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FINAL REPORT, PART I

# STABILITY CRITERIA FOR THE OAO COARSE POINTING MODE


BY

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SUMMARY

As will often be the case, the attitude control of the Orbiting Astronomical Observatory shall be maintained by means of two of the six star trackers and the momentum wheels which react to differences between gimbal pickoff and command information.

The requirements of the closed-loop feedback system include a high degree of stability with respect to the four command gimbal angles. It is the purpose of this investigation to isolate sufficient conditions that ensure a stable asymptotic response of the satellite's system to an impulsive disturbance.

There are three sources of nonlinearities, two of which stem from possible saturation of multicomponent signals in the Digitalizer Logic Unit, and in the motor which drives the wheels. The remaining nonlinearity can be attributed to large gimbal angle errors. A model, which includes motor saturation, is analyzed in detail. The effects from both saturations, as well as those from motor saturation and large gimbal angle errors, are formulated separately.

Generally, the study of stability is equivalent to a search for sets of stable parametric values (steady-state and those of command) associated with allowable error signals. If linearity is assumed throughout, the investigation reduces to a parametric examination. For this purpose the Routh-Hurwitz criterion is available. Two closed areas result (one for each tracker) whose points have the coordinates, inner and outer gimbal angles. Any two points, one from each region, may be used; and the dynamic behavior, viz., the damping which is contingent on their locations, will be asymptotically stable. There are no restrictions to the initial conditions since the state of stability for the linear model is also global. When considering a more realistic control loop that includes motor saturation, we find that in a Jordan decomposition of the error space, two principal state vectors arise, decoupled in the linear portion of the nonlinear system of equations. The first vector is stable, i.e., it has a linear equation that is globally asymptotically stable. More-

over, it is so for all command values because the equation is independent of gimbal angles. If the nonlinearities are small, globality persists. On the other hand, the behavior of the second vector is completely determined by the nonlinearities, no matter how insignificant, because its linear solution is a constant which classifies it critical.

A Liapunov technique is developed, consistent with the vector decomposition, and based on: two selected positive definite diagonal matrices corresponding to the two state error vectors, a measure of the nonlinearities, and a positive parameter which introduces a positive definite integral of the nonlinearities. It also consists of two forms: quadratic in each of the state vectors and the controlling vector whose components are the nonlinearities. The optimum Liapunov matrix is found analytically. It maximizes, at zenith, positive definiteness of the quadratic which is in terms of the stable vector so that we obtain the largest parametric regions in which the stability of the null or equilibrium solution is independent of this vector. The measure of maximum definiteness is shown to be the reciprocal of that for the nonlinearities, while the parameter of the integral cannot be less than unity, thus confirming the conclusion concerning local behavior near linearity. In application, the requirement for positive definiteness proved to be too severe, i.e., one must use orientations near zenith for definiteness. Consequently, the criterion for the quadratic is weakened to just positiveness, introducing dependence of the magnitude of the vector on parametric values for stability. The other quadratic which is in terms of the critical vector cannot be positive definite, and therefore dependence is immanent. Its Liapunov matrix is selected so that the behavior of the quadratic is identical to that of the sufficient condition: the negative of the inner product of the vector's derivative and the vector itself must be positive. Since this quadratic is by far the more sensitive one, a practical criterion for an asymptotically stable response is the positiveness of this single number.

The weakened criterion of positiveness is tantamount to the modified requirement that the time interval over which stability is defined be bounded.

# INTRODUCTION

The purpose of this three-part investigation is to find criteria that ensure a stable response of the vehicle to an impulsive disturbance.

Section 1 deals with a physical description of the closed-loop multidimensional feedback system, the definition of an asymptotically stable response,and similarities within distinct groups of two-tracker cases.

Basic equations which simulate the dynamics in the elements are presented in Section 2. They include sufficient generality to accommodate all nonlinear mathematical models. In particular, the section discusses stability for the linear model, and criterion represented by closed regions in gimbal angle space.

Section 3 considers the nonlinear model that includes only motor saturation. Physical meanings of the mathematical manipulations are given throughout. First, the differential equations are examined to reveal local behavior near linearity, and then a Liapunov technique is used to shed light on the question of globality. Criteria are sought in terms of inequalities.

The remaining two nonlinear models (DLU and motor saturations and motor saturation with large gimbal angle errors) are formulated in the Appendix.

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## PHYSICAL DESCRIPTION

### 1.1 PERFORMANCE

The Orbiting Astronomical Observatory (OAO) is a satellite which will explore the heavens. It is necessary for the vehicle to accurately point its astronomical telescope in a given direction and maintain this orientation over a predetermined time interval. The capability of such control requires a precise closed-loop feedback system that possesses a high degree of stability. In the coarse pointing mode, attitude control is achieved by means of star trackers and fine reaction wheels.

Each of the two active star trackers has two degrees of rotational freedom, an outer and inner gimbal angle. Differences between gimbal pickoff and command information are determined in the Digitalizer Logic Unit (DLU). The gimbal error signal which has four components is averaged in the Processor so that a three-component attitude error signal emerges. This is transmitted to the Controller where, through compensation, it is transformed and applied to the motor that drives the reaction wheels. The subsequent exchange of momentum between the vehicle and wheels results in rotational accelerations about the control axes, and the attitude is brought more in line with the command orientation. It is important to note that even at this attitude there are errors, but they represent a steady dynamical state in which there are no rotations.

When the input to an element of control exceeds the limit of normal or linear operation, saturation of the output signal occurs, i.e. it is approximately constant. In the DLU the linear range is very narrow; and consequently, a saturated signal is frequently processed. The most common saturation, however, appears in the power output stage of the motor.

Quite another type of nonlinearity stems from large gimbal angle errors. The basis on which these errors are processed involves nonlinear transcendental functions of the errors themselves; however, if sufficiently small, the equation relating gimbal and attitude errors can be assumed linear. In addition, from geometric considerations, the rate at which gimbal errors are eliminated is a similar nonlinear function, but of control rates and errors. Neglecting these errors is sufficient to linearize this relation as well.

## 1.2 STATE OF STABILITY

An asymptotically stable response to an impulsive disturbance is the complete restoration of the vehicle's attitude to its previous state of dynamic equilibrium-the steady state- in which there are no rotations about the control axes. It is global when the initial values can be of any magnitude and absolute if, in addition, it is also independent of nonlinearities within a large class. The greater the generality, however, the more difficult it is to realize in practice.

The nature of any behavior, stable or unstable, depends on the errors induced in the control loop while the vehicle is in the state of dynamic equilibrium, the description of which is given by the momentum level and the command values of the gimbal angles. Hence, an investigation into the stable properties of an attitude control system is, in essence, a search for sets of parametric values (command and those of the steady state) in conjunction with allowable limits for the initial conditions or errors.

## 1.3 TRACKER CASES

Based on the manner in which all six trackers are mounted (as described in Table 1) there exists three distinct groups of two-tracker combinations. These groupings are itemized in Table 2.

Table 1. Direction of Tracker Axes

| Tracker Axis | Tracker Number | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Optical | $\xi$ | $-\xi$ | $\eta$ | $-\eta$ | $\zeta$ | $-\zeta$ |
| Inner | $-\zeta$ | $\eta$ | $-\xi$ | $-\zeta$ | $-\xi$ | $\eta$ |
| Outer | $\eta$ | $-\zeta$ | $\zeta$ | $\xi$ | $-\eta$ | $\xi$ |

Note: $\xi$= roll axis direction; $\eta$= pitch axis direction; $\zeta$= yaw axis direction

Table 2. Groups of Two-Tracker Cases

| Group | Trackers | Property |
|---|---|---|
| I | 1-3<br>1-6<br>2-4<br>2-5<br>3-6<br>4-5 | None |
| II | 3-5<br>1-4<br>2-6 | Inner gimbal angle axes are parallel. |
| III | 4-6<br>1-5<br>2-3 | Outer gimbal angle axes are parallel. |

Note: The three remaining cases have parallel optical axes and are consequently of little interest. These cases are 1-2, 3-4, and 5-6.

## SECTION 2

## STABILITY ANALYSIS FOR THE LINEAR MODEL

### 2.1 DIFFERENTIAL EQUATIONS

With few exceptions, lower case letters shall denote vectors, upper case shall signify matrices, and Greek symbols shall mark scalars and ordinary numbers. The differences

$$\beta_i - \beta_{i_c} = \beta_{i_e} \qquad \gamma_i - \gamma_{i_c} = \gamma_{i_e} \tag{1}$$

which are computed in the DLU, represent gimbal angle errors for the ith tracker. Instead of the fourth order error vector, b, formed from

$$b^T = \left(\beta_{1_e}, \beta_{2_e}, \beta_{3_e}, \beta_{4_e}, \beta_{5_e}, \beta_{6_e}, \gamma_{1_e}, \gamma_{2_e}, \gamma_{3_e}, \gamma_{4_e}, \gamma_{5_e}, \gamma_{6_e}\right)$$

by suppressing inapplicable components, the saturated signal, a, is used whenever the magnitude of any one component is not within the linear range, $\pm 21$ arc minutes. For this purpose, there exists a functional relationship that is continuous and single-valued:

$$a = a\ (b). \tag{2}$$

The averaging process within the Processor yields the attitude error signal, e:

$$e = E\,a \tag{3}$$

The appropriate 3 x 4 matrix, E, comes from

$$2\,E \equiv \begin{bmatrix} \sin\gamma_1 & -\sin\gamma_2 & -\cos\gamma_3 & 0 & -\cos\gamma_5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & \cos\gamma_2 & \sin\gamma_3 & -\sin\gamma_4 & 0 & \cos\gamma_6 & 1 & 0 & 0 & 0 & -1 & 0 \\ -\cos\gamma_1 & 0 & 0 & -\cos\gamma_4 & \sin\gamma_5 & -\sin\gamma_6 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

by suppressing all columns except those that refer to the two active trackers with which we are dealing.

In the Controller, the dynamics are described by

$$\tau_2 \dot{v} - \varkappa_1 (e + \tau_1 \dot{e}) + v = 0 \tag{4}$$

where $\tau_1$ and $\tau_2$ are time constants, $\varkappa_1$ is the amplifier constant and v is the motor voltage, provided saturation which occurs at $\pm 26$v is not present. Otherwise, w is available from a continuous and single-valued relationship

$$w = w(v). \tag{5}$$

The dot signifies differentiation with respect to time $\tau$.

Since the directions of principal inertia are parallel to the control axes, the components of inertia are essentially the same because of symmetry, and the external disturbances are impulsive rather than continuous. considerable simplification results in the expressions for the momentum exchange between the wheels and rigid body. Let the components of p be the roll, pitch, and yaw motions, and suppose $h_o$ denotes the angular momentum of the entire system, then it is conserved, $h_o = h_w + h_b$ = constant, and the rate of change of angular momentum for the body is just $\dot{h}_b = K_b \dot{p}$, in which a non-zero element of the inertia tensor $K_b$ is $\rho$. Thus, the rate of momentum change for the wheels satisfies $\dot{h}_w + K_b \dot{p} = 0$, which when substituted into the modified motor - torque equation $\dot{h}_w = \varkappa_5 (w - h_w/\varkappa_6 \rho_w)$, involving the wheels' inertia $\rho_w$, motor torque constant $\varkappa_5$, and velocity constant $\varkappa_6$, yields

$$\dot{p} = \varkappa_7 (h_o - \rho p - \varkappa_6 \rho_w w) \tag{6}$$

where $\varkappa_7 \equiv \varkappa_5/\varkappa_6 \rho \rho_w$.

Rotations about the inner and outer gimbal angle axes correspond to motions about the control axes, as given by

$$\dot{b} = Fp \tag{7}$$

The 4 x 3 matrix, F, is taken from

$$F \equiv \begin{bmatrix} \sin \gamma_1 & 0 & -\cos \gamma_1 \\ -\sin \gamma_2 & \cos \gamma_2 & 0 \\ -\cos \gamma_3 & \sin \gamma_3 & 0 \\ 0 & -\sin \gamma_4 & -\cos \gamma_4 \\ -\cos \gamma_5 & 0 & \sin \gamma_5 \\ 0 & \cos \gamma_6 & -\sin \gamma_6 \\ -\tan \beta_1 \cos \gamma_1 & 1 & -\tan \beta_1 \sin \gamma_1 \\ \tan \beta_2 \cos \gamma_2 & \tan \beta_2 \sin \gamma_2 & -1 \\ -\tan \beta_3 \sin \gamma_3 & -\tan \beta_3 \cos \gamma_3 & 1 \\ 1 & \tan \beta_4 \cos \gamma_4 & -\tan \beta_4 \sin \gamma_4 \\ -\tan \beta_5 \sin \gamma_5 & -1 & -\tan \beta_5 \cos \gamma_5 \\ 1 & \tan \beta_6 \sin \gamma_6 & \tan \beta_6 \cos \gamma_6 \end{bmatrix}$$

by deleting those terms with inapplicable indices.

## 2.2 CONSTANTS

The numerical values of the constants already mentioned are listed in Table 3.

## 2.3 LINEAR MODEL

With the restrictions $a \equiv b$, $v \equiv w$, and the approximation of neglecting gimbal errors in matrices E and F so that they are constant, we obtain the linear model.

Table 3. Numerical Values of Constants

| Constant | Numerical Value |
|---|---|
| $\tau_1$ | 5.0 secs |
| $\tau_2$ | 0.5 sec |
| $\rho$ | 1500 slug-ft$^2$ |
| $\rho_w$ | 0.0219 slug-ft$^2$ |
| $\varkappa_1$ | $2.685 \times 10^5$ volts/rad |
| $\varkappa_5$ | $1.002 \times 10^{-3}$ ft-lb/volt |
| $\varkappa_6$ | 3.512 rad/volt-sec |
| $\varkappa_7$ | $8.681 \times 10^{-6}$ (ft-lb-sec$^3$)$^{-1}$ |

Combining Equation 3 with Equation 7, there results

$$\dot{e} = G p \tag{8}$$

where $G \equiv EF$. Substituting into Equation 4, one gets

$$\dot{v} = \varkappa_2 e + \varkappa_3 G p + \varkappa_4 v \tag{9}$$

for which the additional constants $\varkappa_4 \equiv -1/\tau_2$, $\varkappa_2 \equiv -\varkappa_1 \varkappa_4$, and $\varkappa_3 \equiv \varkappa_2 \tau_1$ are defined.

At equilibrium and a given momentum level $h_o$, or, what is equivalent, wheel speed since $p \equiv 0$, Equation 6 gives $v_s = h_o/\varkappa_6 \rho_w$, Equation 4 asserts that $e_s = v_s/\varkappa_1$. From Equation 3 we obtain all admissible solutions for $b_s$, which has one degree of arbitrariness. These are the errors that exist at the steady state.

Let x represent the ninth order vector

$$x = \begin{pmatrix} v \\ p \\ e \end{pmatrix}$$

In terms of the errors different from those of steady state, $q \equiv x - x_s$, the linear system, Equations 6 (with $w \equiv v$), 8, and 9, is given by

$$\dot{q} = Q q \tag{10}$$

where

$$Q \equiv \begin{bmatrix} \varkappa_4 I & \varkappa_3 G & \varkappa_2 I \\ -\varkappa_5 I/\rho & -\rho \varkappa_7 I & 0 \\ 0 & G & 0 \end{bmatrix},$$

I being the 3 x 3 unit matrix. The null solution of Equation 10 corresponds to the state of dynamic equilibrium.

## 2.4 STABILITY

It follows from the definitions already given for the state of dynamic equilibrium, null solution, and an asymptotically stable response, that the null solution is asymptotically stable if the response is to dynamic equilibrium.

An important consequence of the autonomy of Equation 10 is that if its null solution is asymptotically stable, it is uniformly asymptotically stable (Reference 1). Moreover, the state of stability does not depend on the errors, q, i. e., it is also globally asymptotically stable. It is, however, a function of the selected parametric values.

A useful criterion, in this connection, is the group of well-known Routh-Hurwitz conditions, which are inequalities involving coefficients in the characteristic polynomial of Q, $|Q - \lambda I| = \sum_0^9 \mu_j \lambda^{9-j} = 0$ with $\mu_o = 1$. These coefficients are (Reference 2) as follows:

$$\mu_1 = 3 (\rho \varkappa_7 - \varkappa_4)$$

$$\mu_2 = 3 (\rho \varkappa_7 - \varkappa_4)^2 - 3\rho \varkappa_7 \varkappa_4 + 2 \frac{\varkappa_3 \varkappa_5}{\rho}$$

$$\mu_3 = (\rho \varkappa_7 - \varkappa_4)^3 - 6 \rho \varkappa_7 \varkappa_4 (\rho \varkappa_7 - \varkappa_4)$$

$$+ 2 \frac{\varkappa_3 \varkappa_5}{\rho} \left[ 2 (\rho \varkappa_7 - \varkappa_4) + \frac{\varkappa_2}{\varkappa_3} \right]$$

$$\mu_4 = - 3 \varkappa_4 \rho \varkappa_7 \left[ (\rho \varkappa_7 - \varkappa_4)^2 - \rho \varkappa_7 \varkappa_4 \right]$$

$$+ 4 \frac{\varkappa_3 \varkappa_5}{\rho} \left( \frac{\varkappa_2}{\varkappa_3} \right) (\rho \varkappa_7 - \varkappa_4)$$

$$+ 2 \frac{\varkappa_3 \varkappa_5}{\rho} \left[ (\rho \varkappa_7 - \varkappa_4)^2 - 2 \rho \varkappa_7 \varkappa_4 \right]$$

$$+ \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right)^2 (1 + |G|)$$

$$\mu_5 = 3 (\rho \varkappa_7 \varkappa_4)^2 (\rho \varkappa_7 - \varkappa_4)$$

$$+ 2 \frac{\varkappa_3 \varkappa_5}{\rho} \left( \frac{\varkappa_2}{\varkappa_3} \right) \left( \varkappa_4^2 - 4 \rho \varkappa_7 \varkappa_4 + \rho^2 \varkappa_7^2 \right)$$

$$- 4 \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right) \rho \varkappa_7 \varkappa_4 (\rho \varkappa_7 - \varkappa_4)$$

$$+ \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right)^2 \left( \rho \varkappa_7 - \varkappa_4 + 2 \frac{\varkappa_2}{\varkappa_3} \right) (1 + |G|)$$

$$\mu_6 = - (\rho \varkappa_7 \varkappa_4)^3$$

$$- 2 \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right) \rho \varkappa_7 \varkappa_4 \left[ 2 \frac{\varkappa_2}{\varkappa_3} (\rho \varkappa_7 - \varkappa_4) - \rho \varkappa_7 \varkappa_4 \right]$$

$$+ 2 \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right)^2 \frac{\varkappa_2}{\varkappa_3} (\rho \varkappa_7 - \varkappa_4) (1 + |G|)$$

$$+ \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right)^2 \left( \frac{\varkappa_2^2}{\varkappa_3^2} - \rho \varkappa_7 \varkappa_4 \right) (1 + |G|)$$

$$+ |G| \left( \frac{\varkappa_3 \varkappa_5}{\rho} \right)^3$$

Group II

$$
\begin{aligned}
|G| = \frac{1}{8} & \left[\cos^2\gamma_{3_c}\,(1+\sin^2\gamma_{5_c}) + \cos^2\gamma_{5_c}\,(1+\sin^2\gamma_{3_c})\right] \\
& + \frac{1}{4}\left[-\sin\gamma_{3_c}\cos\gamma_{5_c}\sin\gamma_{5_c}\right]\tan\beta_{3_c} \\
& + \frac{1}{4}\left[\cos\gamma_{3_c}\sin\gamma_{3_c}\sin\gamma_{5_c}\right]\tan\beta_{5_c} \\
& + \frac{1}{4}\left[\cos\gamma_{3_c}\cos\gamma_{5_c}\right]\tan\beta_{3_c}\tan\beta_{5_c}
\end{aligned}
\qquad (11b)
$$

Group III

$$|G| = \frac{1}{4}\cos^2(\gamma_{4_c} - \gamma_{6_c}) \qquad \text{No Inner Gimbal Angles Appear.} \qquad (11c)$$

Table 4. Gimbal Angle Equivalents

| Group | Tracker Case | Gimbal Angles: First Tracker Inner | Gimbal Angles: First Tracker Outer | Gimbal Angles: Second Tracker Inner | Gimbal Angles: Second Tracker Outer |
|---|---|---|---|---|---|
| I | *1-3 | $\beta_{1_c}$ | $\gamma_{1_c}$ | $\beta_{3_c}$ | $\gamma_{3_c}$ |
| | 1-6 | $-\beta_{6_c}$ | $\gamma_{6_c}$ | $\beta_{1_c}$ | $\gamma_{1_c}$ |
| | 2-4 | $-\beta_{4_c}$ | $-\gamma_{4_c}$ | $\beta_{2_c}$ | $\gamma_{2_c}$ |
| | 2-5 | $\beta_{2_c}$ | $\gamma_{2_c}$ | $-\beta_{5_c}$ | $\gamma_{5_c}$ |
| | 3-6 | $\beta_{3_c}$ | $\gamma_{3_c}$ | $-\beta_{6_c}$ | $\gamma_{6_c}$ |
| | 4-5 | $-\beta_{5_c}$ | $\gamma_{5_c}$ | $-\beta_{4_c}$ | $-\gamma_{4_c}$ |

*The principal case

Table 4. Gimbal Angle Equivalents (Cont'd)

| Group | Tracker Case | Gimbal Angles | | | |
|---|---|---|---|---|---|
| | | First Tracker | | Second Tracker | |
| | | Inner | Outer | Inner | Outer |
| II | *3-5 | $\beta_{3_c}$ | $\gamma_{3_c}$ | $\beta_{5_c}$ | $\gamma_{5_c}$ |
| | 1-4 | $\beta_{1_c}$ | $\gamma_{1_c}$ | $\beta_{4_c}$ | $-\gamma_{4_c}$ |
| | 2-6 | $\beta_{2_c}$ | $\gamma_{2_c}$ | $\beta_{6_c}$ | $\gamma_{6_c}$ |
| III | *4-6 | $\beta_{4_c}$ | $\gamma_{4_c}$ | $\beta_{6_c}$ | $\gamma_{6_c}$ |
| | 1-5 | $\beta_{5_c}$ | $-\gamma_{5_c}$ | $-\beta_{1_c}$ | $\gamma_{1_c}$ |
| | 2-3 | $-\beta_{2_c}$ | $-\gamma_{2_c}$ | $-\beta_{3_c}$ | $\gamma_{3_c}$ |

*The principal case

Equation 11a can be derived from Equation 11b if $\gamma_{3_c}^{\,II} \rightarrow \frac{\pi}{2} - \gamma_{1_c}^{\,I}$, $\gamma_{5_c}^{\,II} \rightarrow \gamma_{3_c}^{\,I}$, $\beta_{3_c}^{\,II} \rightarrow \beta_{1_c}^{\,I}$, and $\beta_{5_c}^{\,II} \rightarrow -\beta_{3_c}^{\,I}$.

The Routh-Hurwitz criterion is equivalent to positive definiteness of the matrix

$$\begin{bmatrix}
\mu_1 & \mu_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_3 & \mu_2 & \mu_1 & \mu_o & 0 & 0 & 0 & 0 & 0 \\
\mu_5 & \mu_4 & \mu_3 & \mu_2 & \mu_1 & \mu_o & 0 & 0 & 0 \\
\mu_7 & \mu_6 & \mu_5 & \mu_4 & \mu_3 & \mu_2 & \mu_1 & \mu_o & 0 \\
\mu_9 & \mu_8 & \mu_7 & \mu_6 & \mu_5 & \mu_4 & \mu_3 & \mu_2 & \mu_1 \\
0 & 0 & \mu_9 & \mu_8 & \mu_7 & \mu_6 & \mu_5 & \mu_4 & \mu_3 \\
0 & 0 & 0 & 0 & \mu_9 & \mu_8 & \mu_7 & \mu_6 & \mu_5 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_9 & \mu_8 & \mu_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_9
\end{bmatrix}$$

Let the positive range of the ith successive principal minor be $\mathcal{R}_i$, then of interest is the greatest range that is common to all $\mathcal{R}_i$, i.e., $\mathcal{R}_1 \cap \mathcal{R}_2 \; \ldots\ldots \cap \mathcal{R}_9 \equiv \mathcal{H}$. This is found to be (0, 0.77), (Reference 2). Now suppose we are considering the tracker pair m-n. Equations 11a and 11b are of the form

$$\phi\left(\beta_{m_c}, \gamma_{m_c}, \beta_{n_c}, \gamma_{n_c}; |G|\right) = 0 \tag{12}$$

If $|G| \in \mathcal{H}$, solutions of Equation 12 are consequently parametric values that lead to stability, which we term stable. What is required, however, is some explicit representation for all solutions. One method is to plot $|G|$ contours for various values of $\mathcal{H}$ in the finite portion of the plane $(\beta_{m_c}, \gamma_{m_c})$ in which the travel of a tracker is restricted to $\pm 40$ degrees (Reference 3), and with respect to a fixed point of $(\beta_{n_c}, \gamma_{n_c})$. As a result, the area covered by these contours contains only stable points. It follows that a useful stable parametric region can be generated from the intersection of areas associated with all physically admissible points in $(\beta_{n_c}, \gamma_{n_c})$; or what is more practical, with points on the boundary. In this regard, we need only to look at the first quadrant because of the symmetries summarized in Table 5. All admissible parametric values for cases of Group III are stable and consequently are not shown.

Table 5. Symmetries for $|G|$ Relations

| Group | Axes to Which Symmetry is Considered | |
|---|---|---|
| I | $\beta_{3_c}$ | $\gamma_{1_c}, \beta_{1_c}$ |
| | $\gamma_{3_c}, \beta_{3_c}$ | $\gamma_{1_c}$ |
| II | $\gamma_{5_c}$ | $\gamma_{3_c}, \beta_{3_c}$ |
| | $\beta_{5_c}$ | $\beta_{3_c}$ |

To illustrate this method, we choose Trackers 1 and 3, the principal case of Group I, together with $\gamma_{3_c} = 5.2^\circ$ and $\beta_{3_c} = -29.5^\circ$, and plot the contours depicted in Figure 1. For Trackers 3 and 5 of Group II and $\gamma_{5_c} = 38.1^\circ$ and $\beta_{5_c} = 12.4^\circ$, the contours are those of Figure 2, with the property that $|G| = 1/4$ always passes through the zenith of Tracker 3. In both examples, the entire interval of $\mathcal{H}$ was not needed within the prescribed physical limits of tracker-travel.

The parametric regions for stability, whose points represent solutions of Equation 12, are shown in Figures 3 and 4 for Groups I and II, respectively. The boundary separating stability from instability was governed by $|G| = 0$, in all cases, and by the extremes of allowable travel.

## 2.5 CONCLUSIONS

Suppose the satellite is controlled by Trackers m-n. From Table 4 we identify the group to which the pair belongs. The equivalents are also noted; they indicate how to interpret the command gimbal angles for the principal case in terms of those for m-n. Figure 3 or Figure 4 is then used, depending on whether the pair, m-n, is a member of Group I or Group II, respectively. If it belongs to Group III, then all parametric values are stable. Within the areas shown, any two points may be chosen, one for each tracker, and the response will be asymptotically stable.

Any error or initial conditions may exist, i.e., the state of stability is also global. The parameters do not include those that describe the equilibrium or steady state.

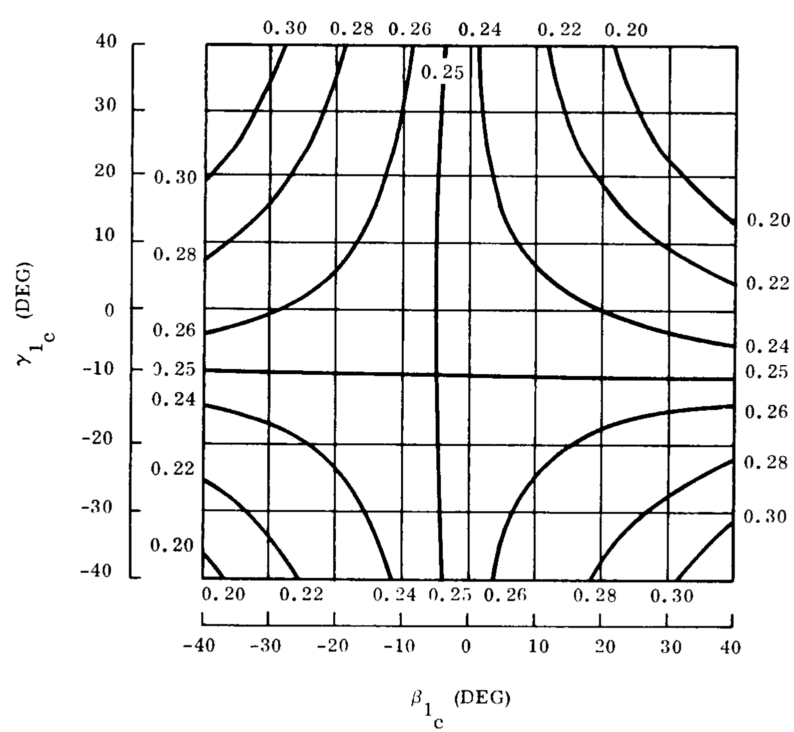


Figure 1. | G | Contours for Trackers 1 and 3, (Group I), and $\gamma_3 = 5.2^o$, $\beta_3 = -29.5^o$; (Reference 4)

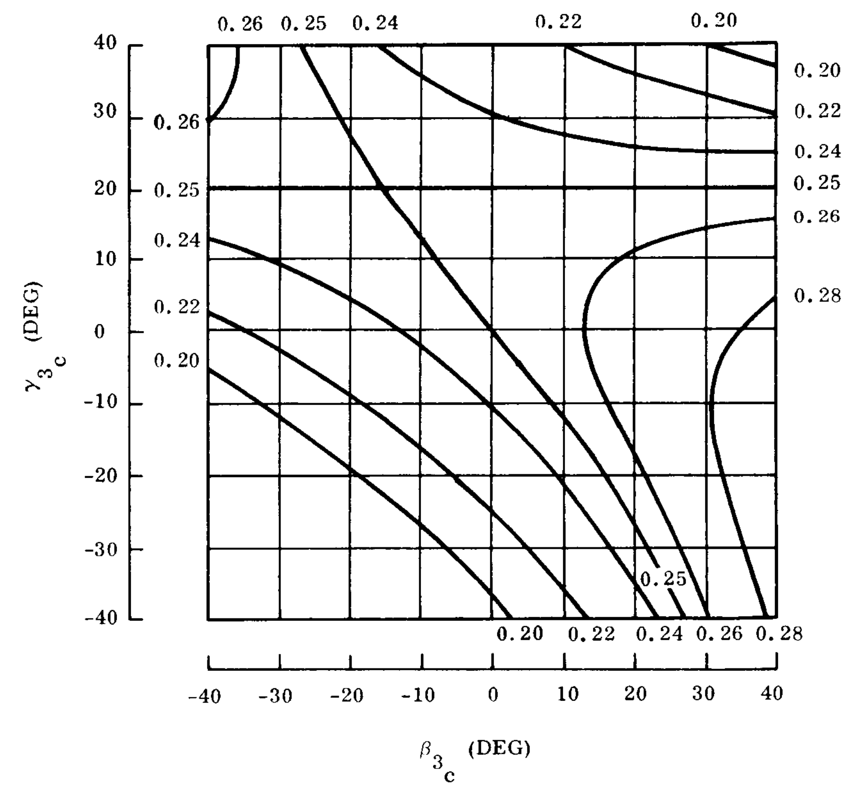


Figure 2. | G | Contours for Trackers 3 and 5, (Group II), and $\gamma_5 = 38.1^o$, $\beta_5 = 12.4^o$; (Reference 4)

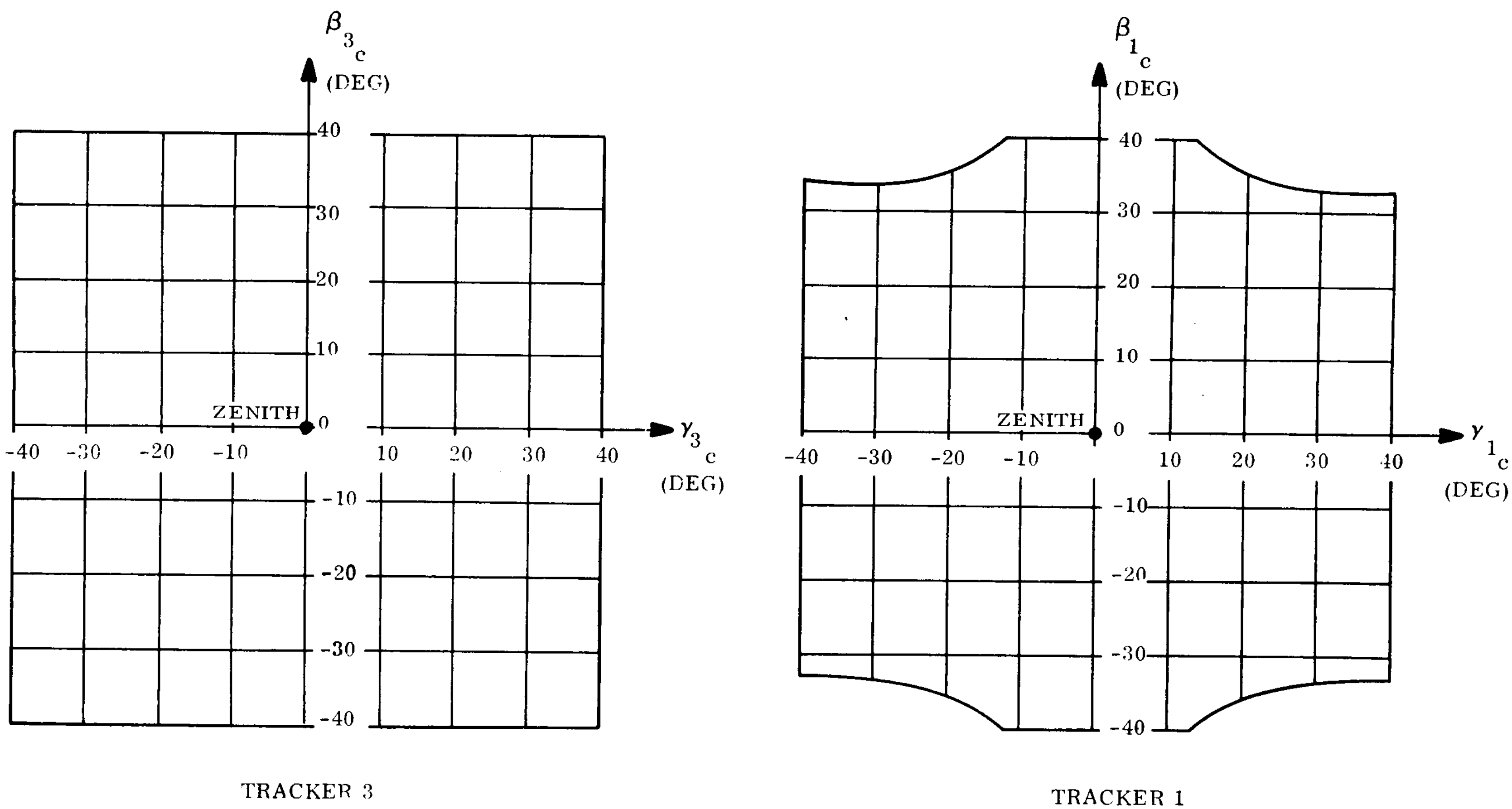


Figure 3. Stable Parametric Regions for Group I

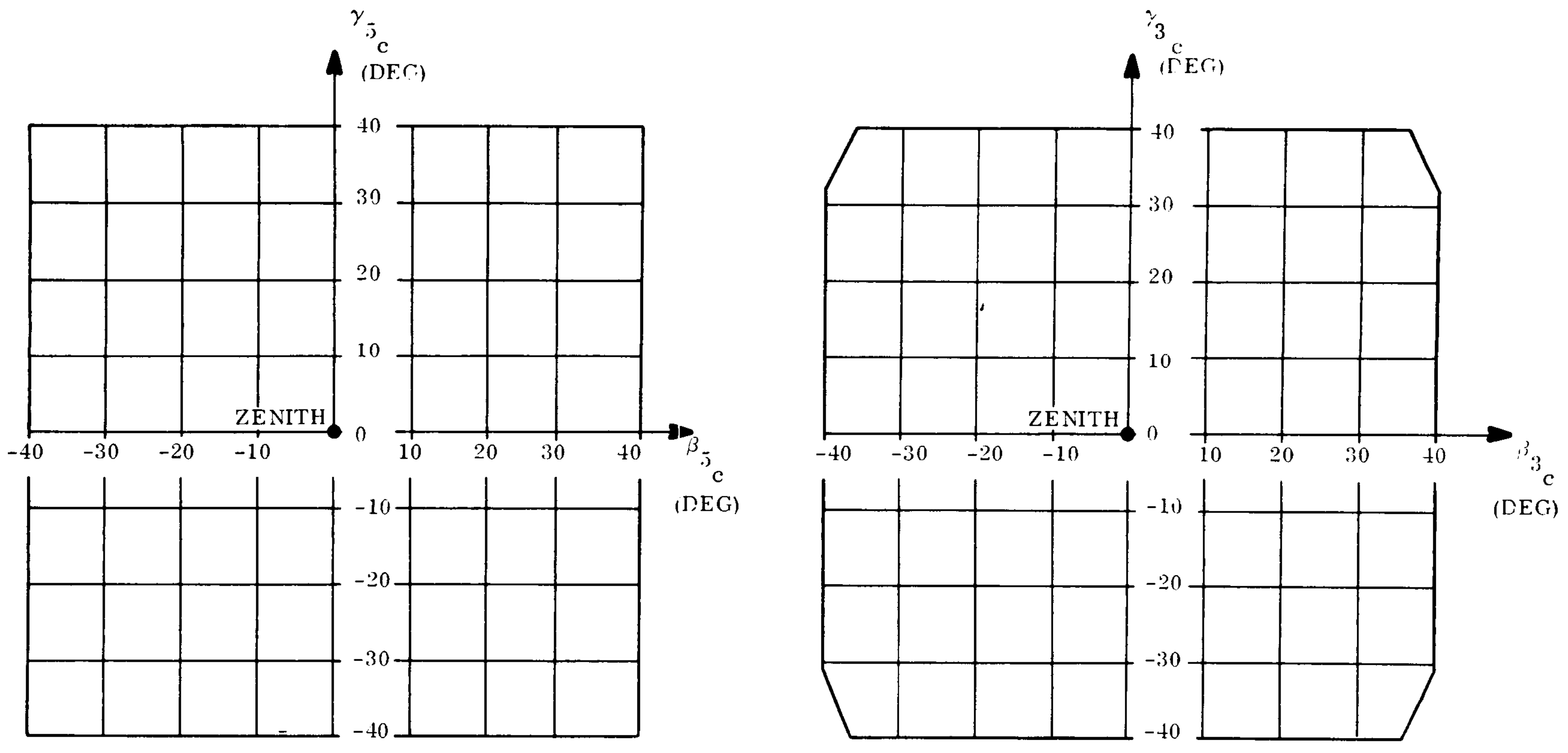


Figure 4. Stable Parametric Regions for Group II

## SECTION 3
## STABILITY ANALYSIS FOR A NONLINEAR MODEL

### 3.1 NONLINEAR MODEL

We shall now consider a model that has nonlinearities originating from motor saturation. Matrices E and F are assumed constant, and $a \equiv b$. It is convenient to nondimensionalize Equations 6, 8, and 9, with the aid of the transformations $e = \tilde{e}$, $p = \rho\varkappa_7 \tilde{p}$, $v = \varkappa_8 \tilde{v}$, and $d/d\tau = \rho\varkappa_7\, d/d\sigma$; constant $\varkappa_8$ is the voltage at saturation. A standard form for direct nonlinear control (Reference 5), results

$$\tilde{q}' = A\tilde{q} + B\tilde{g} \tag{13a}$$

$$\tilde{g} = \tilde{g}(\tilde{r}) \text{ with } \tilde{g}(0) = 0 \tag{13b}$$

$$\tilde{r} = C\tilde{q} \tag{13c}$$

$$A \equiv \begin{bmatrix} \varkappa_4 I/\rho\varkappa_7 & \varkappa_3 G/\varkappa_8 & \varkappa_2 I/\rho\varkappa_7 \varkappa_8 \\ 0 & -I & 0 \\ 0 & G & 0 \end{bmatrix}$$

$$B \equiv \begin{bmatrix} 0 \\ -\varkappa_5 \varkappa_8 I/\rho(\rho\varkappa_7)^2 \\ 0 \end{bmatrix}, \quad C \equiv \begin{bmatrix} I & 0 & 0 \end{bmatrix}$$

$$\tilde{q} = \begin{pmatrix} \tilde{v} - \tilde{v}_s \\ \tilde{p} \\ \tilde{e} - \tilde{e}_s \end{pmatrix}, \quad \tilde{g} = \tilde{w} - \tilde{w}_s$$

The tilde denotes dimensionless quantities, and the prime signifies differentiation with respect to $\sigma$. Matrices A, B, and C are of the orders 9 x 9, 9 x 3, and 3 x 9, respectively.

A differs from the remaining constant matrices since it contains parameters through Submatrix G.

The three components of feedback signal, $\tilde{r}$, are determined as linear combinations of the nine components of error, $\tilde{q}$, the coefficients of which are the elements of C. Proportional to the feedback, controlling signal, $\tilde{g}$, is introduced by B as the nonlinearity. In this way, the asymptotic behavior is closely associated with parametric quantities, command values of the gimbal angles, and the steady state $\tilde{w}_s = h_o/\varkappa_6 \rho_w \varkappa_8$, as well as the errors, including those that are derived from a nonlinear function, Equation 5.

At equilibrium, there exist errors given by $\tilde{w}_s$, $\tilde{v}_s = \tilde{v}_s(\tilde{w}_s)$, and $\tilde{e}_s = \varkappa_8 \tilde{v}_s/\varkappa_1$, and from Equation 3 (with a ≡ b) admissible values for $\tilde{b}_s$.

## 3.2 STABILITY

The null solution of Equation 13 is asymptoically stable over the interval $[0, \infty]$ if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that any other solution $\tilde{q}(\sigma)$ which fulfills $\|\tilde{q}(0)\| < \delta$ also satisfies $\|\tilde{q}(\sigma)\| < \epsilon$ for all $\sigma \geq 0$ and, in addition, $\|\tilde{q}(\sigma)\| \rightarrow 0$ as $\sigma \rightarrow \infty$ (Reference 6). If $\delta$ can be the entire space of $\tilde{q}$, then the null solution is globally asymptotically stable.

A classical Liapunov criterion is available (for analytic differential equations) which when satisfied will ensure behavior such as that described by the above definition. Briefly, the origin or null solution is asymptotically stable if there exists a positive definite function, $\Psi(\tilde{q})$, whose derivative, $\Psi'(\tilde{q})$, is negative definite. If, in addition, $\Psi(\tilde{q}) \rightarrow \infty$ as $\|\tilde{q}\| \rightarrow \infty$, then all solutions tend to this point (Reference 7).

We shall investigate the properties of stability with Liapunov functions that are a natural extension of the Lurie-type for one-dimensional nonlinearities, i.e., the sum of a quadratic form and an integral of the nonlinearity (Reference 8). The functions, which are for multi-dimensional feedback systems, shall have the restriction that the coefficient matrix of the quadratic form be diagonal.

It is first necessary to inspect the linear system $\tilde{q}' = A\tilde{q}$ which describes the dynamics in the open-loop of error sensing and evaluation, from which the wheel and rotational control are decoupled. The characteristic polynomial $|A - \lambda I| = (\varkappa_4/\rho\varkappa_7 - \lambda)^3 (-1 - \lambda)^3 \lambda^3 = 0$ states that the eigenvalues are $\varkappa_4/\rho\varkappa_7$, $\varkappa_4/\rho\varkappa_7$, $\varkappa_4/\rho\varkappa_7$, -1, -1, -1, 0, 0, 0. In contrast with the eigenvalues of Matrix Q of the linear model, Equation 10, these are independent of $|G|$, implying a linear behavior that is unaffected by commands and the nonexistence of restrictions on parametric values, such as the Routh-Hurwitz inequalities. Moreover, A is not stable, i.e., not all eigenvalues have negative real parts. The critical situation of three zeros asserts that there exist three combinations of the components of $\tilde{q}$ whose behavior is uncertain from linear considerations. In order to examine this state more closely, the vector space is now decomposed by means of a Jordan normal form of Matrix A.

Let

$$\tilde{q} = Ls \tag{14}$$

$$L \equiv \begin{bmatrix} I & (\varkappa_2 - \rho\varkappa_7\varkappa_3)G/\varkappa_8(\rho\varkappa_7 + \varkappa_4) & -\varkappa_2 I/\varkappa_4\varkappa_8 \\ 0 & I & 0 \\ 0 & -G & I \end{bmatrix}$$

which is obviously nonsingular, and which yields $L^{-1} A L = J$ where (Reference 9),

$$J \equiv \begin{bmatrix} \varkappa_4 I/\rho\varkappa_7 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$= J_0 \oplus J_1, \; J_1 \equiv \begin{bmatrix} 0 \end{bmatrix} .$$

Suppose $\overline{B} \equiv L^{-1}B = \overline{B}_o \oplus \overline{B}_1$, $\overline{C} \equiv CL = \overline{C}_o \oplus \overline{C}_1$ and correspondingly $s^T = (s_o, s_1)$, then the decomposed system of equations is

$$s_o{}' = J_o s_o + \overline{B}_o \tilde{g} \tag{15a}$$

$$s_1{}' = \overline{B}_1 \tilde{g} \tag{15b}$$

$$\tilde{g} = \tilde{g}(\tilde{r}) \qquad \tilde{g}(0) = 0 \tag{15c}$$

$$\tilde{r} = \overline{C}_o s_o + \overline{C}_1 s_1 \tag{15d}$$

If $s_o$ is decomposed as $s_o{}^T = (s_{o1}, s_{o2})$, $s_{o1}$ is a combination of the components of $\tilde{q}$, but $s_{o2} \equiv \tilde{p}$. The meaning of $s_1$ is $\tilde{e}' + (\tilde{e} - \tilde{e}_s)$.

The two variables $s_o$ and $s_1$ are decoupled in the linear system, and consequently their solutions $s_o(\sigma) = e^{\sigma J_o} s_o(0)$, $s_1(\sigma) \equiv s_1(0)$, are completely independent. Moreover, command and steady state values are not involved, in contrast with the linear behavior of $(\tilde{v} - \tilde{v}_s)$ or $(\tilde{e} - \tilde{e}_s)$.

Obviously, $s_o \equiv 0$ is globally asymptotically stable. We expect that for sufficiently small nonlinearities, this tendency will be maintained. Since the dynamics of $s_1$ are solely governed by nonlinearities, Equation 15b and the null solution $s \equiv 0$ of the nonlinear system, Equations 15a through d, will be unstable in $s_1$ before it is in $s_o$. These conjectures must be borne out by a Liapunov analysis.

Suppose $k_i$ is the slope of a line in the plane containing the $i^{\underline{th}}$ component of $\tilde{g}$ and $\tilde{r}$ and which passes through the origin. From the physical nature of a saturated signal, we can further restrict Equation 5 by requiring

$$0 \leq \tilde{g}_i / \tilde{r}_i \leq k_i \ , \ \tilde{r}_i \neq 0 \tag{16}$$

The Liapunov function is

$$\Psi(s) = s_o^T D_o s_o + s_1^T D_1 s_1 + \nu\,\Gamma(s) \qquad (17)$$

for which $\Gamma(\tilde{r}) \equiv \int_o^{\tilde{r}} \tilde{g}^T(t)\,dt.$

The integral is positive definite by virtue of Equation 16, $\nu > 0$, and the quadratic forms, $s_o^T D_o s_o$ and $s_1^T D_1 s_1$, are positive definite. Thus, $\Psi(s)$ is positive definite.

Taking the negative of the derivative of Equation 17, we obtain

$$-\Psi'(s) = y_o^T Y_o y_o + y_1^T Y_1 y_1 + \tilde{g}^T\left(\bar{C}s - \frac{\tilde{g}}{\|k\|}\right) \qquad (18)$$

where if

$$-H_o \equiv 2\,J_o D_o \quad \text{and} \quad \bar{V} \equiv \bar{B}^T D + \frac{\nu}{2}\,\bar{C} J = \bar{V}_o \oplus \bar{V}_1,$$

$$Y_o \equiv \begin{bmatrix} H_o & -\bar{V}_o^T \\ -(\bar{V}_o + \bar{C}_o) & \dfrac{I}{\|k\|} \end{bmatrix} \qquad y_o \equiv \begin{pmatrix} s_o \\ \tilde{g} \end{pmatrix}$$

$$Y_1 \equiv \begin{bmatrix} 0 & -\bar{V}_1^T \\ -(\bar{V}_1 + \bar{C}_1) & 0 \end{bmatrix} \qquad y_1 \equiv \begin{pmatrix} s_1 \\ \tilde{g} \end{pmatrix}$$

$$\bar{V}_o^T \equiv \begin{bmatrix} -\dfrac{\varkappa_5\varkappa_8}{\rho(\rho\varkappa_7)^2}\begin{bmatrix}\dfrac{\rho\varkappa_7}{\varkappa_4} & \dfrac{\varkappa_2 + \varkappa_3\varkappa_4}{\varkappa_8(\rho\varkappa_7 + \varkappa_4)}\end{bmatrix} D_{o1}\,G + \dfrac{\nu}{2}\,\dfrac{\varkappa_4}{\rho\varkappa_7}\,I \\ -\dfrac{\varkappa_5\varkappa_8}{\rho(\rho\varkappa_7)^2}\,D_{o2} - \dfrac{\nu}{2}\,\dfrac{\varkappa_2 - \rho\varkappa_7\varkappa_3}{\varkappa_8(\rho\varkappa_7 + \varkappa_4)}\,G^T \end{bmatrix}$$

$$\bar{V}_1^T \equiv -\frac{\varkappa_5\varkappa_8}{\rho(\rho\varkappa_7)^2}\,D_1 G \quad \text{with} \quad D_o \equiv D_{o1} \oplus D_{o2}$$

$$\bar{C}_o^T \equiv \begin{bmatrix} I \\ \dfrac{\varkappa_2 - \rho\varkappa_7\varkappa_3}{\varkappa_8(\rho\varkappa_7 + \varkappa_4)}\,G^T \end{bmatrix}$$

$$\overline{C}_1 \equiv -\frac{\varkappa_2}{\varkappa_4 \varkappa_8} I$$

It can be shown from Equation 16 that $\tilde{g}^T \left( \overline{C} s - \frac{g}{\|k\|} \right)$ is positive definite, and as a consequence we need to be concerned only with the two quadratic forms.

The definiteness of $s_o^T D_o s_o$ and $s_1^T D_1 s_1$ are not affected by parameters. Thus, the positive definiteness of $\Psi(s)$ is not a function of commands, or for that matter, errors. On the other hand, the definiteness of $y_o^T Y_o y_o$ does depend on parameters because $\overline{V}_o$ involves G. This means that only certain command values are associated with definiteness, in which case the stability of the null solution $s \equiv 0$ or the positiveness of $-\Psi'(s)$ is independent of $s_o$. If we use parametric values different from, but in the neighborhood of, those that correspond to definiteness, $y_o^T Y_o y_o$ would still be positive. Obviously, $y_1^T Y_1 y_1$ cannot be definite, which implies that the state of stability of $s \equiv 0$ and the positiveness of $-\Psi'(s)$ are contingent on command quantities because of $\overline{V}_1$, errors due to $s_1$ and steady state values introduced by $\tilde{g}$.

A Liapunov approach is used because it may give information with regard to globality over and above that which can be inferred from the differential equations. It, however, introduces additional degrees of an arbitrary selection within the definiteness of a function, and any conclusion would therefore be influenced by the choice. In our case, the positive definite Matrices $H_o$ and $D_1$ are available. It is desirable to select that $H_o$ which maximizes the extent of definiteness of $y_o^T Y_o y_o$ and a $D_1$ which is consistent with some physical significance of $y_1^T Y_1 y_1$.

## 3.3 OPTIMIZED LIAPUNOV FUNCTION

The quadratic form $y_o^T Y_o y_o$ is positive definite if the successive principal minors of the symmetric coefficient matrix are positive. Suppose $H_o$ = Diagonal $(\chi_1, \chi_2, \cdots \chi_6)$, $\chi_j >$ 0, $j = 1, 2, \ldots, 6$;then,the minors beyond $|H_o|$ are $|H_o| \Delta_1$, $|H_o| \Delta_1 \Delta_2$, and $|H_o| \Delta_1 \Delta_2 \Delta_3$

$= |Y|_o .$

We shall optimize $\Delta_i$, $i = 1,2,3$, with respect to $\chi_j$ for the zenith orientation, $\gamma_m = \beta_m = \gamma_n = \beta_n = 0$ $(m \neq n)$, at which G = Diagonal $(\Omega_1, \Omega_2, \Omega_3)$; $\Omega_1 = \Omega_2 = 1/2$, $\Omega_3 = 1$ in Group I and $\Omega_1 = 1, \Omega_2 = \Omega_3 = 1/2$ in Groups II and III. Now

$$\begin{aligned}\Delta_i = \frac{1}{\|k\|} &- \left(\frac{\alpha_1}{2\alpha_2}\right)^2 \Omega_i^{\,2}\, \chi_i \\ &+ \left(\frac{\alpha_1}{2\alpha_2}\right)(1+\nu\alpha_2)\,\Omega_i \\ &- \left[\frac{1}{2}(1+\nu\alpha_2)\right]^2 \chi_i^{-1} \\ &- \left(\frac{\alpha_3}{2}\right)^2 \chi_{i+3} \\ &+ \left(\frac{\alpha_3\alpha_4}{2}\right)(1-\nu)\,\Omega_i \\ &- \left[\frac{\alpha_4}{2}(1-\nu)\,\Omega_i\right]^2 \chi_{i+3}^{-1} \qquad i = 1,2,3,\end{aligned} \tag{19}$$

where $$\alpha_1 = -\frac{\varkappa_5\varkappa_8}{\rho(\rho\varkappa_7)^2}\left[\frac{\rho\varkappa_7}{\varkappa_4} - \frac{\varkappa_2 + \varkappa_3\varkappa_4}{\varkappa_8(\rho\varkappa_7+\varkappa_4)}\right]$$

$$\alpha_2 = \frac{\varkappa_4}{\rho\varkappa_7}, \qquad \alpha_3 = -\frac{\varkappa_5\varkappa_8}{\rho(\rho\varkappa_7)^2}$$

$$\alpha_4 = -\frac{\varkappa_2 - \rho\varkappa_7\varkappa_3}{\varkappa_8(\rho\varkappa_7+\varkappa_4)},$$

and consequently the optimum values are taken from

$$\chi_i = \pm\left(\frac{\alpha_2}{\alpha_1}\right)(1+\nu\alpha_2)\,\Omega_i^{-1} \tag{20a}$$

$$\chi_{i+3} = \pm\left(\frac{\alpha_4}{\alpha_3}\right)(1-\nu)\,\Omega_i, \tag{20b}$$

and summarized in Table 6.

It is interesting to note that although Equation 20 involves parametric values and $\nu$, the maximum $\Delta_i$ is a function only of the sector within which the nonlinearities lie. In addition, this relation

$$\text{Max } \Delta_i = \frac{1}{\|k\|} \tag{21}$$

is valid for all $\nu \geq 1$.

Table 6. Optimum $H_o$ at Zenith

| Element | Group | Numerical Value x $10^4$ |
|---|---|---|
| $\chi_1$ | I<br>II & III | 0.15077273<br>0.07538637 |
| $\chi_2$ | I<br>II & III | 0.15077273<br>0.15077273 |
| $\chi_3$ | I<br>II & III | 0.07538637<br>0.15077273 |
| $\chi_4$ | I<br>II & III | 4.74363957<br>9.48727915 |
| $\chi_5$ | I<br>II & III | 4.74363957<br>4.74363957 |
| $\chi_6$ | I<br>II & III | 9.48727915<br>4.74363957 |

Note: $H_o$ = Diagonal $(\chi_1, \chi_2, \ldots, \chi_6)$

At zenith and with $\chi_j$ from Equation 20, $y_o^T Y_o y_o$ is positive definite; therefore, $s_o$ can be of any magnitude without affecting the stability of the null solution. As $\|k\| \to \infty$, the sector broadens, the class of admissible nonlinearities is increased, and the measure of positive definiteness $\|k\|^{-1}$ weakens, finally becoming critical beyond which globality of $s_o$ is no longer present. Any other set of $\chi_j$ would not give more favorable results since the values from Equation 20 are optimum. As $\|k\| \to 0$, the linear problem is approached, i.e., $\tilde{g} \equiv 0$, and the measure of positive definiteness increases without limit, which only underscores the inherent properties of stability of the linear solution $s_o = 0$.

Maximizing the measure of positive definiteness of the quadratic form is equivalent to maximizing the parametric region within which definiteness is present. In other words, as the command values change from zero, and G is no longer diagonal, the persistence of globality for $s_o$ will be longest for the optimum $H_o$. As it turns out, this neighborhood is too restrictive, being $\pm$ 3 degrees or smaller for all pairs of trackers when $\|k\| = \sqrt{3}$ (Reference 10); and therefore, the requirement of definiteness is too severe. The more realistic stability criterion is: for given errors, $s_o$, steady state values in $\tilde{g}$, and command values $(\gamma_{m_c}, \theta_{m_c}, \gamma_{n_c}, \beta_{n_c})$, $y_o^T Y_o y_o$ must be positive.

### 3.4 PHYSICAL INTERPRETATION OF $y_1^T Y_1 y_1$ AT ZENITH

The Liapunov function, Equation 17, includes a positive definite integral; and as a result, $\overline{C}_1$ enters $y_1^T Y_1 y_1$. This makes the proper selection of the positive definite quadratic form $s_1^T D_1 s_1$ difficult to resolve because $D_1$ does not just scale the variables as it would if the Liapunov function were simply quadratic.

We shall require that $D_1$ = Diagonal $(\theta_1, \theta_2, \theta_3)$ satisfy

$$y_1^T Y_1 y_1 \equiv - s_1'^T s_1 \tag{22}$$

which has a well-known physical interpretation as a sufficient condition for asymptotic stability. Comparing the coefficients, we observe that the equality can be effected if $\overline{C}\ \overline{B} = 0$; and the differential equation does not contain a linear term. Both conditions are of course fulfilled for $s_1$. The elements of $D_1$ are found to be

$$\theta_i = \frac{1}{2}\left[1 - \frac{\varkappa_2}{\varkappa_4 \varkappa_8} \frac{\rho(\rho\varkappa_7)^2}{\varkappa_5 \varkappa_8} \Omega_i^{-1}\right] \qquad i = 1,2,3 \tag{23}$$

whose numerical values are listed in Table 7.

Table 7. $D_1$ for $y_1^T Y_1 y_1 \equiv - s_1'^T s_1$

| Element | Group | Numerical Value x $10^5$ |
|---|---|---|
| $\theta_1$ | I<br>II & III | 1.0081972<br>0.50410111 |
| $\theta_2$ | I<br>II & III | 1.0081972<br>1.0081972 |
| $\theta_3$ | I<br>II & III | 0.50410111<br>1.0081972 |

Note: $D_1$ = Diagonal $(\theta_1, \theta_2, \theta_3)$

## 3.5 CONCLUSIONS

Decomposing the vector space, the transformed variables are decoupled, and parameters are no longer present in the linear portion of the nonlinear system. Since the linear solution $s_o \equiv 0$ is globally asymptotically stable, the behavior of any other linear solution, $s_o(\sigma)$, is independent of command values and initial conditions. When the nonlinearities are sufficiently small, this global asymptotic character will persist for all realizable gimbal angles. On the other hand, $s_1(\sigma)$ is completely determined by the nonlinearities, no matter how insignificant, because its linear solution is $s_1 \equiv$constant.

A Liapunov technique is used to gain some insight into this behavior for large nonlinearities. Conclusions inferred from the approach are based on two selected positive definite diagonal matrices, $H_o$ and $D_1$, a measure of the magnitude of the nonlinearities $\|k\|$, and a positive parameter $\nu$, which admits a positive definite integral of the nonlinearities. The analysis also includes the positiveness of two quadratic forms, $y_o^T Y_o y_o$ and $y_1^T Y_1 y_1$, for which the first involves $s_o$, the second $s_1$, and both contain parameters.

Selecting the optimum $H_o$, which maximizes the positive definiteness of $y_o^T Y_o y_o$ at zenith (zero commands), the measure of maximum definiteness is shown to be $\|k\|^{-1}$ with $\nu > 1$. Thus, as $\|k\| \rightarrow 0$ and the nonlinearities are more confined, the neighborhood of zenith, within which definiteness exists, increases without limit. This confirms the previous observation that for small nonlinearities the global asymptotic behavior of $s_o$

is sustained regardless of what parametric values are used. We must, however, refer to the inclusive parametric region within which, for given errors, $y_o^T Y_o y_o$ is just positive. This became necessary when the requirement of definiteness proved to be too severe for $\|k\| = \sqrt{3}$. The positiveness of $y_o^T Y_o y_o$ is not as sensitive to changes in parametric and error values as is $y_1^T Y_1 y_1$. For this reason, the second quadratic is the more important one, and therefore becomes our measure for determining stability of the nine dimensional nonlinear system. The simplicity of a single number representation was however achieved with some compromise; for instead of positive definiteness, only definiteness of quadratic forms is assured. As a result, definition which we term "initial asymptotic stability" and which is a modification of the classical definition already cited, is: For a given $\epsilon > o$ there exists a $\delta = \delta(\epsilon) > o$ and $\Sigma < \infty$ such that any solution $s(\sigma)$ other than the trivial one which satisfies $\|s(o)\| < \delta$ also fulfills $\|s(\sigma)\| < \epsilon$ for all $\sigma \in [o, \Sigma]$.

Matrix $D_1$ is chosen so that $y_1^T Y_1 y_1$ is equivalent to the sufficient condition at zenith for asymptotic stability involving $-s_1'^T s_1$. Even for small nonlinearities, the errors and parametric values affect this quantity, as was already noted. Thus, for most practical purposes, the criterion for asymptotic stability of the null solution of the nonlinear system is that the single number $y_1^T Y_1 y_1$ be positive. This investigation has not been carried far enough to show the existence of a globally asymptotically stable $s_o$ together with a simple dynamical behavior for $s_1$ outside a small neighborhood of zenith.

## APPENDIX

## OTHER NONLINEAR MODELS

### A.1 DLU AND MOTOR SATURATIONS

Here there are two mathematical sources of nonlinearity. The first stems from $E\dot{a}$, which represents $\dot{e}$, and the other directly from motor saturation already discussed. In this regard, we must stipulate that $\dot{a}$ is unambiguously defined and that $E \equiv \left[\omega_{ki}\right]$ and $F \equiv \left[\psi_{ij}\right]$ are constant. It is also convenient to recognize a physical simplification of Equation 2, $a_i = a_i\,(b_i)$.

Suppose the three coefficient matrices $X_k$, $k = 1, 2, 3$, be defined $\left[\psi_{ij}\ \omega_{ki}\right]$. Then each is of the order, 4 x 3.

The formulation for the control system is

$$\tilde{q}' = A\tilde{q} + B\tilde{g} \tag{25a}$$

$$\tilde{g} = \tilde{g}(\tilde{r}) \quad \text{with } \tilde{g}(0) = 0 \tag{25b}$$

$$\tilde{r} = C\tilde{q} \tag{25c}$$

where

$$A \equiv \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & \varkappa_4 I/\rho\varkappa_7 & 0 & \varkappa_2 I/\rho\varkappa_7\varkappa_8 \\ F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B \equiv \begin{bmatrix} -\varkappa_5\varkappa_8\, I/\rho\, (\rho\varkappa_7)^2 & 0 & 0 \\ 0 & \varkappa_3 I/\varkappa_8 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

$$C \equiv \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

$$\tilde{r} \equiv \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = \begin{pmatrix} C_1 \tilde{q} \\ C_2 \tilde{q} \\ C_3 \tilde{q} \end{pmatrix}$$

$$\tilde{q} \equiv \begin{pmatrix} \tilde{p} \\ \tilde{v} - \tilde{v}_s \\ \tilde{b} - \tilde{b}_s \\ \tilde{e} - \tilde{e}_s \end{pmatrix} \qquad \tilde{g}(\tilde{r}) \equiv \begin{pmatrix} \tilde{w}(\tilde{r}_1 + \tilde{v}_s) - \tilde{w}_s \\ \left[\dfrac{d\tilde{a}}{d\tilde{b}}(\tilde{r}_3 + \tilde{b}_s)\right]^T X_k \tilde{r}_2 \\ 0 \end{pmatrix}$$

and for which the notation $d\tilde{a}/d\tilde{b}$ means the vector $(d\tilde{a}_i/d\tilde{b}_i)$.

If, for example, $\tilde{a} = \tanh \tilde{b}$ and $\tilde{w} = \tanh \tilde{v}$, then $\tilde{w}(\tilde{r}_1 + \tilde{v}_s) - \tilde{w}_s = \tanh(\tilde{r}_1 + \tilde{v}_s) - \tanh \tilde{v}_s$ and $\left(\frac{d\tilde{a}}{d\tilde{b}}\right)^T X_k \tilde{r}_2 = \left[\operatorname{sech}^2(\tilde{r}_3 + \tilde{b}_s)\right]^T X_k \tilde{r}_2$.

A complication of the DLU saturation, which is not true of motor saturation, is that the signal a = a(b) does not form components of controlling vector $\tilde{g}$, but rather the derived quantity $E\dot{a}$ does. This has an immediate effect of violating Equation 16, which implies $\Gamma(\tilde{r}) = \int_0^{\tilde{r}} \tilde{g}^T(t)\, dt$ is no longer positive definite. It is therefore necessary to determine sets of parametric values and errors different from the steady state for which $\Gamma$ is just positive. The additional restriction will have the effect of reducing the ranges of command gimbal angles that lead to stability.

## A.2 LARGE GIMBAL ERRORS AND MOTOR SATURATION

Another type of nonlinearity arises from the variation of E and F. We consider this question by first expanding these matrices about the command values, and then by investigating an effect terms of the second degree have on the state of stability when in the presence of motor saturation.

The control system is described as

$$\tilde{q}' = A\tilde{q} + B\tilde{g} \tag{26a}$$

$$\tilde{g} = \tilde{g}(\tilde{r}) \quad \text{with } \tilde{g}(0) = 0 \tag{26b}$$

$$\tilde{r} = C\,\tilde{q} \tag{26c}$$

for which

$$A \equiv \begin{bmatrix} -I & 0 & 0 & 0 \\ \varkappa_3 G(0)/\varkappa_8 & \varkappa_4 I/\rho\varkappa_7 & 0 & \varkappa_2 I/\rho\varkappa_7\varkappa_8 \\ F(0) & 0 & 0 & 0 \\ G(0) & 0 & 0 & 0 \end{bmatrix}$$

$$B \equiv \begin{bmatrix} -\varkappa_5\varkappa_8 I/\rho(\rho\varkappa_7)^2 & 0 & 0 \\ 0 & \varkappa_3 I/\varkappa_8 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$C \equiv \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$\tilde{r} \equiv \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} = \begin{pmatrix} C_1\tilde{q} \\ C_2\tilde{q} \end{pmatrix}$$

$$\tilde{q} \equiv \begin{pmatrix} \tilde{p} \\ \tilde{v} - \tilde{v}_s \\ \tilde{b} - \tilde{b}_s \\ \tilde{e} - \tilde{e}_s \end{pmatrix} \qquad \tilde{g}(\tilde{r}) \equiv \begin{pmatrix} \tilde{w}(\tilde{r}_1 + \tilde{v}_s) - \tilde{w}_s \\ \\ \tilde{r}_2^{\ T} Z_k \tilde{r}_2 + 2 z_s^{\ T} Z_k \tilde{r}_2 \end{pmatrix}$$

$$z_s = \begin{pmatrix} 0 \\ \tilde{v}_s \\ \tilde{b}_s \end{pmatrix} \quad .$$

The first and last three components of the ten symmetric coefficient matrices $Z_k$, each of which is 10 x 10, are taken from Equation 27; whereas, the middle four come from Equation 28. In all components, terms possessing inapplicable indices, with respect to the three principal tracker pairs, must be deleted.

For k = 1, 2, 3 and 8, 9, 10 we find $Z_k$ from $\quad -\tilde{r}_2^{\,T} Z_k \tilde{r}_2 - 2\,\tilde{r}_2^{\,T} Z_k z_s =$

$$\left(\begin{array}{l}
p_\xi \gamma_{1_e}(-\sin 2\gamma_{1_c}) + p_\xi \gamma_{2_e}(-\sin 2\gamma_{2_c}) + p_\xi \gamma_{3_e}(\sin 2\gamma_{3_c}) + p_\xi \gamma_{5_e}(\sin 2\gamma_{5_c}) \\
+ p_\eta \gamma_{2_e}(\cos 2\gamma_{2_c}) + p_\eta \gamma_{3_e}(\cos 2\gamma_{3_c}) + p_\eta \gamma_{4_e}(\sin \gamma_{4_c} \tan \beta_{4_c}) + p_\eta \gamma_{6_e}(-\cos \gamma_{6_c} \tan \beta_{6_c}) \\
+ p_\zeta \gamma_{1_e}(\cos 2\gamma_{1_c}) + p_\zeta \gamma_{4_e}(\cos \gamma_{4_c} \tan \beta_{4_c}) + p_\zeta \gamma_{5_e}(\cos 2\gamma_{5_c}) + p_\zeta \gamma_{6_e}(\sin \gamma_{6_c} \tan \beta_{6_c}) \\
+ p_\xi \beta_{1_e}(\cos^2 \gamma_{1_c} \tan \beta_{1_c}) + p_\xi \beta_{2_e}(\cos^2 \gamma_{2_c} \tan \beta_{2_c}) + p_\xi \beta_{3_e}(\sin^2 \gamma_{3_c} \tan \beta_{3_c}) \\
+ p_\xi \beta_{5_e}(\sin^2 \gamma_{5_c} \tan \beta_{5_c}) + p_\eta \beta_{1_e}(-\cos \gamma_{1_c}) + p_\eta \beta_{2_e}(\tfrac{1}{2} \sin 2\gamma_{2_c} \tan \beta_{2_c}) \\
+ p_\eta \beta_{3_e}(\tfrac{1}{2} \sin 2\gamma_{3_c} \tan \beta_{3_c}) + p_\eta \beta_{4_e}(-\cos \gamma_{4_c} \sec^2 \beta_{4_c}) + p_\eta \beta_{5_e}(\sin \gamma_{5_c}) \\
+ p_\eta \beta_{6_e}(-\sin \gamma_{6_c} \sec^2 \beta_{6_c}) + p_\zeta \beta_{1_e}(\tfrac{1}{2} \sin 2\gamma_{1_c} \tan \beta_{1_c}) + p_\zeta \beta_{2_e}(-\cos \gamma_{2_c}) + p_\zeta \beta_{3_e}(-\sin \gamma_{3_c}) \\
+ p_\zeta \beta_{4_e}(\sin \gamma_{4_c} \sec^2 \beta_{4_c}) + p_\zeta \beta_{5_e}(\tfrac{1}{2} \sin 2\gamma_{5_c} \tan \beta_{5_c}) + p_\zeta \beta_{6_e}(-\cos \gamma_{6_c} \sec^2 \beta_{6_c}) \\
\\
p_\xi \gamma_{1_e}(-\sin \gamma_{1_c} \tan \beta_{1_c}) + p_\xi \gamma_{2_e}(\cos 2\gamma_{2_c}) + p_\xi \gamma_{3_e}(\cos 2\gamma_{3_c} + p_\xi \gamma_{5_e}(-\cos \gamma_{5_c} \tan \beta_{5_c}) \\
+ p_\eta \gamma_{2_e}(\sin 2\gamma_{2_c}) + p_\eta \gamma_{3_e}(-\sin 2\gamma_{3_c}) + p_\eta \gamma_{4_e}(-\sin 2\gamma_{4_c}) + p_\eta \gamma_{6_e}(\sin 2\gamma_{6_c}) \\
+ p_\zeta \gamma_{1_e}(\cos \gamma_{1_c} \tan \beta_{1_c}) + p_\zeta \gamma_{4_e}(-\cos 2\gamma_{4_c}) + p_\zeta \gamma_{5_e}(\sin \gamma_{5_c} \tan \beta_{5_c}) + p_\zeta \gamma_{6_e}(\cos 2\gamma_{6_c}) \\
+ p_\xi \beta_{1_e}(\cos \gamma_{1_c} \sec^2 \beta_{1_c}) + p_\xi \beta_{2_e}(\tfrac{1}{2} \sin 2\gamma_{2_c} \tan \beta_{2_c}) + p_\xi \beta_{3_e}(\tfrac{1}{2} \sin 2\gamma_{3_c} \tan \beta_{3_c}) \\
+ p_\xi \beta_{4_e}(\cos \gamma_{4_c}) + p_\xi \beta_{5_e}(-\sin \gamma_{5_c} \sec^2 \beta_{5_c}) + p_\xi \beta_{6_e}(\sin \gamma_{6_c}) + p_\eta \beta_{2_e}(\sin^2 \gamma_{2_c} \tan \beta_{2_c}) \\
+ p_\eta \beta_{3_e}(\cos^2 \gamma_{3_c} \tan \beta_{3_c}) + p_\eta \beta_{4_e}(\cos^2 \gamma_{4_c} \tan \beta_{4_c}) + p_\eta \beta_{6_e}(\sin^2 \gamma_{6_c} \tan \beta_{6_c}) \\
+ p_\zeta \beta_{1_e}(\sin \gamma_{1_c} \sec^2 \beta_{1_c}) + p_\zeta \beta_{2_e}(-\sin \gamma_{2_c}) + p_\zeta \beta_{3_e}(-\cos \gamma_{3_c}) \\
+ p_\zeta \beta_{4_e}(-\tfrac{1}{2} \sin 2\gamma_{4_c} \tan \beta_{4_c}) + p_\zeta \beta_{5_e}(-\cos \gamma_{5_c} \sec^2 \beta_{5_c}) + p_\zeta \beta_{6_e}(\tfrac{1}{2} \sin 2\gamma_{6_c} \tan \beta_{6_c}) \\
\\
p_\xi \gamma_{1_e}(\cos 2\gamma_{1_c}) + p_\xi \gamma_{2_e}(-\sin \gamma_{2_c} \tan \beta_{2_c}) + p_\xi \gamma_{3_e}(\cos \gamma_{3_c} \tan \beta_{3_c}) + p_\xi \gamma_{5_e}(\cos 2\gamma_{5_c}) \\
+ p_\eta \gamma_{2_e}(\cos \gamma_{2_c} \tan \beta_{2_c}) + p_\eta \gamma_{3_e}(-\sin \gamma_{3_c} \tan \beta_{3_c}) + p_\eta \gamma_{4_e}(-\cos 2\gamma_{4_c}) + p_\eta \gamma_{6_e}(\cos 2\gamma_{6_c}) \\
+ p_\zeta \gamma_{1_e}(\sin 2\gamma_{1_c}) + p_\zeta \gamma_{4_e}(\sin 2\gamma_{4_c}) + p_\zeta \gamma_{5_e}(-\sin 2\gamma_{5_c}) + p_\zeta \gamma_{6_e}(-\sin 2\gamma_{6_c}) \\
+ p_\xi \beta_{1_e}(\tfrac{1}{2} \sin 2\gamma_{1_c} \tan \beta_{1_c}) + p_\xi \beta_{2_e}(\cos \gamma_{2_c} \sec^2 \beta_{2_c}) + p_\xi \beta_{3_e}(\sin \gamma_{3_c} \sec^2 \beta_{3_c}) \\
+ p_\xi \beta_{4_e}(-\sin \gamma_{4_c}) + p_\xi \beta_{5_e}(\tfrac{1}{2} \sin 2\gamma_{5_c} \tan \beta_{5_c}) + p_\xi \beta_{6_e}(\cos \gamma_{6_c}) + p_\eta \beta_{1_e}(-\sin \gamma_{1_c}) \\
+ p_\eta \beta_{2_e}(\sin \gamma_{2_c} \sec^2 \beta_{2_c}) + p_\eta \beta_{3_e}(\cos \gamma_{3_c} \sec^2 \beta_{3_c}) + p_\eta \beta_{4_e}(-\tfrac{1}{2} \sin 2\gamma_{4_c} \tan \beta_{4_c}) \\
+ p_\eta \beta_{5_e}(\cos \gamma_{5_c}) + p_\eta \beta_{6_e}(\tfrac{1}{2} \sin 2\gamma_{6_c} \tan \beta_{6_c}) + p_\zeta \beta_{1_e}(\sin^2 \gamma_{1_c} \tan \beta_{1_c}) \\
+ p_\zeta \beta_{4_e}(\sin^2 \gamma_{4_c} \tan \beta_{4_c}) + p_\zeta \beta_{5_e}(\cos^2 \gamma_{5_c} \tan \beta_{5_c}) + p_\zeta \beta_{6_e}(\cos^2 \gamma_{6_c} \tan \beta_{6_c})
\end{array}\right) \tag{27}$$

The $Z_k$, k = 4, 5, 6, 7, stems from

$$
-\tilde{r}_2^{T} Z_k \tilde{r}_s - 2\,\tilde{r}_2^{T} Z_k z_s = \begin{pmatrix}
-p_{\xi}\gamma_{1_e}(\cos\gamma_{1_c}) - p_{\zeta}\gamma_{1_e}(\sin\gamma_{1_c}) \\
p_{\xi}\gamma_{2_e}(\cos\gamma_{2_c}) + p_{\eta}\gamma_{2_e}(\sin\gamma_{2_c}) \\
-p_{\xi}\gamma_{3_e}(\sin\gamma_{3_c}) - p_{\eta}\gamma_{3_e}(\cos\gamma_{3_c}) \\
p_{\eta}\gamma_{4_e}(\cos\gamma_{4_c}) - p_{\zeta}\gamma_{4_e}(\sin\gamma_{4_c}) \\
-p_{\xi}\gamma_{5_e}(\sin\gamma_{5_c}) - p_{\zeta}\gamma_{5_e}(\cos\gamma_{5_c}) \\
p_{\eta}\gamma_{6_e}(\sin\gamma_{6_c}) + p_{\zeta}\gamma_{6_e}(\cos\gamma_{6_c}) \\
\begin{array}{l} p_{\xi}\gamma_{1_e}(-\sin\gamma_{1_c}\tan\beta_{1_c}) + p_{\xi}\beta_{1_e}(\cos\gamma_{1_c}\sec^2\beta_{1_c}) \\ +p_{\zeta}\beta_{1_e}(\sin\gamma_{1_c}\sec^2\beta_{1_c}) + p_{\zeta}\gamma_{1_e}(\cos\gamma_{1_c}\tan\beta_{1_c}) \end{array} \\
\begin{array}{l} p_{\xi}\gamma_{2_e}(\sin\gamma_{2_c}\tan\beta_{2_c}) + p_{\xi}\beta_{2_e}(-\cos\gamma_{2_c}\sec^2\beta_{2_c}) \\ +p_{\eta}\beta_{2_e}(-\sin\gamma_{2_c}\sec^2\beta_{2_c}) + p_{\eta}\gamma_{2_e}(-\cos\gamma_{2_c}\tan\beta_{2_c}) \end{array} \\
\begin{array}{l} p_{\xi}\gamma_{3_e}(\cos\gamma_{3_c}\tan\beta_{3_c}) + p_{\xi}\beta_{3_e}(\sin\gamma_{3_c}\sec^2\beta_{3_c}) \\ +p_{\eta}\beta_{3_e}(\cos\gamma_{3_c}\sec^2\beta_{3_c}) + p_{\eta}\gamma_{3_e}(-\sin\gamma_{3_c}\tan\beta_{3_c}) \end{array} \\
\begin{array}{l} p_{\eta}\gamma_{4_e}(\sin\gamma_{4_c}\tan\beta_{4_c}) + p_{\eta}\beta_{4_e}(-\cos\gamma_{4_c}\sec^2\beta_{4_c}) \\ +p_{\zeta}\beta_{4_e}(\sin\gamma_{4_c}\sec^2\beta_{4_c}) + p_{\zeta}\gamma_{4_e}(\cos\gamma_{4_c}\tan\beta_{4_c}) \end{array} \\
\begin{array}{l} p_{\xi}\gamma_{5_e}(\cos\gamma_{5_c}\tan\beta_{5_c}) + p_{\xi}\beta_{5_e}(\sin\gamma_{5_c}\sec^2\beta_{5_c}) \\ +p_{\zeta}\beta_{5_e}(\cos\gamma_{5_c}\sec^2\beta_{5_c}) + p_{\zeta}\gamma_{5_e}(-\sin\gamma_{5_c}\tan\beta_{5_c}) \end{array} \\
\begin{array}{l} p_{\eta}\gamma_{6_e}(-\cos\gamma_{6_c}\tan\beta_{6_c}) + p_{\eta}\beta_{6_e}(-\sin\gamma_{6_c}\sec^2\beta_{6_c}) \\ +p_{\zeta}\beta_{6_e}(-\cos\gamma_{6_c}\sec^2\beta_{6_c}) + p_{\zeta}\gamma_{6_e}(\sin\gamma_{6_c}\tan\beta_{6_c}) \end{array}
\end{pmatrix} \tag{28}
$$

The components $(\tilde{p}_\xi, \tilde{p}_\eta, \tilde{p}_\zeta)$ are projections onto the control axes. Equation 16 is again violated, and consequently $\Gamma(\tilde{r}) = \int_0^{\tilde{r}} \tilde{g}^T(t)\, dt$ and $\tilde{g}^T \left(C\tilde{q} - \frac{g}{\|k\|}\right)$ cease to be positive definite. We shall, therefore, use explicit forms, based on the approximation $\tilde{w} = \tanh \tilde{v}$, as two additional sufficient conditions for positiveness, (Reference 11).

$$
\begin{aligned}
\Gamma(\tilde{r}) &\equiv \int_0^{\tilde{r}} \tilde{g}^T(t)\, dt \\
&= \sum_{j=1}^{3} \ln\cosh(\tilde{r} + \tilde{v}_{s_j}) - \ln\cosh \tilde{v}_{s_j} - \tilde{r}_j \tanh \tilde{v}_{s_j}\, . \\
&- \sum_{j=5}^{13} \sum_{i=4}^{j-1} \mathcal{C}_{ijj}\, \tilde{r}_i \tilde{r}_j^2 \\
&- \sum_{j=6}^{13} \sum_{i=4}^{j-1} \sum_{\substack{h=4 \\ i \neq h}}^{j-1} \mathcal{C}_{ihj}\, \tilde{r}_i \tilde{r}_h \tilde{r}_j \\
&- \sum_{j=5}^{13} \sum_{i=4}^{j-1} 2\mathcal{D}_{ij}\, \tilde{r}_i \tilde{r}_j - \sum_{j=4}^{13} \mathcal{D}_{jj}\, \tilde{r}_j^2
\end{aligned}
\qquad (29)
$$

where , if $(\Pi_{ihk}) = Z_k$

$$\mathcal{C}_{ijj} = \Pi_{(i-3)\,(j-3)\,(j-3)} \qquad \mathcal{C}_{ihj} = \Pi_{(i-3)\,(h-3)\,(j-3)}$$

$$\mathcal{D}_{ij} = \sum_{h=1}^{10} \Pi_{(i-3)\,h\,(j-3)}\, z_{s_h}$$

$$\mathcal{D}_{jj} = \sum_{h=1}^{10} \Pi_{(j-3)\,h\,(j-3)}\, z_{s_h}$$

$$
\begin{aligned}
\Lambda &\equiv \tilde{g}^T \left(\tilde{r} - \frac{\tilde{g}}{\| k \|}\right) \\
&= \sum_{j=1}^{3} \left\{ \tilde{r}_{1_j} - \| k \|^{-1} \left[ \tanh\,(\tilde{r}_1 + \tilde{v}_s)_j - \tanh \tilde{v}_{s_j} \right] \right\} \\
&\qquad \left\{ \tanh\,(\tilde{r}_1 + \tilde{v}_s)_j - \tanh\, \tilde{v}_{s_j} \right\} \\
&+ \sum_{j=1}^{10} \left\{ \tilde{r}_{2_j} - \|k\|^{-1} \left[ \tilde{r}_2^{\,T} Z_j\, \tilde{r}_2 + 2\, \tilde{r}_2^{\,T} Z_j\, z_s \right] \right\} \\
&\qquad \left\{ \tilde{r}_2^{\,T}\, Z_j\, \tilde{r}_2 + 2\, \tilde{r}_2^{\,T}\, Z_j\, z_s \right\}
\end{aligned}
\tag{30}
$$

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