2m} C_{DL} \sin u + p_2 \frac{\beta R S \rho v}{2m} C_{LO} \cos u$$

$$\frac{\partial f_8}{\partial u} = 0$$

$$\frac{\partial u}{\partial v} = - \frac{\sin u \cos u}{v}$$

$$\frac{\partial u}{\partial \gamma} = 0$$

$$\frac{\partial u}{\partial \xi} = 0$$

$$\frac{\partial u}{\partial \zeta} = 0$$

$$\frac{\partial u}{\partial p_1} = - \frac{\sin u \cos u}{p_1}$$

$$\frac{\partial u}{\partial p_2} = \frac{\sin u \cos u}{p_2}$$

$$\frac{\partial u}{\partial p_3} = 0$$

$$\frac{\partial u}{\partial p_4} = 0$$

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APPENDIX B

REENTRY PROBLEM CONSTANTS

Constant		Case 1	Case 2	Units
Gravitational constant	g_o =	32.172	32.172	ft/sec^2
Air density at sea level	ρ_o =	$.2704 \times 10^{-2}$	0.23769×10^{-2}	$slug/ft^3$
Air density exponential constant	β =	0.426×10^{-4}	0.4255×10^{-4}	1/ft
Earth Radius	R =	20.9×10^6	20.035	ft
Frontal area of vehicle	S =	1.064	1.0	ft^2
Mass of vehicle	m =	2.0	2.0	slug
Radius of vehicle nose	N =	4.0	4.0	ft
Convective heating	c =	0.2×10^{-7}	0.2×10^{-7}	$\frac{BTU\ sec}{ft^3\ lb^{\frac{1}{2}}}$
Radiative heating	c^1 =	0	7.5×10^{-50}	$\frac{BTU\ sec^{11.5}}{ft^{15.5}}$
Control Constants				
	C_{DO}	1.174	0.88	
	C_{DL}	-0.9	0.52	
	C_{LO}	0.6	-0.505	