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Tests of Hypotheses and Estimation of the Correlation Coefficient Using Six and Eight Quantiles

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Abstract

This report extends previous results obtained from the investigation into the use of quantiles for data compression of space telemetry. Tests of hypotheses are given, using six and eight optimum sample quantiles. Tests A and $\bar{\text{A}}$ test the mean of a normal population. In Test A the variance is assumed to be known and in Test $\bar{\text{A}}$ it is assumed to be unknown. Test B tests the variance of a normal population when the mean is unknown. Tests D and $\bar{\text{D}}$ decide whether the unknown means of two independent normal populations are identical under different assumptions on the values of the parameters. Test $\bar{\text{E}}$ decides whether the unknown variances of two independent normal populations are identical when their common mean is unknown. Tests F and $\bar{\text{F}}$ decide whether or not two normal populations are independent. In addition, estimators of the correlation coefficient are constructed. Suboptimum test statistics and estimators are also given. In all cases, the sample sizes are assumed to be large.

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I. Introduction

Data compression of space telemetry is desirable because there is a limit to the total amount of information that can be transmitted through a communications channel in a given time and, therefore, a limit to the number of experiments that can be performed simultaneously aboard a spacecraft for given sample sizes if all the observations are to be transmitted back to earth in that same time. The aim of a data compression system is to transmit only "useful" information, discarding the remainder of the data. If the data compression ratio (the ratio of the number of observations taken to the number transmitted) is high enough, it will then be possible to perform additional experiments with a relatively small increase in cost.

The criterion used to determine whether information is useful or not usually depends upon the type of information desired. For example, given an initial observation, the next useful bit of information might be the first subsequent observation which differs from the initial one by more than some previously prescribed amount. If the quantity under observation is changing slowly, only a small fraction of the total number of observations would be defined, under this criterion, as being useful. In other words, one may be interested primarily in sufficiently large changes in the observations, rather than in the observations themselves, and a data compression scheme that chooses and transmits only those observations which indicate these changes may achieve large compression ratios with little or no loss of useful information. On the other hand, if one loses, say, $1/n$ of the useful information contained in n observations by deleting any one of them, any attempt to achieve data compression in this case is useless.

In those instances where the data are to be used to draw statistical conclusions from a *histogram*, it may be possible to achieve significant amounts of data compression with only a small loss of useful information by transmitting a small number of sample *quantiles* instead of all the sample values when the sample size is large. The sample quantiles can then be used to perform the identical statistical analyses for which the histogram was originally intended.

The uncertainty that invariably accompanies statistical conclusions usually decreases as the sample size increases when standard statistical techniques are used. Because the variances of sample quantiles are, asymptotically, inversely proportional to the sample size n, the same reduction in uncertainty, when n is increased, follows from the use of these *order statistics* as from the use of nonordered ones. Thus, the principal advantage of a large sample size is not sacrificed by this form of data compression. Consequently, an investigation into the use of sample quantiles to achieve data compression of space telemetry has been in effect for several years at the Jet Propulsion Laboratory (JPL) and is still continuing.

Previous results of this investigation are given in three JPL Technical Reports, Refs. 1, 2, and 3. Reference 1 deals with the problem of efficiently estimating the parameters of a normal distribution, using up to 20 quantiles, and also describes two goodness-of-fit tests, each using four quantiles. References 2 and 3 are concerned with hypothesis testing and the estimation of the correlation coefficient of a bivariate normal distribution, using up to four sample quantiles. The present report extends most of the results derived in Refs. 2 and 3 to six and eight quantiles.

For comparison purposes, the test designations here will be the same as those in Refs. 1–3.

In Tests A and $\bar{A}$, it is assumed that we are given n independent observations from a normal population; the tests are designed to decide whether the mean, μ, has a value of μ_1 or μ_2. In Test A, the variance, σ^2, is assumed to be known, while in Test $\bar{A}$, no such assumption is made.

In Test B, we test whether σ has a value of σ_1 and σ_2. When an even number of quantiles are used, it is not necessary to assume that μ is known.

In Tests D, $\bar{D}$, and $\bar{E}$, it is assumed that we are given sets of independent sample values taken from two independent, normally distributed populations, with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. In Test D, it is assumed that $\sigma = \sigma_1 = \sigma_2$ is known and μ_1 is unknown, and we test whether $\mu_2 = \mu_1$ or $\mu_2 = \mu_1 + \theta$, $\theta \neq 0$. In Test $\bar{D}$, the assumption that σ is known is not used. In Test $\bar{E}$, it is assumed that $\mu = \mu_1 = \mu_2$ is unknown, σ_1 is unknown, and we test whether $\sigma_2 = \sigma_1$ or $\sigma_2 = \theta\sigma_1$, $\theta > 0$.

In Tests F and $\bar{F}$, we are given n independent pairs of observations taken from two normally distributed populations. In Test F, we assume that μ_1, μ_2, σ_1, and σ_2 are known and test whether $\rho = 0$ or $\rho \neq 0$. In Test $\bar{F}$, we assume that both $\mu = \mu_1 = \mu_2$ and $\sigma = \sigma_1 = \sigma_2$ are unknown and again test whether $\rho = 0$ or $\rho \neq 0$.

In estimating ρ, it will first be assumed that the conditions of Test F hold. This estimator will be denoted by $\hat{\rho}_1$. For the second estimator, $\hat{\rho}_2$, it will be assumed that $\mu = \mu_1 = \mu_2$ is unknown and that σ_1 and σ_2 are known.

Table 1 summarizes the hypotheses and assumptions above. The statement "$g(x) = N(\mu, \sigma)$" will mean that the random variable under consideration is normally distributed with mean μ and variance σ^2 and has the density function $g(x)$ associated with it.

The power functions P_0 of the quantile tests are derived, and the power function P_0' of the best test using all the sample values will also be given. The efficiencies of the quantile tests, defined as P_0/P_0', are determined. The efficiencies, $\mathrm{var}(r)/\mathrm{var}(\hat{\rho}_1)$ and $\mathrm{var}(r)/\mathrm{var}(\hat{\rho}_2)$ of $\hat{\rho}_1$ and $\hat{\rho}_2$, respectively, are also determined for the special case $\rho = 0$, where r is the sample correlation coefficient.

Test A_i will denote Test A using i quantiles, Test $\bar{A}_i$ will denote Test $\bar{A}$ using i quantiles, and so on. In all cases, the sample sizes are assumed to be large ($\geqq 200$).

The efficiency of a test is a measure of the loss of information that results from applying the test using a test statistic other than the one that maximizes the power of the test. For each of the tests discussed in this report,

Table 1. Hypotheses and assumptions relating to the tests, and assumptions relating to estimating ρ_1 and ρ_2

Test	Null hypothesis	Alternative hypothesis	Assumptions
A $\bar{A}$	$g(x) = N(\mu_1, \sigma)$	$g(x) = N(\mu_2, \sigma)$	σ known σ unknown
B	$g(x) = N(\mu, \sigma_1)$	$g(x) = N(\mu, \sigma_2)$	μ unknown
D $\bar{D}$	$g_1(x) = N(\mu, \sigma)$ $g_2(y) = N(\mu, \sigma)$	$g_1(x) = N(\mu, \sigma)$ $g_2(y) = N(\mu + \theta, \sigma)$ $\theta \neq 0$	x and y independent; σ known, μ unknown x and y independent; μ and σ unknown
$\bar{E}$	$g_1(x) = N(\mu, \sigma)$ $g_2(y) = N(\mu, \sigma)$	$g_1(x) = N(\mu, \sigma)$ $g_2(y) = N(\mu, \theta\sigma)$ $\theta > 0$	x and y independent; μ and σ unknown
F $\bar{F}$	$g_1(x) = N(\mu_1, \sigma_1)$ $g_2(y) = N(\mu_2, \sigma_2)$ $\rho = 0$	$g_1(x) = N(\mu_1, \sigma_1)$ $g_2(y) = N(\mu_2, \sigma_2)$ $\rho \neq 0$	$\mu_1, \mu_2, \sigma_1, \sigma_2$ known $\mu = \mu_1 = \mu_2$ and $\sigma = \sigma_1 = \sigma_2$ unknown
Estimating $\hat{\rho}_1$			$\mu_1, \mu_2, \sigma_1, \sigma_2$, known
Estimating $\hat{\rho}_2$			σ_1 and σ_2 known $\mu_1 = \mu_2 = \mu$ unknown

it is well known that the test statistic that maximizes the power of the test is a function of all the sample values. By using test statistics that are, instead, functions of a small number of sample quantiles, a large data compression ratio naturally results when the sample size is large. However, the important question still remains as to how much information is lost by this substitution. The following is a summary of the minimum efficiencies of each test for a sample size of 200 and a significance level of 0.01. Details are given in the appropriate tables.

The efficiencies of Tests A and $\bar{A}$ depend upon $\mu_2 - \mu_1/\sigma$. The minimum efficiencies for Test A are 0.955 using six quantiles and 0.971 using eight quantiles. The minimum efficiencies for Test $\bar{A}$ are 0.938 using six quantiles and 0.955 using eight quantiles.

The efficiencies of Test B depend upon σ_2/σ_1. The minimum efficiencies for Test B are 0.875 using six quantiles and 0.907 using eight quantiles.

The efficiencies of Tests D and $\bar{D}$ depend upon θ/σ. The minimum efficiencies for Test D are 0.954 using six quantiles and 0.971 using eight quantiles. The minimum efficiencies for Test $\bar{D}$ are 0.946 using six quantiles and 0.963 using eight quantiles.

The efficiencies of Test $\bar{\bar{E}}$ depend upon θ, and the minimum efficiencies are 0.873 using six quantiles and 0.901 using eight quantiles.

The efficiencies of Tests F and $\bar{F}$ depend upon ρ. The minimum efficiencies for Test F are 0.920 using six quantiles and 0.949 using eight quantiles. The minimum efficiencies for Test $\bar{F}$ are 0.905 using six quantiles and 0.933 using eight quantiles.

The efficiencies of $\hat{\rho}_1$ are 0.869 using six quantiles and 0.895 using eight quantiles, while the efficiencies of $\hat{\rho}_2$ are 0.862 using six quantiles and 0.886 using eight quantiles.

The summary above is a clear indication that the high data compression ratios that can be achieved by using six and eight quantiles instead of all the sample values for the tests and estimators of ρ are not accompanied by an excessive loss in information.

II. Review of Quantiles

To define a quantile, consider a sample of n independent sample values, $x_1, x_2, \cdots, x_n$, taken from a distribution of a continuous type with distribution function $G(x)$ and density function $g(x)$. The pth quantile, or the quantile of order p of the distribution or population, denoted by ζ_p^*, is defined as the root of the equation $G(\zeta_p^*) = p$; that is,

$$p = \int_{-\infty}^{\zeta_p^*} dG(x) = \int_{-\infty}^{\zeta_p^*} g(x)\,dx$$

The corresponding *sample* quantile z_p is defined as follows: If the sample values are arranged in nondecreasing order of magnitude

$$x_{(1)} \leqq x_{(2)} \leqq \cdots \leqq x_{(n)}$$

then $x_{(i)}$ is called the ith order statistic and

$$z_p = x_{([np]+1)}$$

where $[np]$ is the greatest integer $\leqq np$.

If $g(x)$ is differentiable in some neighborhood of each quantile considered, it has been shown (Ref. 4) that the joint distribution of any number of quantiles is asymptotically normal as $n \to \infty$ and that, asymptotically,

$$E(z_p) = \zeta_p^*$$

$$\mathrm{var}(z_p) = \frac{p(1-p)}{ng^2(\zeta_p^*)}$$

$$\rho_{12} = \left[\frac{p_1(1-p_2)}{p_2(1-p_1)}\right]^{1/2}$$

where ρ_{12} is the correlation between z_{p_1} and z_{p_2} and where $p_1 < p_2$.

Throughout this report we will denote by $F(x)$ and $f(x) = F'(x)$ the distribution function and density function, respectively, of the standard normal distribution; that is,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

where

$$f(x) = \frac{1}{(2\pi)^{1/2}} \exp(-1/2 x^2)$$

Denoting by ζ_p the pth quantile of the standard normal distribution, one has

$$p = \int_{-\infty}^{\zeta^*} g(x)\,dx = \int_{-\infty}^{(\zeta_p^* - \mu)/\sigma} f(x)\,dx = \int_{-\infty}^{\zeta_p} f(x)\,dx$$

Hence, one sees that, asymptotically,

$$\mathrm{E}(z_p) = \zeta_p^* = \sigma\zeta_p + \mu$$

and, since $g(\zeta_p^*) = f(\zeta_p)/\sigma$

$$\mathrm{var}(z_p) = \frac{\sigma^2 F(\zeta_p)[1 - F(\zeta_p)]}{nf^2(\zeta_p)}$$

so that the moments of the sample quantiles of normal distributions are expressible in terms of the standard normal distribution. When m quantiles are being considered, the sample quantiles will be denoted as z_i of order p_i, $i = 1, 2, \cdots, m$, and $p_i < p_j$ for $i < j$. Here ζ_i will denote the corresponding population quantiles of the standard normal. Since n is assumed to be large, the statistical analyses to be given in the sequel will be based on the asymptotic normal distribution of the sample quantiles.

III. Tests A and $\bar{\mathrm{A}}$: Testing the Mean of a Normal Distribution Using Six and Eight Quantiles

A. Test A_6

We test here the simple null hypothesis

$$\mathrm{H}_0: g(x) = g_1(x) = N(\mu_1, \sigma)$$

against the simple alternative hypothesis

$$\mathrm{H}_1: g(x) = g_2(x) = N(\mu_2, \sigma)$$

where $\mu_2 > \mu_1$ ($\mu_2 < \mu_1$).

Let z_i, $i = 1, 2, \cdots, 6$, denote six sample quantiles such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$. Using these six quantiles for the test, it is easy to deduce from previous results obtained using one and two pairs of symmetric quantiles that the best critical (or rejection) region is that for which

$$\begin{aligned} y_6 = \alpha_1(z_1 + z_6) + \alpha_2(z_2 + z_5) \\ + \alpha_3(z_3 + z_4) \gtrless k, \qquad \mu_2 \gtrless \mu_1 \end{aligned}$$

where $2(\alpha_1 + \alpha_2 + \alpha_3) = 1$. We determine k such that, under H_0, $\mathrm{pr}(y \gtrless k) = \varepsilon$, the significance level of the test. Under H_0,

$$\begin{aligned} \mathrm{E}(y_6) &= 2\mu_1(\alpha_1 + \alpha_2 + \alpha_3) = \mu_1 \\ \mathrm{var}(y_6) &= \frac{2\sigma^2}{n}[\alpha_1^2 a_6^2(1 + \rho_{16}) + \alpha_2^2 a_5^2(1 + \rho_{25}) + \alpha_3^2 a_4^2(1 + \rho_{34}) \\ &\quad + 2\alpha_1\alpha_2 a_5 a_6(\rho_{12} + \rho_{15}) + 2\alpha_1\alpha_3 a_4 a_6(\rho_{13} + \rho_{14}) + 2\alpha_2\alpha_3 a_4 a_5(\rho_{23} + \rho_{24})] \\ &= \frac{2\sigma^2}{n}\gamma_1^2 \end{aligned}$$

where

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{f^2(\zeta_i)}, \qquad i = 4, 5, 6$$

and ρ_{ij} denotes the correlation between z_i and z_j. Under H_1,

$$\mathrm{E}(y_6) = \mu_2$$

$$\mathrm{var}(y_6) = \frac{2\sigma^2}{n}\gamma_1^2$$

To determine the value of k, one has, for $\mu_2 > \mu_1$, under H_0,

$$\mathrm{pr}(y_6 < k) = F\left[\frac{k - \mu_1}{\left(\frac{2}{n}\right)^{1/2}\sigma\gamma_1}\right] = F(b) = 1 - \varepsilon$$

Therefore,

$$k = \left(\frac{2}{n}\right)^{1/2}\sigma\gamma_1 b + \mu_1$$

The power function P_0 of the test is determined as follows: under H_1,

$$\mathrm{pr}\,(y_6 < k) = F\left[\frac{k-\mu_2}{\left(\frac{2}{n}\right)^{1/2}\sigma\gamma_1}\right] = F\left(b - \frac{n^{1/2}}{2^{1/2}\gamma_1}\,\frac{\mu_2-\mu_1}{\sigma}\right)$$

$$= 1 - P_0 \tag{1}$$

From Eq. (1) it can be seen that in order to maximize P_0, the values of the α_i (subject to the condition $2\sum_{i=1}^{3}\alpha_i = 1$) and the orders of the quantiles should be chosen so as to minimize γ_1. From the results given in Ref. 1 relating to parameter estimation, these values are

$p_1 = 0.0540$	$p_6 = 0.9460$	$\alpha_1 = 0.0968$
$p_2 = 0.1915$	$p_5 = 0.8085$	$\alpha_2 = 0.1787$
$p_3 = 0.3898$	$p_4 = 0.6102$	$\alpha_3 = 0.2245$

Using these values, one has

$$k = \mu_1 + \frac{1.0228b\sigma}{n^{1/2}}$$

$$P_0 = 1 - F\left(b - 0.9778\,n^{1/2}\frac{\mu_2-\mu_1}{\sigma}\right)$$

For $\mu_2 < \mu_1$,

$$k = \mu_1 - \frac{1.0228b\sigma}{n^{1/2}}$$

Therefore, Test A_6 can now be stated as follows: if

$$\begin{aligned} y_6 &= 0.0968\,[z(0.0540) + z(0.9460)] \\ &\quad + 0.1787\,[z(0.1915) + z(0.8085)] \\ &\quad + 0.2245\,[z(0.3898) + z(0.6102)] \\ &\lessgtr \mu_1 \pm \frac{1.0228b\sigma}{n^{1/2}}, \qquad \mu_2 \gtrless \mu_1 \end{aligned} \tag{2}$$

accept H_0. Otherwise, reject H_0. The decision will be made at a significance level of $\varepsilon = 1 - F(b)$.

B. Test $\bar{A}_6$

In this test we are assuming that σ is unknown. Hence, a rejection region of the form defined by Ineq. (2) cannot be used because of the dependence on σ. However, since an estimate of σ can be obtained using six quantiles of the form $\hat{\sigma} = c\,(z_6 - z_1 + z_5 - z_2 + z_4 - z_3)$, we can substitute $\hat{\sigma}$ for σ in Ineq. (2), which results, for $\mu_2 > \mu_1$, in a rejection region of the form

$$\begin{aligned} \bar{y}_6 &= (\alpha_1 + \alpha)\,z_1 + (\alpha_1 - \alpha)\,z_6 + (\alpha_2 + \alpha)\,z_2 + (\alpha_2 - \alpha)\,z_5 \\ &\quad + (\alpha_3 + \alpha)\,z_3 + (\alpha_3 - \alpha)\,z_4 > \mu_1 \end{aligned} \tag{3}$$

where α must be determined such that the probability of Ineq. (3) occurring is equal to ε when σ is unknown. Under H_0,

$$\begin{aligned} \mathrm{E}\,(\bar{y}_6) &= (\alpha_1 + \alpha)(\mu_1 - \sigma\zeta_6) + (\alpha_1 - \alpha)(\mu_1 + \sigma\zeta_6) \\ &\quad + (\alpha_2 + \alpha)(\mu_1 - \sigma\zeta_5) + (\alpha_2 - \alpha)(\mu_1 + \sigma\zeta_5) \\ &\quad + (\alpha_3 + \alpha)(\mu_1 - \sigma\zeta_4) + (\alpha_3 - \alpha)(\mu_1 + \sigma\zeta_4) \\ &= \mu_1 - 2\alpha\sigma\,(\zeta_4 + \zeta_5 + \zeta_6) \end{aligned}$$

$$\begin{aligned} \mathrm{var}\,(\bar{y}_6) &= \frac{2\sigma^2}{n}\{a_6^2\,[\alpha_1^2(1+\rho_{16}) + \alpha^2(1-\rho_{16})] + a_5^2\,[\alpha_2^2(1+\rho_{25}) + \alpha^2(1-\rho_{25})] \\ &\quad + a_4^2\,[\alpha_3^2(1+\rho_{34}) + \alpha^2(1-\rho_{34})] + 2a_5a_6\,[(\alpha_1\alpha_2 + \alpha^2)\,\rho_{12} + (\alpha_1\alpha_2 - \alpha^2)\,\rho_{15}] \\ &\quad + 2a_4a_6\,[(\alpha_1\alpha_3 + \alpha^2)\,\rho_{13} + (\alpha_1\alpha_3 - \alpha^2)\,\rho_{14}] + 2a_4a_5\,[(\alpha_2\alpha_3 + \alpha^2)\,\rho_{23} + (\alpha_2\alpha_3 - \alpha^2)\,\rho_{24}]\} \\ &= \frac{2\sigma^2}{n}\gamma_2^2 \end{aligned}$$

where

$$a_i^2 = \frac{F(\zeta_i)\,[1 - F(\zeta_i)]}{f^2(\zeta_i)}, \qquad i = 4, 5, 6$$

and ρ_{ij} denotes the correlation between z_i and z_j;

$$\mathrm{pr}(\bar{y}_6 < \mu_1) = F\left[\frac{\mu_1 - \mu_1 + 2\alpha\sigma(\zeta_4 + \zeta_5 + \zeta_6)}{\left(\frac{2}{n}\right)^{1/2}\sigma\gamma_2}\right] = F\left[\frac{(2n)^{1/2}\alpha(\zeta_4 + \zeta_5 + \zeta_6)}{\gamma_2}\right] = F(b) = 1 - \varepsilon$$

It is thus seen that the probability of Ineq. (3) occurring is indeed independent of σ, and the orders of the sample quantiles are subject only to the restriction that

$$\alpha^2 = \frac{b^2\left[\alpha_1^2 a_6^2(1+\rho_{16}) + \alpha_2^2 a_5^2(1+\rho_{25}) + \alpha_3^2 a_4^2(1+\rho_{34}) + 2\alpha_1\alpha_2 a_5 a_6(\rho_{12}+\rho_{15}) + 2\alpha_1\alpha_3 a_4 a_6(\rho_{13}+\rho_{14}) + 2\alpha_2\alpha_3 a_4 a_5(\rho_{23}+\rho_{24})\right]}{2n(\zeta_4+\zeta_5+\zeta_6)^2 - b^2\left[a_6^2(1-\rho_{16}) + a_5^2(1-\rho_{25}) + a_4^2(1-\rho_{34}) + 2a_5a_6(\rho_{12}-\rho_{15}) + 2a_4a_6(\rho_{13}-\rho_{14}) + 2a_4a_5(\rho_{23}-\rho_{24})\right]} > 0$$

To determine P_0, one has, under H_1,

$$\mathrm{E}(\bar{y}_6) = \mu_2 - 2\alpha\sigma(\zeta_4 + \zeta_5 + \zeta_6)$$

$$\mathrm{var}(\bar{y}_6) = \frac{2\sigma^2}{n}\gamma_2^2$$

$$\mathrm{pr}(\bar{y}_6 < \mu_1) = F\left[\frac{\mu_1 - \mu_2 + 2\alpha\sigma(\zeta_4 + \zeta_5 + \zeta_6)}{\left(\frac{2}{n}\right)^{1/2}\sigma\gamma_2}\right] = F\left(b - \frac{n^{1/2}}{2^{1/2}\gamma_2}\frac{\mu_2 - \mu_1}{\sigma}\right) = 1 - P_0$$

Although if one uses in Test $\bar{\mathrm{A}}_6$ the quantiles and values of the α_i which maximize the power of Test A_6 the result will not be strictly optimum, it is shown in Ref. 3 that when this procedure was adopted in the two-quantile case, the loss in power was negligible. Hence, we will continue this practice in Test $\bar{\mathrm{A}}_6$ and Test $\bar{\mathrm{A}}_8$. Thus, using in Test $\bar{\mathrm{A}}_6$ the same quantiles and values of the α_i as were used in Test A_6, one has

$$\alpha^2 = \frac{b^2}{29.1169n - 18.6964b^2}, \qquad F(b) = 1 - \varepsilon \tag{4}$$

and one sees that even for moderate sample sizes, $\alpha^2 > 0$ for all realistic values of ε. Since ζ_4, ζ_5, and ζ_6 are all positive and, for the usual small values of ε, $b \gtrless 0$ when $\mu_2 \gtrless \mu_1$, the positive root of α^2 must be used when $\mu_2 > \mu_1$ and the negative root used when $\mu_2 < \mu_1$. Thus Test $\bar{\mathrm{A}}_6$ can now be stated as follows: if

$$\begin{aligned}\bar{y}_6 = {} & (0.0968 \pm\alpha)\,z(0.0540) + (0.0968 \mp\alpha)\,z(0.9460) \\ & + (0.1787 \pm\alpha)\,z(0.1915) \\ & + (0.1787 \mp\alpha)\,z(0.8085) + (0.2245 \pm\alpha)\,z(0.3898) \\ & + (0.2245 \mp\alpha)\,z(0.6102) \\ & \lesseqgtr \mu_1, \qquad \mu_2 \gtrless \mu_1\end{aligned}$$

accept H_0. Otherwise, reject H_0. Here α^2 is given by Eq. (4).

To compare the efficiency of Test A_6 and Test $\bar{\mathrm{A}}_6$, we take, as an example, $n = 200$ and $\varepsilon = 0.01$. The power functions of the tests then become

$$P_0(\text{Test A}_6) = 1 - F\left(2.326 - 13.829\frac{\mu_2 - \mu_1}{\sigma}\right)$$

$$P_0(\text{Test }\bar{\text{A}}_6) = 1 - F\left(2.326 - 13.708\frac{\mu_2 - \mu_1}{\sigma}\right)$$

The power function P_0' of the best test using all the sample values is given by

$$\begin{aligned}P_0' &= 1 - F\left(b - n^{1/2}\frac{\mu_2 - \mu_1}{\sigma}\right) \\ &= 1 - F\left(2.326 - 14.142\frac{\mu_2 - \mu_1}{\sigma}\right)\end{aligned}$$

If one compares the coefficient of $(\mu_2 - \mu_1)/\sigma$ in P_0 (Test $\bar{\mathrm{A}}_6$) with that in P_0 (Test A_6), it is readily seen that the loss in power by using in Test $\bar{\mathrm{A}}_6$ the quantiles and values of α_i which maximize P_0 (Test A_6) is indeed negligible for $n = 200$. Under the best of circumstances the coefficient in P_0 (Test $\bar{A}_6$) must always be less than that in P_0 (Test A_6); the fact that they are, under these conditions, almost identical is a strong indication that Test $\bar{A}_6$, as given, is very near optimum.

Table 2 gives the power and efficiency of Test A_6 and the efficiency of Test $\bar{A}_6$ for $n = 200$ and $\varepsilon = 0.01$. The efficiencies of both tests are always greater than 0.93.

Table 2. Power and efficiency of Test A_6 and efficiency of Test $\bar{A}_6$ for $n = 200$, $\varepsilon = 0.01$

$\left\|\frac{\mu_2 - \mu_1}{\sigma}\right\|$	Test A_6 P_0	Test A_6 Efficiency	Test $\bar{A}_6$, Efficiency
0.01	0.0143	0.9924	0.9896
0.05	0.0510	0.9681	0.9564
0.10	0.1727	0.9551	0.9381
01.5	0.4005	0.9564	0.9397
0.20	0.6698	0.9677	0.9550
0.25	0.8709	0.9822	0.9750
0.30	0.9658	0.9933	0.9903
0.35	0.9940	0.9984	0.9976

C. Tests A_8 and $\bar{A}_8$

The procedure in Tests A_8 and $\bar{A}_8$ is identical with those used in Tests A_6 and $\bar{A}_6$. Let z_i, $i = 1, 2, \cdots, 8$, be eight sample quantiles such that $p_1 + p_8 = p_2 + p_7 = p_3 + p_6 = p_4 + p_5 = 1$. For Test A_8 the rejection region is given by

$$y_8 = \alpha_1 (z_1 + z_8) + \alpha_2 (z_2 + z_7) + \alpha_3 (z_3 + z_6) + \alpha_4 (z_4 + z_5) \gtrless k, \qquad \mu_2 \gtrless \mu_1$$

The value of k and the power function are again given by

$$k = \mu_1 \pm \left(\frac{2}{n}\right)^{1/2} \gamma_3 \sigma b, \qquad \mu_2 \gtrless \mu_1$$

$$P_0 = 1 - F\left(b - \frac{n^{1/2}}{2^{1/2}\gamma_3} \frac{\mu_2 - \mu_1}{\sigma}\right)$$

where

$$\begin{aligned}\gamma_3^2 = {} & \alpha_1^2 a_8^2 (1 + \rho_{18}) + \alpha_2^2 a_7^2 (1 + \rho_{27}) + \alpha_3^2 a_6^2 (1 + \rho_{36}) \\ & + \alpha_4^2 a_5^2 (1 + \rho_{45}) + 2\alpha_1\alpha_2 a_7 a_8 (\rho_{12} + \rho_{17}) \\ & + 2\alpha_1\alpha_3 a_6 a_8 (\rho_{13} + \rho_{16}) + 2\alpha_1\alpha_4 a_5 a_8 (\rho_{14} + \rho_{15}) \\ & + 2\alpha_2\alpha_3 a_6 a_7 (\rho_{23} + \rho_{26}) + 2\alpha_2\alpha_4 a_5 a_7 (\rho_{24} + \rho_{25}) \\ & + 2\alpha_3\alpha_4 a_5 a_6 (\rho_{34} + \rho_{35})\end{aligned}$$

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{f^2(\zeta_i)}, \qquad i = 5, 6, 7, 8$$

and ρ_{ij} denotes the correlation between z_i and z_j. The orders of the quantiles and the values of the α_i which maximize P_0 are

$$\begin{array}{lll} p_1 = 0.0310 & p_8 = 0.9690 & \alpha_1 = 0.0559 \\ p_2 = 0.1154 & p_7 = 0.8846 & \alpha_2 = 0.1119 \\ p_3 = 0.2481 & p_6 = 0.7519 & \alpha_3 = 0.1550 \\ p_4 = 0.4126 & p_5 = 0.5874 & \alpha_4 = 0.1772 \end{array}$$

Using the values above, one obtains

$$k = \mu_1 \pm \frac{1.0142 b\sigma}{n^{1/2}}, \qquad \mu_2 \gtrless \mu_1$$

and, for $\mu_2 > \mu_1$,

$$P_0 = 1 - F\left(b - 0.9859\, n^{1/2} \frac{\mu_2 - \mu_1}{\sigma}\right), \qquad F(b) = 1 - \varepsilon$$

Test A_8 can now be stated as follows: if

$$\begin{aligned}y_8 = {} & 0.0559\,[z(0.0310) + z(0.9690)] \\ & + 0.1119\,[z(0.1154) + z(0.8846)] \\ & + 0.1550\,[z(0.2481) + z(0.7519)] \\ & + 0.1772\,[z(0.4126) + z(0.5874)] \\ & \lesseqgtr \mu_1 \pm \frac{1.0142 b\sigma}{n^{1/2}}, \qquad \mu_2 \gtrless \mu_1\end{aligned}$$

accept H_0. Otherwise, reject H_0.

In Test $\bar{A}_8$, the rejection region is of the form

$$\bar{y}_8 = (\alpha_1 \pm \alpha) z_1 + (\alpha_1 \mp \alpha) z_8 + (\alpha_2 \pm \alpha) z_2 + (\alpha_2 \mp \alpha) z_7 + (\alpha_3 \pm \alpha) z_3 + (\alpha_3 \mp \alpha) z_6 + (\alpha_4 \pm \alpha) z_4 + (\alpha_4 \mp \alpha) z_5 \gtrless \mu_1$$

for $\mu_2 \gtrless \mu_1$, where

$$\alpha^2 = \frac{\begin{aligned} b^2 [\alpha_1^2 a_8^2 (1+\rho_{18}) + \alpha_2^2 a_7^2 (1+\rho_{27}) + \alpha_3^2 a_6^2 (1+\rho_{36}) + \alpha_4^2 a_5^2 (1+\rho_{45}) + 2\alpha_1\alpha_2 a_7 a_8 (\rho_{12}+\rho_{17}) \\ + 2\alpha_1\alpha_3 a_6 a_8 (\rho_{13}+\rho_{16}) + 2\alpha_1\alpha_4 a_5 a_8 (\rho_{14}+\rho_{15}) + 2\alpha_2\alpha_3 a_6 a_7 (\rho_{23}+\rho_{26}) \\ + 2\alpha_2\alpha_4 a_5 a_7 (\rho_{24}+\rho_{25}) + 2\alpha_3\alpha_4 a_5 a_6 (\rho_{34}+\rho_{35})] \end{aligned}}{\begin{aligned} 2n(\zeta_5+\zeta_6+\zeta_7+\zeta_8)^2 + b^2 [a_8^2(1-\rho_{18}) + a_7^2(1-\rho_{27}) + a_6^2(1-\rho_{36}) + a_5^2(1-\rho_{45}) \\ + 2a_7a_8(\rho_{12}-\rho_{17}) + 2a_6a_8(\rho_{13}-\rho_{16}) + 2a_5a_8(\rho_{14}-\rho_{15}) \\ + 2a_6a_7(\rho_{23}-\rho_{26}) + 2a_5a_7(\rho_{24}-\rho_{25}) + 2a_5a_6(\rho_{34}-\rho_{35})] \end{aligned}}$$

The power function for $\mu_2 > \mu_1$ is given by

$$P_0 = 1 - F\left(b - \frac{n^{1/2}}{2^{1/2}\gamma_4}\,\frac{\mu_2-\mu_1}{\sigma}\right), \qquad F(b) = 1 - \varepsilon$$

where

$$\begin{aligned} \gamma_4^2 = {} & a_8^2[\alpha_1^2(1+\rho_{18}) + \alpha^2(1-\rho_{18})] + a_7^2[\alpha_2^2(1+\rho_{27}) + \alpha^2(1-\rho_{27})] \\ & + a_6^2[\alpha_3^2(1+\rho_{36}) + \alpha^2(1-\rho_{36})] + a_5^2[\alpha_4^2(1+\rho_{45}) + \alpha^2(1-\rho_{45})] \\ & + 2a_7a_8[(\alpha_1\alpha_2+\alpha^2)\rho_{12} + (\alpha_1\alpha_2-\alpha^2)\rho_{17}] + 2a_6a_8[(\alpha_1\alpha_3+\alpha^2)\rho_{13} + (\alpha_1\alpha_3-\alpha^2)\rho_{16}] \\ & + 2a_5a_8[(\alpha_1\alpha_4+\alpha^2)\rho_{14} + (\alpha_1\alpha_4-\alpha^2)\rho_{15}] + 2a_6a_7[(\alpha_2\alpha_3+\alpha^2)\rho_{23} + (\alpha_2\alpha_3-\alpha^2)\rho_{26}] \\ & + 2a_5a_7[(\alpha_2\alpha_4+\alpha^2)\rho_{24} + (\alpha_2\alpha_4-\alpha^2)\rho_{25}] + 2a_5a_6[(\alpha_3\alpha_4+\alpha^2)\rho_{34} + (\alpha_3\alpha_4-\alpha^2)\rho_{35}] \end{aligned}$$

Using in Test $\overline{\mathrm{A}}_8$ the quantiles and values of the α_i used in Test A_8 gives

$$\alpha^2 = \frac{b^2}{61.1625n - 35.8901b^2} \tag{5}$$

Test $\overline{\mathrm{A}}_8$ can now be stated as follows: if

$$\begin{aligned} \bar{y}_8 = {} & (0.0559 \pm \alpha)\,z(0.0310) + (0.0559 \mp \alpha)\,z(0.9690) + (0.1119 \pm \alpha)\,z(0.1154) + (0.1119 \mp \alpha)\,z(0.8846) \\ & + (0.1550 \pm \alpha)\,z(0.2481) + (0.1550 \mp \alpha)\,z(0.7519) + (0.1772 \pm \alpha)\,z(0.4126) + (0.1772 \mp \alpha)\,z(0.5874) \\ & \lesseqgtr \mu_1, \qquad \mu_2 \gtrless \mu_1 \end{aligned}$$

accept H_0. Otherwise, reject H_0. Here α^2 is given by Eq. (5).

For $n = 200$ and $\varepsilon = 0.01$, the power functions of Tests A_8 and $\overline{\mathrm{A}}_8$ are given by

$$\left.\begin{aligned} P_0(\text{Test } \mathrm{A}_8) &= 1 - F\left(2.326 - 13.943\,\frac{\mu_2-\mu_1}{\sigma}\right) \\ P_0(\text{Test } \overline{\mathrm{A}}_8) &= 1 - F\left(2.326 - 13.832\,\frac{\mu_2-\mu_1}{\sigma}\right) \end{aligned}\right\} \qquad \mu_2 > \mu_1$$

Table 3 gives the power and efficiency of Test A_8 and the efficiency of Test $\overline{\mathrm{A}}_8$ for $n = 200$ and $\varepsilon = 0.01$. The efficiencies of both tests are always greater than 0.95.

Table 3. Power and efficiency of Test A_8 and efficiency of Test $\bar{A}_8$ for $n = 200$, $\varepsilon = 0.01$

$\left\lvert \frac{\mu_2 - \mu_1}{\sigma} \right\rvert$	Test A_8		Test $\bar{A}_8$, Efficiency
	P_0	Efficiency	
0.01	0.0144	0.9952	0.9924
0.05	0.0516	0.9799	0.9687
0.10	0.1756	0.9713	0.9555
0.15	0.4071	0.9723	0.9568
0.20	0.6780	0.9796	0.9680
0.25	0.8769	0.9889	0.9824
0.30	0.9683	0.9959	0.9933
0.35	0.9947	0.9990	0.9984

IV. Test B: Testing the Standard Deviation of a Normal Distribution Using Six and Eight Quantiles

Since σ was assumed to be the same in both hypotheses of Test A, the test statistics turned out to be linear functions of the sample quantiles. As a result, the best tests using quantiles were all one-sided. In the present test, however, we wish to discriminate between $\sigma = \sigma_1$ and $\sigma = \sigma_2$ and, as a consequence of using pairs of symmetric quantiles, it is not necessary to assume that μ is known. Moreover, it will be seen that the best tests using quantiles are not one-sided, but it will be shown that these can be closely approximated by a one-sided test with a negligible loss in power. Consequently, only these one-sided tests will be given.

We test the simple null hypothesis

$$H_0\colon g(x) = g_1(x) = N(\mu, \sigma_1)$$

against the simple alternative hypothesis

$$H_1\colon g(x) = g_2(x) = N(\mu, \sigma_2)$$

where μ is not necessarily known and where

$$\sigma_2 > \sigma_1 \, (\sigma_2 < \sigma_1)$$

Let z_i, $i = 1, 2, \cdots, z_6$, denote six sample quantiles such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$. From previous results we can deduce that one should base the test on the statistic

$$y_6 = \alpha_1(z_6 - z_1) + \alpha_2(z_5 - z_2) + \alpha_3(z_4 - z_3)$$

where $2(\alpha_1 \zeta_6 + \alpha_2 \zeta_5 + \alpha_3 \zeta_4) = 1$. We also deduce that the best critical region, for $\sigma_2 > \sigma_1$, is given by

$$\left(y_6 - \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2}\right)^2 > k^2 \tag{6}$$

where k is determined such that the probabilty of Ineq. (6) occurring is equal to ε.

To determine k and P_0, one has, under H_0,

$$E(y_6) = \alpha_1(\mu + \sigma_1\zeta_6 - \mu + \sigma_1\zeta_6) + \alpha_2(\mu + \sigma_1\zeta_5 - \mu + \sigma_1\zeta_5) + \alpha_3(\mu + \sigma_1\zeta_4 - \mu + \sigma_1\zeta_4) = \sigma_1$$

$$\begin{aligned} \operatorname{var}(y_6) = \frac{2\sigma_1^2}{n} & [\alpha_1^2 a_6^2 (1 - \rho_{16}) + \alpha_2^2 a_5^2 (1 - \rho_{25}) + \alpha_3^2 a_4^2 (1 - \rho_{34}) \\ & + 2\alpha_1\alpha_2 a_5 a_6 (\rho_{12} - \rho_{15}) + 2\alpha_1\alpha_3 a_4 a_6 (\rho_{13} - \rho_{14}) + 2\alpha_2\alpha_3 a_4 a_5 (\rho_{23} - \rho_{24})] = \frac{2\sigma_1^2 \gamma_5^2}{n} \end{aligned}$$

where

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{f^2(\zeta_i)}, \qquad i = 4, 5, 6$$

and ρ_{ij} is the correlation between z_i and z_j;

$$\mathrm{pr}\left[\left(y_6 - \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right)^2 < k^2\right] = \mathrm{pr}\left[-k + \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} < y_6 < k + \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right]$$

$$= F\left[\frac{k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_1}{\sigma_1\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right] - F\left[\frac{-k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_1}{\sigma_1\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right] = F(b) - F(c) = 1 - \varepsilon \tag{7}$$

$$k = \sigma_1\gamma_5 b\left(\frac{2}{n}\right)^{1/2} + \sigma_1 - \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} = -\sigma_1\gamma_5 c\left(\frac{2}{n}\right)^{1/2} - \sigma_1 + \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} \tag{8}$$

Under H_1,

$$E(y_6) = \sigma_2$$

$$\mathrm{var}(y_6) = \frac{2\sigma_2^2}{n}\gamma_5^2$$

$$\mathrm{pr}\left[\left(y_6 - \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right)^2 < k^2\right] = F\left[\frac{k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_2}{\sigma_2\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right] - F\left[\frac{-k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_2}{\sigma_2\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right]$$

$$= F\left[\frac{\sigma_1}{\sigma_2}b - \frac{n^{1/2}}{\gamma_5 2^{1/2}}\left(1 - \frac{\sigma_1}{\sigma_2}\right)\right] - F\left[\frac{\sigma_1}{\sigma_2}c - \frac{n^{1/2}}{\gamma_5 2^{1/2}}\left(1 - \frac{\sigma_1}{\sigma_2}\right)\right] = 1 - P_0 \tag{9}$$

Theoretically, the values of b and c depend upon σ_1 and σ_2 as well as upon ε. However, if one determines the value of b in Eq. (7) by the relation $F(b) = 1 - \varepsilon$ and neglects as negligible the second term of the left-hand side of Eq. (9), P_0 will be a maximum if the orders of the quantiles and the values of the α_i are chosen so as to minimize γ_5. These values are known to be

$p_1 = 0.0104$	$p_6 = 0.9896$	$\alpha_1 = 0.0549$
$p_2 = 0.0548$	$p_5 = 0.9452$	$\alpha_2 = 0.1244$
$p_3 = 0.1696$	$p_4 = 0.8304$	$\alpha_3 = 0.1825$

For these values, Eq. (9) becomes

$$P_0 = 1 - F\left[\frac{\sigma_1}{\sigma_2}b - 1.3363\, n^{1/2}\left(1 - \frac{\sigma_1}{\sigma_2}\right)\right] + F\left[\frac{\sigma_1}{\sigma_2}c - 1.3363\, n^{1/2}\left(1 - \frac{\sigma_1}{\sigma_2}\right)\right] \tag{10}$$

Now, noting from Eq. (8) that

$$c = -b + \frac{2}{\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\left(\frac{\sigma_2}{\sigma_1} - 1\right)$$

one has, for the argument of the last term of the right-hand side of Eq. (10),

$$\frac{-\sigma_1}{\sigma_2}c - 1.3363\,n^{1/2}\left(1-\frac{\sigma_1}{\sigma_2}\right) = \frac{-\sigma_1}{\sigma_2}b - 2.6726\,n^{1/2}\frac{\sigma_1}{\sigma_2}\left(\frac{1}{1+\dfrac{\sigma_2}{\sigma_1}}\right) - 1.3363\,n^{1/2}\left(1-\frac{\sigma_1}{\sigma_2}\right)$$

$$\leqq \frac{-\sigma_1}{\sigma_2}b - 0.8248\,n^{1/2} < -0.8284\,n^{1/2}$$

(since $b > 0$ for small values of ε), a result obtained by maximizing

$$-2.6726\,n^{1/2}\frac{\sigma_1}{\sigma_2}\left(\frac{1}{1+\dfrac{\sigma_2}{\sigma_1}}\right) - 1.3363\,n^{1/2}\left(1-\frac{\sigma_1}{\sigma_2}\right)$$

over all values of $\sigma_2/\sigma_1 > 1$. If $n \geqq 200$, then

$$F\left[\frac{\sigma_1}{\sigma_2}c - 1.3363\,n^{1/2}\left(1-\frac{\sigma_1}{\sigma_2}\right)\right] < F(-11.715) \cong 0$$

and hence contributes nothing to the power of the test, verifying the negligibility of this term. If one determines b by the relation $F(b) = 1 - \varepsilon$, the test then becomes the best one-sided test and has the advantage of being independent of σ_2^2.

In effect we have set $c = -\infty$ when $\sigma_2 > \sigma_1$. For $\sigma_2 < \sigma_1$, since the rejection region becomes

$$\left(y_6 - \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right)^2 < k^2$$

one has

$$F\left[\frac{k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_1}{\sigma_1\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right] - F\left[\frac{-k + \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2} - \sigma_1}{\sigma_1\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\right] = F(b') - F(c') = \varepsilon$$

$$F\left[\frac{\sigma_1}{\sigma_2}b' + \frac{n^{1/2}}{\gamma_5\,2^{1/2}}\left(\frac{\sigma_1}{\sigma_2}-1\right)\right] - F\left[\frac{\sigma_1}{\sigma_2}c' + \frac{n^{1/2}}{\gamma_5\,2^{1/2}}\left(\frac{\sigma_1}{\sigma_2}-1\right)\right] = P_0$$

For this case, we determine b' such that $F(b') = \varepsilon$, resulting in $b' = -b$, $c' = c = -\infty$, and

$$P_0 = F\left[\frac{-\sigma_1}{\sigma_2}b + \frac{1}{\gamma_5\left(\dfrac{2}{n}\right)^{1/2}}\left(\frac{\sigma_1}{\sigma_2}-1\right)\right]$$

The best one-sided Test B_6 can now be stated as follows: if

$$y_6 = 0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)] + 0.1825\,[z(0.8304) - z(0.1696)]$$

$$\lesseqgtr \sigma_1\left(1.0 \pm \frac{0.7484b}{n^{1/2}}\right), \qquad \sigma_2 \gtreqless \sigma_1$$

accept H_0. Otherwise, reject H_0.

With respect to Test B_8, let z_i, $i = 1, 2, \cdots, 8$, denote eight sample quantiles such that $p_i + p_{8-i+1} = 1$, $i = 1, 2, 3, 4$. The test statistic and best rejection region for this test are given by

$$y_8 = \sum_{i=1}^{4} \alpha_i \left(z_{8-i+1} - z_i\right), \qquad 2 \sum_{i=1}^{4} \alpha_i \zeta_{8-i+1} = 1$$

$$\left(y_8 - \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2}\right)^2 \gtrless k^2, \qquad \sigma_2 \gtrless \sigma_1$$

Under H_0,

$$E(y_8) = \sigma_1$$

$$\begin{aligned} \operatorname{var}(y_8) &= \frac{2\sigma_1^2}{n} \left[\alpha_1^2 a_8^2 (1 - \rho_{18}) + \alpha_2^2 a_7^2 (1 - \rho_{27}) + \alpha_3^2 a_6^2 (1 - \rho_{36}) + \alpha_4^2 a_5^2 (1 - \rho_{45})\right. \\ &\quad + 2\alpha_1\alpha_2 a_7 a_8 (\rho_{12} - \rho_{17}) + 2\alpha_1\alpha_3 a_6 a_8 (\rho_{13} - \rho_{16}) + 2\alpha_1\alpha_4 a_5 a_8 (\rho_{14} - \rho_{15}) \\ &\quad \left. + 2\alpha_2\alpha_3 a_6 a_7 (\rho_{23} - \rho_{26}) + 2\alpha_2\alpha_4 a_5 a_7 (\rho_{24} - \rho_{25}) + 2\alpha_3\alpha_4 a_5 a_6 (\rho_{34} - \rho_{35})\right] \\ &= \frac{2\sigma_1^2}{n} \gamma_6^2 \end{aligned}$$

where

$$a_i^2 = \frac{F(\zeta_i)\,[1 - F(\zeta_i)]}{f^2(\zeta_i)}, \qquad i = 5, 6, 7, 8$$

and ρ_{ij} is the correlation between z_i and z_j. Under H_1,

$$E(y_8) = \sigma_2$$

$$\operatorname{var}(y_8) = \frac{2\sigma_2^2}{n} \gamma_6^2$$

As in Test B_6, we determine b by the relation $F(b) = 1 - \varepsilon$, for $\sigma_2 > \sigma_1$, and $F(-b) = \varepsilon$ for $\sigma_2 < \sigma_1$. The orders of the quantiles and the values of the α_i which maximize the power of this one-sided best test are given by

$p_1 = 0.00549$	$p_8 = 0.99451$	$\alpha_1 = 0.0307$
$p_2 = 0.0286$	$p_7 = 0.9714$	$\alpha_2 = 0.0730$
$p_3 = 0.0851$	$p_6 = 0.9149$	$\alpha_3 = 0.1168$
$p_4 = 0.2017$	$p_5 = 0.7983$	$\alpha_4 = 0.1477$

Using these values, the power functions of the test are given by

$$P_0 = F\left[-\frac{\sigma_1}{\sigma_2} b + 1.3568\, n^{1/2} \left(\pm 1 \mp \frac{\sigma_1}{\sigma_2}\right)\right], \qquad \sigma_2 \gtrless \sigma_1$$

Test B_8 can now be stated as follows: if

$$\begin{aligned} y_8 &= 0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)] \\ &\quad + 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)] \\ &\lessgtr \sigma_1\left(1.0 \pm \frac{0.7370b}{n^{1/2}}\right), \qquad \sigma_2 \gtrless \sigma_1 \end{aligned}$$

accept H_0. Otherwise, reject H_0. The form of the power functions P_0' (Test B) are identical with those of Test B_6 and Test B_8. The coefficient of $(\pm 1 \mp \sigma_1/\sigma_2)$ is $(2n)^{1/2}$. Table 4 shows P_0' (Test B) and the efficiency of Test B_6 and Test B_8, for $n = 200$ and $\varepsilon = 0.01$.

We used symmetric quantiles in Test A because they have been shown to be the optimum spacing for estimating the mean of a normal distribution with an even number of quantiles. Although the use of symmetric quantiles for Test B has not yet been proved to be the optimum procedure, we have used them because we conjecture that this is the optimum thing to do, and also because the tests can be performed with no knowledge of μ.

Table 4. P_0' (Test B) and efficiency of Tests B_6 and B_8 for $n = 200$, $\varepsilon = 0.01$

σ_2/σ_1	P_0' (Test B)	Efficiency	
		Test B_6	Test B_8
0.80	0.9818	0.9833	0.9886
0.85	0.7860	0.9225	0.9439
0.90	0.3584	0.8755	0.9068
0.95	0.0813	0.8968	0.9233
0.99	0.0159	0.9742	0.9792
1.01	0.0176	0.9733	0.9801
1.05	0.1033	0.9117	0.9343
1.10	0.3834	0.9018	0.9274
1.15	0.7210	0.9304	0.9492
1.20	0.9184	0.9658	0.9756
1.25	0.9838	0.9885	0.9920
1.30	0.9976	0.9973	0.9982

V. Tests D, $\bar{\text{D}}$, and $\bar{\text{E}}$: Two-Sample Tests

A. Tests D_6 and $\bar{D}_6$

In this section, it is assumed that we are given sets of independent sample values taken from two independent, normally distributed populations with density functions $g_1(x)$ and $g_2(y)$ and consider the test

$$H_0\colon g_1(x) = N(\mu, \sigma) \qquad g_2(y) = N(\mu, \sigma)$$

$$H_1\colon g_1(x) = N(\mu, \sigma) \qquad g_2(y) = N(\mu + \theta, \sigma), \qquad \theta \neq 0$$

For Test D, we assume that σ is known and μ is unknown, while in Test $\bar{D}$, we assume that both μ and σ are unknown. Sample sizes n_1 and n_2 are assumed, where n_1 and n_2 are both large ($\geqq 200$).

Beginning with Test D_6, let z_i, $i = 1, 2, \cdots, 6$, be six sample quantiles of the first sample, such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$, and let z_i' be the corresponding sample quantiles of the second sample. Form

$$w_i = z_i - z_i', \qquad i = 1, 2, \cdots, 6$$

The test will be made on the statistic given by the linear combination

$$y_6 = \alpha_1(w_1 + w_6) + \alpha_2(w_2 + w_5) + \alpha_3(w_3 + w_4)$$

where $2\sum_{i=1}^{3}\alpha_i = 1$. The best rejection region is that for which

$$y_6 \lessgtr k, \qquad \theta \gtrless 0$$

Under H_0,

$$E(y_6) = 0$$

$$\operatorname{var}(y_6) = 2\sigma^2\gamma_1^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

For $\theta < 0$,

$$\operatorname{pr}(y_6 < k) = F\left[\frac{k}{2^{1/2}\sigma\gamma_1\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{1/2}}\right] = F(b) = 1 - \varepsilon$$

$$k = 2^{1/2}\sigma\gamma_1 b\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{1/2}$$

Under H_1,

$$E(y_6) = -\theta$$

$$\mathrm{var}(y_6) = 2\sigma^2\gamma_1^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

$$\mathrm{pr}(y_i < k) = F\left[\frac{k+\theta}{2^{1/2}\sigma\gamma_1\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^{1/2}}\right]$$

$$= F\left[b + \frac{1}{2^{1/2}\gamma_1\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^{1/2}}\frac{\theta}{\sigma}\right]$$

$$= 1 - P_0 \qquad (11)$$

The order of the quantiles and the values of the α_i which maximize P_0 are the same as those which maximize the power of Test A_6, namely,

$$p_1 = 0.0540 \qquad p_6 = 0.9460 \qquad \alpha_1 = 0.0968$$

$$p_2 = 0.1915 \qquad p_5 = 0.8085 \qquad \alpha_2 = 0.1787$$

$$p_3 = 0.3898 \qquad p_4 = 0.6102 \qquad \alpha_3 = 0.2245$$

Through the use of these values, Eq. (11) becomes

$$P_0 = F\left[-b \pm 0.9778\frac{\theta}{\sigma}\left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2}\right], \qquad \theta \gtrless 0$$

The power of the best test is given by

$$P_0' = F\left[-b \pm \frac{\theta}{\sigma}\left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2}\right], \qquad \theta \gtrless 0$$

Test D_6 can now be stated as follows: if

$$\begin{aligned} y_6 = {} & 0.0968\,[z(0.0540) - z'(0.0540) + z(0.9460) - z'(0.9460)] \\ & + 0.1787\,[z(0.1915) - z'(0.1915) + z(0.8025) - z'(0.8025)] \\ & + 0.2245\,[z(0.3898) - z'(0.3898) + z(0.6102) - z'(0.6102)] \\ & \lessgtr \pm 1.0228 b\sigma\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^{1/2}, \qquad \theta \lessgtr 0 \end{aligned}$$

accept H_0. Otherwise, reject H_0.

The procedure for Test $\bar{D}_6$ is similar to that employed for Test $\bar{A}_6$. The test statistics and rejection regions are given by

$$\begin{aligned} \bar{y}_6 = {} & (\alpha_1 \pm \alpha)(z_6 - z_1') + (\alpha_1 \mp \alpha)(z_1 - z_6') \\ & + (\alpha_2 \pm \alpha)(z_5 - z_2') + (\alpha_2 \mp \alpha)(z_2 - z_5') \\ & + (\alpha_3 \pm \alpha)(z_4 - z_3') + (\alpha_3 \mp \alpha)(z_3 - z_4') \\ & \lessgtr 0, \qquad \theta \gtrless 0 \end{aligned}$$

Under H_0,

$$E(\bar{y}_6) = 4\alpha\sigma(\zeta_4 + \zeta_5 + \zeta_6)$$

$$\mathrm{var}(\bar{y}_6) = 2\sigma^2\gamma_2^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

Under H_1,

$$E(\bar{y}_6) = 4\alpha\sigma(\zeta_4 + \zeta_5 + \zeta_6) - \theta$$

$$\mathrm{var}(\bar{y}_6) = 2\sigma^2\gamma_2^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

Using in Test $\bar{D}_6$ the same quantiles and values of the α_i as were used in Test D_6, one has

$$\alpha^2 = \frac{b^2}{116.4672\dfrac{n_1 n_2}{n_1+n_2} - 18.6964 b^2}$$

and if one sets $n_1 = n_2 = 200$ and $\varepsilon = 0.01$, the power functions of Test D_6 and Test $\overline{D}_6$ are given by

$$\left.\begin{aligned} P_0\,(\text{Test } D_6) &= F\left(-2.326 \pm 9.7780\,\frac{\theta}{\sigma}\right) \\ P_0\,(\text{Test } \overline{D}_6) &= F\left(-2.326 \pm 9.7357\,\frac{\theta}{\sigma}\right) \end{aligned}\right\} \qquad \theta \gtrless 0$$

Test $\overline{D}_6$ can now be stated as follows: if

$$\begin{aligned} \overline{y}_6 = {} & (0.0968 \pm \alpha)\,[z\,(0.9460) - z'\,(0.0540)] + (0.0968 \mp \alpha)\,[z\,(0.0540) - z'\,(0.9460)] \\ & + (0.1787 \pm \alpha)\,[z\,(0.8025) - z'\,(0.1915)] + (0.1787 \mp \alpha)\,[z\,(0.1915) - z'\,(0.8025)] \\ & + (0.2245 \pm \alpha)\,[z\,(0.6102) - z'\,(0.3898)] + (0.2245 \mp \alpha)\,[z\,(0.3898) - z'\,(0.6102)] \\ & \gtrless 0, \qquad \theta \gtrless 0 \end{aligned}$$

accept H_0. Otherwise, reject H_0.

B. Tests D_8 and $\overline{D}_8$

Let z_i, $i = 1, 2, \cdots, 8$, be eight sample quantiles of the first sample such that $z_i + z_{8-i+1} = 1$, and let z'_i be the corresponding sample quantiles of the second sample. For Test D_8, the test statistic and rejection regions are given by

$$y_8 = \sum_{i=1}^{4} \alpha_i\,(z_i - z'_i + z_{8-i+1} - z'_{8-i+1}) \lessgtr k, \qquad \theta \gtrless 0$$

Under H_0,

$$E\,(y_8) = 0$$

$$\text{var}\,(y_8) = 2\,\sigma^2\gamma_3^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$k = 2^{1/2}\,\gamma_3 b\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{1/2}$$

Under H_1,

$$E\,(y_8) = -\theta$$

$$\text{var}\,(y_8) = 2\,\sigma^2\gamma_3^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$P_0 = F\left[-b \pm \frac{1}{2^{1/2}\,\gamma_3\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)^{1/2}}\,\frac{\theta}{\sigma}\right],$$

$$\theta \gtrless 0, F\,(b) = 1 - \varepsilon \tag{12}$$

The order of the quantiles and the values of the α_i which maximize P_0 are given by

$$
\begin{array}{lll}
p_1 = 0.0310 & p_8 = 0.9690 & \alpha_1 = 0.0559 \\
p_2 = 0.1154 & p_7 = 0.8846 & \alpha_2 = 0.1119 \\
p_3 = 0.2481 & p_6 = 0.7519 & \alpha_3 = 0.1550 \\
p_4 = 0.4126 & p_5 = 0.5874 & \alpha_4 = 0.1772
\end{array}
$$

Through the use of these values, Eq. (12) becomes

$$P_0 = F\left[-b \pm 0.9860 \frac{\theta}{\sigma} \left(\frac{n_1 n_2}{n_1 + n_2} \right)^{1/2} \right], \qquad \theta \gtrless 0$$

Test D_8 can now be stated as follows: if

$$
\begin{aligned}
y_8 = {} & 0.0559\,[z(0.0310) - z'(0.0310) + z(0.9690) - z'(0.9690)] \\
& + 0.1119\,[z(0.1154) - z'(0.1154) + z(0.8846) - z'(0.8846)] \\
& + 0.1550\,[z(0.2481) - z'(0.2481) + z(0.7519) - z'(0.7519)] \\
& + 0.1772\,[z(0.4126) - z'(0.4126) + z(0.5874) - z'(0.5874)] \\
& \lesseqgtr \pm 1.0142 b\sigma \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^{1/2}, \qquad \theta \lesseqgtr 0
\end{aligned}
$$

accept H_0. Otherwise, reject H_0.

In Test $\overline{D}_8$, the test statistics and rejection regions are given by

$$\bar{y}_8 = \sum_{i=1}^{4} (\alpha_i \pm \alpha)(z_{8-i+1} - z'_i) + (\alpha_i \mp \alpha)(z_i - z'_{8-i+1}) \lesseqgtr 0, \qquad \theta \gtrless 0$$

Under H_0,

$$E(\bar{y}_8) = 4\alpha\sigma \sum_{i=5}^{8} \zeta_i$$

$$\operatorname{var}(\bar{y}_8) = 2\sigma^2 \gamma_4^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)$$

Under H_1,

$$E(\bar{y}_8) = 4\alpha\sigma \sum_{i=5}^{8} \zeta_i - \theta$$

$$\operatorname{var}(\bar{y}_8) = 2\sigma^2 \gamma_4 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)$$

Using in Test $\overline{\text{D}}_8$ the sample quantiles and values of the α_i that were used in Test D_8, one has

$$\alpha^2 = \frac{b^2}{244.650\left(\dfrac{n_1 n_2}{n_1 + n_2}\right) - 35.8901 b^2}$$

If we let $n_1 = n_2 = 200$ and $\varepsilon = 0.01$, the power functions of Test D_8 and Test $\overline{\text{D}}_8$ are given by

$$\left.\begin{aligned} P_0\,(\text{Test D}_8) &= F\left(-2.326 \pm 9.8600\frac{\theta}{\sigma}\right) \\ P_0\,(\text{Test }\overline{\text{D}}_8) &= F\left(-2.326 \pm 9.8202\frac{\theta}{\sigma}\right) \end{aligned}\right\} \qquad \theta \gtrless 0$$

Table 5 gives the power and efficiency of Test D_6 and the efficiency of Test $\overline{\text{D}}_6$ for $n_1 = n_2 = 200$, $\varepsilon = 0.01$. Table 6 provides the same information for Test D_8 and Test $\overline{\text{D}}_8$.

Table 5. Power and efficiency of Test D_6 and efficiency of Test $\overline{\text{D}}_6$ for $n_1 = n_2 = 200$, $\varepsilon = 0.01$

$\lvert\theta/\sigma\rvert$	Test D_6		Test $\overline{\text{D}}_6$, Efficiency
	P_0	Efficiency	
0.01	0.0129	0.9938	0.9931
0.05	0.0331	0.9752	0.9708
0.10	0.0887	0.9603	0.9535
0.15	0.1949	0.9539	0.9460
0.20	0.3552	0.9546	0.9467
0.25	0.5467	0.9610	0.9541
0.30	0.7278	0.9708	0.9655
0.35	0.8632	0.9813	0.9778

Table 6. Power and efficiency of Test D_8 and efficiency of Test $\overline{\text{D}}_8$ for $n_1 = n_2 = 200$, $\varepsilon = 0.01$

$\lvert\theta/\sigma\rvert$	Test D_8		Test $\overline{\text{D}}_8$, Efficiency
	P_0	Efficiency	
0.01	0.0130	0.9962	0.9954
0.05	0.0334	0.9844	0.9799
0.10	0.0901	0.9752	0.9682
0.15	0.1984	0.9711	0.9629
0.20	0.3615	0.9717	0.9636
0.25	0.5551	0.9758	0.9689
0.30	0.7362	0.9819	0.9767
0.35	0.8696	0.9885	0.9851

C. Tests $\overline{\text{E}}_6$ and $\overline{\text{E}}_8$

The hypotheses of these tests are given by

$$\text{H}_0\!: g_1(x) = N(\mu, \sigma), \qquad g_2(y) = N(\mu, \sigma)$$

$$\text{H}_1\!: g_1(x) = N(\mu, \sigma), \qquad g_2(y) = N(\mu, \theta\sigma), \qquad \theta > 0$$

where μ and σ are unknown. For simplicity, we will first assume that $n_1 = n_2 = n$.

For Test $\overline{\text{E}}_6$, let z_i, $i = 1, 2, \cdots, z_6$, be six sample quantiles such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$. Let z_i' be corresponding sample quantiles of the second sample. In order to eliminate dependence on μ and σ, we take as the test statistic the linear combinations

$$\begin{aligned}\overline{y}_6 = (1+\alpha)\,[\alpha_1(z_6 - z_1) + \alpha_2(z_5 - z_2) + \alpha_3(z_4 - z_3)] \\ + (1-\alpha)\,[\alpha_1(z_6' - z_1') + \alpha_2(z_5' - z_2') + \alpha_3(z_4' - z_3')]\end{aligned}$$

where $2(\alpha_1\zeta_6 + \alpha_2\zeta_5 + \alpha_3\zeta_4) = 1$ and the rejection region will be taken as $\overline{y}_6 < 0$.

Under H_0,

$$\text{E}(\overline{y}_6) = 2\sigma$$

$$\begin{aligned}\text{var}(\overline{y}_6) &= \frac{1}{n}[2(1+\alpha)^2\sigma^2\gamma_5^2 + 2(1-\alpha)^2\sigma^2\gamma_5^2] \\ &= \frac{4(1+\alpha^2)\,\sigma^2\gamma_5^2}{n}\end{aligned}$$

$$\begin{aligned}\text{pr}(\overline{y}_6 < 0) &= F\left[\frac{-2\sigma n^{1/2}}{2(1+\alpha^2)^{1/2}\sigma\gamma_5}\right] \\ &= F\left[\frac{-n^{1/2}}{(1+\alpha^2)^{1/2}\gamma_5}\right] = F(-b) = \varepsilon\end{aligned}$$

where $F(b) = 1 - \varepsilon$ and

$$\alpha^2 = \frac{n}{b^2\gamma_5^2} - 1 \tag{13}$$

Under H_1,

$$\text{E}(\overline{y}_6) = (1+\alpha)\,\sigma + (1-\alpha)\,\theta\sigma = \sigma[1 + \alpha + \theta(1-\alpha)]$$

$$\text{var}(\overline{y}_6) = \frac{2\sigma^2}{n}\gamma_5^2[(1+\alpha)^2 + (1-\alpha)^2\theta^2]$$

$$\begin{aligned}\text{pr}(\overline{y}_6 < 0) &= F\left\{\frac{-[1+\alpha+\theta(1-\alpha)]\,n^{1/2}}{2^{1/2}\gamma_5[(1+\alpha)^2 + (1-\alpha)^2]^{1/2}}\right\} \\ &= F\left\{\frac{-b(1+\alpha^2)^{1/2}[1+\alpha+\theta(1-\alpha)]}{2^{1/2}[(1+\alpha)^2 + (1-\alpha)^2\theta^2]^{1/2}}\right\} = P_0\end{aligned}$$

As in Tests $\overline{\mathrm{E}}_2$ and $\overline{\mathrm{E}}_4$, we will use in Test $\overline{\mathrm{E}}_6$ the orders of the quantiles and values of the α_i which minimize the variance of the quantile estimator of σ from a single set of sample values, namely,

$$p_1 = 0.0104 \qquad p_6 = 0.9896 \qquad \alpha_1 = 0.0549$$
$$p_2 = 0.0548 \qquad p_5 = 0.9452 \qquad \alpha_2 = 0.1244$$
$$p_3 = 0.1696 \qquad p_4 = 0.8304 \qquad \alpha_3 = 0.1825$$

Through the use of these values, Eq. (13) becomes

$$\alpha^2 = \frac{3.5173\,n}{b^2} - 1 \tag{14}$$

Now, for $\theta > 1$, since $b > 0$, P_0 will increase as θ increases from $\theta = 1$ if $\alpha > 1$, and for $\theta < 1$, P_0 will increase as θ decreases from $\theta = 1$ if $\alpha < -1$. From Eq. (14) one sees that for realistic values of ε and even moderate values of n, $\alpha^2 > 1$. For example, for $n = 200$ and $\varepsilon = 0.01$, $\alpha^2 = 131.018$. Thus, for $\theta > 1$, one uses the positive root of Eq. (14) and, for $\theta < 1$, one uses the negative root.

Test $\overline{\mathrm{E}}_6$ can now be stated as follows: if

$$\begin{aligned}\overline{y}_6 = {} & (1 \pm \alpha)\,\{0.0549\,[z\,(0.9896) - z\,(0.0104)] + 0.1244\,[z\,(0.9452) - z\,(0.0548)] + 0.1825\,[z\,(0.8304) - z\,(0.1696)]\} \\ & + (1 \mp \alpha)\,\{0.0549\,[z'\,(0.9896) - z'\,(0.0104)] + 0.1244\,[z'\,(0.9452) - z'\,(0.0548)] + 0.1825\,[z'\,(0.8304) - z'\,(0.1696)]\} \\ & > 0, \qquad \theta \gtrless 1\end{aligned}$$

accept H_0. Otherwise, reject H_0. Here α^2 is given by Eq. (14).

If $n_1 \neq n_2$, α is determined by the relation

$$0.14\left[(1 + \alpha^2)\left(\frac{1}{n_1} + \frac{1}{n_2}\right) + 2\alpha\left(\frac{1}{n_1} - \frac{1}{n_2}\right)\right] - \frac{1}{b^2} = 0 \tag{15}$$

Then the positive root of Eq. (15) is used for $\theta > 1$ and the negative root is used for $\theta < 1$.

With respect to Test $\overline{\mathrm{E}}_8$, let z_i, $i = 1, 2, \cdots, 8$, denote eight sample quantiles of the first sample such that $p_i + p_{8-i+1} = 1$, and let z'_i denote the corresponding sample quantiles of the second sample. The test statistic for the test is given by

$$\begin{aligned}\overline{y}_8 = {} & (1 + \alpha)\,[\alpha_1 (z_8 - z_1) + \alpha_2 (z_7 - z_2) + \alpha_3 (z_6 - z_3) + \alpha_4 (z_5 - z_4)] \\ & + (1 - \alpha)\,[\alpha_1 (z'_8 - z'_1) + \alpha_2 (z'_7 - z'_2) + \alpha_3 (z'_6 - z'_3) + \alpha_4 (z'_5 - z'_4)]\end{aligned}$$

where

$$2 \sum_{i=1}^{4} \alpha_i \zeta_{8-i+1} = 1$$

The rejection region is given by $\overline{y}_8 < 0$. Omitting the details, which are analogous to those of Test $\overline{\mathrm{E}}_6$, for $n_1 = n_2 = n$, α is determined by the relation

$$\alpha^2 = \frac{n}{b^2 \gamma_6^2} - 1 \tag{16}$$

where $F(b) = 1 = \varepsilon$. The orders of the quantiles and the values of the α_i used in the test are given by

$$
\begin{array}{lll}
p_1 = 0.00549 & p_8 = 0.99451 & \alpha_1 = 0.0307 \\
p_2 = 0.0286 & p_7 = 0.9714 & \alpha_2 = 0.0730 \\
p_3 = 0.0851 & p_6 = 0.9149 & \alpha_3 = 0.1168 \\
p_4 = 0.2017 & p_5 = 0.7983 & \alpha_4 = 0.1477
\end{array}
$$

Through the use of these values, Eq. (16) becomes

$$\alpha^2 = \frac{3.6817n}{b^2} - 1 \tag{17}$$

Since the positive root of Eq. (17) is used for $\theta > 1$ and the negative root used for $\theta < 1$, Test $\overline{\mathrm{E}}_8$ can now be stated as follows: if

$$
\begin{aligned}
\bar{y}_{8i} = {} & (1 \pm \alpha)\{0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)] \\
& + 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)]\} \\
& + (1 \mp \alpha)\{0.0307\,[z'(0.99451) - z'(0.00549)] + 0.0730\,[z'(0.9714) - z'(0.0286)] \\
& + 0.1168\,[z'(0.9149) - z'(0.0851)] + 0.1477\,[z'(0.7983) - z'(0.2017)]\} \\
& > 0, \qquad \theta \gtrless 1
\end{aligned}
$$

accept H_0. Otherwise, reject H_0.

The power function of Test $\overline{\mathrm{E}}_8$ is identical in form with that of Test $\overline{\mathrm{E}}_6$ and is given by

$$P_0 = F\left\{\frac{-b(1+\alpha^2)^{1/2}[1+\alpha+\theta(1-\alpha)]}{2^{1/2}[(1+\alpha)^2+(1-\alpha)^2\theta^2]^{1/2}}\right\}$$

If $n_1 \neq n_2$, α is determined by the relation

$$0.1358\left[(1+\alpha^2)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+2\alpha\left(\frac{1}{n_1}-\frac{1}{n_2}\right)\right]-\frac{1}{b^2}=0 \tag{18}$$

The positive root of Eq. (18) is used for $\theta > 1$ and the negative root is used for $\theta < 1$.

The power function P_0' of the best test using all the sample values is given (Ref. 2, p. 27) by

$$P_0' = F\left(-b + \frac{n^{1/2}}{2}\ln\theta^2\right), \qquad \theta > 1$$

$$P_0' = F\left(-b + \frac{n^{1/2}}{2}\ln\frac{1}{\theta^2}\right), \qquad \theta < 1$$

It is readily seen that $P_0'(\theta) = P_0'(1/\theta)$ and $P_0(\theta) = P_0(1/\theta)$ in Tests $\overline{\mathrm{E}}_6$ and $\overline{\mathrm{E}}_8$.

Table 7 shows P_0' (Test $\overline{\mathrm{E}}$) and the efficiency of Test $\overline{\mathrm{E}}_6$ and Test $\overline{\mathrm{E}}_8$, for $n_1 = n_2 = 200$, $\varepsilon = 0.01$.

Table 7. P_0' (Test $\overline{\mathrm{E}}$) and efficiency of Tests $\overline{\mathrm{E}}_6$ and $\overline{\mathrm{E}}_8$ for $n_1 = n_2 = 200$, $\varepsilon = 0.01$

θ	P_0' (Test $\overline{\mathrm{E}}$)	Efficiency	
		Test $\overline{\mathrm{E}}_6$	Test $\overline{\mathrm{E}}_8$
1.025	0.0240	0.9433	0.9563
1.05	0.0509	0.9069	0.9263
1.10	0.1640	0.8735	0.9013
1.15	0.3635	0.8750	0.9038
1.20	0.5995	0.8975	0.9225
1.25	0.7976	0.9267	0.9453
1.30	0.9168	0.9572	0.9688
1.35	0.9724	0.9789	0.9850
1.40	0.9925	0.9914	0.9940

VI. Tests F and $\bar{F}$: Tests of Independence and Estimation of the Correlation Coefficient ρ

A. Statement of the Problem

Given a set of n independent pairs of observations $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ taken from two normally distributed populations with means μ_1 and μ_2 and variances σ_1 and σ_2, one is often interested in the answers to the following two questions:

(1) Can we assert that the set of observations

$$x = \{x_1, x_2, \cdots, x_n\}$$

is independent of the set of observations

$$y = \{y_1, y_2, \cdots, y_n\}\ ?$$

(2) What can be said about the correlation between them, if any?

To answer the first question, the problem of testing the null hypothesis

$$\mathrm{H}_0\colon g_1(x) = N(\mu_1, \sigma_1), \qquad g_2(y) = N(\mu_2, \sigma_2), \qquad \rho = 0$$

against the alternative hypothesis

$$\mathrm{H}_1\colon g_1(x) = N(\mu_1, \sigma_1), \qquad g_2(y) = N(\mu_2, \sigma_2), \qquad \rho \neq 0$$

will be considered under two different assumptions. In Test F, since we will assume that the means and variances are known, we can, without loss of generality, assume standard normal distribution, that is, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$. In Test $\bar{F}$ we will assume that $\mu = \mu_1 = \mu_2$ and $\sigma = \sigma_1 = \sigma_2$ are both unknown.

To answer the second question, unbiased estimators, $\hat{\rho}_1$ and $\hat{\rho}_2$ of ρ, will be constructed. For $\hat{\rho}_1$, the assumptions of Test F will be retained. For $\hat{\rho}_2$, we will assume that $\mu = \mu_1 = \mu_2$ is unknown and that σ_1 and σ_2 are known and hence can be put equal to 1.

B. Tests F_6 and F_8 and Estimators $\hat{\rho}_1$

At this point, it is necessary to form two new sets of values $\{u_i\}$ and $\{v_i\}$ from the sample values $\{x_i\}$ and $\{y_i\}$ by means of the linear transformations

$$u_i = \frac{2^{1/2}}{2}(x_i + y_i)$$

$$v_i = \frac{2^{1/2}}{2}(-x_i + y_i)$$

It is easily verified that, under H_0,

$$\mathrm{E}(u_i) = \mathrm{E}(v_i) = 0$$

$$\mathrm{var}(u_i) = \mathrm{var}(v_i) = 1$$

$$\mathrm{E}(u_i v_i) = 0$$

and, under H_1,

$$\mathrm{E}(u_i) = \mathrm{E}(v_i) = 0$$

$$\mathrm{var}(u_i) = 1 + \rho, \qquad \mathrm{var}(v_i) = 1 - \rho$$

$$\mathrm{E}(u_i v_i) = 0$$

Therefore, the set of values $\{u_i\}$ *is independent of the set of values* $\{v_i\}$ *under both hypotheses.* Tests F_6 and F_8 and $\hat{\rho}_1$ will be based on the quantiles of the transformed sets of variable $\{u_i\}$ and $\{v_i\}$, which are all normally distributed.

Beginning with Test F_6, let z_i, $i = 1, 2, \cdots, 6$, be six sample quantiles of the $\{u_i\}$ such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$, and let z'_i be the corresponding sample quantiles of the $\{v_i\}$. The test statistic is given by

$$y_6 = \alpha_1[z_6 - z'_6 - (z_1 - z'_1)] + \alpha_2[z_5 - z'_5 - (z_2 - z'_2)] + \alpha_3[z_4 - z'_4 - (z_3 - z'_3)]$$

where $2(\alpha_1\zeta_6 + \alpha_2\zeta_5 + \alpha_3\zeta_4) = 1$. The rejection region is

$$y_6 \gtrless k, \qquad \rho \gtrless 0$$

Under H_0,

$$\mathrm{E}(y_6) = 0$$

$$\mathrm{var}(y_6) = \frac{4\gamma_5^2}{n}$$

For $\rho > 0$,

$$\mathrm{pr}\,(y_6 < k) = F\left(\frac{k\,n^{1/2}}{2\gamma_5}\right) = F\,(b) = 1 - \varepsilon$$

$$k = \frac{2\gamma_5 b}{n^{1/2}}$$

Under H_1,

$$\mathrm{E}\,(y_6) = (1+\rho)^{1/2} - (1-\rho)^{1/2}$$

$$\mathrm{var}\,(y_6) = \frac{4\gamma_5^2}{n}$$

For $\rho > 0$,

$$\mathrm{pr}\,(y_6 < k) = F\left\{\frac{k - [(1+\rho)^{1/2} - (1-\rho)]^{1/2}}{\dfrac{2\gamma_5}{n^{1/2}}}\right\}$$

$$= F\left[b - \frac{(1+\rho)^{1/2} - (1-\rho)^{1/2}}{\dfrac{2\gamma_5}{n^{1/2}}}\right] = 1 - P_0$$

Test F_6 can now be stated as follows: if

$$\begin{aligned} y_6 = {} & 0.0549\,[z\,(0.9896) - z'\,(0.9896) - z\,(0.0104) + z'\,(0.0104)] \\ & + 0.1244\,[z\,(0.9452) - z'\,(0.9452) - z\,(0.0548) + z'\,(0.0548)] \\ & + 0.1825\,[z\,(0.8304) - z'\,(0.8304) - z\,(0.1696) + z'\,(0.1696)] \\ & \lesseqgtr \pm \frac{1.0583b}{n^{1/2}}, \qquad \rho \gtrless 0 \end{aligned}$$

accept H_0. Otherwise, reject H_0.

The orders of the quantiles and the values of the α_i which maximize P_0 are the same as those which maximize P_0 (Test B_6), namely,

$p_1 = 0.0104$	$p_6 = 0.9896$	$\alpha_1 = 0.0549$
$p_2 = 0.0548$	$p_5 = 0.9452$	$\alpha_2 = 0.1244$
$p_3 = 0.1696$	$p_4 = 0.8304$	$\alpha_3 = 0.1825$

Using these values, one has

$$k = \pm\frac{1.0583b}{n^{1/2}}, \qquad \rho \gtrless 0, \qquad F\,(b) = 1 - \varepsilon$$

$$P_0 = F\,\{-b + 0.9449\,n^{1/2}\,[\pm\,(1+\rho)^{1/2} \mp (1-\rho)^{1/2}]\}, \qquad \rho \gtrless 0$$

The power functions of the best test using all the transformed values $\{u_i\}$ and $\{v_i\}$ are given by

$$P_0' = F\left\{\frac{1}{1-\rho^2}\,[-b\,(1+\rho^2)^{1/2} \pm \rho\,n^{1/2}]\right\}, \qquad \rho \gtrless 0$$

It is readily seen that $P_0'\,(\rho_1) = P_0'\,(-\rho_1)$ and that $P_0\,(\rho_1) = P_0\,(-\rho_1)$.

In Test F_8, omitting many of the details, the order of the quantiles and the values of the α_i which maximize the power of the test are

$p_1 = 0.00549$	$p_8 = 0.99451$	$\alpha_1 = 0.0307$
$p_2 = 0.0286$	$p_7 = 0.9714$	$\alpha_2 = 0.0730$
$p_3 = 0.0851$	$p_6 = 0.9149$	$\alpha_3 = 0.1168$
$p_4 = 0.2017$	$p_5 = 0.7983$	$\alpha_4 = 0.1477$

For these values,

$$k = \pm\frac{1.0423b}{n^{1/2}}, \qquad \rho \gtrless 0, \qquad F\,(b) = 1 - \varepsilon$$

$$P_0 = F\,\{-b + 0.9594\,n^{1/2}\,[\pm(1+\rho)^{1/2} \mp (1-\rho)^{1/2}]\}, \qquad \rho \gtrless 0$$

Test F_8 can now be stated as follows: if

$$\begin{aligned} y_8 = {} & 0.0307\,[z(0.99451) - z'(0.99451) - z(0.00549) + z'(0.00549)] \\ & + 0.0730\,[z(0.9714) - z'(0.9714) - z(0.0286) + z'(0.0286)] \\ & + 0.1168\,[z(0.9149) - z'(0.9149) - z(0.0851) + z'(0.0851)] \\ & + 0.1477\,[z(0.7983) - z'(0.7983) - z(0.2017) + z'(0.2017)] \\ & \lessgtr \pm \frac{1.04236}{n^{1/2}}, \qquad \rho \gtrless 0 \end{aligned}$$

accept H_0. Otherwise, reject H_0.

Unbiased estimators of ρ, denoted by $\hat{\rho}_1$, will now be constructed using six and eight pairs of sample quantiles. The efficiencies of these estimators will be determined relative to the sample correlation coefficient r, the minimum variance unbiased estimator of ρ, given by

$$r = \frac{\sum_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v})}{\left[\sum_{i=1}^{n} (u_i - \bar{u})^2 \sum_{i=1}^{n} (v_i - \bar{v})^2\right]^{1/2}}$$

where

$$\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i, \qquad \bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i$$

for the special case $\rho = 0$. Since the asymptotic variance, var $(r|\rho = 0)$, is $1/(n-1)$ (Ref. 5), the efficiency will be defined as

$$\text{eff}\,(\hat{\rho}_1) = \frac{\text{var}\,(r|\rho = 0)}{\text{var}\,(\hat{\rho}_1|\rho = 0)} = \frac{1}{(n-1)\,\text{var}\,(\hat{\rho}_1|\rho = 0)}$$

When estimators of ρ were previously constructed using one and two pairs of sample quantiles, it was shown that the use of the optimum quantiles in Test F was very nearly optimum with respect to maximizing the efficiency of the estimators. Consequently, this procedure was adopted for estimating ρ using four pairs of quantiles and will also be used here.

Let z_i and z'_i, $i = 1, 2, \cdots, 6$, be defined as in Test F_6. Then an unbiased estimator of ρ using six pairs of sample quantiles is given by

$$\hat{\rho}_1 = \frac{\alpha_1\,[z_1^2 + z_6^2 - (z_1'^2 + z_6'^2)] + \alpha_2\,[z_2^2 + z_5^2 - (z_2'^2 + z_5'^2)] + \alpha_3\,[z_3^2 + z_4^2 - (z_3'^2 + z_4'^2)]}{4\,[\alpha_1\,(a_6^2 + \zeta_6^2) + \alpha_2\,(a_5^2 + \zeta_5^2) + \alpha_3\,(a_4^2 + \zeta_4^2)]}$$

where

$$a_i^2 = \frac{F(\zeta_i)\,[1 - F(\zeta_i)]}{n f^2(\zeta_i)}, \qquad i = 4, 5, 6$$

It is easily verified that $E(\hat{\rho}_1) = \rho$. The variance of the estimator is given by

$$\begin{aligned}\operatorname{var}(\hat{\rho}_1) = \frac{1}{2\left[\alpha_1(a_6^2+\zeta_6^2)+\alpha_2(a_5^2+\zeta_5^2)+\alpha_3(a_4^2+\zeta_4^2)\right]^2}&\{\alpha_1^2\left[a_6^4(1+\rho_{16}^2)+2a_6^2\zeta_6^2(1-\rho_{16})\right]\\ &+\alpha_2^2\left[a_5^4(1+\rho_{25}^2)+2a_5^2\zeta_5^2(1-\rho_{25})\right]\\ &+\alpha_3^2\left[a_4^4(1-\rho_{34}^2)+2a_4^2\zeta_4^2(1-\rho_{34})\right]\\ &+2\alpha_1\alpha_2\left[2a_5a_6\zeta_5\zeta_6(\rho_{12}-\rho_{15})+a_5^2\zeta_5^2(\rho_{12}^2+\rho_{15}^2)\right]\\ &+2\alpha_1\alpha_3\left[2a_4a_6\zeta_4\zeta_6(\rho_{13}-\rho_{14})+a_4^2a_6^2(\rho_{13}^2+\rho_{14}^2)\right]\\ &+2\alpha_2\alpha_3\left[2a_4a_5\zeta_4\zeta_5(\rho_{23}-\rho_{24})+a_4^2a_5^2(\rho_{23}^2+\rho_{24}^2)\right]\}(1+\rho^2)\end{aligned}$$

where ρ_{ij} denotes the correlation between z_i and z_j as well as the correlation between z'_i and z'_j. Using the optimum quantiles and the values of the α_i used in Test F_6 results in the following:

$$\hat{\rho}_1 = \frac{\begin{array}{l}0.0549\{[z(0.0104)]^2+[z(0.9896)]^2-[z'(0.0104)]^2-[z'(0.9896)]^2\}\\ +0.1244\{[z(0.0548)]^2+[z(0.9452)]^2-[z'(0.0548)]^2-[z'(0.9452)]^2\}\\ +0.1825\{[z(0.1696)]^2+[z(0.8304)]^2-[z'(0.1696)]^2-[z'(0.8304)]^2\}\end{array}}{\dfrac{6.6777}{n}+3.1143}$$

$$\operatorname{var}(\hat{\rho}_1 \mid \rho = 0) = \frac{1.2947+1.4252n}{5.5740+5.1991n+1.2123n^2}$$

For $n = 200$,

$$\operatorname{var}(\hat{\rho}_1 \mid \rho = 0) = 0.005780$$
$$\operatorname{eff}(\hat{\rho}_1) = 0.8694$$

Now let z_i and z'_i, $i = 1, 2, \cdots, 8$, be defined as in Test F_8. Then, omitting the details, using the optimum quantiles and values of the α_i of Test F_8 results in the following unbiased estimate of ρ and its variance, using eight pairs of sample quantiles:

$$\hat{\rho}_1 = \frac{\begin{array}{l}0.0307\{[z(0.00549)]^2+[z(0.99451)]^2-[z'(0.00549)]^2-[z'(0.99451)]^2\}\\ +0.0730\{[z(0.0286)]^2+[z(0.9714)]^2-[z'(0.0286)]^2-[z'(0.9714)]^2\}\\ +0.1168\{[z(0.0851)]^2+[z(0.9149)]^2-[z'(0.0851)]^2-[z'(0.9149)]^2\}\\ +0.1477\{[z(0.2017)]^2+[z(0.7983)]^2-[z'(0.2017)]^2-[z'(0.7983)]^2\}\end{array}}{\dfrac{7.2993}{n}+3.1397}$$

$$\operatorname{var}(\hat{\rho}_1 \mid \rho = 0) = \frac{1.3135+1.4097n}{6.6599+5.7194n+1.2322n^2}$$

For $n = 200$,

$$\operatorname{var}(\hat{\rho}_1 \mid \rho = 0) = 0.005615$$
$$\operatorname{eff}(\hat{\rho}_1) = 0.8949$$

C. Tests $\overline{F}_6$ and $\overline{F}_8$ and Estimators $\hat{\rho}_2$

Under the assumptions of Test $\overline{F}$, the linear transformations

$$u_i = \frac{2^{1/2}}{2}(x_i + y_i)$$

$$v_i = \frac{2^{1/2}}{2}(-x_i + y_i)$$

give the following results:

Under H_0,

$$E(u_i) = \mu 2^{1/2}, \qquad E(v_i) = 0$$

$$\mathrm{var}(u_i) = \mathrm{var}(v_i) = \sigma^2$$

$$E(u_i v_i) = 0$$

Under H_1,

$$E(u_i) = u 2^{1/2}, \qquad E(v_i) = 0$$

$$\mathrm{var}(u_i) = \sigma^2(1+\rho), \qquad \mathrm{var}(v_i) = \sigma^2(1-\rho)$$

$$E(u_i v_i) = 0$$

Now let z_i, $i = 1, 2, \cdots, 6$, be six sample quantiles of the $\{u_i\}$ such that $p_1 + p_6 = p_2 + p_5 = p_3 + p_4 = 1$ and let z_i' be the corresponding quantiles of the $\{v_i\}$. To eliminate dependence on μ and σ, the statistic that will be used for the test is given by

$$\begin{aligned}\overline{y}_6 = (1+\alpha)\,[\alpha_1(z_6 - z_1) + \alpha_2(z_5 - z_2) + \alpha_3(z_4 - z_3)] \\ + (1-\alpha)\,[\alpha_1(z_6' - z_1') + \alpha_2(z_5' - z_2') + \alpha_3(z_4' - z_3')]\end{aligned}$$

where $2(\alpha_1\zeta_6 + \alpha_2\zeta_5 + \alpha_3\zeta_4) = 1$, and the rejection region will be taken as $\overline{y}_6 < 0$.

Under H_0,

$$E(\overline{y}_6) = 2\sigma$$

$$\mathrm{var}(\overline{y}_6) = \frac{4(1+\alpha^2)}{n}\sigma^2\gamma_5^2$$

$$\mathrm{pr}(\overline{y}_6 < 0) = F\left(\frac{-n^{1/2}}{(1+\alpha^2)^{1/2}\gamma_5}\right) = F(-b) = \varepsilon$$

where $F(b) = 1 - \varepsilon$ and

$$\alpha^2 = \frac{n}{b^2\gamma_5^2} - 1 \tag{19}$$

Under H_1,

$$E(\overline{y}_6) = \sigma\,[(1+\alpha)(1+\rho)^{1/2} + (1-\alpha)(1-\rho)^{1/2}]$$

$$\mathrm{var}(\overline{y}_6) = \frac{4\sigma^2\gamma_5^2}{n}(1 + 2\alpha\rho + \alpha^2)$$

$$\mathrm{pr}(\overline{y}_6 < 0) = F\left[\frac{-b(1+\alpha^2)^{1/2}[(1+\alpha)(1+\rho)^{1/2} + (1-\alpha)(1-\rho)^{1/2}]}{2(1+2\alpha\rho+\alpha^2)^{1/2}}\right] = P_0 \tag{20}$$

If one uses in Test $\overline{F}_6$ the same quantiles and values of the α_i as were used in Test F_6, Eq. (19) becomes

$$\alpha^2 = \frac{3.5173n}{b^2} - 1 \tag{21}$$

In order to maximize P_0, it can be seen that the negative root of α^2 should be used for $\rho > 0$ and the positive root for $\rho < 0$. Thus Test $\overline{F}_6$ can be stated as follows: if

$$\begin{aligned}\overline{y}_6 = (1 \mp \alpha)\,\{&0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)] + 0.1825\,[z(0.8304) - z(0.1696)]\} \\ + (1 \pm \alpha)\,\{&0.0549\,[z'(0.9896) - z'(0.0104)] + 0.1244\,[z'(0.9452) - z'(0.0548)] + 0.1825\,[z'(0.8304) - z'(0.1696)]\} \\ > 0, \qquad & \rho \gtrless 0\end{aligned}$$

accept H_0. Otherwise, reject H_0. Here α is determined from Eq. (21).

With respect to Test $\overline{F}_8$, the use of the same quantiles and values of the α_i as were used in Test F_8 results in the following:

$$\alpha^2 = \frac{n}{b^2\gamma_6^2} - 1 = \frac{3.6817n}{b^2} - 1$$

where $F(b) = 1 - \varepsilon$. The power function of the test is identical in form with that given in Eq. (20), and Test $\overline{F}_8$ can be finally stated as follows: if

$$\begin{aligned}\bar{y}_8 = {} & (1 \mp \alpha)\{0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)] \\ & + 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)]\} \\ & + (1 \pm \alpha)\{0.0307\,[z'(0.99451) - z'(0.00549)] + 0.0730\,[z'(0.9714) - z'(0.0286)] \\ & + 0.1168\,[z'(0.9149) - z'(0.0851)] + 0.1477\,[z'(0.7983) - z'(0.2017)]\} \\ & > 0, \qquad \rho \gtrless 0\end{aligned}$$

accept H_0. Otherwise, reject H_0.

Table 8 gives P_0' (Test F) and the efficiency of Tests F_6, $\overline{F}_6$, F_8, and $\overline{F}_8$, for $n = 200$, $\varepsilon = 0.01$.

Under the assumption that $\mu = \mu_1 = \mu_2$ is unknown and $\sigma_1 = \sigma_2 = 1$, unbiased estimators of ρ, denoted by $\hat{\rho}_2$, will now be constructed using six and eight pairs of sample quantiles from the $\{u_i\}$ and $\{v_i\}$. The efficiencies of $\hat{\rho}_2$ will also be defined as

$$\text{eff}\,(\hat{\rho}_2) = \frac{1}{(n-1)\,\text{var}\,(\rho_2\,|\,\rho = 0)}$$

Let z_i and z_i', $i = 1, 2, \cdots, 6$, be defined as in Test $\overline{F}_6$. Then an unbiased estimator of ρ using six pairs of sample quantiles is given by

$$\hat{\rho}_2 = \frac{\alpha_1\,[(z_6 - z_1)^2 - (z_6' - z_1')^2] - \alpha_2\,[(z_5 - z_2)^2 - (z_5' - z_2')^2] + \alpha_3\,[(z_4 - z_3)^2 - (z_4' - z_3')^2]}{4D}$$

Table 8. P_0' (Test F) and efficiency of Tests F_6, $\overline{F}_6$, F_8, and $\overline{F}_8$, for $n = 200$, $\varepsilon = 0.01$

$\pm\rho$	P_0' (Test F)	Efficiency			
		Test F_6	Test $\overline{F}_6$	Test F_8	Test $\overline{F}_8$
0.010	0.01445	0.9806	0.9751	0.9862	0.9799
0.025	0.02419	0.9587	0.9450	0.9702	0.9582
0.050	0.05198	0.9369	0.9194	0.9569	0.9394
0.075	0.1004	0.9250	0.9071	0.9508	0.9330
0.100	0.1755	0.9206	0.9055	0.9495	0.9346
0.150	0.4067	0.9246	0.9197	0.9535	0.9491
0.200	0.6828	0.9383	0.9427	0.9607	0.9655
0.250	0.8876	0.9590	0.9665	0.9721	0.9795
0.300	0.9769	0.9808	0.9859	0.9862	0.9909
0.350	0.9972	0.9952	0.9970	0.9966	0.9983

where

$$D = \alpha_1 [a_6^2 (1 - \rho_{16}) + 2\zeta_6^2] + \alpha_2 [a_5^2 (1 - \rho_{25}) + 2\zeta_5^2] + \alpha_3 [a_3^2 (1 - \rho_{34}) + 2\zeta_4^2]$$

$$a_i^2 = \frac{F(\zeta_i)[1 - F(\zeta_i)]}{nf^2(\zeta_i)}, \qquad i = 4, 5, 6$$

and ρ_{ij} is the correlation between z_i and z_j as well as between z'_i and z'_j. It is not difficult to verify that $E(\hat{\rho}_2) = \rho$. The variance of $\hat{\rho}_2$ is given by

$$\begin{aligned} \mathrm{var}(\hat{\rho}_2) = \frac{1}{D^2} \{ & \alpha_1^2 [a_6^4 (1 - \rho_{16})^2 + 4a_6^2\zeta_6^2 (1 - \rho_{16})] + \alpha_2^2 [a_5^4 (1 - \rho_{25})^2 + 4a_5^2\zeta_5^2 (1 - \rho_{25})] \\ & + \alpha_3^2 [a_4^4 (1 - \rho_{34})^2 + 4a_4^2\zeta_4^2 (1 - \rho_{34})] + 2\alpha_1\alpha_2 [a_5^2 a_6^2 (\rho_{12} - \rho_{15})^2 + 4a_5 a_6 \zeta_5 \zeta_6 (\rho_{12} - \rho_{15})] \\ & + 2\alpha_1\alpha_3 [a_4^2 a_6^2 (\rho_{13} - \rho_{14})^2 + 4a_4 a_6 \zeta_4 \zeta_6 (\rho_{13} - \rho_{14})] + 2\alpha_2\alpha_3 [a_4^2 a_5^2 (\rho_{23} - \rho_{24})^2 + 4a_4 a_5 \zeta_4 \zeta_5 (\rho_{23} - \rho_{24})]\} (1 + \rho^2) \end{aligned}$$

The use of the same quantiles and values of the α_i as were used in Test $\overline{\mathrm{F}}_6$ results in the following:

$$\hat{\rho}_2 = \frac{\begin{array}{l} 0.0549 \{[z(0.9896) - z(0.0104)]^2 - [z'(0.9896) - z'(0.0104)]^2\} \\ + 0.1244 \{[z(0.9452) - z(0.0548)]^2 - [z'(0.9452) - z'(0.0548)]^2\} \\ + 0.1825 \{[z(0.8304) - z(0.1696)]^2 - [z'(0.8304) - z'(0.1696)]^2\} \end{array}}{\dfrac{6.1961}{n} + 6.2286}$$

$$\mathrm{var}(\hat{\rho}_2 | \rho = 0) = \frac{1.1072 + 2.8505n}{2.3995 + 4.8241n + 2.4247n^2}$$

For $n = 200$,

$$\mathrm{var}(\hat{\rho}_2 | \rho = 0) = 0.005831$$

$$\mathrm{eff}(\hat{\rho}_2) = 0.8617$$

Now let z_i and z'_i, $i = 1, 2, \cdots, 8$, be defined as in Test $\overline{\mathrm{F}}_8$. Then, using the same quantiles and values of the α_i as were used in Test $\overline{\mathrm{F}}_8$, and omitting the details, one has

$$\hat{\rho}_2 = \frac{\begin{array}{l} 0.0307 \{[z(0.99451) - z(0.00549)]^2 - [z'(0.99451) - z'(0.00549)]^2\} \\ + 0.0730 \{[z(0.9714) - z(0.0286)]^2 - [z'(0.9714) - z'(0.0286)]^2\} \\ + 0.1168 \{[z(0.9149) - z(0.0851)]^2 - [z'(0.9149) - z'(0.0851)]^2\} \\ + 0.1477 \{[z(0.7983) - z(0.2017)]^2 - [z'(0.7983) - z'(0.2017)]^2\} \end{array}}{\dfrac{6.8466}{n} + 6.2794}$$

$$\mathrm{var}(\hat{\rho}_2 | \rho = 0) = \frac{1.1360 + 2.8194n}{2.9298 + 5.3740n + 2.4644n^2}$$

For $n = 200$,

$$\mathrm{var}(\hat{\rho}_2 | \rho = 0) = 0.005668$$

$$\mathrm{eff}(\hat{\rho}_2) = 0.8863$$

VII. Applying the Tests

Two sets of samples, each containing 200 sample values, were drawn from a table of random numbers (Ref. 6) in which the entries are distributed $N(0, 1)$. Hence, the sets of sample values can be considered as samples of two independent normal random variables, x and y, with means $\mu_x = \mu_y = 0$ and variances $\sigma_x^2 = \sigma_y^2 = 1$. The sample quantiles (denoted by $z(p)$ and $z'(p)$, respectively) necessary to perform the tests, as well as those used for the estimation of ρ, were determined. All the tests were performed at a significance level of 0.01. From the sample values of x, the following quantiles were obtained:

$z(0.0540) = -1.782$	$z(0.0104) = -2.112$
$z(0.1915) = -0.979$	$z(0.0548) = -1.782$
$z(0.3898) = -0.247$	$z(0.1696) = -1.125$
$z(0.6102) = 0.246$	$z(0.8304) = 0.875$
$z(0.8085) = 0.812$	$z(0.9452) = 1.500$
$z(0.9460) = 1.500$	$z(0.9896) = 2.358$

$z(0.0310) = -1.912$	$z(0.00549) = -2.444$
$z(0.1154) = -1.316$	$z(0.0286) = -1.939$
$z(0.2481) = -0.689$	$z(0.0851) = -1.616$
$z(0.4126) = -0.211$	$z(0.2017) = -0.957$
$z(0.5874) = 0.200$	$z(0.7985) = 0.790$
$z(0.7519) = 0.681$	$z(0.9149) = 1.241$
$z(0.8846) = 1.129$	$z(0.9714) = 1.808$
$z(0.9690) = 1.719$	$z(0.99451) = 2.462$

From the sample values of y, the following quantiles were also obtained:

$z'(0.0540) = -1.569$	$z'(0.0104) = -2.205$
$z'(0.1915) = -0.765$	$z'(0.0546) = -1.569$
$z'(0.3898) = -0.285$	$z'(0.1696) = -0.820$
$z'(0.6102) = 0.246$	$z'(0.8304) = 0.949$
$z'(0.8085) = 0.923$	$z'(0.9452) = 1.407$
$z'(0.9460) = 1.407$	$z'(0.9896) = 2.358$

$z'(0.0310) = -1.753$	$z'(0.00549) = -2.273$
$z'(0.1154) = -1.086$	$z'(0.0286) = -1.893$
$z'(0.2481) = -0.544$	$z'(0.0851) = -1.244$
$z'(0.4126) = -0.185$	$z'(0.2017) = -0.734$
$z'(0.5874) = 0.200$	$z'(0.7983) = 0.790$
$z'(0.7519) = 0.681$	$z'(0.9149) = 1.246$
$z'(0.8846) = 1.129$	$z'(0.9714) = 1.646$
$z'(0.9690) = 1.719$	$z'(0.99451) = 2.320$

The sample means and sample standard deviations were computed and found to be

$$\bar{x} = -0.0557 \qquad s_x = 0.9994$$

$$\bar{y} = 0.0345 \qquad s_y = 0.9372$$

The corresponding estimates using six and eight optimal sample quantiles were also found to be

$$\hat{\mu}_x = -0.0574 \qquad \hat{\sigma}_x = 1.0187$$

$$\hat{\mu}_x = -0.0349 \qquad \hat{\sigma}_x = 1.0159$$

$$\hat{\mu}_y = 0.0168 \qquad \hat{\sigma}_y = 0.9370$$

$$\hat{\mu}_y = 0.0258 \qquad \hat{\sigma}_y = 0.9201$$

Estimates of $\rho = 0$ using six and eight pairs of quantiles, as well as the sample correlation, were also computed and were found to be

$$\hat{\rho}_1 = 0.0680 \qquad \hat{\rho}_2 = 0.0655 \qquad r = 0.0245$$

$$\hat{\rho}_1 = 0.0850 \qquad \hat{\rho}_2 = 0.0830$$

Tests A, $\bar{\text{A}}$, and B, using six and eight sample quantiles, were performed on both sets of samples, with H_0 being true. In all six tests, H_0 was accepted. For Tests D, $\bar{\text{D}}$, and $\bar{\text{E}}$, which require sample quantiles from both sets of samples for each test, H_0 was accepted in all six tests when H_0 was true. For Tests F and $\bar{\text{F}}$, it was assumed that the given sets of sample values were actually transformed values $\{u_i\}$ and $\{v_i\}$ obtained from sets $\{x_i\}$ and $\{y_i\}$ taken from two normal distributions with $\rho = 0$. With H_0 being true, in each of the four tests H_0 was accepted.

Now, if x is distributed $N(\mu, \sigma)$, then $x' = ax + b$, $a > 0$, is distributed $N(a\mu + b, a\sigma)$. If the transformations above were applied to all the sample values taken from a population distributed $N(\mu, \sigma)$, one sees that not only would the new sample values be distributed $N(a\mu + b, a\sigma)$ but the *order* of the samples would remain unchanged; that is, if $x_i < x_j$, then $x_i' < x_j'$. Hence, if $z(p)$ were the quantile of order p of the $\{x_i\}$, then $az(p) + b$ would be the quantile of order p of the $\{x'_i\}$. This fact permits us to perform the tests when H_0 is *not true* by simply performing a linear transformation on the sample quantiles of the x_i and y_i. These tests will be given in detail. The best tests using all the sample values will also be given.

In Tests A and $\bar{A}$, by adding 0.20 to each quantile $z(p)$ and $z'(p)$, one can assume in each case that $\mu_1 = 0$, $\mu_2 = 0.20$, $\sigma = 1.0$, and H_1 is true. The results of each test and the decision are as follows ($\mathfrak{z}$ and $\mathfrak{z}'$ will denote the sample quantile *after* the transformation):

Test A_6

$$0.0968(\mathfrak{z}_1 + \mathfrak{z}_6) + 0.1787(\mathfrak{z}_2 + \mathfrak{z}_5) + 0.2245(\mathfrak{z}_3 + \mathfrak{z}_4) = 0.1468 < 0.1682, \quad \text{accept } H_0$$

$$0.0968(\mathfrak{z}'_1 + \mathfrak{z}'_6) + 0.1787(\mathfrak{z}'_2 + \mathfrak{z}'_5) + 0.2245(\mathfrak{z}'_3 + \mathfrak{z}'_4) = 0.2168 > 0.1682, \quad \text{reject } H_0$$

Test A_8

$$0.0559(\mathfrak{z}_1 + \mathfrak{z}_8) + 0.1119(\mathfrak{z}_2 + \mathfrak{z}_7) + 0.1550(\mathfrak{z}_3 + \mathfrak{z}_6) + 0.2245(\mathfrak{z}_4 + \mathfrak{z}_5) = 0.1653 < 0.1668, \quad \text{accept } H_0$$

$$0.0559(\mathfrak{z}'_1 + \mathfrak{z}'_8) + 0.1119(\mathfrak{z}'_2 + \mathfrak{z}'_7) + 0.1550(\mathfrak{z}'_3 + \mathfrak{z}'_6) + 0.2245(\mathfrak{z}'_4 + \mathfrak{z}'_5) = 0.2258 > 0.1668, \quad \text{reject } H_0$$

Test $\bar{A}_6$

$$0.1276\mathfrak{z}_1 + 0.0660\mathfrak{z}_6 + 0.2095\mathfrak{z}_2 + 0.1479\mathfrak{z}_5 + 0.2553\mathfrak{z}_3 + 0.1937\mathfrak{z}_4 = -0.0288 < \mu_1, \quad \text{accept } H_0$$

$$0.1276\mathfrak{z}'_1 + 0.0660\mathfrak{z}'_6 + 0.2095\mathfrak{z}'_2 + 0.1479\mathfrak{z}'_5 + 0.2553\mathfrak{z}'_3 + 0.1937\mathfrak{z}'_4 = 0.0550 > \mu_1, \quad \text{reject } H_0$$

Test $\bar{A}_8$

$$0.0771\mathfrak{z}_1 + 0.0347\mathfrak{z}_8 + 0.1331\mathfrak{z}_2 + 0.0907\mathfrak{z}_7 + 0.1762\mathfrak{z}_3 + 0.1338\mathfrak{z}_6 + 0.1984\mathfrak{z}_4 + 0.1560\mathfrak{z}_5 = -0.0015 < \mu_1, \quad \text{accept } H_0$$

$$0.0771\mathfrak{z}'_1 + 0.0347\mathfrak{z}'_8 + 0.1331\mathfrak{z}'_2 + 0.0907\mathfrak{z}'_7 + 0.1762\mathfrak{z}'_3 + 0.1338\mathfrak{z}'_6 + 0.1984\mathfrak{z}'_4 + 0.1560\mathfrak{z}'_5 = 0.0730 > \mu_1, \quad \text{reject } H_0$$

Adding 0.20 to each sample value and applying Test A to all the sample values results in

$$\frac{1}{n}\sum_{i=1}^{n} x_i = 0.1443 < 0.1645, \quad \text{accept } H_0$$

$$\frac{1}{n}\sum_{i=1}^{n} y_i = 0.2345 > 0.1645, \quad \text{reject } H_0$$

In Test B, by multiplying each quantile by 1.2, one can assume that $\mu = 0$, $\sigma_1 = 1.0$, $\sigma_2 = 1.2$, and H_1 is true. Then one has

Test B_6

$$0.0549(\mathfrak{z}_6 - \mathfrak{z}_1) + 0.1244(\mathfrak{z}_5 - \mathfrak{z}_2) + 0.1825(\mathfrak{z}_4 - \mathfrak{z}_3) = 1.2224 > 1.1230, \quad \text{reject } H_0$$

$$0.0549(\mathfrak{z}'_6 - \mathfrak{z}'_1) + 0.1244(\mathfrak{z}'_5 - \mathfrak{z}'_2) + 0.1825(\mathfrak{z}'_4 - \mathfrak{z}'_3) = 1.2444 > 1.1230, \quad \text{reject } H_0$$

Test B_8

$$0.0307(\mathfrak{z}_8 - \mathfrak{z}_1) + 0.0730(\mathfrak{z}_7 - \mathfrak{z}_2) + 0.1168(\mathfrak{z}_6 - \mathfrak{z}_3) + 0.1477(\mathfrak{z}_5 - \mathfrak{z}_4) = 1.1041 < 1.1207, \quad \text{accept } H_0$$

$$0.0307(\mathfrak{z}'_8 - \mathfrak{z}'_1) + 0.0730(\mathfrak{z}'_7 - \mathfrak{z}'_2) + 0.1168(\mathfrak{z}'_6 - \mathfrak{z}'_3) + 0.1477(\mathfrak{z}'_5 - \mathfrak{z}'_4) = 1.2191 > 1.1207, \quad \text{reject } H_0$$

Multiplying each sample value by 1.2 and then applying Test B to all the sample values results in

$$2\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = 23.963 > 22.326, \quad \text{reject } H_0$$

$$2\left(\sum_{i=1}^{n} y_i^2\right)^{1/2} = 22.451 > 22.326, \quad \text{reject } H_0$$

In Tests D and $\overline{\mathrm{D}}$, by putting $\theta = 0.20$ and hence adding 0.20 to each $z'(p)$ and leaving each $z(p)$ unchanged, one can assume that $\theta = 0.20$, $\mu_2 = \mu_1 + 0.20$, $\sigma_1 = \sigma_2 = 1.0$, and H_1 is true. Then one has

Test D_6

$$0.0968\,(z_1 - \tilde{z}'_1 + z_6 - \tilde{z}'_6) + 0.1787\,(z_2 - \tilde{z}'_2 + z_5 - \tilde{z}'_5) + 0.2245\,(z_3 - \tilde{z}'_3 + z_4 - \tilde{z}'_4) = -0.2751 < -0.2379, \quad \text{reject } \mathrm{H}_0$$

Test D_8

$$\begin{aligned} &0.0559\,(z_1 - \tilde{z}'_1 + z_8 - \tilde{z}'_8) + 0.1119\,(z_2 - \tilde{z}'_2 + z_7 - \tilde{z}'_7) + 0.1550\,(z_3 - \tilde{z}'_3 + z_6 - \tilde{z}'_6) \\ &+ 0.1772\,(z_4 - \tilde{z}'_4 + z_5 - \tilde{z}'_5) = -0.2607 < -0.2359, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Test $\overline{\mathrm{D}}_6$

$$\begin{aligned} &0.1184\,(z_6 - \tilde{z}'_1) + 0.0752\,(z_1 - \tilde{z}'_6) + 0.2003\,(z_5 - \tilde{z}'_2) + 0.1571\,(z_2 - \tilde{z}'_5) \\ &+ 0.2461\,(z_4 - \tilde{z}'_3) + 0.2029\,(z_3 - \tilde{z}'_4) = -0.0405 < 0, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Test $\overline{\mathrm{D}}_8$

$$\begin{aligned} &0.0708\,(z_8 - \tilde{z}'_1) + 0.0410\,(z_1 - \tilde{z}'_8) + 0.1268\,(z_7 - \tilde{z}'_2) + 0.0970\,(z_2 - \tilde{z}'_7) + 0.1699\,(z_6 - \tilde{z}'_3) \\ &+ 0.1401\,(z_3 - \tilde{z}'_6) + 0.1921\,(z_5 - \tilde{z}'_4) + 0.1623\,(z_4 - \tilde{z}'_5) = -0.0360 < 0, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Adding 0.20 to each y_i, leaving each x_i unchanged, and then applying Test D to all the sample values results in

$$\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} y_i = -0.2902 < -0.2326, \quad \text{reject } \mathrm{H}_0$$

In Test $\overline{\mathrm{E}}$, by multiplying each $z'(p)$ by 1.30 and leaving each $z(p)$ unchanged, one can assume that $\theta = 1.30$, $\mu_1 = \mu_2 = 0$, $\sigma_2 = 1.30\,\sigma_1$, and H_1 is true. Then one has

Test $\overline{\mathrm{E}}_6$

$$\begin{aligned} &12.4463\,[0.0549\,(z_6 - z_1) + 0.1244\,(z_5 - z_2) + 0.1825\,(z_4 - z_3)] \\ &-10.4463\,[0.0549\,(\tilde{z}'_6 - \tilde{z}'_1) + 0.1244\,(\tilde{z}'_5 - \tilde{z}'_2) + 0.1825\,(\tilde{z}'_4 - \tilde{z}'_3)] = -0.0456 < 0, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Test $\overline{\mathrm{E}}_8$

$$\begin{aligned} &12.6233\,[0.0307\,(z_8 - z_1) + 0.0730\,(z_7 - z_2) + 0.1168\,(z_6 - z_3) + 0.1477\,(z_5 - z_4)] \\ &+10.6233\,[0.0307\,(\tilde{z}'_8 - \tilde{z}'_1) + 0.0730\,(\tilde{z}'_7 - \tilde{z}'_2) + 0.1168\,(\tilde{z}'_6 - \tilde{z}'_3) + 0.1477\,(\tilde{z}'_5 - \tilde{z}'_4)] = 0.1172 > 0, \quad \text{accept } \mathrm{H}_0 \end{aligned}$$

Multiplying each y_i by 1.30, leaving each x_i unchanged, and then applying Test $\overline{\mathrm{E}}$ to all the sample values results in

$$\frac{1}{2}\ln\left(\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} y_i^2}\right) = -0.0856 > -0.1645, \quad \text{accept } \mathrm{H}_0$$

In Tests F and $\overline{\mathrm{F}}$, by multiplying each $z(p)$ by $1.2^{1/2}$ and each $z'(p)$ by $0.80^{1/2}$, it can be assumed that each $\tilde{z}(p)$ is the quantile of order p of a transformed set of variables $\{\mu_i\}$ distributed $N[0, (1+\rho)^{1/2}]$, that each $\tilde{z}'(p)$ is the quantile of order p of a transformed set $\{v_i\}$, distributed $N[0, (1-\rho)^{1/2}]$, and that the transformations were applied to correlated sets $\{x_i\}$ and $\{y_i\}$, each distributed $N(0,1)$ with $\rho = 0.2$. Hence, for Tests F and $\overline{\mathrm{F}}$, one has $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0.20$, and H_1 is true. The results of the tests are

Test F_6

$$0.0549\,(\tilde{z}_6 - \tilde{z}'_6 - \tilde{z}_1 + \tilde{z}'_1) + 0.1244\,(\tilde{z}_5 - \tilde{z}'_5 - \tilde{z}_2 + \tilde{z}'_2) + 0.1825\,(\tilde{z}_4 - \tilde{z}'_4 - \tilde{z}_3 + \tilde{z}'_3) = 0.2741 > 0.1741, \quad \text{reject } \mathrm{H}_0$$

Test F_8

$$\begin{aligned} &0.0307\,(\tilde{z}_8 - \tilde{z}'_8 - \tilde{z}_1 + \tilde{z}'_1) + 0.0730\,(\tilde{z}_7 - \tilde{z}'_7 - \tilde{z}_2 + \tilde{z}'_2) + 0.1168\,(\tilde{z}_6 - \tilde{z}'_6 - \tilde{z}_3 + \tilde{z}'_3) \\ &+ 0.1477\,(\tilde{z}_5 - \tilde{z}'_5 - \tilde{z}_4 - \tilde{z}'_4) = 0.2899 > 0.1714, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Test $\overline{\mathrm{F}}_6$

$$\begin{aligned} &-10.4463\,[0.0549\,(\tilde{z}_6 - \tilde{z}_1) + 0.1244\,(\tilde{z}_5 - \tilde{z}_2) + 0.1825\,(\tilde{z}_4 - \tilde{z}_3)] \\ &+ 12.4463\,[0.0549\,(\tilde{z}'_6 - \tilde{z}'_1) + 0.1244\,(\tilde{z}'_5 - \tilde{z}'_2) + 0.1825\,(\tilde{z}'_4 - \tilde{z}'_3)] = -1.2264 < 0, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Test $\overline{\mathrm{F}}_8$

$$\begin{aligned} &-10.6233\,[0.0307\,(\tilde{z}_8 - \tilde{z}_1) + 0.0730\,(\tilde{z}_7 - \tilde{z}_2) + 0.1163\,(\tilde{z}_6 - \tilde{z}_3) + 0.1477\,(\tilde{z}_5 - \tilde{z}_4)] \\ &+ 12.6233\,[0.0307\,(\tilde{z}'_8 - \tilde{z}'_1) + 0.0730\,(\tilde{z}'_7 - \tilde{z}'_2) + 0.1163\,(\tilde{z}'_6 - \tilde{z}'_3) + 0.1477\,(\tilde{z}'_5 - \tilde{z}'_4)] = -1.4338 < 0, \quad \text{reject } \mathrm{H}_0 \end{aligned}$$

Multiplying each x_i by $1.2^{1/2}$ and each y_i by $0.80^{1/2}$ and applying Test F to all the sample values results in

$$\sum_{i=1}^{n} (\rho - 1)\, u_i^2 + \sum_{i=1}^{n} (\rho + 1)\, v_i^2 = -23.390 < 12.908, \quad \text{reject } \mathrm{H}_0$$

VIII. Suboptimum Test Statistics

Tables 9 through 12 give the test statistics and acceptance regions to be used in Tests A, B, D, and F. The tests are given as functions of n for $\varepsilon = 0.01$ and $\varepsilon = 0.05$. Tables 13 through 16 give the test statistics and acceptance regions to be used in Tests $\overline{\mathrm{A}}$, $\overline{\mathrm{D}}$, $\overline{\mathrm{E}}$, and $\overline{\mathrm{F}}$ as functions of n and ε.

However, in order to apply the results developed here to statistical experiments performed aboard a spacecraft, it may be necessary to specify the order of the quantiles in advance. For maximum data compression, only one set of k quantiles should be so specified for a k quantile test or estimator, regardless of which test or estimator is required. Since a set of quantiles that is optimum for one test is not, as we have seen, necessarily optimum for another, it is obvious that a compromise is required, based on some reasonable criterion. This problem was encountered in our previous investigations into the use of quantiles for data compression, and hence a proposed solution is at hand and will be presented here.

It has, no doubt, been noted that only two sets of k quantiles or k pairs of quantiles (as well as the values of the α_i) have been used for the tests and for estimating ρ. The sets and values of the α_i used in Tests A, $\overline{\mathrm{A}}$, D, and $\overline{\mathrm{D}}$ are those which provide the asymptotically unbiased estimators of the mean of a single normal population with minimum variance. The sets and values of the α_i used in Tests B, $\overline{\mathrm{E}}$, F, and $\overline{\mathrm{F}}$ and for estimating ρ are those which provide the asymptotically unbiased estimators of the standard deviation with minimum variance. Thus we are faced with the problem of effecting a compromise between two sets of quantiles and values of the α_i, one that minimizes $\mathrm{var}\,(\hat{\mu})$ and another that minimizes $\mathrm{var}\,(\hat{\sigma})$. The compromise we now propose is one that was adopted previously for estimating μ and σ and for the tests using four quantiles. Determine the orders of the set of $k/2$ pairs of symmetric quantiles and weights α_i and β_i, $i = 1, 2, \cdots, k/2$, such that unbiased estimators of μ and σ are given by

$$\hat{\mu} = \sum_{i=1}^{k/2} \alpha_i\,(z_1 + z_{k-i+1})$$

$$\hat{\sigma} = \sum_{i=1}^{k/2} \beta_i\,(z_{k-i+1} - z_i)$$

and for which the linear combination $\mathrm{var}\,(\hat{\mu}) + C\,\mathrm{var}\,(\hat{\sigma})$ is a minimum, $C = 1, 2, \cdots$.

The same sets of quantiles are to be used in all tests and for estimating ρ. Weights α_i are to be used in the test statistics of Tests A, A, D, and D, while weights β_i are to be used in the test statistics of Tests B, E, F, and F and for estimating ρ.

In Ref. 1, sets of six and eight quantiles as well as the values of the α_i and β_i are given which meet the above conditions for $C = 1, 2, 3$. Test statistics (designated as suboptimum) and acceptance regions were calculated for these sets of quantiles and values of the α_i and β_i and are given in Tables 17 through 22 as functions of n and ε. Finally, Tables 23 through 26 give the near-optimum and suboptimum estimators of ρ. In all cases, the loss in efficiency in going from optimum or near-optimum to suboptimum conditions is not excessive.

Table 9. Test statistics and acceptance regions for Tests A_6 and A_8

H_0: $g(x) = g_1(x) = N(\mu_1, \sigma)$

H_1: $g(x) = g_2(x) = N(\mu_2, \sigma)$, σ known

Conditions	Acceptance regions
$\mu_2 \gtrless \mu_1$ $\varepsilon = 0.01$	$0.0968\,[z(0.0540) + z(0.9460)] + 0.1787\,[z(0.1915) + z(0.8085)] + 0.2245\,[z(0.3898) + z(0.6102)] \lessgtr \mu_1 \pm \frac{2.3793\sigma}{n^{1/2}}$ $0.0059\,[z(0.0310) + z(0.9690)] + 0.1119\,[z(0.1154) + z(0.8846)] + 0.1550\,[z(0.2481) + z(0.7519)] + 0.1772\,[z(0.4126) + z(0.5874)] \lessgtr \mu_1 \pm \frac{2.3594\sigma}{n^{1/2}}$
$\mu_2 \gtrless \mu_1$ $\varepsilon = 0.05$	$0.0968\,[z(0.0540) + z(0.9460)] + 0.1787\,[z(0.1915) + z(0.8085)] + 0.2245\,[z(0.3898) + z(0.6102)] \lessgtr \mu_1 \pm \frac{1.6822\sigma}{n^{1/2}}$ $0.0559\,[z(0.0310) + z(0.9690)] + 0.1119\,[z(0.1154) + z(0.8846)] + 0.1550\,[z(0.2481) + z(0.7519)] + 0.1772\,[z(0.4126) + z(0.5874)] \lessgtr \mu_1 \pm \frac{1.6681\sigma}{n^{1/2}}$

Table 10. Test statistics and acceptance regions for Tests B_6 and B_8

H_0: $g(x) = g_1(x) = N(\mu, \sigma_1)$

H_1: $g(x) = g_2(x) = N(\mu, \sigma_2)$, μ unknown

Conditions	Acceptance regions
$\sigma_2 \gtrless \sigma_1$ $\varepsilon = 0.01$	$0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)] + 0.1825\,[z(0.8304) - z(0.1696)] \lessgtr \sigma_1\left(1.0 \pm \frac{1.7411}{n^{1/2}}\right)$ $0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)] + 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)] \lessgtr \sigma_1\left(1.0 \pm \frac{1.7146}{n^{1/2}}\right)$
$\sigma_2 \gtrless \sigma_1$ $\varepsilon = 0.05$	$0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)] + 0.1825\,[z(0.8304) - z(0.1696)] \lessgtr \sigma_1\left(1.0 \pm \frac{1.2309}{n^{1/2}}\right)$ $0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)] + 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)] \lessgtr \sigma_1\left(1.0 \pm \frac{1.2121}{n^{1/2}}\right)$

Table 11. Test statistics and acceptance regions for Tests D_6 and D_8

H_0: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \sigma)$, σ known, μ unknown

H_1: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu + \theta, \sigma)$, $\theta \neq 0$

Conditions	Acceptance regions
$\theta \gtrless 0$ $\varepsilon = 0.01$	$0.0968\,[z(0.0540) - z'(0.0540) + z(0.9460) - z'(0.9460)]$ $+\,0.1787\,[z(0.1915) - z'(0.1915) + z(0.8085) - z'(0.8085)]$ $+\,0.2245\,[z(0.3898) - z'(0.3898) + z(0.6102) - z'(0.6102)]$ $\gtrless \mp 2.3793\,\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$ $0.0559\,[z(0.0310) - z'(0.0310) + z(0.9690) - z'(0.9690]$ $+\,0.1119\,[z(0.1154) - z'(0.1154) + z(0.8846) - z'(0.8846)]$ $+\,0.1550\,[z(0.2481) - z'(0.2481) + z(0.7519) - z'(0.7519)]$ $+\,0.1772\,[z(0.4126) - z'(0.4126) + z(0.5874) - z'(0.5874)]$ $\gtrless \mp 2.3594\,\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$
$\theta \gtrless 0$ $\varepsilon = 0.05$	$0.0968\,[z(0.0540) - z'(0.0540) + z(0.9460) - z'(0.9460)]$ $+\,0.1787\,[z(0.1915) - z'(0.1915) + z(0.8085) - z'(0.8085)]$ $+\,0.2245\,[z(0.3898) - z'(0.3898) + z(0.6102) - z'(0.6102)]$ $\gtrless \mp 1.6822\,\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$ $0.0559\,[z(0.0310) - z'(0.0310) + z(0.9690) - z'(0.9690)]$ $+\,0.1119\,[z(0.1154) - z'(0.1154) + z(0.8846) - z'(0.8846)]$ $+\,0.1550\,[z(0.2481) - z'(0.2481) + z(0.7519) - z'(0.7519)]$ $+\,0.1772\,[z(0.4126) - z'(0.4126) + z(0.5874) - z'(0.5874)]$ $\gtrless \mp 1.6681\,\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$

Table 12. Test statistics and acceptance regions for Tests F_6 and F_8

H_0: $g_1(x) = N(0, 1)$, $g_2(y) = N(0, 1)$, $\rho = 0$

H_1: $g_1(x) = N(0, 1)$, $g_2(y) = N(0, 1)$, $\rho \neq 0$

Conditions	Acceptance regions
$\rho \gtrless 0$ $\varepsilon = 0.01$	$0.0549\,[z(0.9896) - z'(0.9896) - z(0.0104) + z'(0.0104)] + 0.1244\,[z(0.9452) - z'(0.9452) - z(0.0548) + z'(0.0548)] + 0.1825\,[z(0.8304) - z'(0.8304) - z(0.1696) + z'(0.1696)] \lesseqgtr \pm \frac{2.4620}{n^{1/2}}$ $0.0307\,[z(0.99451) - z'(0.99451) - z(0.00549) + z'(0.00549)] + 0.0730\,[z(0.9714) - z'(0.9714) - z(0.0286) + z'(0.0286)] + 0.1168\,[z(0.9149) - z'(0.9149) - z(0.0851) + z'(0.0851)] + 0.1477\,[z(0.7983) - z'(0.7983) - z(0.2016) + z'(0.2016)] \lesseqgtr \pm \frac{2.4248}{n^{1/2}}$
$\rho \gtrless 0$ $\varepsilon = 0.05$	$0.0549\,[z(0.9896) - z'(0.9896) - z(0.0104) + z'(0.0104)] + 0.1244\,[z(0.9452) - z'(0.9452) - z(0.0548) + z'(0.0548)] + 0.1825\,[z(0.8304) - z'(0.8304) - z(0.1696) + z'(0.1696)] \lesseqgtr \pm \frac{1.7406}{n^{1/2}}$ $0.0307\,[z(0.99451) - z'(0.99451) - z(0.00549) + z'(0.00549)] + 0.0730\,[z(0.9714) - z'(0.9714) - z(0.0286) + z'(0.0286)] + 0.1168\,[z(0.9149) - z'(0.9149) - z(0.0851) + z'(0.0851)] + 0.1477\,[z(0.7983) - z'(0.7983) - z(0.2016) + z'(0.2016)] \lesseqgtr \pm \frac{1.7143}{n^{1/2}}$

Table 13. Test statistics and acceptance regions for Tests $\bar{A}_6$ and $\bar{A}_8$, using near-optimum quantiles

H_0: $g(x) = g_1(x) = N(\mu_1, \sigma)$

H_1: $g(x) = g_2(x) = N(\mu_2, \sigma)$, σ unknown

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test $\bar{A}_6$ $(0.0968 \pm \alpha)\,z(0.0540) + (0.0968 \mp \alpha)\,z(0.9460) + (0.1787 \pm \alpha)\,z(0.1915) + (0.1787 \mp \alpha)\,z(0.8085) + (0.2245 \pm \alpha)\,z(0.3898) + (0.2245 \mp \alpha)\,z(0.6102) \lesseqgtr \mu_1$	$\alpha^2 = \frac{b^2}{29.117\,n - 18.696\,b^2}$ $F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\bar{A}_8$ $(0.0559 \pm \alpha)\,z(0.0310) + (0.0559 \mp \alpha)\,z(0.9690) + (0.1119 \pm \alpha)\,z(0.1154) + (0.1119 \mp \alpha)\,z(0.8846) + (0.1550 \pm \alpha)\,z(0.2481) + (0.1550 \mp \alpha)\,z(0.7519) + (0.1772 \pm \alpha)\,z(0.4126) + (0.1772 \mp \alpha)\,z(0.5874) \lesseqgtr \mu_1$	$\alpha^2 = \frac{b^2}{61.163\,n - 35.890\,b^2}$ $F(b) = 1 - \varepsilon$

Table 14. Test statistics and acceptance regions for Tests $\bar{D}_6$ and $\bar{D}_8$, using near-optimum quantiles

H_0: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \sigma)$, μ and σ unknown
H_1: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu + \theta, \sigma)$, $\theta \neq 0$

Conditions	Acceptance regions	Constraints
$\theta \gtrless 0$	Test $\bar{D}_6$ $(0.0968 \pm \alpha)\,[z(0.9460) - z'(0.0540)] + (0.0968 \mp \alpha)\,[z(0.0540) - z'(0.9460)]$ $+ (0.1787 \pm \alpha)\,[z(0.8085) - z'(0.1915)] + (0.1787 \mp \alpha)\,[z(0.1915) - z'(0.8085)]$ $+ (0.2245 \pm \alpha)\,[z(0.6102) - z'(0.3898)] + (0.2245 \mp \alpha)\,[z(0.3898) - z'(0.6102)]$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{58.234\,n - 18.696\,b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\bar{D}_8$ $(0.0559 \pm \alpha)\,[z(0.9690) - z'(0.0310)] + (0.0559 \mp \alpha)\,[z(0.0310) - z'(0.9690)]$ $+ (0.1119 \pm \alpha)\,[z(0.8846) - z'(0.1154)] + (0.1119 \mp \alpha)\,[z(0.1154) - z'(0.8846)]$ $+ (0.1550 \pm \alpha)\,[z(0.7519) - z'(0.2481)] + (0.1550 \mp \alpha)\,[z(0.2481) - z'(0.7519)]$ $+ (0.1772 \pm \alpha)\,[z(0.5874) - z'(0.4126)] + (0.1772 \mp \alpha)\,[z(0.4126) - z'(0.5874)]$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{122.325\,n - 35.890\,b^2}$ $F(b) = 1 - \varepsilon$

Table 15. Test statistics and acceptance regions for Tests $\bar{E}_6$ and $\bar{E}_8$, using near-optimum quantiles

H_0: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \sigma)$, μ and σ unknown
H_1: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \theta\sigma)$, $\theta > 0$.

Conditions	Acceptance regions	Constraints
$\theta \gtrless 1$	Test $\bar{E}_6$ $(1 \pm \alpha)\,\{0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)]$ $+ 0.1825\,[z(0.8304) - z(0.1696)]\} + (1 \mp \alpha)\,\{0.0549\,[z'(0.9896) - z'(0.0104)]$ $+ 0.1244\,[z'(0.9452) - z'(0.0548)] + 0.1825\,[z'(0.8304) - z'(0.1696)]\}$ > 0	$\alpha^2 = \dfrac{3.5713\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\bar{E}_8$ $(1 \pm \alpha)\,\{0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)]$ $+ 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)]\}$ $+ (1 \mp \alpha)\,\{0.0307\,[z'(0.99451) - z'(0.00549)] + 0.0730\,[z'(0.9714) - z'(0.0286)]$ $+ 0.1168\,[z'(0.9149) - z'(0.0851)] + 0.1477\,[z'(0.7983) - z'(0.2017)]\}$ > 0	$\alpha^2 = \dfrac{3.6817\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 16. Test statistics and acceptance regions for Tests $\bar{F}_6$ and $\bar{F}_8$, using near-optimum quantiles

H_0: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \sigma)$, $\rho = 0$
H_1: $g_1(x) = N(\mu, \sigma)$, $g_2(y) = N(\mu, \sigma)$, $\rho \neq 0$, μ and σ unknown

Conditions	Acceptance regions	Constraints
$\rho \gtrless 0$	Test $\bar{F}_6$ $(1 \mp \alpha)\,\{0.0549\,[z(0.9896) - z(0.0104)] + 0.1244\,[z(0.9452) - z(0.0548)]$ $+ 0.1825\,[z(0.8304) - z(0.1696)]\} + (1 \pm \alpha)\,\{0.0549\,[z'(0.9896) - z(0.0104)]$ $+ 0.1244\,[z'(0.9452) - z'(0.0548)] + 0.1825\,[z'(0.8304) - z'(0.1696)]\}$ > 0	$\alpha^2 = \dfrac{3.5713\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\bar{F}_8$ $(1 \mp \alpha)\,\{0.0307\,[z(0.99451) - z(0.00549)] + 0.0730\,[z(0.9714) - z(0.0286)]$ $+ 0.1168\,[z(0.9149) - z(0.0851)] + 0.1477\,[z(0.7983) - z(0.2017)]\}$ $+ (1 \pm \alpha)\,\{0.0307\,[z'(0.99451) - z'(0.00549)] + 0.0730\,[z'(0.9714) - z'(0.0286)]$ $+ 0.1168\,[z'(0.9149) - z'(0.0851)] + 0.1477\,[z'(0.7983) - z'(0.2017)]\}$ > 0	$\alpha^2 = \dfrac{3.6817\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 17. Suboptimum test statistics and acceptance regions for $k = 6$, $C = 1$

$p_1 = 0.0231 \qquad p_6 = 0.9769$

$p_2 = 0.1180 \qquad p_5 = 0.8820$

$p_3 = 0.3369 \qquad p_4 = 0.6631$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_6 $0.0497\,(z_1 + z_6) + 0.1550\,(z_2 + z_5) + 0.2953\,(z_3 + z_4)$ $\gtrless \mu_1 \pm \dfrac{1.0282\, b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\bar{A}_6$ $(0.0497 \pm \alpha)\, z_1 + (0.0497 \mp \alpha)\, z_6 + (0.1550 \pm \alpha)\, z_2 + (0.1550 \mp \alpha)\, z_5$ $+ (0.2953 \pm \alpha)\, z_3 + (0.2953 \mp \alpha)\, z_4$ $\lesseqgtr \mu_1$	$\alpha^2 = \dfrac{b^2}{49.033\, n - 29.516\, b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_6 $0.1088\,(z_6 - z_1) + 0.1951\,(z_5 - z_2) + 0.1228\,(z_4 - z_3)$ $\lesseqgtr \sigma_1 \left(1.0 \pm \dfrac{0.7650\, b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_6 $0.0497\,(z_1 - z_1' + z_6 - z_6') + 0.1550\,(z_2 - z_2' + z_5 - z_5')$ $+ 0.2953\,(z_3 - z_3' + z_4 - z_4')$ $\gtrless \mp 1.0282\, b\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\bar{D}_6$ $(0.0497 \pm \alpha)\,(z_6 - z_1') + (0.0497 \mp \alpha)\,(z_1 - z_6') + (0.1550 \pm \alpha)\,(z_5 - z_2') + (0.1550 \mp \alpha)\,(z_2 - z_5')$ $+ (0.2953 \pm \alpha)\,(z_4 - z_3') + (0.2953 \mp \alpha)\,(z_3 - z_4')$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{98.065\, n - 29.516\, b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\bar{E}_6$ $(1 \pm \alpha)\,[0.1088\,(z_6 - z_1) + 0.1951\,(z_5 - z_2) + (0.1228\,(z_4 - z_3)]$ $+ (1 \mp \alpha)\,[0.1088\,(z_6' - z_1') + 0.1951\,(z_5' - z_2') + 0.1228\,(z_4' - z_3')]$ > 0	$\alpha^2 = \dfrac{3.4176\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_6 $0.1088\,(z_6 - z_6' - z_1 + z_1') + 0.1951\,(z_5 - z_5' - z_2 + z_2')$ $+ 0.1228\,(z_4 - z_4' - z_3 + z_3')$ $\lesseqgtr \pm \dfrac{1.0819\, b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\bar{F}_6$ $(1 \mp \alpha)\,[0.1088\,(z_6 - z_1) + 0.1951\,(z_5 - z_2) + 0.1228\,(z_4 - z_3)]$ $+ (1 \pm \alpha)\,[0.1088\,(z_6' - z_1') + 0.1951\,(z_5' - z_2') + 0.1228\,(z_4' - z_3')]$ > 0	$\alpha^2 = \dfrac{3.4176\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 18. Suboptimum test statistics and acceptance regions for $k = 6$, $C = 2$

$p_1 = 0.0193 \qquad p_6 = 0.9807$

$p_2 = 0.1009 \qquad p_5 = 0.8991$

$p_3 = 0.3071 \qquad p_4 = 0.6929$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_6 $0.0424\,(z_1 + z_6) + 0.1401\,(z_2 + z_5) + 0.3175\,(z_3 + z_4)$ $\lessgtr \mu_1 \pm \frac{1.0348\, b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\overline{A}_6$ $(0.0424 \pm \alpha)\, z_1 + (0.0424 \mp \alpha)\, z_6 + (0.1401 \pm \alpha)\, z_2 + (0.1401 \mp \alpha)\, z_5$ $+ (0.3175 \pm \alpha)\, z_3 + (0.3175 \mp \alpha)\, z_4$ $\lessgtr \mu_1$	$\alpha^2 = \frac{b^2}{55.344\, n - 34.559\, b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_6 $0.0940\,(z_6 - z_1) + 0.1847\,(z_5 - z_2) + 0.1387\,(z_4 - z_3)$ $\lessgtr \sigma_1 \left(1.0 \pm \frac{0.7606\, b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_6 $0.0424\,(z_1 - z'_1 + z_6 - z'_6) + 0.1401\,(z_2 - z'_2 + z_5 - z'_5)$ $+ 0.3175\,(z_3 - z'_3 + z_4 - z'_4)$ $\gtrless \mp 1.0348\, b\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\overline{D}_6$ $(0.0424 \pm \alpha)\,(z_6 - z'_1) + (0.0424 \mp \alpha)\,(z_1 - z'_6) + (0.1401 \pm \alpha)\,(z_5 - z'_2)$ $+ (0.1401 \mp \alpha)\,(z_2 - z'_5) + (0.3175 \pm \alpha)\,(z_4 - z'_3) + (0.3175 \mp \alpha)\,(z_3 - z'_4)$ $\gtrless 0$	$\alpha^2 = \frac{b^2}{110.688\, n - 34.559\, b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\overline{E}_6$ $(1 \pm \alpha)\,[0.0940\,(z_6 - z_1) + 0.1847\,(z_5 - z_2) + 0.1387\,(z_4 - z_3)]$ $+ (1 \mp \alpha)\,[0.0940\,(z'_6 - z'_1) + 0.1847\,(z'_5 - z'_2) + 0.1387\,(z'_4 - z'_3)]$ > 0	$\alpha^2 = \frac{3.4572\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_6 $0.0940\,(z_6 - z'_6 - z_1 + z'_1) + 0.1847\,(z_5 - z'_5 - z_2 + z'_2)$ $+ 0.1387\,(z_4 - z'_4 - z_3 + z'_3)$ $\lessgtr \pm \frac{1.0757\, b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\overline{F}_6$ $(1 \mp \alpha)\,[0.0940\,(z_6 - z_1) + 0.1847\,(z_5 - z_2) + 0.1387\,(z_4 - z'_3)]$ $+ (1 \pm \alpha)\,[0.0940\,(z'_6 - z'_1) + 0.1847\,(z'_5 - z'_2) + 0.1387\,(z'_4 - z'_3)]$ > 0	$\alpha^2 = \frac{3.4572\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 19. Suboptimum test statistics and acceptance regions for $k = 6$, $C = 3$

$p_1 = 0.0175$	$p_6 = 0.9825$
$p_2 = 0.0922$	$p_5 = 0.9078$
$p_3 = 0.2858$	$p_4 = 0.7142$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_6 $0.0389(z_1 + z_6) + 0.1306(z_2 + z_5) + 0.3305(z_3 + z_4)$ $\lessgtr \mu_1 \pm \frac{1.0383\, b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\overline{A}_6$ $(0.0389 \pm \alpha) z_1 + (0.0389 \mp \alpha) z_6 + (0.1306 \pm \alpha) z_2 + (0.1306 \mp \alpha) z_5$ $+ (0.3305 \pm \alpha) z_3 + (0.3305 \mp \alpha) z_4$ $\lessgtr \mu_1$	$\alpha^2 = \frac{b^2}{59.420\, n - 35.816\, b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_6 $0.0865(z_6 - z_1) + 0.1764(z_5 - z_2) + 0.1478(z_4 - z_3)$ $\lessgtr \sigma_1 \left(1.0 \pm \frac{0.7580\, b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_6 $0.0389(z_1 - z'_1 + z_6 - z'_6) + 0.1306(z_2 - z'_2 + z_5 - z'_5)$ $+ 0.3305(z_3 - z'_3 + z_4 - z'_4)$ $\gtrless \mp 1.0383\, b\sigma \left(\frac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\overline{D}_6$ $(0.0389 \pm \alpha)(z_6 - z'_1) + (0.0389 \mp \alpha)(z_1 - z'_6) + (0.1306 \pm \alpha)(z_5 - z'_2)$ $+ (0.1306 \mp \alpha)(z_2 - z'_5) + (0.3305 \pm \alpha)(z_4 - z'_3) + (0.3305 \mp \alpha)(z_3 - z'_4)$ $\gtrless 0$	$\alpha^2 = \frac{b^2}{118.839\, n - 35.816\, b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\overline{E}_6$ $(1 \pm \alpha)[0.0865(z_6 - z_1) + 0.1764(z_5 - z_2) + 0.1478(z_4 - z_3)]$ $+ (1 \mp \alpha)[0.0865(z'_6 - z'_1) + 0.1764(z'_5 - z'_2) + 0.1478(z'_4 - z'_3)]$ > 0	$\alpha^2 = \frac{3.4758\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_6 $0.0865(z_6 - z'_6 - z_1 + z'_1) + 0.1764(z_5 - z'_5 - z_2 + z'_2)$ $+ 0.1478(z_4 - z'_4 - z_3 + z'_3)$ $\lessgtr \pm \frac{1.0720\, b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\overline{F}_6$ $(1 \mp \alpha)[0.0865(z_6 - z_1) + 0.1764(z_5 - z_2) + 0.1478(z_4 - z_3)]$ $+ (1 \pm \alpha)[0.0865(z'_6 - z'_1) + 0.1764(z'_5 - z'_2) + 0.1478(z'_4 - z'_3)]$ > 0	$\alpha^2 = \frac{3.4758\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 20. Suboptimum test statistics and acceptance regions for $k = 8$, $C = 1$

$p_1 = 0.0119$ $\quad p_2 = 0.0604$ $\quad p_3 = 0.1721$ $\quad p_4 = 0.3711$

$p_8 = 0.9881$ $\quad p_7 = 0.9396$ $\quad p_6 = 0.8279$ $\quad p_5 = 0.6289$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_8 $0.0249\,(z_1 + z_8) + 0.0764\,(z_2 + z_7) + 0.1568\,(z_3 + z_6) + 0.2419\,(z_4 + z_5)$ $\lessgtr \mu_1 \pm \dfrac{1.0175\, b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\overline{A}_8$ $(0.0249 \pm \alpha)\, z_1 + (0.0249 \mp \alpha)\, z_8 + (0.0764 \pm \alpha)\, z_2 + (0.0764 \mp \alpha)\, z_7$ $+ (0.1568 \pm \alpha)\, z_3 + (0.1568 \mp \alpha)\, z_6 + (0.2419 \pm \alpha)\, z_4 + (0.2419 \mp \alpha)\, z_5$ $\lessgtr \mu_1$	$\alpha^2 = \dfrac{b^2}{99.965\, n - 58.972\, b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_8 $0.0600\,(z_8 - z_1) + 0.1249\,(z_7 - z_2) + 0.1528\,(z_6 - z_3) + 0.0789\,(z_5 - z_4)$ $\lessgtr \sigma_1 \left(1.0 \pm \dfrac{0.7432\, b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_8 $0.0249\,(z_1 - z'_1 + z_8 - z'_8) + 0.0764\,(z_2 - z'_2 + z_7 - z'_7) + 0.1568\,(z_3 - z'_3 + z_6 - z'_6)$ $+ 0.2419\,(z_4 - z'_4 + z_5 - z'_5)$ $\gtrless \mp 1.0175\, b\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\overline{D}_8$ $(0.0249 \pm \alpha)\,(z_8 - z'_1) + (0.0249 \mp \alpha)\,(z_1 - z'_8) + (0.0764 \pm \alpha)\,(z_7 - z'_2)$ $+ (0.0764 \mp \alpha)\,(z_2 - z'_7) + (0.1568 \pm \alpha)\,(z_6 - z'_3) + (0.1568 \mp \alpha)\,(z_3 - z'_6)$ $+ (0.2419 \pm \alpha)\,(z_5 - z'_4) + (0.2419 \mp \alpha)\,(z_4 - z'_5)$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{199.931\, n - 58.972\, b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\overline{E}_8$ $(1 \pm \alpha)\,[0.0600\,(z_8 - z_1) + 0.1249\,(z_7 - z_2) + 0.1528\,(z_6 - z_3) + 0.0789\,(z_5 - z_4)]$ $+ (1 \mp \alpha)\,[0.0600\,(z'_8 - z'_1) + 0.1249\,(z'_7 - z'_2) + 0.1528\,(z'_6 - z'_3) + 0.789\,(z'_5 - z'_4)]$ > 0	$\alpha^2 = \dfrac{3.6206\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_8 $0.0600\,(z_8 - z'_8 - z_1 + z'_1) + 0.1249\,(z_7 - z'_7 - z_2 + z'_2)$ $+ 0.1528\,(z_6 - z'_6 - z_3 + z'_3) + 0.0789\,(z_5 - z'_5 - z_4 + z'_4)$ $\lessgtr \pm \dfrac{1.0510\, b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\overline{F}_8$ $(1 \mp \alpha)\,[0.0600\,(z_8 - z_1) + 0.1249\,(z_7 - z_2) + 0.1528\,(z_6 - z_3) + 0.0789\,(z_5 - z_4)]$ $+ (1 \pm \alpha)\,[0.0600\,(z'_8 - z'_1) + 0.1249\,(z'_7 - z'_2) + 0.1528\,(z'_6 - z'_3) + 0.0789\,(z'_5 - z'_4)]$ > 0	$\alpha^2 = \dfrac{3.6206\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 21. Suboptimum test statistics and acceptance regions for $k = 8$, C = 2

$p_1 = 0.00998$ $p_2 = 0.0515$ $p_3 = 0.1511$ $p_4 = 0.3481$
$p_8 = 0.99002$ $p_7 = 0.9485$ $p_6 = 0.8489$ $p_5 = 0.6519$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_8 $0.0212(z_1 + z_8) + 0.0668(z_2 + z_7) + 0.1473(z_3 + z_6) + 0.2647(z_4 + z_5)$ $\lesseqgtr \mu_1 \pm \dfrac{1.0201\, b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\bar{A}_8$ $(0.0212 \pm \alpha) z_1 + (0.0212 \mp \alpha) z_8 + (0.0668 \pm \alpha) z_2 + (0.0668 \mp \alpha) z_7$ $+ (0.1473 \pm \alpha) z_3 + (0.1473 \mp \alpha) z_6 + (0.2647 \pm \alpha) z_4 + (0.2647 \mp \alpha) z_5$ $\lesseqgtr \mu_1$	$\alpha^2 = \dfrac{b^2}{111.246\, n - 66.247\, b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_8 $0.0518(z_8 - z_1) + 0.1134(z_7 - z_2) + 0.1534(z_6 - z_3)$ $+ 0.0925(z_5 - z_4)$ $\lesseqgtr \sigma_1 \left(1.0 \pm \dfrac{0.7407\, b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_8 $0.0212(z_1 - z_1' + z_8 - z_8') + 0.0668(z_2 - z_2' + z_7 - z_7') + 0.1473(z_3 - z_3' + z_6 - z_6')$ $+ 0.2647(z_4 - z_4' + z_5 - z_5')$ $\gtrless \mp 1.0201\, b\sigma \left(\dfrac{n_1 + n_2}{n_1 n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\bar{D}_8$ $(0.0212 \pm \alpha)(z_8 - z_1') + (0.0212 \mp \alpha)(z_1 - z_8') + (0.0668 \pm \alpha)(z_7 - z_2')$ $+ (0.0668 \mp \alpha)(z_2 - z_7') + (0.1473 \pm \alpha)(z_6 - z_3') + (0.1473 \mp \alpha)(z_3 - z_6')$ $+ (0.2647 \pm \alpha)(z_5 - z_4') + (0.2647 \mp \alpha)(z_4 - z_5')$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{222.493\, n - 66.247\, b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\bar{E}_8$ $(1 \pm \alpha)[0.0518(z_8 - z_1) + 0.1134(z_7 - z_2) + 0.1534(z_6 - z_3) + 0.0925(z_5 - z_4)]$ $+ (1 \mp \alpha)[0.0518(z_8' - z_1') + 0.1134(z_7' - z_2') + 0.1534(z_6' - z_3') + 0.0925(z_5' - z_4')]$ > 0	$\alpha^2 = \dfrac{3.6456\, n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_8 $0.0518(z_8 - z_8' - z_1 + z_1') + 0.1134(z_7 - z_7' - z_2 + z_2')$ $+ 0.1534(z_6 - z_6' - z_3 + z_3') + 0.0925(z_5 - z_5' - z_4 + z_4')$ $\lesseqgtr \pm \dfrac{1.0475\, b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\bar{F}_8$ $(1 \mp \alpha)[0.0518(z_8 - z_1) + 0.1134(z_7 - z_2) + 0.1534(z_6 - z_3) + 0.0925(z_5 - z_4)]$ $+ (1 \pm \alpha)[0.0518(z_8' - z_1') + 0.1134(z_7' - z_2') + 0.1534(z_6' - z_3') + 0.0925(z_5' - z_4')]$ > 0	$\alpha^2 = \dfrac{3.6456\, n}{b^2} - 1$

Table 22. Suboptimum test statistics and acceptance regions for $k = 8$, $C = 3$

$p_1 = 0.00921$ $\quad p_2 = 0.0476$ $\quad p_3 = 0.1407$ $\quad p_4 = 0.3314$

$p_8 = 0.99079$ $\quad p_7 = 0.9524$ $\quad p_6 = 0.8593$ $\quad p_5 = 0.6686$

Conditions	Acceptance regions	Constraints
$\mu_2 \gtrless \mu_1$	Test A_8 $0.0196\,(z_1 + z_8) + 0.0626\,(z_2 + z_7) + 0.1404\,(z_3 + z_6) + 0.2774\,(z_4 + z_5)$ $\lessgtr \mu_1 \pm \dfrac{1.0223\,b\sigma}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\mu_2 \gtrless \mu_1$	Test $\bar{\bar{A}}_8$ $(0.0196 \pm \alpha)\,z_1 + (0.0196 \mp \alpha)\,z_8 + (0.0626 \pm \alpha)\,z_2 + (0.0626 \mp \alpha)\,z_7$ $+ (0.1404 \pm \alpha)\,z_3 + (0.1404 \mp \alpha)\,z_6 + (0.2774 \pm \alpha)\,z_4 + (0.2774 \mp \alpha)\,z_5$ $\lessgtr \mu_1$	$\alpha^2 = \dfrac{b^2}{117.400\,n - 70.729\,b^2}$ $F(b) = 1 - \varepsilon$
$\sigma_2 \gtrless \sigma_1$	Test B_8 $0.0484\,(z_8 - z_1) + 0.1075\,(z_7 - z_2) + 0.1512\,(z_6 - z_3) + 0.1004\,(z_5 - z_4)$ $\lessgtr \sigma_1 \left(1.0 \pm \dfrac{0.7397\,b}{n^{1/2}}\right)$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test D_8 $0.0196\,(z_1 - z'_1 + z_8 - z'_8) + 0.0626\,(z_2 - z'_2 + z_7 - z'_7) + 0.1404\,(z_3 - z'_3 + z_6 - z'_6)$ $+ 0.2774\,(z_4 - z'_4 + z_5 - z'_5)$ $\gtrless \mp 1.0223\,b\sigma \left(\dfrac{n_1 + n_2}{n_1\,n_2}\right)^{1/2}$	$F(b) = 1 - \varepsilon$
$\theta \gtrless 0$	Test $\bar{\bar{D}}_8$ $(0.0196 \pm \alpha)\,(z_8 - z'_1) + (0.0196 \mp \alpha)\,(z_1 - z'_8) + (0.0626 \pm \alpha)\,(z_7 - z'_2)$ $+ (0.0626 \mp \alpha)\,(z_2 - z'_7) + (0.1404 \pm \alpha)\,(z_6 - z'_3) + (0.1404 \mp \alpha)\,(z_3 - z'_6)$ $+ (0.2774 \pm \alpha)\,(z_5 - z'_4) + (0.2774 \mp \alpha)\,(z_4 - z'_5)$ $\gtrless 0$	$\alpha^2 = \dfrac{b^2}{234.800\,n - 70.729\,b^2}$ $F(b) = 1 - \varepsilon$
$\theta \gtrless 1$	Test $\bar{\bar{E}}_8$ $(1 \pm \alpha)\,[0.0484\,(z_8 - z_1) + 0.1075\,(z_7 - z_2) + 0.1512\,(z_6 - z_3) + 0.1004\,(z_5 - z_4)]$ $+ (1 \mp \alpha)\,[0.0484\,(z'_8 - z'_1) + 0.1075\,(z'_7 - z'_2) + 0.1512\,(z'_6 - z'_3) + 0.1004\,(z'_5 - z'_4)]$ > 0	$\alpha^2 = \dfrac{3.6550\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test F_8 $0.0484\,(z_8 - z'_8 - z_1 + z'_1) + 0.1075\,(z_7 - z'_7 - z_2 + z'_2)$ $+ 0.1512\,(z_6 - z'_6 - z_3 + z'_3) + 0.1004\,(z_5 - z'_5 - z_4 + z'_4)$ $\lessgtr \pm \dfrac{1.0461\,b}{n^{1/2}}$	$F(b) = 1 - \varepsilon$
$\rho \gtrless 0$	Test $\bar{\bar{F}}_8$ $(1 \mp \alpha)\,[0.0484\,(z_8 - z_1) + 0.1075\,(z_7 - z_2) + 0.1512\,(z_6 - z_3) + 0.1004\,(z_5 - z_4)]$ $+ (1 \pm \alpha)\,[0.0484\,(z'_8 - z'_1) + 0.1075\,(z'_7 - z'_2) + 0.1512\,(z'_6 - z'_3) + 0.1004\,(z'_5 - z'_4)]$ > 0	$\alpha^2 = \dfrac{3.6550\,n}{b^2} - 1$ $F(b) = 1 - \varepsilon$

Table 23. Estimators of ρ under near-optimum and suboptimum conditions for $k = 6$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$

Conditions	Estimators of ρ
Near-optimum	$p_1 = 0.0104$ $p_2 = 0.0548$ $p_3 = 0.1696$ $p_6 = 0.9896$ $p_5 = 0.9452$ $p_4 = 0.8304$ $\hat{\rho}_1 = \dfrac{0.0549\,(z_1^2 + z_6^2 - z_1'^2 - z_6'^2) + 0.1244\,(z_2^2 + z_5^2 - z_2'^2 - z_5'^2) + 0.1825\,(z_3^2 + z_4^2 - z_3'^2 - z_4'^2)}{\dfrac{6.6777}{n} + 3.1143}$
$C = 1$	$p_1 = 0.0231$ $p_2 = 0.1180$ $p_3 = 0.3369$ $p_6 = 0.9769$ $p_5 = 0.8820$ $p_4 = 0.6631$ $\hat{\rho}_1 = \dfrac{0.1088\,(z_1^2 + z_6^2 - z_1'^2 - z_6'^2) + 0.1951\,(z_2^2 + z_5^2 - z_2'^2 - z_5'^2) + 0.1228\,(z_3^2 + z_4^2 - z_3'^2 - z_4'^2)}{\dfrac{6.1915}{n} + 2.9132}$
$C = 2$	$p_1 = 0.0193$ $p_2 = 0.1009$ $p_3 = 0.3071$ $p_6 = 0.9807$ $p_5 = 0.8991$ $p_4 = 0.6929$ $\hat{\rho}_1 = \dfrac{0.0940\,(z_1^2 + z_6^2 - z_1'^2 - z_6'^2) + 0.1847\,(z_2^2 + z_5^2 - z_2'^2 - z_5'^2) + 0.1387\,(z_3^2 + z_4^2 - z_3'^2 - z_4'^2)}{\dfrac{6.3320}{n} + 2.9536}$
$C = 3$	$p_1 = 0.0175$ $p_2 = 0.0922$ $p_3 = 0.2858$ $p_6 = 0.9825$ $p_5 = 0.9078$ $p_4 = 0.7142$ $\hat{\rho}_1 = \dfrac{0.0865\,(z_1^2 + z_6^2 - z_1'^2 - z_6'^2) + 0.1764\,(z_2^2 + z_5^2 - z_2'^2 - z_5'^2) + 0.1478\,(z_3^2 + z_4^2 - z_3'^2 - z_4'^2)}{\dfrac{6.3978}{n} + 2.9707}$

Table 24. Estimators of ρ under near-optimum and suboptimum conditions for $k = 8$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$

Conditions	Estimators of ρ
Near-optimum	$p_1 = 0.00549$ $p_2 = 0.0286$ $p_3 = 0.0851$ $p_4 = 0.2017$ $p_8 = 0.99451$ $p_7 = 0.9714$ $p_6 = 0.9149$ $p_5 = 0.7983$ $\hat{\rho}_1 = \dfrac{0.0307\,(z_1^2 + z_8^2 - z_1'^2 - z_8'^2) + 0.0730\,(z_2^2 + z_7^2 - z_2'^2 - z_7'^2) + 0.1168\,(z_3^2 + z_6^2 - z_3'^2 - z_6'^2) + 0.1477\,(z_4^2 + z_5^2 - z_4'^2 - z_5'^2)}{\dfrac{7.2993}{n} + 3.1397}$
$C = 1$	$p_1 = 0.0119$ $p_2 = 0.0604$ $p_3 = 0.1721$ $p_4 = 0.3711$ $p_8 = 0.9881$ $p_7 = 0.9396$ $p_6 = 0.8279$ $p_5 = 0.6289$ $\hat{\rho}_1 = \dfrac{0.0600\,(z_1^2 + z_8^2 - z_1'^2 - z_8'^2) + 0.1249\,(z_2^2 + z_7^2 - z_2'^2 - z_7'^2) + 0.1528\,(z_3^2 + z_6^2 - z_3'^2 - z_6'^2) + 0.0789\,(z_4^2 + z_5^2 - z_4'^2 - z_5'^2)}{\dfrac{6.7687}{n} + 3.0100}$
$C = 2$	$p_1 = 0.00998$ $p_2 = 0.0515$ $p_3 = 0.1511$ $p_4 = 0.3481$ $p_8 = 0.99002$ $p_7 = 0.9485$ $p_6 = 0.8489$ $p_5 = 0.6519$ $\hat{\rho}_1 = \dfrac{0.0518\,(z_1^2 + z_8^2 - z_1'^2 - z_8'^2) + 0.1134\,(z_2^2 + z_7^2 - z_2'^2 - z_7'^2) + 0.1534\,(z_3^2 + z_6^2 - z_3'^2 - z_6'^2) + 0.0925\,(z_4^2 + z_5^2 - z_4'^2 - z_5'^2)}{\dfrac{6.9271}{n} + 3.0374}$
$C = 3$	$p_1 = 0.00921$ $p_2 = 0.0476$ $p_3 = 0.1407$ $p_4 = 0.3314$ $p_8 = 0.99079$ $p_7 = 0.9524$ $p_6 = 0.8593$ $p_5 = 0.6686$ $\hat{\rho}_1 = \dfrac{0.0484\,(z_1^2 + z_8^2 - z_1'^2 - z_8'^2) + 0.1075\,(z_2^2 + z_7^2 - z_2'^2 - z_7'^2) + 0.1512\,(z_3^2 + z_6^2 - z_3'^2 - z_6'^2) + 0.1004\,(z_4^2 + z_5^2 - z_4'^2 - z_5'^2)}{\dfrac{6.9961}{n} + 3.0506}$

Table 25. Estimators of ρ under near-optimum and suboptimum conditions for $k = 6$, $\mu = \mu_1 = \mu_2$ unknown, $\sigma_1 = \sigma_2 = 1$

Conditions	Estimators of ρ
Near-optimum	$p_1 = 0.0104$ $p_2 = 0.0548$ $p_3 = 0.1696$ $p_6 = 0.9896$ $p_5 = 0.9452$ $p_4 = 0.8304$ $\hat{\rho}_2 = \dfrac{0.0549\,[(z_6 - z_1)^2 - (z_6' - z_1')^2] + 0.1244\,[(z_5 - z_2)^2 - (z_5' - z_2')^2] + 0.1825\,[(z_4 - z_3)^2 - (z_4' - z_3')^2]}{\dfrac{6.1961}{n} + 6.2286}$
C = 1	$p_1 = 0.0231$ $p_2 = 0.1180$ $p_3 = 0.3369$ $p_6 = 0.9769$ $p_5 = 0.8820$ $p_4 = 0.6631$ $\hat{\rho}_2 = \dfrac{0.1088\,[(z_6 - z_1)^2 - (z_6' - z_1')^2] + 0.1951\,[(z_5 - z_2)^2 - (z_5' - z_2')^2] + 0.1228\,[(z_4 - z_3)^2 - (z_4' - z_3')^2]}{\dfrac{5.4173}{n} + 5.8264}$
C = 2	$p_1 = 0.0193$ $p_2 = 0.1009$ $p_3 = 0.3071$ $p_6 = 0.9807$ $p_5 = 0.8991$ $p_4 = 0.6929$ $\hat{\rho}_2 = \dfrac{0.0940\,[(z_6 - z_1)^2 - (z_6' - z_1')^2] + 0.1847\,[(z_5 - z_2)^2 - (z_5' - z_2')^2] + 0.1387\,[(z_4 - z_3)^2 - (z_4' - z_3')^2]}{\dfrac{5.6036}{n} + 5.9072}$
C = 3	$p_1 = 0.0175$ $p_2 = 0.0922$ $p_3 = 0.2858$ $p_6 = 0.9825$ $p_5 = 0.9078$ $p_4 = 0.7142$ $\hat{\rho}_2 = \dfrac{0.0865\,[(z_6 - z_1)^2 - (z_6' - z_1')^2] + 0.1764\,[(z_5 - z_2)^2 - (z_5' - z_2')^2] + 0.1478\,[(z_4 - z_3)^2 - (z_4' - z_3')^2]}{\dfrac{5.7037}{n} + 5.9414}$

Table 26. Estimators of ρ under near-optimum and suboptimum conditions for $k = 8$, $\mu = \mu_1 = \mu_2$ unknown, $\sigma_1 = \sigma_2 = 1$

Conditions	Estimators of ρ
Near-optimum	$p_1 = 0.00549$ $p_2 = 0.0286$ $p_3 = 0.0851$ $p_4 = 0.2017$ $p_8 = 0.99451$ $p_7 = 0.9714$ $p_6 = 0.9149$ $p_5 = 0.7982$ $\hat{\rho}_2 = \dfrac{0.0307\,[(z_8 - z_1)^2 - (z_8' - z_1')^2] + 0.0730\,[(z_7 - z_2)^2 - (z_7' - z_2')^2] + 0.1168\,[(z_6 - z_3)^2 - (z_6' - z_3')^2] + 0.1477\,[(z_5 - z_4)^2 - (z_5' - z_4')^2]}{\dfrac{6.8466}{n} + 6.2794}$
C = 1	$p_1 = 0.0119$ $p_2 = 0.0604$ $p_3 = 0.1721$ $p_4 = 0.3711$ $p_8 = 0.9881$ $p_7 = 0.9396$ $p_6 = 0.8279$ $p_5 = 0.6289$ $\hat{\rho}_2 = \dfrac{0.0600\,[(z_8 - z_1)^2 - (z_8' - z_1')^2] + 0.1249\,[(z_7 - z_2)^2 - (z_7' - z_2')^2] + 0.1528\,[(z_6 - z_3)^2 - (z_6' - z_3')^2] + 0.0789\,[(z_5 - z_4)^2 - (z_5' - z_4')^2]}{\dfrac{6.0237}{n} + 6.0200}$
C = 2	$p_1 = 0.00998$ $p_2 = 0.0515$ $p_3 = 0.1511$ $p_4 = 0.3481$ $p_8 = 0.99002$ $p_7 = 0.9485$ $p_6 = 0.8489$ $p_5 = 0.6519$ $\hat{\rho}_2 = \dfrac{0.0518\,[(z_8 - z_1)^2 - (z_8' - z_1')^2] + 0.1134\,[(z_7 - z_2)^2 - (z_7' - z_2')^2] + 0.1534\,[(z_6 - z_3)^2 - (z_6' - z_3')^2] + 0.0925\,[(z_5 - z_4)^2 - (z_5' - z_4')^2]}{\dfrac{6.2067}{n} + 6.0748}$
C = 3	$p_1 = 0.00921$ $p_2 = 0.0476$ $p_3 = 0.1407$ $p_4 = 0.3314$ $p_8 = 0.99079$ $p_7 = 0.9524$ $p_6 = 0.8593$ $p_5 = 0.6686$ $\hat{\rho}_2 = \dfrac{0.0484\,[(z_8 - z_1)^2 - (z_8' - z_1')^2] + 0.1075\,[(z_7 - z_2)^2 - (z_7' - z_2')^2] + 0.1512\,[(z_6 - z_3)^2 - (z_6' - z_3')^2] + 0.1004\,[(z_5 - z_4)^2 - (z_5' - z_4')^2]}{\dfrac{6.2952}{n} + 6.1012}$

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