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ABSTRACT. Sequencies of orthogonally blocked statistical designs of experiments are presented for optimum seeking. The sequences are such that observations from the first block can be used to estimate the coefficients of a simple model and then be retained and combined with observations from new blocks so that all acquired observations are used cumulatively to estimate models of successively greater generality. Such blocks are said to form a "telescoping" sequence. Specific choices were motivated by the problem of optimum seeking experiments in alloy development.

The designs consist of full and fractionally replicated two-level factorial experiments with four to eight factors. The sizes of the experiments include 8, 16, 32, and 64 treatments.

INTRODUCTION. Optimum seeking experiments have been conducted by NASA in developing improved engine materials for the supersonic transport. The use of the designs presented herewith for optimum seeking has been discussed in reference 1. In addition to optimum seeking, the designs could be used in many situations where the experimenting begins without prior knowledge of the complexity needed for the model.

The designs consist of two level fractional factorial experiments performed as sequences of blocks. The designs are to be such that the first block will be a small fraction of the full factorial, but large enough for estimating the parameters of a first degree model. Successive blocks are to be such that all acquired data can be used cumulatively to estimate models of successively greater generality, with block effects being uncorrelated with the parameter estimates. The sequences terminate in designs that give estimates of first degree and two factor interaction coefficients and the estimates are free of aliases with other second degree or lower order coefficients. Without considering blocking, Steve Webb in reference 2 applied the terms expansible and contractible to related sequences of designs.

Sequences of regular fractions were discussed in reference 3 by Cuthbert Daniel. Sequences of irregular fractions were discussed by

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Peter John in reference 4. The general subject was explored further by Sidney Adelman in reference 5.

Box and Hunter in reference 6 recommended the use of sequences of rotatable orthogonally blocked designs for optimum seeking. These properties require that the fractions be regular fractions, that is, the number of treatments is $1/2^h$ times the number of treatments in a full factorial experiment, where h is an integer. The designs to be presented are all regular fractions.

SYMBOLS.

b	number of blocks
$E(\ )$	value of ( ) if averaged over infinite number of observations
g	number of independent variables (factors)
h	fractional replicate contains $1/2^h$ times number of treatments performed in full two-level factorial experiment
i	index number for trials
j, k	index number for independent variables
ℓ	$g - h$
R	resolution level
X_j	vector giving levels of x_{ij}, $i = 1, \ldots, n$
x_{ij}	standardized level of ξ_j
y	response (observed variate)
β	unknown population parameter
ϵ	error
ξ_j	independent variable, $j = 1, \ldots, g$
σ^2	variance of ϵ

SIZES OF EXPERIMENTS.

Degrees of Freedom for Lack of Fit. Consider the fitting of a model equation to a 2^3 full factorial experiment. The appropriate equation is as follows:

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3 \qquad (1)$$

The equation illustrates the notation. Main effects are designated by symbols such as β_1 and β_2. Two factor interactions are represented by symbols such as β_{12}. The independent variates are represented by lower case symbols such as x_1 and x_2.

The number of treatments minus the number of parameters estimated is the degrees of freedom for lack of fit. The 2^3 experiment contains 8 treatments, but the optimum seeking begins with a first degree equation containing only four parameters, leaving four degrees of freedom for lack of fit. The final stage of optimum seeking includes the two factor interactions so that only one degree of freedom would remain for lack of fit (eq. (1)).

Some information on the lack of fit is always desirable. The degrees of freedom for lack of fit of the designs to be presented vary from 0 to 35, and designs are provided for numbers of factors varying from 4 to 8. With 9 factors the use of a regular fraction requires 128 treatments of which 66 represent degrees of freedom for lack of fit. In other words, an insistence on the use of regular fractions does not seem to be unduly extravagant unless there are 9 or more factors. The use of irregular fractions seems to be appropriate in situations involving 9 or more factors or for lesser numbers of factors, where the experimenting is very expensive, and where the relative error is known to be small.

Resolution Levels. The factorial experiment with conditions fixed at just two levels of g independent variables (factors) permits the estimation of parameters representing the grand mean over the experiment, the first-order effects of the factors, and the results of factors interacting two at a time, three at a time, and in all combinations up to g at a time. If a fraction $1/2^h$ of this experiment is performed, not all these parameters can be estimated. True response functions in physical investigations are typically smooth enough that the higher order coefficients of an approximating polynomial may be assumed to be negligible over a small enough range of the experimentation. Accordingly, only the lower order coefficients need be estimated; however, they are allowed to be biased by (aliased with) coefficients of higher order interactions because such coefficients are assumed to be negligible.

Let the number of factors in the highest order interaction requiring estimation be e, and let the number of factors in the lowest order interaction with which it is allowed to be aliased be c; then the required resolution R of the design is defined (ref. 7) to be

$$R = e + c$$

As a minimum requirement on the first-order experiments, the coefficients will be allowed to be aliased with only the coefficients of two-factor or higher order interactions. This requires that $R = e + c = 1 + 2 = 3$. A somewhat improved design occurs if the first-order coefficients are estimated clear of two-factor interactions. This requires that $R = e + c = 1 + 3 = 4$.

For the interaction experiments, the estimates of two factor interaction coefficients should be allowed to be aliased only with higher order interaction coefficients. This requires that $R = e + c = 2 + 3 = 5$.

The design of the interaction experiment (of resolution 5) is now specified to be blocked into b blocks such that any one block will provide a design of resolution 3 for the first-degree model. As a consequence of this requirement, the experimenter may switch at any time from the method of steepest ascents to the method of local exploration by completing the $b - 1$ uncompleted blocks of the resolution 5 experiment.

Occasions could arise in which the experimenter would not wish to proceed immediately from a minimum-size first-degree design to the design for estimating all interaction coefficients. For example, a design of only eight treatments hardly provides enough information to test the validity of the first-degree model. The performance of a second block of eight treatments could lead to a much better decision. Also, the experimenter may have prior knowledge that certain interactions are negligible so that he can stop short of the experiment that estimates all two-factor interactions. For these reasons, the designs and parameter estimates are given for such intermediate size experiments.

Numbers of Factors and Block Sizes. The assumption was made that a sequence of blocks should not terminate in a total experiment that contained less than 16 treatments, that is, the assumption was made that a completed experiment containing less than 16 experimental units is too small for any statistical assessment of validity. With 16 treatments, the smallest number of factors in the (efficient) unreplicated experiment is four, and therefore no designs were investigated having less than four independent variables.

As was shown in reference 3, the degrees of freedom efficiency of regular fractions of two level factorial experiments of resolution 5 becomes and remains poor, and the experiment sizes become enormous, if the number of factors exceeds 8. The investigation was therefore limited to 4, 5, 6, 7, and 8 factors.

The regular fractional factorial first degree experiment on four factors requires a minimum of 8 treatments, whereas the regular fractional factorial first degree experiment with eight factors requires a minimum of 16 treatments. Correspondingly, the sizes of the blocks are limited to 8 and 16 treatments.

So that the experimenter will always get results on his "standard conditions" first, the principal block will always be given as the first block.

CONSTRUCTION OF DESIGNS AND ESTIMATES OF PARAMETERS.

Defining Contrasts. The mixed usage of Yates' notation for treatments and the special notation of the present work is illustrated by table 1. The treatments are listed in the familiar Yates' notation and Yates' order in the first column. The resulting dependent variates are listed in the corresponding order in the second column. Lower case symbols like x_1 had been used for the independent variates. The full set of levels of such a variate is a column vector of plus and minus ones and is represented by the corresponding upper case symbol as shown by the column headings. A column heading showing a product means that elements from identical rows have been multiplied to produce a new column with the same number of rows.

This rule of multiplication leads to such relations as

$$(X_1X_2)(X_2X_3X_4) = X_1X_0X_3X_4 = X_1X_3X_4$$

These operations are similar to the more popular terminology in which:

$$(AB)(BCD) = AICD = ACD$$

The present usage of symbols such as β_0, β_{12}, X_0, X_1X_2 avoids such ambiguities as I standing for both the grand mean and the identity vector, and AB standing for both the interaction parameter β_{12} and the contrast vector X_1X_2.

The general rules for sequences of blocked designs were given in reference 3. Given now are rules that are much more narrowly stated. The purpose of the narrow statement is to quickly and easily arrive at a list of treatments and aliased parameters that will be in Yates' order Thus, if the responses are listed in Yates' order then Yates' computational procedure will give estimates that will be in the order of easily identified sets of aliased parameters. Actually, this narrowly stated procedure results in no loss of generality, because the experimenter is free to assign the symbols $x_1, x_2, \ldots$ to his physical variables in any order he chooses.

Although designs are given for numbers of factors from 4 to 8 and block sizes of 8 and 16, their construction will be illustrated by only an example with 6 factors and a block size of 8. For this block size the first 8 rows of table 1 give treatment levels that can be used for the factors x_1, x_2, and x_3. The design must be completed with orthogonal levels of x_4, x_5, and x_6. For orthogonality the levels can only be levels that already occur for columns from X_1 to the product $X_1X_2X_3$. Then multiplying the elements of a new column by the elements from its equal among the old columns will result in a column of plus ones, namely, the X_0 column.

The first block is to be a $1/2^3$ replicate of the 2^6 design. The fractional replication is characterized by 2^3 defining contrasts of which 3 are independent, and the telescoping requires that some constraints be placed on the 3 independent defining contrasts. From among the columns from X_1 to the product $X_1X_2X_3$ select 3 (as yet unspecified) columns and call them U, V, and W. Then

$$X_4 = U \qquad X_5 = V \qquad X_6 = W$$

$$\underline{UX_4} = X_4^2 = X_0; \ \underline{VX_5} = X_5^2 = X_0; \ \underline{WX_6} = X_6^2 = X_0$$

The underlined items are the defining contrasts. Because they each contain a column not contained in the others, they are independent, and because there are three of them, they are all of the $h = 3$ independent defining contrasts. The group of defining contrasts is found by forming the products of the independent contrasts in all possible combinations:

UX_4
VX_5
WX_6
UVX_4X_5
UWX_4X_6
VWX_5X_6
$UVWX_4X_5X_6$
$UX_4UX_4 = X_0$

The fact that a sequence of telescoping designs is desired will impose some constraints on the choice of U, V, and W in terms of X_1, X_2, and X_3.

Defining contrasts are now to be considered for the two blocks that will constitute a 2/8 replicate. The 16 treatment levels for x_1, x_2, x_3, and x_4 are given in Yates' order by table 1. The columns of levels of x_5 and x_6 need to be identical with two of the columns from X_1 to $X_1X_2X_3X_4$ of table 1. Let these columns (as yet unspecified) be called Y and Z, that is, $X_5 = Y$, $X_6 = Z$ so that the independent defining contrasts for the 2/8 replicate are YX_5 and ZX_6. The complete group of defining contrasts is:

X_0
YX_5
ZX_6
YZX_5X_6

In the case of the 4/8 replicate, X_6 is set equal to one of the product columns of a 2^5 experiment. The defining contrast is symbolized by TX_6.

In summary, the groups of as yet, incompletely specified defining contrasts are:

1/8 replicate	2/8 replicate	4/8 replicate
X_0	X_0	X_0
UX_4	YX_5	TX_6
VX_5	ZX_6	
WX_6	YZX_5X_6	
UVX_4X_5		
UWX_4X_6		
VWX_5X_6		
$UVWX_4X_5X_6$		

Some of the constraints of the design problem are that one of the blocks of the 2/8 replicate must be identical to the 1/8 replicate, and two of the blocks of the 4/8 replicate must be identical to those of the 2/8 replicate. Thus, for example, the treatment levels of X_1, X_2, and X_3, associated with X_5 of the 2/8 replicate must have 8 points of identity with the treatment levels of X_1, X_2, and X_3 associated with X_5 in the 1/8 replicate.

These identities are achieved by setting

$$Y = V$$

or

$$Y = UVX_4$$

and also

$$Z = W$$

or

$$Z = UWX_4$$

For the 4/8 replicate, a necessary condition is that

$$T = Z$$

or that

$$T = YZX_5$$

Among the preceding constraints, desirable choices would result in TX_6 having at least 5 symbols so that the 4/8 replicate would be of resolution 5. Also, because each stage must be of resolution 3, all defining contrasts must contain at least 3 symbols. The choices of U, V, W, Y, and Z should be consistent with these objectives.

So that the first block will be a principle block (so that it will contain a treatment with all factors at their "low" levels) the defining contrasts must be negative if they contain an odd number of symbols, and positive if they contain an even number of symbols.

Suppose that $U = -X_1X_2$, $V = -X_2X_3$ and $W = X_1X_2X_3$. Multiplying the resulting defining contrasts together in all combinations gives the group for the 1/8 replicate as listed in table 8. The contrasts with the larger numbers of symbols are desirable for the 2/8 replicate. They are attained by selecting $Y = UVX_4$, and $Z = W$, and the defining contrasts for the 2/8 replicate are:

$$YX_5 = UVX_4X_5 = X_1X_3X_4X_5$$

$$ZX_6 = WX_6 = X_1X_2X_3X_6$$

$$YZX_5X_6 = UVWX_4X_5X_6 = X_2X_4X_5X_6$$

and these contrasts are listed as the 2/8 replicate in table 8. For the 4/8 replicate the choice was $T = Z$ so that

$$TX_6 = ZX_6 = WX_6 = X_1X_2X_3X_6$$

and the 4/8 replicate fails to be of resolution 5. The question arises as to whether a better choice could have been made for the defining contrasts of the 1/8 replicate.

Achievement of the highest possible resolution number at each stage of a sequence of telescoping designs would be helped if the total number of symbols in the group of defining contrasts were as large as possible. For a $1/2^h$ fraction with g factors the maximum number of symbols was given in reference 5 as

$$A = g2^{h-1}$$

For the example of six factors with blocks of size 8, this number is:

Replicate	1/8	2/8	4/8
A	24	12	6

If a resolution 5 design is to be achieved at the 4/8 replicate, then TX_6 must contain at least 5 symbols. From the preceding table, the number cannot exceed 6. The maximum total number of symbols for the 2/8 replicate is 12 so that the numbers of symbols might be distributed among the contrasts as follows:

YX_5,	ZX_6,	YZX_5X_6
3	3	5
3	4	5
3	3	6

To have a resolution 3 design for the 1/8 replicate, all 7 defining contrasts must contain at least 3 symbols, but the total number cannot exceed 24. For the telescoping, three of the 7 defining contrasts must be distributed according to one of the three preceding distributions of symbols. Considering only the upper limit of 24, the possibilities are:

(3, 3, 3, 3, 3, 4, 5)

or

(3, 3, 3, 3, 3, 3, 6)

The multiplication of two defining contrasts each containing 3 symbols could result in defining contrasts of length 2, 4, or 6. Contrasts of length 2 would violate the condition that the design must be of resolution 3. If 3 contrasts are of length three, the multiplication of all pairwise combinations results in 3 contrasts at least of length 4. Therefore the preceding combinations are not attainable, that is a telescoping sequence cannot lead from a 1/8 replicate of resolution 3 to a 4/8 replicate of resolution 5. The sequence must be continued to the full replicate.

Identification of Parameters Estimated by Yates' Contrasts. The manner in which defining contrasts can be obtained for telescoping sequences of orthogonal blocks has been illustrated. Reference 1 shows how the defining contrasts were used to determine the detailed treatments in Yates' order. Reference 1 also shows how the results of the Yates' computation are identified with the appropriate sets of aliased parameters.

In the case of the first-degree experiments, if a two-factor interaction coefficient is aliased with a single-factor coefficient (if the sum of a two-factor coefficient and a single-factor coefficient is estimated by a single contrast), then the two-factor coefficient is assumed to be zero. If a contrast does not estimate any combination of two-factor or lower order coefficients, the contrast will be given a name by listing the lowest order set of interaction coefficients that it does estimate. For example, table 17 lists a treatment bcde, and the Yates' computation would give an estimator of β_{234} in the same row. From table 15 the full set of aliased parameters can be shown to be β_{234}, $-\beta_{1245}$, β_{147}, β_{126}, $-\beta_{3457}$, $-\beta_{2356}$, β_{367}, and $-\beta_{1567}$ of which the lowest order set is β_{234}, $+\beta_{147}$, $+\beta_{126}$, $+\beta_{367}$. Those parameters, the estimates of which are confounded with block effects, will be identified by attaching an asterisk to the parameters.

PROPERTIES OF RECOMMENDED DESIGNS. The designs are identified by code numbers. For example, Plan 1/8; 7f, 8t/b; 2b means that the design is a 1/8 replicate of a full factorial experiment with 7 factors, employing 8 treatments per block, and using 2 blocks. The order of presentation of the designs (tables 2 to 29) is the order of increasing numbers of factors. For a given number of factors, a sequence of designs with blocks of 8 treatments is presented first, followed by a sequence of designs with blocks of 16 treatments. Within any sequence, the order is the order of increasing numbers of blocks. The properties of the designs are summarized in table 30 and therefore table 30 serves as a "Table of Contents" for the designs.

Use of Resolution 4 Designs in Fitting First-Order Model. In general, the use of the first-order model as a prediction equation, with coefficients estimated from an experiment, requires the assumption that all second-order parameters are zero. However, circumstances might arise where the experimenter desired an approximate first-order predicting equation and ignored the existence of possible nonzero two-factor interactions. He might then prefer a resolution 4 design to a resolution 3 design because the estimates of the first-order coefficients would not be aliased with (biased by) two-factor interactions.

Minimum-size designs of resolution 4 are shown for 4 factors by table 2, for 5 factors by table 5, and for 6 factors by table 10. Minimum-size designs of resolution 4 for 7 and 8 factors were given by Natrella (ref. 8, p. 12-18), and these designs are also given in tables 28 and 29. Unfortunately, no success was achieved in trying to include the designs of tables 28 and 29 in the telescoping sequences of 7- and 8-factor blocked designs, that is, tables 21 to 27. However, the designs of tables 28 and 29 might be used for the very first trial of a Box-Wilson procedure, when the experimenter believed that he would be so far from an optimum condition that a first-order model would be a good enough approximation.

After such a trial he could move to a new design center and then elect a design capable of being sequentially expanded by blocks into designs of higher order, that is, the designs of tables 21 or 25.

Conditions for Using Resolution 3 and Resolution 4 Designs in Estimating the Second-Order Model. If the experimenter has prior knowledge that some of the two-factor interactions are zero, he may be able to choose the labels for his factors so that the nonzero interaction parameters can be estimated from designs of less than resolution 5. The specific cases are listed:

Table 2. - Plan 1/2; 4f; 8t/b; 1b. - If one of the factors (for example x_1) does not interact with the other factors, then all the remaining interactions are estimable (table 2). If x_1 is noninteracting, the estimated parameters are β_0, β_1, β_2, β_{34}, β_3, β_{24}, β_{23}, and β_4.

Table 5. - Plan 1/2; 5f; 8t/b; 2b. - The factor believed most likely to interact with other factors should be labeled x_4 because the plan (table 5) gives unconfounded estimates of β_{14}, β_{24}, β_{34}, and β_{45}. If any one of x_1, x_2, x_3, or x_5 does not interact with the others (for example, x_1) then all the remaining two-factor interactions are estimable and the estimated parameters are β_0, β_1, β_2, β_{35}, β_3, β_{25}, β_{23}, β_5, β_4, β_{14}, β_{24}, β_{345}, β_{34}, β_{245}, $(\beta^*_{234} + \beta^*_{145})$, and β_{45}. Under previously stated assumptions, the estimates of β_{14}, β_{345}, and β_{245} are assumed to be nothing more than random error.

Table 10. - Plan 1/4; 6f; 8t/b; 2b. - If x_1 does not interact with any other factor, and if x_2 does not interact with x_4, x_5, and x_6, then the parameters estimated are as follows: β_0, β_1, β_2, β_{36}, β_3, β_{45}, β_{23}, β_6, β_4, β_{35}, β_{56}, $(\beta^*_{124} + \beta^*_{156} + \beta^*_{235} + \beta^*_{346})$, β_{34}, β_5, β_{46}, and the estimate of $(\beta_{125} + \beta_{146} + \beta_{234} + \beta_{356})$ is assumed to be random error (table 10).

Table 11. - Plan 1/2; 6f; 8t/b; 4b. - If the label x_1 had been given to the most likely noninteracting factor in the design of table 10, the performance of the two augmenting blocks of table 11 would result in a design with all interactions estimable under the minimal assumptions that β_{12}, β_{13}, and β_{16} are zero.

Table 13. - Plan 1/4; 6f; 16t/b; 1b. - Assume that there are two groups of three factors and that each factor does not interact within its group. Give the factors within one group the labels x_1, x_2, and x_6 and label the factors of the other group x_3, x_4, and x_5. Then all the nonzero two-factor interaction coefficients (one factor from each group) are

estimable and are β_{13}, β_{14}, β_{15}, β_{23}, β_{24}, β_{25}, β_{36}, β_{46}, and β_{56} (table 13).

Table 18. - Plan 1/4; 7f; 8t/b; 4b. - This plan (table 18) becomes a suitable second-order design under the assumptions that x_1 does not interact with x_3, x_4 or x_6, and that x_2, x_5, and x_7 do not interact with each other.

Table 21. - Plan 1/8; 7f; 16t/b; 1b. - This plan (table 21) estimates two-factor interactions if x_1 is noninteracting, if x_2 is noninteracting ~~with x_3, x_4, x_5, and x_7~~, and if x_5 is noninteracting with x_4 and x_6.

Table 22. - Plan 1/4; 7f; 16t/b; 2b. - This plan (table 22) estimates all two-factor interactions if any one of x_1, x_2, x_4, or x_6 does not interact with the other factors of this group.

Table 26. - Plan 1/8; 8f; 16t/b; 2b. - This plan (table 26) estimates all interactions if x_8 is noninteracting with x_1, x_2, x_3, x_5, and x_7, and if x_3 is noninteracting with x_1, x_2, x_4, and x_6. Thus the label x_8 should be given to the least interacting variable, the label x_3 should be given to the next least interacting variable, the labels x_3, x_5, and x_7 should be given to the variables least likely to interact with x_8, and the labels x_4 and x_6 should be given to the variables least likely to interact with x_3.

CHOICE OF BLOCK SIZE. The present investigation assumes that the experimenter will wish to perform a block of treatments, analyze the data, and then perform another block of treatments, and that the block effects arise during the interruption of the experimenting for analyzing data (furnaces are overhauled, instruments are newly calibrated, etc.). Under these assumptions, block sizes 8 and 16 are particularly appropriate for experiments on 4 to 8 factors. On the other hand, the physical situation could limit the experimenter to smaller block sizes. Under such limitations, other designs would have to be synthesized, and the synthesis could be done according to rules already presented.

Another reason for using small block sizes is to protect against the hazard of missing values. If through accident, the observations from one or more treatments are missing from a block, the whole block could be rerun, especially if it is small. On the other hand, only the missing treatments need be run, if the experimenter can say that no block effect will arise between the new runs and the block from which observations are missing. If the design is not severely fractionated (if the number of treatments is significantly larger than the number of parameters estimated), methods of estimating for missing values may be used (ref. 9 or 10).

Some attributes of the proposed designs are summarized in table 30. In the case of 4 factors, all coefficients are estimable from two blocks of size 8 and a single block of size 16 is of no advantage in estimating the parameters of a second-degree model. In the case of 7 factors, the attainment of a resolution 5 design requires 64 treatments for either blocks of size 8 or size 16, so that there is no clear advantage is using blocks of size 16. With 8 factors, the minimum first-order design requires 16 treatments, and this is the only block size presented for the problem with 8 factors. In the cases of 5 and 6 factors, the choice of a block size of 8 or 16 is particularly complex.

A comparison of the number of experimental units required in experimenting with block sizes of 8 and 16 for 5 and 6 factors is given in table 31. The column headed "Total number of units required" shows that for five factors, the break-even point for the two block sizes occurs at three repetitions of the first-order experiments. For six factors, the break-even point occurs for five repetitions of the first-degree experiments. In other words, if the experimenter believes that he will perform many cycles of experimenting with the method of steepest ascents, he should use a block size of 8 because it uses a relatively smaller number of experimental units. On the other hand, the block of size 16 uses a relatively smaller number of experimental units in the method of local exploration. The block size of 16 should be used if the experimenter believes he will spend relatively few cycles of experiments with the method of steepest ascents, less than three cycles with 5 factors or less than five cycles with 6 factors.

Maximum economy could be sought with a mixed strategy. The experimenter could use the block of size 8 until his intuition told him that the first-degree model might not be appropriate. He could then switch to the block of size 16. Its greater number of degrees of freedom for "lack of fit" would provide better information about the validity of the first-degree model, and on switching to the method of local exploration, fewer experimental units would be needed to complete the interaction model than if the smaller block had been used. Thus with five factors, one or two experiments of the method of steepest ascents should be performed with the small block size followed by a switch to the larger block. With six factors, the break-even point is not reached until the fifth design center. Furthermore, two blocks of size 8 (table 10) provide a resolution 4 design, whereas the single block of size 16 (table 13) is only a resolution 3 design. With six factors, the best strategy might consist of using blocks of size 8 (table 9) until interactions were suspected, at which point the design could be enlarged to that of table 10. If no new design center were desired, the design could then be augmented to that of table 11. If the design of table 10 had not shown significant interactions, experimenting at a new design center could continue with the design of table 9, but if significant interactions had been shown, the new experimenting should begin with the design of table 13.

CONCLUDING REMARKS. Sequences of blocked designs of experiments have been presented that are telescoping, in the sense that the first block is a design for which main effects are measurable, and that subsequent blocks, as they are added to the design, allow models of successively greater generality to be fitted to all acquired observations at each stage. The sequences terminate in designs for which all two factor interactions are measurable.

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TABLE 1. - FULL 2^4 EXPERIMENT

Treat-ment	Response	Matrix of independent variables															
		X_0	X_1	X_2	X_1X_2	X_3	X_1X_3	X_2X_3	$X_1X_2X_3$	X_4	X_1X_4	X_2X_4	$X_1X_2X_4$	X_3X_4	$X_1X_3X_4$	$X_2X_3X_4$	$X_1X_2X_3X_4$
(1)	y_1	+1	-1	-1	+1	-1	+1	+1	-1	-1	+1	+1	-1	+1	-1	-1	+1
a	y_2	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1	+1	+1	+1	+1	-1	-1
b	y_3	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1	+1	-1	+1	-1
ab	y_4	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1
c	y_5	+1	-1	-1	+1	+1	-1	-1	+1	-1	+1	+1	-1	-1	+1	+1	-1
ac	y_6	+1	+1	-1	-1	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1	+1	+1
bc	y_7	+1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1
abc	y_8	+1	+1	+1	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1
d	y_9	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1
ad	y_{10}	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1
bd	y_{11}	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1
abd	y_{12}	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1
cd	y_{13}	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1
acd	y_{14}	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1
bcd	y_{15}	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1
abcd	y_{16}	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1

Table 2.[a] - PLAN 1/2; 4f; 8t/b; 1b -

R = 4

$[X_0 = X_1X_2X_3X_4.]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	ad	β_1
1	bd	β_2
1	ab	$\beta_{12} + \beta_{34}$
1	cd	β_3
1	ac	$\beta_{13} + \beta_{24}$
1	bc	$\beta_{23} + \beta_{14}$
1	abcd	β_4

[a]Refs. ~~12~~ 9 (p. 484) and ~~13~~ 8 (p. 12-16).

TABLE 3.[a] - PLAN 1; 4f; 8t/b; 2b -

R = 5

[Block confounding, $X_1X_2X_3X_4$.]

Block	Treatment	Estimated effects (b)
1	(1)	β_0
2	a	β_1
2	b	β_2
1	ab	β_{12}
2	c	β_3
1	ac	β_{13}
1	bc	β_{23}
2	abc	β_{123}
2	d	β_4
1	ad	β_{14}
1	bd	β_{24}
2	abd	β_{124}
1	cd	β_{34}
2	acd	β_{134}
2	bcd	β_{234}
1	abcd	β^{*}_{1234}

[a]Refs. ~~12~~ 9 (p. 429) and ~~13~~ 8 (p. 12-10).
[b]Asterisk denotes confounding with blocks.

TABLE 4. - PLAN 1/4; 5f; 8t/b; 1b -

R = 3

$[X_0 = -X_2X_3X_4 = X_1X_2X_3X_5$
$= -X_1X_4X_5.]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	ae	$\beta_1 - \beta_{45}$
1	bde	$\beta_2 - \beta_{34}$
1	abd	$\beta_{12} + \beta_{35}$
1	cde	$\beta_3 - \beta_{24}$
1	acd	$\beta_{13} + \beta_{25}$
1	bc	$-\beta_4 + \beta_{23} + \beta_{15}$
1	abce	$\beta_5 - \beta_{14}$

TABLE 5. - PLAN 1/2; 5f; 8t/b; 2b -

R = 4

$[X_0 = X_1X_2X_3X_5$; block confounding,
$-X_2X_3X_4.]$

Block	Treatment	Estimated effects (a)
1	(1)	β_0
1	ae	β_1
2	be	β_2
2	ab	$\beta_{12} + \beta_{35}$
2	ce	β_3
2	ac	$\beta_{13} + \beta_{25}$
1	bc	$\beta_{23} + \beta_{15}$
1	abce	β_5
2	d	β_4
2	ade	β_{14}
1	bde	β_{24}
1	abd	$\beta_{124} + \beta_{345}$
1	cde	β_{34}
1	acd	$\beta_{134} + \beta_{245}$
2	bcd	$\beta^*_{234} + \beta^*_{145}$
2	abcde	β_{45}

[a]Asterisk denotes confounding with blocks.

TABLE 6. - PLAN 1; 5f; 8t/b; 4b

[Block confounding, $-X_2X_3X_4$, $-X_1X_4X_5$, $X_1X_2X_3X_5$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	e	β_5
4	a	β_1	1	ae	β_{15}
3	b	β_2	2	be	β_{25}
2	ab	β_{12}	3	abe	β_{125}
3	c	β_3	2	ce	β_{35}
2	ac	β_{13}	3	ace	β_{135}
1	bc	β_{23}	4	bce	β_{235}
4	abc	β_{123}	1	abce	β^*_{1235}
2	d	β_4	3	de	β_{45}
3	ad	β_{14}	2	ade	β^*_{145}
4	bd	β_{24}	1	bde	β_{245}
1	abd	β_{124}	4	abde	β_{1245}
4	cd	β_{34}	1	cde	β_{345}
1	acd	β_{134}	4	acde	β_{1345}
2	bcd	β^*_{234}	3	bcde	β_{2345}
3	abcd	β_{1234}	2	abcde	β_{12345}

[a]Asterisk denotes confounding with blocks.

TABLE 7.[a] - PLAN 1/2; 5f; 16t/b; 1b -

R = 5

[$X_0 = -X_1X_2X_3X_4X_5$.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	ae	β_1
1	be	β_2
1	ab	β_{12}
1	ce	β_3
1	ac	β_{13}
1	bc	β_{23}
1	abce	$-\beta_{45}$
1	de	β_4
1	ad	β_{14}
1	bd	β_{24}
1	abde	$-\beta_{35}$
1	cd	β_{34}
1	acde	$-\beta_{25}$
1	bcde	$-\beta_{15}$
1	abcd	$-\beta_5$

[a]Refs. 9 (p. 485) and 6 (p. 12-16).

TABLE 8. - DEFINING CONTRASTS, 6 FACTORS ON

BLOCKS OF 8 TREATMENTS

Source	Defining contrasts		
	1/8 Replicate	1/4 Replicate	1/2 Replicate
X_4^2	$-X_1X_2X_4$		
X_5^2	$-X_2X_3X_5$		
X_6^2	$X_1X_2X_3X_6$	$X_1X_2X_3X_6$	$X_1X_2X_3X_6$
$X_4^2X_5^2$	$X_1X_3X_4X_5$	$X_1X_3X_4X_5$	
$X_4^2X_6^2$	$-X_3X_4X_6$		
$X_5^2X_6^2$	$-X_1X_5X_6$		
$X_4^2X_5^2X_6^2$	$X_2X_4X_5X_6$	$X_2X_4X_5X_6$	

TABLE 9. - PLAN 1/8; 6f; 8t/b; 1b -

R = 3

$$\left[X_0 = -X_1X_2X_4 = -X_2X_3X_5 = X_1X_2X_3X_6 = X_1X_3X_4X_5 = -X_3X_4X_6 = -X_1X_5X_6 = X_2X_4X_5X_6.\right]$$

Block	Treatment	Estimated effects
1	(1)	β_0
1	adf	$\beta_1 - \beta_{24} - \beta_{56}$
1	bdef	$\beta_2 - \beta_{35} - \beta_{14}$
1	abe	$-\beta_4 + \beta_{12} + \beta_{36}$
1	cef	$\beta_3 - \beta_{25} - \beta_{46}$
1	acde	$\beta_{13} + \beta_{26} + \beta_{45}$
1	bcd	$-\beta_5 + \beta_{23} + \beta_{16}$
1	abcf	$\beta_6 - \beta_{15} - \beta_{34}$

TABLE 10. - PLAN 1/4; 6f; 8t/b; 2b - R = 4

[$X_0 = X_1X_2X_3X_6 = X_1X_3X_4X_5 = X_2X_4X_5X_6$; block confounding, $-X_1X_2X_4$.]

Block	Treatment	Estimated effects (a)
1	(1)	β_0
2	aef	β_1
2	bf	β_2
1	abe	$\beta_{12} + \beta_{36}$
1	cef	β_3
2	ac	$\beta_{13} + \beta_{45} + \beta_{26}$
2	bce	$\beta_{23} + \beta_{16}$
1	abcf	β_6
2	de	β_4
1	adf	$\beta_{14} + \beta_{35}$
1	bdef	$\beta_{24} + \beta_{56}$
2	abd	$\beta^*_{124} + \beta^*_{156} + \beta^*_{235} + \beta^*_{346}$
2	cdf	$\beta_{15} + \beta_{34}$
1	acde	β_5
1	bcd	$\beta_{125} + \beta_{146} + \beta_{234} + \beta_{356}$
2	abcdef	$\beta_{25} + \beta_{46}$

[a]Asterisk denotes confounding with blocks.

TABLE 11. - PLAN 1/2; 6f; 8t/b; 4b - R = 4

[$X_0 = X_1X_2X_3X_6$; block confounding, $-X_1X_2X_4$, $-X_2X_3X_5$, $X_1X_3X_4X_5$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	e	β_5
3	af	β_1	2	aef	β_{15}
2	bf	β_2	3	bef	β_{25}
4	ab	$\beta_{12} + \beta_{36}$	1	abe	$\beta_{125} + \beta_{356}$
4	cf	β_3	1	cef	β_{35}
2	ac	$\beta_{13} + \beta_{26}$	3	ace	$\beta_{135} + \beta_{256}$
3	bc	$\beta_{23} + \beta_{16}$	2	bce	$\beta^*_{235} + \beta^*_{156}$
1	abcf	β_6	4	abcef	β_{56}
3	d	β_4	2	de	β_{45}
1	adf	β_{14}	4	adef	β_{145}
4	bdf	β_{24}	1	bdef	β_{245}
2	abd	$\beta^*_{124} + \beta^*_{346}$	3	abde	$\beta_{1245} + \beta_{3456}$
2	cdf	β_{34}	3	cdef	β_{345}
4	acd	$\beta_{134} + \beta_{246}$	1	acde	$\beta^*_{1345} + \beta^*_{2456}$
1	bcd	$\beta_{234} + \beta_{146}$	4	bcde	$\beta_{2345} + \beta_{1456}$
3	abcdf	β_{46}	2	abcdef	β_{456}

[a]Asterisk denotes confounding with blocks.

TABLE 12. - PLAN 1; 6f; 8t/b; 8b

[Block confounding, $-X_1X_2X_4$, $-X_2X_3X_5$, $X_1X_2X_3X_6$, $X_1X_3X_4X_5$, $-X_3X_4X_6$, $-X_1X_5X_6$, $X_2X_4X_5X_6$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	1	β_0	4	e	β_5	8	f	β_6	7	ef	β_{56}
5	a	β_1	6	ae	β_{15}	3	af	β_{16}	2	aef	β^*_{156}
6	b	β_2	5	be	β_{25}	2	bf	β_{26}	3	bef	β_{256}
4	ab	β_{12}	1	abe	β_{125}	7	abf	β_{126}	8	abef	β_{1256}
7	c	β_3	8	ce	β_{35}	4	cf	β_{36}	1	cef	β_{356}
2	ac	β_{13}	3	ace	β_{135}	6	acf	β_{136}	5	acef	β_{1356}
3	bc	β_{23}	2	bce	β^*_{235}	5	bcf	β_{236}	6	bcef	β_{2356}
8	abc	β_{123}	7	abce	β_{1235}	1	abcf	β^*_{1236}	4	abcef	β_{12356}
3	d	β_4	2	de	β_{45}	5	df	β_{46}	6	def	β_{456}
8	ad	β_{14}	7	ade	β_{145}	1	adf	β_{146}	4	adef	β_{1456}
7	bd	β_{24}	8	bde	β_{245}	4	bdf	β_{246}	1	bdef	β^*_{2456}
2	abd	β^*_{124}	3	abde	β_{1245}	6	abdf	β_{1246}	5	abdef	β_{12456}
6	cd	β_{34}	5	cde	β_{345}	2	cdf	β^*_{346}	3	cdef	β_{3456}
4	acd	β_{134}	1	acde	β^*_{1345}	7	acdf	β_{1346}	8	acdef	β_{13456}
1	bcd	β_{234}	4	bcde	β_{2345}	8	bcdf	β_{2346}	7	bcdef	β_{23456}
5	abcd	β_{1234}	6	abcde	β_{12345}	3	abcdf	β_{12346}	2	abcdef	β_{123456}

[a]Asterisk denotes confounding with blocks.

TABLE 13. - PLAN 1/4; 6f; 16t/b; 1b -

R = 3

$[X_0 = -X_3X_4X_5 = -X_1X_2X_6 = X_1X_2X_3X_4X_5X_6.]$

Block	Treatment	Estimated effects
1	(1)	β_0
1	af	$\beta_1 - \beta_{26}$
1	bf	$\beta_2 - \beta_{16}$
1	ab	$-\beta_6 + \beta_{12}$
1	ce	$\beta_3 - \beta_{45}$
1	acef	β_{13}
1	bcef	β_{23}
1	abce	$-\beta_{36}$
1	de	$\beta_4 - \beta_{35}$
1	adef	β_{14}
1	bdef	β_{24}
1	abde	$-\beta_{46}$
1	cd	$-\beta_5 + \beta_{34}$
1	acdf	$-\beta_{15}$
1	bcdf	$-\beta_{25}$
1	abcd	β_{56}

TABLE 14.[a] - PLAN 1/2; 6f; 16t/b; 2b - R = 5

$[X_0 = X_1X_2X_3X_4X_5X_6;$ block confounding, $-X_3X_4X_5.]$

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	ef	β_5
1	af	β_1	2	ae	β_{15}
1	bf	β_2	2	be	β_{25}
1	ab	β_{12}	2	abef	$\beta_{125} + \beta_{346}$
2	cf	β_3	1	ce	β_{35}
2	ac	β_{13}	1	acef	$\beta_{135} + \beta_{246}$
2	bc	β_{23}	1	bcef	$\beta_{235} + \beta_{146}$
2	abcf	$\beta_{123} + \beta_{456}$	1	abce	β_{46}
2	df	β_4	1	de	β_{45}
2	ad	β_{14}	1	adef	$\beta_{145} + \beta_{236}$
2	bd	β_{24}	1	bdef	$\beta_{245} + \beta_{136}$
2	abdf	$\beta_{124} + \beta_{356}$	1	abde	β_{36}
1	cd	β_{34}	2	cdef	$\beta^*_{345} + \beta^*_{126}$
1	acdf	$\beta_{134} + \beta_{256}$	2	acde	β_{26}
1	bcdf	$\beta_{234} + \beta_{156}$	2	bcde	β_{16}
1	abcd	β_{56}	2	abcdef	β_6

[a] [redacted]).
[b] Asterisk denotes confounding with blocks.

TABLE 15. - DEFINING CONTRASTS WITH 7 FACTORS ON BLOCKS OF 8 TREATMENTS

Source	Defining contrasts			
	1/16 Replicate	1/8 Replicate	1/4 Replicate	1/2 Replicate
X_4^2	$-X_1X_2X_4$			
X_5^2	$-X_1X_3X_5$	$-X_1X_3X_5$		
X_6^2	$-X_2X_3X_6$			
X_7^2	$X_1X_2X_3X_7$	$X_1X_2X_3X_7$		
$X_4^2X_5^2$	$X_2X_3X_4X_5$			
$X_4^2X_6^2$	$X_1X_3X_4X_6$	$X_1X_3X_4X_6$	$X_1X_3X_4X_6$	
$X_4^2X_7^2$	$-X_3X_4X_7$			
$X_5^2X_6^2$	$X_1X_2X_5X_6$			
$X_5^2X_7^2$	$-X_2X_5X_7$	$-X_2X_5X_7$	$-X_2X_5X_7$	
$X_6^2X_7^2$	$-X_1X_6X_7$			
$X_4^2X_5^2X_6^2$	$-X_4X_5X_6$	$-X_4X_5X_6$		
$X_4^2X_5^2X_7^2$	$X_1X_4X_5X_7$			
$X_4^2X_6^2X_7^2$	$X_2X_4X_6X_7$	$X_2X_4X_6X_7$		
$X_5^2X_6^2X_7^2$	$X_3X_5X_6X_7$			
$X_4^2X_5^2X_6^2X_7^2$	$-X_1X_2X_3X_4X_5X_6X_7$	$-X_1X_2X_3X_4X_5X_6X_7$	$-X_1X_2X_3X_4X_5X_6X_7$	$-X_1X_2X_3X_4X_5X_6X_7$

TABLE 16. - PLAN 1/16; 7f; 8t/b; 1b -

R = 3

[Defining contrasts given by table 15.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	adeg	$\beta_1 - \beta_{24} - \beta_{35} - \beta_{67}$
1	bdfg	$\beta_2 - \beta_{14} - \beta_{36} - \beta_{57}$
1	abef	$-\beta_4 + \beta_{12} + \beta_{37} + \beta_{56}$
1	cefg	$\beta_3 - \beta_{15} - \beta_{26} - \beta_{47}$
1	acdf	$-\beta_5 + \beta_{13} + \beta_{46} + \beta_{27}$
1	bcde	$-\beta_6 + \beta_{23} + \beta_{17} + \beta_{45}$
1	abcg	$\beta_7 - \beta_{34} - \beta_{25} - \beta_{16}$

TABLE 17. - PLAN 1/8; 7f; 8t/b; 2b - R = 3

[Defining contrasts given by table 15; block confounding, $-X_1X_2X_4$.]

Block	Treatment	Estimated effects (a)
1	(1)	β_0
2	aefg	$\beta_1 - \beta_{35}$
2	bg	$\beta_2 - \beta_{57}$
1	abef	$\beta_{12} + \beta_{37}$
1	cefg	$\beta_3 - \beta_{15}$
2	ac	$-\beta_5 + \beta_{13} + \beta_{46} + \beta_{27}$
2	bcef	$\beta_{23} + \beta_{17}$
1	abcg	$\beta_7 - \beta_{25}$
2	df	$\beta_4 - \beta_{56}$
1	adeg	$\beta_{14} + \beta_{36}$
1	bdfg	$\beta_{24} + \beta_{67}$
2	abde	$\beta^*_{124} + \beta^*_{347} + \beta^*_{236} + \beta^*_{167}$
2	cdeg	$\beta_{34} + \beta_{16}$
1	acdf	$\beta_6 - \beta_{45}$
1	bcde	$\beta_{147} + \beta_{126} + \beta_{367} + \beta_{234}$
2	abcdfg	$\beta_{47} + \beta_{26}$

[a]Asterisk denotes confounding with blocks.

TABLE 18. - PLAN 1/4; 7f; 8t/b; 4b - R = 3

[Defining contrasts given by table 15; block confounding, $-X_1X_2X_4$, $-X_1X_3X_5$, $X_2X_3X_4X_5$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	eg	$\beta_5 - \beta_{27}$
3	af	β_1	2	aefg	β_{15}
2	bg	$\beta_2 - \beta_{57}$	3	be	$-\beta_7 + \beta_{25}$
4	abfg	β_{12}	1	abef	$-\beta_{17}$
4	cf	β_3	1	cefg	β_{35}
2	ac	$\beta_{13} + \beta_{46}$	3	aceg	$\beta^*_{135} + \beta^*_{456}$
3	bcfg	β_{23}	2	bcef	$-\beta_{37}$
1	abcg	$\beta_{123} + \beta_{246}$	4	abce	$-\beta_{137} - \beta_{467}$
2	df	β_4	3	defg	β_{45}
4	ad	$\beta_{14} + \beta_{36}$	1	adeg	$\beta_{145} + \beta_{356}$
1	bdfg	β_{24}	4	bdef	$-\beta_{47}$
3	abdg	$\beta^*_{124} + \beta^*_{236}$	2	abde	$-\beta_{147} - \beta_{367}$
3	cd	$\beta_{34} + \beta_{16}$	2	cdeg	$\beta_{345} + \beta_{156}$
1	acdf	β_6	4	acdefg	β_{56}
4	bcdg	$\beta_{234} + \beta_{126}$	1	bcde	$-\beta^*_{347} - \beta^*_{167}$
2	abcdfg	β_{26}	3	abcdef	$-\beta_{67}$

[a]Asterisk denotes confounding with blocks.

TABLE 19. - PLAN 1/2; 7f; 8t/b, 8b

[$X_0 = -X_1X_2X_3X_4X_5X_6X_7$; block confounding, $-X_1X_2X_4$, $-X_1X_3X_5$, $-X_2X_3X_6$, $X_2X_3X_4X_5$, $X_1X_3X_4X_6$, $X_1X_2X_5X_6$, $-X_4X_5X_6$.]

Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	4	eg	β_5	7	fg	β_6	5	ef	β_{56}
6	ag	β_1	8	ae	β_{15}	3	af	β_{16}	2	aefg	β_{156}
2	bg	β_2	3	be	β_{25}	8	bf	β_{26}	6	befg	β_{256}
5	ab	β_{12}	7	abeg	β_{125}	4	abfg	β_{126}	1	abef	$-\beta^*_{347}$
5	cg	β_3	7	ce	β_{35}	4	cf	β_{36}	1	cefg	β_{356}
2	ac	β_{13}	3	aceg	β^*_{135}	8	acfg	β_{136}	6	acef	$-\beta_{247}$
6	bc	β_{23}	8	bceg	β_{235}	3	bcfg	β^*_{236}	2	bcef	$-\beta_{147}$
1	abcg	β_{123}	4	abce	$-\beta_{467}$	7	abcf	$-\beta_{457}$	5	abcefg	$-\beta_{47}$
8	dg	β_4	6	de	β_{45}	2	df	β_{46}	3	defg	β^*_{456}
4	ad	β_{14}	1	adeg	β_{145}	5	adfg	β_{146}	7	adef	$-\beta_{237}$
7	bd	β_{24}	5	bdeg	β_{245}	1	bdfg	β_{246}	4	bdef	$-\beta_{137}$
3	abdg	β^*_{124}	2	abde	$-\beta_{367}$	6	abdf	$-\beta_{357}$	8	abdefg	$-\beta_{37}$
3	cd	β_{34}	2	cdeg	β_{345}	6	cdfg	β_{346}	8	cdef	$-\beta_{127}$
7	acdg	β_{134}	5	acde	$-\beta_{267}$	1	acdf	$-\beta^*_{257}$	4	acdefg	$-\beta_{27}$
4	bcdg	β_{234}	1	bcde	$-\beta^*_{167}$	5	bcdf	$-\beta_{157}$	7	bcdefg	$-\beta_{17}$
8	abcd	$-\beta_{567}$	6	abcdeg	$-\beta_{67}$	2	abcdfg	$-\beta_{57}$	3	abcdef	$-\beta_7$

[a] Asterisk denotes confounding with blocks.

TABLE 20. - DEFINING CONTRASTS WITH 7 FACTORS

ON BLOCKS OF 16 TREATMENTS

Source	Defining contrasts		
	1/8 Replicate	1/4 Replicate	1/2 Replicate
X_5^2	$-X_1X_4X_5$		
X_6^2	$X_1X_2X_4X_6$	$X_1X_2X_4X_6$	
X_7^2	$X_2X_3X_4X_7$		
$X_5^2X_6^2$	$-X_2X_5X_6$		
$X_5^2X_7^2$	$-X_1X_2X_3X_5X_7$	$-X_1X_2X_3X_5X_7$	
$X_6^2X_7^2$	$X_1X_3X_6X_7$		
$X_5^2X_6^2X_7^2$	$-X_3X_4X_5X_6X_7$	$-X_3X_4X_5X_6X_7$	$-X_3X_4X_5X_6X_7$

TABLE 21. - PLAN 1/8; 7f; 16t/b; 1b -

R = 3

[Defining contrasts given in table 20.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	aef	$\beta_1 - \beta_{45}$
1	bfg	$\beta_2 - \beta_{56}$
1	abeg	$\beta_{12} + \beta_{46}$
1	cg	β_3
1	acefg	$\beta_{13} + \beta_{67}$
1	bcf	$\beta_{23} + \beta_{47}$
1	abce	$-\beta_{57}$
1	defg	$\beta_4 - \beta_{15}$
1	adg	$-\beta_5 + \beta_{14} + \beta_{26}$
1	bde	$\beta_{24} + \beta_{16} + \beta_{37}$
1	abdf	$\beta_6 - \beta_{25}$
1	cdef	$\beta_{34} + \beta_{27}$
1	acd	$-\beta_{35}$
1	bcdeg	β_7
1	abcdfg	$\beta_{36} + \beta_{17}$

TABLE 22.[a] - PLAN 1/4; 7f; 16t/b; 2b - R = 4

[Defining contrasts given in table 20; block confounding, $-X_1X_4X_5$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	eg	β_5
2	afg	β_1	1	aef	β_{15}
1	bfg	β_2	2	bef	β_{25}
2	ab	$\beta_{12} + \beta_{46}$	1	abeg	$-\beta_{37}$
1	cg	β_3	2	ce	β_{35}
2	acf	β_{13}	1	acefg	$-\beta_{27}$
1	bcf	β_{23}	2	bcefg	$-\beta_{17}$
2	abcg	$-\beta_{57}$	1	abce	$-\beta_7$
2	df	β_4	1	defg	β_{45}
1	adg	$\beta_{14} + \beta_{26}$	2	ade	$\beta^*_{145} + \beta^*_{256}$
2	bdg	$\beta_{24} + \beta_{16}$	1	bde	$\beta_{245} + \beta_{156}$
1	abdf	β_6	2	abdefg	β_{56}
2	cdfg	β_{34}	1	cdef	$-\beta_{67}$
1	acd	$\beta_{134} + \beta_{236}$	2	acdeg	$-\beta_{247} - \beta_{167}$
2	bcd	$\beta_{234} + \beta_{136}$	1	bcdeg	$-\beta_{147} - \beta_{267}$
1	abcdfg	β_{36}	2	abcdef	$-\beta_{47}$

[a] ~~Ref. 4 (p. 20)~~.

[b] Asterisk denotes confounding with blocks.

TABLE 23. - PLAN 1/2; 7f; 16t/b; 4b - R = 5

[$X_0 = -X_3X_4X_5X_6X_7$; block confounding, $-X_1X_4X_5$, $-X_2X_5X_6$, $X_1X_2X_4X_6$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)	Block	Treatment	Estimated effects (a)
1	(1)	β_0	2	eg	β_5	4	fg	β_6	3	ef	β_{56}
3	a	β_1	4	aeg	β_{15}	2	afg	β_{16}	1	aef	β_{156}
4	b	β_2	3	beg	β_{25}	1	bfg	β_{26}	2	bef	β^*_{256}
2	ab	β_{12}	1	abeg	β_{125}	3	abfg	β_{126}	4	abef	β_{1256}
1	cg	β_3	2	ce	β_{35}	4	cf	β_{36}	3	cefg	$-\beta_{47}$
3	acg	β_{13}	4	ace	β_{135}	2	acf	β_{136}	1	acefg	$-\beta_{147}$
4	bcg	β_{23}	3	bce	β_{235}	1	bcf	β_{236}	2	bcefg	$-\beta_{247}$
2	abcg	β_{123}	1	abce	β_{1235}	3	abcf	β_{1236}	4	abcefg	$-\beta_{1247}$
3	dg	β_4	4	de	β_{45}	2	df	β_{46}	1	defg	$-\beta_{37}$
1	adg	β_{14}	2	ade	β^*_{145}	4	adf	β_{146}	3	adefg	$-\beta_{137}$
2	bdg	β_{24}	1	bde	β_{245}	3	bdf	β_{246}	4	bdefg	$-\beta_{237}$
4	abdg	β_{124}	3	abde	β_{1245}	1	abdf	β^*_{1246}	2	abdefg	$-\beta_{1237}$
3	cd	β_{34}	4	cdeg	$-\beta_{67}$	2	cdfg	$-\beta_{57}$	1	cdef	$-\beta_7$
1	acd	β_{134}	2	acdeg	$-\beta_{167}$	4	acdfg	$-\beta_{157}$	3	acdef	$-\beta_{17}$
2	bcd	β_{234}	1	bcdeg	$-\beta_{267}$	3	bcdfg	$-\beta_{257}$	4	bcdef	$-\beta_{27}$
4	abcd	β_{1234}	3	abcdeg	$-\beta_{1267}$	1	abcdfg	$-\beta_{1257}$	2	abcdef	$-\beta_{127}$

[a]Asterisk denotes confounding with blocks.

TABLE 24. - DEFINING CONTRASTS WITH 8 FACTORS

ON BLOCKS OF 16 TREATMENTS

Source	Defining contrasts		
	1/16 Replicate	1/8 Replicate	1/4 Replicate
X_5^2	$-X_1X_4X_5$		
X_6^2	$X_1X_3X_4X_6$	$X_1X_3X_4X_6$	
X_7^2	$X_2X_3X_4X_7$		
X_8^2	$-X_2X_3X_8$	$-X_2X_3X_8$	
$X_5^2X_6^2$	$-X_3X_5X_6$		
$X_5^2X_7^2$	$-X_1X_2X_3X_5X_7$	$-X_1X_2X_3X_5X_7$	$-X_1X_2X_3X_5X_7$
$X_5^2X_8^2$	$X_1X_2X_3X_4X_5X_8$		
$X_6^2X_7^2$	$X_1X_2X_6X_7$		
$X_6^2X_8^2$	$-X_1X_2X_4X_6X_8$	$-X_1X_2X_4X_6X_8$	$-X_1X_2X_4X_6X_8$
$X_7^2X_8^2$	$-X_4X_7X_8$		
$X_5^2X_6^2X_7^2$	$-X_2X_4X_5X_6X_7$	$-X_2X_4X_5X_6X_7$	
$X_5^2X_6^2X_8^2$	$X_2X_5X_6X_8$		
$X_5^2X_7^2X_8^2$	$X_1X_5X_7X_8$	$X_1X_5X_7X_8$	
$X_6^2X_7^2X_8^2$	$-X_1X_3X_6X_7X_8$		
$X_5^2X_6^2X_7^2X_8^2$	$X_3X_4X_5X_6X_7X_8$	$X_3X_4X_5X_6X_7X_8$	$X_3X_4X_5X_6X_7X_8$

TABLE 25. - PLAN 1/16; 8f; 16t/b; 1b -

R = 3

[Defining contrasts given in table 24.]

Block	Treatment	Estimated effects
1	(1)	β_0
1	aef	$\beta_1 - \beta_{45}$
1	bgh	$\beta_2 - \beta_{38}$
1	abefgh	$\beta_{12} + \beta_{67}$
1	cfgh	$\beta_3 - \beta_{28} - \beta_{56}$
1	acegh	$\beta_{13} + \beta_{46}$
1	bcf	$-\beta_8 + \beta_{23} + \beta_{47}$
1	abce	$-\beta_{57} - \beta_{18}$
1	defg	$\beta_4 - \beta_{15} - \beta_{78}$
1	adg	$-\beta_5 + \beta_{14} + \beta_{36}$
1	bdefh	$\beta_{24} + \beta_{37}$
1	abdh	$-\beta_{68} - \beta_{25}$
1	cdeh	$\beta_{34} + \beta_{16} + \beta_{27}$
1	acdfh	$\beta_6 - \beta_{35}$
1	bcdeg	$\beta_7 - \beta_{48}$
1	abcdfg	$\beta_{58} + \beta_{17} + \beta_{26}$

TABLE 26. - PLAN 1/8; 8f; 16t/b; 2b - R = 3

[Defining contrasts given in table 24; block confounding, $-X_1X_4X_5$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (a)
1	(1)	β_0	2	eg	β_5
2	afg	β_1	1	aef	$\beta_{15} + \beta_{78}$
1	bgh	$\beta_2 - \beta_{38}$	2	beh	β_{25}
2	abfh	β_{12}	1	abefgh	$-\beta_{37}$
1	cfgh	$\beta_3 - \beta_{28}$	2	cefh	β_{35}
2	ach	$\beta_{13} + \beta_{46}$	1	acegh	$-\beta_{27}$
1	bcf	$-\beta_8 + \beta_{23}$	2	bcefg	$-\beta_{58} - \beta_{17}$
2	abcg	$-\beta_{57} - \beta_{18}$	1	abce	$-\beta_7$
2	df	β_4	1	defg	β_{45}
1	adg	$\beta_{14} + \beta_{36}$	2	ade	$\beta^*_{145} + \beta^*_{356} + \beta^*_{478}$
2	bdfgh	β_{24}	1	bdefh	$-\beta_{67}$
1	abdh	$-\beta_{68}$	2	abdegh	$-\beta_{347} - \beta_{568} - \beta_{167}$
2	cdgh	$\beta_{34} + \beta_{16}$	1	cdeh	$\beta_{345} + \beta_{156} + \beta_{678}$
1	acdfh	β_6	2	acdefgh	β_{56}
2	bcd	$-\beta_{48}$	1	bcdeg	$-\beta_{147} - \beta_{367} - \beta_{458}$
1	abcdfg	β_{26}	2	abcdef	$-\beta_{47}$

[a]Asterisk denotes confounding with blocks.

TABLE 27.[a] - PLAN 1/4; 8f; 16t/b; 4b - R = 5

[Defining contrasts given in table 24; block confounding, $-X_1X_4X_5$, $X_1X_3X_4X_6$, $-X_3X_5X_6$.]

Block	Treatment	Estimated effects	Block	Treatment	Estimated effects (b)	Block	Treatment	Estimated effects (b)	Block	Treatment	Estimated effects (b)
1	(1)	β_0	2	eg	β_5	4	fh	β_6	3	efgh	β_{56}
3	agh	β_1	4	aeh	β_{15}	2	afg	β_{16}	1	aef	β_{156}
1	bgh	β_2	2	beh	β_{25}	4	bfg	β_{26}	3	bef	β_{256}
3	ab	β_{12}	4	abeg	$-\beta_{37}$	2	abfh	$-\beta_{48}$	1	abefgh	$-\beta_{367} - \beta_{458}$
4	cg	β_3	3	ce	β_{35}	1	cfgh	β_{36}	2	cefh	$\beta^*_{356} + \beta^*_{478}$
2	ach	β_{13}	1	acegh	$-\beta_{27}$	3	acf	β_{136}	4	acefg	$-\beta_{267}$
4	bch	β_{23}	3	bcegh	$-\beta_{17}$	1	bcf	β_{236}	2	bcefg	$-\beta_{167}$
2	abcg	$-\beta_{57}$	1	abce	$-\beta_7$	3	abcfgh	$-\beta_{567} - \beta_{348}$	4	abcefh	$-\beta_{67}$
3	dh	β_4	4	degh	β_{45}	2	df	β_{46}	1	defg	$\beta_{456} + \beta_{378}$
1	adg	β_{14}	2	ade	β^*_{145}	4	adfgh	$-\beta_{28}$	3	adefh	$-\beta_{258}$
3	bdg	β_{24}	4	bde	β_{245}	2	bdfgh	$-\beta_{18}$	1	bdefh	$-\beta_{158}$
1	abdh	$-\beta_{68}$	2	abdegh	$-\beta_{347} - \beta_{568}$	4	abdf	$-\beta_8$	3	abdefg	$-\beta_{58}$
2	cdgh	β_{34}	1	cdeh	$\beta_{345} + \beta_{678}$	3	cdfg	$\beta_{346} + \beta_{578}$	4	cdef	β_{78}
4	acd	β_{134}	3	acdeg	$-\beta_{247}$	1	acdfh	$-\beta^*_{238}$	2	acdefgh	β_{178}
2	bcd	β_{234}	1	bcdeg	$-\beta_{147}$	3	bcdfh	$-\beta_{138}$	4	bcdefgh	β_{278}
4	abcdgh	$-\beta_{457} - \beta_{368}$	3	abcdeh	$-\beta_{47}$	1	abcdfg	$-\beta_{38}$	2	abcdef	$-\beta_{467} - \beta_{358}$

[a] ~~Ref. 4 (p. 21)~~.
[b] Asterisk denotes confounding with blocks.

TABLE 28.[a] - PLAN 1/8; 7f; 16t/b; 1b - R = 4

$$[X_0 = X_1X_2X_3X_5 = X_2X_3X_4X_6 = X_1X_2X_4X_7 = X_1X_4X_5X_6 = X_3X_4X_5X_7 = X_1X_3X_6X_7 = X_2X_5X_6X_7.]$$

Block	Treatment	Estimated effects
1	(1)	β_0
1	aeg	β_1
1	befg	β_2
1	abf	$\beta_{12} + \beta_{35} + \beta_{47}$
1	cef	β_3
1	acfg	$\beta_{13} + \beta_{25} + \beta_{67}$
1	bcg	$\beta_{23} + \beta_{15} + \beta_{46}$
1	abce	β_5
1	dfg	β_4
1	adef	$\beta_{14} + \beta_{56} + \beta_{27}$
1	bde	$\beta_{24} + \beta_{36} + \beta_{17}$
1	abdg	β_7
1	cdeg	$\beta_{26} + \beta_{34} + \beta_{57}$
1	acd	$\beta_{134} + \beta_{126} + \beta_{237} + \beta_{245} + \beta_{356} + \beta_{467} + \beta_{157}$
1	bcdf	β_6
1	abcdefg	$\beta_{45} + \beta_{16} + \beta_{37}$

[a]Refs. 9 (p. 486) and 8 (p. 12-18).

TABLE 29.[a] - PLAN 1/16; 8f; 16t/b; 1b - R = 4

$$[X_0 = X_1X_2X_3X_5 = X_2X_3X_4X_6 = X_1X_2X_4X_7 = X_1X_3X_4X_8 = X_1X_4X_5X_6 = X_3X_4X_5X_7 = X_2X_4X_5X_8 = X_1X_3X_6X_7 = X_1X_2X_6X_8 = X_2X_3X_7X_8 = X_2X_5X_6X_7 = X_3X_5X_6X_8 = X_1X_5X_7X_8 = X_4X_6X_7X_8 = X_1X_2X_3X_4X_5X_6X_7X_8$$

Block	Treatment	Estimated effects
1	(1)	β_0
1	aegh	β_1
1	befg	β_2
1	abfh	$\beta_{12} + \beta_{35} + \beta_{47} + \beta_{68}$
1	cefh	β_3
1	acfg	$\beta_{13} + \beta_{25} + \beta_{48} + \beta_{67}$
1	bcgh	$\beta_{23} + \beta_{15} + \beta_{46} + \beta_{78}$
1	abce	β_5
1	dfgh	β_4
1	adef	$\beta_{14} + \beta_{27} + \beta_{38} + \beta_{56}$
1	bdeh	$\beta_{24} + \beta_{36} + \beta_{17} + \beta_{58}$
1	abdg	β_7
1	cdeg	$\beta_{34} + \beta_{26} + \beta_{18} + \beta_{57}$
1	acdh	β_8
1	bcdf	β_6
1	abcdefgh	$\beta_{45} + \beta_{16} + \beta_{37} + \beta_{28}$

[a]Refs. 9 (p. 486) and 8 (p. 12-18).

TABLE 30. - ATTRIBUTES OF RECOMMENDED DESIGNS

Table	Replication	Factors, g	Treatments per block	Number of blocks	Resolution, R	Number of two-factor interactions, g(g - 1)/2	Number of estimable two-factor interactions (a)
2	1/2	4	8	1	4	6	0
3	Full	4	8	2	5	6	6
4	1/4	5	8	1	3	10	0
5	1/2	5	8	2	4	10	4
6	Full	5	8	4	5	10	10
7	1/2	5	16	1	5	10	10
9	1/8	6	8	1	3	15	0
10	1/4	6	8	2	4	15	0
11	1/2	6	8	4	4	15	9
12	Full	6	8	8	5	15	15
13	1/4	6	16	1	3	15	9
14	1/2	6	16	2	5	15	15
16	1/16	7	8	1	3	21	0
17	1/8	7	8	2	3	21	0
18	1/4	7	8	4	3	21	11
19	1/2	7	8	8	5	21	21
21	1/8	7	16	1	3	21	1
22	1/4	7	16	2	4	21	15
23	1/2	7	16	4	5	21	21
25	1/16	8	16	1	3	28	0
26	1/8	8	16	2	3	28	11
27	1/4	8	16	4	5	28	28
28	1/8	7	16	1	4	21	0
29	1/16	8	16	1	4	28	0

[a]Only unconfounded two-factor interaction estimators are counted.

TABLE 31. - COMPARISON OF TOTAL TREATMENTS (EXPERIMENTAL UNITS) REQUIRED WHEN FIRST BLOCK IS PERFORMED TO ESTIMATE FIRST-ORDER MODEL AT STATED NUMBER OF DESIGN CENTERS AND INTERACTION EXPERIMENT IS PERFORMED ONLY AT FINAL DESIGN CENTER

Factors	Design centers for first-order model	Treatments for first-order model		Treatments for completion of interaction model		Total number of units required	
		Blocks of size 8	Blocks of size 16	Blocks of size 8	Blocks of size 16	Blocks of size 8	Blocks of size 16
5	1	8	16	24	0	32	16
5	2	16	32	24	0	40	32
5	3	24	48	24	0	48	48
5	4	32	64	24	0	56	64
6	1	8	16	56	16	64	32
6	2	16	32	56	16	72	48
6	3	24	48	56	16	80	64
6	4	32	64	56	16	88	80
6	5	40	80	56	16	96	96
6	6	48	96	56	16	104	112